Introduction to quantum groups

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Book available from Springer, 1st edition (January 2023), 435 pages

2010 *Mathematics Subject Classification.* 46L65

Key words and phrases. Quantum space, Quantum group

Abstract. This is an introduction to the quantum groups, or rather to the simplest quantum groups. The idea is that the unitary group U_N has a free analogue U_N^+, whose standard coordinates $u_{ij}\in C(U_N^+)$ are allowed to be free, and the closed subgroups $G\subset U_N^+$ can be thought of as being the compact quantum Lie groups. There are many interesting examples of such quantum groups, for the most designed in order to help with questions in quantum mechanics and statistical mechanics, and some general theory available as well, including Peter-Weyl theory, Tannakian duality, Brauer theorems and Weingarten integration. We discuss here the basic aspects of all this.

Preface

A quantum group is something similar to a group, except for the fact that the functions on it $f : G \to \mathbb{C}$ do not necessarily commute, $fg \neq gf$. As the name indicates, quantum groups are meant to have something with do with quantum physics. To be more precise, $fg \neq gf$, which mathematically might sound like some kind of pathology, is in fact something beautiful, reminding Heisenberg's uncertainty principle, and which puts the quantum groups in good position of describing the "symmetries" of quantum systems.

Such ideas go back to the work of Faddeev and the Leningrad School of physics [**55**], from the late 70s. Later on, during the 80s, Drinfeld [**54**] and Jimbo [**64**] on one hand, and Woronowicz [**99**], [**100**] on the other, came up with some precise mathematics for the quantum groups. This mathematics has become increasingly popular during the 90s and 00s, to the point that we have now all sorts of classes of quantum groups, with some of them having nothing much to do with the original physics motivations.

Which reminds a bit the story of particle physics, in its golden era. Things back then used to be quite wild, to the point that Willis Lamb started his Nobel Prize acceptance speech in 1955 by saying that "while the finder of a new elementary particle used to be rewarded by receiving a Nobel Prize, one should now be punished by a $10,000 fine".

So, what are the good and useful quantum groups? No one really knows the answer here, but personally I would put my money on the "simplest". In all honesty, I don't believe that God is a bad person, and I'm convinced that he created this world such that simple mathematics corresponds to simple physics, and vice versa.

But what are then the simplest quantum groups? This question is far more tricky. You would say that the simplest groups are the compact Lie groups $G \subset U_N$, and so that a quantum group should be something similar, namely some kind of "smooth compact noncommutative manifold, endowed with a group-type structure".

And is this correct or not. Mathematically speaking, this sounds good, but if you're really passioned by physics, as I personally am, there is a problem here. Physics tells us that smoothness is some sort of miracle, appearing via complicated $N \to \infty$ limiting procedures, and only in the classical, macroscopic setting. Indeed, isn't that true for everything thermodynamics, where smoothness comes from collisions, in the $N \to \infty$ limit.

And for electrodynamics too, where smoothness comes from certain things happening at the QED level, once again with $N \to \infty$. And finally, probably for classical mechanics too, because who really knows what gravity looks like at the Planck scale, and so again, the smoothness in classical mechanics might well come from a $N \to \infty$ procedure.

Of course, all this is a bit subjective, but you have to agree with me that, if we want the quantum groups to do their intended job, namely be of help in quantum mechanics, shall we really go head-first into Lie theory and smoothness, or be a bit more philosophers, and look for something else. Ideally, some mixture of algebra and probability, since basic quantum mechanics itself is after all a cocktail of algebra and probability.

And fortunately, math comes to the rescue. Forgetting now about physics, and about anything advanced, let us just look at the unitary group U_N. The standard coordinates $u_{ij} \in C(U_N)$ obviously commute, but if we allow them to be free, we obtain a certain algebra $C(U_N^+)$, and so a certain quantum group U_N^+. And then, in analogy with the fact that any compact Lie group appears as a closed subgroup $G \subset U_N$, we can say that the closed quantum subgroups $G \subset U_N^+$ are the "compact quantum Lie groups".

The present book is an introduction to such quantum groups, $G \subset U_N^+$. We will see that there are many interesting examples, worth studying, and also, following Woronowicz [99], [100] and others, that some substantial general theory can be developed for such quantum groups, including an existence result for the Haar measure, Peter-Weyl theory, Tannakian duality, Brauer theorems and Weingarten integration.

We will insist on examples, and more specifically on examples designed in order to help with questions in quantum mechanics, and statistical mechanics.

The presentation will be elementary, with the present book being a standard, first year graduate level textbook. More advanced aspects are discussed in my "Quantum permutation groups" research monograph [10]. As for the applications to physics, these will be discussed in a series of physics books, the first of which, "Introduction to quantum mechanics", having actually no quantum groups inside, is in preparation [11].

The mathematics in this book will be based on a number of research papers, starting with those of Woronowicz from the late 80s. I was personally involved in all this, during the last 30 years, and it is a pleasure to thank my coworkers, and particularly Julien Bichon, Benoît Collins, Steve Curran and Roland Speicher, for substantial joint work on the subject. Many thanks go as well to my cats, for sharing with me some of their quantum mechanical knowledge, cat common sense, and other skills.

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