Almost classical geometries

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Abstract. This is an introduction to the various geometries which are close to our usual geometry, with particular attention to the half-classical geometry, and the twisted geometry. We first discuss axiomatic aspects, with the aim of understanding what almost classical, for an abstract geometry, should mean. This leads us to the conclusion that the half-liberation operation should be investigated first, and we go on, with a detailed study of the half-classical geometry. Afterwards, with motivation and input coming from anticommutation, we present a similar study, for the twisted geometry. Finally, we discuss a number of more advanced aspects, in relation with various axiomatic questions, and with the geometric and analytic study of the manifolds that we found.

Preface

Classical geometry comes in many flavors, namely real and complex, affine and projective, algebraic and differential, and so on. In all cases what we have to do is to extend our knowledge about $\mathbb{R}^N$, learned the hard way in multivariable calculus, to certain classes of manifolds X. And things can be quite tricky here, with the mathematics of the points $x \in X$ being sometimes replaced, for the better, by that of the functions $f : X \to \mathbb{R}$, or that of the many other more specialized algebraic objects associated to X.

But you surely know all this, and you surely know too that classical geometry has always been very useful for understanding classical mechanics, and its various ramifications, such as classical electrodynamics, classical thermodynamics, and so on. In modern times, as a main success of classical geometry, we have for instance the Einstein unification of space $\mathbb{R}^3$ and time $\mathbb{R}$, into something which is a manifold, curved version of $\mathbb{R}^4$.

As bad news, however, and with this twisting the minds of mathematicians and physicists since the 1920s, classical geometry does not describe well quantum mechanics. The damn little particles there are not totally wild, and seem to obey to some sort of geometry, but this geometry is of a new type, having built in some contradictions, with for instance some of the functions there $f, g : X \to \mathbb{R}$ being subject to non-commutation:

$$fg \neq gf$$

And so, what to do. In answer, develop of course all sorts of geometric theories not based on $fg = gf$, and as a consequence, not based on points $x \in X$ either, and expect a lot of tricky algebra here, mixed with some tricky analysis too, and with all this by keeping an eye on physics and objectives, with the aim of reaching for the jackpot. With the jackpot being a good understanding of the Standard Model of particle physics, say by saying something new on the various masses and constants involved there.

So, this was for the problems coming from quantum mechanics, and in what regards the solutions, in the lack of anything spectacular so far, despite long decades of hard work by mathematicians and physicists, but let us remain however optimistic, we are still free to do what we want, for the very reason that we are in the dark.

We will present in this book some speculations about this, new geometries that are believed to be relevant to questions in physics. Our belief is that these geometries fall into two classes, namely herbivore, with $fg = gf$ being not totally negated, but rather replaced by something weaker, of type $fgh = hgf$, or $fg = \pm gf$, or $fgh = \pm hgf$ and so on, and then carnivore, with $fg = gf$ being totally negated, and replaced by freeness.

Go with the carnivore way, you would say, but in practice, not that quick. Things are difficult, free geometry is hard to develop, and among others, we need some practice in hunting herbivores, before casually talking to carnivores. So, getting now to the contents of the present book, this will be mostly about herbivore geometries, that is, about those which are close to our usual geometry. The book is organized in 4 parts, as follows:

(1) We will first discuss axiomatic aspects, with the aim of understanding what almost classical, for an abstract geometry, should mean.

(2) This will lead us to the conclusion that half-commutation, $fgh = hgf$, should be investigated first, and we will go on a detailed study of half-classical geometry.

(3) Afterwards, with motivation and input coming from anticommutation, $fg = \pm gf$, we will present a similar study, for the twisted geometry.

(4) Finally, we will discuss some more advanced aspects, in relation with $fgh = \pm hgf$, with axiomatics, and with the geometric study of the manifolds that we found.

All in all, this will be some sort of introduction to noncommutative geometry, getting you familiar with the main concepts and techniques, by staying not very far from classical geometry, and with focus on some geometries that are definitely something basic and fundamental, mathematically speaking, and believed to be physically relevant too.

The material in this book basically goes back to research work from the late 2000s and early 2010s, and I am grateful to Julien Bichon, Benoît Collins, Debashish Goswami and my other coworkers, for substantial joint work on the subject. Thanks as well to my cats, for their daily teachings on how to catch a mouse, practicing hard every day, one day I will get to that. In the meantime, in the hope that you will like this book.

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Part I

Axiomatics

I get my kicks from watching people
Running to and fro
And if you ask them where they're going
Half of them don't know

CHAPTER 1

Operator algebras

1a. Linear operators

There are several ways of getting into noncommutative geometry, and we will follow here the most standard way, namely linear operators, followed by operator algebras, and then quantum spaces, and more complicated things. It is possible of course to take some shortcuts instead, but with these guaranteeing you not to understand anything. Always remember, indeed, that we are interested in quantum mechanics, and linear operators.

So, we would like to first discuss the theory of linear operators $T:H\to H$ over a complex Hilbert space H, usually taken separable. You might probably ask at this point, what is H, and in answer, for the simplest quantum mechanical system, which is the hydrogen atom, this is the space $L^2(\mathbb R^3)$ of wave functions of the electron. Or rather, this is $L^2(\mathbb R^3)$, summed with a copy of $\mathbb C^2$, in order to account for the electron spin.

But more on quantum mechanics later, for the moment we will take our Hilbert spaces H as mentioned above, namely abstract, complex, usually infinite dimensional, and usually taken separable. Let us start with a basic result about operators, as follows:

THEOREM 1.1. *Given a Hilbert space H, consider the linear operators $T:H\to H$, and for each such operator define its norm by the following formula:*

$$||T||=\sup_{||x||=1}||Tx||$$

The operators which are bounded, $||T||<\infty$, form then a complex algebra $B(H)$, which is complete with respect to $||.||$. When H comes with a basis $\{e_i\}_{i\in I}$, we have

$$B(H)\subset\mathcal L(H)\subset M_I(\mathbb C)$$

where $\mathcal L(H)$ is the algebra of all linear operators $T:H\to H$, and $\mathcal L(H)\subset M_I(\mathbb C)$ is the correspondence $T\to M$ obtained via the usual linear algebra formulae, namely:

$$T(x)=Mx\quad,\quad M_{ij}=<Te_j,e_i>$$

In infinite dimensions, none of the above two inclusions is an equality.

PROOF. This is something straightforward, the idea being as follows:

(1) The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S+T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(2) Regarding now the completness assertion, if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x) + T_n(y) \\ &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) \\ &= T(x) + T(y) \end{aligned}$$

Similarly, we have $T(\lambda x) = \lambda T(x)$, and we conclude that $T \in \mathcal{L}(H)$.

(3) With this done, it remains to prove now that we have $T \in B(H)$, and that $T_n \to T$ in norm. For this purpose, observe that we have:

$$\begin{aligned} ||T_n - T_m|| \leq \varepsilon\ ,\ \forall n,m \geq N &\implies ||T_n x - T_m x|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n,m \geq N \\ &\implies ||T_n x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n \geq N \\ &\implies ||T_N x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1 \\ &\implies ||T_N - T|| \leq \varepsilon \end{aligned}$$

But this gives both $T \in B(H)$, and $T_N \to T$ in norm, and we are done.

(4) Regarding the embeddings, the correspondence $T \to M$ in the statement is indeed linear, and its kernel is $\{0\}$, so we have indeed an embedding as follows, as claimed:

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

In finite dimensions we have an isomorphism, because any $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \to \mathbb{C}^N$, given by $< Te_j, e_i >= M_{ij}$. However, in infinite dimensions, we have matrices not producing operators, as for instance the all-one matrix.

(5) As for the examples of linear operators which are not bounded, these are more complicated, coming from logic, and we will not really need them in what follows. □

You surely know from linear algebra, which corresponds to taking $H = \mathbb{C}^N$ with $N < \infty$ in Theorem 1.1, that the linear maps $T : H \to H$ do not come alone, but rather in pairs (T, T^*), with $T^* : H \to H$ being the adjoint map. In our setting, we have:

THEOREM 1.2. *Each operator $T\in B(H)$ has an adjoint $T^*\in B(H)$, given by:*
$$<Tx,y>=<x,T^*y>$$
The operation $T\to T^*$ is antilinear, antimultiplicative, involutive, and satisfies:
$$||T||=||T^*||\quad,\quad ||TT^*||=||T||^2$$
When H comes with a basis $\{e_i\}_{i\in I}$, the operation $T\to T^*$ corresponds to
$$(M^*)_{ij}=\overline{M}_{ji}$$
at the level of the associated matrices $M\in M_I(\mathbb C)$.

PROOF. This is standard too, and can be proved in 3 steps, as follows:

(1) The existence of the adjoint operator T^*, given by the formula in the statement, comes from the fact that the function $\varphi(x)=<Tx,y>$ being a linear map $H\to\mathbb C$, we must have a formula as follows, for a certain vector $T^*y\in H$:
$$\varphi(x)=<x,T^*y>$$
Moreover, since this vector is unique, T^* is unique too, and we have as well:
$$(S+T)^*=S^*+T^*\quad,\quad(\lambda T)^*=\bar\lambda T^*\quad,\quad(ST)^*=T^*S^*\quad,\quad(T^*)^*=T$$
Observe also that we have indeed $T^*\in B(H)$, because:
$$\begin{aligned}||T||&=\sup_{||x||=1}\sup_{||y||=1}<Tx,y>\\&=\sup_{||y||=1}\sup_{||x||=1}<x,T^*y>\\&=||T^*||\end{aligned}$$
(2) Regarding now $||TT^*||=||T||^2$, which is a key formula, observe that we have:
$$||TT^*||\leq||T||\cdot||T^*||=||T||^2$$
On the other hand, we have as well the following estimate:
$$\begin{aligned}||T||^2&=\sup_{||x||=1}|<Tx,Tx>|\\&=\sup_{||x||=1}|<x,T^*Tx>|\\&\leq||T^*T||\end{aligned}$$
By replacing $T\to T^*$ we obtain from this $||T||^2\leq||TT^*||$, as desired.

(3) Finally, when H comes with a basis, the formula $<Tx,y>=<x,T^*y>$ applied with $x=e_i$, $y=e_j$ translates into the formula $(M^*)_{ij}=\overline{M}_{ji}$, as desired. □

Let us discuss now the diagonalization problem for the operators $T\in B(H)$, in analogy with the diagonalization problem for the usual matrices $A\in M_N(\mathbb C)$. As a first observation, we can talk about eigenvalues and eigenvectors, as follows:

Definition 1.3. *Given an operator $T\in B(H)$, assuming that we have*

$$Tx=\lambda x$$

we say that $x\in H$ is an eigenvector of T, with eigenvalue $\lambda\in\mathbb C$.

We know many things about eigenvalues and eigenvectors, in the finite dimensional case. However, most of these will not extend to the infinite dimensional case, or at least not extend in a straightforward way, due to a number of reasons:

(1) Most of basic linear algebra is based on the fact that $Tx=\lambda x$ is equivalent to $(T-\lambda)x=0$, so that λ is an eigenvalue when $T-\lambda$ is not invertible. In the infinite dimensional setting $T-\lambda$ might be injective and not surjective, or vice versa, or invertible with $(T-\lambda)^{-1}$ not bounded, and so on.

(2) Also, in linear algebra $T-\lambda$ is not invertible when $\det(T-\lambda)=0$, and with this leading to most of the advanced results about eigenvalues and eigenvectors. In infinite dimensions, however, it is impossible to construct a determinant function $\det:B(H)\to\mathbb C$, and this even for the diagonal operators on $l^2(\mathbb N)$.

Summarizing, we are in trouble. Forgetting about (2), which obviously leads nowhere, let us focus on the difficulties in (1). In order to cut short the discussion there, regarding the various properties of $T-\lambda$, we can just say that $T-\lambda$ is either invertible with bounded inverse, the "good case", or not. We are led in this way to the following definition:

Definition 1.4. *The spectrum of an operator $T\in B(H)$ is the set*

$$\sigma(T)=\left\{\lambda\in\mathbb C\Big|T-\lambda\notin B(H)^{-1}\right\}$$

where $B(H)^{-1}\subset B(H)$ is the set of invertible operators.

As a basic example, in the finite dimensional case, $H=\mathbb C^N$, the spectrum of a usual matrix $A\in M_N(\mathbb C)$ is the collection of its eigenvalues, taken without multiplicities. We will see many other examples. In general, the spectrum has the following properties:

Proposition 1.5. *The spectrum of $T\in B(H)$ contains the eigenvalue set*

$$\varepsilon(T)=\left\{\lambda\in\mathbb C\Big|\ker(T-\lambda)\neq\{0\}\right\}$$

and $\varepsilon(T)\subset\sigma(T)$ is an equality in finite dimensions, but not in infinite dimensions.

Proof. We have several assertions here, the idea being as follows:

(1) First of all, the eigenvalue set is indeed the one in the statement, because $Tx=\lambda x$ tells us precisely that $T-\lambda$ must be not injective. The fact that we have $\varepsilon(T)\subset\sigma(T)$ is clear as well, because if $T-\lambda$ is not injective, it is not bijective.

(2) In finite dimensions we have $\varepsilon(T)=\sigma(T)$, because $T-\lambda$ is injective if and only if it is bijective, with the boundedness of the inverse being automatic.

(3) In infinite dimensions we can assume $H = l^2(\mathbb{N})$, and the shift operator $S(e_i) = e_{i+1}$ is injective but not surjective. Thus $0 \in \sigma(T) - \varepsilon(T)$. □

Philosophically, the best way of thinking at this is as follows: the numbers $\lambda \notin \sigma(T)$ are good, because we can invert $T - \lambda$, the numbers $\lambda \in \sigma(T) - \varepsilon(T)$ are bad, because so they are, and the eigenvalues $\lambda \in \varepsilon(T)$ are evil. Welcome to operator theory.

Let us develop now some general theory for the spectra. We first have:

PROPOSITION 1.6. *We have the "polynomial functional calculus" formula*

$$\sigma(P(T)) = P(\sigma(T))$$

valid for any polynomial $P \in \mathbb{C}[X]$, and any operator $T \in B(H)$.

PROOF. We pick a scalar $\lambda \in \mathbb{C}$, and we decompose the polynomial $P - \lambda$:

$$P(X) - \lambda = c(X - r_1) \ldots (X - r_n)$$

We have then the following equivalences:

$$\begin{aligned}
\lambda \notin \sigma(P(T)) &\iff P(T) - \lambda \in B(H)^{-1} \\
&\iff c(T - r_1) \ldots (T - r_n) \in B(H)^{-1} \\
&\iff T - r_1, \ldots, T - r_n \in B(H)^{-1} \\
&\iff r_1, \ldots, r_n \notin \sigma(T) \\
&\iff \lambda \notin P(\sigma(T))
\end{aligned}$$

Thus, we are led to the formula in the statement. □

More generally, we have the following result, extending Proposition 1.6:

THEOREM 1.7. *We have the "rational functional calculus" formula*

$$\sigma(f(T)) = f(\sigma(T))$$

valid for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(T)$.

PROOF. We pick a scalar $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set:

$$F = P - \lambda Q$$

By using now Proposition 1.6, for this polynomial, we obtain:

$$\begin{aligned}
\lambda \in \sigma(f(T)) &\iff F(T) \notin B(H)^{-1} \\
&\iff 0 \in \sigma(F(T)) \\
&\iff 0 \in F(\sigma(T)) \\
&\iff \exists \mu \in \sigma(T), F(\mu) = 0 \\
&\iff \lambda \in f(\sigma(T))
\end{aligned}$$

Thus, we are led to the formula in the statement. □

In order to formulate our next result, we will need the following notion:

Definition 1.8. *Given an operator $T\in B(H)$, its spectral radius*

$$\rho(T)\in\big[0,||T||\big]$$

is the radius of the smallest disk centered at 0 containing $\sigma(T)$.

Now with this notion in hand, we have the following key result:

Theorem 1.9. *The spectral radius of an operator $T\in B(H)$ is given by*

$$\rho(T)=\lim_{n\to\infty}||T^n||^{1/n}$$

and in this formula, we can replace the limit by an inf.

Proof. We have several things to be proved, the idea being as follows:

(1) Our first claim is that the numbers $u_n=||T^n||^{1/n}$ satisfy:

$$(n+m)u_{n+m}\leq nu_n+mu_m$$

Indeed, we have the following estimate, using the Young inequality $ab\leq a^p/p+b^q/q$, with exponents $p=(n+m)/n$ and $q=(n+m)/m$:

$$\begin{aligned}
u_{n+m}&=&||T^{n+m}||^{1/(n+m)}\\
&\leq&||T^n||^{1/(n+m)}||T^m||^{1/(n+m)}\\
&\leq&||T^n||^{1/n}\cdot\frac{n}{n+m}+||T^m||^{1/m}\cdot\frac{m}{n+m}\\
&=&\frac{nu_n+mu_m}{n+m}
\end{aligned}$$

(2) Our second claim is that the second assertion holds, namely:

$$\lim_{n\to\infty}||T^n||^{1/n}=\inf_n||T^n||^{1/n}$$

For this purpose, we just need the inequality found in (1). Indeed, fix $m\geq1$, let $n\geq1$, and write $n=lm+r$ with $0\leq r\leq m-1$. By using twice $u_{ab}\leq u_b$, we get:

$$\begin{aligned}
u_n&\leq&\frac{1}{n}(lmu_{lm}+ru_r)\\
&\leq&\frac{1}{n}(lmu_m+ru_1)\\
&\leq&u_m+\frac{r}{n}u_1
\end{aligned}$$

It follows that we have $\limsup_n u_n\leq u_m$, which proves our claim.

(3) Summarizing, we are left with proving the main formula, which is as follows, and with the remark that we already know that the sequence on the right converges:

$$\rho(T)=\lim_{n\to\infty}||T^n||^{1/n}$$

In one sense, we can use the polynomial calculus formula $\sigma(T^n)=\sigma(T)^n$. Indeed, this gives the following estimate, valid for any n, as desired:

$$\begin{aligned}\rho(T) &= \sup_{\lambda\in\sigma(T)}|\lambda|\\ &= \sup_{\rho\in\sigma(T)^n}|\rho|^{1/n}\\ &= \sup_{\rho\in\sigma(T^n)}|\rho|^{1/n}\\ &= \rho(T^n)^{1/n}\\ &\leq ||T^n||^{1/n}\end{aligned}$$

(4) For the reverse inequality, we fix a number $\rho>\rho(T)$, and we want to prove that we have $\rho\geq\lim_{n\to\infty}||T^n||^{1/n}$. By using the Cauchy formula, we have:

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=\rho}\frac{z^n}{z-T}\,dz &= \frac{1}{2\pi i}\int_{|z|=\rho}\sum_{k=0}^\infty z^{n-k-1}T^k\,dz\\ &= \sum_{k=0}^\infty\frac{1}{2\pi i}\left(\int_{|z|=\rho}z^{n-k-1}dz\right)T^k\\ &= \sum_{k=0}^\infty\delta_{n,k+1}T^k\\ &= T^{n-1}\end{aligned}$$

By applying the norm we obtain from this formula:

$$||T^{n-1}||\leq\frac{1}{2\pi}\int_{|z|=\rho}\left|\left|\frac{z^n}{z-T}\right|\right|\,dz\leq\rho^n\cdot\sup_{|z|=\rho}\left|\left|\frac{1}{z-T}\right|\right|$$

Since the sup does not depend on n, by taking n-th roots, we obtain in the limit:

$$\rho\geq\lim_{n\to\infty}||T^n||^{1/n}$$

Now recall that ρ was by definition an arbitrary number satisfying $\rho>\rho(T)$. Thus, we have obtained the following estimate, valid for any $T\in B(H)$:

$$\rho(T)\geq\lim_{n\to\infty}||T^n||^{1/n}$$

Thus, we are led to the conclusion in the statement. □

1b. Normal operators

In the case of the normal elements, we have the following finer result:

THEOREM 1.10. *The spectral radius of a normal element,*

$$TT^* = T^*T$$

is equal to its norm.

PROOF. We can proceed in two steps, as follows:

Step 1. In the case $T = T^*$ we have $||T^n|| = ||T||^n$ for any exponent of the form $n = 2^k$, by using the formula $||TT^*|| = ||T||^2$, and by taking n-th roots we get:

$$\rho(T) \geq ||T||$$

Thus, we are done with the self-adjoint case, with the result $\rho(T) = ||T||$.

Step 2. In the general normal case $TT^* = T^*T$ we have $T^n(T^n)^* = (TT^*)^n$, and by using this, along with the result from Step 1, applied to TT^*, we obtain:

$$\begin{aligned}
\rho(T) &= \lim_{n\to\infty} ||T^n||^{1/n} \\
&= \sqrt{\lim_{n\to\infty} ||T^n(T^n)^*||^{1/n}} \\
&= \sqrt{\lim_{n\to\infty} ||(TT^*)^n||^{1/n}} \\
&= \sqrt{\rho(TT^*)} \\
&= \sqrt{||T||^2} \\
&= ||T||
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

By using Theorem 1.10 we can say a number of non-trivial things about the normal operators, commonly known as "spectral theorem for normal operators". As a first result here, we can improve the polynomial functional calculus formula, as follows:

THEOREM 1.11. *Given $T \in B(H)$ normal, we have a morphism of algebras*

$$\mathbb{C}[X] \to B(H) \quad , \quad P \to P(T)$$

having the properties $||P(T)|| = ||P_{|\sigma(T)}||$, and $\sigma(P(T)) = P(\sigma(T))$.

PROOF. This is an improvement of Proposition 1.6 in the normal case, with the extra assertion being the norm estimate. But the element $P(T)$ being normal, we can apply to it the spectral radius formula for normal elements, and we obtain:

$$\begin{aligned}
||P(T)|| &= \rho(P(T)) \\
&= \sup_{\lambda\in\sigma(P(T))} |\lambda| \\
&= \sup_{\lambda\in P(\sigma(T))} |\lambda| \\
&= ||P_{|\sigma(T)}||
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

We can improve as well the rational calculus formula, and the holomorphic calculus formula, in the same way. Importantly now, at a more advanced level, we have:

THEOREM 1.12. *Given $T \in B(H)$ normal, we have a morphism of algebras*

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

which is isometric, $||f(T)|| = ||f||$, and has the property $\sigma(f(T)) = f(\sigma(T))$.

PROOF. The idea here is to "complete" the morphism in Theorem 1.11, namely:

$$\mathbb{C}[X] \to B(H) \quad , \quad P \to P(T)$$

Indeed, we know from Theorem 1.11 that this morphism is continuous, and is in fact isometric, when regarding the polynomials $P \in \mathbb{C}[X]$ as functions on $\sigma(T)$:

$$||P(T)|| = ||P_{|\sigma(T)}||$$

Thus, by Stone-Weierstrass, we have a unique isometric extension, as follows:

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

It remains to prove $\sigma(f(T)) = f(\sigma(T))$, and we can do this by double inclusion:

"$\subset$" Given a continuous function $f \in C(\sigma(T))$, we must prove that we have:

$$\lambda \notin f(\sigma(T)) \implies \lambda \notin \sigma(f(T))$$

For this purpose, consider the following function, which is well-defined:

$$\frac{1}{f-\lambda} \in C(\sigma(T))$$

We can therefore apply this function to T, and we obtain:

$$\left(\frac{1}{f-\lambda}\right)T = \frac{1}{f(T)-\lambda}$$

In particular $f(T)-\lambda$ is invertible, so $\lambda \notin \sigma(f(T))$, as desired.

"$\supset$" Given a continuous function $f \in C(\sigma(T))$, we must prove that we have:

$$\lambda \in f(\sigma(T)) \implies \lambda \in \sigma(f(T))$$

But this is the same as proving that we have:

$$\mu \in \sigma(T) \implies f(\mu) \in \sigma(f(T))$$

For this purpose, we approximate our function by polynomials, $P_n \to f$, and we examine the following convergence, which follows from $P_n \to f$:

$$P_n(T) - P_n(\mu) \to f(T) - f(\mu)$$

We know from polynomial functional calculus that we have:

$$P_n(\mu) \in P_n(\sigma(T)) = \sigma(P_n(T))$$

Thus, the operators $P_n(T) - P_n(\mu)$ are not invertible. On the other hand, we know that the set formed by the invertible operators is open, so its complement is closed. Thus the limit $f(T) - f(\mu)$ is not invertible either, and so $f(\mu) \in \sigma(f(T))$, as desired. □

We can further extend Theorem 1.12 to the measurable functions, as follows:

THEOREM 1.13. *Given $T \in B(H)$ normal, we have a morphism of algebras as follows, with L^∞ standing for abstract measurable functions, or Borel functions,*

$$L^\infty(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

which is isometric, $||f(T)|| = ||f||$, and has the property $\sigma(f(T)) = f(\sigma(T))$.

PROOF. As before, the idea will be that of "completing" what we have. To be more precise, we can use the Riesz theorem and a polarization trick, as follows:

(1) Given a vector $x \in H$, consider the following functional:

$$C(\sigma(T)) \to \mathbb{C} \quad , \quad g \to < g(T)x, x >$$

By the Riesz theorem, this functional must be the integration with respect to a certain measure μ on the space $\sigma(T)$. Thus, we have a formula as follows:

$$< g(T)x, x > = \int_{\sigma(T)} g(z) d\mu(z)$$

Now given an arbitrary Borel function $f \in L^\infty(\sigma(T))$, as in the statement, we can define a number $< f(T)x, x > \in \mathbb{C}$, by using exactly the same formula, namely:

$$< f(T)x, x > = \int_{\sigma(T)} f(z) d\mu(z)$$

Thus, we have managed to define numbers $< f(T)x, x > \in \mathbb{C}$, for all vectors $x \in H$, and in addition we can recover these numbers as follows, with $g_n \in C(\sigma(T))$:

$$< f(T)x, x > = \lim_{g_n \to f} < g_n(T)x, x >$$

(2) In order to define now numbers $< f(T)x, y > \in \mathbb{C}$, for all vectors $x, y \in H$, we can use a polarization trick. Indeed, for any operator $S \in B(H)$ we have:

$$\begin{aligned} < S(x+y), x+y > &= < Sx, x > + < Sy, y > \\ &\quad + < Sx, y > + < Sy, x > \end{aligned}$$

By replacing $y \to iy$, we have as well the following formula:

$$\begin{aligned} < S(x+iy), x+iy > &= < Sx, x > + < Sy, y > \\ &\quad -i < Sx, y > +i < Sy, x > \end{aligned}$$

By multiplying this latter formula by i, we obtain the following formula:

$$\begin{aligned} i < S(x+iy), x+iy > &= i < Sx, x > +i < Sy, y > \\ &\quad + < Sx, y > - < Sy, x > \end{aligned}$$

Now by summing this latter formula with the first one, we obtain:

$$\begin{aligned} < S(x+y), x+y > +i < S(x+iy), x+iy > &= (1+i)[< Sx, x > + < Sy, y >] \\ &\quad +2 < Sx, y > \end{aligned}$$

(3) But with this, we can now finish. Indeed, by combining (1,2), given a Borel function $f \in L^\infty(\sigma(T))$, we can define numbers $< f(T)x, y > \in \mathbb{C}$ for any $x, y \in H$, and it is routine to check, by using approximation by continuous functions $g_n \to f$ as in (1), that we obtain in this way an operator $f(T) \in B(H)$, having all the desired properties. □

We can now formulate the spectral theorem for normal operators, as follows:

THEOREM 1.14. *The following happen:*

(1) *Any self-adjoint operator, $T = T^*$, is diagonalizable.*
(2) *In fact, any family $\{T_i\}$ of commuting self-adjoint operators is diagonalizable.*
(3) *On the other hand, any normal operator, $TT^* = T^*T$, is diagonalizable too.*
(4) *In fact, any family $\{T_i\}$ of commuting normal operators is diagonalizable.*

PROOF. This is certainly a tough theorem, the idea being as follows:

(1) This comes by further building on Theorem 1.13, in the self-adjoint case.

(2) This is something straightforward, appearing as a technical version of (1).

(3) This follows from (2), by using the $T = Re(T) + i \cdot Im(T)$ decomposition.

(4) This is again straightforward, appearing as a technical version of (3). □

Many other things can be said about operators, as a continuation of the above. For our purposes here, which are rather introductory, the above will do.

1c. Operator algebras

Good news, we can now talk about operator algebras. Let us start with the following broad definition, obtained by imposing the "minimal" set of reasonable axioms:

DEFINITION 1.15. *An operator algebra is an algebra of bounded operators $A \subset B(H)$ which contains the unit, is closed under taking adjoints,*

$$T \in A \implies T^* \in A$$

and is closed as well under the norm.

Here, as before, H is an arbitrary Hilbert space, with the case that we are mostly interested in being the separable one. Also as before, $B(H)$ is the algebra of linear operators $T : H \to H$ which are bounded, in the sense that $||T|| = \sup_{||x||=1} ||Tx||$ is finite. This algebra has an involution $T \to T^*$, with the adjoint operator $T^* \in B(H)$ being defined by the formula $< Tx, y >=< x, T^*y >$, and in the above definition, the assumption $T \in A \implies T^* \in A$ refers to this involution. Thus, A must be a $*$-algebra.

As a first result now regarding the operator algebras, in relation with the normal operators, where most of the non-trivial results that we have so far are, we have:

THEOREM 1.16. *The operator algebra $< T >\subset B(H)$ generated by a normal operator $T \in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(\sigma(T))$$

where $\sigma(T) \subset \mathbb{C}$ denotes as usual the spectrum of T.

PROOF. We know that we have a continuous morphism of $*$-algebras, as follows:

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the continuous calculus, also established in the above, this morphism is injective, and its image is the norm closed algebra $< T >$ generated by T, T^*. Thus, we obtain the isomorphism in the statement. □

The above result is very nice, and it is possible to further build on it, as follows:

THEOREM 1.17. *The operator algebra $< T_i >\subset B(H)$ generated by a family of normal operators $T_i \in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(X)$$

where $X \subset \mathbb{C}$ is a certain compact space associated to the family $\{T_i\}$. Equivalently, any commutative operator algebra $A \subset B(H)$ is of the form $A = C(X)$.

PROOF. We have two assertions here, the idea being as follows:

(1) Regarding the first assertion, this follows exactly as in the proof of Theorem 1.16, by using this time the spectral theorem for families of normal operators.

(2) As for the second assertion, this is clear from the first one, because any commutative algebra $A \subset B(H)$ is generated by its elements $T \in A$, which are all normal. □

All this is good to know, but Theorem 1.16 and Theorem 1.17 remain something quite heavy, based on the spectral theorem. We would like to present now an alternative proof for these results, which is rather elementary, and has the advantage of reconstructing the compact space X directly from the knowledge of the algebra A. We will need:

Theorem 1.18. *Given an operator $T\in A\subset B(H)$, define its spectrum as:*

$$\sigma(T)=\left\{\lambda\in\mathbb C\Big|T-\lambda\notin A^{-1}\right\}$$

The following spectral theory results hold, exactly as in the $A=B(H)$ case:

1. *We have $\sigma(ST)\cup\{0\}=\sigma(TS)\cup\{0\}$.*
2. *We have polynomial, rational and holomorphic calculus.*
3. *As a consequence, the spectra are compact and non-empty.*
4. *The spectra of unitaries ($U^*=U^{-1}$) and self-adjoints ($T=T^*$) are on $\mathbb T,\mathbb R$.*
5. *The spectral radius of normal elements ($TT^*=T^*T$) is given by $\rho(T)=||T||$.*

In addition, assuming $T\in A\subset B$, the spectra of T with respect to A and to B coincide.

Proof. This is something that we basically know from before, in the case $A=B(H)$. In general the proof is similar, the idea being as follows:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra $A=B(H)$. All this is standard material, and in fact, things before were written in such a way as for their extension now, to the general operator algebra setting, to be obvious.

(2) Regarding the last assertion, the inclusion $\sigma_B(T)\subset\sigma_A(T)$ is clear. For the converse, assume $T-\lambda\in B^{-1}$, and consider the following self-adjoint element:

$$S=(T-\lambda)^*(T-\lambda)$$

The difference between the two spectra of $S\in A\subset B$ is then given by:

$$\sigma_A(S)-\sigma_B(S)=\left\{\mu\in\mathbb C-\sigma_B(S)\Big|(S-\mu)^{-1}\in B-A\right\}$$

Thus this difference in an open subset of $\mathbb C$. On the other hand S being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of S are equal, and in particular S is invertible in A, and so $T-\lambda\in A^{-1}$, as desired.

(3) As an observation, the last assertion applied with $B=B(H)$ shows that the spectrum $\sigma(T)$ as constructed in the statement coincides with the spectrum $\sigma(T)$ as constructed and studied before, so the fact that (1-5) hold indeed is no surprise.

(4) Finally, I can hear you screaming that I should have concieved this book differently, matter of not proving the same things twice. Good point, with my distinguished colleague Bourbaki saying the same, and in answer, wait for the next section, where we will prove exactly the same things a third time. We can discuss pedagogy at that time. □

We can now get back to the commutative algebras, and we have the following result, due to Gelfand, which provides an alternative to Theorem 1.16 and Theorem 1.17:

THEOREM 1.19. *Any commutative operator algebra $A\subset B(H)$ is of the form*

$$A=C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi:A\to\mathbb C$, with topology making continuous the evaluation maps $ev_T:\chi\to\chi(T)$.

PROOF. Given a commutative operator algebra A, we can define X as in the statement. Then X is compact, and $T\to ev_T$ is a morphism of algebras, as follows:

$$ev:A\to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the $z=Re(z)+iIm(z)$ formula for the usual complex numbers:

$$T=\frac{T+T^*}{2}+i\cdot\frac{T-T^*}{2i}$$

Thus it is enough to prove the equality $ev_{T^*}=ev_T^*$ for self-adjoint elements T. But this is the same as proving that $T=T^*$ implies that ev_T is a real function, which is in turn true, because $ev_T(\chi)=\chi(T)$ is an element of $\sigma(T)$, contained in $\mathbb R$.

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_T||=\rho(T)=||T||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

1d. Normed algebras

We have been talking so far about the general operator $*$-algebras $A\subset B(H)$, closed with respect to the norm. But this suggests formulating the following definition:

DEFINITION 1.20. *A C^*-algebra is an complex algebra A, given with:*

1. *A norm $a\to||a||$, making it into a Banach algebra.*
2. *An involution $a\to a^*$, related to the norm by the formula $||aa^*||=||a||^2$.*

Here by Banach algebra we mean a complex algebra with a norm satisfying all the conditions for a vector space norm, along with $||ab||\leq||a||\cdot||b||$ and $||1||=1$, and which is such that our algebra is complete, in the sense that the Cauchy sequences converge. As for the involution, this must be antilinear, antimultiplicative, and satisfying $a^{**}=a$.

As basic examples, we have the operator algebra $B(H)$, for any Hilbert space H, and more generally, the norm closed $*$-subalgebras $A\subset B(H)$. It is possible to prove that any C^*-algebra appears in this way, but this is a non-trivial result, called GNS theorem, and more on this later. Note in passing that this result tells us that there is no need to memorize the above axioms for the C^*-algebras, because these are simply the obvious things that can be said about $B(H)$, and its norm closed $*$-subalgebras $A\subset B(H)$.

As a second class of basic examples, which are of great interest for us, we have:

PROPOSITION 1.21. *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a C^*-algebra, with the usual norm and involution, namely:*

$$||f|| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

This algebra is commutative, in the sense that $fg = gf$, for any $f, g \in C(X)$.

PROOF. All this is clear from definitions. Observe that we have indeed:

$$||ff^*|| = \sup_{x \in X} |f(x)|^2 = ||f||^2$$

Thus, the axioms are satisfied, and finally $fg = gf$ is clear. □

In general, the C^*-algebras can be thought of as being algebras of operators, over some Hilbert space which is not present. By using this philosophy, one can emulate spectral theory in this setting, with extensions of our previous results. Let us start with:

DEFINITION 1.22. *Given element $a \in A$ of a C^*-algebra, define its spectrum as:*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \Big| a - \lambda \notin A^{-1} \right\}$$

Also, we call spectral radius of $a \in A$ the number $\rho(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$.

In what regards the examples, for $A = B(H)$ what we have here is the usual notion of spectrum, from before. More generally, as explained in Theorem 1.18, in the case $A \subset B(H)$ we obtain the same spectra as those in the case $A = B(H)$. Finally, in the case $A = C(X)$, as in Proposition 1.21, the spectrum of a function is its image:

$$\sigma(f) = Im(f)$$

Now with the above notion of spectrum in hand, we have the following result:

THEOREM 1.23. *The following results hold, exactly as in the $A \subset B(H)$ case:*

(1) *We have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.*
(2) *We have polynomial, rational and holomorphic calculus.*
(3) *As a consequence, the spectra are compact and non-empty.*
(4) *The spectra of unitaries ($u^* = u^{-1}$) and self-adjoints ($a = a^*$) are on $\mathbb{T}, \mathbb{R}$.*
(5) *The spectral radius of normal elements ($aa^* = a^*a$) is given by $\rho(a) = ||a||$.*

In addition, assuming $a \in A \subset B$, the spectra of a with respect to A and to B coincide.

PROOF. This is something that we know from before, in the case $A = B(H)$, and then from Theorem 1.18, in the case $A \subset B(H)$. In general, the proof is similar:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra $A = B(H)$. All

this is standard material, and in fact, things before were written in such a way as for their extension now, to the general C^*-algebra setting, to be obvious.

(2) Regarding the last assertion, we know this from before for $A \subset B \subset B(H)$, and the proof in general is similar. Indeed, the inclusion $\sigma_B(a) \subset \sigma_A(a)$ is clear. For the converse, assume $a - \lambda \in B^{-1}$, and consider the following self-adjoint element:

$$b = (a-\lambda)^*(a-\lambda)$$

The difference between the two spectra of $b \in A \subset B$ is then given by:

$$\sigma_A(b) - \sigma_B(b) = \left\{\mu \in \mathbb{C} - \sigma_B(b) \Big| (b-\mu)^{-1} \in B - A\right\}$$

Thus this difference in an open subset of $\mathbb{C}$. On the other hand b being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of b are equal, and in particular b is invertible in A, and so $a - \lambda \in A^{-1}$, as desired. □

We can get back now to the commutative algebras, and we have the following result, due to Gelfand, which will be of crucial importance for us:

THEOREM 1.24. *The commutative C^*-algebras are exactly the algebras of the form*

$$A = C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi : A \to \mathbb{C}$, with topology making continuous the evaluation maps $ev_a : \chi \to \chi(a)$.

PROOF. This is something that we basically know from before, but always good to talk about it again. Given a commutative C^*-algebra A, we can define X as in the statement. Then X is compact, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for the usual complex numbers:

$$a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

In view of the Gelfand theorem, we can formulate the following key definition:

Definition 1.25. *Given an arbitrary C^*-algebra A, we write*

$$A = C(X)$$

and call X a compact quantum space.

This might look like something informal, but it is not. Indeed, we can define the category of compact quantum spaces to be the category of the C^*-algebras, with the arrows reversed. When A is commutative, the above space X exists indeed, as a Gelfand spectrum, $X = Spec(A)$. In general, X is something rather abstract, and our philosophy here will be that of studying of course A, but formulating our results in terms of X. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

Let us also mention that, technically speaking, we will see later that the above formalism has its limitations, and needs a fix. But more on this later.

As a first concrete consequence now of the Gelfand theorem, we have:

Theorem 1.26. *Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.*

1. *We can define $f(a) \in A$, with $f \to f(a)$ being a morphism of C^*-algebras.*
2. *We have the "continuous functional calculus" formula $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Since a is normal, the C^*-algebra $< a >$ that is generates is commutative, so if we denote by X the space formed by the characters $\chi :< a >\to \mathbb{C}$, we have:

$$< a >= C(X)$$

Now since the map $X \to \sigma(a)$ given by evaluation at a is bijective, we obtain:

$$< a >= C(\sigma(a))$$

Thus, we are dealing with usual functions, and this gives all the assertions. □

As another consequence of the Gelfand theorem, we have:

Theorem 1.27. *For a normal element $a \in A$, the following are equivalent:*

1. *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
2. *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
3. *$a = cc^*$, for some $c \in A$.*

Proof. This is very standard, exactly as in $A = B(H)$ case, as follows:

(1) $\implies$ (2) Since $f(z) = \sqrt{z}$ is well-defined on $\sigma(a) \subset [0, \infty)$, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can set $c = b$.

(3) $\implies$ (1) We proceed by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting the fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, that we know to hold for $A = B(H)$, and whose proof in general is similar. □

In order to develop some general theory, let us start by investigating the finite dimensional case. Here the ambient algebra is $B(H) = M_N(\mathbb{C})$, any linear subspace $A \subset B(H)$ is automatically closed, for the norm topology, and we have the following result:

THEOREM 1.28. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $n_1, \ldots, n_k \in \mathbb{N}$ satisfying $n_1 + \ldots + n_k = N$, we have indeed an obvious embedding of $*$-algebras, via matrix blocks, as follows:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \to UMU^*$$

(2) In the other sense now, consider a $*$-algebra $A \subset M_N(\mathbb{C})$. It is elementary to prove that the center $Z(A) = A \cap A'$, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \ldots, e_k \in \mathbb{C}^k$, and let $p_1, \ldots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \ldots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \ldots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2 = p_i^* = p_i$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \ldots + p_k = 1$ and the projection equations $p_i^2 = p_i^* = p_i$, we conclude that we have the needed generation property, namely:

$$A_1 + \ldots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \ldots + p_k = 1$, and from the projection equations $p_i^2 = p_i^* = p_i$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i = rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{n_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Moreover, a more careful look at the isomorphisms established in Step 3 shows that at the global level, that of the algebra A itself, the above isomorphism simply comes by twisting the following standard multimatrix embedding, discussed in the beginning of the proof, (1) above, by a certain unitary matrix $U \in U_N$:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

Now by putting everything together, we obtain the result. □

In terms of our usual C^*-algebra formalism, the above result tells us that we have:

Theorem 1.29. *The finite dimensional C^*-algebras are exactly the algebras*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with norm $||(a_1, \ldots, a_k)|| = \sup_i ||a_i||$, and involution $(a_1, \ldots, a_k)^* = (a_1^*, \ldots, a_k^*)$.

Proof. This is indeed a reformulation of what we know from Theorem 1.28, in terms of our usual C^*-algebra formalism, from Definition 1.20. □

Let us record as well the quantum space formulation of our result:

Theorem 1.30. *The finite quantum spaces are exactly the disjoint unions of type*

$$X = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

where M_n is the finite quantum space given by $C(M_n) = M_n(\mathbb{C})$.

Proof. This is a reformulation of Theorem 1.29, by using the quantum space philosophy. Indeed, for a compact quantum space X, coming from a C^*-algebra A via the formula $A = C(X)$, being finite can only mean that the following number is finite:

$$|X| = \dim_{\mathbb{C}} A < \infty$$

Thus, by using Theorem 1.29, we are led to the conclusion that we must have:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

But since direct sums of algebras A correspond to disjoint unions of quantum spaces X, via the correspondence $A = C(X)$, this leads to the conclusion in the statement. □

Let us discuss now a key result, called GNS representation theorem, stating that any C^*-algebra appears as an operator algebra. As a first result here, we have:

Proposition 1.31. *Let A be a commutative C^*-algebra, write $A = C(X)$, with X being a compact space, and let μ be a positive measure on X. We have then*

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to the operator $g \to fg$.

Proof. Given a continuous function $f \in C(X)$, consider the operator $T_f(g) = fg$, on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^*-algebra embedding $A \subset B(H)$, as claimed. □

In order to prove the GNS representation theorem, we must extend the above construction, to the case where A is not necessarily commutative. Let us start with:

Definition 1.32. *Consider a C^*-algebra A.*

(1) *$\varphi : A \to \mathbb{C}$ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.*
(2) *$\varphi : A \to \mathbb{C}$ is called faithful and positive when $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.*

In the commutative case, $A = C(X)$, the positive elements are the positive functions, $f : X \to [0,\infty)$. As for the positive linear forms $\varphi : A \to \mathbb{C}$, these appear as follows, with μ being positive, and strictly positive if we want φ to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying "positive measures". We can use them as follows:

PROPOSITION 1.33. *Let $\varphi : A \to \mathbb{C}$ be a positive linear form.*

(1) *$< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.*
(2) *By separating and completing we obtain a Hilbert space H.*
(3) *$\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.*
(4) *If φ is faithful in the above sense, then π is faithful.*

PROOF. Almost everything here is straightforward, as follows:

(1) This is clear from definitions, and from the basic properties of the positive elements $a \geq 0$, which can be established exactly as in the $A = B(H)$ case.

(2) This is a standard procedure, which works for any scalar product, the idea being that of dividing by the vectors satisfying $< x, x >= 0$, then completing.

(3) All the verifications here are standard algebraic computations, in analogy with what we have seen many times, for multiplication operators, or group algebras.

(4) Assuming that we have $a \neq 0$, we have then $\pi(aa^*) \neq 0$, which in turn implies by faithfulness that we have $\pi(a) \neq 0$, which gives the result. □

In order to establish the embedding theorem, it remains to prove that any C^*-algebra has a faithful positive linear form $\varphi : A \to \mathbb{C}$. This is something more technical:

PROPOSITION 1.34. *Let A be a C^*-algebra.*

(1) *Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.*
(2) *A linear form φ is positive iff there is a norm one $h \in A_+$ such that $||\varphi|| = \varphi(h)$.*
(3) *For any $a \in A$ there exists a positive norm one form φ such that $\varphi(aa^*) = ||a||^2$.*
(4) *If A is separable there is a faithful positive form $\varphi : A \to \mathbb{C}$.*

PROOF. The proof here is quite technical, inspired from the existence proof of the probability measures on abstract compact spaces, the idea being as follows:

(1) This follows from Proposition 1.33, via the following estimate:

$$|\varphi(a)| \leq ||\pi(a)||\varphi(1) \leq ||a||\varphi(1)$$

(2) In one sense we can take $h = 1$. Conversely, let $a \in A_+$, $||a|| \leq 1$. We have:

$$|\varphi(h) - \varphi(a)| \leq ||\varphi|| \cdot ||h - a|| \leq \varphi(h)$$

Thus we have $Re(\varphi(a)) \geq 0$, and with $a = 1 - h$ we obtain:

$$Re(\varphi(1-h)) \geq 0$$

Thus $Re(\varphi(1)) \geq ||\varphi||$, and so $\varphi(1) = ||\varphi||$, so we can assume $h = 1$. Now observe that for any self-adjoint element a, and any $t \in \mathbb{R}$ we have, with $\varphi(a) = x + iy$:

$$\begin{aligned}
\varphi(1)^2(1 + t^2||a||^2) &\geq \varphi(1)^2||1 + t^2a^2|| \\
&= ||\varphi||^2 \cdot ||1 + ita||^2 \\
&\geq |\varphi(1 + ita)|^2 \\
&= |\varphi(1) - ty + itx| \\
&\geq (\varphi(1) - ty)^2
\end{aligned}$$

Thus we have $y = 0$, and this finishes the proof of our remaining claim.

(3) We can set $\varphi(\lambda aa^*) = \lambda||a||^2$ on the linear space spanned by aa^*, then extend this functional by Hahn-Banach, to the whole A. The positivity follows from (2).

(4) This is standard, by starting with a dense sequence (a_n), and taking the Cesàro limit of the functionals constructed in (3). We have $\varphi(aa^*) > 0$, and we are done. □

With these ingredients in hand, we can now state and prove:

THEOREM 1.35. *Any C^*-algebra appears as a norm closed $*$-algebra of operators*

$$A \subset B(H)$$

over a certain Hilbert space H. When A is separable, H can be taken to be separable.

PROOF. This result, called called GNS representation theorem after Gelfand, Naimark and Segal, follows indeed by combining Proposition 1.33 with Proposition 1.34. □

1e. Exercises

Exercises:

EXERCISE 1.36.

EXERCISE 1.37.

EXERCISE 1.38.

EXERCISE 1.39.

EXERCISE 1.40.

EXERCISE 1.41.

EXERCISE 1.42.

EXERCISE 1.43.

Bonus exercise.

CHAPTER 2

Quantum groups

2a. Quantum groups

We know from chapter 1 what the compact quantum spaces are, abstractly speaking. In order to get some intuition about such spaces, a good idea is that of working out first, with full details, the quantum group case. That is, we would like to know what the compact quantum groups really are, and what are their main properties.

Let us start with the finite case, which is elementary, and easy to explain. The idea will be that of calling finite quantum groups the compact quantum spaces G appearing via a formula of type $A = C(G)$, with the algebra A being finite dimensional, and having some suitable extra structure. In order to simplify the presentation, we use:

Definition 2.1. *Given a finite dimensional C^*-algebra A, any morphisms of type*

$$\Delta : A \to A \otimes A \quad , \quad \varepsilon : A \to \mathbb{C} \quad , \quad S : A \to A^{opp}$$

will be called comultiplication, counit and antipode.

The terminology comes from the fact that in the commutative case, $A = C(X)$, the morphisms Δ, ε, S are transpose to group-type operations, as follows:

$$m : X \times X \to X \quad , \quad u : \{.\} \to X \quad , \quad i : X \to X$$

The reasons for using the opposite algebra A^{opp} instead of A will become clear in a moment. Now with these conventions in hand, we can formulate:

Definition 2.2. *A finite dimensional Hopf algebra is a finite dimensional C^*-algebra A, with a comultiplication, counit and antipode, satisfying the conditions*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

along with the condition $S^2 = id$. Given such an algebra we write $A = C(G) = C^*(H)$, and call G, H finite quantum groups, dual to each other.

In this definition everything is standard, except for our usual choice to use C^*-algebras in all that we are doing, and also for the last axiom, $S^2 = id$. This axiom corresponds to the fact that, in the corresponding quantum group, we have:

$$(g^{-1})^{-1} = g$$

It is possible to prove that this condition is automatic, in the present C^*-algebra setting. However, this is something non-trivial, and since all this is just a preliminary discussion, not needed later, we have opted for including $S^2 = id$ in our axioms.

For reasons that will become clear in a moment, we say that a Hopf algebra A as above is cocommutative if, with $\Sigma(a \otimes b) = b \otimes a$ being the flip map, we have:

$$\Sigma\Delta = \Delta$$

With this convention made, we have the following result, which summarizes the basic theory of finite quantum groups, and justifies the terminology and axioms:

THEOREM 2.3. *The following happen:*

(1) *If G is a finite group then $C(G)$ is a commutative Hopf algebra, with*

$$\Delta(\varphi) = (g,h) \to \varphi(gh) \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S(\varphi) = g \to \varphi(g^{-1})$$

as structural maps. Any commutative Hopf algebra is of this form.

(2) *If H is a finite group then $C^*(H)$ is a cocommutative Hopf algebra, with*

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

as structural maps. Any cocommutative Hopf algebra is of this form.

(3) *If G, H are finite abelian groups, dual to each other via Pontrjagin duality, then we have an identification of Hopf algebras $C(G) = C^*(H)$.*

PROOF. These results are all elementary, the idea being as follows:

(1) The fact that Δ, ε, S satisfy the axioms is clear from definitions, and the converse follows from the Gelfand theorem, by working out the details, regarding Δ, ε, S.

(2) Once again, the fact that Δ, ε, S satisfy the axioms is clear from definitions, with the remark that the use of the opposite multiplication $(a, b) \to a \cdot b$ in really needed here, in order for the antipode S to be an algebra morphism, as shown by:

$$\begin{aligned} S(gh) &= (gh)^{-1} \\ &= h^{-1}g^{-1} \\ &= g^{-1} \cdot h^{-1} \\ &= S(g) \cdot S(h) \end{aligned}$$

For the converse, we use a trick. Let A be an arbitrary Hopf algebra, as in Definition 2.2, and consider its multiplication, unit, comultiplication, counit and antipode:

$$m:A\otimes A\to A\quad,\quad u:\mathbb C\to A$$
$$\Delta:A\to A\otimes A\quad,\quad \varepsilon:A\to\mathbb C$$
$$S:A\to A^{opp}$$

The transposes of these maps are then linear maps as follows:

$$\Delta^t:A^*\otimes A^*\to A^*\quad,\quad \varepsilon^t:\mathbb C\to A^*$$
$$m^t:A^*\to A^*\otimes A^*\quad,\quad u^t:A^*\to\mathbb C$$
$$S^t:A^*\to(A^*)^{opp}$$

It is routine to check that these maps make A^* into a Hopf algebra. Now assuming that A is cocommutative, it follows that A^* is commutative, so by (1) we obtain $A^*=C(G)$ for a certain finite group G, which in turn gives $A=C^*(G)$, as desired.

(3) This follows from the discussion in the proof of (2) above. □

This was for the basics of finite quantum groups, under the above axioms. It is possible to further build on this, but we will discuss this directly in the compact setting. So, getting now to the compact case, here is what we need to know:

Fact 2.4. *The compact Lie groups are exactly the closed subgroups $G\subset U_N$, and for such a closed subgroup the multiplication, unit and inverse operation are given by*

$$(UV)_{ij}=\sum_kU_{ik}V_{kj}\quad,\quad(1_N)_{ij}=\delta_{ij}\quad,\quad(U^{-1})_{ij}=U_{ji}^*$$

that is, the usual formulae for unitary matrices.

Now assuming that we are a bit familiar with Gelfand duality, and so are we, it should not be hard from this to axiomatize the algebras of type $A=C(G)$, with $G\subset U_N$ being a closed subgroup, and then lift the commutativity assumption on A.

Getting directly to the answer, and with the Gelfand duality details in the classical case, $G\subset U_N$, to be explained in a moment, in the proof of Proposition 2.6 below, we are led in this way to the following definition, due to Woronowicz [99]:

Definition 2.5. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that we have morphisms of C^*-algebras*

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to\mathbb C\quad,\quad S:A\to A^{opp}$$

given by the following formulae, on the standard generators u_{ij}:

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

In this case, we write $A=C(G)$, and call G a compact quantum Lie group.

All this is quite subtle, and there are countless comments to be made. Generally speaking, we will defer these comments for a bit later, once we'll know at least the basic examples, and also some basic theory. As some quick comments, however:

(1) In the above definition $A \otimes A$ can be any topological tensor product of A with itself, meaning C^*-algebraic completion of the usual algebraic tensor product, and with the choice of the exact $\otimes$ operation being irrelevant, because we will divide later the class of Woronowicz algebras by a certain equivalence relation, making the choice of $\otimes$ to be irrelevant. In short, good news, no troubles with $\otimes$, and more on this later.

(2) Generally speaking, the above definition is motivated by Fact 2.4, and a bit of Gelfand duality thinking, and we will see details in a moment, in the proof of Proposition 2.6 below. The morphisms Δ, ε, S are called comultiplication, counit and antipode. Observe that if these morphisms exist, they are unique. This is analogous to the fact that a closed set of unitary matrices $G \subset U_N$ is either a compact group, or not.

So, getting started now, and taking Definition 2.5 as it is, mysterious new thing, that we will have to explore, we first have the following result:

Proposition 2.6. *Given a closed subgroup $G \subset U_N$, the algebra $A = C(G)$, with the matrix formed by the standard coordinates $u_{ij}(g) = g_{ij}$, is a Woronowicz algebra, and:*

1. *For this algebra, the morphisms Δ, ε, S appear as functional analytic transposes of the multiplication, unit and inverse maps m, u, i of the group G.*
2. *This Woronowicz algebra is commutative, and conversely, any Woronowicz algebra which is commutative appears in this way.*

Proof. Since we have $G \subset U_N$, the matrix $u = (u_{ij})$ is unitary. Also, since the coordinates u_{ij} separate the points of G, by the Stone-Weierstrass theorem we obtain that the $*$-subalgebra $\mathcal{A} \subset C(G)$ generated by them is dense. Finally, the fact that we have morphisms Δ, ε, S as in Definition 2.5 follows from the proof of (1) below.

(1) We use the formulae for U_N from Fact 2.4. The fact that the transpose of the multiplication m^t satisfies the condition in Definition 2.5 follows from:

$$\begin{aligned} m^t(u_{ij})(U \otimes V) &= (UV)_{ij} \\ &= \sum_k U_{ik} V_{kj} \\ &= \sum_k (u_{ik} \otimes u_{kj})(U \otimes V) \end{aligned}$$

Regarding now the transpose of the unit map u^t, the verification of the condition in Definition 2.5 is trivial, coming from the following equalities:

$$u^t(u_{ij}) = 1_{ij} = \delta_{ij}$$

Finally, the transpose of the inversion map i^t verifies the condition in Definition 2.5, because we have the following computation, valid for any $U\in G$:

$$i^t(u_{ij})(U)=(U^{-1})_{ij}=\bar{U}_{ji}=u_{ji}^*(U)$$

(2) Assume that A is commutative. By using the Gelfand theorem, we can write $A=C(G)$, with G being a certain compact space. By using now the coordinates u_{ij}, we obtain an embedding $G\subset U_N$. Finally, by using Δ,ε,S, it follows that the subspace $G\subset U_N$ that we have obtained is in fact a closed subgroup, and we are done. □

Let us go back now to the general setting of Definition 2.5. According to Proposition 2.6, and to the general C^*-algebra philosophy, the morphisms Δ,ε,S can be thought of as coming from a multiplication, unit map and inverse map, as follows:

$$m:G\times G\to G\quad,\quad u:\{.\}\to G\quad,\quad i:G\to G$$

Here is a first result of this type, expressing in terms of Δ,ε,S the fact that the underlying maps m,u,i should satisfy the usual group theory axioms:

Proposition 2.7. *The comultiplication, counit and antipode have the following properties, on the dense $*$-subalgebra $\mathcal{A}\subset A$ generated by the variables u_{ij}:*

1. *Coassociativity: $(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$.*
2. *Counitality: $(id\otimes\varepsilon)\Delta=(\varepsilon\otimes id)\Delta=id$.*
3. *Coinversality: $m(id\otimes S)\Delta=m(S\otimes id)\Delta=\varepsilon(.)1$.*

In addition, the square of the antipode is the identity, $S^2=id$.

Proof. Observe first that the result holds in the case where A is commutative. Indeed, by using Proposition 2.6 we can write:

$$\Delta=m^t\quad,\quad\varepsilon=u^t\quad,\quad S=i^t$$

The above 3 conditions come then by transposition from the basic 3 group theory conditions satisfied by m,u,i, which are as follows, with $\delta(g)=(g,g)$:

$$m(m\times id)=m(id\times m)$$

$$m(id\times u)=m(u\times id)=id$$

$$m(id\times i)\delta=m(i\times id)\delta=1$$

Observe that $S^2=id$ is satisfied as well, coming from $i^2=id$, which is a consequence of the group axioms. In general now, the proof goes as follows:

(1) We have indeed the following computation:

$$(\Delta\otimes id)\Delta(u_{ij})=\sum_l\Delta(u_{il})\otimes u_{lj}=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

On the other hand, we have as well the following computation:

$$(id\otimes\Delta)\Delta(u_{ij})=\sum_k u_{ik}\otimes\Delta(u_{kj})=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

(2) The proof here is quite similar. We first have the following computation:

$$(id\otimes\varepsilon)\Delta(u_{ij})=\sum_k u_{ik}\otimes\varepsilon(u_{kj})=u_{ij}$$

On the other hand, we have as well the following computation:

$$(\varepsilon\otimes id)\Delta(u_{ij})=\sum_k \varepsilon(u_{ik})\otimes u_{kj}=u_{ij}$$

(3) By using the fact that the matrix $u=(u_{ij})$ is unitary, we obtain:

$$\begin{aligned}
m(id\otimes S)\Delta(u_{ij}) &= \sum_k u_{ik}S(u_{kj})\\
&= \sum_k u_{ik}u_{jk}^*\\
&= (uu^*)_{ij}\\
&= \delta_{ij}
\end{aligned}$$

Similarly, we have the following computation:

$$\begin{aligned}
m(S\otimes id)\Delta(u_{ij}) &= \sum_k S(u_{ik})u_{kj}\\
&= \sum_k u_{ki}^*u_{kj}\\
&= (u^*u)_{ij}\\
&= \delta_{ij}
\end{aligned}$$

Finally, the formula $S^2=id$ holds as well on the generators, and we are done. □

Still at the theoretical level, we have as well the following useful result:

PROPOSITION 2.8. *Given a Woronowicz algebra (A,u), we have*

$$u^t=\bar{u}^{-1}$$

so the matrix $u=(u_{ij})$ is a biunitary, meaning unitary, with unitary transpose.

PROOF. The idea is that $u^t=\bar{u}^{-1}$ comes from $u^*=u^{-1}$, by applying the antipode. Indeed, by denoting $(a,b)\to a\cdot b$ the multiplication of A^{opp}, we have:

$$\begin{aligned}(uu^*)_{ij}=\delta_{ij}\quad&\implies\quad\sum_k u_{ik}u_{jk}^*=\delta_{ij}\\&\implies\quad\sum_k S(u_{ik})\cdot S(u_{jk}^*)=\delta_{ij}\\&\implies\quad\sum_k u_{kj}u_{ki}^*=\delta_{ij}\\&\implies\quad(u^t\bar{u})_{ji}=\delta_{ij}\end{aligned}$$

Similarly, we have the following computation:

$$\begin{aligned}(u^*u)_{ij}=\delta_{ij}\quad&\implies\quad\sum_k u_{ki}^*u_{kj}=\delta_{ij}\\&\implies\quad\sum_k S(u_{ki}^*)\cdot S(u_{kj})=\delta_{ij}\\&\implies\quad\sum_k u_{jk}^*u_{ik}=\delta_{ij}\\&\implies\quad(\bar{u}u^t)_{ji}=\delta_{ij}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Let us discuss now another class of basic examples, namely the group duals:

PROPOSITION 2.9. *Given a finitely generated discrete group $\Gamma=<g_1,\ldots,g_N>$, the group algebra $A=C^*(\Gamma)$, together with the diagonal matrix formed by the standard generators, $u=diag(g_1,\ldots,g_N)$, is a Woronowicz algebra, with Δ,ε,S given by:*

$$\Delta(g)=g\otimes g\quad,\quad\varepsilon(g)=1\quad,\quad S(g)=g^{-1}$$

This Woronowicz algebra is cocommutative, in the sense that $\Sigma\Delta=\Delta$.

PROOF. Since the involution on $C^*(\Gamma)$ is given by $g^*=g^{-1}$, the standard generators $g_1,\ldots,g_N$ are unitaries, and so must be the diagonal matrix $u=diag(g_1,\ldots,g_N)$ formed by them. Also, since $g_1,\ldots,g_N$ generate Γ, these elements generate the group algebra $C^*(\Gamma)$ as well, in the algebraic sense. Let us verify now the axioms in Definition 2.5:

(1) Consider the following map, which is a unitary representation:

$$\Gamma\to C^*(\Gamma)\otimes C^*(\Gamma)\quad,\quad g\to g\otimes g$$

This representation extends, as desired, into a morphism of algebras, as follows:

$$\Delta:C^*(\Gamma)\to C^*(\Gamma)\otimes C^*(\Gamma)\quad,\quad\Delta(g)=g\otimes g$$

(2) The situation for ε is similar, because this comes from the trivial representation:

$$\Gamma \to \{1\} \quad , \quad g \to 1$$

(3) Finally, the antipode S comes from the following unitary representation:

$$\Gamma \to C^*(\Gamma)^{opp} \quad , \quad g \to g^{-1}$$

Summarizing, we have shown that we have a Woronowicz algebra, with Δ, ε, S being as in the statement. Regarding now the last assertion, observe that we have:

$$\Sigma\Delta(g) = \Sigma(g \otimes g) = g \otimes g = \Delta(g)$$

Thus $\Sigma\Delta = \Delta$ holds on the group elements $g \in \Gamma$, and by linearity and continuity, this formula must hold on the whole algebra $C^*(\Gamma)$, as desired. $\square$

We will see later that any cocommutative Woronowicz algebra appears as above, up to a standard equivalence relation for such algebras, and with this being something non-trivial. In the abelian group case now, we have a more precise result, as follows:

PROPOSITION 2.10. *Assume that Γ as above is abelian, and let $G = \widehat{\Gamma}$ be its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$. The canonical isomorphism*

$$C^*(\Gamma) \simeq C(G)$$

transforms the comultiplication, counit and antipode of $C^*(\Gamma)$ into the comultiplication, counit and antipode of $C(G)$, and so is a compact quantum group isomorphism.

PROOF. Assume indeed that $\Gamma =< g_1, \ldots, g_N >$ is abelian. Then with $G = \widehat{\Gamma}$ we have a group embedding $G \subset U_N$, constructed as follows:

$$\chi \to \begin{pmatrix} \chi(g_1) & & \\ & \ddots & \\ & & \chi(g_N) \end{pmatrix}$$

Thus, we have two Woronowicz algebras to be compared, namely $C(G)$, constructed as in Proposition 2.6, and $C^*(\Gamma)$, constructed as in Proposition 2.9. We already know from chapter 1 that the underlying C^*-algebras are isomorphic. Now since Δ, ε, S agree on $g_1, \ldots, g_N$, they agree everywhere, and we are led to the above conclusions. $\square$

As a conclusion to this, we can supplement Definition 2.5 with:

DEFINITION 2.11. *Given a Woronowicz algebra $A = C(G)$, we write as well*

$$A = C^*(\Gamma)$$

and call $\Gamma = \widehat{G}$ a finitely generated discrete quantum group.

However, things are still not over with this, axiomatically, because in view of various amenability issues that can appear, we must make as well the following convention:

Definition 2.12. *Given two Woronowicz algebras (A, u) and (B, v), we write*

$$A \simeq B$$

and identify the corresponding quantum groups, when we have an isomorphism

$$< u_{ij} > \simeq < v_{ij} >$$

of $*$-algebras, mapping standard coordinates to standard coordinates.

With this convention, the functoriality problem is fixed, any compact or discrete quantum group corresponding to a unique Woronowicz algebra, up to equivalence.

As another comment, we can now see why in Definition 2.5 the choice of the exact topological tensor product $\otimes$ is irrelevant. Indeed, no matter what tensor product $\otimes$ we use there, we end up with the same Woronowicz algebra, and the same compact and discrete quantum groups, up to equivalence. In practice, we will use in what follows the simplest such tensor product $\otimes$, which is the so-called maximal one, obtained as completion of the usual algebraic tensor product with respect to the biggest C^*-norm. With the remark that this maximal tensor product is something rather algebraic and abstract, and so can be treated, in practice, as a usual algebraic tensor product.

We will be back to this later, with a number of supplementary comments, and some further results on the subject, when talking about amenability.

2b. Representations

In order to reach to some more advanced insight into the structure of the compact quantum groups, we can use representation theory. We follow Woronowicz's paper [99], with a few simplifications coming from our $S^2 = id$ formalism. We first have:

Definition 2.13. *A corepresentation of a Woronowicz algebra (A, u) is a unitary matrix $v \in M_n(\mathcal{A})$ over the dense $*$-algebra $\mathcal{A} =< u_{ij} >$, satisfying:*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

That is, v must satisfy the same conditions as u.

As basic examples here, we have the trivial corepresentation, having dimension 1, as well as the fundamental corepresentation, and its adjoint:

$$1 = (1) \quad , \quad u = (u_{ij}) \quad , \quad \bar{u} = (u_{ij}^*)$$

In the classical case, we recover in this way the usual representations of G:

Proposition 2.14. *Given a closed subgroup $G \subset U_N$, the corepresentations of the associated Woronowicz algebra $C(G)$ are in one-to-one correspondence, given by*

$$\pi(g) = \begin{pmatrix} v_{11}(g) & \dots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \dots & v_{nn}(g) \end{pmatrix}$$

with the finite dimensional unitary smooth representations of G.

Proof. We have indeed a correspondence $v \leftrightarrow \pi$, between the unitary matrices satisfying the equations in Definition 2.13, and the finite dimensional unitary representations of G. Regarding now the smoothness part, this is something more subtle, which requires some knowledge of Lie theory. The point is that any closed subgroup $G \subset U_N$ is a Lie group, and since the coefficient functions $u_{ij} : G \to \mathbb{C}$ are smooth, we have:

$$\mathcal{A} \subset C^\infty(G)$$

Thus, when assuming $v \in M_n(\mathcal{A})$, the corresponding representation $\pi : G \to U_n$ is smooth, and the converse of this fact is known to hold as well. □

In general now, we have the following operations on the corepresentations:

Proposition 2.15. *The corepresentations are subject to the following operations:*

1. *Making sums, $v + w = diag(v, w)$.*
2. *Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.*
3. *Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.*
4. *Spinning by unitaries, $w = UvU^*$.*

Proof. Observe that the result holds in the commutative case, where we obtain the usual operations on the representations of the corresponding group. In general now:

(1) Everything here is clear from definitions.

(2) First of all, the matrix $v \otimes w$ is unitary. Indeed, we have:

$$\begin{aligned} \sum_{jb} (v \otimes w)_{ia,jb} (v \otimes w)^*_{kc,jb} &= \sum_{jb} v_{ij} w_{ab} w^*_{cb} v^*_{kj} \\ &= \delta_{ac} \sum_j v_{ij} v^*_{kj} \\ &= \delta_{ik} \delta_{ac} \end{aligned}$$

In the other sense, the computation is similar, as follows:

$$\begin{aligned}\sum_{jb}(v\otimes w)^*_{jb,ia}(v\otimes w)_{jb,kc} &= \sum_{jb}w^*_{ba}v^*_{ji}v_{jk}w_{bc}\\ &= \delta_{ik}\sum_b w^*_{ba}w_{bc}\\ &= \delta_{ik}\delta_{ac}\end{aligned}$$

The comultiplicativity condition follows from the following computation:

$$\begin{aligned}\Delta((v\otimes w)_{ia,jb}) &= \sum_{kc}v_{ik}w_{ac}\otimes v_{kj}w_{cb}\\ &= \sum_{kc}(v\otimes w)_{ia,kc}\otimes(v\otimes w)_{kc,jb}\end{aligned}$$

The proof of the counitality condition is similar, as follows:

$$\varepsilon((v\otimes w)_{ia,jb})=\delta_{ij}\delta_{ab}=\delta_{ia,jb}$$

As for the condition involving the antipode, this can be checked as follows:

$$S((v\otimes w)_{ia,jb})=w^*_{ba}v^*_{ji}=(v\otimes w)^*_{jb,ia}$$

(3) In order to check that $\bar{v}$ is unitary, we can use the antipode, exactly as we did before, for $\bar{u}$. As for the comultiplicativity axioms, these are all clear.

(4) The matrix $w=UvU^*$ is unitary, and its comultiplicativity properties can be checked by doing some computations. Here is however another proof of this fact, using a trick. In the context of Definition 2.13, if we write $v\in M_n(\mathbb{C})\otimes A$, the axioms read:

$$(id\otimes\Delta)v=v_{12}v_{13}\quad,\quad(id\otimes\varepsilon)v=1\quad,\quad(id\otimes S)v=v^*$$

Here we use standard tensor calculus conventions. Now when spinning by a unitary the matrix that we obtain, with these conventions, is $w=U_1vU_1^*$, and we have:

$$\begin{aligned}(id\otimes\Delta)w &= U_1v_{12}v_{13}U_1^*\\ &= U_1v_{12}U_1^*\cdot U_1v_{13}U_1^*\\ &= w_{12}w_{13}\end{aligned}$$

The proof of the counitality condition is similar, as follows:

$$(id\otimes\varepsilon)w=U\cdot 1\cdot U=1$$

Finally, the last condition, involving the antipode, can be checked as follows:

$$(id\otimes S)w=U_1v^*U_1^*=w^*$$

Thus, with usual notations, $w=UvU^*$ is a corepresentation, as claimed. □

In the group dual case, we have the following result:

PROPOSITION 2.16. *Assume $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$.*

1. *Any group element $h \in \Gamma$ is a 1-dimensional corepresentation of A, and the operations on corepresentations are the usual ones on group elements.*
2. *Any diagonal matrix of type $v = diag(h_1, \ldots, h_n)$, with $n \in \mathbb{N}$ arbitrary, and with $h_1, \ldots, h_n \in \Gamma$, is a corepresentation of A.*
3. *More generally, any matrix of type $w = U diag(h_1, \ldots, h_n) U^*$ with $h_1, \ldots, h_n \in \Gamma$ and with $U \in U_n$, is a corepresentation of A.*

PROOF. These assertions are all elementary, as follows:

(1) The first assertion is clear from definitions and from the comultiplication, counit and antipode formulae for the discrete group algebras, namely:

$$\Delta(h) = h \otimes h \quad , \quad \varepsilon(h) = 1 \quad , \quad S(h) = h^{-1}$$

The assertion on the operations is clear too, because we have:

$$(g) \otimes (h) = (gh) \quad , \quad \overline{(g)} = (g^{-1})$$

(2) This follows from (1) by performing sums, as in Proposition 2.15.

(3) This follows from (2), again via Proposition 2.15. As a comment here, we will see later that all corepresentations of $C^*(\Gamma)$ appear in this way. □

Let us develop now Peter-Weyl theory. Let us start with:

DEFINITION 2.17. *Given two corepresentations $v \in M_n(A), w \in M_m(A)$, we set*

$$Hom(v, w) = \left\{ T \in M_{m \times n}(\mathbb{C}) \Big| Tv = wT \right\}$$

and we use the following conventions:

1. *We use the notations $Fix(v) = Hom(1, v)$, and $End(v) = Hom(v, v)$.*
2. *We write $v \sim w$ when $Hom(v, w)$ contains an invertible element.*
3. *We say that v is irreducible, and write $v \in Irr(G)$, when $End(v) = \mathbb{C}1$.*

In the classical case $A = C(G)$ we obtain the usual notions concerning the representations. Observe also that in the group dual case we have:

$$g \sim h \iff g = h$$

Finally, observe that $v \sim w$ means that v, w are conjugated by an invertible matrix. Here are a few basic results, regarding the above Hom spaces:

PROPOSITION 2.18. *We have the following results:*

1. $T \in Hom(u, v), S \in Hom(v, w) \implies ST \in Hom(u, w)$.
2. $S \in Hom(p, q), T \in Hom(v, w) \implies S \otimes T \in Hom(p \otimes v, q \otimes w)$.
3. $T \in Hom(v, w) \implies T^* \in Hom(w, v)$.

In other words, the Hom spaces form a tensor $*$-category.

PROOF. These assertions are all elementary, as follows:

(1) By using our assumptions $Tu = vT$ and $Sv = Ws$ we obtain, as desired:

$$STu = SvT = wST$$

(2) Assume indeed that we have $Sp = qS$ and $Tv = wT$. We have then:

$$\begin{aligned}(S \otimes T)(p \otimes v) &= S_1T_2p_{13}v_{23} \\ &= (Sp)_{13}(Tv)_{23}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}(q \otimes w)(S \otimes T) &= q_{13}w_{23}S_1T_2 \\ &= (qS)_{13}(wT)_{23}\end{aligned}$$

The quantities on the right being equal, this gives the result.

(3) By conjugating, and then using the unitarity of v, w, we obtain, as desired:

$$\begin{aligned}Tv = wT &\implies v^*T^* = T^*w^* \\ &\implies vv^*T^*w = vT^*w^*w \\ &\implies T^*w = vT^*\end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3) above, the Hom spaces are linear spaces, and contain the units. In short, this is just a theoretical remark, that will be used only later on. □

Still following Woronowicz [99], we have the following key result:

THEOREM 2.19. *Any Woronowicz algebra $A = C(G)$ has a Haar integration,*

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

which can be constructed by starting with any faithful positive form $\varphi \in A^*$, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation $v \in M_n(\mathbb{C}) \otimes A$ we have

$$\left(id \otimes \int_G\right)v = P$$

where P is the orthogonal projection onto $Fix(v) = \left\{\xi \in \mathbb{C}^n \middle| v\xi = \xi\right\}$.

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a=(\tau\otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v)=\left\{\xi\in\mathbb{C}^n\Big|v\xi=\xi\right\}$$

The point now is that when $\varphi\in A^*$ is faithful, by using a standard positivity trick, one can prove that we have $P_\varphi=P$. Assume indeed $P_\varphi\xi=\xi$, and let us set:

$$a=\sum_i\left(\sum_j v_{ij}\xi_j-\xi_i\right)\left(\sum_k v_{ik}\xi_k-\xi_i\right)^*$$

We must prove that we have $a=0$. Since v is biunitary, we have:

$$\begin{aligned}
a&=\sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar{\xi}_k-\frac{1}{N}\bar{\xi}_i\right)\right)\\
&=\sum_{ijk}v_{ij}v_{ik}^*\xi_j\bar{\xi}_k-\frac{1}{N}v_{ij}\xi_j\bar{\xi}_i-\frac{1}{N}v_{ik}^*\xi_i\bar{\xi}_k+\frac{1}{N^2}\xi_i\bar{\xi}_i\\
&=\sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar{\xi}_i-\sum_{ik}v_{ik}^*\xi_i\bar{\xi}_k+\sum_i|\xi_i|^2\\
&=||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&=2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $P_\varphi\xi=\xi$, we obtain from this:

$$\begin{aligned}
\varphi(a)&=2\varphi(||\xi||^2-Re(<v\xi,\xi>))\\
&=2(||\xi||^2-Re(<P_\varphi\xi,\xi>))\\
&=2(||\xi||^2-||\xi||^2)\\
&=0
\end{aligned}$$

Now since φ is faithful, this gives $a = 0$, and so $v\xi = \xi$. Thus $\int_\varphi$ is independent of φ, and is given on coefficients $a = (\tau \otimes id)v$ by the following formula:

$$\left(id \otimes \int_\varphi\right) v = P$$

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

Consider the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

Also, given a corepresentation $v \in M_n(\mathcal{A})$, define its character as being its trace:

$$\chi_v = \sum_{i=1}^n v_{ii} \in \mathcal{A}$$

Still following Woronowicz [99], we have the following result:

THEOREM 2.20. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. *$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [99], the idea being as follows:

(1) Given a corepresentation $v \in M_n(A)$, consider its intertwiner algebra:

$$End(v) = \left\{ T \in M_n(\mathbb{C}) \Big| Tv = vT \right\}$$

According to the results in Proposition 2.18 this is a finite dimensional C^*-algebra, and we conclude from this that we have a decomposition as follows:

$$End(v) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1 = p_1 + \ldots + p_k$$

But this decomposition allows us to define subcorepresentations $v_i \subset v$, which are irreducible, so we obtain, as desired, a decomposition $v = v_1 + \ldots + v_k$.

(2) To any corepresentation $v\in M_n(A)$ we associate its space of coefficients, given by $C(v)=span(v_{ij})$. The construction $v\to C(v)$ is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A}=\sum_{k\in\mathbb{N}*\mathbb{N}}C(u^{\otimes k})$$

Now given an arbitrary corepresentation $v\in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v)\subset\mathcal{A}$, and so we must have, for certain positive integers $k_1,\ldots,k_p$, an inclusion of vector spaces, as follows:

$$C(v)\subset C(u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v\subset u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p}$$

Thus, by using (1), we are led to the conclusion in the statement.

(3) By using (1) and (2), we obtain a linear space decomposition as follows:

$$\mathcal{A}=\sum_{v\in Irr(A)}C(v)=\sum_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$$

In order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C(v)\perp C(w)$$

As a first observation, which follows from an elementary computation, for any two corepresentations v,w we have a Frobenius type isomorphism, as follows:

$$Hom(v,w)\simeq Fix(\bar{v}\otimes w)$$

Now let us set $P_{ia,jb}=\int_G v_{ij}w_{ab}^*$. According to Theorem 2.19, the matrix P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(\bar{v},\bar{w})=\{0\}$$

Thus we have $P=0$, and so $C(v)\perp C(w)$, which gives the result.

(4) The algebra $\mathcal{A}_{central}$ contains indeed all the characters, because we have:

$$\Sigma\Delta(\chi_v)=\sum_{ij}v_{ji}\otimes v_{ij}=\Delta(\chi_v)$$

The fact that the characters span $\mathcal{A}_{central}$, and form an orthogonal basis of it, follow from (3). Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl}=\int_G v_{ij}v_{kl}^*$$

We know from Theorem 2.19 that these integrals form the orthogonal projection onto $Fix(v\otimes\bar{v})\simeq End(\bar{v})=\mathbb{C}1$. By using this fact, we obtain the following formula:

$$\int_G\chi_v\chi_v^*=\sum_{ij}\int_G v_{ii}v_{jj}^*=\sum_i\frac{1}{N}=1$$

Thus the characters have indeed norm 1, and we are done. □

As an application of Theorem 2.20, when A is cocommutative the irreducible corepresentations must be all 1-dimensional, and so $A=C^*(\Gamma)$, for some discrete group Γ, and with this clarifying some questions that we were having before.

At a more technical level now, we have the following key result:

THEOREM 2.21. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full}\to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon:A_{full}\to\mathbb{C}$ factorizes through A_{red}.*
4. *We have $N\in\sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A=C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon:A_{red}\to\mathbb{C}$ produces an isomorphism $A_{red}\to A_{full}$, via a formula of type $(\varepsilon\otimes id)\Phi$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N-Re(\chi(u)))=0$, and the converse can be proved by doing some standard functional analysis. □

2c. Free rotations

Time now to discuss some key new examples. Following Wang, we have:

THEOREM 2.22. *The following constructions produce compact quantum groups,*

$$C(O_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)$$
$$C(U_N^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

which appear respectively as liberations of the groups O_N and U_N.

PROOF. This is something quite elementary, the idea being as follows:

(1) The first assertion follows from the standard fact that if a matrix $u=(u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u^\Delta_{ij}=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u^\varepsilon_{ij}=\delta_{ij}\quad,\quad u^S_{ij}=u^*_{ji}$$

(2) Indeed, the biunitarity of u^Δ can be checked by a direct computation. Regarding now the matrix $u^\varepsilon=1_N$, this is clearly biunitary. Also, regarding the matrix u^S, there is nothing to prove here either, because its unitarity its clear too. And finally, observe that if u has self-adjoint entries, then so do the above matrices $u^\Delta,u^\varepsilon,u^S$.

(3) Thus our claim is proved, and we can define morphisms Δ,ε,S as in Definition 2.5, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$.

(4) As for the second assertion, this follows from the Gelfand theorem. □

The basic properties of O_N^+,U_N^+ can be summarized as follows:

THEOREM 2.23. *The quantum groups O_N^+,U_N^+ have the following properties:*

1. *The closed subgroups $G\subset U_N^+$ are exactly the $N\times N$ compact quantum groups. As for the closed subgroups $G\subset O_N^+$, these are those satisfying $u=\bar{u}$.*
2. *We have liberation embeddings $O_N\subset O_N^+$ and $U_N\subset U_N^+$, obtained by dividing the algebras $C(O_N^+),C(U_N^+)$ by their respective commutator ideals.*
3. *We have as well embeddings $\widehat{L}_N\subset O_N^+$ and $\widehat{F}_N\subset U_N^+$, where L_N is the free product of N copies of $\mathbb{Z}_2$, and where F_N is the free group on N generators.*

PROOF. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 2.5, the formula $S(u_{ij})=u^*_{ji}$ shows that the matrix $\bar{u}$ must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N),C(U_N)$, similar to those in Theorem 2.22, but with the commutativity between the standard coordinates and their adjoints added:

$$\begin{aligned}C(O_N)&=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)\\ C(U_N)&=C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)\end{aligned}$$

Thus, we are led to the conclusion in the statement.

(3) This follows indeed from (1) and from Proposition 2.9, with the remark that with $u=diag(g_1,\ldots,g_N)$, the condition $u=\bar{u}$ is equivalent to $g_i^2=1$, for any i. □

The last assertion in Theorem 2.23 suggests the following construction:

PROPOSITION 2.24. *Given a closed subgroup $G \subset U_N^+$, consider its diagonal torus, which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

PROOF. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

With this notion in hand, Theorem 2.23 (3) reformulates as follows:

THEOREM 2.25. *The diagonal tori of the basic unitary groups are the basic tori:*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

In particular, the basic unitary groups are all distinct.

PROOF. This is something clear and well-known in the classical case, and in the free case, this is a reformulation of Theorem 2.23 (3), which tells us that the diagonal tori of O_N^+, U_N^+, in the sense of Proposition 2.24, are the group duals $\widehat{L}_N, \widehat{F}_N$. □

2d. Free reflections

In order to introduce now quantum reflection groups, things are more tricky, involving quantum permutation groups. Following Wang, let us start with:

THEOREM 2.26. *The following construction, where "magic" means formed of projections, which sum up to 1 on each row and column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \Big| u = \text{magic} \right)$$

produces a quantum group liberation of S_N. Moreover, the inclusion

$$S_N \subset S_N^+$$

is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. We have several things to be proved, the idea being as follows:

(1) The quantum group assertion follows by using the same arguments as those in the proof of Theorem 2.22. Consider indeed the following matrix:

$$U_{ij}=\sum_k u_{ik}\otimes u_{kj}$$

As a first observation, the entries of this matrix are self-adjoint, $U_{ij}=U_{ij}^*$. In fact the entries U_{ij} are orthogonal projections, because we have as well:

$$U_{ij}^2=\sum_{kl}u_{ik}u_{il}\otimes u_{kj}u_{lj}=\sum_k u_{ik}\otimes u_{kj}=U_{ij}$$

In order to prove now that the matrix $U=(U_{ij})$ is magic, it remains to verify that the sums on the rows and columns are 1. For the rows, this can be checked as follows:

$$\sum_j U_{ij}=\sum_{jk}u_{ik}\otimes u_{kj}=\sum_k u_{ik}\otimes 1=1\otimes 1$$

For the columns the computation is similar, as follows:

$$\sum_i U_{ij}=\sum_{ik}u_{ik}\otimes u_{kj}=\sum_k 1\otimes u_{kj}=1\otimes 1$$

Thus the $U=(U_{ij})$ is magic, and so we can define a comultiplication map by using the universality property of $C(S_N^+)$, by setting $\Delta(u_{ij})=U_{ij}$. By using a similar reasoning, we can define as well a counit map by $\varepsilon(u_{ij})=\delta_{ij}$, and an antipode map by $S(u_{ij})=u_{ji}$. Thus the Woronowicz algebra axioms from Definition 2.5 are satisfied, and this finishes the proof of the first assertion, stating that S_N^+ is indeed a compact quantum group.

(2) Observe now that we have an embedding of compact quantum groups $S_N\subset S_N^+$, obtained by using the standard coordinates of S_N, viewed as an algebraic group:

$$u_{ij}=\chi\left(\sigma\in S_N\Big|\sigma(j)=i\right)$$

By using the Gelfand theorem and working out the details, as we did with the free spheres are free unitary groups, the embedding $S_N\subset S_N^+$ is indeed a liberation.

(3) Finally, regarding the last assertion, the study here is as follows:

Case $N=2$. The result here is trivial, the 2×2 magic matrices being by definition as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of a 2×2 magic matrix must pairwise commute, and so the algebra $C(S_2^+)$ follows to be commutative, which gives the result.

Case $N = 3$. By using the same argument as in the $N = 2$ case, and permuting rows and columns, it is enough to check that u_{11}, u_{22} commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, this gives $u_{22}u_{11} = u_{11}u_{22}u_{11}$, and then $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N = 4$. In order to prove our various claims about S_4^+, consider the following matrix, with p, q being projections, on some infinite dimensional Hilbert space:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and if we choose p, q as for the algebra $< p, q >$ to be not commutative, and infinite dimensional, we conclude that $C(S_4^+)$ is not commutative and infinite dimensional as well, and in particular is not isomorphic to $C(S_4)$.

Case $N \geq 5$. Here we can use the standard embedding $S_4^+ \subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u \to \begin{pmatrix} u & 0 \\ 0 & 1_{N-4} \end{pmatrix}$$

Indeed, with this embedding in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N \geq 5$ has these two properties as well. $\square$

The above result came as a surprise, in the late 1990s, and there has been a lot of work since, in order to understand what the quantum permutations really are, at $N \geq 4$. We will be back to this, on several occasions. For the moment, let us just record the following alternative approach to S_N^+, which shows that we are not wrong with our formalism:

Proposition 2.27. *The quantum group S_N^+ acts on the set $X = \{1, \ldots, N\}$, the corresponding coaction map $\Phi : C(X) \to C(X) \otimes C(S_N^+)$ being given by:*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

In fact, S_N^+ is the biggest compact quantum group acting on X, by leaving the counting measure invariant, in the sense that $(tr \otimes id)\Phi = tr(.)1$, where $tr(e_i) = \frac{1}{N}, \forall i$.

PROOF. Our claim is that given a compact matrix quantum group G, the following formula defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u = (u_{ij})$ is a magic corepresentation of $C(G)$:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Indeed, let us first determine when Φ is multiplicative. We have:

$$\Phi(e_i)\Phi(e_k) = \sum_{jl} e_j e_l \otimes u_{ji}u_{lk} = \sum_j e_j \otimes u_{ji}u_{jk}$$

On the other hand, we have as well the following computation:

$$\Phi(e_i e_k) = \delta_{ik}\Phi(e_i) = \delta_{ik}\sum_j e_j \otimes u_{ji}$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ji}u_{jk} = \delta_{ik}u_{ji} \quad , \quad \forall i, j, k$$

Regarding now the unitality of Φ, we have the following formula:

$$\Phi(1) = \sum_i \Phi(e_i) = \sum_{ij} e_j \otimes u_{ji} = \sum_j e_j \otimes \left(\sum_i u_{ji}\right)$$

Thus Φ is unital when $\sum_i u_{ji} = 1$, $\forall j$. Finally, the fact that Φ is a $*$-morphism translates into $u_{ij} = u_{ij}^*$, $\forall i, j$. Summing up, in order for $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ to be a morphism of C^*-algebras, the elements u_{ij} must be projections, summing up to 1 on each row of u. Regarding now the preservation of the trace condition, observe that we have:

$$(tr \otimes id)\Phi(e_i) = \frac{1}{N}\sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the columns of u. We conclude from this that $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ is a morphism of C^*-algebras preserving the trace precisely when u is magic, and since the coaction conditions on Φ are equivalent to the fact that u must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. □

With the above results in hand, we can now introduce the quantum reflections. Following Bichon, let us start with the following standard construction:

PROPOSITION 2.28. *Given closed subgroups $G \subset U_N^+$, $H \subset S_k^+$, with fundamental corepresentations u, v, the following construction produces a closed subgroup of U_{Nk}^+:*

$$C(G \wr_* H) = (C(G)^{*k} * C(H)) / < [u_{ij}^{(a)}, v_{ab}] = 0 >$$

In the case where G, H are classical, the classical version of $G \wr_* H$ is the usual wreath product $G \wr H$. Also, when G is a quantum permutation group, so is $G \wr_* H$.

PROOF. Consider indeed the matrix $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$, over the quotient algebra in the statement. It is routine to check that w is unitary, and in the case $G \subset S_N^+$, our claim is that this matrix is magic. Indeed, the entries are projections, because they appear as products of commuting projections, and the row sums are as follows:

$$\begin{aligned}\sum_{jb} w_{ia,jb} &= \sum_{jb} u_{ij}^{(a)} v_{ab}\\ &= \sum_b v_{ab} \sum_j u_{ij}^{(a)}\\ &= 1\end{aligned}$$

As for the column sums, these are as follows:

$$\begin{aligned}\sum_{ia} w_{ia,jb} &= \sum_{ia} u_{ij}^{(a)} v_{ab}\\ &= \sum_a v_{ab} \sum_i u_{ij}^{(a)}\\ &= 1\end{aligned}$$

With these observations in hand, it is routine to check that $G \wr_* H$ is indeed a quantum group, with fundamental corepresentation w, by constructing maps Δ, ε, S as in section 1, and in the case $G \subset S_N^+$, we obtain in this way a closed subgroup of S_{Nk}^+. Finally, the assertion regarding the classical version is standard as well. □

We can now introduce the quantum reflection groups, as follows:

THEOREM 2.29. *The following constructions produce compact quantum groups,*

$$\begin{aligned}C(H_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u_{ij} = u_{ij}^*,\ (u_{ij}^2) = \text{magic}\right)\\ C(K_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| [u_{ij}, u_{ij}^*] = 0,\ (u_{ij}u_{ij}^*) = \text{magic}\right)\end{aligned}$$

which appear as liberations of the reflection groups H_N and K_N, and we have

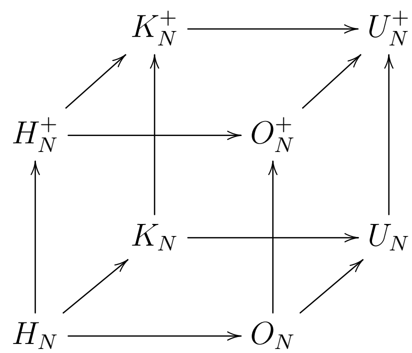
$$H_N^+ = \mathbb{Z}_2 \wr_* S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_* S_N^+$$

in analogy with the wreath product decompositions $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$.

PROOF. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, with the second assertion, regarding the liberation claim, coming via Gelfand, as in the free rotation case, and with the third assertion being something straightforward too. □

We have the following result, collecting our main examples of quantum groups, and which refines the various liberation statements formulated above:

THEOREM 2.30. *The quantum unitary and reflection groups are as follows,*



and in this diagram, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

PROOF. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For further details on this, we refer to the literature. □

2e. Exercises

Exercises:

EXERCISE 2.31.

EXERCISE 2.32.

EXERCISE 2.33.

EXERCISE 2.34.

EXERCISE 2.35.

EXERCISE 2.36.

EXERCISE 2.37.

EXERCISE 2.38.

Bonus exercise.

CHAPTER 3

Diagrams, easiness

3a. Tensor categories

Generally speaking, Tannakian duality amounts in recovering a Woronowicz algebra (A,u) from the tensor category formed by its corepresentations. In what follows we will present a soft form of this duality, which uses the following smaller category:

Definition 3.1. *The Tannakian category associated to a Woronowicz algebra (A,u) is the collection $C=(C(k,l))$ of vector spaces*

$$C(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

where the corepresentations $u^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer, defined by

$$u^{\otimes\emptyset}=1\quad,\quad u^{\otimes\circ}=u\quad,\quad u^{\otimes\bullet}=\bar{u}$$

and multiplicativity, $u^{\otimes kl}=u^{\otimes k}\otimes u^{\otimes l}$, are the Peter-Weyl corepresentations.

We know from chapter 2 that C is a tensor $*$-category. To be more precise, if we denote by $H=\mathbb{C}^N$ the Hilbert space where $u\in M_N(A)$ coacts, then C is a tensor $*$-subcategory of the tensor $*$-category formed by the following linear spaces:

$$E(k,l)=\mathcal{L}(H^{\otimes k},H^{\otimes l})$$

Here the tensor powers $H^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer are those where the corepresentations $u^{\otimes k}$ act, defined by the following formulae, and multiplicativity:

$$H^{\otimes\emptyset}=\mathbb{C}\quad,\quad H^{\otimes\circ}=H\quad,\quad H^{\otimes\bullet}=\bar{H}\simeq H$$

Our purpose in what follows will be that of reconstructing (A,u) in terms of the category $C=(C(k,l))$. We will see afterwards that this method has many applications.

As a first, elementary result on the subject, we have:

Proposition 3.2. *Given a morphism $\pi:(A,u)\to(B,v)$ we have inclusions*

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(v^{\otimes k},v^{\otimes l})$$

for any k,l, and if these inclusions are all equalities, π is an isomorphism.

Proof. The fact that we have indeed inclusions as in the statement is clear from definitions. As for the last assertion, this follows from the Peter-Weyl theory. Indeed, if we assume that π is not an isomorphism, then one of the irreducible corepresentations of A must become reducible as a corepresentation of B. But the irreducible corepresentations being subcorepresentations of the Peter-Weyl corepresentations $u^{\otimes k}$, one of the spaces $End(u^{\otimes k})$ must therefore increase strictly, and this gives the desired contradiction. □

The Tannakian duality result that we want to prove states, in a simplified form, that in what concerns the last conclusion in the above statement, the assumption that we have a morphism $\pi : (A, u) \to (B, v)$ is not needed. In other words, if we know that the Tannakian categories of A, B are different, then A, B themselves must be different.

In order to get started, our first goal will be that of gaining some familiarity with the notion of Tannakian category. And here, we have to use the only general fact that we know about u, namely that this matrix is biunitary. We have:

Proposition 3.3. *Consider the operator $R : \mathbb{C} \to \mathbb{C}^N \otimes \mathbb{C}^N$ given by:*

$$R(1) = \sum_i e_i \otimes e_i$$

An abstract matrix $u \in M_N(A)$ is then a biunitary precisely when the conditions

$$R \in Hom(1, u \otimes \bar{u}) \quad , \quad R \in Hom(1, \bar{u} \otimes u)$$

$$R^* \in Hom(u \otimes \bar{u}, 1) \quad , \quad R^* \in Hom(\bar{u} \otimes u, 1)$$

are all satisfied, in a formal sense, as suitable commutation relations.

Proof. Let us first recall that, in the Woronowicz algebra setting, the definition of the Hom space between two corepresentations $v \in M_n(A)$, $w \in M_m(A)$ is as follows:

$$Hom(v, w) = \left\{ T \in M_{m \times n}(\mathbb{C}) \Big| Tv = wT \right\}$$

But this is something that makes no reference to the Woronowicz algebra structure of A, or to the fact that v, w are indeed corepresentations. Thus, this notation can be formally formally used for any two matrices $v \in M_n(A)$, $w \in M_m(A)$, over an arbitrary C^*-algebra A, and so our statement, as formulated, makes sense indeed. Now with the operator R being as in the statement, we have the following computation:

$$\begin{aligned}(u \otimes \bar{u})(R(1) \otimes 1) &= \sum_{ijk} e_i \otimes e_k \otimes u_{ij} u_{kj}^* \\ &= \sum_{ik} e_i \otimes e_k \otimes (uu^*)_{ik}\end{aligned}$$

We conclude from this that we have the following equivalence:

$$R \in Hom(1, u \otimes \bar{u}) \iff uu^* = 1$$

Consider now the adjoint operator $R^* : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}$, which is given by:

$$R^*(e_i \otimes e_j) = \delta_{ij}$$

With this formula in hand, we have then the following computation:

$$(R^* \otimes id)(u \otimes \bar{u})(e_j \otimes e_l \otimes 1) = \sum_i u_{ij} u_{il}^* = (u^t \bar{u})_{jl}$$

We conclude from this that we have the following equivalence:

$$R^* \in Hom(u \otimes \bar{u}, 1) \iff u^t \bar{u} = 1$$

Similarly, or simply by replacing u in the above two conclusions with its conjugate $\bar{u}$, which is a corepresentation too, we have as well the following two equivalences:

$$R \in Hom(1, \bar{u} \otimes u) \iff \bar{u} u^t = 1$$

$$R^* \in Hom(\bar{u} \otimes u, 1) \iff u^* u = 1$$

Thus, we are led to the biunitarity conditions, and we are done. □

As a consequence of this computation, we have the following result:

Proposition 3.4. *The Tannakian category $C = (C(k,l))$ associated to a Woronowicz algebra (A,u) must contain the operators*

$$R : 1 \to \sum_i e_i \otimes e_i \quad , \quad R^*(e_i \otimes e_j) = \delta_{ij}$$

in the sense that we must have:

$$R \in C(\emptyset, \circ\bullet) \quad , \quad R \in C(\emptyset, \bullet\circ)$$

$$R^* \in C(\circ\bullet, \emptyset) \quad , \quad R^* \in C(\bullet\circ, \emptyset)$$

In fact, C must contain the whole tensor category $< R, R^* >$ generated by R, R^*.

Proof. The first assertion is clear from the above result. As for the second assertion, this is clear from definitions, because $C = (C(k,l))$ is indeed a tensor category. □

Let us formulate now the following key definition:

Definition 3.5. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C = (C(k,l))$ of subspaces*

$$C(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

(1) *$S, T \in C$ implies $S \otimes T \in C$.*
(2) *If $S, T \in C$ are composable, then $ST \in C$.*
(3) *$T \in C$ implies $T^* \in C$.*
(4) *Each $C(k,k)$ contains the identity operator.*
(5) *$C(\emptyset, \circ\bullet)$ and $C(\emptyset, \bullet\circ)$ contain the operator $R : 1 \to \sum_i e_i \otimes e_i$.*

As a basic example here, the collection of the vector spaces $\mathcal{L}(H^{\otimes k}, H^{\otimes l})$ is of course a tensor category over H. There are many other concrete examples, which can be constructed by using various combinatorial methods, and we will discuss this later on.

In relation with the quantum groups, this formalism generalizes the Tannakian category formalism from Definition 3.1, because we have the following result:

PROPOSITION 3.6. *Let (A,u) be a Woronowicz algebra, with fundamental corepresentation $u\in M_N(A)$. The associated Tannakian category $C=(C(k,l))$, given by*

$$C(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

is then a tensor category over the Hilbert space $H=\mathbb{C}^N$.

PROOF. The fact that the above axioms (1-5) are indeed satisfied is clear, as follows:

(1) This follows from our results from chapter 2.

(2) Once again, this follows from our results from chapter 2.

(3) This again follows from our results from chapter 2.

(4) This is something which is clear from definitions.

(5) This follows indeed from what we have in Proposition 3.4. □

Our purpose in what follows will be that of proving that the converse of the above statement holds. That is, we would like to prove that any tensor category in the sense of Definition 3.5 must appear as a Tannakian category.

As a first result on this subject, providing us with a correspondence $C\to A_C$, which is complementary to the correspondence $A\to C_A$ from Proposition 3.6, we have:

PROPOSITION 3.7. *Given a tensor category $C=(C(k,l))$, the following algebra, with u being the fundamental corepresentation of $C(U_N^+)$, is a Woronowicz algebra:*

$$A_C=C(U_N^+)\Big/\left\langle T\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall T\in C(k,l)\right\rangle$$

When C comes from a Woronowicz algebra (A,v), we have a quotient map as follows:

$$A_C\to A$$

Moreover, this map is an isomorphism in the discrete group algebra case.

PROOF. Given two colored integers k,l and a linear operator $T\in\mathcal{L}(H^{\otimes k},H^{\otimes l})$, consider the following $*$-ideal of the algebra $C(U_N^+)$:

$$I=\left\langle T\in Hom(u^{\otimes k},u^{\otimes l})\right\rangle$$

Our claim is that I is a Hopf ideal. Indeed, let us set:

$$U=\sum_k u_{ik}\otimes u_{kj}$$

We have then the following implication, which is something elementary, coming from a standard algebraic computation with indices, and which proves our claim:

$$T\in Hom(u^{\otimes k},u^{\otimes l})\implies T\in Hom(U^{\otimes k},U^{\otimes l})$$

With this claim in hand, the algebra A_C appears from $C(U_N^+)$ by dividing by a certain collection of Hopf ideals, and is therefore a Woronowicz algebra. Since the relations defining A_C are satisfied in A, we have a quotient map as in the statement, namely:

$$A_C\to A$$

Regarding now the last assertion, assume that we are in the case $A=C^*(\Gamma)$, with $\Gamma=<g_1,\ldots,g_N>$ being a finitely generated discrete group. If we denote by $\mathcal{R}$ the complete collection of relations between the generators, then we have:

$$\Gamma=F_N/\mathcal{R}$$

By using now the basic functoriality properties of the group algebra construction, we deduce from this that we have an identification as follows:

$$A_C=C^*\left(F_N\Big/\Big\langle\mathcal{R}\Big\rangle\right)$$

Thus the quotient map $A_C\to A$ is indeed an isomorphism, as claimed. □

With the above two constructions in hand, from Proposition 3.6 and Proposition 3.7, we are now in position of formulating a clear objective. To be more precise, the theorem that we want to prove states that the following operations are inverse to each other:

$$A\to A_C\quad,\quad C\to C_A$$

We have the following result, to start with, which simplifies our work:

PROPOSITION 3.8. *Consider the following conditions:*

(1) $C=C_{A_C}$, *for any Tannakian category* C.
(2) $A=A_{C_A}$, *for any Woronowicz algebra* (A,u).

We have then $(1)\implies(2)$. *Also,* $C\subset C_{A_C}$ *is automatic.*

PROOF. Given a Woronowicz algebra (A,u), let us set $C=C_A$. By using (1) we have $C_A=C_{A_{C_A}}$. On the other hand, by Proposition 3.7 we have an arrow as follows:

$$A_{C_A}\to A$$

Thus Proposition 3.2 applies, and gives an isomorphism $A_{C_A}=A$, as desired. Finally, the fact that we have an inclusion $C\subset C_{A_C}$ is clear from definitions. □

Summarizing, in order to establish the Tannakian duality correspondence, it is enough to prove that we have $C_{A_C}\subset C$, for any Tannakian category C. Let us start with:

Proposition 3.9. *Given a tensor category $C=C((k,l))$ over a Hilbert space H,*

$$E_C=\bigoplus_{k,l}C(k,l)\subset\bigoplus_{k,l}B(H^{\otimes k},H^{\otimes l})\subset B\left(\bigoplus_kH^{\otimes k}\right)$$

is a finite dimensional C^*-subalgebra. Also,

$$E_C^{(s)}=\bigoplus_{|k|,|l|\leq s}C(k,l)\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

is a closed $*$-subalgebra.

Proof. This is clear indeed from the categorical axioms from Definition 3.5, via the standard embeddings and isomorphisms in the statement. □

Now back to our reconstruction question, given a tensor category $C=(C(k,l))$, we want to prove that we have $C=C_{A_C}$, which is the same as proving that we have:

$$E_C=E_{C_{A_C}}$$

Equivalently, we want to prove that we have equalities as follows, for any $s\in\mathbb N$:

$$E_C^{(s)}=E_{C_{A_C}}^{(s)}$$

The problem, however, is that these equalities are not easy to establish directly. In order to solve this question, we will use a standard commutant trick, as follows:

Proposition 3.10. *For any C^*-algebra $B\subset M_n(\mathbb C)$ we have the formula*

$$B=B''$$

where prime denotes the commutant, computed inside $M_n(\mathbb C)$.

Proof. This is a particular case of von Neumann's bicommutant theorem, which follows in our case from the explicit description of B given in chapter 1. To be more precise, let us decompose B as there, as a direct sum of matrix algebras:

$$B=M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb C$ corresponding to a matrix block:

$$B'=\mathbb C\oplus\ldots\oplus\mathbb C$$

By taking once again the commutant, and computing over the matrix blocks, we obtain the algebra B itself, and this gives the formula in the statement. □

Now back to our questions, we recall that we want to prove that we have $C = C_{A_C}$, for any Tannakian category C. By using the bicommutant trick, we have:

THEOREM 3.11. *Given a Tannakian category C, the following are equivalent:*

(1) $C = C_{A_C}$.
(2) $E_C = E_{C_{A_C}}$.
(3) $E_C^{(s)} = E_{C_{A_C}}^{(s)}$, *for any* $s \in \mathbb{N}$.
(4) $E_C^{(s)'} = E_{C_{A_C}}^{(s)'}$, *for any* $s \in \mathbb{N}$.

In addition, the inclusions $\subset, \subset, \subset, \supset$ *are automatically satisfied.*

PROOF. This follows from the above results, as follows:

(1) $\iff$ (2) This is clear from definitions.

(2) $\iff$ (3) This is clear from definitions as well.

(3) $\iff$ (4) This comes from the bicommutant theorem. As for the last assertion, we have indeed $C \subset C_{A_C}$ from Proposition 4.8, and this gives the result. □

3b. Tannakian duality

In order to etaablish Tannakian duality, given a tensor category $C = (C(k,l))$, we would like to prove that we have inclusions as follows, for any $s \in \mathbb{N}$:

$$E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$$

Let us first study the commutant on the right. As a first observation, we have:

PROPOSITION 3.12. *Given a Woronowicz algebra (A,u), we have*

$$E_{C_A}^{(s)} = End\left(\bigoplus_{|k|\leq s} u^{\otimes k}\right)$$

as subalgebras of $B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$.

PROOF. According to the various identifications in Proposition 3.9, we have:

$$\begin{aligned} E_{C_A}^{(s)} &= \bigoplus_{|k|,|l|\leq s} Hom(u^{\otimes k}, u^{\otimes l}) \\ &\subset \bigoplus_{|k|,|l|\leq s} B(H^{\otimes k}, H^{\otimes l}) \\ &= B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right) \end{aligned}$$

On the other hand, the algebra of intertwiners of $\bigoplus_{|k|\leq s}u^{\otimes k}$ is given by:

$$\begin{aligned}End\left(\bigoplus_{|k|\leq s}u^{\otimes k}\right)&=\bigoplus_{|k|,|l|\leq s}Hom(u^{\otimes k},u^{\otimes l})\\&\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})\\&=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)\end{aligned}$$

Thus we have indeed the same algebra, and we are done. □

In practice now, we have to compute the commutant of the above algebra. And for this purpose, we can use the following general result:

Proposition 3.13. *Given a corepresentation $v\in M_n(A)$, we have a representation*

$$\pi_v:A^*\to M_n(\mathbb C)\quad,\quad\varphi\to(\varphi(v_{ij}))_{ij}$$

whose image is given by $Im(\pi_v)=End(v)'$.

Proof. The first assertion is clear, with the multiplicativity claim coming from:

$$\begin{aligned}(\pi_v(\varphi*\psi))_{ij}&=(\varphi\otimes\psi)\Delta(v_{ij})\\&=\sum_k\varphi(v_{ik})\psi(v_{kj})\\&=\sum_k(\pi_v(\varphi))_{ik}(\pi_v(\psi))_{kj}\\&=(\pi_v(\varphi)\pi_v(\psi))_{ij}\end{aligned}$$

Let us first prove the inclusion $\subset$. Given $\varphi\in A^*$ and $T\in End(v)$, we have:

$$\begin{aligned}[\pi_v(\varphi),T]=0&\iff\sum_k\varphi(v_{ik})T_{kj}=\sum_kT_{ik}\varphi(v_{kj}),\forall i,j\\&\iff\varphi\left(\sum_kv_{ik}T_{kj}\right)=\varphi\left(\sum_kT_{ik}v_{kj}\right),\forall i,j\\&\iff\varphi((vT)_{ij})=\varphi((Tv)_{ij}),\forall i,j\end{aligned}$$

But this latter formula is true, because $T\in End(v)$ means that we have:

$$vT=Tv$$

As for the converse inclusion $\supset$, the proof is quite similar. Indeed, by using the bicommutant theorem, this is the same as proving that we have:

$$Im(\pi_v)' \subset End(v)$$

But, by using the above equivalences, we have the following computation:

$$\begin{aligned} T \in Im(\pi_v)' &\iff [\pi_v(\varphi), T] = 0, \forall\varphi \\ &\iff \varphi((vT)_{ij}) = \varphi((Tv)_{ij}), \forall\varphi, i, j \\ &\iff vT = Tv \end{aligned}$$

Thus, we have obtained the desired inclusion, and we are done. □

By combining now the above results, we obtain:

THEOREM 3.14. *Given a Woronowicz algebra (A, u), we have*

$$E_{C_A}^{(s)'} = Im(\pi_v)$$

as subalgebras of the following algebra,

$$B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$$

where $v = \bigoplus_{|k|\leq s} u^{\otimes k}$, and where $\pi_v : A^* \to M_n(\mathbb{C})$ is given by $\varphi \to (\varphi(v_{ij}))_{ij}$.

PROOF. This follows indeed from Proposition 3.12 and Proposition 3.13. □

We recall that we want to prove that we have $E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$, for any $s \in \mathbb{N}$. For this purpose, we must first refine Theorem 3.14, in the case $A = A_C$. In order to do so, we will use an explicit model for A_C. In order to construct such a model, let $< u_{ij} >$ be the free $*$-algebra over $\dim(H)^2$ variables, with comultiplication and counit as follows:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij}$$

Following Malacarne, we can model this $*$-bialgebra, in the following way:

PROPOSITION 3.15. *Consider the following pair of dual vector spaces,*

$$F = \bigoplus_k B\left(H^{\otimes k}\right) \quad , \quad F^* = \bigoplus_k B\left(H^{\otimes k}\right)^*$$

and let $f_{ij}, f_{ij}^* \in F^*$ be the standard generators of $B(H)^*, B(\bar{H})^*$.

1. *F^* is a $*$-algebra, with multiplication $\otimes$ and involution $f_{ij} \leftrightarrow f_{ij}^*$.*
2. *F^* is a $*$-bialgebra, with $\Delta(f_{ij}) = \sum_k f_{ik} \otimes f_{kj}$ and $\varepsilon(f_{ij}) = \delta_{ij}$.*
3. *We have a $*$-bialgebra isomorphism $< u_{ij} > \simeq F^*$, given by $u_{ij} \to f_{ij}$.*

PROOF. Since F^* is spanned by the various tensor products between the variables f_{ij}, f_{ij}^*, we have a vector space isomorphism as follows, given by $u_{ij} \to f_{ij}, u_{ij}^* \to f_{ij}^*$:

$$< u_{ij} > \simeq F^*$$

The corresponding $*$-bialgebra structure on F^* is then the one in the statement. □

Now back to our algebra A_C, we have the following modeling result for it:

PROPOSITION 3.16. *The smooth part of the algebra A_C is given by*

$$\mathcal{A}_C \simeq F^*/J$$

where $J \subset F^*$ is the ideal coming from the following relations,

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} f_{p_1j_1} \otimes \ldots \otimes f_{p_kj_k} = \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} f_{i_1q_1} \otimes \ldots \otimes f_{i_lq_l}$$

one for each pair of colored integers k, l, and each $T \in C(k, l)$.

PROOF. By Proposition 4.3, the algebra A_C appears as enveloping C^*-algebra of the following universal $*$-algebra, where $u = (u_{ij})$ is regarded as a formal corepresentation:

$$\mathcal{A}_C = \left\langle (u_{ij})_{i,j=1,\ldots,N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall k, l, \forall T \in C(k, l) \right\rangle$$

Now with this in hand, the conclusion is that we have a formula as follows, where I is the ideal coming from the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C(k, l)$:

$$\mathcal{A}_C =< u_{ij} > /I$$

Now if we denote by $J \subset F^*$ the image of the ideal I via the $*$-algebra isomorphism $< u_{ij} > \simeq F^*$ from Proposition 3.15, we obtain an identification as follows:

$$\mathcal{A}_C \simeq F^*/J$$

In order to compute J, let us go back to I. With standard multi-index notations, and by assuming that $k, l \in \mathbb{N}$ are usual integers, for simplifying, a relation of type $T \in Hom(u^{\otimes k}, u^{\otimes l})$ inside $< u_{ij} >$ is equivalent to the following conditions:

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} u_{p_1j_1} \ldots u_{p_kj_k} = \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} u_{i_1q_1} \ldots u_{i_lq_l}$$

Now by recalling that the isomorphism of $*$-algebras $< u_{ij} > \to F^*$ is given by $u_{ij} \to f_{ij}$, and that the multiplication operation of F^* corresponds to the tensor product operation $\otimes$, we conclude that $J \subset F^*$ is the ideal from the statement. □

With the above result in hand, let us go back to Theorem 3.14. We have:

Proposition 3.17. *The linear space $\mathcal{A}_C^*$ is given by the formula*

$$\mathcal{A}_C^* = \left\{a \in F \Big| Ta_k = a_l T, \forall T \in C(k,l)\right\}$$

and the representation

$$\pi_v : \mathcal{A}_C^* \to B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$$

appears diagonally, by truncating, via $\pi_v : a \to (a_k)_{kk}$.

Proof. We know from Proposition 3.16 that we have:

$$\mathcal{A}_C \simeq F^*/J$$

But this gives a quotient map $F^* \to \mathcal{A}_C$, and so an inclusion as follows:

$$\mathcal{A}_C^* \subset F$$

To be more precise, we have the following formula:

$$\mathcal{A}_C^* = \left\{a \in F \Big| f(a) = 0, \forall f \in J\right\}$$

Now since $J =< f_T >$, where f_T are the relations in Proposition 3.16, we obtain:

$$\mathcal{A}_C^* = \left\{a \in F \Big| f_T(a) = 0, \forall T \in C\right\}$$

Given $T \in C(k,l)$, for an arbitrary element $a = (a_k)$, we have:

$$\begin{aligned}
& f_T(a) = 0 \\
\iff & \sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k}(a_k)_{p_1\ldots p_k,j_1\ldots j_k} = \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k}(a_l)_{i_1\ldots i_l,q_1\ldots q_l}, \forall i,j \\
\iff & (Ta_k)_{i_1\ldots i_l,j_1\ldots j_k} = (a_l T)_{i_1\ldots i_l,j_1\ldots j_k}, \forall i,j \\
\iff & Ta_k = a_l T
\end{aligned}$$

Thus, the dual space $\mathcal{A}_C^*$ is given by the formula in the statement. It remains to compute the representation π_v, which appears as follows:

$$\pi_v : \mathcal{A}_C^* \to B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$$

With $a = (a_k)$, we have the following computation:

$$\begin{aligned}
\pi_v(a)_{i_1\ldots i_k,j_1\ldots j_k} &= a(v_{i_1\ldots i_k,j_1\ldots j_k}) \\
&= (f_{i_1j_1} \otimes \ldots \otimes f_{i_kj_k})(a) \\
&= (a_k)_{i_1\ldots i_k,j_1\ldots j_k}
\end{aligned}$$

Thus, our representation π_v appears diagonally, by truncating, as claimed. $\square$

In order to further advance, consider the following vector spaces:

$$F_s=\bigoplus_{|k|\leq s}B\left(H^{\otimes k}\right)\quad,\quad F_s^*=\bigoplus_{|k|\leq s}B\left(H^{\otimes k}\right)^*$$

We denote by $a\to a_s$ the truncation operation $F\to F_s$. We have:

Proposition 3.18. *The following hold:*

(1) $E_C^{(s)'}\subset F_s$.
(2) $E_C'\subset F$.
(3) $\mathcal{A}_C^*=E_C'$.
(4) $Im(\pi_v)=(E_C')_s$.

Proof. These results basically follow from what we have, as follows:

(1) We have an inclusion as follows, as a diagonal subalgebra:

$$F_s\subset B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

The commutant of this algebra is given by:

$$F_s'=\left\{b\in F_s\Big|b=(b_k),b_k\in\mathbb{C},\forall k\right\}$$

On the other hand, we know from the identity axiom for the category C that this algebra is contained inside $E_C^{(s)}$:

$$F_s'\subset E_C^{(s)}$$

Thus, our result follows from the bicommutant theorem, as follows:

$$F_s'\subset E_C^{(s)}\implies F_s\supset E_C^{(s)'}$$

(2) This follows from (1), by taking inductive limits.

(3) With the present notations, the formula of $\mathcal{A}_C^*$ from Proposition 3.17 reads:

$$\mathcal{A}_C^*=F\cap E_C'$$

Now since by (2) we have $E_C'\subset F$, we obtain from this $\mathcal{A}_C^*=E_C'$.

(4) This follows from (3), and from the formula of π_v in Proposition 3.17. □

We can now state and prove our main duality result, as follows:

Theorem 3.19. *The Tannakian duality constructions*

$$C\to A_C\quad,\quad A\to C_A$$

are inverse to each other, modulo identifying full and reduced versions.

Proof. According to Proposition 3.8, Theorem 3.11, Theorem 3.14 and Proposition 3.18, we have to prove that, for any Tannakian category C, and any $s\in\mathbb{N}$:

$$E_C^{(s)'}\subset(E_C')_s$$

By taking duals, this is the same as proving that we have:

$$\left\{f\in F_s^*\Big|f_{|(E_C')_s}=0\right\}\subset\left\{f\in F_s^*\Big|f_{|E_C^{(s)'}}=0\right\}$$

For this purpose, we use the following formula, coming from Proposition 3.18:

$$\mathcal{A}_C^*=E_C'$$

We know as well that we have the following formula:

$$\mathcal{A}_C=F^*/J$$

We conclude that the ideal J is given by the following formula:

$$J=\left\{f\in F^*\Big|f_{|E_C'}=0\right\}$$

Our claim is that we have the following formula, for any $s\in\mathbb{N}$:

$$J\cap F_s^*=\left\{f\in F_s^*\Big|f_{|E_C^{(s)'}}=0\right\}$$

Indeed, let us denote by X_s the spaces on the right. The categorical axioms for C show that these spaces are increasing, that their union $X=\cup_sX_s$ is an ideal, and that:

$$X_s=X\cap F_s^*$$

We must prove that we have $J=X$, and this can be done as follows:

"$\subset$" This follows from the following fact, for any $T\in C(k,l)$ with $|k|,|l|\leq s$:

$$\begin{aligned}(f_T)_{|\{T\}'}=0&\implies (f_T)_{|E_C^{(s)'}}=0\\&\implies f_T\in X_s\end{aligned}$$

"$\supset$" This follows from our description of J, because from $E_C^{(s)}\subset E_C$ we obtain:

$$f_{|E_C^{(s)'}}=0\implies f_{|E_C'}=0$$

Summarizing, we have proved our claim. On the other hand, we have:

$$\begin{aligned}J\cap F_s^*&=\left\{f\in F^*\Big|f_{|E_C'}=0\right\}\cap F_s^*\\&=\left\{f\in F_s^*\Big|f_{|E_C'}=0\right\}\\&=\left\{f\in F_s^*\Big|f_{|(E_C')_s}=0\right\}\end{aligned}$$

Thus, our claim is exactly the inclusion that we wanted to prove, and we are done. □

Summarizing, we have proved Tannakian duality. As already mentioned in the beginning of this chapter, there are many other forms of Tannakian duality for the compact quantum groups, and we refer here to Woronowicz's original paper [99], which contains a full discussion of the subject, and to the subsequent literature.

3c. Diagrams, easiness

In order to efficiently deal with the various quantum rotation and reflection groups introduced in chapter 2, we will need some specialized Tannakian duality results, in the spirit of the Brauer theorems for O_N, U_N. Let us start with the following definition:

Definition 3.20. *Associated to any partition $\pi\in P(k,l)$ between an upper row of k points and a lower row of l points is the linear map $T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$ given by*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not.

To be more precise here, we agree to put the two multi-indices i,j on the two rows of points of our partition π, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi=1$ when all the strings of π join equal indices, and by $\delta_\pi=0$ otherwise.

Here are a few basic examples of such linear maps:

Proposition 3.21. *The correspondence $\pi\to T_\pi$ has the following properties:*

1. $T_\cap=R$.
2. $T_\cup=R^*$.
3. $T_{||\ldots||}=id$.
4. $T_{\slash\!\!\!\backslash}=\Sigma$.

Proof. All this comes from definitions, with the computations going as follows:

(1) We have $\cap\in P_2(\emptyset,\circ\circ)$, and so the corresponding operator is a certain linear map $T_\cap:\mathbb C\to\mathbb C^N\otimes\mathbb C^N$. The formula of this map is as follows:

$$\begin{aligned}T_\cap(1)&=\sum_{ij}\delta_\cap(i\ j)e_i\otimes e_j\\&=\sum_{ij}\delta_{ij}e_i\otimes e_j\\&=\sum_i e_i\otimes e_i\end{aligned}$$

We recognize here the formula of $R(1)$, and so we have $T_\cap=R$, as claimed.

(2) Here we have $\cup \in P_2(\circ\circ, \emptyset)$, and so the corresponding operator is a certain linear form $T_\cap : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}$. The formula of this linear form is as follows:

$$\begin{aligned} T_\cap(e_i \otimes e_j) &= \delta_\cap(i\ j) \\ &= \delta_{ij} \end{aligned}$$

Since this is the same as $R^*(e_i \otimes e_j)$, we have $T_\cup = R^*$, as claimed.

(3) Consider indeed the identity pairing $|| \ldots || \in P_2(k,k)$, with $k = \circ\circ \ldots \circ\circ$. The corresponding linear map is then the identity, because we have:

$$\begin{aligned} T_{||\ldots||}(e_{i_1} \otimes \ldots \otimes e_{i_k}) &= \sum_{j_1 \ldots j_k} \delta_{||\ldots||}\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_k} \\ &= \sum_{j_1 \ldots j_k} \delta_{i_1 j_1} \ldots \delta_{i_k j_k} e_{j_1} \otimes \ldots \otimes e_{j_k} \\ &= e_{i_1} \otimes \ldots \otimes e_{i_k} \end{aligned}$$

(4) In the case of the basic crossing $\times \in P_2(\circ\circ, \circ\circ)$, the corresponding linear map $T_\times : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ can be computed as follows:

$$\begin{aligned} T_\times(e_i \otimes e_j) &= \sum_{kl} \delta_\times \begin{pmatrix} i & j \\ k & l \end{pmatrix} e_k \otimes e_l \\ &= \sum_{kl} \delta_{il}\delta_{jk} e_k \otimes e_l \\ &= e_j \otimes e_i \end{aligned}$$

Thus we obtain the flip operator $\Sigma(a \otimes b) = b \otimes a$, as claimed. □

Summarizing, the correspondence $\pi \to T_\pi$ provides us with some simple formulae for the operators R, R^* that we used before, and for other important operators too, such as the flip $\Sigma(a \otimes b) = b \otimes a$, and has as well some interesting categorical properties.

Let us further explore now these categorial properties, and make the link with the Tannakian categories. We have the following key result:

PROPOSITION 3.22. *The assignement $\pi \to T_\pi$ is categorical, in the sense that we have*

$$T_\pi \otimes T_\sigma = T_{[\pi\sigma]} \quad , \quad T_\pi T_\sigma = N^{c(\pi,\sigma)} T_{[^\sigma_\pi]} \quad , \quad T_\pi^* = T_{\pi^*}$$

where $c(\pi, \sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. This follows from some routine computations, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}
& (T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
= & \sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
= & \sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
= & T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
= & \sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
= & \sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
= & N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
= & \sum_{i_1\ldots i_p}< T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}),e_{i_1}\otimes\ldots\otimes e_{i_p}> e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

In analogy with the Tannakian categories, we have the following notion:

DEFINITION 3.23. *A collection of sets $D=\bigsqcup_{k,l}D(k,l)$ with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*

As a basic example, the set $D = P$ itself, formed by all partitions, is a category of partitions. The same goes for the category of pairings $P_2 \subset P$. There are many other examples, and we will gradually explore them, in what follows.

Generally speaking, the axioms in Definition 3.23 can be thought of as being a "delinearized version" of the categorical conditions which are verified by the Tannakian categories, from the beginning of this chapter. We have in fact the following result:

THEOREM 3.24. *Each category of partitions $D = (D(k,l))$ produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the formula*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

which produces a Tannakian category, and therefore a closed subgroup $G_N \subset U_N^+$. We call easy the quantum groups which appear in this way.

PROOF. This follows indeed from Woronowicz's Tannakian duality, in its soft form, as formulated in Theorem 3.19. Indeed, let us set:

$$C(k,l) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

By using the axioms in Definition 3.23, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 3.22, we deduce that $C = (C(k,l))$ is a Tannakian category. Thus the Tannakian duality result applies, and gives the result. □

As a first application, we can formulate a general Brauer theorem, as follows:

THEOREM 3.25. *The basic classical and quantum rotation groups are all easy,*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad : \quad \begin{matrix} NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & \mathcal{P}_2 \end{matrix}$$

with the quantum groups on the left corresponding to the categories on the right.

PROOF. This is something that we know from Brauer for O_N, U_N, but since these results follow easily from those for O_N^+, U_N^+, let us just prove everything, as follows:

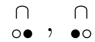
(1) The quantum group U_N^+ is defined via the following relations:

$$u^* = u^{-1} \quad , \quad u^t = \bar{u}^{-1}$$

But, via our correspondence between partitions and maps, these relations tell us that the following two operators must be in the associated Tannakian category C:

$$T_\pi \quad , \quad \pi = {\cap \atop \circ\bullet} \ , \ {\cap \atop \bullet\circ}$$

Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =<$$  $$>= \mathcal{NC}_2$$

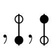
(2) The quantum group $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad , \quad \pi =$$ 

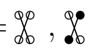
Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =< \mathcal{NC}_2,$$  $$>= NC_2$$

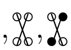
(3) The group $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0 \quad , \quad [u_{ij}, \bar{u}_{kl}] = 0$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad , \quad \pi =$$ 

Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =< \mathcal{NC}_2,$$  $$>= \mathcal{P}_2$$

(4) In order to deal now with O_N, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

Indeed, at the categorical level, this formula tells us that the associated Tannakian category is given by $C = span(T_\pi|\pi \in D)$, with:

$$D =< NC_2, \mathcal{P}_2 >= P_2$$

Thus, we are led to the conclusions in the statement. □

3d. The standard cube

Getting now into permutations, we have the following result, which provides a more reasonable explanation for the liberation operation $S_N \to S_N^+$, and its mysteries:

THEOREM 3.26. *The following hold:*

1. *The quantum groups S_N, S_N^+ are both easy, coming respectively from the categories P, NC of partitions, and noncrossing partitions.*
2. *Thus, $S_N \to S_N^+$ is just a regular easy quantum group liberation, coming from $D \to D \cap NC$ at the level of the associated categories of partitions.*

PROOF. We already know the result for S_N, so we just need to prove the result for S_N^+. In order to do so, recall that the subgroup $S_N^+\subset O_N^+$ appears as follows:

$$C(S_N^+)=C(O_N^+)\Big/\Big\langle u=\text{magic}\Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y\in P(2,1)$$

Given a corepresentation u, we have the following formulae:

$$(T_Yu^{\otimes 2})_{i,jk}=\sum_{lm}(T_Y)_{i,lm}(u^{\otimes 2})_{lm,jk}=u_{ij}u_{ik}$$

$$(uT_Y)_{i,jk}=\sum_l u_{il}(T_Y)_{l,jk}=\delta_{jk}u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y\in Hom(u^{\otimes 2},u)\iff u_{ij}u_{ik}=\delta_{jk}u_{ij},\forall i,j,k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+)=C(O_N^+)\Big/\Big\langle T_Y\in Hom(u^{\otimes 2},u)\Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D=<Y>=NC$$

Finally, observe that this proves the result for S_N too, because from the formula $S_N=S_N^+\cap O_N$ we obtain that the group S_N is easy, coming from the category of partitions $D=<NC,P_2>=P$. Thus, we are led to the conclusions in the statement. □

In order to deal now with the quantum reflections, let us start with H_N,H_N^+. As a continuation of the material from chapter 2, we have the following result:

PROPOSITION 3.27. *The algebra $C(H_N^+)$ can be presented in two ways, as follows:*

1. *As the universal algebra generated by the entries of a $2N\times 2N$ magic unitary having the "sudoku" pattern $w=\begin{pmatrix}a&b\\b&a\end{pmatrix}$, with a,b being square matrices.*
2. *As the universal algebra generated by the entries of a $N\times N$ orthogonal matrix which is "cubic", in the sense that $u_{ij}u_{ik}=u_{ji}u_{ki}=0$, for any $j\neq k$.*

As for $C(H_N)$, this has similar presentations, among the commutative algebras.

PROOF. We must prove that the algebras A_s,A_c coming from (1,2) coincide. We can define a morphism $A_c\to A_s$ by the following formula:

$$\varphi(u_{ij})=a_{ij}-b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\alpha_{ij}=\frac{u_{ij}^2+u_{ij}}{2}\quad,\quad\beta_{ij}=\frac{u_{ij}^2-u_{ij}}{2}$$

These are projections, and the following matrix is a sudoku unitary:

$$M = \begin{pmatrix} (\alpha_{ij}) & (\beta_{ij}) \\ (\beta_{ij}) & (\alpha_{ij}) \end{pmatrix}$$

Thus we can define a morphism $A_s \to A_c$ by the following formula:

$$\psi(a_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2} \quad , \quad \psi(b_{ij}) = \frac{u_{ij}^2 - u_{ij}}{2}$$

We check now the fact that ψ, φ are indeed inverse morphisms:

$$\psi\varphi(u_{ij}) = \psi(a_{ij} - b_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2} - \frac{u_{ij}^2 - u_{ij}}{2} = u_{ij}$$

As for the other composition, we have the following computation:

$$\varphi\psi(a_{ij}) = \varphi\left(\frac{u_{ij}^2 + u_{ij}}{2}\right) = \frac{(a_{ij} - b_{ij})^2 + (a_{ij} - b_{ij})}{2} = a_{ij}$$

A similar computation gives $\varphi\psi(b_{ij}) = b_{ij}$, as desired. As for the final assertion, regarding $C(H_N)$, this follows from the above results, by taking classical versions. $\square$

We can now work out the easiness property of H_N, H_N^+, with respect to the cubic representations, and we are led to the following result:

Theorem 3.28. *The quantum groups H_N, H_N^+ are both easy, as follows:*

(1) *H_N corresponds to the category P_{even}.*
(2) *H_N^+ corresponds to the category NC_{even}.*

Proof. This is something quite routine, the idea being as follows:

(1) We know that $H_N^+ \subset O_N^+$ appears via the cubic relations, namely:

$$u_{ij}u_{ik} = u_{ji}u_{ki} = 0 \quad , \quad \forall j \neq k$$

Our claim is that, in Tannakian terms, these relations reformulate as follows, with $H \in P(2,2)$ being the 1-block partition, joining all 4 points:

$$T_H \in End(u^{\otimes 2})$$

(2) In order to prove our claim, observe first that we have, by definition of T_H:

$$T_H(e_i \otimes e_j) = \delta_{ij} e_i \otimes e_i$$

With this formula in hand, we have the following computation:

$$\begin{aligned}
T_H u^{\otimes 2}(e_i \otimes e_j \otimes 1) &= T_H \left(\sum_{abij} e_{ai} \otimes e_{bj} \otimes u_{ai} u_{bj} \right) (e_i \otimes e_j \otimes 1) \\
&= T_H \sum_{ab} e_a \otimes e_b \otimes u_{ai} u_{bj} \\
&= \sum_a e_a \otimes e_a \otimes u_{ai} u_{aj}
\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}
u^{\otimes 2} T_H(e_i \otimes e_j \otimes 1) &= \delta_{ij} u^{\otimes 2}(e_i \otimes e_j \otimes 1) \\
&= \delta_{ij} \left(\sum_{abij} e_{ai} \otimes e_{bj} \otimes u_{ai} u_{bj} \right) (e_i \otimes e_j \otimes 1) \\
&= \delta_{ij} \sum_{ab} e_a \otimes e_b \otimes u_{ai} u_{bi}
\end{aligned}$$

We conclude that $T_H u^{\otimes 2} = u^{\otimes 2} T_H$ means that u is cubic, as desired.

(3) With our claim proved, we can go back to H_N^+. Indeed, it follows from Tannakian duality that this quantum group is easy, coming from the following category:

$$D =< H >= NC_{even}$$

(4) But this proves as well the result for H_N. Indeed, since this group is the classical version of H_N^+, we have as desired easiness, the corresponding category being:

$$E =< NC_{even}, \slash\!\!\!\backslash >= P_{even}$$

Thus, we are led to the conclusions in the statement. □

The reflection groups H_N and their liberations H_N^+ belong in fact to two remarkable series, depending on a parameter $s \in \mathbb{N} \cup \{\infty\}$, constructed as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N \quad , \quad H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

To be more precise, the free analogues of the reflection groups H_N^s, that we already met in the above at the special values $s = 1, 2, \infty$, can be constructed as follows:

DEFINITION 3.29. *The algebra $C(H_N^{s+})$ is the universal C^*-algebra generated by N^2 normal elements u_{ij}, subject to the following relations,*

1. *$u = (u_{ij})$ is unitary,*
2. *$u^t = (u_{ji})$ is unitary,*
3. *$p_{ij} = u_{ij} u_{ij}^*$ is a projection,*
4. *$u_{ij}^s = p_{ij}$,*

with Woronowicz algebra maps Δ, ε, S constructed by universality.

Here we allow the value $s = \infty$, with the convention that the last axiom simply disappears in this case. Observe that at $s < \infty$ the normality condition is actually redundant. This is because a partial isometry a subject to the relation $aa^* = a^s$ is normal. In analogy with the results from the real case, we have the following result:

PROPOSITION 3.30. *The algebras $C(H_N^{s+})$ with $s = 1, 2, \infty$, and their presentation relations in terms of the entries of the matrix $u = (u_{ij})$, are as follows:*

(1) *For $C(H_N^{1+}) = C(S_N^+)$, the matrix u is magic: all its entries are projections, summing up to 1 on each row and column.*
(2) *For $C(H_N^{2+}) = C(H_N^+)$ the matrix u is cubic: it is orthogonal, and the products of pairs of distinct entries on the same row or the same column vanish.*
(3) *For $C(H_N^{\infty+}) = C(K_N^+)$ the matrix u is unitary, its transpose is unitary, and all its entries are normal partial isometries.*

PROOF. This is something elementary, the idea being as follows:

(1) This follows from definitions and from standard operator algebra tricks.

(2) This follows as well from definitions and standard operator algebra tricks.

(3) This is just a translation of the definition of $C(H_N^{s+})$, at $s = \infty$. □

Let us prove now that H_N^{s+} with $s < \infty$ is a quantum permutation group. For this purpose, we must change the fundamental representation. Let us start with:

DEFINITION 3.31. *A (s, N)-sudoku matrix is a magic unitary of size sN, of the form*

$$m = \begin{pmatrix} a^0 & a^1 & \dots & a^{s-1} \\ a^{s-1} & a^0 & \dots & a^{s-2} \\ \vdots & \vdots & & \vdots \\ a^1 & a^2 & \dots & a^0 \end{pmatrix}$$

where $a^0, \dots, a^{s-1}$ are $N \times N$ matrices.

The basic examples of such matrices come from the group H_n^s. Indeed, with $w = e^{2\pi i/s}$, each of the N^2 matrix coordinates $u_{ij} : H_N^s \to \mathbb{C}$ takes values in the following set:

$$S = \{0\} \cup \{1, w, \dots, w^{s-1}\}$$

Thus, this coordinate function $u_{ij} : H_N^s \to \mathbb{C}$ decomposes as follows:

$$u_{ij} = \sum_{r=0}^{s-1} w^r a_{ij}^r$$

Here each a_{ij}^r is a function taking values in $\{0, 1\}$, and so a projection in the C^*-algebra sense, and it follows from definitions that these projections form a sudoku matrix. Now with this notion in hand, we have the following result:

THEOREM 3.32. *The following happen:*

(1) *The algebra $C(H_N^s)$ is isomorphic to the universal commutative C^*-algebra generated by the entries of a (s,N)-sudoku matrix.*
(2) *The algebra $C(H_N^{s+})$ is isomorphic to the universal C^*-algebra generated by the entries of a (s,N)-sudoku matrix.*

PROOF. The first assertion follows from the second one. In order to prove the second assertion, consider the universal algebra in the statement, namely:

$$A = C^*\left(a_{ij}^p \,\middle|\, \left(a_{ij}^{q-p}\right)_{pi,qj} = (s,N) - \text{sudoku}\right)$$

Consider also the algebra $C(H_N^{s+})$. According to Definition 3.29, this is presented by certain relations R, that we will call here level s cubic conditions:

$$C(H_N^{s+}) = C^*\left(u_{ij} \,\middle|\, u = N\times N \text{ level } s \text{ cubic}\right)$$

We will construct a pair of inverse morphisms between these algebras.

(1) Our first claim is that $U_{ij} = \sum_p w^{-p} a_{ij}^p$ is a level s cubic unitary. Indeed, by using the sudoku condition, the verification of (1-4) in Definition 3.29 is routine.

(2) Our second claim is that the elements $A_{ij}^p = \frac{1}{s}\sum_r w^{rp} u_{ij}^r$, with the convention $u_{ij}^0 = p_{ij}$, form a level s sudoku unitary. Once again, the proof here is routine.

(3) According to the above, we can define a morphism $\Phi : C(H_N^{s+}) \to A$ by the formula $\Phi(u_{ij}) = U_{ij}$, and a morphism $\Psi : A \to C(H_N^{s+})$ by the formula $\Psi(a_{ij}^p) = A_{ij}^p$.

(4) It is easy to check, as in the proof of Proposition 3.27, that Φ, Ψ are indeed inverse morphisms. Thus we have an isomorphism $C(H_N^{s+}) = A$, as claimed. □

Regarding now the easiness property of H_N^s, H_N^{s+}, we already know that this happens at $s = 1, 2$. The point is that this happens in general, the result being as follows:

THEOREM 3.33. *The quantum groups H_N^s, H_N^{s+} are easy, the corresponding categories*

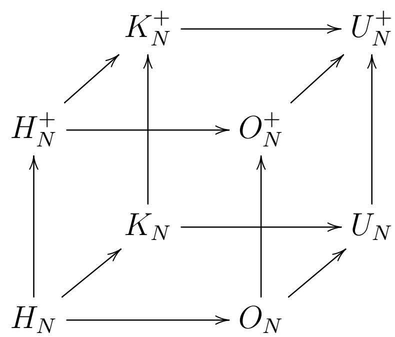
$$P^s \subset P \quad , \quad NC^s \subset NC$$

consisting of partitions satisfying $\#\circ = \#\bullet\,(s)$, as a weighted sum, in each block.

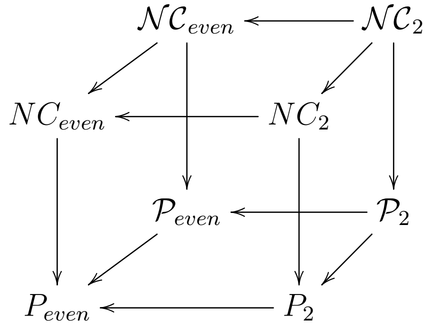
PROOF. The result holds at $s = 1$, trivially, and then at $s = 2$ as well, where our condition is equivalent to $\#\circ = \#\bullet(2)$ in each block, as found in Theorem 3.28. In general, this follows as in the case of H_N, H_N^+, by using the one-block partition in $P(s,s)$. □

Good news, we can now complete our cube, as follows:

Theorem 3.34. *We have quantum rotation and reflection groups, as follows,*



which are all easy, the corresponding categories of partitions being as follows,



with on top, the symbol NC standing everywhere for noncrossing partitions.

Proof. This follows indeed from the above results. □

3e. Exercises

Exercises:

Exercise 3.35.

Exercise 3.36.

Exercise 3.37.

Exercise 3.38.

Exercise 3.39.

Exercise 3.40.

Exercise 3.41.

Exercise 3.42.

Bonus exercise.

CHAPTER 4

Quantum manifolds

4a. Spheres and tori

We are now ready, or almost, to develop some basic noncommutative geometry. The idea will be that of further building on the material from chapter 3, by enlarging the class of compact quantum groups studied there, with the consideration of quantum homogeneous spaces, $X = G/H$, and with classical and free probability as our main tools.

But let us start with something intuitive, namely basic algebraic geometry, in a basic sense. The simplest compact manifolds that we know are the spheres, and if we want to have free analogues of these spheres, there are not many choices here, and we have:

Definition 4.1. *We have compact quantum spaces, constructed as follows,*

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1,\ldots,x_N\Big|x_i = x_i^*,\sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

called respectively free real sphere, and free complex sphere.

Observe that our spheres are indeed well-defined, due to the following estimate:

$$||x_i||^2 = ||x_ix_i^*|| \leq \left|\left|\sum_i x_ix_i^*\right|\right| = 1$$

Given a compact quantum space X, meaning as usual the abstract spectrum of a C^*-algebra, we define its classical version to be the classical space X_{class} obtained by dividing $C(X)$ by its commutator ideal, then applying the Gelfand theorem:

$$C(X_{class}) = C(X)/I \quad , \quad I =< [a,b] >$$

Observe that we have an embedding of compact quantum spaces $X_{class} \subset X$. In this situation, we also say that X appears as a "liberation" of X. We have:

Theorem 4.2. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc}
S^{N-1}_{\mathbb{R},+} & \to & S^{N-1}_{\mathbb{C},+}\\
\uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}} & \to & S^{N-1}_{\mathbb{C}}
\end{array}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{array}{ccc}
C(S^{N-1}_{\mathbb{R},+}) & \leftarrow & C(S^{N-1}_{\mathbb{C},+})\\
\downarrow & & \downarrow\\
C(S^{N-1}_{\mathbb{R}}) & \leftarrow & C(S^{N-1}_{\mathbb{C}})
\end{array}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*,\sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

It is enough to establish the second isomorphism. So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_{\mathbb{C}}$ satisfy the defining relations for A, we have a quotient map of as follows:

$$A \to C(S^{N-1}_{\mathbb{C}})$$

Conversely, let us write $A = C(S)$, by using the Gelfand theorem. Then $x_1,\ldots,x_N$ become in this way true coordinates, providing us with an embedding as follows:

$$S \subset \mathbb{C}^N$$

Also, the quadratic relations become $\sum_i |x_i|^2 = 1$, so we have $S \subset S^{N-1}_{\mathbb{C}}$. Thus, we have a quotient map $C(S^{N-1}_{\mathbb{C}}) \to A$, as desired, and this gives all the results. □

By using the free spheres constructed above, we can now formulate:

Definition 4.3. *A real algebraic submanifold $X\subset S^{N-1}_{\mathbb C,+}$ is a closed quantum space defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain noncommutative polynomials $f_i\in\mathbb C<X_1,\ldots,X_N>$. We identify two such manifolds, $X\simeq Y$, when we have an isomorphism of $*$-algebras of coordinates

$$\mathcal C(X)\simeq\mathcal C(Y)$$

mapping standard coordinates to standard coordinates.

As a basic example here, we have the free real sphere $S^{N-1}_{\mathbb R,+}$. The classical spheres $S^{N-1}_{\mathbb C},S^{N-1}_{\mathbb R}$, and their real submanifolds, are covered as well by this formalism.

In fact, while our assumption $X\subset S^{N-1}_{\mathbb C,+}$ looks like something technical, we are not losing much when imposing it, and we have the following list of examples:

Theorem 4.4. *The following are algebraic submanifolds $X\subset S^{N-1}_{\mathbb C,+}$:*

1. *The spheres $S^{N-1}_{\mathbb R}\subset S^{N-1}_{\mathbb C},S^{N-1}_{\mathbb R,+}\subset S^{N-1}_{\mathbb C,+}$.*
2. *Any compact Lie group, $G\subset U_n$, with $N=n^2$.*
3. *The duals $\widehat{\Gamma}$ of finitely generated groups, $\Gamma=<g_1,\ldots,g_N>$.*
4. *More generally, the closed subgroups $G\subset U_n^+$, with $N=n^2$.*

Proof. These facts are all well-known, the proofs being as follows:

(1) This is indeed true by definition of our various spheres.

(2) Given a closed subgroup $G\subset U_n$, we have an embedding $G\subset S^{N-1}_{\mathbb C}$, with $N=n^2$, given in double indices by $x_{ij}=u_{ij}/\sqrt n$, that we can further compose with the standard embedding $S^{N-1}_{\mathbb C}\subset S^{N-1}_{\mathbb C,+}$. As for the fact that we obtain indeed a real algebraic manifold, this is standard too, coming either from Lie theory or from Tannakian duality.

(3) Given a group $\Gamma=<g_1,\ldots,g_N>$, consider the variables $x_i=g_i/\sqrt N$. These variables satisfy then the quadratic relations $\sum_ix_ix_i^*=\sum_ix_i^*x_i=1$ defining $S^{N-1}_{\mathbb C,+}$, and the algebricity claim for the manifold $\widehat{\Gamma}\subset S^{N-1}_{\mathbb C,+}$ is clear.

(4) Given a closed subgroup $G\subset U_n^+$, we have indeed an embedding $G\subset S^{N-1}_{\mathbb C,+}$, with $N=n^2$, given by $x_{ij}=u_{ij}/\sqrt n$. As for the fact that we obtain indeed a real algebraic manifold, this comes from the Tannakian duality results from chapter 3. □

Summarizing, what we have in Definition 4.3 is something quite fruitful, covering many interesting examples. In addition, all this is nice too at the axiomatic level, because the equivalence relation for our algebraic manifolds, as formulated in Definition 4.3, fixes in a quite clever way the functoriality issues of the Gelfand correspondence.

At the level of the general theory now, as a first tool that we can use, for the study of our manifolds, we have the following version of the Gelfand theorem:

THEOREM 4.5. *Assuming that $X\subset S^{N-1}_{\mathbb C,+}$ is an algebraic manifold, given by*

$$C(X)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain noncommutative polynomials $f_i\in\mathbb C<X_1,\ldots,X_N>$, we have

$$X_{class}=\left\{x\in S^{N-1}_{\mathbb C}\Big|f_i(x_1,\ldots,x_N)=0\right\}$$

and X itself appears as a liberation of X_{class}.

PROOF. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X_{class}' the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class})\to C(X_{class}')$$

Conversely now, from $X\subset S^{N-1}_{\mathbb C,+}$ we obtain $X_{class}\subset S^{N-1}_{\mathbb C}$. Now since the relations defining X_{class}' are satisfied by X_{class}, we obtain an inclusion $X_{class}\subset X_{class}'$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}')\to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Now back to the tori, as constructed before, we know that these are algebraic manifolds, in the sense of Definition 4.3. In fact, we have the following result:

THEOREM 4.6. *The four main quantum spheres produce the main quantum tori*

$$\begin{matrix}
S^{N-1}_{\mathbb R,+}&\to&S^{N-1}_{\mathbb C,+}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb R}&\to&S^{N-1}_{\mathbb C}
\end{matrix}\qquad\to\qquad
\begin{matrix}
T_N^+&\to&\mathbb T_N^+\\
\uparrow&&\uparrow\\
T_N&\to&\mathbb T_N
\end{matrix}$$

via the formula $T=S\cap\mathbb T_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb C,+}$.

PROOF. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads:

$$\mathbb T_N^+=S^{N-1}_{\mathbb C,+}\cap\mathbb T_N^+$$

But this is something trivial, because we have $\mathbb T_N^+\subset S^{N-1}_{\mathbb C,+}$.

(2) Free real case. Here the formula in the statement reads:

$$T_N^+ = S^{N-1}_{\mathbb{R},+} \cap \mathbb{T}_N^+$$

But this is clear as well, the real version of $\mathbb{T}_N^+$ being T_N^+.

(3) Classical complex case. Here the formula in the statement reads:

$$\mathbb{T}_N = S^{N-1}_{\mathbb{C}} \cap \mathbb{T}_N^+$$

But this is clear as well, the classical version of $\mathbb{T}_N^+$ being $\mathbb{T}_N$.

(4) Classical real case. Here the formula in the statement reads:

$$T_N = S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+$$

But this follows by intersecting the formulae from the proof of (2) and (3). □

In order to discuss now the relation with the free rotations, which can only come via some sort of "isometric actions", let us start with the following standard fact:

THEOREM 4.7. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

defines a morphism of C^*-algebras $\Phi : C(X) \to C(X) \otimes C(G)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

PROOF. Assume indeed that our manifold $X \subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

In order to prove the result, consider the following variables:

$$X_i = \sum_j x_j \otimes u_{ji} \in C(X) \otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G = G^+(X)$ appears as:

$$C(G) = C(U_N^+) \Big/ \Big\langle f_\alpha(X_1, \ldots, X_N) = 0 \Big\rangle$$

In order to prove this, pick one of the defining polynomials, and write it as follows:

$$f_\alpha(x_1, \ldots, x_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \cdot x_{i_1^r} \ldots x_{i_{s_r}^r}$$

With $X_i = \sum_j x_j \otimes u_{ji}$ as above, we have the following formula:

$$f_\alpha(X_1, \ldots, X_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \sum_{j_1^r \ldots j_{s_r}^r} x_{j_1^r} \ldots x_{j_{s_r}^r} \otimes u_{j_1^r i_1^r} \ldots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f_\alpha(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a closed subspace $G\subset U_N^+$, with a universal map, as follows:

$$\Phi: C(X)\to C(X)\otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider as well the following elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f_\alpha(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f_\alpha(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f_\alpha(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. $\square$

We can now formulate a result about spheres and rotations, as follows:

THEOREM 4.8. *The quantum isometry groups of the basic spheres are*

$$\begin{matrix}
S^{N-1}_{\mathbb R,+}&\to&S^{N-1}_{\mathbb C,+}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb R}&\to&S^{N-1}_{\mathbb C}
\end{matrix}
\qquad\to\qquad
\begin{matrix}
O_N^+&\to&U_N^+\\
\uparrow&&\uparrow\\
O_N&\to&U_N
\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

PROOF. We have 4 results to be proved, the idea being as follows:

$\underline{S^{N-1}_{\mathbb C,+}}$. Let us first construct an action $U_N^+\curvearrowright S^{N-1}_{\mathbb C,+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb C,+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

By using the biunitarity of u, we have the following computation:

$$\sum_i X_iX_i^*=\sum_{ijk}x_jx_k^*\otimes u_{ji}u_{ki}^*=\sum_j x_jx_j^*\otimes 1=1\otimes 1$$

Once again by using the biunitarity of u, we have as well:

$$\sum_i X_i^*X_i=\sum_{ijk}x_j^*x_k\otimes u_{ji}^*u_{ki}=\sum_j x_j^*x_j\otimes 1=1\otimes 1$$

Thus we have an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$, which gives $G^+(S^{N-1}_{\mathbb{C},+}) = U_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R},+}}$. Let us first construct an action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$. We already know that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, so we just have to check that these variables are self-adjoint. But this is clear from $u = \bar{u}$, as follows:

$$X_i^* = \sum_j x_j^* \otimes u_{ji}^* = \sum_j x_j \otimes u_{ji} = X_i$$

Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{R},+}$, with $G \subset U_N^+$. The variables $X_i = \sum_j x_j \otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u = \bar{u}$. Thus our quantum group must satisfy $G \subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{C}}}$. The fact that we have an action $U_N \curvearrowright S^{N-1}_{\mathbb{C}}$ is clear. Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{C}}$, with $G \subset U_N^+$. We must prove that this implies $G \subset U_N$, and we will use a standard trick of Bhowmick-Goswami. We have:

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_i x_k) = \sum_{jl} x_j x_l \otimes u_{ji} u_{lk}$$

$$\Phi(x_k x_i) = \sum_{jl} x_l x_j \otimes u_{lk} u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{ji} u_{lk} + u_{li} u_{jk}) + \sum_j x_j^2 \otimes u_{ji} u_{jk}$$

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{lk} u_{ji} + u_{jk} u_{li}) + \sum_j x_j^2 \otimes u_{jk} u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji} u_{lk} + u_{li} u_{jk} = u_{lk} u_{ji} + u_{jk} u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^* u_{ij}^* + u_{kj}^* u_{il}^* = u_{ij}^* u_{kl}^* + u_{il}^* u_{kj}^*$$

$$u_{ij} u_{kl} + u_{il} u_{kj} = u_{kl} u_{ij} + u_{kj} u_{il}$$

$$u_{ji} u_{lk} + u_{jk} u_{li} = u_{lk} u_{ji} + u_{li} u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li} u_{jk} = u_{jk} u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_i x_k^*) = \sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^*$$

$$\Phi(x_k^* x_i) = \sum_{jl} x_l^* x_j \otimes u_{lk}^* u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl} x_j x_l^* \otimes u_{ji} u_{lk}^* = \sum_{jl} x_j x_l^* \otimes u_{lk}^* u_{ji}$$

Thus we have $u_{ji} u_{lk}^* = u_{lk}^* u_{ji}$, and so $G \subset U_N$, as claimed.

$\underline{S^{N-1}_\mathbb{R}}$. The fact that we have an action $O_N \curvearrowright S^{N-1}_\mathbb{R}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned} G \curvearrowright S^{N-1}_\mathbb{R} &\implies G \curvearrowright S^{N-1}_{\mathbb{R},+}, S^{N-1}_\mathbb{C} \\ &\implies G \subset O_N^+, U_N \\ &\implies G \subset O_N^+ \cap U_N = O_N \end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_\mathbb{R}) = O_N$, as desired. □

4b. Fine structure

Let us discuss now the correspondence $U \to S$. In the classical case the situation is very simple, because the sphere $S = S^{N-1}$ appears by rotating the point $x = (1, 0, \ldots, 0)$ by the isometries in $U = U_N$. Moreover, the stabilizer of this action is the subgroup $U_{N-1} \subset U_N$ acting on the last $N-1$ coordinates, and so the sphere $S = S^{N-1}$ appears from the corresponding rotation group $U = U_N$ as an homogeneous space, as follows:

$$S^{N-1} = U_N / U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U \to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \to u_{1i}$$

In general now, the straightforward homogeneous space interpretation of S as above fails. However, we can have some theory going by using the functional analytic viewpoint, with an embedding $x_i \to u_{1i}$ as above. Let us start with the following result:

PROPOSITION 4.9. *For the basic spheres, we have a diagram as follows,*

$$\begin{array}{ccc} C(S) & \xrightarrow{\Phi} & C(S)\otimes C(U)\\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id}\\ C(U) & \xrightarrow{\Delta} & C(U)\otimes C(U) \end{array}$$

where on top $\Phi(x_i)=\sum_j x_j\otimes u_{ji}$*, and on the left* $\alpha(x_i)=u_{1i}$.

PROOF. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows, on these coordinates:

$$\begin{array}{ccc} x_i & \longrightarrow & \sum_j x_j\otimes u_{ji}\\ \downarrow & & \downarrow\\ u_{1i} & \longrightarrow & \sum_j u_{1j}\otimes u_{ji} \end{array}$$

Thus by linearity and multiplicativity, the whole the diagram commutes. □

As a consequence of the above result, we can now formulate:

PROPOSITION 4.10. *We have a quotient map and an inclusion as follows,*

$$U\to S_U\subset S$$

with S_U *being the first row space of* U*, given by*

$$C(S_U)=<u_{1i}>\subset C(U)$$

at the level of the corresponding algebras of functions.

PROOF. At the algebra level, we have an inclusion and a quotient map as follows:

$$C(S)\to C(S_U)\subset C(U)$$

Thus, we obtain the result, by transposing. □

The above result is all that we need, for getting started with our study, and we will prove in what follows that the inclusion $S_U\subset S$ constructed above is an isomorphism. This will produce the correspondence $U\to S$ that we are currently looking for.

In order to do so, we will use the uniform integration over S, which can be introduced, in analogy with what happens in the classical case, in the following way:

Definition 4.11. *We endow each of the algebras $C(S)$ with its integration functional*

$$\int_S : C(S) \to C(U) \to \mathbb{C}$$

obtained by composing the morphism $x_i \to u_{1i}$ with the Haar integration of $C(U)$.

In order to efficiently integrate over the sphere S, and in the lack of some trick like spherical coordinates, we need to know how to efficiently integrate over the corresponding quantum isometry group U. As before in the classical case, we have:

Theorem 4.12. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

for any indices $i_r, j_r \in \{1,\ldots,N\}$ and any exponents $e_r \in \{\emptyset, *\}$, where δ are the usual Kronecker type symbols, and where

$$W_{kN} = G_{kN}^{-1}$$

is the inverse of the matrix $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. Let us arrange indeed all the integrals to be computed, at a fixed value of the exponent $k = (e_1 \ldots e_k)$, into a single matrix, of size $N^k \times N^k$, as follows:

$$P_{i_1 \ldots i_k, j_1 \ldots j_k} = \int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k}$$

According to the construction of the Haar measure of Woronowicz, explained in the above, this matrix P is the orthogonal projection onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

In order to compute this projection, consider the following linear map:

$$E(x) = \sum_{\pi \in D(k)} < x, \xi_\pi > \xi_\pi$$

Consider as well the inverse W of the restriction of E to the following space:

$$span\left(T_\pi \Big| \pi \in D(k)\right)$$

By a standard linear algebra computation, it follows that we have:

$$P = WE$$

But the restriction of E is the linear map corresponding to G_{kN}, so W is the linear map corresponding to W_{kN}, and this gives the result. □

With this in hand, we can now integrate over the spheres S, as follows:

THEOREM 4.13. *The integration over the basic spheres is given by*

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, *where* $W_{kN}=G_{kN}^{-1}$ *is the inverse of* $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

PROOF. According to our conventions, the integration over S is a particular case of the integration over U, via $x_i=u_{1i}$. By using now Theorem 4.12, we obtain:

$$\begin{aligned}\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\int_U u_{1i_1}^{e_1}\ldots u_{1i_k}^{e_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Again with some inspiration from the classical case, we have the following key result:

THEOREM 4.14. *The integration functional of S has the ergodicity property*

$$\left(id\otimes\int_U\right)\Phi(x)=\int_S x$$

where $\Phi:C(S)\to C(S)\otimes C(U)$ *is the universal affine coaction map.*

PROOF. In the real case, $x_i=x_i^*$, it is enough to check the equality in the statement on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})&=\sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_U u_{j_1i_1}\ldots u_{j_ki_k}\\&=\sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D(k)}\delta_\pi(j)\delta_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\delta_\pi(j)x_{j_1}\ldots x_{j_k}\end{aligned}$$

Let us look now at the last sum on the right. The situation is as follows:

– In the free case we have to sum quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given noncrossing pairing π, and just by using the condition $\sum_i x_i^2=1$, we conclude that the sum is 1.

– The same happens in the classical case. Indeed, our pairing π can now be crossing, but we can use the commutation relations $x_ix_j=x_jx_i$, and the sum is again 1.

Thus the sum on the right is 1, in all cases, and we obtain:

$$\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}\int_S x_{i_1}\ldots x_{i_k}&=\int_U u_{1i_1}\ldots u_{1i_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are done with the proof of the result, in the real case. In the complex case the proof is similar, by adding exponents everywhere. □

We can now deduce a useful characterization of the integration, as follows:

Theorem 4.15. *There is a unique positive unital trace $tr:C(S)\to\mathbb{C}$ satisfying*

$$(tr\otimes id)\Phi(x)=tr(x)1$$

where Φ is the coaction map of the corresponding quantum isometry group,

$$\Phi:C(S)\to C(S)\otimes C(U)$$

and this is the canonical integration, as constructed in Definition 4.11.

Proof. First of all, it follows from the Haar integral invariance condition for U that the canonical integration has indeed the invariance property in the statement, namely:

$$(tr\otimes id)\Phi(x)=tr(x)1$$

In order to prove now the uniqueness, let tr be as in the statement. We have:

$$\begin{aligned}tr\left(id\otimes\int_U\right)\Phi(x)&=\int_U(tr\otimes id)\Phi(x)\\&=\int_U(tr(x)1)\\&=tr(x)\end{aligned}$$

On the other hand, according to Theorem 4.14, we have as well:

$$tr\left(id\otimes\int_U\right)\Phi(x)=tr\left(\int_S x\right)=\int_S x$$

We therefore conclude that tr equals the standard integration, as claimed. □

Getting back now to our axiomatization questions, we have:

THEOREM 4.16. *The operation $S\to S_U$ produces a correspondence as follows,*

$$\begin{matrix}S^{N-1}_{\mathbb R,+}&\to&S^{N-1}_{\mathbb C,+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb R}&\to&S^{N-1}_{\mathbb C}\end{matrix}\quad\to\quad\begin{matrix}O_N^+&\to&U_N^+\\ \uparrow&&\uparrow\\ O_N&\to&U_N\end{matrix}$$

between basic unitary groups and the basic noncommutative spheres.

PROOF. We use the ergodicity formula from Theorem 4.14, namely:

$$\left(id\otimes\int_U\right)\Phi=\int_S$$

We know that $\int_U$ is faithful on $C(U)$, and that we have:

$$(id\otimes\varepsilon)\Phi=id$$

The coaction map Φ follows to be faithful as well. Thus for any $x\in C(S)$ we have:

$$\int_S xx^*=0\implies x=0$$

Thus $\int_S$ is faithful on $C(S)$. But this shows that we have:

$$S=S_U$$

Thus, we are led to the conclusion in the statement. □

4c. Partial isometries

Our goal now will be that of finding a suitable collection of "free homogeneous spaces", generalizing at the same time the free spheres S, and the free unitary groups U. This can be done at several levels of generality, and central here is the construction of the free spaces of partial isometries, which can be done in fact for any easy quantum group. In order to explain this, let us start with the classical case. We have here:

DEFINITION 4.17. *Associated to any integers $L\leq M,N$ are the spaces*

$$O_{MN}^L=\left\{T:E\to F\text{ isometry}\Big|E\subset\mathbb R^N,F\subset\mathbb R^M,\dim_{\mathbb R}E=L\right\}$$

$$U_{MN}^L=\left\{T:E\to F\text{ isometry}\Big|E\subset\mathbb C^N,F\subset\mathbb C^M,\dim_{\mathbb C}E=L\right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L=M=N$ we obtain the groups O_N, U_N:

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L=M=1$. Here the elements of O_{1N}^1 are the isometries $T: E \to \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$, and so our results here are:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1} \quad , \quad U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L=N=1$. Here the elements of O_{1N}^1 are the isometries $T: \mathbb{R} \to F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely determined by $T(1) \in \mathbb{R}^M$, which must belong to $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$, and so our results here are:

$$O_{M1}^1 = S_{\mathbb{R}}^{M-1} \quad , \quad U_{M1}^1 = S_{\mathbb{C}}^{M-1}$$

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 4.18. *We have identifications of compact spaces*

$$O_{MN}^L \simeq \left\{ U \in M_{M \times N}(\mathbb{R}) \Big| UU^t = \text{projection of trace } L \right\}$$

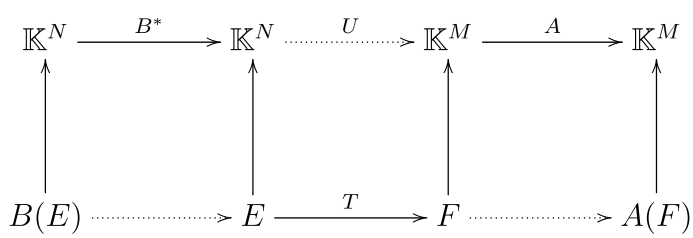
$$U_{MN}^L \simeq \left\{ U \in M_{M \times N}(\mathbb{C}) \Big| UU^* = \text{projection of trace } L \right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T: E \to F$ with their corresponding extensions $U: \mathbb{R}^N \to \mathbb{R}^M$, $U: \mathbb{C}^N \to \mathbb{C}^M$, obtained by setting $U_{E^\perp} = 0$. Then, we can identify these latter maps U with the corresponding rectangular matrices. □

As an illustration, at $L=M=N$ we recover in this way the usual matrix description of O_N, U_N. Also, at $L=M=1$ we obtain the usual description of $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$, as row spaces over the corresponding groups O_N, U_N. Finally, at $L=N=1$ we obtain the usual description of $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$, as column spaces over the corresponding groups O_N, U_N.

Now back to the general case, observe that the isometries $T: E \to F$, or rather their extensions $U: \mathbb{K}^N \to \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:



With the identifications in Proposition 4.18 made, the precise statement here is:

Proposition 4.19. *We have action maps as follows, which are both transitive,*

$$O_M \times O_N \curvearrowright O_{MN}^L \quad , \quad (A,B)U = AUB^t$$

$$U_M \times U_N \curvearrowright U_{MN}^L \quad , \quad (A,B)U = AUB^*$$

whose stabilizers are respectively $O_L \times O_{M-L} \times O_{N-L}$ *and* $U_L \times U_{M-L} \times U_{N-L}$.

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A,B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix} x & * \\ 0 & a \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & 0 \\ * & b \end{pmatrix}$$

Now since A,B are unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{ (A,B) \Big| A = \begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, B = \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix} \right\}$$

The stabilizer of U is parametrized by triples (x,a,b) belonging to $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$, and we are led to the conclusion in the statement. □

Finally, let us work out the quotient space description of O_{MN}^L, U_{MN}^L. We have here:

Theorem 4.20. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned} O_{MN}^L &= (O_M \times O_N)/(O_L \times O_{M-L} \times O_{N-L}) \\ U_{MN}^L &= (U_M \times U_N)/(U_L \times U_{M-L} \times U_{N-L}) \end{aligned}$$

with the quotient maps being given by $(A,B) \to AUB^*$, *where* $U = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$.

Proof. This is just a reformulation of Proposition 4.19, by taking into account the fact that the fixed point used in the proof there was $U = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$. □

Once again, the basic examples here come from the cases $L = M = N$ and $L = M = 1$. At $L = M = N$ the quotient spaces at right are respectively:

$$O_N \quad , \quad U_N$$

At $L = M = 1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1} \quad , \quad U_N/U_{N-1}$$

In fact, in the general $L = M$ case we obtain the following spaces:

$$O_{MN}^M = O_N/O_{N-M} \quad , \quad U_{MN}^M = U_N/U_{N-M}$$

Similarly, the examples coming from the cases $L=M=N$ and $L=N=1$ are particular cases of the general $L=N$ case, where we obtain the following spaces:

$$O_{MN}^N=O_N/O_{M-N}\quad,\quad U_{MN}^N=U_N/U_{M-N}$$

Summarizing, we have here some basic homogeneous spaces, unifying the spheres with the rotation groups. The point now is that we can liberate these spaces, as follows:

Definition 4.21. *Associated to any integers $L\leq M,N$ are the algebras*

$$\begin{aligned}C(O_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace } L\right)\\ C(U_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace } L\right)\end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij}u_{ij}u_{ij}^*=\sum_{ij}u_{ij}^*u_{ij}=L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc}O_{MN}^{L+}&\longrightarrow&U_{MN}^{L+}\\ \uparrow&&\uparrow\\ O_{MN}^L&\longrightarrow&U_{MN}^L\end{array}$$

At the level of basic examples now, at $L=M=1$ and at $L=N=1$ we obtain the following diagrams, showing that our formalism covers indeed the free spheres:

$$\begin{array}{ccc}S^{N-1}_{\mathbb{R},+}&\longrightarrow&S^{N-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb{R}}&\longrightarrow&S^{N-1}_{\mathbb{C}}\end{array}\qquad\qquad\begin{array}{ccc}S^{M-1}_{\mathbb{R},+}&\longrightarrow&S^{M-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{M-1}_{\mathbb{R}}&\longrightarrow&S^{M-1}_{\mathbb{C}}\end{array}$$

We have as well the following result, in relation with the free rotation groups:

Proposition 4.22. *At $L=M=N$ we obtain the diagram*

$$\begin{array}{ccc}
O_N^+ & \longrightarrow & U_N^+\\
\uparrow & & \uparrow\\
O_N & \longrightarrow & U_N
\end{array}$$

consisting of the groups O_N,U_N, and their liberations.

Proof. We recall that the various quantum groups in the statement are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$C(O_N^\times) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=u^tu=1\right)$$
$$C(U_N^\times) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1,\bar{u}u^t=u^t\bar{u}=1\right)$$

On the other hand, according to Proposition 4.18 and to Definition 4.21, we have the following presentation results:

$$C(O_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace }N\right)$$
$$C(U_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace }N\right)$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. We use as well the following formulae:

$$Tr(uu^*)=Tr(u^t\bar{u})\quad,\quad Tr(\bar{u}u^t)=Tr(u^*u)$$

We therefore obtain the following formulae:

$$C(O_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t,u^tu=\text{projections of trace }N\right)$$
$$C(U_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,u^*u,\bar{u}u^t,u^t\bar{u}=\text{projections of trace }N\right)$$

Now observe that, in tensor product notation, the conditions at right are all of the form $(tr\otimes id)p=1$. Thus, p must be follows, for the above conditions:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

We therefore obtain that, for any faithful state φ, we have $(tr\otimes\varphi)(1-p)=0$. It follows from this that the following projections must be all equal to the identity:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

But this leads to the conclusion in the statement. $\square$

Regarding now the homogeneous space structure of $O_{MN}^{L\times}, U_{MN}^{L\times}$, the situation here is a bit more complicated in the free case than in the classical case, due to a number of algebraic and analytic issues. We first have the following result:

PROPOSITION 4.23. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times \times U_N^\times \curvearrowright U_{MN}^{L\times}$, given by $u_{ij} \to \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$.*
(2) *We have a map $U_M^\times \times U_N^\times \to U_{MN}^{L\times}$, given by $u_{ij} \to \sum_{r\leq L} a_{ri} \otimes b_{rj}^*$.*

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

PROOF. In the classical case, consider the following action and quotient maps:

$$U_M \times U_N \curvearrowright U_{MN}^L \quad , \quad U_M \times U_N \to U_{MN}^L$$

The transposes of these two maps are as follows, where $J = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi &\to ((U,A,B) \to \varphi(AUB^*))\\ \varphi &\to ((A,B) \to \varphi(AJB^*))\end{aligned}$$

But with $\varphi = u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u = u$, let us set:

$$U_{ij} = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

We have then the following computation:

$$\begin{aligned}(UU^*U)_{ij} &= \sum_{pq}\sum_{klmnst} u_{kl}u_{mn}^*u_{st} \otimes a_{ki}a_{mq}^*a_{sq} \otimes b_{lp}^*b_{np}b_{tj}^*\\ &= \sum_{klmt} u_{kl}u_{ml}^*u_{mt} \otimes a_{ki} \otimes b_{tj}^*\\ &= \sum_{kt} u_{kt} \otimes a_{ki} \otimes b_{tj}^*\\ &= U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}\sum_{ij} U_{ij}U_{ij}^* &= \sum_{ij}\sum_{klst} u_{kl}u_{st}^* \otimes a_{ki}a_{si}^* \otimes b_{lj}^*b_{tj}\\ &= \sum_{kl} u_{kl}u_{kl}^* \otimes 1 \otimes 1\\ &= L\end{aligned}$$

(2) Assuming $uu^*u=u$, let us set:

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We have then the following computation:

$$\begin{aligned}(VV^*V)_{ij} &= \sum_{pq}\sum_{x,y,z\leq L}a_{xi}a_{yq}^*a_{zq}\otimes b_{xp}^*b_{yp}b_{zj}^*\\ &= \sum_{x\leq L}a_{xi}\otimes b_{xj}^*\\ &= V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}V_{ij}V_{ij}^* &= \sum_{ij}\sum_{r,s\leq L}a_{ri}a_{si}^*\otimes b_{rj}^*b_{sj}\\ &= \sum_{l\leq L}1\\ &= L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. $\square$

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X=G/H$, the associated action and quotient maps are given by:

$$\begin{cases}a:X\times G\to X & : \quad (Hg,h)\to Hgh\\ p:G\to X & : \quad g\to Hg\end{cases}$$

Thus we have $a(p(g),h)=p(gh)$. In our context, a similar result holds:

Theorem 4.24. *With $G=G_M\times G_N$ and $X=G_{MN}^L$, where $G_N=O_N^\times,U_N^\times$, we have*

$$\begin{array}{ccc}G\times G & \xrightarrow{\ m\ } & G\\ \downarrow{\scriptstyle p\times id} & & \downarrow{\scriptstyle p}\\ X\times G & \xrightarrow{\ a\ } & X\end{array}$$

where a,p are the action map and the map constructed in Proposition 4.23.

PROOF. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ,α are morphisms of algebras induced by a,p:

$$\begin{array}{ccc}
C(X) & \xrightarrow{\ \ \Phi\ \ } & C(X\times G)\\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id}\\
C(G) & \xrightarrow{\ \ \Delta\ \ } & C(G\times G)
\end{array}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned}
(\alpha\otimes id)\Phi(u_{ij}) &= (\alpha\otimes id)\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\\
&= \sum_{kl}\sum_{r\leq L}a_{rk}\otimes b_{rl}^*\otimes a_{ki}\otimes b_{lj}^*
\end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned}
\Delta\pi(u_{ij}) &= F_{23}(\Delta\otimes\Delta)\sum_{r\leq L}a_{ri}\otimes b_{rj}^*\\
&= F_{23}\left(\sum_{r\leq L}\sum_{kl}a_{rk}\otimes a_{ki}\otimes b_{rl}^*\otimes b_{lj}^*\right)
\end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. $\square$

Many other things can be said, as a continuation of the above, notably with some explicit integration results, in the spirit of those for the rotation groups, and the spheres, and applications of these. We will be back to this, later in this book.

4d. Discrete versions

Let us discuss now some discrete versions of the above constructions, which are something quite interesting as well, for various reasons. We can use here:

DEFINITION 4.25. *Associated to a partial permutation, $\sigma:I\simeq J$ with $I\subset\{1,\ldots,N\}$ and $J\subset\{1,\ldots,M\}$, is the real/complex partial isometry*

$$T_\sigma:span\left(e_i\Big|i\in I\right)\to span\left(e_j\Big|j\in J\right)$$

given on the standard basis elements by $T_\sigma(e_i)=e_{\sigma(i)}$.

Let S_{MN}^L be the set of partial permutations $\sigma:I\simeq J$ as above, with range $I\subset\{1,\ldots,N\}$ and target $J\subset\{1,\ldots,M\}$, and with $L=|I|=|J|$. We have:

Proposition 4.26. *The space of partial permutations signed by elements of $\mathbb{Z}_s$,*

$$H_{MN}^{sL}=\left\{T(e_i)=w_ie_{\sigma(i)}\Big|\sigma\in S_{MN}^L,w_i\in\mathbb{Z}_s\right\}$$

is isomorphic to the quotient space

$$(H_M^s\times H_N^s)/(H_L^s\times H_{M-L}^s\times H_{N-L}^s)$$

via a standard isomorphism.

Proof. This follows by adapting the computations in the proof of Proposition 4.19 and Theorem 4.20. Indeed, we have an action map as follows, which is transitive:

$$H_M^s\times H_N^s\to H_{MN}^{sL}\quad,\quad (A,B)U=AUB^*$$

Consider now the following point:

$$U=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

The stabilizer of this point follows to be the following group:

$$H_L^s\times H_{M-L}^s\times H_{N-L}^s$$

To be more precise, this group is embedded via:

$$(x,a,b)\to\left[\begin{pmatrix}x&0\\0&a\end{pmatrix},\begin{pmatrix}x&0\\0&b\end{pmatrix}\right]$$

But this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+}=\mathbb{Z}_s\wr_*S_N^+$. The result here is as follows:

Proposition 4.27. *The compact quantum space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+})=C(U_{MN}^{L+})\Big/\left\langle u_{ij}u_{ij}^*=u_{ij}^*u_{ij}=p_{ij}=\text{projections},u_{ij}^s=p_{ij}\right\rangle$$

has an action map, and is the target of a quotient map, as in Theorem 4.24.

Proof. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\quad,\quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We use the fact that the standard coordinates a_{ij},b_{ij} on the quantum groups H_M^{s+},H_N^{s+} satisfy the following relations, for any $x\neq y$ on the same row or column of a,b:

$$xy=xy^*=0$$

We obtain, by using these relations, the following formula:

$$U_{ij}U_{ij}^*=\sum_{klmn}u_{kl}u_{mn}^*\otimes a_{ki}a_{mi}^*\otimes b_{lj}^*b_{mj}=\sum_{kl}u_{kl}u_{kl}^*\otimes a_{ki}a_{ki}^*\otimes b_{lj}^*b_{lj}$$

On the other hand, we have as well the following formula:

$$V_{ij}V_{ij}^*=\sum_{r,t\leq L}a_{ri}a_{ti}^*\otimes b_{rj}^*b_{tj}=\sum_{r\leq L}a_{ri}a_{ri}^*\otimes b_{rj}^*b_{rj}$$

In terms of the projections $x_{ij}=a_{ij}a_{ij}^*$, $y_{ij}=b_{ij}b_{ij}^*$, $p_{ij}=u_{ij}u_{ij}^*$, we have:

$$U_{ij}U_{ij}^*=\sum_{kl}p_{kl}\otimes x_{ki}\otimes y_{lj}\quad,\quad V_{ij}V_{ij}^*=\sum_{r\leq L}x_{ri}\otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij},V_{ij}^*V_{ij}$ are given by the same formulae. Finally, once again by using the relations of type $xy=xy^*=0$, we have:

$$U_{ij}^s=\sum_{k_rl_r}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i}\ldots a_{k_si}\otimes b_{l_1j}^*\ldots b_{l_sj}^*=\sum_{kl}u_{kl}^s\otimes a_{ki}^s\otimes(b_{lj}^*)^s$$

On the other hand, we have as well the following formula:

$$V_{ij}^s=\sum_{r_l\leq L}a_{r_1i}\ldots a_{r_si}\otimes b_{r_1j}^*\ldots b_{r_sj}^*=\sum_{r\leq L}a_{ri}^s\otimes(b_{rj}^*)^s$$

Thus the conditions of type $u_{ij}^s=p_{ij}$ are satisfied as well, and we are done. □

Let us discuss now the general case. We have the following result:

Proposition 4.28. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with $s=(e_1,\ldots,e_s)$ ranging over all the colored integers, and with $\pi,\sigma\in D(0,s)$.

Proof. According to the various constructions above, the relations defining the quantum space G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D:

$$\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=\delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned}\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s} &= \sum_{i_1\ldots i_s}\delta_\pi(i)\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}\\ &= \sum_{i_1\ldots i_s}\delta_\pi(i)\delta_\sigma(i)\\ &= L^{|\pi\vee\sigma|}\end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. □

In the general case now, where $G=(G_N)$ is an arbitary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

THEOREM 4.29. *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with π,σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 4.24.

PROOF. We proceed as in the proof of Proposition 4.27. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\quad,\quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Regarding the variables U_{ij}, the computation here goes as follows:

$$\begin{aligned}&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\ =&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\otimes\delta_\pi(i)\delta_\sigma(j)a_{k_1i_1}^{e_1}\ldots a_{k_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\ =&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\\ =&\ L^{|\pi\vee\sigma|}\end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned}
& \sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
= & \sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{l_1i_1}^{e_1}\ldots a_{l_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
= & \sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)\\
= & L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 4.27, and the commutation of the diagram in Theorem 4.24 is then trivial. □

Many other things can be said, as a continuation of this.

4e. Exercises

Exercises:

Exercise 4.30.

Exercise 4.31.

Exercise 4.32.

Exercise 4.33.

Exercise 4.34.

Exercise 4.35.

Exercise 4.36.

Exercise 4.37.

Bonus exercise.

Part II

Half-liberation

Rise up
Rise up out of the fire
And forge yourself
Once more

CHAPTER 5

Half-liberation

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

EXERCISE 5.7.

EXERCISE 5.8.

Bonus exercise.

CHAPTER 6

Algebraic theory

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

EXERCISE 6.6.

EXERCISE 6.7.

EXERCISE 6.8.

Bonus exercise.

CHAPTER 7

Matrix models

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

Tangent spaces

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.

Part III

Twisted geometry

Come - around a little closer
Love - my energy it's fun
Run - your body through the motions
Shine - tonight and you'll be mine

CHAPTER 9

Anticommutation

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

Exercise 9.1.

Exercise 9.2.

Exercise 9.3.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

Exercise 9.7.

Exercise 9.8.

Bonus exercise.

CHAPTER 10

Twisted symmetries

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

EXERCISE 10.8.

Bonus exercise.

CHAPTER 11

Twisted geometry

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.

Exercise 11.4.

Exercise 11.5.

Exercise 11.6.

Exercise 11.7.

Exercise 11.8.

Bonus exercise.

CHAPTER 12

Modeling questions

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Advanced aspects

Watching every motion
In my foolish lover's game
On this endless ocean
Finally lovers know no shame

CHAPTER 13

Heavy algebra

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Bonus exercise.

CHAPTER 14

Calculus, geometry

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

Into amenability

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

Towards freeness

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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