Analysis on quantum groups

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2010 *Mathematics Subject Classification.* 81R50

Key words and phrases. Quantum group, Fourier transform

Abstract. This is an introduction to the finite, compact and discrete quantum groups, chosen functional analytic in nature, and to Fourier analysis over them.

Preface

This is an introduction to the finite, compact and discrete quantum groups, chosen functional analytic in nature, and to Fourier analysis over them.

Many thanks to my colleagues and collaborators, for substantial joint work on all this. Thanks as well to my cats, for some help with computing Haar integrals.

Cergy, February 2026
Teo Banica

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I got it one piece at a time
And it wouldn't cost me a dime
You'll know it's me
When I come through your town

CHAPTER 1

Hopf algebras

1a. Hopf algebras

The classical spaces X, such as the Lie groups, homogeneous spaces, or more general manifolds, are described by various algebras A, defined over various fields F. These algebras A typically satisfy a commutativity type condition, such as $fg = gf$ when A is a usual algebra of functions, and the idea of quantum algebra is that of lifting this commutativity condition, and calling quantum spaces the underlying space-like objects X. With the hope that these quantum spaces X can be useful in physics.

In this chapter we start developing quantum algebra, with some inspiration from classical group theory. We would like to develop a theory of suitable algebras A, which are not necessarily commutative, corresponding to quantum groups G.

In what regards the classical constructions $G \to A$ that we have in mind, in view of a noncommutative extension, there are three of them, all well-known and widely used in group theory, which can be informally described, modulo several details, as follows:

Fact 1.1. *A group G can be typically described by several algebras:*

1. *We have the algebra $F(G)$ of functions $f : G \to F$, with the usual, pointwise product of functions. This algebra is commutative, $fg = gf$.*
2. *We have the algebra $F[G]$ of functions $f : G \to F$, with the convolution product of functions. This algebra is commutative when G is abelian.*
3. *We have the algebra $U\mathfrak{g}$ generated by the functions $f : G \to F$ infinitesimally defined around 1. This algebra is commutative when G is abelian.*

As already mentioned, this is something quite informal, just meant to help us in order to start this book, and do not worry, we will come back to this later, with details. At the present stage of things, the comments to be made on this are as follows:

(1) The construction there is something quite simple and solid, and makes sense as stated, with the remark however that when G is a topological group, things get more complicated, because we can further ask for the functions $f : G \to F$ that we use to be continuous, or even smooth, or vanishing at ∞, or be measurable, and so on, leading to several interesting versions of $F(G)$. Also, the recovery of G from the algebra $F(G)$, or one of its versions, when G is topological, is usually a non-trivial question.

(2) In what regards the construction there, pretty much the same comments apply, the point being that when G is a topological group, things get more complicated, again leading to several interesting versions of the algebra $F[G]$, and with the recovery of the group G itself, out of these algebras, being usually a non-trivial question. We will be back to all this later, with details, and in the meantime, you can simply consider, by using Dirac masses, that $F[G]$ is the formal span of the group elements $g \in G$.

(3) What we said there is definitely informal, the idea being that, when G is a Lie group, we can consider its tangent space at the origin, or Lie algebra $\mathfrak{g} = T_1(G)$, consisting of the functions $f : G \to F$ infinitesimally defined around 1, and then the corresponding enveloping Lie algebra $U\mathfrak{g}$, with product such that the Lie bracket is given by $[x, y] = xy - yx$. All this is quite non-trivial, notably with a discussion in relation with the field F being needed, but do not worry, we will come back to it, with details.

Looking now at our list (1,2,3) above, it looks like (1) is the simplest construction, and the most adapted to our noncommutative goals, followed by (2), followed by (3). So, let us formulate the following goal, for the theory that we want to develop:

Goal 1.2. *We want to develop a theory of associative algebras A over a given field F, with some extra structure, as follows:*

(1) *As main and motivating examples, we want to have the algebras $F(G)$.*
(2) *We also want our theory to include, later, the algebras $F[G]$ and $U\mathfrak{g}$.*
(3) *And we also want, later, to discuss what happens when G is topological.*

Needless to say, this goal is formulated quite informally, but this is just a goal, and if at this point you can see right away a complete and rigorous theory doing the job, that would be of course very welcome, and I will look myself for something else to do.

Getting started now, we would first like to have a look at the algebras $F(G)$ that we want to generalize. But before that, let us have a closer look at the groups G themselves, with algebraic motivations in mind, in relation with the algebras $F(G)$. As our first result in this book, we have the following frightening reformulation of the group axioms:

Proposition 1.3. *A group is a set G with operations as follows,*

$$m : G \times G \to G \quad , \quad u : \{.\} \to G \quad , \quad i : G \to G$$

which are subject to the following axioms, with $\delta(g) = (g, g)$:

$$m(m \times id) = m(id \times m)$$

$$m(u \times id) = m(id \times u) = id$$

$$m(i \times id)\delta = m(id \times i)\delta = 1$$

In addition, the inverse map i satisfies $i^2 = id$.

PROOF. Our claim is that the formulae in the statement correspond to the axioms satisfied by the multiplication, unit and inverse map of G, given by:

$$m(g,h)=gh \quad , \quad u(.)=1 \quad , \quad i(g)=g^{-1}$$

Indeed, let us start with the group axioms for G, which are as follows:

$$(gh)k=g(hk)$$

$$1g=g1=g$$

$$g^{-1}g=gg^{-1}=1$$

With $\delta(g)=(g,g)$ being as in the statement, these group axioms read:

$$m(m\times id)(g,h,k)=m(id\times m)(g,h,k)$$

$$m(u\times id)(g)=m(id\times u)(g)=g$$

$$m(i\times id)\delta(g)=m(id\times i)\delta(g)=1$$

Now since these must hold for any g,h,k, they are equivalent, as claimed, to:

$$m(m\times id)=m(id\times m)$$

$$m(u\times id)=m(id\times u)=id$$

$$m(i\times id)\delta=m(id\times i)\delta=1$$

As for $i^2=id$, this is something which holds too, coming from $(g^{-1})^{-1}=1$:

$$(g^{-1})^{-1}=1\iff i^2(g)=g\iff i^2=id$$

Thus, we are led to the various conclusions in the statement. □

The above result does not look very healthy, and might make Sophus Lie, Felix Klein and the others turn in their graves, but for our purposes here, this is exactly what we need. Indeed, turning now to the algebra $F(G)$, we have the following result:

THEOREM 1.4. *Given a finite group G, the functional transposes of the structural maps m,u,i, called comultiplication, counit and antipode, are as follows,*

$$\Delta:A\to A\otimes A \quad , \quad \varepsilon:A\to F \quad , \quad S:A\to A$$

with $A=F(G)$ being the algebra of functions $\varphi:G\to F$. The group axioms read:

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

In addition, the square of the antipode is the identity, $S^2=id$.

PROOF. This is something which is clear from Proposition 1.3, and from the properties of the functional transpose, with no computations needed. However, since the formalism of the functional transpose might be new to you, here is a detailed proof:

(1) Let us first recall that, given a map between two sets $f : X \to Y$, its functional transpose is the morphism of algebras $f^t : F(Y) \to F(X)$ given by:

$$f^t(\varphi) = [x \to \varphi(f(x))]$$

To be more precise, this map f^t is indeed well-defined, and the fact that we obtain in this way a morphism of algebras is clear. Indeed, for the addition, we have:

$$\begin{aligned} f^t(\varphi + \psi) &= [x \to (\varphi + \psi)(f(x))] \\ &= [x \to (\varphi(f(x)) + \psi(f(x)))] \\ &= [x \to (\varphi(f(x))] + [x \to \psi(f(x))] \\ &= f^t(\varphi) + f^t(\psi) \end{aligned}$$

As for the multiplication, the verification here is similar, as follows:

$$\begin{aligned} f^t(\varphi\psi) &= [x \to (\varphi\psi)(f(x))] \\ &= [x \to (\varphi(f(x))\psi(f(x)))] \\ &= [x \to (\varphi(f(x))] \cdot [x \to \psi(f(x))] \\ &= f^t(\varphi)f^t(\psi) \end{aligned}$$

(2) Observe now that the operation $f \to f^t$ is by definition contravariant, in the sense that it reverses the direction of the arrows. Also, we have the following formula:

$$(fg)^t = g^t f^t$$

In order to check this, consider indeed two composable maps, as follows:

$$g : X \to Y \quad , \quad f : Y \to Z$$

The transpose of the composed map $fg : X \to Z$ is then given by:

$$\begin{aligned} (fg)^t(\varphi) &= [x \to \varphi(fg(x))] \\ &= [x \to (\varphi f)(g(x))] \\ &= [x \to (f^t(\varphi))(g(x))] \\ &= g^t f^t(\varphi) \end{aligned}$$

Thus, we have indeed $(fg)^t = g^t f^t$, as claimed above. It is of course possible to use this iteratively, with the general formula, that we will often use, being as follows:

$$(f_1 \ldots f_n)^t = f_n^t \ldots f_1^t$$

(3) As a second piece of preliminaries, we will need a bit of tensor product calculus too. In case you are familiar with this, say from physics classes, that is good news, and if not, here is a crash course on this. Let us start with something familiar, namely:

$$F^{M+N} = F^M \oplus F^N$$

As a consequence of this, that my students quite often tend to forget, we have:

$$F^{MN} \neq F^M \oplus F^N$$

And so, question for us now, what can be the new, mysterious operation $\otimes$, which is definitely not the direct sum $\oplus$, making the following formula work:

$$F^{MN} = F^M \otimes F^N$$

(4) In answer, let us look at the standard bases of these three vector spaces. We can denote by $\{f_1, \ldots, f_M\}$ the standard basis of F^M, and by $\{g_1, \ldots, g_N\}$ the standard basis of F^N. As for the space F^{MN} on the left, here you would probably say to use the notation $\{e_1, \ldots, e_{MN}\}$, but I have here something better, namely $\{e_{11}, \ldots, e_{MN}\}$, by using double indices. And, with this trick, the solution of our problem becomes clear, namely:

$$e_{ia} = f_i \otimes g_a$$

Thus, as a conclusion, given two vector spaces with bases $\{f_i\}$ and $\{g_a\}$, we can talk about their tensor product, as being the vector space with basis $\{f_i \otimes g_a\}$. And with this, we have the following formula, answering the question raised above:

$$F^{MN} = F^M \otimes F^N$$

(5) As a continuation of this, in the case where our vector spaces are algebras A, B, their tensor product $A \otimes B$ is an algebra too, with multiplication as follows:

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

Indeed, the algebra axioms are easily seen to be satisfied. And as a verification here, the above identification $F^{MN} = F^M \otimes F^N$ is indeed an algebra morphism, due to:

$$\begin{aligned}
e_{ia}e_{jb} &= (f_i \otimes g_a)(f_j \otimes g_b) \\
&= f_if_j \otimes g_ag_b \\
&= \delta_{ij}f_i \otimes \delta_{ab}g_a \\
&= \delta_{ia,jb}f_i \otimes g_a \\
&= \delta_{ia,jb}e_{ia}
\end{aligned}$$

(6) Moving ahead, still in relation with tensor products, we can say, more abstractly, that given two finite sets X, Y, we have an isomorphism of algebras as follows:

$$F(X \times Y) = F(X) \otimes F(Y)$$

To be more precise, we have a morphism from right to left, constructed as follows:

$$\varphi\otimes\psi\to[(x,y)\to\varphi(x)\psi(y)]$$

Now since this morphism is injective, and since the dimensions of the domain and range match, both being equal to $|X|\cdot|Y|$, this morphism is an isomorphism.

(7) In what follows, the above formula from (6) will be what we will mostly need, in relation with the tensor products. As a last comment here, observe that this is in fact the formula $F^{MN}=F^M\otimes F^N$ that we started with. Indeed, when writing $X=\{1,\ldots,M\}$ and $Y=\{1,\ldots,N\}$, the above formula from (6) takes the following form:

$$F\left(\{1,\ldots,M\}\times\{1,\ldots,N\}\right)=F\left(\{1,\ldots,M\}\right)\otimes F\left(\{1,\ldots,N\}\right)$$

But this obviously translates into the following formula, as claimed:

$$F^{MN}=F^M\otimes F^N$$

And we will end with our preliminaries here. In case all this was not fully clear, may the Gods of Algebra forgive us, and we recommend on one hand reading about tensor products from a solid algebra book, such as Lang [67], and on the other hand, doing some physics computations with tensor products, nothing can replace those either.

(8) Still with me, I hope, and time now to prove our theorem? With the above ingredients in hand, let us go back indeed to our group theory problems. To start with, the structural maps m,u,i of our group G are maps as follows:

$$m:G\times G\to G\quad,\quad u:\{.\}\to G\quad,\quad i:G\to G$$

Thus, with $A=F(G)$ being the algebra of functions $\varphi:G\to F$, their functional transposes are morphisms of algebras Δ,ε,S as follows:

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to F\quad,\quad S:A\to A$$

(9) Regarding now the formulae of these transposed maps, we know that the structural maps m,u,i of our group G are given by the following formulae:

$$m(g,h)=gh\quad,\quad u(.)=1\quad,\quad i(g)=g^{-1}$$

Thus, the functional transposes Δ,ε,S are given by the following formulae:

$$\Delta(\varphi)=[(g,h)\to\varphi(gh)]\quad,\quad\varepsilon(\varphi)=\varphi(1)\quad,\quad S(\varphi)=[g\to\varphi(g^{-1})]$$

(10) Regarding the group axioms, we know from Proposition 1.3 that in terms of the structural maps m,u,i, these are as follows, with $\delta(g)=(g,g)$:

$$m(m\times id)=m(id\times m)$$

$$m(u\times id)=m(id\times u)=id$$

$$m(i\times id)\delta=m(id\times i)\delta=1$$

In terms of the functional transposes Δ,ε,S, these axioms read:

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

Finally, the formula $S^2=id$ comes by transposing $i^2=id$. □

Good news, Theorem 1.4 is all we need, or almost, in order to fulfill Goal 1.2 (1). Indeed, based on what we found above, we can formulate the following definition:

Definition 1.5. *A Hopf algebra is an algebra A, with morphisms of algebras*

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to F\quad,\quad S:A\to A^{opp}$$

called comultiplication, counit and antipode, satisfying the following conditions:

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

If the square of the antipode is the identity, $S^2=id$, we say that A is undeformed. Otherwise, in the case $S^2\neq id$, we say that A is deformed.

Here everything is standard, based on what we found in Theorem 1.4, we just copied the formulae there, with a different banner for them, except for two points, namely:

(1) We chose to have the antipode S as being a morphism of algebras $S:A\to A^{opp}$, instead of being a morphism $S:A\to A$, as Theorem 1.4 might suggest. Indeed, since the algebra $A=F(G)$ in Theorem 1.4 is commutative, we have $A=A^{opp}$ in that case, so we can make this choice. And, we will see in a moment that $S:A\to A^{opp}$ is indeed the good choice, with this coming from some further examples, that we want our formalism to cover, and more specifically, coming from the algebras $F[G]$, from Goal 1.2 (2).

(2) We also chose the antipode S not to be subject to the condition $S^2=id$. However, this is something debatable, because in the usual group setting $i^2=id$, while formally not being a group axiom, is something so trivial and familiar, that it is "almost" a group axiom. We will be back to this issue, on several occasions. In fact, clarifying the relation between Hopf algebras axiomatized with $S^2=id$, and Hopf algebras axiomatized without $S^2=id$, will be a main theme of discussion, throughout this book.

Finally, observe also that we chose not to impose any finite dimensionality condition on our Hopf algebra A, and this in contrast with Theorem 1.4, where the group G there is finite. Again, this is something subtle, to be discussed more in detail later on.

1b. Basic examples

There are several basic examples of Hopf algebras, which are all undeformed. We first have the following result, which provides a good motivation for our theory:

THEOREM 1.6. *Given a finite group G, the algebra of the F-valued functions on it, $F(G) = \{\varphi : G \to F\}$, with the usual pointwise product of functions,*

$$(\varphi\psi)(g) = \varphi(g)\psi(g)$$

is a Hopf algebra, with comultiplication, counit and antipode as follows:

$$\Delta(\varphi) = [(g,h) \to \varphi(gh)]$$
$$\varepsilon(\varphi) = \varphi(1)$$
$$S(\varphi) = [g \to \varphi(g^{-1})]$$

This Hopf algebra is finite dimensional, commutative, and undeformed.

PROOF. This is a reformulation of Theorem 1.4, by taking into account Definition 1.5, with the remark, already made above, that we have $A = A^{opp}$ in this case, due to the commutativity of $A = F(G)$, and with the last assertion being something clear. □

In view of the above result, we can make the following speculation:

SPECULATION 1.7. *We can think of any finite dimensional Hopf algebra A as being of the following form, with G being a finite quantum group:*

$$A = F(G)$$

That is, we can define the category of finite quantum groups G to be the category of finite dimensional Hopf algebras A, with the arrows reversed.

Observe that, from the perspective of pure mathematics, all this is not that speculatory, because what we said in the end is something categorical and rigorous, perfectly making sense, and with the category of the usual finite groups G embedding covariantly into the category of the finite quantum groups G, thanks to Theorem 1.6.

However, still mathematically speaking, there are some bugs with this. One problem is whether we want to include or not $S^2 = id$ in our axioms, and in the lack of $S^2 \neq id$ examples, at this stage of things, we are in the dark. Another problem is that, even when assuming $S^2 = id$, nothing guarantees that a finite dimensional commutative Hopf algebra must be of the form $A = F(G)$, which would be something desirable to have.

As for the perspective brought by applied mathematics, here things are harsher, because the use of the word "quantum" would normally assume that our notion of Hopf algebra has something to do with quantum physics, and this is certainly not the case, now that we are into chapter 1 of the present book. Long way to go here, trust me.

In short, Speculation 1.7 remains a speculation, with our comments on it being:

COMMENT 1.8. *The above $A = F(G)$ picture is something very useful, definitely worth to be kept in mind, but we will have to work some more on our axioms for Hopf algebras A, as for the corresponding objects G to deserve the name "quantum groups".*

Finally, still in connection with all this, axiomatics, we would like if possible the construction in Theorem 1.6 to cover other groups as well, infinite this time, such as the discrete ones, or the compact ones, or, ideally, the locally compact ones.

The problem with this, however, is that in the framework of Definition 1.5 this is not exactly possible, due to the fact that the comultiplication Δ would have to land in the algebra $F(G \times G)$, and for infinite groups G, H, we have:

$$F(G \times H) \neq F(G) \otimes F(H)$$

However, there are several tricks in order to overcome this, either by allowing $\otimes$ to be a topological tensor product, or by using Lie algebras. We will be back to this question, which is not trivial to solve, on several occassions, in what follows.

Moving ahead now, let us say that a Hopf algebra A as axiomatized above is cocommutative if, with $\Sigma(a \otimes b) = b \otimes a$ being the flip map, we have the following formula:

$$\Sigma\Delta = \Delta$$

With this notion in hand, we have the following result, providing us with more examples, and that we will soon see to be "dual" to Theorem 1.6, in a suitable sense:

THEOREM 1.9. *Given a group H, which can be finite or not, its group algebra*

$$F[H] = span(H)$$

is a Hopf algebra, with structural maps given on group elements as follows:

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

This Hopf algebra is cocommutative, and undeformed.

PROOF. This is something elementary, the idea being as follows:

(1) As a first observation, we can define indeed linear maps Δ, ε, S as in the statement, by linearity, and the maps Δ, ε are obviously morphisms of algebras. As for the antipode $S : A \to A^{opp}$, this is a morphism of algebras too, due to the following computation, crucially using the opposite multiplication $(a, b) \to a \cdot b$ on the target algebra:

$$\begin{aligned} S(gh) &= (gh)^{-1} \\ &= h^{-1}g^{-1} \\ &= g^{-1} \cdot h^{-1} \\ &= S(g) \cdot S(h) \end{aligned}$$

(2) We have to verify now that Δ,ε,S satisfy the axioms in Definition 1.5, and the verifications here, performed on generators, are as follows:

$$(\Delta\otimes id)\Delta(g)=(id\otimes\Delta)\Delta(g)=g\otimes g\otimes g$$

$$(\varepsilon\otimes id)\Delta(g)=(id\otimes\varepsilon)\Delta(g)=g$$

$$m(S\otimes id)\Delta(g)=m(id\otimes S)\Delta(g)=\varepsilon(g)1=1$$

(3) Finally, it is clear from definitions that our Hopf algebra satisfies indeed the cocommutativity condition $\Sigma\Delta=\Delta$, as well as the condition $S^2=id$. □

The fact that the group H in the above can be infinite comes as good news, and it is tempting to jump on this, and formulate, in analogy with Speculation 1.7:

Speculation 1.10. *We can think of any Hopf algebra A as being of the following form, with H being a quantum group:*

$$A=F[H]$$

That is, we can define the category of quantum groups H to be the category of Hopf algebras A.

However, as before with Speculation 1.7, while this being something useful, providing us with some intuition on what a Hopf algebra is, when looking more in detail at this, there are countless problems with it, which are both purely mathematical, of algebraic and analytic nature, and applied mathematical, in relation with quantum physics, which is certainly something more complicated than what we did in the above.

Be said in passing, observe that, while both Speculation 1.7 and Speculation 1.10 formally make sense, from a pure mathematics perspective, their joint presence does not make much sense, mathematically, at least with the results that we have so far, because nothing guarantees that the category of finite quantum groups from Speculation 1.7 is indeed a subcategory of the category of quantum groups from Speculation 1.10.

Thinking a bit more at all this, we are led into the following question:

Question 1.11. *What is the precise relation between Theorems 1.6 and 1.9, in the finite group case, and can this make peace between Speculations 1.7 and 1.10?*

And the point now is that, despite its informal look, this question appears to be well-defined, and quite interesting, and answering it will be our next objective.

For this purpose, we first need to see what happens to Theorem 1.9, when assuming that the group H there is finite. And here, we have the following statement:

THEOREM 1.12. *Given a finite group H, the algebra of the F-valued functions on it $F[H]=\{\varphi:H\to F\}$, with the convolution product of functions,*

$$(\varphi*\psi)(g)=\sum_{g=hk}\varphi(h)\psi(k)$$

is a Hopf algebra, with structural maps given on Dirac masses as follows:

$$\Delta(\delta_g)=\delta_g\otimes\delta_g\quad,\quad\varepsilon(\delta_g)=1\quad,\quad S(\delta_g)=\delta_{g^{-1}}$$

This Hopf algebra, which coincides with the previous $F[H]$, in the finite group case, is finite dimensional, cocommutative, and undeformed.

PROOF. This is what comes from Theorem 1.9, when the group H there is finite. Indeed, in this case the vector space $F[H]=span(H)$ from Theorem 1.9 coincides with the vector space $F[H]=\{\varphi:H\to F\}$ in the statement, with the correspondence being given on the standard generators $g\in span(H)$ by the following formula:

$$g\to\delta_g$$

Regarding now the product operation, the product on $F[H]=span(H)$ from Theorem 1.9 corresponds to the above convolution product on $F[H]=\{\varphi:H\to F\}$, because:

$$\begin{aligned}(\delta_r*\delta_s)(g)&=\sum_{g=hk}\delta_r(h)\delta_s(k)\\&=\delta_{g,rs}\\&=\delta_{rs}(g)\end{aligned}$$

Thus $\delta_r*\delta_s=\delta_{rs}$, as desired. We conclude that the algebra $F[H]$ from Theorem 1.9 coincides with the algebra $F[H]$ constructed here, and this gives the result. □

In practice now, the above statement has a weakness, coming from the fact that our formulae for Δ,ε,S are in terms of the Dirac masses. Here is a better version of it:

THEOREM 1.13. *Given a finite group H, the algebra of the F-valued functions on it $F[H]=\{\varphi:H\to F\}$, with the convolution product of functions,*

$$(\varphi*\psi)(g)=\sum_{g=hk}\varphi(h)\psi(k)$$

is a Hopf algebra, with structural maps constructed as follows:

$$\Delta(\varphi)=[(g,h)\to\delta_{gh}\varphi(g)]$$

$$\varepsilon(\varphi)=\sum_{g\in H}\varphi(g)$$

$$S(\varphi)=[g\to\varphi(g^{-1})]$$

This Hopf algebra, which coincides with the previous $F[H]$, in the finite group case, is finite dimensional, cocommutative, and undeformed.

Proof. This is what comes from Theorem 1.12, by linearity. Indeed, according to our formula of Δ there, on the Dirac masses, we have:

$$\begin{aligned}\Delta(\varphi)(g,h) &= \Delta\left(\sum_{k\in H}\varphi(k)\delta_k\right)(g,h)\\ &= \sum_{k\in H}\varphi(k)\Delta(\delta_k)(g,h)\\ &= \sum_{k\in H}\varphi(k)(\delta_k\otimes\delta_k)(g,h)\\ &= \sum_{k\in H}\varphi(k)\delta_{kg}\delta_{kh}\\ &= \delta_{gh}\varphi(g)\end{aligned}$$

Also, according to our formula of ε there, on the Dirac masses, we have:

$$\begin{aligned}\varepsilon(\varphi) &= \varepsilon\left(\sum_{g\in H}\varphi(g)\delta_g\right)\\ &= \sum_{g\in H}\varphi(g)\varepsilon(\delta_g)\\ &= \sum_{g\in H}\varphi(g)\end{aligned}$$

Finally, according to our formula of S there, on the Dirac masses, we have:

$$\begin{aligned}S(\varphi)(g) &= S\left(\sum_{h\in H}\varphi(h)\delta_h\right)(g)\\ &= \sum_{h\in H}\varphi(h)S(\delta_h)(g)\\ &= \sum_{h\in H}\varphi(h)\delta_{h^{-1}}(g)\\ &= \varphi(g^{-1})\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a comment here, the proof of the above result relies on Theorem 1.12, which itself relies on Theorem 1.9, and in this type of situation, when things pile up, it is better to work out a new, direct proof, matter of doublechecking everything, and also matter of better understanding what is going on. So, let us do this, as an instructive exercise.

The problem is that of checking the Hopf algebra axioms directly, starting from the formulae of Δ,ε,S from Theorem 1.13, and we have here the following result:

PROPOSITION 1.14. *We have the following formulae over the algebra $F[H]$,*

$$(\Delta\otimes id)\Delta(\varphi)=(id\otimes\Delta)\Delta(\varphi)$$

$$(\varepsilon\otimes id)\Delta(\varphi)=(id\otimes\varepsilon)\Delta(\varphi)=\varphi$$

$$m(S\otimes id)\Delta(\varphi)=m(id\otimes S)\Delta(\varphi)=\varepsilon(\varphi)1$$

guaranteeing that $F[H]$ is indeed a Hopf algebra, for any finite group H.

PROOF. As mentioned, we already know this, as a consequence of Theorem 1.13, coming as consequence of Theorem 1.12, coming itself as consequence of Theorem 1.9, but time now to prove this directly as well, by using the formulae of Δ,ε,S, namely:

$$\Delta(\varphi)=[(g,h)\to\delta_{gh}\varphi(g)]$$

$$\varepsilon(\varphi)=\sum_{g\in H}\varphi(g)$$

$$S(\varphi)=[g\to\varphi(g^{-1})]$$

We have to prove the following formulae, for any group elements g,h,k:

$$(\Delta\otimes id)\Delta(\varphi)(g,h,k)=(id\otimes\Delta)\Delta(\varphi)(g,h,k)$$

$$(\varepsilon\otimes id)\Delta(\varphi)(g)=(id\otimes\varepsilon)\Delta(\varphi)(g)=\varphi(g)$$

$$m(S\otimes id)\Delta(\varphi)(g)=m(id\otimes S)\Delta(\varphi)(g)=\varepsilon(\varphi)1$$

In what regards the first formula, this is clear, because the second iteration $\Delta^{(2)}$ of the comultiplication, no matter how computed, will by given the following formula:

$$\Delta^{(2)}(\varphi)(g,h,k)=\delta_{ghk}\varphi(g)$$

Regarding the second formula, this is again clear, because when applying either of the maps $E_1=\varepsilon\otimes id$ and $E_2=id\otimes\varepsilon$ to the quantity $\Delta(\varphi)$, what we get is:

$$E_i\Delta(\varphi)(g)=\sum_{g=h}\varphi(h)=\varphi(g)$$

As for the third formula, this is similar, because when applying either of the maps $T_1=m(S\otimes id)$ and $T_2=m(id\otimes S)$ to the quantity $\Delta(\varphi)$, what we get is:

$$T_i\Delta(\varphi)(g)=\sum_{g\in H}\varphi(g)1=\varepsilon(\varphi)1$$

Thus, we are led to the conclusions in the statement. □

1c. Abelian groups

With the above done, let us try now to understand the relation between the algebras $F(G)$ from Theorem 1.6, and the algebras $F[H]$ from Theorem 1.12, or Theorem 1.13.

For this purpose, we must first talk about abelian groups, and their duals. And as a starting point here, we first have the following elementary result:

THEOREM 1.15. *Given a finite group G, the multiplicative characters*

$$\chi : G \to F^*$$

form a group $\widehat{G}$, called dual group, having the following properties:

(1) *$\widehat{G}$ is finite and abelian, and depends on both G and F.*
(2) *$\widehat{G} = \widehat{G}_{ab}$, where $G_{ab} = G/[G,G]$ is the abelianization of G.*
(3) *We have a morphism $G \to \widehat{\widehat{G}}$, producing a morphism $G_{ab} \to \widehat{\widehat{G}}$.*
(4) *The dual of a product is the product of duals, $\widehat{G \times H} = \widehat{G} \times \widehat{H}$.*

PROOF. Our first claim is that $\widehat{G}$ as constructed above is indeed a group, with the pointwise multiplication of the characters, given by the following formula:

$$(\chi\rho)(g) = \chi(g)\rho(g)$$

Indeed, if χ, ρ are characters, so is $\chi\rho$, and so the multiplication is well-defined on $\widehat{G}$. Regarding the unit, this is the trivial character, constructed as follows:

$$1 : G \to F^* \quad , \quad g \to 1$$

Finally, we have inverses, with the inverse of $\chi : G \to F^*$ being as follows:

$$\chi^{-1} : G \to F^* \quad , \quad g \to \chi(g)^{-1}$$

Thus the dual group $\widehat{G}$ is indeed a group, and regarding now the other assertions:

(1) We have several things to be proved here, the idea being as follows:

– Our first claim is that $\widehat{G}$ is finite. Indeed, given a group element $g \in G$, we can talk about its order, which is smallest integer $N \in \mathbb{N}$ such that $g^N = 1$. Now assuming that we have a character $\chi : G \to F^*$, we must have the following formula:

$$\chi(g)^N = 1$$

Thus $\chi(g)$ must be one of the N-th roots of unity inside F, that is, must be one of the roots over F of the polynomial $X^N - 1$, and in particular, there are finitely many choices for $\chi(g)$. Thus, there are finitely many choices for χ, and so $\widehat{G}$ is finite, as claimed.

– Next, the fact that $\widehat{G}$ is abelian follows from definitions, because the pointwise multiplication of functions, and in particular of characters, is commutative.

– Finally, the group dual $\widehat{G}$ as constructed above certainly depends on G, but the point is that it can depend on the ground field F too. Indeed, for an illustration here, consider the cyclic group $G = \mathbb{Z}_N$. A character $\chi : \mathbb{Z}_N \to F^*$ is then uniquely determined by its value $z = \chi(g)$ on the standard generator $g \in \mathbb{Z}_N$, and this value must satisfy:

$$z^N = 1$$

Now over the complex numbers, $F = \mathbb{C}$, the solutions here are the usual N-th roots of unity, so we have $|\widehat{\mathbb{Z}}_N| = N$. Moreover, by thinking a bit, we have in fact:

$$\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$$

In contrast, over the real numbers, $F = \mathbb{R}$, the possible solutions of $z^N = 1$ must be among $z = \pm 1$, and we conclude that in this case, the dual is given by:

$$\widehat{\mathbb{Z}}_N = \begin{cases} \{1\} & (N \text{ odd}) \\ \mathbb{Z}_2 & (N \text{ even}) \end{cases}$$

There are of course far more things that can be said here, with all this being related to the structure of the group of N-th roots of unity inside F, but for our present purposes, what we have so far, namely the above illustration using $F = \mathbb{R}, \mathbb{C}$, will do.

(2) Let us prove now the second assertion, $\widehat{G} = \widehat{G}_{ab}$. We recall that given a group G, its commutator subgroup $[G, G] \subset G$ is constructed as follows:

$$[G, G] = \left\{ ghg^{-1}h^{-1} \Big| g, h \in G \right\}$$

This subgroup is then normal, and we can define the abelianization of G as being:

$$G_{ab} = G/[G, G]$$

Now given a character $\chi : G \to F^*$, we have the following formula, for any $g, h \in G$, based of course on the fact that the multiplicative group F^* is abelian:

$$\chi(ghg^{-1}h^{-1}) = 1$$

Thus, our character factorizes as follows, into a character of the group G_{ab}:

$$\chi : G \to G_{ab} \to F^*$$

Summarizing, we have constructed an identification $\widehat{G} = \widehat{G}_{ab}$, as claimed.

(3) Regarding now the third assertion, as a first remark, given a finite group G we have indeed a morphism $I : G \to \widehat{\widehat{G}}$, appearing via evaluation maps, as follows:

$$I : G \to \widehat{\widehat{G}} \quad , \quad g \to [\chi \to \chi(g)]$$

Since the group duals, and so the group double duals too, are always abelian, we cannot expect I to be injective, in general. In fact, due to $\chi(ghg^{-1}h^{-1}) = 1$, we have:

$$[G, G] \subset \ker I$$

Thus, the morphism $I:G\to\widehat{\widehat{G}}$ constructed above factorizes as follows:

$$I:G\to G_{ab}\to\widehat{\widehat{G}}$$

And with this, third assertion proved. There are of course some further things that can be said here, in relation with the factorized morphism $G_{ab}\to\widehat{\widehat{G}}$, but the examples in (1) above, where G itself was abelian, show that, when using an arbitrary field F, as we are currently doing, this factorized morphism is not an isomorphism, in general.

(4) Finally, in what regards the fourth assertion, $\widehat{G\times H}=\widehat{G}\times\widehat{H}$, observe that a character of a product of groups $\chi:G\times H\to F^*$ must satisfy:

$$\begin{aligned}\chi(g,h)&=\chi\left[(g,1)(1,h)\right]\\&=\chi(g,1)\chi(1,h)\\&=\chi_{|G}(g)\chi_{|H}(h)\end{aligned}$$

Thus χ must appear as the product of its restrictions $\chi_{|G},\chi_{|H}$, which must be both characters, and this gives the identification in the statement. □

As a conclusion to what we have above, certainly some interesting things, but with the overall situation being not very good, due to the fact that the group dual $\widehat{G}$ depends on the ground field F, and with this preventing many interesting things to happen.

So, good time to temporarily break with our policy so far of using an arbitrary field F. By choosing our ground field to be the smartest one around, namely $F=\mathbb{C}$, and by assuming as well that G is abelian, in view of what we found above, we are led to:

Theorem 1.16. *Given a finite abelian group G, its complex characters*

$$\chi:G\to\mathbb{T}$$

form a finite abelian group $\widehat{G}$, called Pontrjagin dual, and the following happen:

1. *The dual of a cyclic group is the group itself, $\widehat{\mathbb{Z}}_N=\mathbb{Z}_N$.*
2. *The dual of a product is the product of duals, $\widehat{G\times H}=\widehat{G}\times\widehat{H}$.*
3. *Any product of cyclic groups $G=\mathbb{Z}_{N_1}\times\ldots\times\mathbb{Z}_{N_k}$ is self-dual, $G=\widehat{G}$.*

Proof. We already know some of these things from Theorem 1.15, but since we are here simplifying that, the best is to start all over again. We have indeed a finite abelian group, as stated, and regarding the other assertions, their proof goes as follows:

(1) A character $\chi:\mathbb{Z}_N\to\mathbb{T}$ is uniquely determined by its value $z=\chi(g)$ on the standard generator $g\in\mathbb{Z}_N$. But this value must satisfy:

$$z^N=1$$

Thus we must have $z \in \mathbb{Z}_N$, with the cyclic group $\mathbb{Z}_N$ being regarded this time as being the group of N-th roots of unity. Now conversely, any N-th root of unity $z \in \mathbb{Z}_N$ defines a character $\chi : \mathbb{Z}_N \to \mathbb{T}$, by setting, for any $r \in \mathbb{N}$:

$$\chi(g^r) = z^r$$

Thus we have an identification $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$, as claimed.

(2) A character of a product of groups $\chi : G \times H \to \mathbb{T}$ must satisfy:

$$\chi(g,h) = \chi\left[(g,1)(1,h)\right] = \chi(g,1)\chi(1,h)$$

Thus χ must appear as the product of its restrictions $\chi_{|G}, \chi_{|H}$, which must be both characters, and this gives the identification in the statement.

(3) This follows from (1) and (2). Alternatively, any character $\chi : G \to \mathbb{T}$ is uniquely determined by its values $\chi(g_1), \ldots, \chi(g_k)$ on the standard generators of $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_k}$, which must belong to $\mathbb{Z}_{N_1}, \ldots, \mathbb{Z}_{N_k} \subset \mathbb{T}$, and this gives $\widehat{G} = G$, as claimed. □

Let us discuss now the structure result for the finite abelian groups. This is something which is more advanced, requiring good knowledge of group theory, as follows:

THEOREM 1.17. *The finite abelian groups are the following groups,*

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

and these groups are all self-dual, $G = \widehat{G}$.

PROOF. This is something quite tricky, the idea being as follows:

(1) In order to prove our result, assume that G is finite and abelian. For any prime number $p \in \mathbb{N}$, let us define $G_p \subset G$ to be the subset of elements having as order a power of p. Equivalently, this subset $G_p \subset G$ can be defined as follows:

$$G_p = \left\{ g \in G \Big| \exists k \in \mathbb{N}, g^{p^k} = 1 \right\}$$

(2) It is then routine to check, based on definitions, that each G_p is a subgroup. Our claim now is that we have a direct product decomposition as follows:

$$G = \prod_p G_p$$

(3) Indeed, by using the fact that our group G is abelian, we have a morphism as follows, with the order of the factors when computing $\prod_p g_p$ being irrelevant:

$$\prod_p G_p \to G \quad , \quad (g_p) \to \prod_p g_p$$

Moreover, it is routine to check that this morphism is both injective and surjective, via some simple manipulations, so we have our group decomposition, as in (2).

(4) Thus, we are left with proving that each component G_p decomposes as a product of cyclic groups, having as orders powers of p, as follows:

$$G_p = \mathbb{Z}_{p^{r_1}} \times \ldots \times \mathbb{Z}_{p^{r_s}}$$

But this is something that can be checked by recurrence on $|G_p|$, via some routine computations, and so we are led to the conclusions in the statement.

(5) So, this was for the quick story of the present result, structure theorem for the finite abelian groups, and for more on all this, technical details, and some useful generalizations too, we recommend learning this from a solid algebra book, such as Lang [67]. □

Getting back now to the Hopf algebras, we have the following result:

THEOREM 1.18. *If G, H are finite abelian groups, dual to each other via Pontrjagin duality, in the sense that each of them is the character group of the other,*

$$G = \left\{\chi : H \to \mathbb{T}\right\} \quad , \quad H = \left\{\rho : G \to \mathbb{T}\right\}$$

we have an identification of Hopf algebras as follows:

$$F(G) = F[H]$$

In the case $G = H = \mathbb{Z}_N$, this identification is the usual discrete Fourier transform isomorphism. In general, we obtain a tensor product of such Fourier transforms.

PROOF. All this is standard Fourier analysis, the idea being as follows:

(1) In the simplest case, where our groups are $G = H = \mathbb{Z}_N$, we have indeed an identification of algebras as above, which is a Hopf algebra isomorphism, given by the usual discrete Fourier transform isomorphism, whose matrix with respect to the standard basis on each side is the following matrix, with $w = e^{2\pi i/N}$, called Fourier matrix:

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{N-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \ldots & w^{(N-1)^2} \end{pmatrix}$$

(2) In the general case now, we can invoke the stucture theorem for the abelian groups, which tells us that G must appear as a product of cyclic groups, as follows:

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

Indeed, due to the functorial properties of the Pontrjagin duality, we have as well:

$$H = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

Thus, we are led to the conclusions in the statement, with the corresponding isomorphism of Hopf algebras being obtained by tensoring the isomorphisms from (1), and corresponding to the following matrix, called generalized Fourier matrix:

$$F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$$

(3) Alternatively, it is possible to be more direct on all this, and short-circuiting the heavy results, simply by viewing the identification $F(G) = F[H]$ as appearing by complexifying the characters from the definition of the dual group, namely:

$$H = \left\{ \chi : G \to \mathbb{T} \right\} \quad \text{or} \quad G = \left\{ \chi : H \to \mathbb{T} \right\}$$

Indeed, with this approach, which relies only on the definition of the Pontrjagin dual, there is no need for the computations for $\mathbb{Z}_N$, or for the structure theorem for the finite abelian groups. Again, we will leave the details here as an instructive exercise. □

As a comment here, we can feel that Theorem 1.18 is related to Fourier analysis, and this is indeed the case. The point is that we have 3 types of Fourier analysis in life:

(1) We first have the "standard" one, corresponding to $G = \mathbb{R}$, that you probably know well, and which can be learned from any advanced analysis book.

(2) Then we have another one, called the "Fourier series" one, which is also something popular and useful, corresponding to $G = \mathbb{Z}, \mathbb{T}$, that you probably know well too.

(3) And finally we have the "discrete" one, as above, over $G = \mathbb{Z}_N$ and other finite abelian groups. We will be back to this, on several occasions, in this book.

1d. Duality theory

Quite remarkably, the Pontrjagin duality for finite abelian groups can be extended to the general finite group case, in the context of the Hopf algebras. To be more precise, we have the following result, which is something truly remarkable, solving many questions, and which will be our first general theorem on the Hopf algebras:

THEOREM 1.19. *Given a finite dimensional Hopf algebra A, its dual space*

$$A^* = \left\{ \varphi : A \to F \text{ linear} \right\}$$

is also a finite dimensional Hopf algebra, with multiplication and unit as follows,

$$\Delta^t : A^* \otimes A^* \to A^* \quad , \quad \varepsilon^t : F \to A^*$$

and with comultiplication, counit and antipode as follows:

$$m^t : A^* \to A^* \otimes A^* \quad , \quad u^t : A^* \to F \quad , \quad S^t : A^* \to A^*$$

This duality makes correspond the commutative algebras to the cocommutative algebras. Also, this duality makes correspond $F(G)$ to $F[G]$, for any finite group G.

PROOF. At the first glance, we can only expect here something more complicated than for Theorem 1.18, that our result generalizes. However, by the power of abstract algebra, where precise formulations matter a lot, things are in fact quite simple:

(1) To start with, we know that A is a Hopf algebra. Thus, as an associative algebra, A has a multiplication map m, and a unit map u, which are as follows:

$$m : A \otimes A \to A \quad , \quad u : F \to A$$

Also, A has a comultiplication Δ, counit ε and antipode S, which are as follows:

$$\Delta : A \to A \otimes A \quad , \quad \varepsilon : A \to F \quad , \quad S : A \to A^{opp}$$

(2) By taking now the functional transposes of these 5 maps, we obtain 5 other maps, whose domains and ranges are as in the statement. Moreover, it is routine to check that these latter 5 maps are all morphisms of algebras, with this being actually clear for all the maps involved, except for S^t, which requires some thinking at opposite algebras.

(3) Regarding now the axioms, since A is, before anything, an associative algebra, its multiplication and unit maps m, u are subject to the following axioms:

$$m(m \otimes id) = m(id \otimes m)$$

$$m(u \otimes id) = m(id \otimes u) = id$$

We also know that A is a Hopf algebra, so the following are satisfied too:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(4) The point now is that the collection of the above 8 formulae is "self-dual", in the sense that when transposing, we obtain exactly the same 8 formulae. Indeed, the transposes of the first two formulae are as follows:

$$(m^t \otimes id)m^t = (id \otimes m^t)m^t$$

$$(u^t \otimes id)m^t = (id \otimes u^t)m^t = id$$

As for the transposes of the last three formulae, these are as follows:

$$\Delta^t(\Delta^t \otimes id) = \Delta^t(id \otimes \Delta^t)$$

$$\Delta^t(\varepsilon^t \otimes id) = \Delta^t(id \otimes \varepsilon^t) = id$$

$$\Delta^t(S^t \otimes id)m^t = \Delta^t(id \otimes S^t)m^t = u^t(.)1$$

But, we recognize here the full axioms for Hopf algebras, including those for associative algebras. Thus A^*, as constructed in the statement, is indeed a Hopf algebra.

(5) Observe now, as a complement to what is said in the statement, and which is something that is useful to know, in practice, that the operation $A \to A^*$ is indeed a duality, because if we dualize one more time, we obtain A itself:

$$A^{**} = A$$

(6) Regarding the assertion about commutative and cocommutative algebras, this is clear from definitions, because we have the following equivalences:

$$\begin{aligned}
\Sigma m^t = m^t &\iff \Sigma m^t(\varphi) = m^t(\varphi), \forall\varphi \\
&\iff \Sigma m^t(\varphi)(a,b) = m^t(\varphi)(a,b), \forall\varphi, a, b \\
&\iff \varphi(ba) = \varphi(ab), \forall\varphi, a, b \\
&\iff ba = ab, \forall a, b
\end{aligned}$$

Indeed, this computation gives the result in one sense, and in the other sense, this follows either via a similar computation, or just by dualizing, and using (5).

(7) Finally, the last assertion, regarding the group algebras, is clear from definitions too, after a quick comparison with Theorem 1.6 and Theorem 1.13.

(8) Indeed, the point is that we have dual vector spaces, and the Hopf algebra maps from Theorem 1.6 are given by the following formulae:

$$(\varphi\psi)(g) = \varphi(g)\psi(g)$$

$$1 = [g \to 1]$$

$$\Delta(\varphi) = [(g,h) \to \varphi(gh)]$$

$$\varepsilon(\varphi) = \varphi(1)$$

$$S(\varphi) = [g \to \varphi(g^{-1})]$$

(9) As for the Hopf algebra maps from Theorem 1.13, these are as follows:

$$(\varphi * \psi)(g) = \sum_{g=hk} \varphi(h)\psi(k)$$

$$1 = [g \to \delta_{g1}]$$

$$\Delta(\varphi) = [(g,h) \to \delta_{gh}\varphi(g)]$$

$$\varepsilon(\varphi) = \sum_{g\in H} \varphi(g)$$

$$S(\varphi) = [g \to \varphi(g^{-1})]$$

Thus, we have indeed a pair of dual Hopf algebras, as stated.

(10) As a last observation, in the case where the finite group G is abelian, we recover in this way what we know from Theorem 1.18, proved there the hard way. But, as mentioned there, there are in fact several proofs for that result, with going via the heavy theorems for the finite abelian groups being something practical, but not really necessary. □

As a conclusion to this, we can now answer Question 1.11 in the affirmative, by merging Speculation 1.7 and Speculation 1.10 in the finite case, in the following way:

SPECULATION 1.20. *We can think of any finite dimensional Hopf algebra A, not necessarily commutative or cocommutative, as being of the form*

$$A = F(G) = F[H]$$

with G, H being finite quantum groups, related by a generalized Pontrjagin duality. And with this generalizing what we know about the abelian groups.

All this is very nice, and we will leave some further categorical thinking and clarification of all this, if needed, depending on taste, as an exercise.

As a second comment, despite the above being something nice, and quite deep, the criticisms formulated on the occasion of Speculation 1.7 and Speculation 1.10 remain. One problem, as usual, is whether we want to further include the condition $S^2 = id$ in our axioms for the quantum groups. Another problem is that, even when assuming $S^2 = id$, nothing guarantees that a finite dimensional commutative Hopf algebra must be of the form $A = F(G)$. And also, once again when assuming $S^2 = id$, nothing guarantees that a finite dimensional cocommutative Hopf algebra must be of the form $A = F[H]$. And finally, we have the big question regarding the relation of all this, mathematics developed in a few pages, with quantum physics, which is certainly something more complicated. All these are good problems, and we will be back to them, in what follows.

As a third comment, all the above concerns the finite dimensional case, and when trying to do such things in the infinite dimensional setting, which must be topological, as per the usual Pontrjagin duality for infinite groups, there are plenty of difficulties. Again, these are all good questions, and we will be back to them, later in this book.

1e. Exercises

We had a lot of interesting algebra in this chapter, sometimes going towards basic functional analysis, or differential geometry, and as exercises, we have:

EXERCISE 1.21. *Learn everything about tensor products, from a good algebra book.*

EXERCISE 1.22. *Do as well some physics computations with $\otimes$, matter of loving it.*

EXERCISE 1.23. *Clarify all details of the functional transpose operation $f \to f^t$.*

EXERCISE 1.24. *Learn everything about abelian groups, and Pontrjagin duality.*

EXERCISE 1.25. *Learn about the various types of Fourier transforms and series.*

EXERCISE 1.26. *Work out all details for the duality theorem for Hopf algebras.*

As bonus exercise, learn some functional analysis, which is obviously related to all this. The more you will know here, in advance, the better that will be.

CHAPTER 2

Basic theory

2a. The antipode

With the Hopf algebras axiomatized, the basic examples $F(G)$ and $F[H]$ discussed, and the duality theory discussed too, in the finite dimensional case, what is next? Many things, and as a list of pressing topics to be discussed, for this chapter, we have:

(1) We would first like to have a more detailed look at the Hopf algebra axioms, and what can be done with them. And notably, know more about the antipode S.

(2) Next, at the level of the basic examples, we have unfinished business, or rather unstarted business, with the enveloping Lie algebras $U\mathfrak{g}$. On our to-do list, too.

(3) Then, with the knowledge of $F(G)$, $F[H]$, $U\mathfrak{g}$, we can try to emulate, as a continuation of (1), some advanced group technology, inside the arbitrary Hopf algebras A.

(4) And finally, again with some inspiration from $F(G)$, $F[H]$, $U\mathfrak{g}$, which will be our main input here, we can have a discussion about Haar integration.

Which sounds quite good, and with none of the above questions (1,2,3,4) being easy to solve, via some instant thinking, expect lots of new and interesting things, to come.

Getting started, as a first topic for this chapter, let us go back to the arbitrary Hopf algebras, as axiomatized in chapter 1, and have a more detailed look at their antipode S. The definition and basic properties of the antipode can be summarized as follows:

Theorem 2.1. *Given a Hopf algebra A, its antipode is the morphism of algebras*

$$S : A \to A^{opp}$$

A^{opp} being the opposite algebra, with product $a \cdot b = ba$, subject to the following axiom:

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

For $F(G)$ the antipode is the transpose of the inversion map $i : G \to G$. For $F[H]$, the antipode is given by $S(g) = g^{-1}$. In both these cases, the above axiom corresponds to

$$g^{-1}g = gg^{-1} = 1$$

and the extra condition $S^2 = id$, coming from $(g^{-1})^{-1} = g$, is satisfied.

Proof. This is something that we know well from chapter 1, and for full details about all this, along with slightly more about S, we refer to the material there. $\square$

In relation with this, as a first question that you might have, are there Hopf algebras with $S^2\neq id$? Here is a key example, due to Sweedler, which is the simplest one:

Theorem 2.2. *The Sweedler algebra, $A=span(1,c,x,cx)$ with the relations*

$$c^2=1\quad,\quad x^2=0\quad,\quad cx=-xc$$

and with Hopf algebra structure given by

$$\Delta(c)=c\otimes c\quad,\quad\Delta(x)=1\otimes x+x\otimes c\quad,\quad\Delta(cx)=c\otimes cx+cx\otimes 1$$

$$\varepsilon(c)=1\quad,\quad\varepsilon(x)=0\quad,\quad\varepsilon(cx)=0$$

$$S(c)=c\quad,\quad S(x)=cx\quad,\quad S(cx)=-x$$

is not commutative, nor cocommutative, and has the property $S^4=id$, but $S^2\neq id$.

Proof. This is something quite tricky, the idea being as follows:

(1) Consider the 4-dimensional vector space $A=span(1,c,x,cx)$, with $1,c,x,cx$ being some abstract variables. We can make then A into an associative algebra, with unit 1, by declaring that we have indeed $c\cdot x=cx$, and by imposing the following rules:

$$c^2=1\quad,\quad x^2=0\quad,\quad cx=-xc$$

(2) Next, by using the universal property of A, we can define a morphism of algebras $\Delta:A\to A\otimes A$, according to the following formulae, on the algebra generators c,x:

$$\Delta(c)=c\otimes c\quad,\quad\Delta(x)=1\otimes x+x\otimes c$$

Observe that by multiplying we have as well $\Delta(cx)=c\otimes cx+cx\otimes 1$.

(3) Now let us try to prove that A is a Hopf algebra. By using the Hopf algebra axioms, we conclude that ε,S can only be given on c,x by the following formulae:

$$\varepsilon(c)=1\quad,\quad\varepsilon(x)=0$$

$$S(c)=c\quad,\quad S(x)=cx$$

But, and here comes the point, we can define indeed such morphisms, $\varepsilon:A\to F$ and $S:A\to A^{opp}$, via the above formulae, by using the universality property of A. Observe that by multiplying, we obtain as well $\varepsilon(cx)=0$ and $S(cx)=-x$.

(4) Summarizing, we have our Hopf algebra, which is clearly not commutative, not cocommutative either, and whose antipode satisfies $S^4=id$, but $S^2\neq id$. $\square$

Getting back now to the general case, in order to further build on Theorem 2.1, our main source of inspiration will be what happens for $A=F(G)$, where the antipode appears as the functional analytic transpose, $S=i^t$, of the inversion map $i(g)=g^{-1}$.

In view of this, we have the following natural question, which appears:

QUESTION 2.3. *In group theory we have many elementary formulae involving products, units and inverses, all coming from the group axioms, such as*

$$1^{-1}=1 \quad , \quad (g^{-1})^{-1}=g \quad , \quad (gh)^{-1}=h^{-1}g^{-1} \quad , \quad \ldots$$

which can be reformulated in terms of m,u,i, and then transposed, leading to formulae as follows, involving Δ,ε,S, inside the algebras $A=F(G)$:

$$\varepsilon S=\varepsilon \quad , \quad S^2=id \quad , \quad \Delta S=\Sigma(S\otimes S)\Delta \quad , \quad \ldots$$

But, which of these latter formulae hold in general, inside any Hopf algebra A?

And isn't this a good question. Indeed, as we know well from chapter 1, the answer to the above question is definitely "yes" for the group axioms themselves, which are as follows, and which reformulate, by definition, into the Hopf algebra axioms:

$$(gh)k=g(hk) \quad , \quad 1g=g1=g \quad , \quad g^{-1}g=gg^{-1}=1$$

However, in what regards for instance the group theory formula $(g^{-1})^{-1}=g$, this reformulates in Hopf algebra terms as $S^2=id$, and we have seen in Theorem 2.2 that we have counterexamples to this. Thus, Question 2.3 definitely makes sense.

In order to get familiar with this, let us first study $1^{-1}=1$. We have here:

THEOREM 2.4. *We have the following formula, valid over any Hopf algebra,*

$$\varepsilon S=\varepsilon$$

and with this coming from $1^{-1}=1$ for $A=F(G)$, and being trivial for $A=F[H]$.

PROOF. This is something elementary, the idea being as follows:

(1) In order to establish the above formula, we can use the Hopf algebra axiom for S. Indeed, by applying the counit to this axiom, we obtain the following formula:

$$\varepsilon m(S\otimes id)\Delta=\varepsilon m(id\otimes S)\Delta=\varepsilon$$

Let us compute the map on the left. By using the counit axiom, we have:

$$\begin{aligned}\varepsilon m(S\otimes id)\Delta &= (\varepsilon\otimes\varepsilon)(S\otimes id)\Delta\\ &= (\varepsilon S\otimes\varepsilon)\Delta\\ &= \varepsilon S(id\otimes\varepsilon)\Delta\\ &= \varepsilon S\circ id\\ &= \varepsilon S\end{aligned}$$

Similarly, and although not needed here, the map appearing in the middle in the above formula is εS too. Thus, our above equality of maps reads $\varepsilon S=\varepsilon S=\varepsilon$, as desired.

(2) Regarding now the case $A = F(G)$, here our condition is as follows, as claimed:

$$\begin{aligned}\varepsilon S = \varepsilon \quad &\iff \quad \varepsilon S(\varphi) = \varepsilon(\varphi)\\ &\iff \quad \varphi(1^{-1}) = \varphi(1)\\ &\iff \quad 1^{-1} = 1\end{aligned}$$

(3) On the opposite, in the case $A = F[H]$ our condition is trivial, coming from:

$$\begin{aligned}\varepsilon S = \varepsilon \quad &\iff \quad \varepsilon S(g) = \varepsilon(g)\\ &\iff \quad \varepsilon(g^{-1}) = \varepsilon(g)\\ &\iff \quad 1 = 1\end{aligned}$$

(4) Thus, result proved, with all this being quite trivial, but with the remark that what we have at the end, that $F(G)$ vs $F[H]$ dissymmetry, is however quite interesting. □

Next on our list, still coming from Question 2.3, let us have a look at $(gh)^{-1} = h^{-1}g^{-1}$. Things here are more tricky, and as a first result on the subject, we have:

PROPOSITION 2.5. *The antipode of both the algebras $F(G)$ and $F[H]$ satisfies*

$$\Delta S = \Sigma(S \otimes S)\Delta$$

with this coming from $(gh)^{-1} = h^{-1}g^{-1}$ for $A = F(G)$, and being trivial for $A = F[H]$.

PROOF. For $A = F(G)$ the proof goes as follows, with $\sigma(g,h) = (h,g)$:

$$\begin{aligned}(gh)^{-1} = h^{-1}g^{-1} \quad &\iff \quad im(g,h) = m(i \times i)\sigma(g,h)\\ &\iff \quad im = m(i \times i)\sigma\\ &\iff \quad m^t i^t = \sigma^t(i^t \otimes i^t)m^t\\ &\iff \quad \Delta S = \Sigma(S \otimes S)\Delta\end{aligned}$$

As for the algebra $A = F[H]$, here the verification is trivial, as follows:

$$\begin{aligned}\Delta S = \Sigma(S \otimes S)\Delta \quad &\iff \quad \Delta S(g) = \Sigma(S \otimes S)\Delta(g)\\ &\iff \quad \Delta(g^{-1}) = \Sigma(S \otimes S)(g \otimes g)\\ &\iff \quad \Delta(g^{-1}) = \Sigma(g^{-1} \otimes g^{-1})\\ &\iff \quad \Delta(g^{-1}) = g^{-1} \otimes g^{-1}\end{aligned}$$

Observe that, again, we have an interesting $F(G)$ vs $F[H]$ dissymmetry here. □

Before getting further with our study of $(gh)^{-1} = h^{-1}g^{-1}$, that we will eventually show to hold over any Hopf algebra, let us go back to Question 2.3. We have only recorded 3 possible relations there, but there are infinitely many more, and in relation with this, we have the following result, related to $(gh)^{-1} = h^{-1}g^{-1}$, making a bit of cleanup:

PROPOSITION 2.6. *We have the following implication, over any Hopf algebra,*

$$\Delta S=\Sigma(S\otimes S)\Delta\quad\Longrightarrow\quad\Delta^{(2)}S=\Sigma^{(2)}(S\otimes S\otimes S)\Delta^{(2)}$$

corresponding in the case $A=F(G)$ to the following implication, at the group level:

$$(gh)^{-1}=h^{-1}g^{-1}\quad\Longrightarrow\quad(ghk)^{-1}=k^{-1}h^{-1}g^{-1}$$

Moreover, we can iterate this observation, as many times as we want to.

PROOF. We are presently into uncharted territory, so take the above statement as it is, as something a bit informal. The point with all this is that, in the group theory setting, it is quite obvious that by iterating, we have the following series of implications:

$$\begin{aligned}(gh)^{-1}=h^{-1}g^{-1}\quad&\Longrightarrow\quad(ghk)^{-1}=k^{-1}h^{-1}g^{-1}\\&\Longrightarrow\quad(ghkl)^{-1}=l^{-1}k^{-1}h^{-1}g^{-1}\\&\Longrightarrow\quad\ldots\end{aligned}$$

And the question is that, can we have these implications going, I mean these implications only, not the validity of the formulae themselves, in the Hopf algebra setting. With the answer to this latter question being definitely yes, for the first implication, with the computation being as follows, using twice the condition $\Delta S=\Sigma(S\otimes S)\Delta$:

$$\begin{aligned}\Delta^{(2)}S&=(\Delta\otimes id)\Delta S\\&=(\Delta\otimes id)\Sigma(S\otimes S)\Delta\\&=\Sigma_{\leftarrow}(id\otimes\Delta)(S\otimes S)\Delta\\&=\Sigma_{\leftarrow}(S\otimes\Delta S)\Delta\\&=\Sigma_{\leftarrow}(S\otimes\Sigma(S\otimes S)\Delta)\Delta\\&=\Sigma_{\leftarrow}(id\otimes\Sigma)(S\otimes S\otimes S)(id\otimes\Delta)\Delta\\&=\Sigma^{(2)}(S\otimes S\otimes S)\Delta^{(2)}\end{aligned}$$

Thus, result proved, and I will leave it to you to figure out what the various versions of Σ,Δ used above exactly mean. In what regards the last assertion, exercise as well. □

Summarizing, things are quite tricky. In order to further discuss this, we will need some abstract algebraic preliminaries. Let us start with something standard, namely:

THEOREM 2.7. *If we define the convolution of linear maps $\varphi,\psi:A\to A$ by*

$$\varphi*\psi=m(\varphi\otimes\psi)\Delta$$

then the Hopf algebra axiom for the antipode reads

$$S*id=id*S=\varepsilon(.)1$$

with the map on the right, $\varepsilon(.)1$, being the unit for the operation $*$.

PROOF. This is something which comes from the axioms, as follows:

(1) The first assertion is clear from the Hopf algebra axiom for the antipode, as formulated in chapter 1, or in Theorem 2.1, which was as follows:

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

Indeed, in terms of the convolution operation from the statement, $\varphi*\psi=m(\varphi\otimes\psi)\Delta$, this axiom takes the following more conceptual form, as indicated above:

$$S*id=id*S=\varepsilon(.)1$$

(2) Regarding the second assertion, this follows from the counit axiom, namely:

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

Indeed, given a linear map $\varphi:A\to A$, we have the following computation:

$$\begin{aligned}\varphi*\varepsilon(.)1&=m(\varphi\otimes\varepsilon(.)1)\Delta\\&=\varphi(id\otimes\varepsilon)\Delta\\&=\varphi\circ id\\&=\varphi\end{aligned}$$

Similarly, again for any linear map $\varphi:A\to A$, we have the following computation:

$$\begin{aligned}\varepsilon(.)1*\varphi&=m(\varepsilon(.)1\otimes\varphi)\Delta\\&=\varphi(\varepsilon\otimes id)\Delta\\&=\varphi\circ id\\&=\varphi\end{aligned}$$

Thus the linear map $\varepsilon(.)1$ is indeed the unit for the operation $*$, as claimed. □

In order to do our next antipode computations, which will be sometimes quite tough, we will need as well the following useful convention, due to Sweedler:

DEFINITION 2.8. *We use the Sweedler notation for the comultiplication Δ,*

$$\Delta(x)=\sum x_1\otimes x_2$$

with the sum on the right being understood to correspond to the tensor expansion of $\Delta(x)$.

And in the hope that this will sound quite nice and clever to you, when seeing it for the first time. As illustrations for this Sweedler notation, or rather as a first piece of advertisement for it, let us have a look at the Hopf algebra axioms, namely:

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

We know these axioms since the beginning of chapter 1, and we certainly have some knowledge in dealing with them. However, the point is that by using the Sweedler notation above, these axioms take the following form, which is even more digest:

$$\sum \Delta(x_1)\otimes x_2=\sum x_1\otimes\Delta(x_2)$$

$$\sum \varepsilon(x_1)x_2=\sum x_1\varepsilon(x_2)=x$$

$$\sum S(x_1)x_2=\sum x_1S(x_2)=\varepsilon(x)$$

With this discussed, let us go back now to Theorem 2.7, which is our main theoretical result, so far. With a bit more work, we can further improve this result, as follows:

Theorem 2.9. *Given an algebra A, with morphisms of algebras*

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to F$$

satisfying the usual axioms for a comultiplication and an antipode, namely

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

this is a Hopf algebra precisely when $id:A\to A$ is invertible with respect to

$$\varphi*\psi=m(\varphi\otimes\psi)\Delta$$

and in this case, the convolution inverse $S=id^{-1}$ is the antipode.

Proof. This follows indeed from what we have in Theorem 2.7, and from a few extra computations, best done by using the Sweedler notation, the idea being as follows:

(1) Assume first that A is a Hopf algebra. According to Theorem 2.7 we have indeed $S=id^{-1}$, inverse with respect to convolution, so the only thing that we have to prove is that this inverse is unique. But this is something purely algebraic, which is valid under very general circumstances, because for any associative multiplication $\cdot$ we have:

$$ab=ba=1,\ ac=ca=1\quad\Longrightarrow\quad b=bac=c$$

(2) Conversely now, assume that the identity map $id:A\to A$ is invertible with respect to the convolution operation $*$, with inverse $S=id^{-1}$. As explained in Theorem 2.7, the condition $S=id^{-1}$ tells us that S satisfies the usual antipode axiom, namely:

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

However, we are not done yet, because our map $S:A\to A$ is just a linear map, that we still have to prove to be a morphism, when regarded as map $S:A\to A^{opp}$. Thus, with $S=id^{-1}$ regarded as it comes, as linear map $S:A\to A$, we must prove that we have:

$$S(ab)=S(b)S(a)$$

(3) In order to prove this formula, consider the following three linear maps:

$$m(a\otimes b)=ab\quad,\quad p(a\otimes b)=S(ab)\quad,\quad q(a\otimes b)=S(b)S(a)$$

We have then the following computation, involving the maps m,p:

$$\begin{aligned}(p*m)(a\otimes b)&=\sum p\left((a\otimes b)_1\right)m\left((a\otimes b)_2\right)\\&=\sum p(a_1\otimes b_1)m(a_2\otimes b_2)\\&=\sum S(a_1b_1)a_2b_2\\&=\sum S\left((ab)_1\right)(ab)_2\\&=(S*1)(ab)\\&=\varepsilon(ab)\\&=\varepsilon(a)\varepsilon(b)\\&=(\varepsilon\otimes\varepsilon)(a\otimes b)\end{aligned}$$

On the other hand, we have as well the following computation, involving m,q:

$$\begin{aligned}(m*q)(a\otimes b)&=\sum m(a_1\otimes b_1)q(a_2\otimes b_2)\\&=\sum a_1b_1S(b_2)S(a_2)\\&=\sum a_1\varepsilon(b)S(a_2)\\&=(1*S)(a)\cdot\varepsilon(b)\\&=\varepsilon(a)\varepsilon(b)\\&=(\varepsilon\otimes\varepsilon)(a\otimes b)\end{aligned}$$

Summarizing, we have proved that we have the following formulae:

$$p*m=m*q=(\varepsilon\otimes\varepsilon)(.)1\otimes 1$$

(4) But with this, we can finish our proof, in the following way:

$$\begin{aligned}p&=p*((\varepsilon\otimes\varepsilon)(.)1\otimes 1)\\&=p*(m*q)\\&=(p*m)*q\\&=((\varepsilon\otimes\varepsilon)(.)1\otimes 1)*q\\&=q\end{aligned}$$

Thus we have $p=q$, which means $S(ab)=S(b)S(a)$, as desired. □

With all these preliminaries discussed, time now for our first true theorem. We can indeed formulate something non-trivial regarding the antipode, as follows:

THEOREM 2.10. *The antipode of a Hopf algebra $S:A\to A^{opp}$ satisfies:*

(1) $\varepsilon S=\varepsilon$.
(2) $\Delta S=\Sigma(S\otimes S)\Delta$.
(3) $S^2=id$, *when A is commutative or cocommutative.*

PROOF. This is something quite tricky, the idea being as follows:

(1) This is something that we already know, from Theorem 2.4.

(2) We have, by using the Sweedler notation from Definition 2.8:

$$\begin{aligned}(\Delta S*\Delta)(x) &= \sum\Delta S(x_1)\Delta(x_2)\\ &= \Delta\left(\sum S(x_1)x_2\right)\\ &= \Delta(\varepsilon(x)1)\\ &= \varepsilon(x)\cdot 1\otimes 1\end{aligned}$$

On the other hand, we have as well, by using the Sweedler notation, iterated:

$$\begin{aligned}(\Delta*\Sigma(S\otimes S)\Delta)(x) &= \sum(x_1\otimes x_2)(S(x_4)\otimes S(x_3))\\ &= \sum x_1S(x_4)\otimes x_2S(x_3)\\ &= \sum x_1S(x_3)\otimes\varepsilon(x_2)1\\ &= \sum x_1S(x_3)\varepsilon(x_2)\otimes 1\\ &= \sum x_1S(x_2)\otimes 1\\ &= \varepsilon(x)\cdot 1\otimes 1\end{aligned}$$

As a conclusion to this, we have proved the following equalities:

$$\Delta S*\Delta=\Delta*\Sigma(S\otimes S)\Delta=\varepsilon(.)1\otimes 1$$

Now by using Theorem 2.7, we obtain from this, as desired:

$$\begin{aligned}\Delta S &= \Delta S*(\varepsilon(.)1\otimes 1)\\ &= \Delta S*(\Delta*\Sigma(S\otimes S)\Delta)\\ &= (\Delta S*\Delta)*\Sigma(S\otimes S)\Delta\\ &= (\varepsilon(.)1\otimes 1)*\Sigma(S\otimes S)\Delta\\ &= \Sigma(S\otimes S)\Delta\end{aligned}$$

(3) Our first claim is that when A is commutative or cocommutative, we have:

$$\sum S(x_2)x_1=\varepsilon(x)$$

Indeed, in the commutative case this follows from the Hopf algebra axiom for S, in Sweedler notation, which reads, as explained before:

$$\sum x_1 S(x_2) = \varepsilon(x)$$

As for the cocommutative case, here we can use again the axiom for S, namely:

$$m(S \otimes id)\Delta(x) = \varepsilon(x)1$$

Indeed, by replacing $\Delta(x)$ with $\Sigma\Delta(x)$, for flipping the tensors, we obtain the formula claimed in the above. Thus, claim proved, and with this in hand, we have:

$$\begin{aligned} m(S \otimes S^2)\Delta(x) &= m(S \otimes S^2)\left(\sum x_1 \otimes x_2\right) \\ &= \sum S(x_1)S^2(x_2) \\ &= S\left(\sum S(x_2)x_1\right) \\ &= S(\varepsilon(x)1) \\ &= \varepsilon(x)1 \end{aligned}$$

Now by using Theorem 2.9, we obtain from this $S^2 = id$, as claimed. □

As a first comment on the above result, (2) there can be used in conjunction with Proposition 2.6, and shows that much more is true. However, Question 2.3 as formulated is still there, and knowing what group theory type relations hold, and what don't, inside an arbitrary Hopf algebra, remains something which requires some experience and skill.

Many other things can be said, as a continuation of the above, notably with some general theory for the square of the antipode $S^2 : A \to A$, which is more specialized, again based on the general interpretation of the antipode coming from Theorem 2.9. For more on all this, we refer to the specialized Hopf algebra literature.

2b. Lie algebras

As a second task for this chapter, again coming as a continuation of what we did in chapter 1, let us try now to better understand what happens, at the fine level, beyond the finite dimensional case, where we already have some good examples and results.

As already mentioned in chapter 1, when trying to cover various infinite groups, such as the compact, discrete, or more generally locally compact ones, the standard trick is that of modifying a bit the Hopf algebra axioms, by using topological tensor products. But this is something quite technical, and we will discuss this later in this book.

For the moment, let us show that we can cover the compact Lie groups, without changing our axioms, by using a Lie algebra trick. Our claim is as follows:

CLAIM 2.11. *Given a compact Lie group G, with Lie algebra $\mathfrak{g}$, the corresponding enveloping Lie algebra $U\mathfrak{g}$ is a Hopf algebra, which is cocommutative.*

Obviously, many non-trivial notions involved here, with this being not your routine abstract algebra statement, that you can understand right away, armed with Love for algebra only. So, we will explain in what follows the various notions involved, namely the Lie groups G, the Lie algebras $\mathfrak{g}$, and the enveloping Lie algebras $U\mathfrak{g}$, and then we will get of course to what the above claim says, and even provide a proof for it.

Getting started now, a Lie group is by definition a group which is a smooth manifold. So, let us start our discussion with this, smooth manifolds. Here is their definition:

DEFINITION 2.12. *A smooth manifold is a space X which is locally isomorphic to $\mathbb{R}^N$. To be more precise, this space X must be covered by charts, bijectively mapping open pieces of it to open pieces of $\mathbb{R}^N$, with the changes of charts being C^∞ functions.*

As basic examples of smooth manifolds, we have of course $\mathbb{R}^N$ itself, or any open subset $X \subset \mathbb{R}^N$, with only 1 chart being needed here. Other basic examples, in the plane, at $N = 2$, include the circle, or various curves like ellipses and so on, somehow for obvious reasons. Here is a more precise statement in this sense, covering the conics:

PROPOSITION 2.13. *The following are smooth manifolds, in the plane:*

1. *The circles.*
2. *The ellipses.*
3. *The non-degenerate conics.*
4. *Smooth deformations of these.*

PROOF. All this is quite intuitive, the idea being as follows:

(1) Consider the unit circle, $x^2 + y^2 = 1$. We can write then $x = \cos t$, $y = \sin t$, with $t \in [0, 2\pi)$, and we seem to have here the solution to our problem, just using 1 chart. But this is of course wrong, because $[0, 2\pi)$ is not open, and we have a problem at 0. In practice we need to use 2 such charts, say with the first one being with $t \in (0, 3\pi/2)$, and the second one being with $t \in (\pi, 5\pi/2)$. As for the fact that the change of charts is indeed smooth, this comes by writing down the formulae, or just thinking a bit, and arguing that this change of chart being actually a translation, it is automatically linear.

(2) This follows from (1), by pulling the circle in both the Ox and Oy directions, and the formulae here, based on the standard formulae for ellipses, are left to you reader.

(3) We already have the ellipses, and the case of the parabolas and hyperbolas is elementary as well, and in fact simpler than the case of the ellipses. Indeed, a parablola is clearly homeomorphic to $\mathbb{R}$, and a hyperbola, to two copies of $\mathbb{R}$.

(4) This is something which is clear too, depending of course on what exactly we mean by "smooth deformation", and by using a bit of multivariable calculus if needed. □

In higher dimensions we have as basic examples the spheres, and I will leave it to you to find a proof, using spherical coordinates, or the stereographic projection. Exercise as well, to find higher dimensional analogues of the other assertions in Proposition 2.13.

Getting now to what we wanted to do here, Lie groups, let us start with:

DEFINITION 2.14. *A Lie group is a group G which is a smooth manifold, with the corresponding multiplication and inverse maps*

$$m : G \times G \to G \quad , \quad i : G \to G$$

being assumed to be smooth. The tangent space at the origin $1 \in G$ is denoted

$$\mathfrak{g} = T_1 G$$

and is called Lie algebra of G.

So, this is our definition, and as a first observation, the examples of Lie groups abound, with the circle $\mathbb{T}$ and with the higher dimensional tori $\mathbb{T}^N$ being the standard examples. For these, the Lie algebra is obviously equal to $\mathbb{R}$ and $\mathbb{R}^N$, respectively. There are of course many other examples, all very interesting, and more on this in a moment.

Before getting into examples, let us discuss a basic question, that you surely have in mind, namely why calling the tangent space $\mathfrak{g} = T_1 G$ an algebra. In answer, since G is a group, with a certain multiplication map $m : G \times G \to G$, we can normally expect this map m to produce some sort of "algebra structure" on the tangent space $\mathfrak{g} = T_1 G$.

This was for the general idea, but in practice, things are more complicated than this, because even for very simple examples of Lie groups, what we get in this way is not an associative algebra, but rather a new type of beast, called Lie algebra.

So, coming as a continuation and complement to Definition 2.14, we have:

DEFINITION 2.15. *A Lie algebra is a vector space $\mathfrak{g}$ with an operation $(x, y) \to [x, y]$, called Lie bracket, subject to the following conditions:*

1. $[x + y, z] = [x, z] + [y, z]$, $[x, y + z] = [x, y] + [x, z]$.
2. $[\lambda x, y] = [x, \lambda y] = \lambda [x, y]$.
3. $[x, x] = 0$.
4. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

As a basic example, consider a usual, associative algebra A. We can define then the Lie bracket on it as being the usual commutator, namely:

$$[x, y] = xy - yx$$

The above axioms (1,2,3) are then clearly satisfied, and in what regards axiom (4), called Jacobi identity, this is satisfied too, the verification being as follows:

$$\begin{aligned}
& [[x,y],z]+[[y,z],x]+[[z,x],y] \\
= & [xy-yx,z]+[yz-zy,x]+[zx-xz,y] \\
= & xyz-yxz-zxy+zyx+yzx-zyx-xyz+xzy+zxy-xzy-yzx+yxz \\
= & 0
\end{aligned}$$

We will see in a moment that up to a certain abstract operation $\mathfrak{g} \to U\mathfrak{g}$, called enveloping Lie algebra construction, and which is something quite elementary, any Lie algebra appears in this way, with its Lie bracket being formally given by:

$$[x,y]=xy-yx$$

Before that, however, you might wonder where that Gothic letter $\mathfrak{g}$ in Definition 2.15 comes from. That comes from the following fundamental result, making the connection with the theory of Lie groups from Definition 2.14, denoted as usual by G:

Theorem 2.16. *Given a Lie group G, that is, a group which is a smooth manifold, with the group operations being smooth, the tangent space at the identity*

$$\mathfrak{g}=T_1(G)$$

is a Lie algebra, with its Lie bracket being basically a usual commutator.

Proof. This is something non-trivial, the idea being as follows:

(1) Let us first have a look at the orthogonal and unitary groups O_N, N_N. These are both Lie groups, and the corresponding Lie algebras $\mathfrak{o}_N, \mathfrak{u}_N$ can be computed by differentiating the equations defining O_N, U_N, with the conclusion being as follows:

$$\mathfrak{o}_N=\left\{A\in M_N(\mathbb{R})\Big|A^t=-A\right\}$$

$$\mathfrak{u}_N=\left\{B\in M_N(\mathbb{C})\Big|B^*=-B\right\}$$

This was for the correspondences $O_N\to\mathfrak{o}_N$ and $U_N\to\mathfrak{u}_N$. In the other sense, the correspondences $\mathfrak{o}_N\to O_N$ and $\mathfrak{u}_N\to U_N$ appear by exponentiation, the result here stating that, around 1, the orthogonal matrices can be written as $U=e^A$, with $A\in\mathfrak{o}_N$, and the unitary matrices can be written as $U=e^B$, with $B\in\mathfrak{u}_N$.

(2) Getting now to the Lie bracket, the first observation is that both $\mathfrak{o}_N, \mathfrak{u}_N$ are stable under the usual commutator of the $N\times N$ matrices. Indeed, assuming that $A,B\in M_N(\mathbb{R})$

satisfy $A^t = -A$, $B^t = -B$, their commutator satisfies $[A,B] \in M_N(\mathbb{R})$, and:

$$\begin{aligned}[A,B]^t &= (AB - BA)^t \\ &= B^t A^t - A^t B^t \\ &= BA - AB \\ &= -[A,B]\end{aligned}$$

Similarly, assuming that $A, B \in M_N(\mathbb{C})$ satisfy $A^* = -A$, $B^* = -B$, their commutator $[A,B] \in M_N(\mathbb{C})$ satisfies the condition $[A,B]^* = -[A,B]$.

(3) We conclude from this discussion that both the tangent spaces $\mathfrak{o}_N, \mathfrak{u}_N$ are Lie algebras, with the Lie bracket being the usual commutator of the $N \times N$ matrices.

(4) It remains now to understand how the Lie bracket $[A,B] = AB - BA$ is related to the group commutator $[U,V] = UVU^{-1}V^{-1}$ via the exponentiation map $U = e^A$, and this can be indeed done, by making use of the differential geometry of O_N, U_N, and the situation is quite similar when dealing with an arbitrary Lie group G. □

With this understood, let us go back to the arbitrary Lie algebras, as axiomatized in Definition 2.15. There is an obvious analogy there with the axioms for the usual, associative algebras, and based on this analogy, we can build some abstract algebra theory for the Lie algebras. Let us record some basic results, along these lines:

Proposition 2.17. *Let $\mathfrak{g}$ be a Lie algebra. If we define its ideals as being the vector spaces $\mathfrak{i} \subset \mathfrak{g}$ satisfying the condition*

$$x \in \mathfrak{i}, y \in \mathfrak{g} \implies [x,y] \in \mathfrak{i}$$

then the quotients $\mathfrak{g}/\mathfrak{i}$ are Lie algebras. Also, given a morphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{h}$, its kernel $ker(f) \subset \mathfrak{g}$ is an ideal, and we have $\mathfrak{g}/\ker(f) = Im(f)$.

Proof. All this is very standard, exactly as in the case of the associative algebras, and we will leave the various verifications here as an instructive exercise. □

Getting now to the point, remember our claim from the discussion after Definition 2.15, stating that up to a certain abstract operation $\mathfrak{g} \to U\mathfrak{g}$, called enveloping Lie algebra construction, any Lie algebra appears in fact from the "trivial" associative algebra construction, that is, with its Lie bracket being formally a usual commutator:

$$[x,y] = xy - yx$$

Time now to clarify this. The result here, making as well to the link with the various Lie group considerations from Theorem 2.16 and its proof, is as follows:

THEOREM 2.18. *Given a Lie algebra $\mathfrak{g}$, define its enveloping Lie algebra $U\mathfrak{g}$ as being the quotient of the tensor algebra of $\mathfrak{g}$, namely*

$$T(\mathfrak{g})=\bigoplus_{k=0}^\infty\mathfrak{g}^{\otimes k}$$

by the following associative algebra ideal, with x,y ranging over the elements of $\mathfrak{g}$:

$$I=<x\otimes y-y\otimes x-[x,y]>$$

Then $U\mathfrak{g}$ is an associative algebra, so it is a Lie algebra too, with bracket

$$[x,y]=xy-yx$$

and the standard embedding $\mathfrak{g}\subset U\mathfrak{g}$ is a Lie algebra embedding.

PROOF. This is something which is quite self-explanatory, and in what regards the examples, illustrations, and other things that can be said, for instance in relation with the Lie groups, we will leave some further reading here as an instructive exercise. □

Importantly, the above enveloping Lie algebra construction makes the link with our Hopf algebra considerations, from the present book, via the following result:

THEOREM 2.19. *Given a Lie algebra $\mathfrak{g}$, its enveloping Lie algebra $U\mathfrak{g}$ is a cocommutative Hopf algebra, with comultiplication, counit and antipode given by*

$$\Delta:U\mathfrak{g}\to U(\mathfrak{g}\oplus\mathfrak{g})=U\mathfrak{g}\otimes U\mathfrak{g}\quad,\quad x\to x+x$$

$$\varepsilon:U\mathfrak{g}\to F\quad,\quad x\to 1$$

$$S:U\mathfrak{g}\to U\mathfrak{g}^{opp}=(U\mathfrak{g})^{opp}\quad,\quad x\to -x$$

via various standard identifications, for the various associative algebras involved.

PROOF. Again, this is something quite self-explanatory, and in what regards the examples, illustrations, and other things that can be said, for instance in relation with the Lie groups, we will leave some further reading here as an instructive exercise. □

We will be back to this, and to Lie algebras in general, on several occasions, in what follows. Among others, we will see later in this book how to reconstruct the Lie group G from the knowledge of the enveloping Lie algebra $U\mathfrak{g}$, using representation theory.

2c. Special elements

In view of the above results regarding the enveloping Lie algebras $U\mathfrak{g}$, which are cocommutative, and of the results from chapter 1 too, regarding the group algebras $F[H]$, which are cocommutative too, it makes sense to have a more systematic look at the Hopf algebras A which are cocommutative, in our usual sense, namely:

$$\Sigma\Delta=\Delta$$

We already know a bit about such algebras, in the finite dimensional case, and as a complement to that material, we first have the following result:

THEOREM 2.20. *Let A be a Hopf algebra. The elements satisfying the condition*

$$\Delta(a) = a \otimes a$$

which are called group-like, have the following properties:

(1) *They form a group G_A.*
(2) *They satisfy $\Sigma\Delta(a) = \Delta(a)$.*
(3) *We have a Hopf algebra embedding $F[G_A] \subset A$.*
(4) *For a group algebra $A = F[H]$, this embedding is an isomorphism.*

PROOF. This is something elementary, the idea being as follows:

(1) Let us call indeed group-like the elements $a \in A$ satisfying $\Delta(a) = a \otimes a$, in analogy with the formula $\Delta(g) = g \otimes g$ when $A = F[H]$, and more on this in a moment. The group-like elements are then stable under the product operation, as shown by:

$$\Delta(ab) = \Delta(a)\Delta(b) = (a \otimes a)(b \otimes b) = ab \otimes ab$$

We have as well the stability under taking inverses, with this coming from:

$$\Delta(a^{-1}) = \Delta(a)^{-1} = (a \otimes a)^{-1} = a^{-1} \otimes a^{-1}$$

Finally, the formula $\Delta(1) = 1 \otimes 1$ shows that $1 \in A$ is group-like. Thus, the set of group-like elements $G_A \subset A$ is indeed a multiplicative subgroup, as claimed.

(2) Assuming that $a \in A$ is group-like, we have indeed, as claimed:

$$\Sigma\Delta(a) = \Sigma(a \otimes a) = a \otimes a = \Delta(a)$$

Observe that the converse of what we just proved here does not hold, for instance because in the case of the group algebras $A = F[H]$, which are cocommutative, $\Sigma\Delta = \Delta$, there are many elements $a \in A$ which are not group-like. More on this in a moment.

(3) There are several checks here, all being trivial or routine, the only point being that of proving that the group-like elements are linearly independent. So, let us prove this. Assume that we have a linear combination of group-like elements, as follows:

$$a = \sum_i \lambda_i a_i$$

By applying Δ to this element we obtain, by using the condition $a \in G_A$:

$$\Delta(a) = a \otimes a = \sum_{ij} \lambda_i \lambda_j a_i \otimes a_j$$

On the other hand, also by applying Δ, but by using $a_i \in G_A$, we obtain:

$$\Delta(a) = \sum_i \lambda_i \Delta(a_i) = \sum_i \lambda_i a_i \otimes a_i$$

We conclude that the following equation must be satisfied, when $a \in G_A$:

$$\sum_{ij} \lambda_i \lambda_j a_i \otimes a_j = \sum_i \lambda_i a_i \otimes a_i$$

But this equation shows that we must have $\#\{i\} = \#\{j\} = 1$, as desired. That is, we have proved that the group-like elements are linearly independent, and this gives a Hopf algebra embedding $F[G_A] \subset A$ as in the statement, appearing in the obvious way.

(4) This is indeed clear from (3), because in the case of a group algebra $A = F[H]$ we have $G_A = H$, with $G_A \supset H$ being clear, and with $G_A \subset H$ coming from linear independence. Thus, in this case, the embedding $F[G_A] \subset A$ is an isomorphism. □

Many other things can be said about the group-like elements, and we will leave their study in the case algebras $A = F(G)$, and of the algebras $A = \mathfrak{g}$ too, as an instructive exercise. Moving on, here is another key construction, this time Lie algebra-inspired:

Theorem 2.21. *Let A be a Hopf algebra. The elements satisfying the condition*

$$\Delta(a) = a \otimes 1 + 1 \otimes a$$

which are called primitive, have the following properties:

1. *They form a Lie algebra P_A, with bracket $[a, b] = ab - ba$.*
2. *They automatically satisfy $\Sigma\Delta(a) = \Delta(a)$.*
3. *We have a Hopf algebra embedding $UP_A \subset A$.*
4. *For an enveloping Lie algebra $A = U\mathfrak{g}$, this embedding is an isomorphism.*

Proof. Observe the similarity with Theorem 2.20, and more on this later. Regarding now the proof of the various assertions, this is straightforward, as follows:

(1) There are several things to be checked here, all being trivial or routine, the only point being that of proving that the primitive elements are stable under taking commutators. So, let us prove this. Assuming $a, b \in P_A$, we have the following computation:

$$\begin{aligned}
\Delta([a,b]) &= \Delta a \cdot \Delta b - \Delta b \cdot \Delta a \\
&= (a \otimes 1 + 1 \otimes a)(b \otimes 1 + 1 \otimes b) - (b \otimes 1 + 1 \otimes b)(a \otimes 1 + 1 \otimes a) \\
&= ab \otimes 1 - ba \otimes 1 + 1 \otimes ab - 1 \otimes ba \\
&= [a,b] \otimes 1 + 1 \otimes [a,b]
\end{aligned}$$

Thus we have $[a, b] \in P_A$, as desired, and this gives the assertion.

(2) Assuming that $a \in A$ is primitive, we have indeed, as claimed:

$$\begin{aligned}
\Sigma\Delta(a) &= \Sigma(a \otimes 1 + 1 \otimes a) \\
&= 1 \otimes a + a \otimes 1 \\
&= a \otimes 1 + 1 \otimes a \\
&= \Delta(a)
\end{aligned}$$

Observe that the converse of what we just proved does not hold, for instance because in the case of the envelopoing Lie algebras $A = U\mathfrak{g}$, which are cocommutative, $\Sigma\Delta = \Delta$, there are many elements $a \in A$ which are not primitive. More on this in a moment.

(3) This indeed something quite routine, a bit as before, for the group-likes.

(4) This follows indeed from (3), again a bit as before, for the group-likes. □

Many other things can be said about the primitive elements, and we will leave their study in the case of the function algebras $A = F(G)$, and of the group algebras $A = F[H]$ too, which is something quite routine, as an instructive exercise.

Along the same lines, as a third and last construction now, motivated this time by the function algebras $A = F(G)$, which are commutative, we have:

THEOREM 2.22. *Given a Hopf algebra A, we can talk about its center*

$$Z(A) \subset A$$

which is an associative subalgebra, having the following properties:

1. *For $A = F(G)$, and more generally when A is commutative, $Z(A) = A$.*
2. *For $A = F[H]$, this algebra $Z(A)$ is the algebra of central functions on H.*
3. *In particular, when all conjugacy classes of H are infinite, $Z(A) = F$.*

PROOF. This is something quite self-explanatory, and a bit in analogy with Theorem 2.20, and Theorem 2.21. Consider indeed the central elements of a Hopf algebra A:

$$Z(A) = \left\{ a \in A \Big| ab = ba, \forall b \in A \right\}$$

It is then clear that these central elements form an associative subalgebra, and:

(1) For $A = F(G)$, and more generally when A is commutative, $Z(A) = A$.

(2) For $A = F[H]$, consider a linear combination of group elements, as follows:

$$a = \sum_g \lambda_g g$$

By linearity, this element $a \in F[H]$ belongs to the center of $F[H]$ precisely when it commutes with all the group elements $h \in H$, and this gives:

$$\begin{aligned} a \in Z(A) &\iff ah = ha \\ &\iff \sum_g \lambda_g gh = \sum_g \lambda_g hg \\ &\iff \sum_k \lambda_{kh^{-1}} k = \sum_k \lambda_{h^{-1}k} k \\ &\iff \lambda_{kh^{-1}} = \lambda_{h^{-1}k} \end{aligned}$$

We conclude that λ must satisfy $\lambda_{gh} = \lambda_{hg}$, and so must be a central function on H, as claimed, with the precise conclusion being that the center is given by:

$$Z(A) = \left\{ \sum_g \lambda_g g \Big| \lambda_{gh} = \lambda_{hg} \right\}$$

(3) This is a consequence of what we found in (2), and of the fact that the elements $a \in F[H]$ have by definition finite support. Indeed, when the group H is infinite, having the infinite conjugacy class (ICC) property, there is no central function having finite support, except for scalar multiples of the unit, so we have $Z(A) = F$, as stated. $\square$

Many other things can be said about the central elements, and we will leave their study for the enveloping Lie algebras $A = U\mathfrak{g}$ as an instructive exercise.

So long for special elements, inside an arbitrary Hopf algebra. The above results are in fact just the tip of the iceberg, and we will be back to this on several occasions, in what follows, and notably in chapter 3 below, when doing representation theory.

Finally, for our discussion to be complete, many things can be said about the group-like, primitive and central elements, in relation with the various possible operations for the Hopf algebras. But here, again, we will leave all this material for later.

2d. Haar measure

As a last topic for this chapter, which is something which is of key importance too, let us discuss now Haar integration. Let us formulate, indeed:

Definition 2.23. *Given a Hopf algebra A, a linear form $\int : A \to F$ satisfying*

$$\left(\int \otimes id \right) \Delta = \int (.)1$$

is called a left integral. Similarly, a linear form $\int : A \to F$ satisfying

$$\left(id \otimes \int \right) \Delta = \int (.)1$$

is called a right integral. If both conditions are satisfied, we call $\int : A \to F$ an integral.

These notions are motivated by the Haar integration theory on the various types of groups, such as finite, compact, or locally compact. Among others, and in answer to a question that you might have right now, we have to make the distinction between left and right integrals, because for the generally locally compact groups, these two integrals might differ. But more on such topics, which can be quite technical, later.

As a first illustration, in the case of the function algebras, $A = F(G)$, with G finite group, these notions are all equivalent, and lead to the uniform integration over G:

THEOREM 2.24. *For a function algebra, $A = F(G)$ with G finite group, with the notation $\int \varphi = \int_G \varphi(g)dg$, the left integral condition takes the following form,*

$$\int_G \varphi(gh)dg = \int_G \varphi(g)dg$$

and the right integral condition takes the following form,

$$\int_G \varphi(hg)dg = \int_G \varphi(g)dg$$

and in both cases the unique solution is the uniform integration over G,

$$\int_G \varphi(g)dg = \frac{1}{|G|}\sum_{g\in G}\varphi(g)$$

under the normalization assumption $\int 1 = 1$.

PROOF. This is something quite self-explanatory, the idea being as follows:

(1) With the convention $\int \varphi = \int_g \varphi(g)dg$ from the statement, we have:

$$\begin{aligned}\left(\int \otimes id\right)\Delta\varphi &= \left(\int \otimes id\right)[(g,h)\to\varphi(gh)]\\ &= \int_G \varphi(gh)dg\end{aligned}$$

Thus, the left integral condition reformulates as in the statement.

(2) Again with the convention $\int \varphi = \int_g \varphi(g)dg$ from the statement, we have:

$$\begin{aligned}\left(id\otimes\int\right)\Delta &= \left(id\otimes\int\right)\Delta\,[(h,g)\to\varphi(hg)]\\ &= \int_G \varphi(hg)dg\end{aligned}$$

Thus, the right integral condition reformulates as in the statement.

(3) When looking now for solutions, be that either for left invariant forms, or right invariant forms, by taking as input Dirac masses, $\varphi = \delta_g$ with $g \in G$, we are led to the conclusion that our invariant linear form must satisfy the following condition:

$$\int_G \delta_g = \int_G \delta_h \quad , \quad \forall g,h$$

Thus the correspoding density function must be constant over G, and under the extra assumption $\int 1 = 1$, the only solution is the uniform, mass 1 integration, as stated. □

As a second illustration, for the group algebras, $A = F[H]$, with H arbitrary group, the notions in Definition 2.23 are again equivalent, with unique solution, as follows:

THEOREM 2.25. *For a group algebra, $A = F[H]$, with H arbitrary group, the various invariance notions for a linear form $\int : A \to F$ are equivalent, with the solution being*

$$\int g = \delta_{g,1}$$

under the normalization assumption $\int 1 = 1$. When H is finite and abelian we have

$$\int = \int_{\widehat{H}}$$

with $\widehat{H}$ being the dual finite abelian group.

PROOF. This is again something quite self-explanatory, the idea being as follows:

(1) In what regards the left invariance condition, we have the following computation, using the fact that the group elements $g \in H$ span the group algebra $F[H]$:

$$\begin{aligned}
\left(\int \otimes id\right)\Delta = \int (.)1 \quad &\iff \quad \left(\int \otimes id\right)\Delta(g) = \int g \cdot 1\\
&\iff \quad \left(\int \otimes id\right)(g \otimes g) = \int g \cdot 1\\
&\iff \quad \int g \cdot g = \int g \cdot 1\\
&\iff \quad \left[g \neq 1 \implies \int g = 0\right]
\end{aligned}$$

Thus with the normalization $\int 1 = 1$, the solution is unique, $\int g = \delta_{g,1}$, as stated.

(2) In what regards the right invariance condition, we have the following computation, using again the fact that the group elements $g \in H$ span the group algebra $F[H]$:

$$\begin{aligned}
\left(id \otimes \int\right)\Delta = \int (.)1 \quad &\iff \quad \left(id \otimes \int\right)\Delta(g) = \int g \cdot 1\\
&\iff \quad \left(id \otimes \int\right)(g \otimes g) = \int g \cdot 1\\
&\iff \quad \int g \cdot g = \int g \cdot 1\\
&\iff \quad \left[g \neq 1 \implies \int g = 0\right]
\end{aligned}$$

Thus, with the normalization $\int 1 = 1$, the solution is unique, $\int g = \delta_{g,1}$, as stated.

(3) This is something which follows from the uniqueness of the integral, both from Theorem 2.24 and from here, and which is clear as well from definitions. ☐

Inspired by the above, a number of things can be said about integrals in the finite dimensional algebra case, by using the duality $A\leftrightarrow A^*$ from chapter 1, as follows:

THEOREM 2.26. *For a finite dimensional Hopf algebra A, in the context of the duality $A\leftrightarrow A^*$, the left and right integrals, regarded as elements $\int^t\in A^*$, must satisfy*

$$\int^t\cdot(\varphi-\varepsilon(\varphi))=0\quad,\quad(\varphi-\varepsilon(\varphi))\cdot\int^t=0$$

for any $\varphi\in A^*$. At the level of the main examples, of mass 1, these are as follows:

(1) *For $A=F(G)$ the integral is $\int^t=\frac{1}{|G|}\sum_{g\in G}g$, as element of $A^*=F[G]$.*
(2) *For $A=F[H]$ the integral is $\int^t=\delta_1$, as element of $A^*=F(H)$.*

PROOF. In what regards the left integrals, we have the following equivalences:

$$\begin{aligned}\left(\int\otimes id\right)\Delta=\int(.)1\quad&\iff\quad\left(\int\otimes id\right)\Delta=\int\otimes u\\&\iff\quad\Delta^t\left(\int^t\otimes id\right)=\int^t\otimes u^t\\&\iff\quad\Delta^t\left(\int^t\otimes id\right)\varphi=\left(\int^t\otimes u^t\right)\varphi\\&\iff\quad\int^t\cdot\varphi=\int^t\cdot\varepsilon(\varphi)\end{aligned}$$

Similarly, in what regards the right integrals, we have the following equivalences:

$$\begin{aligned}\left(id\otimes\int\right)\Delta=\int(.)1\quad&\iff\quad\left(id\otimes\int\right)\Delta=\int\otimes u\\&\iff\quad\Delta^t\left(id\otimes\int^t\right)=\int^t\otimes u^t\\&\iff\quad\Delta^t\left(id\otimes\int^t\right)\varphi=\left(\int^t\otimes u^t\right)\varphi\\&\iff\quad\varphi\cdot\int^t=\int^t\cdot\varepsilon(\varphi)\end{aligned}$$

Thus, main assertion proved, and in what regards now the illustrations:

(1) For $A=F(G)$ we have $A^*=F[G]$, with the isomorphism $F[G]\simeq F(G)^*$ coming via $g\to\delta_g$. But this isomorphism maps $\frac{1}{|G|}\sum_{g\in G}g\to\frac{1}{|G|}\sum_{g\in G}\delta_g$, which is exactly the normalized, mass 1 integral of $F(G)$, as computed in Theorem 2.24.

(2) Similarly, for the algebra $A=F[H]$ we have $A^*=F(H)$, with the isomorphism $F(H)\simeq F[H]^*$ coming via $\delta_g\to\delta_g$. But this isomorphism maps $\delta_1\to\delta_1$, which is exactly the normalized, mass 1 integral of $F[H]$, as computed in Theorem 2.25. □

As a comment here, with a bit more algebraic work, there are many other things that can be said, in the finite dimensional case, as a continuation of the above. For more on all this, further theory and examples, we refer to any specialized Hopf algebra book.

In the general case now, observe that the invariance conditions in Definition 2.23 can be written as follows, in terms of the usual convolution operation $\varphi * \psi = (\varphi * \psi)\Delta$:

$$\int * id = id * \int = \int (.)1$$

There is a bit of analogy here with what we did in the beginning of this chapter, in relation with the antipode S, and many things can be said here. We have indeed:

THEOREM 2.27. *Both the left and right integrals $\int : A \to F$, when normalized as to have $\int 1 = 1$, satisfy the following idempotent linear form condition:*

$$\int * \int = \int$$

At the level of the main examples, for this latter condition, these are as follows:

(1) *For $A = F(G)$ this condition is satisfied in fact by the normalized uniform integration form over any subgroup $H \subset G$.*
(2) *For $A = F[H]$ this condition is satisfied in fact by the normalized uniform integration form over any quotient group $H \to K$.*

PROOF. We have several asssertions here, the idea being as follows:

(1) In what regards the first assertion, for a left integral $\int : A \to F$, normalized as to have $\int 1 = 1$, we have indeed the following computation:

$$\begin{aligned}
\int * \int &= \left(\int \otimes \int\right) \Delta \\
&= \int \circ \left[\left(\int \otimes id\right) \Delta\right] \\
&= \int \circ \left[\int (.)1\right] \\
&= \int (.) \int (1) \\
&= \int (.)
\end{aligned}$$

(2) Also in what regards the first assertion, for a right integral $\int : A \to F$, again normalized as to have $\int 1 = 1$, the computation is similar, as follows:

$$\begin{aligned}\int * \int &= \left(\int \otimes \int\right)\Delta \\ &= \int \circ \left[\left(id \otimes \int\right)\Delta\right] \\ &= \int \circ \left[\int (.)1\right] \\ &= \int (.) \int (1) \\ &= \int (.)\end{aligned}$$

(3) Finally, regarding the various generalizations of the above computations, in the special cases $A = F(G)$ and $A = F[H]$, as indicated in the statement, we will leave these as an instructive exercise. We will be back to this in the next chapter, when talking about quantum subgroups in general, and subgroups of group duals in particular. □

Summarizing, the theory of integrals for the Hopf algebras brings us right away into a number of interesting topics, featuring duality, subgroups, quotients, and more. We will be back to this later, and discuss as well later the relation with representation theory.

2e. Exercises

We had a lot of interesting algebra in this chapter, sometimes going towards basic functional analysis, or differential geometry, and as exercises, we have:

Exercise 2.28. *Learn more about the Hopf algebra antipode S.*

Exercise 2.29. *Learn more about the square of the antipode S^2.*

Exercise 2.30. *Clarify the missing details for the group-like elements.*

Exercise 2.31. *Clarify the missing details for the primitive elements.*

Exercise 2.32. *Work out some further examples for the central elements.*

Exercise 2.33. *Compute the Haar integral, for some algebras of your choice.*

As bonus exercise, reiterated, learn some functional analysis, which is obviously related to all this. The more you will know here, in advance, the better that will be.

CHAPTER 3

Product operations

3a. Representations

We have seen so far that some interesting general theory can be developed for the Hopf algebras, in analogy with the basic theory of groups, by using the Hopf algebra maps Δ, ε, S, and the axioms satisfied by them. However, when doing group theory, you won't get very far just by playing with m, u, i, and the situation is pretty much the same with the Hopf algebras, where you won't get very far just by playing with Δ, ε, S.

In order to reach to a more advanced theory, we must talk about actions and coactions, and about representations and corepresentations. Many things can be said here, and in what follows we will present the basics, mostly definitions, that we will use right after for talking about various product operations, and keep for later a more detailed study of this, notably in relation with the notion of semisimplicity, and cosemisimplicity.

Let us begin with something straightforward, namely:

Definition 3.1. *An action, or representation, of a Hopf algebra A on a finite dimensional vector space V is a morphism of associative algebras, as follows:*

$$\pi : A \to \mathcal{L}(V)$$

Equivalently, by using a basis of V, this is the same as having a morphism as follows:

$$\pi : A \to M_N(F)$$

In this latter situation, we write $\pi = (\pi_{ij})$, with $\pi_{ij} : A \to F$ given by $\pi_{ij}(a) = \pi(a)_{ij}$.

Observe that the above notion has nothing to do with the Hopf algebra maps Δ, ε, S, with only the associative algebra structure of A being involved. However, when A is a Hopf algebra, as above, several interesting things can be said, as we will soon discover.

To start with, in the context of Definition 3.1, the number $N = \dim V$ is called dimension of the representation π. The simplest situation, namely $N = 1$, corresponds by definition to a representation as follows, also called character of A:

$$\pi : A \to F$$

So, let us first study these characters, under our assumption that A is a Hopf algebra, as in Definition 3.1. We can say several things here, as follows:

THEOREM 3.2. *The characters of a Hopf algebra $\pi : A \to F$ are as follows:*

(1) *When A is finite dimensional, $\pi \in A^*$ must be a group-like element.*
(2) *When $A = F(G)$ with $|G| < \infty$, we must have $\pi(f) = f(g)$, for some $g \in G$.*
(3) *When $A = F[H]$, our character must come from a group morphism $\rho : H \to F^*$.*

PROOF. This follows from the general Hopf algebra theory that we developed in chapter 1, the details of the proof, and of the statement too, being as follows:

(1) Assuming $\dim A < \infty$, we know from chapter 1 how to construct the dual Hopf algebra A^*, consisting of the linear forms $\pi : A \to F$. So, let us pick such a linear form, and see when this form is a character. But this happens precisely when π is multiplicative, $\pi(ab) = \pi(a)\pi(b)$, and we can process this latter condition as follows:

$$\begin{aligned}
\pi(ab) = \pi(a)\pi(b) &\iff \pi m(a \otimes b) = m(\pi \otimes \pi)(a \otimes b) \\
&\iff \pi m = m(\pi \otimes \pi) \\
&\iff m^t \pi^t = (\pi^t \otimes \pi^t) m^t \\
&\iff m^t \pi^t(1) = (\pi^t \otimes \pi^t) m^t(1) \\
&\iff m^t \pi^t(1) = (\pi^t \otimes \pi^t)(1 \otimes 1) \\
&\iff m^t \pi^t(1) = \pi^t(1) \otimes \pi^t(1)
\end{aligned}$$

Now forgetting about A, and using the notation $\Delta = m^t$ for the comultiplication of A^*, and also by identifying $\pi^t(1) \in A^*$ with $\pi \in A^*$, this condition reads:

$$\Delta(\pi) = \pi \otimes \pi$$

Thus, we are led to the conclusion in the statement.

(2) Assume now $A = F(G)$, with $|G| < \infty$. Our Hopf algebra A being finite dimensional, what we found in (1) above applies, and we conclude that the characters of A correspond to the group-like elements of the following Hopf algebra:

$$F(G)^* = F[G]$$

But the group-like elements of $F[G]$ are very easy to compute, due to:

$$\Delta\left(\sum_{g \in G} \lambda_g \, g\right) = \sum_{g \in G} \lambda_g \, g \otimes g$$

Indeed, this formula shows that the group-like elements of $F[G]$ are precisely the group elements $g \in G$. Thus, as a conclusion, the characters $\pi : A \to F$ must come from the group elements $g \in G$, and now by carefully looking at what we did in the above, we can also say that the connecting formula is the one in the statement, namely:

$$\pi(f) = f(g)$$

(3) Again, this is something which comes from the general theory from chapter 1. Indeed, assuming $A = F[H]$, to any character $\pi : A \to F$ we can associate a group morphism $\rho : H \to F^*$, simply as being the following composition:

$$\rho : H \subset F[H] \to F$$

Conversely now, given a group morphism $\rho : H \to F^*$, we can associate to it a Hopf algebra character $\pi : F[H] \to F$, simply by linearizing, as follows:

$$\pi\left(\sum_{h\in H}\lambda_h\, h\right) = \sum_{h\in H}\lambda_h\,\rho(h)$$

Thus, we have our bijection, as claimed in the statement. □

Many other things can be said, as a continuation of the above. Recall for instance from chapter 1 that to any finite abelian group G we can associate its dual $\widehat{G}$ with respect to a field F, as being the finite abelian group formed by the group characters of G:

$$\widehat{G} = \left\{\rho : G \to F^*\right\}$$

In fact, again following chapter 1, we can perform this construction for any group G, not necessarily finite, or abelian, and we obtain in this way a certain group $\widehat{G}$. Of course, this construction is not always very interesting, for instance because there are non-trivial, and even infinite groups G, for which $\widehat{G} = \{1\}$. However, our construction makes sense, as something rather theoretical, and with this in hand, what we found in Theorem 3.2 (3) says that the characters of $A = F[H]$ come from the following group elements:

$$\rho \in \widehat{H}$$

Moving forward now, in the general context of Definition 3.1, we have so far some good understanding on what happens in the case $\dim V = 1$, coming from Theorem 3.2. In general things can be quite complicated, and as a first result here, regarding the main examples of Hopf algebras, namely $F(G)$, with G finite group, and $F[H]$, with H arbitrary group, we can say a few things about representations, as follows:

Theorem 3.3. *The following happen:*

(1) *Given a finite group G, and elements $g_1, \ldots, g_N \in G$, we have a representation $\pi : F(G) \to M_N(F)$, given by $\pi(f) = diag(f(g_1), \ldots, f(g_N))$.*
(2) *Given a group H, the representations $\pi : F[H] \to M_N(F)$ come by linearization from the group representations $\rho : H \to GL_N(F)$.*

Proof. These assertions come as a continuation of Theorem 3.2 (2) and (3), with their proof, along with a bit more on the subject, being as follows:

(1) Given a finite group G, and a field F, let us try to find representations, in the sense of Definition 3.1, of the corresponding function algebra of G:

$$\pi : F(G) \to M_N(F)$$

Now observe that, since the algebra $F(G)$ is commutative, so must be its image $Im(\pi) \subset M_N(F)$. Thus, as a first question, we must look for commutative subalgebras $A \subset M_N(F)$. But the standard choice here is the algebra of diagonal matrices $\Delta \subset M_N(F)$, and its various subalgebras $A \subset \Delta$, which are all commutative.

(2) With this idea in mind, in order to find basic examples, let us look for representations as follows, with $\Delta \subset M_N(F)$ being the algebra of diagonal matrices:

$$\pi : F(G) \to \Delta$$

But such a representation must be of the special form $\pi = diag(\pi_1, \ldots, \pi_N)$, with $\pi_i : F(G) \to F$ being certain 1-dimensional representations, or characters. Now since such characters must come from group elements, we must have $\pi_i(f) = f(g_i)$, for certain elements $g_i \in G$, and we are led to the formula in the statement, namely:

$$\pi(f) = \begin{pmatrix} f(g_1) & & \\ & \ddots & \\ & & f(g_N) \end{pmatrix}$$

(3) Summarizing, we have proved the result for $F(G)$, along with a bit more, namely the fact that any representation of type $\pi : F(G) \to \Delta \subset M_N(F)$ appears as in the statement. It is possible to say more about this, for instance by spinning our representations with the help of a matrix $U \in GL_N(F)$, but no hurry with this, and we will leave this material for later, when systematically doing representation theory.

(4) With this done, let us discuss now the second situation in the statement, where we have a group H, which can be finite or not, and a field F, and we are looking for representations of the corresponding group algebra, as follows:

$$\pi : F[H] \to M_N(F)$$

By restriction to $H \subset F[H]$, we obtain a certain map, as follows:

$$\rho : H \to M_N(F)$$

(5) Now observe that, since π is a morphism of algebras, this map ρ must be multiplicative with respect to the group structure of H, in the sense that we must have:

$$\rho(g)\rho(h) = \rho(gh)$$

In particular with $h = g^{-1}$ we can see that each $\rho(g)$ must be invertible, and so our map ρ must be in fact a group morphism, as follows:

$$\rho : H \to GL_N(F)$$

(6) But this gives the result. Indeed, by linearity, the representation π is uniquely determined by ρ, and conversely, given a group morphism $\rho : H \to GL_N(F)$ as above, by linearizing we obtain an algebra representation $\pi : F[H] \to M_N(F)$, as desired.

(7) Thus, done with the second assertion too, and as before with the first assertion, several things can be added to this. For instance when H is finite and abelian, we know from chapter 1 that we have an isomorphism as follows, with $G = \widehat{H}$:

$$F(G) \simeq F[H]$$

In view of this, the question is, how do the examples of representations of $F(G)$ constructed in (1) fit with the arbitrary representations of $F[H]$ classified in (2).

(8) In answer, the representations classified in (2) correspond, via Pontrjagin duality, to those constructed in (1), further spinned by a matrix $U \in GL_N(F)$, along the lines suggested in step (3) above. We will leave the clarification of this as an instructive exercise, and we will come back to this subject, with full details, in due time. □

Many other things can be said, as a continuation of the above. We will be back to this, once we will have more general theory to be applied, and examples to be studied.

3b. Corepresentations

Moving forward, in order to reach to a continuation of the above, let us recall that a Hopf algebra $A = (A, m, u, \Delta, \varepsilon, S)$ is a special type of bialgebra $A = (A, m, u, \Delta, \varepsilon)$, which itself is a certain mix of an algebra (A, m, u), and a coalgebra (A, Δ, ε). We have not talked about such things so far, but right now is the good time to do it.

Indeed, with Definition 3.1 being something only related to the algebra structure of A, the question is, what is the "dual" definition, related to the coalgebra structure of A. In answer, such a dual definition exists indeed, as follows:

DEFINITION 3.4. *A coaction, or corepresentation, of a Hopf algebra A on a finite dimensional vector space V is a linear map $\alpha : V \to V \otimes A$ satisfying the condition*

$$(\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha$$

called coassociativity. Equivalently, by using a basis of V, and writing

$$\alpha(e_i) = \sum_j e_j \otimes u_{ji}$$

with $u_{ij} \in A$, the square matrix $u = (u_{ij})$ must satisfy the condition

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

also called coassociativity.

As a first observation, it looks like we forgot here to say something in relation with ε, but that condition is automatic, and more on this in a moment. Also, the fact that the condition at the end is indeed equivalent to that in the beginning is something that must be checked, and this can be done by comparing the following two formulae:

$$(id\otimes\Delta)\alpha(e_j)=(id\otimes\Delta)\sum_i e_i\otimes u_{ij}=\sum_i e_i\otimes\Delta(u_{ij})$$

$$(\alpha\otimes id)\alpha(e_j)=(\alpha\otimes id)\sum_k e_k\otimes u_{kj}=\sum_{ik} e_i\otimes u_{ik}\otimes u_{kj}$$

Getting back to Definition 3.4 as formulated, as already said, this appears as a "dual" of Definition 3.1, and similar comments can be made about it. Let us start with:

PROPOSITION 3.5. *The 1-dimensional corepresentations of A correspond to the elements $a\in A$ satisfying*

$$\Delta(a)=a\otimes a$$

which are the group-like elements of A.

PROOF. This is indeed clear from definitions, because at $N=1$ the corepesentations of A are the 1×1 matrices $u=(a)$, satisfying $\Delta(a)=a\otimes a$. □

Observe the similarity with what we know from Theorem 3.2 (1). However, at the level of the proofs, Theorem 3.2 (1) was something rather complicated, while the above is something trivial. Quite surprising all this, hope you agree with me. In short, we have something interesting here, philosophically speaking, suggesting that Definition 3.4 is something quite magical, when compared to Definition 3.1. Good to know, and because of this, we will be often prefer Definition 3.4 over Definition 3.1, in what follows.

As another comment, Definition 3.4 only involves the comultiplication Δ, and you might wonder about the role of the counit ε and the antipode S, in relation with corepresentations. In answer, at $N=1$ the situation is very simple, because, as we know well from chapter 2, the group-like elements $a\in A$ are subject to the following formulae:

$$\varepsilon(a)=1\quad,\quad S(a)=a^{-1}$$

A similar phenomenon happens in general, with the result here, which can be regarded as being a useful complement to Definition 3.4, being as follows:

THEOREM 3.6. *Given a corepresentation $u\in M_N(A)$, we have:*

$$(id\otimes\varepsilon)u=1\quad,\quad(id\otimes S)u=u^{-1}$$

Also, the associated coaction $\alpha:F^N\to F^N\otimes A$ is counital, $(id\otimes\varepsilon)\alpha=id$.

PROOF. There are several things going on here, the idea being as follows:

(1) Let us first prove the second formula, the one involving the antipode S. For this purpose, we can use the Hopf algebra axiom for the antipode, namely:

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

Indeed, by applying this to u_{ij}, and setting $v=(id\otimes S)u$, we have, as desired:

$$\begin{aligned}
&& m(S\otimes id)\Delta(u_{ij})=m(id\otimes S)\Delta(u_{ij})=\varepsilon(u_{ij})\\
&\implies& \sum_k S(u_{ik})u_{kj}=\sum_k u_{ik}S(u_{kj})=\delta_{ij}\\
&\implies& \sum_k v_{ik}u_{kj}=\sum_k u_{ik}v_{kj}=\delta_{ij}\\
&\implies& (vu)_{ij}=(uv)_{ij}=\delta_{ij}\\
&\implies& v=u^{-1}
\end{aligned}$$

(2) Let us prove now the first formula, the one involving the counit ε. For this purpose, we can use the Hopf algebra axiom for the counit, namely:

$$(\varepsilon\otimes id)\Delta=(id\otimes\varepsilon)\Delta=id$$

Indeed, by applying this to u_{ij}, and setting $E=(id\otimes\varepsilon)u$, we have:

$$\begin{aligned}
&& (\varepsilon\otimes id)\Delta(u_{ij})=(id\otimes\varepsilon)\Delta(u_{ij})=u_{ij}\\
&\implies& \sum_k \varepsilon(u_{ik})u_{kj}=\sum_k u_{ik}\varepsilon(u_{kj})=u_{ij}\\
&\implies& \sum_k E_{ik}u_{kj}=\sum_k u_{ik}E_{kj}=u_{ij}\\
&\implies& (Eu)_{ij}=(uE)_{ij}=u_{ij}\\
&\implies& Eu=uE=u
\end{aligned}$$

Now since u is invertible by (1), we obtain from this, as desired:

$$E=1$$

(3) Regarding now the last assertion, our claim here is that, in the general context of Definition 3.4, the following two counitality conditions are equivalent:

$$(id\otimes\varepsilon)\alpha=id\iff(id\otimes\varepsilon)u=1$$

But this is something which is clear, coming from the following computation:

$$(id\otimes\varepsilon)\alpha(e_i)=(id\otimes\varepsilon)\sum_j e_j\otimes u_{ji}=\sum_j e_j\varepsilon(u_{ji})$$

Indeed, we obtain from this the following equivalences:

$$\begin{aligned}(id\otimes\varepsilon)\alpha &\iff (id\otimes\varepsilon)\alpha(e_i)=e_i\\ &\iff \varepsilon(u_{ji})=\delta_{ij}\\ &\iff (id\otimes\varepsilon)u=1\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Still talking generalities, in the finite dimensional case we have the following result, making it clear that Definition 3.1 and Definition 3.4 are indeed dual to each other:

THEOREM 3.7. *Given a finite dimensional Hopf algebra A:*

(1) *The representations of A correspond to the corepresentations of A^*.*
(2) *The corepresentations of A^* correspond to the representations of A.*

PROOF. In view of the duality result from chapter 1, it is enough to prove one of the assertions, and we will prove the first one. So, consider a linear map, as follows:

$$\pi:A\to M_N(F)$$

As in Definition 3.1, let us construct the coefficients $\pi_{ij}:A\to F$ of this map by the following formula, which must hold for any $a\in A$:

$$\pi_{ij}(a)=\pi(a)_{ij}$$

Now observe that each of these coefficients $\pi_{ij}:A\to F$ can be regarded as an element of the dual algebra A^*. As in Definition 3.4, we denote by u_{ij} these elements:

$$\pi_{ij}=u_{ij}\in A^*$$

With these conventions made, we must prove that π is a representation of A precisely when u is a corepresentation of A^*. But this can be done as follows:

(1) Our first claim is that π is associative precisely when u is coassociative. But this is something straightforward, which can be established as follows:

$$\begin{aligned}\pi(ab)=\pi(a)\pi(b) &\iff \pi_{ij}(ab)=\sum_k\pi_{ik}(a)\pi_{kj}(b)\\ &\iff u_{ij}(ab)=\sum_k u_{ik}(a)u_{kj}(b)\\ &\iff \Delta(u_{ij})(a\otimes b)=\left(\sum_k u_{ik}\otimes u_{kj}\right)(a\otimes b)\\ &\iff \Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\end{aligned}$$

(2) Our second claim is that π is unital precisely when u is counital. But this is again something straightforward, which can be established as follows:

$$\begin{aligned}\pi(1)=1 &\iff \pi_{ij}(1)=\delta_{ij}\\ &\iff u_{ij}(1)=\delta_{ij}\\ &\iff \varepsilon(u_{ij})=\delta_{ij}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Finally, at the level of basic examples, we have the following result, which is in analogy with what we know about representations, from Theorem 3.3:

Theorem 3.8. *The following happen:*

1. *Given a finite group G, the corepresentations $u\in M_N(F(G))$ come, via $u_{ij}(g)=\rho(g)_{ij}$ from the group representations $\rho:G\to GL_N(F)$.*
2. *Given an arbitrary group H, and elements $g_1,\ldots,g_N\in H$, we have a corepresentation $u\in M_N(F[H])$, given by $u=diag(g_1,\ldots,g_N)$.*

Proof. As before with Theorem 3.3, many things can be said here, and we will come back to this, on the several occasions, the idea for now being as follows:

(1) To start with, in the case of the finite groups, which produce finite dimensional Hopf algebras, the result formally follows from Theorem 3.3, via the duality from Theorem 3.7. Thus, done with (1), and with (2) being trivial anyway, done.

(2) However, all this is a bit abstract, so let us check as well (1) directly. Given a finite group G, the question is when $u_{ij}\in F(G)$ satisfy the following conditions:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad \varepsilon(u_{ij})=\delta_{ij}$$

But, by doing exactly the same computations as those in the proof of Theorem 3.7, the answer here is that this happens precisely when the following is a representation:

$$g\to\begin{pmatrix}u_{11}(g)&\ldots&u_{1N}(g)\\ \vdots&&\vdots\\ u_{N1}(g)&\ldots&u_{NN}(g)\end{pmatrix}$$

Thus, we have a bijection with the representations $\rho:G\to GL_N(F)$, as stated.

(3) Now let us look at the second assertion in the statement. Given a group H, we know from Proposition 3.5 that all the group elements $g\in H\in F[H]$ are 1-dimensional corepresentations. Now let us perform a diagonal sum of such corepresentations:

$$u=\begin{pmatrix}g_1&&\\&\ddots&\\&&g_N\end{pmatrix}$$

This matrix is then a corepresentation of $F[H]$, as stated, with both the coassociativity and counitaly axioms being clear. And with the remark that we actually proved a bit more, namely that the diagonal corepresentations of $F[H]$ are those in the statement.

(4) Finally, as before in the proof of Theorem 3.3, some further things can be said here, by conjugating such diagonal corepresentations with matrices $U \in GL_N(F)$, and by making the link with the first assertion, in the abelian group case. But, also as before, on several occasions, we will rather leave this for now as an instructive exercise, and come back to all this later, when systematically doing representation theory. □

As a conclusion to all this, we have now a nice representation and corepresentation theory, up and working, for the Hopf algebras, which is related in a nice way to the duality considerations from chapter 1, and with the main examples, which are all quite illustrating, coming from the group algebras of type $F(G)$ and $F[H]$. However, this remains just the tip of the iceberg, and as questions still to be solved, we have:

(1) Fully clarify the classification of the representations and corepresentations of the group algebras of type $F(G)$ and $F[H]$, in the missing cases.

(2) Clarify as well what happens to the duality between representations of A and corepresentations of A^*, when A is no longer finite dimensional.

(3) And finally, discuss what happens for the enveloping Lie algebras $U\mathfrak{g}$, in relation with the representations of the associated Lie groups G.

These questions are all fundamental, but none being trivial, we will leave them for later, when discussing more systematically representation theory.

3c. Product operations

Let us discuss now some natural operations on the Hopf algebras, inspired by those for the groups. We will heavily rely here on the following speculation, from chapter 1:

SPECULATION 3.9. *We can think of any finite dimensional Hopf algebra A, not necessarily commutative or cocommutative, as being of the form*

$$A = F(G) = F[H]$$

with G, H being finite quantum groups, related by a generalized Pontrjagin duality. And with this generalizing what we know about the finite abelian groups.

As explained in chapter 1, this speculation is here for what it is worth, on one hand encapsulating some non-trivial results regarding the finite abelian groups, and the finite dimensional Hopf algebras, and the duality theory for them, but on the other hand, missing some important aspects of the same theory of finite dimensional Hopf algebras. Thus, interesting speculation that we have here, but to be taken with care.

In relation with various product operations, what we would like to have are product operations for the Hopf algebras $\star$, subject to formulae of the following type:

$$F(G)\star F(H)=F(G\circ H)$$

Equivalently, at the dual level, what we would like to have are product operations for the Hopf algebras $\star$, subject to formulae of the following type:

$$F[G]\star F[H]=F(G\bullet H)$$

But probably too much talking, let us get to work. We first have the tensor products of Hopf algebras, whose construction and main properties are as follows:

THEOREM 3.10. *Given two Hopf algebras A,B, so is their tensor product*

$$C=A\otimes B$$

and as main illustrations for this operation, we have the following formulae:

(1) $F(G\times H)=F(G)\otimes F(H)$.
(2) $F[G\times H]=F[G]\otimes F[H]$.

PROOF. This is something quite self-explanatory, relying on the general theory of the tensor products $\otimes$ explained in chapter 1, the details being as follows:

(1) To start with, given two associative algebras A,B, so is their tensor product as vector spaces $A\otimes B$, with multiplicative structure as follows:

$$(a\otimes b)(a'\otimes b')=aa'\otimes bb'\quad,\quad 1=1\otimes 1$$

Now assume in addition that A,B are Hopf algebras, each coming with its own Δ,ε,S operations. In this case we can define Δ,ε,S operations on $A\otimes B$, as follows:

$$\Delta(a\otimes b)=\Delta(a)_{13}\Delta(b)_{24}$$

$$\varepsilon(a\otimes b)=\varepsilon(a)\varepsilon(b)$$

$$S(a\otimes b)=S(a)\otimes S(b)$$

(2) But with the above formulae in hand, the verification of the Hopf algebra axioms is straightforward. Indeed, in what regards the comultiplication axiom, we have:

$$\begin{aligned}(\Delta\otimes id)\Delta(a\otimes b)&=(\Delta\otimes id\otimes id)(\Delta(a)_{13}\Delta(b)_{24})\\&=[(\Delta\otimes id)\Delta(a)]_{135}[(\Delta\otimes id)\Delta(b)]_{246}\\&=[(id\otimes\Delta)\Delta(a)]_{135}[(id\otimes\Delta)\Delta(b)]_{246}\\&=(id\otimes id\otimes\Delta)(\Delta(a)_{13}\Delta(b)_{24})\\&=(id\otimes\Delta)\Delta(a\otimes b)\end{aligned}$$

As for the counit and antipode axioms, their verification is similar, with no tricks of any kind involved. To be more precise, in what regards the counit axiom, we have:

$$\begin{aligned}(\varepsilon\otimes id)\Delta(a\otimes b) &= (\varepsilon\otimes\varepsilon\otimes id\otimes id)(\Delta(a)_{13}\Delta(b)_{24})\\ &= [(\varepsilon\otimes id)\Delta(a)]_1[(\varepsilon\otimes id)\Delta(b)]_2\\ &= a_1b_2\\ &= a\otimes b\end{aligned}$$

Similarly, we have the following computation, for the other counit axiom:

$$\begin{aligned}(id\otimes\varepsilon)\Delta(a\otimes b) &= (id\otimes id\otimes\varepsilon\otimes\varepsilon)(\Delta(a)_{13}\Delta(b)_{24})\\ &= [(id\otimes\varepsilon)\Delta(a)]_1[(id\otimes\varepsilon)\Delta(b)]_2\\ &= a_1b_2\\ &= a\otimes b\end{aligned}$$

Finally, for the antipode axiom, we have the following computation:

$$\begin{aligned}m(S\otimes id)\Delta(a\otimes b) &= (m_{13}m_{24})(S\otimes S\otimes id\otimes id)(\Delta(a)_{13}\Delta(b)_{24})\\ &= (m(S\otimes id)\Delta(a))_1(m(S\otimes id)\Delta(b))_2\\ &= (\varepsilon(a)1)_1(\varepsilon(b)1)_2\\ &= \varepsilon(a)1\otimes\varepsilon(b)1\\ &= \varepsilon(a\otimes b)1\otimes 1\end{aligned}$$

Similarly, we have the following computation, for the other antipode axiom:

$$\begin{aligned}m(id\otimes S)\Delta(a\otimes b) &= (m_{13}m_{24})(id\otimes id\otimes S\otimes S)(\Delta(a)_{13}\Delta(b)_{24})\\ &= (m(id\otimes S)\Delta(a))_1(m(id\otimes S)\Delta(b))_2\\ &= (\varepsilon(a)1)_1(\varepsilon(b)1)_2\\ &= \varepsilon(a)1\otimes\varepsilon(b)1\\ &= \varepsilon(a\otimes b)1\otimes 1\end{aligned}$$

We conclude that $C=A\otimes B$ is indeed a Hopf algebra, as stated.

(3) In what regards now the formula $F(G\times H)=F(G)\otimes F(H)$, when G,H are finite groups, as well as the formula $F[G\times H]=F[G]\otimes F[H]$, when G,H are arbitrary groups, these are both clear from the definition of the tensor product operation. □

As a continuation of the above, in what regards the special elements, we have:

PROPOSITION 3.11. *The special elements of $A\otimes B$ are as follows:*

(1) $G_{A\otimes B}$ *contains* $G_A\times G_B$.
(2) $P_{A\otimes B}$ *contains* P_A,P_B.
(3) $Z(A\otimes B)=Z(A)\otimes Z(B)$.

PROOF. This is something quite self-explanatory, the idea being as follows:

(1) In what regards the group-like elements, $\Delta(c) = c \otimes c$, assuming $a \in G_A, b \in G_B$ we have the following computation, showing that we have $a \otimes b \in G_{A \otimes B}$:

$$\begin{aligned}\Delta(a \otimes b) &= \Delta(a)_{13}\Delta(b)_{24} \\ &= (a \otimes a)_{13}(b \otimes b)_{24} \\ &= a \otimes b \otimes a \otimes b\end{aligned}$$

But this gives the inclusion in the statement, $G_A \times G_B \subset G_{A \otimes B}$.

(2) In what regards the primitive elements, $\Delta(c) = c \otimes 1 + 1 \otimes c$, assuming $a \in P_A$ we have the following computation, showing that we have $a \otimes 1 \in P_{A \otimes B}$:

$$\begin{aligned}\Delta(a \otimes 1) &= \Delta(a)_{13} \\ &= (a \otimes 1 + 1 \otimes a)_{13} \\ &= a \otimes 1 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes a \otimes 1\end{aligned}$$

Similarly, assuming $b \in P_B$ we have $1 \otimes b \in P_{A \otimes B}$. We therefore conclude that the Lie algebra $P_{A \otimes B}$ contains the Lie algebras P_A, P_B, as stated.

(3) In what regards the center, $Z(A) \otimes Z(B) \subset Z(A \otimes B)$ is clear. Conversely, we have the following computation, assuming that the elements b_i are linearly independent:

$$\begin{aligned}\sum_i a_i \otimes b_i \in Z(A \otimes B) &\implies \left[\sum_i a_i \otimes b_i, a \otimes 1\right] = 0 \\ &\implies \sum_i a_i a \otimes b_i = \sum_i a a_i \otimes b_i \\ &\implies a_i a = a a_i \\ &\implies a_i \in Z(A)\end{aligned}$$

Similarly, assuming that the elements a_i are linearly independent, we have:

$$\sum_i a_i \otimes b_i \in Z(A \otimes B) \implies b_i \in Z(B)$$

Thus we have the reverse inclusion too, so $Z(A \otimes B) = Z(A) \otimes Z(B)$, as stated. □

We have as well a result regarding the Haar integration, as follows:

THEOREM 3.12. *The Haar integral of a tensor product $A \otimes B$ appears as*

$$\int_{A \otimes B} = \int_A \otimes \int_B$$

with this happening for left integrals, right integrals, and integrals.

PROOF. This is again something self-explanatory, the idea being as follows:

(1) In what regards the left integrals, the verification goes as follows:

$$\begin{aligned}
\left(\int_{A\otimes B}\otimes id\right)\Delta(a\otimes b) &= \left(\int_A\otimes\int_B\otimes id\otimes id\right)(\Delta(a)_{13}\Delta(b)_{24})\\
&= \left[\left(\int_A\otimes id\right)\Delta(a)\right]_1\left[\left(\int_B\otimes id\right)\Delta(b)\right]_2\\
&= \left(\int_A a\cdot 1\right)_1\left(\int_B b\cdot 1\right)_2\\
&= \int_A a\cdot\int_B b\cdot 1\otimes 1\\
&= \left(\int_A\otimes\int_B\right)(a\otimes b)\cdot 1\otimes 1\\
&= \int_{A\otimes B}(a\otimes b)\cdot 1\otimes 1
\end{aligned}$$

(2) In what regards the right integrals, the verification is similar, as follows:

$$\begin{aligned}
\left(id\otimes\int_{A\otimes B}\right)\Delta(a\otimes b) &= \left(id\otimes id\otimes\int_A\otimes\int_B\right)(\Delta(a)_{13}\Delta(b)_{24})\\
&= \left[\left(id\otimes\int_A\right)\Delta(a)\right]_1\left[\left(id\otimes\int_B\right)\Delta(b)\right]_2\\
&= \left(\int_A a\cdot 1\right)_1\left(\int_B b\cdot 1\right)_2\\
&= \int_A a\cdot\int_B b\cdot 1\otimes 1\\
&= \left(\int_A\otimes\int_B\right)(a\otimes b)\cdot 1\otimes 1\\
&= \int_{A\otimes B}(a\otimes b)\cdot 1\otimes 1
\end{aligned}$$

(3) Finally, in relation with all this, there is a uniqueness discussion to be made too, which is quite standard, and that we will leave here as an instructive exercise. □

Summarizing, we have now a good understanding of the tensor product operation, with good results all around the spectrum, with respect to the general theory developed in chapters 1-2. Many other things can be said, for instance with some straightforward results regarding the representations and corepresentations, introduced earlier in this chapter. We will be back to tensor products on a regular basis, in what follows.

Moving forward, now that we know about tensor products, we can do exactly the same thing with free products, and we are led in this way to the following result:

THEOREM 3.13. *Given two Hopf algebras A,B, so is their free product*

$$C = A * B$$

and as main illustrations for this operation, we have the following formulae:

(1) $F(G \hat{*} H) = F(G) * F(H)$, *standing as definition for $G \hat{*} H$, as quantum group.*
(2) $F[G * H] = F[G] * F[H]$.

PROOF. This is again something self-explanatory, save for the abstract meaning of the object $G \hat{*} H$ appearing in (1), that we will explain below, the details being as follows:

(1) To start with, given two associative algebras A,B, so is their free product $A * B$, with the multiplicative structure being by definition as follows:

$$(\ldots a_i b_i \ldots)(\ldots a_i' b_i' \ldots) = \ldots a_i b_i \ldots a_i' b_i' \ldots$$

$$1 = 1_A = 1_B$$

Now assume in addition that A,B are Hopf algebras, each coming with its own Δ,ε,S operations. In this case we can define Δ,ε,S operations on $A * B$, as follows:

$$\Delta(\ldots a_i b_i \ldots) = \ldots \Delta(a_i)\Delta(b_i) \ldots$$

$$\varepsilon(\ldots a_i b_i \ldots) = \ldots \varepsilon(a_i)\varepsilon(b_i) \ldots$$

$$S(\ldots a_i b_i \ldots) = \ldots S(b_i)S(a_i) \ldots$$

(2) But with the above formulae in hand, the verification of the Hopf algebra axioms is straightforward. Indeed, in what regards the comultiplication axiom, we have:

$$\begin{aligned}
(\Delta \otimes id)\Delta(\ldots a_i b_i \ldots) &= (\Delta \otimes id)(\ldots \Delta(a_i)\Delta(b_i) \ldots) \\
&= \ldots [(\Delta \otimes id)\Delta(a_i)][(\Delta \otimes id)\Delta(b_i)] \ldots \\
&= \ldots [(id \otimes \Delta)\Delta(a_i)][(id \otimes \Delta)\Delta(b_i)] \ldots \\
&= (id \otimes \Delta)(\ldots \Delta(a_i)\Delta(b_i) \ldots) \\
&= (id \otimes \Delta)\Delta(\ldots a_i b_i \ldots)
\end{aligned}$$

As for the counit and antipode axioms, their verification is similar, with no tricks of any kind involved. To be more precise, in what regards the counit axiom, we have:

$$\begin{aligned}
(\varepsilon \otimes id)\Delta(\ldots a_i b_i \ldots) &= (\varepsilon \otimes id)(\ldots \Delta(a_i)\Delta(b_i) \ldots) \\
&= \ldots (\varepsilon \otimes id)\Delta(a_i)(\varepsilon \otimes id)\Delta(b_i) \ldots \\
&= \ldots a_i b_i \ldots
\end{aligned}$$

Similarly, we have the following computation, for the other counit axiom:

$$\begin{aligned}(id\otimes\varepsilon)\Delta(\ldots a_ib_i\ldots) &= (id\otimes\varepsilon)(\ldots\Delta(a_i)\Delta(b_i)\ldots)\\ &= \ldots(id\otimes\varepsilon)\Delta(a_i)(id\otimes\varepsilon)\Delta(b_i)\ldots\\ &= \ldots a_ib_i\ldots\end{aligned}$$

Finally, for the antipode axiom, we have the following computation:

$$\begin{aligned}m(S\otimes id)\Delta(\ldots a_ib_i\ldots) &= m(S\otimes id)(\ldots\Delta(a_i)\Delta(b_i)\ldots)\\ &= m(\ldots(S\otimes id)\Delta(b_i)(S\otimes id)\Delta(a_i)\ldots)\\ &= \varepsilon(\ldots a_ib_i\ldots)1\end{aligned}$$

Similarly, we have the following computation, for the other antipode axiom:

$$\begin{aligned}m(id\otimes S)\Delta(\ldots a_ib_i\ldots) &= m(id\otimes S)(\ldots\Delta(a_i)\Delta(b_i)\ldots)\\ &= m(\ldots(id\otimes S)\Delta(b_i)(id\otimes S)\Delta(a_i)\ldots)\\ &= \varepsilon(\ldots a_ib_i\ldots)1\end{aligned}$$

We conclude that $C=A*B$ is indeed a Hopf algebra, as stated.

(3) In what regards now the formula $F(G\,\hat{*}\,H)=F(G)*F(H)$, when G,H are finite groups, this stands as a definition for $G\,\hat{*}\,H$, as a quantum group, the point being that, unless G or H is trivial, the Hopf algebra $F(G)*F(H)$ is not commutative, and so cannot be understood as being an algebra of functions. Welcome to noncommutativity.

(4) As for the formula $F[G*H]=F[G]*F[H]$, with here G,H being arbitrary groups, possibly infinite, this is something which is clear from definitions. □

As a continuation of the above, in what regards the special elements, we have:

PROPOSITION 3.14. *The special elements of $A*B$ are as follows:*

1. *G_{A*B} contains G_A*G_B.*
2. *P_{A*B} contains P_A,P_B.*
3. *$Z(A*B)=F$, unless $A=F$, or $B=F$.*

PROOF. This is something quite similar to Proposition 3.11, with the various computations being very similar to those there, the idea being as follows:

(1) As before with tensor products, we have $G_A,G_B\subset G_{A*B}$, which gives the result.

(2) Also as before with tensor products, we have $P_A,P_B\subset P_{A*B}$, as claimed.

(3) Finally, in what regards the center, things are different with respect to Proposition 3.11. Indeed, since the elements $a\in A-F$ cannot commute with the elements $b\in B-F$, by definition of the free product $A*B$, we generically have $Z(A*B)=F$, as claimed. □

We have as well a result regarding the Haar integration, as follows:

THEOREM 3.15. *The Haar integral of a free product $A * B$ appears as*

$$\int_{A*B} = \int_A * \int_B$$

with this happening for left integrals, right integrals, and integrals.

PROOF. This is again something self-explanatory, the idea being as follows:

(1) In what regards the left integrals, the verification goes as follows:

$$\begin{aligned}
\left(\int_{A*B} \otimes id\right)\Delta(\ldots a_i b_i \ldots) &= \left(\int_A * \int_B \otimes id\right)(\ldots \Delta(a_i)\Delta(b_i)\ldots)\\
&= \ldots \left(\int_A \otimes id\right)\Delta(a_i)\left(\int_B \otimes id\right)\Delta(b_i)\ldots\\
&= \ldots \int_A a_i \cdot 1 \int_B b_i \cdot 1 \ldots\\
&= \int_{A*B} (\ldots a_i b_i \ldots)\cdot 1
\end{aligned}$$

(2) In what regards the right integrals, the verification is similar, as follows:

$$\begin{aligned}
\left(id \otimes \int_{A*B}\right)\Delta(\ldots a_i b_i \ldots) &= \left(id \otimes \int_A * \int_B\right)(\ldots \Delta(a_i)\Delta(b_i)\ldots)\\
&= \ldots \left(id \otimes \int_A\right)\Delta(a_i)\left(id \otimes \int_B\right)\Delta(b_i)\ldots\\
&= \ldots \int_A a_i \cdot 1 \int_B b_i \cdot 1 \ldots\\
&= \int_{A*B} (\ldots a_i b_i \ldots)\cdot 1
\end{aligned}$$

(3) Finally, in relation with all this, there is a uniqueness discussion to be made too, which is quite standard, and that we will leave here as an instructive exercise. □

As before with the tensor products, many other things can be said about free products, for instance with some straightforward results regarding their representations and corepresentations. We will be back to free products on a regular basis, in what follows.

As a further comment here, algebrically speaking, there are several other possible products, which are quite natural, between the tensor products and the free products. But things here are quite technical, and we will discuss them later in this book.

3d. Quotients, subalgebras

Another standard operation, that we would like to discuss now, is that of taking quantum subgroups, with the result here, at the algebraic level, being as follows:

THEOREM 3.16. *Given a Hopf algebra A, so is its quotient $B = A/I$, provided that $I \subset A$ is an ideal satisfying the following conditions, called Hopf ideal conditions,*

$$\Delta(I) \subset A \otimes I + I \otimes A \quad , \quad \varepsilon(I) = 0 \quad , \quad S(I) \subset I$$

and as main illustrations for this operation, we have the following formulae:

(1) *$F(G)/I = F(H)$, with $H \subset G$ being a certain subgroup.*
(2) *$F[G]/I = F[H]$, with $G \to H$ being a certain quotient.*

PROOF. As before, this is something self-explanatory, the idea being as follows:

(1) Given an associative algebra A and an ideal $I \subset A$, we can certainly construct the quotient $B = A/I$, which is an associative algebra. Thus, we must just see when the Hopf algebra operations Δ, ε, S correctly factorize, from A to B.

(2) Let us first see when Δ factorizes. If we denote by $\pi : A \to B$ the canonical projection, the factorization diagram that we are looking for is as follows:

$$\begin{CD} A @>\Delta>> A \otimes A \\ @V{\pi}VV @VV{\pi\otimes\pi}V \\ B @>>> B \otimes B \end{CD}$$

We can see that the factorization condition is as follows:

$$\pi(a) = 0 \implies (\pi \otimes \pi)\Delta(a) = 0$$

Thus, in terms of the ideal $I \subset A$, the following condition must be satisfied:

$$a \in I \implies (\pi \otimes \pi)\Delta(a) = 0$$

But, for an element $b \in A \otimes A$, we have the following equivalence:

$$(\pi \otimes \pi)b = 0 \iff b \in A \otimes I + I \otimes A$$

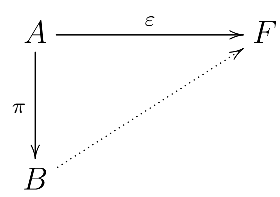
We conclude that the factorization condition for Δ is as follows:

$$a \in I \implies \Delta(a) \in A \otimes I + I \otimes A$$

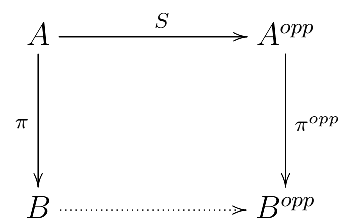
But this is precisely the first condition on I in the statement, namely:

$$\Delta(I) \subset A \otimes I + I \otimes A$$

(3) Similarly, the counit ε factorizes precisely when the condition $\varepsilon(I) = 0$ in the statement is satisfied, with the factorization diagram here being as follows:



(4) Also, the antipode S factorizes precisely when the condition $S(I) \subset I$ in the statement is satisfied, with the factorization diagram here being as follows:



(5) Together with the remark that the maps Δ, ε, S, once factorized, will keep satisfying the Hopf algebra axioms, automatically, we are led to the first assertion.

(6) In the group setting now, the formula $F(G)/I = F(H)$, with $H \subset G$ being a certain subgroup, is something which clear from definitions.

(7) As for the last formula, namely $F[G]/I = F[H]$, with $G \to H$ being a certain quotient, this is something which is clear from definitions too. $\square$

As a continuation of the above, in what regards the special elements, we have:

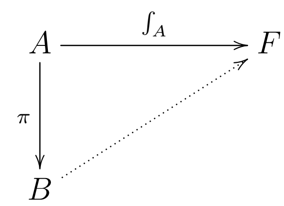
Proposition 3.17. *In what regards the special elements of a quotient $B = A/I$, the quotient map $A \to B$ induces quotient maps as follows:*

(1) $G_A \to G_B$.
(2) $P_A \to P_B$.
(3) $Z(A) \to Z(B)$.

Proof. This is something quite trivial, because in what regards the group-like elements, the primitive elements, and the central elements too, their defining formulae pass to the quotient, in the obvious way. Thus, we are led to the above conclusions. $\square$

We have as well a statement regarding the Haar integration, as follows:

Fact 3.18. *The Haar integral of a quotient $B = A/I$ does not appear as for the other Hopf algebra operations, simply by factorizing the following diagram,*



but is related however to the Haar integral of A, via a number of more technical formulae, and with this happening for left integrals, right integrals, and integrals.

Proof. This is obviously something quite informal, that we included here for the sake of symmetry, with respect to the other operations, the idea being as follows:

(1) The first assertion is certainly something which happens, coming from the intuitive fact that, by taking for instance $B = A/A$, we have certainly not constructed here the integral of A, which is well-known to require some hard work, harder than this.

(2) As for the second assertion, this is something more technical, say involving representations and corepresentations. We will leave working out the details here, based on what happens in the cases $A = F(G)$ and $A = F[H]$, as an instructive exercise, and we will come back to such questions later, under a number of suitable extra assumptions. □

As before with the tensor and free products, many other things can be said, for instance with some straightforward results regarding the representations and corepresentations of quotients. We will be back to quotients on a regular basis, in what follows.

Regarding now taking quotient quantum groups, the result here is quite similar, somehow dual to Theorem 3.16, but technically very straightforward, as follows:

Theorem 3.19. *Given a Hopf algebra A, any subalgebra $B \subset A$ satisfying*

$$\Delta(B) \subset B \otimes B \quad , \quad S(B) \subset B$$

is a Hopf algebra, and as main illustrations, we have the following subalgebras:

1. *$F(H) \subset F(G)$, for any quotient group $G \to H$.*
2. *$F[H] \subset F[G]$, for any subgroup $H \subset G$.*

Proof. The main assertion is clear from definitions, because the Hopf algebra axioms being satisfied over A, they are satisfied as well over the subalgebra $B \subset A$. As for the main illustrations, in the group case, these are as well both clear from definitions. □

The above result is a bit abstract, and as a useful version of it, providing examples, let us record as well the following statement, that will play an important role later:

THEOREM 3.20. *Given a Hopf algebra A, and a finite dimensional corepresentation $u=(u_{ij})$, the subalgebra generated by the coefficients of u,*

$$B=<u_{ij}>\subset A$$

is a Hopf algebra. As main illustrations for this operation, we obtain subalgebras:

(1) *$F(H)\subset F(G)$, with $G\to H$ being certain quotients.*
(2) *$F[H]\subset F[G]$, with $H\subset G$ being certain subgroups.*

PROOF. This is something which follows from Theorem 3.19, and that we will actually fully clarify later in this book, the idea being as follows:

(1) Given a coalgebra A and a corepresentation $u=(u_{ij})$, we can certainly construct the space of coefficients $C_u=<u_{ij}>\subset A$, which is automatically a coalgebra. Indeed, recall that the corepresentations are subject to the following condition:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$$

But, by using this condition, we see that Δ leaves indeed invariant C_u:

$$\Delta(C_u)\subset C_u\otimes C_u$$

(2) In the case now where A is a Hopf algebra, by setting $B=<C>\subset A$, our claim is that we obtain both an algebra and a coalgebra, so that we have a Hopf algebra. Indeed, in what regards Δ, the inclusion found in (1) gives, by multiplicativity:

$$\Delta(B)\subset B\otimes B$$

In what regards now the counit ε, there is no verification needed, because we can simply take the restriction of the counit of A, to the subalgebra $B\subset A$.

(3) Finally, in what regards the antipode S, things here are more tricky. We can use here the following formula, that we know from Theorem 3.6:

$$(id\otimes S)u=u^{-1}$$

Thus, with the convention that our subalgebra $B=<C>\subset A$ contains the inverses of all invertible elements $c\in C$, we see that S satisfies the following condition:

$$S(B)\subset B$$

Alternatively, the fact that our subalgebra $B=<C>\subset A$ contains the inverses of all invertible elements $c\in C$ can come by theorem, under suitable assumptions on the class of algebras involved. We will be back to this point, later on in this book.

(4) As a conclusion, under the assumptions in the statement, the Hopf algebra maps Δ,ε,S restrict to the subalgebra $B=<C>\subset A$. But the Hopf algebras axioms being automatic for these restrictions, we are led to the first assertion.

(5) In the group setting now, the formula $F(H) \subset F(G)$, with $G \to H$ being a certain quotient, is something which is clear from definitions.

(6) As for the last formula, namely $F[H] \subset F[G]$, with $H \subset G$ being a certain subgroup, this is something which is clear from definitions too. $\square$

As a continuation of the above, in what regards the special elements, we have:

PROPOSITION 3.21. *The special elements of a subalgebra $B \subset A$ are as follows,*

(1) $G_B = G_A \cap B$.
(2) $P_B = P_A \cap B$.

and in what regards the center, nothing in particular can be said.

PROOF. This is something trivial, the idea being as follows:

(1) The first two assertions are clear, since the comultiplication of B appears by definition as the restriction of the comultiplication of A. Thus, when it comes to group-like elements, or to primitive elements, we obtain the formulae in the statement.

(2) As for the last assertion, this is something informal, and we will leave some thinking here, with various examples and counterexamples, as an instructive exercise, the idea being that the center of B can be substantially smaller, or larger, than the center of A. $\square$

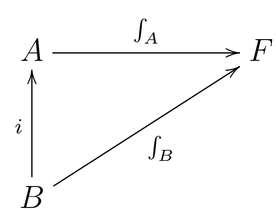
We have as well a result regarding the Haar integration, as follows:

THEOREM 3.22. *The Haar integral of a subalgebra $B \subset A$ appears as a restriction*

$$\int_B = \left(\int_A\right)_B$$

and with this happening for left integrals, right integrals, and integrals.

PROOF. This is again something quite self-explanatory, and clear from definitions, with the corresponding commuting diagram here being as follows:



Thus, we are led to the conclusion in the statement. $\square$

As before with the products and quotients, many other things can be said, for instance with some straightforward results regarding the representations and corepresentations of subalgebras. We will be back to subalgebras on a regular basis, in what follows.

Finally, in relation with the above quotient and subalgebra operations, and with their main illustrations too, there is some discussion to be made, in the finite dimensional case, in the context of the Hopf algebra duality for such Hopf algebras, from chapter 1. To be more precise, we have here the following result, which clarifies the situation:

THEOREM 3.23. *In the context of the duality for finite dimensional Hopf algebras*

$$A \leftrightarrow A^*$$

the operations of taking quotients and subalgebras are dual to each other.

PROOF. This is something straightforward, the idea being as follows:

(1) Consider a finite dimensional Hopf algebra A, with structural maps as follows:

$$m : A \otimes A \to A$$

$$u : F \to A$$

$$\Delta : A \to A \otimes A$$

$$\varepsilon : A \to F$$

$$S : A \to A^{opp}$$

(2) As explained in chapter 1, the dual vector space A^*, consisting of the linear forms $\varphi : A \to F$, is then a Hopf algebra too, with structural maps as follows:

$$\Delta^t : A^* \otimes A^* \to A^*$$

$$\varepsilon^t : F \to A^*$$

$$m^t : A^* \to A^* \otimes A^*$$

$$u^t : A^* \to F$$

$$S^t : A^* \to (A^*)^{opp}$$

(3) But, with these formulae in hand, it is straightforward to check that the Hopf algebra quotients of A correspond to the Hopf subalgebras of A^*, and vice versa:

$$B \subset A \quad \longleftrightarrow \quad C = A^*/I$$

$$B = A/I \quad \longleftrightarrow \quad C \subset A^*$$

(4) Moreover, in the context of the duality between the function algebras $A = F(G)$ and the group algebras $A = F[G]$, with G being a finite group, we obtain in this way generalizations of the well-known fact that the quotients of a finite abelian group G correspond to the subgroups of the dual finite abelian group $\widehat{G}$, and vice versa.

(5) So, this was for the general idea, that everything comes in the end from what we know about the finite abelian groups, and we will leave the proof of the various assertions formulated above, in the precise order that you prefer, as an instructive exercise. □

3e. Exercises

We had a lot of interesting algebra in this chapter, and as exercises, we have:

EXERCISE 3.24. *Further study the representation theory of* $F(G)$.

EXERCISE 3.25. *Further study the representation theory of* $F[H]$.

EXERCISE 3.26. *Fill in the missing details for the* $\otimes$ *operation.*

EXERCISE 3.27. *Fill in the missing details for the* $*$ *operation.*

EXERCISE 3.28. *Clarify the missing details for the quotient operation.*

EXERCISE 3.29. *Clarify the missing details for the subalgebra operation.*

As bonus exercise, figure out what happens to all the above when $F = \mathbb{C}$.

CHAPTER 4

Affine algebras

4a. Affine algebras

We have seen so far the Hopf algebra basics, including the theory of the basic operations for the Hopf algebras. We would like to discuss now a number of more tricky operations on the Hopf algebras, appearing as variations of the above.

For this purpose, let us introduce the following notion, inspired as usual from group theory, that will play a key role in this book, starting from now, and until the end:

Definition 4.1. *We call a Hopf algebra A affine when it is of the form*

$$A = < u_{ij} >$$

with $u \in M_N(A)$ being a corepresentation, called fundamental corepresentation.

As already mentioned, this notion is inspired from group theory, and more specifically, from advanced group theory, our motivation coming from the following facts:

(1) In group theory, at a reasonably advanced level, a natural assumption on a group G is that this appears as an algebraic group, $G \subset GL_N(F)$. This is indeed how most of the examples of groups G appear, in practice, as groups of invertible matrices.

(2) Observe that any finite group G appears as above, thanks to the Cayley embedding theorem, $G \subset S_N$ with $N = |G|$, coupled with the standard embedding $S_N \subset GL_N(F)$ given by the permutation matrices, which give an embedding as follows:

$$G \subset S_N \subset GL_N(F)$$

(3) As another key example, which is more advanced, it is known that any compact Lie group G appears as a group of unitary matrices, $G \subset U_N$, so that we have:

$$G \subset U_N \subset GL_N(\mathbb{C})$$

(4) Now the point is that, save for a few topological issues, the fact that a group G is algebraic is equivalent to the fact that the Hopf algebra $A = F(G)$ is affine, in the sense of Definition 4.1. Thus, we have here a good motivation for Definition 4.1.

In addition to this, getting now to the group dual level, we have some extra motivations for Definition 4.1, again coming from advanced group theory, which are as follows:

(1) Again in group theory, at a reasonably advanced level, but this time with the discrete group theory in mind, instead of the continuous group theory used above, a natural assumption on a group H is that this group is finitely generated:

$$H =< g_1, \ldots, g_N >$$

(2) Indeed, this is how nearly all the interesting discrete groups appear. In fact, many of these groups appear by definition via generators and relations, as follows:

$$H = \left\langle g_1, \ldots, g_N \middle| \mathcal{R} \right\rangle$$

(3) As another comment here, passed the wealth of examples, the fact of being finitely generated is needed for developing the theory, because talking about Cayley graphs and their metric aspects, random walks and so on, always requires the use of generators.

(4) Now the point is that, save for a few topological issues, the fact that a group H is finitely generated is equivalent to the fact that the Hopf algebra $A = F[H]$ is affine, in the sense of Definition 4.1. Thus, we have again a good motivation for Definition 4.1.

Summarizing, in order to reach to a more advanced Hopf algebra theory, we have all good reasons in this world to assume that our algebras are affine, as in Definition 4.1. And for ending this discussion, let us formulate our conclusions as follows:

Conclusion 4.2. *In relation with group advanced theory:*

1. *It makes sense to assume that the groups are algebraic,* $G \subset GL_N(F)$,
2. *Or to assume that the groups are finitely generated,* $H =< g_1, \ldots, g_N >$,
3. *With this basically corresponding to the fact that* $F(G)$ *and* $F[H]$ *are affine,*
4. *So, we will assume in what follows that our Hopf algebras* A *are affine.*

And more on this later. Also, we will see many examples of affine Hopf algebras in what follows, appearing via various operations, the idea being that the affine Hopf algebras are subject to far more operations than those discussed before. More later.

In order to get started now, we need to know more about corepresentations, and in particular, about the fundamental one. As a first result, that we will need in what follows, the corepresentations of an arbitrary Hopf algebra A are subject to a number of operations, exactly as the group representations in the usual group case, as follows:

Proposition 4.3. *The corepresentations of a Hopf algebra* A *are subject to the following operations, in analogy with what happens for the group representations:*

1. *Making sums,* $u + v = diag(u, v)$.
2. *Making tensor products,* $(u \otimes v)_{ia,jb} = u_{ij}v_{ab}$.
3. *Spinning by invertible scalar matrices,* $u \to VuV^{-1}$.

PROOF. Observe first that the result holds indeed for $A = F(G)$, where we obtain the usual operations on the representations of G. In general, the proof goes as follows:

(1) Everything here is clear from definitions.

(2) The comultiplicativity condition follows indeed from the following computation:

$$\begin{aligned}\Delta((u \otimes v)_{ia,jb}) &= \Delta(u_{ij}v_{ab}) \\ &= \Delta(u_{ij})\Delta(v_{ab}) \\ &= \sum_k u_{ik} \otimes u_{kj} \sum_c v_{ac} \otimes v_{cb} \\ &= \sum_{kc} u_{ik}v_{ac} \otimes u_{kj}v_{cb} \\ &= \sum_{kc} (u \otimes v)_{ia,kc} \otimes (u \otimes v)_{kc,jb}\end{aligned}$$

(3) The comultiplicativity property of the matrix $v = VuV^{-1}$ in the statement can be checked by doing a straightforward computation. Alternatively, if we write $u \in M_n(F) \otimes A$, the usual comultiplicativity axiom, as formulated in chapter 3, reads:

$$(id \otimes \Delta)u = u_{12}u_{13}$$

Here we use standard tensor calculus conventions. Now when spinning by a scalar matrix, the matrix that we obtain is $v = V_1uV_1^{-1}$, and we have:

$$\begin{aligned}(id \otimes \Delta)v &= V_1u_{12}u_{13}V_1^{-1} \\ &= V_1u_{12}V_1^{-1} \cdot V_1u_{13}V_1^{-1} \\ &= v_{12}v_{13}\end{aligned}$$

Thus, with usual notations, $v = VuV^{-1}$ is a corepresentation, as claimed. □

As a comment now, the various operations in Proposition 4.3 can be viewed as operations on the class of affine Hopf algebras, the result here being as follows:

PROPOSITION 4.4. *We have the following operations on the affine Hopf algebras, with the convention $A_u = < u_{ij} > \subset A$, for a corepresentation $u \in M_N(A)$:*

(1) $(A_u, A_v) \to A_{u+v}$.
(2) $(A_u, A_v) \to A_{u \otimes v}$.
(3) $(A, u) \to (A, VuV^{-1})$.

PROOF. This is indeed something clear, coming from the various operations on the corepresentations constructed in Proposition 4.3, and with the remark that, in the context of the last assertion, we have indeed $< u_{ij} > = < (VuV^{-1})_{ij} >$, as needed there. □

As a further operation, also inspired from usual group theory, we have:

Theorem 4.5. *Given a Hopf algebra corepresentation $u\in M_n(A)$,*

$$\bar{u}=(t\otimes id)u^{-1}$$

is a corepresentation too, called contragradient, or conjugate to u.

Proof. This is something very standard, the idea being as follows:

(1) To start with, we know from chapter 3 that u is indeed invertible, with inverse given by $u^{-1}=(id\otimes S)u$. Thus, $\bar{u}$ is well-defined, and in addition, we have:

$$\bar{u}=(t\otimes S)u$$

(2) But with this latter formula in hand, the proof of the corepresentation property of $\bar{u}$ goes as follows, by using the formula $\Delta S=\Sigma(S\otimes S)\Delta$ from chapter 2:

$$\begin{aligned}\Delta(\bar{u}_{ij}) &= \Delta S(u_{ji})\\ &= \Sigma(S\otimes S)\Delta(u_{ji})\\ &= \Sigma(S\otimes S)\sum_k u_{jk}\otimes u_{ki}\\ &= \sum_k S(u_{ki})\otimes S(u_{jk})\\ &= \sum_k \bar{u}_{ik}\otimes\bar{u}_{kj}\end{aligned}$$

(3) Alternatively, we can check the corepresentation property of $\bar{u}$ as follows, simply by inverting the corepresentation property of u, written in compact form:

$$\begin{aligned}(id\otimes\Delta)u=u_{12}u_{13} &\implies (id\otimes\Delta)u^{-1}=u_{13}^{-1}u_{12}^{-1}\\ &\implies (t\otimes\Delta)u^{-1}=(t\otimes id)u_{12}^{-1}\cdot(t\otimes id)u_{13}^{-1}\\ &\implies (id\otimes\Delta)\bar{u}=\bar{u}_{12}\bar{u}_{13}\end{aligned}$$

(4) Thus, one way or another, we get the result. As further comments, observe first that $S^2=id$ implies $\bar{\bar{u}}=u$. Observe also that when $F=\mathbb{C}$ and u is unitary, $u^*=u^{-1}$, the matrix $\bar{u}$ is the usual conjugate, given by $\bar{u}_{ij}=u_{ij}^*$. We will be back to this. □

In analogy now with Proposition 4.4, the above result suggests talking about the operation $A_u\to A_{\bar{u}}$ for the affine Hopf algebras, and also investigating the relation between the conditions $A=<u_{ij}>$ and $A=<\bar{u}_{ij}>$. In relation with this, which actually leads to a bit of rethinking of Definition 4.1, let us make the following convention:

Convention 4.6. *Given a Hopf algebra A, with a corepresentation $u\in M_N(A)$,*

(1) *We keep calling A affine when $A=<u_{ij}>$,*
(2) *We call A fully affine when $A=<u_{ij},\bar{u}_{ij}>$,*

with the remark that, when $u\sim\bar{u}$, meaning $u=V\bar{u}V^{-1}$, these notions coincide.

Many further things can be said here, and we will see later in this book that the complex algebra framework, $F=\mathbb{C}$, brings some clarification in relation with all this, by using $*$-algebras and unitary corepresentations, which are subject to $\bar{u}_{ij}=u_{ij}^*$.

However, importantly, we will also see later in this book, when talking $F=\mathbb{C}$, that most of the interesting examples satisfy $u\sim\bar{u}$, and that in fact, up to passing to projective versions, which is something quite natural, we can always assume $u\sim\bar{u}$. Thus, and for closing this discussion, Definition 4.1 as stated is basically the good one.

Getting now to what can be done with an affine Hopf algebra, we will use as source of inspiration what happens for $A=F(G)$. Given an algebraic group $G\subset GL_N(F)$, a natural construction is that of considering its diagonal torus $T\subset G$, which is given by the following formula, $(F^*)^N\subset GL_N(F)$ being the subgroup of diagonal matrices:

$$T=G\cap(F^*)^N$$

The point now is that we can perform in fact this construction in the general affine Hopf algebra context, that of Definition 4.1, with the result being as follows:

THEOREM 4.7. *Given an undeformed affine Hopf algebra (A,u), the quotient*

$$A^\delta=A\Big/\left\langle u_{ij}=\bar{u}_{ij}=0\Big|\forall i\neq j\right\rangle$$

is an affine Hopf algebra too, called diagonal algebra. Its standard generators

$$u_{ii}\in A^\delta$$

are group-like, and the algebra A^δ itself is cocommutative.

PROOF. This is something very standard, the idea being as follows:

(1) We know from chapter 3 that given a Hopf algebra A, so is its quotient $B=A/I$, provided that $I\subset A$ is an ideal satisfying the Hopf ideal conditions, namely:

$$\Delta(I)\subset A\otimes I+I\otimes A\quad,\quad\varepsilon(I)=0\quad,\quad S(I)\subset I$$

In our case, the ideal that we are dividing by is given by the following formula:

$$I=\left\langle u_{ij},\bar{u}_{ij}\Big|i\neq j\right\rangle$$

(2) So, let us check that this is indeed a Hopf ideal. Regarding the condition involving the comultiplication, on the main generators of I we have, as desired:

$$\begin{aligned}\Delta(u_{ij}) &= \sum_k u_{ik}\otimes u_{kj}\\ &= u_{ii}\otimes u_{ij}+\sum_{k\neq i}u_{ik}\otimes u_{kj}\\ &\in A\otimes I+\sum_{k\neq i}I\otimes A\\ &= A\otimes I+I\otimes A\end{aligned}$$

Similarly, in what regards the coefficients of $\bar{u}=(t\otimes id)u^{-1}$, we have:

$$\Delta(\bar{u}_{ij})\in A\otimes I+I\otimes A$$

(3) Next, the condition involving the counit is clear as well, because we have:

$$i\neq j\implies \varepsilon(u_{ij})=\varepsilon(\bar{u}_{ij})=0$$

In what regards now the last condition, involving the antipode, here we can use the following formulae, with the first one being something that we know well, and with the second one coming from it, by using our undeformability assumption $S^2=id$:

$$\bar{u}=(t\otimes S)u\quad,\quad u=(t\otimes S)\bar{u}$$

Indeed, we obtain from these formulae that for any $i\neq j$ we have, as desired:

$$S(u_{ij})=\bar{u}_{ji}\in I\quad,\quad S(\bar{u}_{ij})=u_{ji}\in I$$

(4) We conclude from all this that $A^\delta=A/I$ is indeed an affine Hopf algebra, with its fundamental corepresentation being as follows, with the convention $u_{ii}\in A^\delta$:

$$u^\delta=\begin{pmatrix}u_{11}&&\\&\ddots&\\&&u_{NN}\end{pmatrix}$$

(5) Regarding now the last assertion, this is clear from the above diagonal form of the fundamental corepresentation, but we can check this directly too. We have, inside A^δ:

$$\begin{aligned}\Delta(u_{ii}) &= \sum_k u_{ik}\otimes u_{ki}\\ &= u_{ii}\otimes u_{ii}+\sum_{k\neq i}u_{ik}\otimes u_{ki}\\ &= u_{ii}\otimes u_{ii}+\sum_{k\neq i}0\otimes 0\\ &= u_{ii}\otimes u_{ii}\end{aligned}$$

Thus the standard generators are indeed group-like, and this implies of course that the diagonal algebra A^δ itself is indeed cocommutative, as stated.

(6) Finally, as a complement to this, in relation with what was said before the statement, for $A = F(G)$ with $G \subset GL_N(F)$ we obtain $A^\delta = F(T)$, with $T \subset G$ being the diagonal torus. Also, for $A = F[H]$, with $H =< g_1, \ldots, g_N >$ being a finitely generated group, with fundamental corepresentation $u = diag(g_1, \ldots, g_N)$, we obtain $A^\delta = A$. □

As an interesting variation of the above construction, generalizing it, we have:

THEOREM 4.8. *Given an undeformed affine Hopf algebra (A, u) and $Q \in GL_N(F)$,*

$$A_Q^\delta = A\Big/\left\langle (QuQ^{-1})_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

is an affine Hopf algebra too, called spinned diagonal algebra. Its standard generators

$$(QuQ^{-1})_{ii} \in A_Q^\delta$$

are group-like, and the algebra A_Q^δ itself is cocommutative.

PROOF. This follows indeed from Theorem 4.7 applied to the following affine Hopf algebra, with this latter Hopf algebra coming from Proposition 4.4 (3):

$$(A', u') = (A, QuQ^{-1})$$

Alternatively, we can redo the proof of Theorem 4.7, in the present more general setting, by adding the parameter matrix $Q \in GL_N(F)$, to all the computations there. □

All the above is quite interesting, making a connection with advanced group theory, and more specifically, with the notion of maximal torus, from the Lie group theory. Indeed, at the level of basic examples, we first have the following result:

THEOREM 4.9. *For a function algebra $A = F(G)$, with $G \subset GL_N(F)$, the diagonal algebra $A \to A^\delta$ is the algebra of functions on the diagonal torus $T = G \cap (F^*)^N$:*

$$A^\delta = F(T)$$

More generally, the spinned diagonal algebras $A \to A_Q^\delta$, with $Q \in GL_N(F)$, are the algebras of functions on the spinned diagonal tori, $T_Q = G \cap Q^{-1}(F^*)^N Q$:

$$A_Q^\delta = F(T_Q)$$

Also, when $F = \mathbb{C}$, any abelian subgroup $H \subset G$ appears inside such a torus, $H \subset T_Q$.

PROOF. This is something quite self-explanatory, the idea being as follows:

(1) The first assertion, regarding the diagonal torus, is something routine, coming from definitions, as explained at the end of the proof of Theorem 4.7.

(2) The second assertion, regarding the spinned tori, is clear from definitions too, via the same argument, with the above definition for the spinned tori.

(3) Finally, in what regards the third assertion, given an abelian subgroup $H\subset G$, we can compose this group embedding with the embedding $G\subset GL_N(F)$:

$$H\subset G\subset GL_N(F)$$

Now the group H being assumed to be abelian, when $F=\mathbb{C}$, by representation theory, this embedding must be similar to a diagonal embedding. Thus, if we denote by $Q\in GL_N(F)$ the matrix producing this similarity, we have an embedding as follows:

$$H\subset Q^{-1}(F^N)^*Q$$

But this proves our claim, because by intersecting with G, we obtain:

$$H\subset G\cap Q^{-1}(F^N)^*Q=T_Q$$

So, this was for the general idea, and in practice, we will be back to this later, with full details and explanations, and with some generalizations too. □

Let us discuss as well the case of the group algebras. Here we have:

THEOREM 4.10. *For a group algebra $A=F[H]$, with $H=<g_1,\ldots,g_N>$, the diagonal algebra $A\to A^\delta$ is the group algebra of the following quotient group,*

$$H_Q=H\Big/\Big\langle g_i=g_j\Big|\exists k,Q_{ki}\neq0,Q_{kj}\neq0\Big\rangle$$

with the embedding $T_Q=\widehat{H}_Q\subset G=\widehat{H}$ coming from the quotient map $H\to H_Q$. Also, a similar result holds for $A=F[H]$, with spinned fundamental corepresentation.

PROOF. This is something elementary, the idea being as follows:

(1) Assume first that $H=<g_1,\ldots,g_N>$ is a discrete group, with dual $\widehat{H}$ diagonally embedded, that is, with fundamental corepresentation of $F[H]$ as follows:

$$u=\begin{pmatrix}g_1&&\\&\ldots&\\&&g_N\end{pmatrix}$$

With $v=QuQ^{-1}$, we have then the following computation:

$$\begin{aligned}\sum_s(Q^{-1})_{si}v_{sk}&=\sum_s(Q^{-1})_{si}(QuQ^{-1})_{sk}\\&=\sum_{st}(Q^{-1})_{si}Q_{st}u_{tt}(Q^{-1})_{kt}\\&=\sum_{st}(Q^{-1})_{si}Q_{st}(Q^{-1})_{kt}g_t\\&=\sum_t\delta_{it}(Q^{-1})_{kt}g_t\\&=(Q^{-1})_{ki}g_i\end{aligned}$$

Thus the condition $v_{ij}=0$ for $i\neq j$, used in Theorem 4.8, gives:

$$(Q^{-1})_{ki}v_{kk}=(Q^{-1})_{ki}g_i$$

(2) But this latter condition tells us that we must have:

$$Q_{ki}\neq 0\implies g_i=v_{kk}$$

We conclude from this that we have, as desired:

$$Q_{ki}\neq 0,Q_{kj}\neq 0\implies g_i=g_j$$

(3) In order to finish now, consider the group in the statement. We must prove that the off-diagonal coefficients of QuQ^{-1} vanish. So, let us look at these coefficients:

$$(QuQ^{-1})_{ij}=\sum_kQ_{ik}u_{kk}(Q^{-1})_{kj}=\sum_kQ_{ik}(Q^{-1})_{kj}g_k$$

In this sum k ranges over the set $S=\{1,\ldots,N\}$, but we can restrict the attention to the subset S' of indices having the property $Q_{ik}(Q^{-1})_{kj}\neq 0$. But for these latter indices the elements g_k are all equal, say to an element g, and we obtain, as desired:

$$\begin{aligned}(QuQ^{-1})_{ij}&=\left(\sum_{k\in S'}Q_{ik}(Q^{-1})_{kj}\right)g\\&=\left(\sum_{k\in S}Q_{ik}(Q^{-1})_{kj}\right)g\\&=(QQ^{-1})_{ij}g\\&=\delta_{ij}g\end{aligned}$$

(4) Finally, in what regards the last assertion, this is again elementary, obtained by adding an extra matrix parameter to the above computations, spinning the fundamental corepresentation of $F[H]$, and we will leave the computations here as an exercise. □

Summarizing, we have here some beginning of Lie theory, for the affine Hopf algebras, going beyond what we know from chapter 2. According to the above results, we can expect the collection of tori $\{T_Q|Q\in GL_N(F)\}$ to encode various algebraic and analytic properties of G. We will discuss this later, with a number of results and conjectures.

4b. Projective versions

As already mentioned in the beginning of this chapter, the affine Hopf algebras are subject to far more operations than those discussed in chapter 3, for the arbitrary Hopf algebras, and this makes the world of affine Hopf algebras quite interesting.

In fact, we have already seen a number of such new operations, in Theorem 4.7 and Theorem 4.8, making an interesting link with the advanced Lie group theory. So, let us further explore this subject, what operations can be applied to the affine Hopf algebras. We can first talk about complexifications, in an abstract sense, as follows:

THEOREM 4.11. *Given an affine Hopf algebra (A, u), we can construct its complexification (A^c, u^c) as follows,*

$$A^c =< u^c >\subset F[\mathbb{Z}] \otimes A \quad , \quad u^c = zu$$

with $z = 1 \in F[\mathbb{Z}]$. As main illustrations for this operation, we have:

1. *$F(G)^c = F(G^c)$, for a certain group quotient $\mathbb{T} \times G \to G^c$.*
2. *$F[H]^c = F[H^c]$, with $H^c \subset \mathbb{Z} \times H$ being constructed similarly.*

PROOF. This is something quite routine, the idea being as follows:

(1) Regarding the Hopf algebra assertion, this follows from our general results regarding the tensor products and subalgebras, from chapter 3. Indeed, we have:

$$\begin{aligned}
\Delta(u_{ij}^c) &= \Delta(z)\Delta(u_{ij}) \\
&= (z \otimes z) \sum_k u_{ik} \otimes u_{kj} \\
&= \sum_k zu_{ik} \otimes zu_{kj} \\
&= \sum_k u_{ik}^c \otimes u_{kj}^c
\end{aligned}$$

Thus $\tilde{u} = zu$ is indeed a corepresentation of the tensor product algebra $F[\mathbb{Z}] \otimes F(G)$, as constructed in chapter 3, so the results there apply, and gives the result.

(2) As a comment here, by Fourier transform, we can define alternatively the complexification A^c as follows, with $z = id \in F[\mathbb{T}]$ being the standard generator:

$$A^c =< u^c >\subset F(\mathbb{T}) \otimes A \quad , \quad u^c = zu$$

(3) In what regards now the formula $F(G)^c = F(G^c)$, with $\mathbb{T} \times G \to G^c$, many things can be said here, and we will leave some study here as an exercise.

(4) As for the formula $F[H]^c = F[H^c]$, with the subgroup $H^c \subset \mathbb{Z} \times H$ being constructed similarly, by multiplying the generators by z, this is something self-explanatory too. Again, many things can be said here, and we will leave this as an exercise. □

As a comment here, the above construction is particularly relevant when the ground field is $F = \mathbb{C}$. We will be back to this later, with comments and illustrations.

Now still at the general level, we have a result regarding the Haar integration of the complexifications, which is something quite straightforward, as follows:

THEOREM 4.12. *The Haar integral of a complexification*

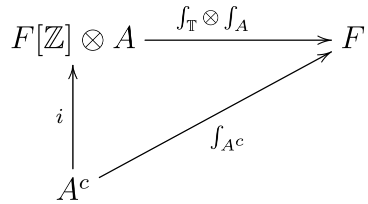
$$A^c =< u^c >\subset F[\mathbb{Z}]\otimes A \quad , \quad u^c = zu$$

appears as a restriction of a tensor product, as follows,

$$\int_{A^c} = \left(\int_{\mathbb{T}}\otimes\int_A\right)_{A^c}$$

and with this happening for left integrals, right integrals, and integrals.

PROOF. This is something quite self-explanatory, and clear from our various results from chapter 3, with the corresponding commuting diagram here being as follows:



Thus, we are led to the conclusion in the statement. □

Moving on, along the same lines, we have as well the following construction:

THEOREM 4.13. *Given an affine Hopf algebra (A, u), we can construct its free complexification $(\widetilde{A}, \tilde{u})$ as follows,*

$$\widetilde{A} =< \tilde{u} >\subset F[\mathbb{Z}] * A \quad , \quad \tilde{u} = zu$$

with $z = 1 \in F[\mathbb{Z}]$. As main illustrations for this operation, we have:

1. *$\widetilde{F}(G) = F(\widetilde{G})$, standing as definition for $\widetilde{G}$, as quantum group.*
2. *$\widetilde{F}[H] = F[\widetilde{H}]$, with $\widetilde{H} \subset \mathbb{Z} * H$ being constructed similarly.*

PROOF. This is something quite similar, the idea being as follows:

(1) Regarding the Hopf algebra assertion, this follows from our general results regarding free products and subalgebras, from chapter 3. Indeed, we have:

$$\begin{aligned}
\Delta(\tilde{u}_{ij}) &= \Delta(z)\Delta(u_{ij})\\
&= (z\otimes z)\sum_k u_{ik}\otimes u_{kj}\\
&= \sum_k zu_{ik}\otimes zu_{kj}\\
&= \sum_k \tilde{u}_{ik}\otimes\tilde{u}_{kj}
\end{aligned}$$

Thus $\tilde{u} = zu$ is indeed a corepresentation of the free product algebra $F[\mathbb{Z}] * F(G)$, as constructed in chapter 3, so the results there apply, and gives the result.

(2) As a comment here, by Fourier transform, we can define alternatively the free complexification $\widetilde{A}$ as follows, with $z = id \in F[\mathbb{T}]$ being the standard generator:

$$\widetilde{A} =< \tilde{u} >\subset F(\mathbb{T}) * A \quad , \quad \tilde{u} = zu$$

(3) In what regards now $\widetilde{F}(G) = F(\widetilde{G})$, this stands as a definition for $\widetilde{G}$, as a quantum group, the point being that, unless G is trivial, the algebra $F(\widetilde{G})$ is not commutative. We will be back to this, with some examples and illustrations, in what follows.

(4) As for the formula $\widetilde{F}[H] = F[\widetilde{H}]$, with the subgroup $\widetilde{H} \subset \mathbb{Z} * H$ being constructed similarly, by multiplying the generators by z, this is something self-explanatory. Again, we will be back to this, with some examples and illustrations, in what follows. ☐

As a comment here, the above construction is particularly relevant when the ground field is $F = \mathbb{C}$. We will be back to this later in this book, with comments and illustrations. Now still at the general level, we have a result regarding the Haar integration, which is quite similar to what we had before for the usual complexifications, as follows:

THEOREM 4.14. *The Haar integral of a free complexification*

$$\widetilde{A} =< \tilde{u} >\subset F[\mathbb{Z}] * A \quad , \quad \tilde{u} = zu$$

appears as a restriction of a free product, as follows,

$$\int_{\widetilde{A}} = \left(\int_{\mathbb{T}} * \int_A\right)_{\widetilde{A}}$$

and with this happening for left integrals, right integrals, and integrals.

PROOF. This is something quite self-explanatory, and clear from our various results from chapter 3, with the corresponding commuting diagram here being as follows:

$$\begin{array}{ccc} F[\mathbb{Z}] * A & \xrightarrow{\int_{\mathbb{T}} * \int_A} & F \\ \uparrow{\scriptstyle i} & \nearrow{\scriptstyle \int_{\widetilde{A}}} & \\ \widetilde{A} & & \end{array}$$

Thus, exactly as before, when talking about the usual complexifications, and their Haar functionals, we are led to the conclusion in the statement. ☐

The above results raise the question of understanding what are the "intermediate" complexifications, lying between the usual one, and the free one. More on this later.

Moving ahead, we can introduce now our next basic operation on the affine Hopf algebras, which is something quite fundamental, as follows:

THEOREM 4.15. *Given an affine Hopf algebra (A,u), we can construct its projective version (PA,v) by setting*

$$PA=<v_{ia,jb}>\subset A\quad,\quad v_{ia,jb}=u_{ij}\bar{u}_{ab}$$

and as main illustrations for this construction, we have the following formulae:

(1) *$PF(G)=F(PG)$, with $PG=G/(G\cap\mathbb{T}^N)$, when $F=\mathbb{C}$.*
(2) *$PF[H]=F[PH]$, with $PH=<g_ig_j^{-1}>$, assuming $H=<g_i>$.*

PROOF. As before, this is something self-explanatory, the idea being as follows:

(1) Our first claim is that the matrix v in the statement is a corepresentation. But this is something standard, the computation being as follows:

$$\begin{aligned}\Delta(v_{ia,jb}) &= \Delta(u_{ij}\bar{u}_{ab})\\ &= \Delta(u_{ij})\Delta(\bar{u}_{ab})\\ &= \sum_k u_{ik}\otimes u_{kj}\sum_c \bar{u}_{ac}\otimes\bar{u}_{cb}\\ &= \sum_{kc} u_{ik}\bar{u}_{ac}\otimes u_{kj}\bar{u}_{cb}\\ &= \sum_{kc} v_{ia,kc}\otimes v_{kc,jb}\end{aligned}$$

(2) Thus, v is indeed a corepresentation, and so, by the results in chapter 3, the projective version PA, as constructed in the statement, is indeed a Hopf subalgebra.

(3) Before going further with our study, let us mention that the construction in the statement is that of the standard, left projective version. It is possible to talk as well about right projective versions, constructed by using the following corepresentation:

$$w_{ia,jb}=\bar{u}_{ij}u_{ab}$$

In general, the left and right projective versions do not coincide, as one can see with examples coming from algebras of type $A=F[H]$, discussed below. Many other things can be said here, but with the subject being quite technical, we will basically restrict the attention in what follows to the left projective versions, from the statement.

(4) Regarding now the formula $PF(G)=F(PG)$, with $PG=G/(G\cap\mathbb{T}^N)$, this follows from the elementary fact that, via Gelfand duality, the matrix v in the statement is the matrix of coefficients of the adjoint representation of G, whose kernel is the subgroup $G\cap\mathbb{T}^N$, where $\mathbb{T}^N\subset U_N$ denotes the subgroup formed by the diagonal matrices.

(5) So, this was for the idea with $PF(G)=F(PG)$, and in practice, we will leave some study here, both over $F=\mathbb{C}$ and in general, as an instructive exercise.

(6) As for the last formula, namely $PF[H]=F[PH]$, with $PH=<g_ig_j^{-1}>$, assuming $H=<g_i>$, this is something trivial, which comes from definitions. □

We have as well a result regarding the Haar integration, as follows:

THEOREM 4.16. *The Haar integral of a projective version*

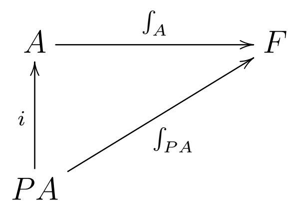
$$PA = < v_{ia,jb} > \subset A \quad , \quad v_{ia,jb} = u_{ij}\bar{u}_{ab}$$

appears as a restriction, as follows,

$$\int_{PA} = \left(\int_A\right)_{PA}$$

and with this happening for left integrals, right integrals, and integrals.

PROOF. This is again something quite self-explanatory, and clear from our results from chapter 3, with the corresponding commuting diagram here being as follows:



Thus, we are led to the conclusion in the statement. □

Importantly now, let us mention that there is an interesting relation here with the notion of free complexification. Indeed, in the context of Theorem 4.13, we have:

THEOREM 4.17. *Given an affine Hopf algebra (A, u), construct its free complexification $(\widetilde{A}, \tilde{u})$, with $\tilde{u} = zu$. We have then an identification*

$$P\widetilde{A} = PA$$

and the same happens at the level of right projective versions.

PROOF. This is something coming from definitions, the idea being as follows:

(1) Let us construct indeed the free complexification $(\widetilde{A}, \tilde{u})$, with $\tilde{u} = zu$, as in Theorem 4.13. The conjugate of $\bar{u}$ is then given by the following formula:

$$\bar{\tilde{u}} = \bar{u}z^{-1}$$

Thus, the adjoint of $\bar{u}$ is given by the following formula:

$$\tilde{v} = zvz^{-1}$$

But this gives an identification $P\widetilde{A} = PA$ as in the statement, by conjugating by z.

(2) Regarding now the right projective versions, as constructed in the proof of Theorem 4.15, things here are in fact even simpler. Indeed, the right adjoint of $\bar{u}$ is given by:

$$\tilde{w} = w$$

Thus, we have a plain equality of right projective versions $P\widetilde{A} = PA$.

(3) Finally, let us mention that the story is not over here, quite the opposite. Indeed, an interesting subsequent question, of categorical and algebraic geometry flavor, is that of understanding if the free complexification $\widetilde{A}$ is the biggest affine Hopf algebra having the same projective version as A. We will discuss this question later in this book, when systematically discussing the relation between affine and projective geometry. □

So long for the basic operations on the Hopf algebras. Many more things can be said, and we will be back to this later, when discussing the representation theory and Haar integration, for the various products of Hopf algebras constructed above.

Finally, let us mention that it is possible to talk as well about wreath products, and free wreath products, in the Hopf algebra setting. However, this is something more tricky, requiring talking about quantum permutations first, and we will do this later in this book, after developing some more theory, in order to talk about quantum permutations.

4c. Intersection, generation

We would like to discuss now a number of further basic operations on the Hopf algebras, again coming from group theory, but which are more subtle, and related to each other, namely the intersection operation, generation operation, and Hopf image operation.

As before with the usual operations, it is convenient in what follows to have in mind the following informal formula, for an arbitrary Hopf algebra A, in terms of certain underlying quantum groups G and H, related by some sort of generalized Pontrjagin duality:

$$A = F(G) = F[H]$$

In terms of such quantum groups, and more specifically of the first ones, $A = F(G)$, the questions that we would like to solve, which are quite natural, are as follows:

PROBLEM 4.18. *Given two quantum subgroups $H, K \subset G$ of a quantum group:*

(1) *How to define their intersection, $H \cap K$?*
(2) *What about the subgroup that they generate, $< H, K >$?*

In order to answer the first question, the best is to start by drawing some diagrams. In the classical case, given a group G and two subgroups $H, K \subset G$, we can indeed intersect these subgroups, and the relevant diagram, which gives $H \cap K$, is as follows:

$$\begin{array}{ccc} G & \longleftarrow & H \\ \uparrow & & \uparrow \\ K & \longleftarrow & H \cap K \end{array}$$

Now at the level of the corresponding algebras of functions, the diagram, having as arrows surjections which are dual to the above inclusions, is as follows:

$$\begin{array}{ccc} F(G) & \longrightarrow & F(H) \\ \downarrow & & \downarrow \\ F(K) & \longrightarrow & F(H\cap K) \end{array}$$

But this is exactly what we need, in order to solve our intersection problem formulated above. Indeed, based on this, we can come up with the following solution:

THEOREM 4.19. *Given two quotient Hopf algebras $A\to B,C$, we can construct the universal Hopf algebra quotient $A\to B\sqcap C$ producing the following diagram:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B\sqcap C \end{array}$$

As an illustration for this, in the group algebra case we have the formula

$$F(H)\sqcap F(K)=F(H\cap K)$$

for any two subgroups of a given group, $H,K\subset G$.

PROOF. We must prove that the universal Hopf algebra in the statement exists indeed. For this purpose, let us pick writings as follows, with I,J being Hopf ideals:

$$B=A/I\quad,\quad C=A/J$$

We can then construct our universal Hopf algebra, as follows:

$$B\sqcap C=A/<I,J>$$

Thus, we are led to the conclusions in the statement. □

In the affine Hopf algebra setting now, that of Definition 4.1, the operation $\sqcap$ constructed above can be usually computed by using the following simple fact:

PROPOSITION 4.20. *Assuming $A\to B,C$, the intersection $B\sqcap C$ is given by*

$$B\sqcap C=A/\{\mathcal{R},\mathcal{P}\}$$

whenever we have writings as follows,

$$B=A/\mathcal{R}\quad,\quad C=A/\mathcal{P}$$

with $\mathcal{R},\mathcal{P}$ being certain sets of polynomial relations between the coordinates u_{ij}.

PROOF. This follows from Theorem 4.19, or rather from its proof, and from the following trivial fact, regarding relations and ideals:

$$I =< \mathcal{R} > \ , \ J =< \mathcal{P} > \quad \implies \quad < I, J >=< \mathcal{R}, \mathcal{P} >$$

Thus, we are led to the conclusion in the statement. □

Finally, let us record as well what happens in the group algebra case:

THEOREM 4.21. *Given two quotient groups $G \to H, K$, we have the formula*

$$F[H] \sqcap F[K] = F[H \sqcap K]$$

with the quotient $G \to [H \sqcap K]$ being the one producing the following diagram:

$$\begin{array}{ccc} G & \longrightarrow & H \\ \downarrow & & \downarrow \\ K & \longrightarrow & H \sqcap K \end{array}$$

Alternatively, we have $[H \sqcap K] = G/ < \ker(G \to H), \ker(G \to K) >$.

PROOF. This is indeed something self-explanatory, with the first assertion coming from Theorem 4.19, and with the second assertion coming from Proposition 4.20. □

Moving on, regarding now the generation operation question, from Problem 4.18 (2), the theory here is quite similar. In the classical case, given a group G and two subgroups $H, K \subset G$, the relevant diagram, which gives the subgroup $< H, K >$, is as follows:

$$\begin{array}{ccc} G & \longleftarrow & H \\ \uparrow & & \downarrow \\ K & \longrightarrow & < H, K > \end{array}$$

Now at the level of the corresponding algebras of functions, the diagram, having as arrows surjections which are dual to the above inclusions, is as follows:

$$\begin{array}{ccc} F(G) & \longrightarrow & F(H) \\ \downarrow & & \uparrow \\ F(K) & \longleftarrow & F(< H, K >) \end{array}$$

But this is exactly what we need, in order to solve our generation problem formulated above. Indeed, based on this, we can come up with the following solution:

THEOREM 4.22. *Given two quotient Hopf algebras $A \to B, C$, we can construct the universal Hopf algebra quotient $A \to [B, C]$ producing the following diagram:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \uparrow \\ C & \longleftarrow & [B,C] \end{array}$$

As an illustration for this, in the group algebra case we have the formula

$$[F(H), F(K)] = F(< H, K >)$$

for any two subgroups of a given group, $H, K \subset G$.

PROOF. We must prove that the universal Hopf algebra in the statement exists indeed. For this purpose, let us pick writings as follows, with I, J being Hopf ideals:

$$B = A/I \quad , \quad C = A/J$$

We can then construct our universal Hopf algebra, as follows:

$$B \sqcap C = A/(I \cap J)$$

Thus, we are led to the conclusions in the statement. □

As a complement to this, let us record as well what happens for group algebras:

THEOREM 4.23. *Given two quotient groups $G \to H, K$, we have the formula*

$$\big[F[H], F[K]\big] = F\big[[H, K]\big]$$

with the quotient $G \to [H, K]$ being the one producing the following diagram:

$$\begin{array}{ccc} G & \longrightarrow & H \\ \downarrow & & \uparrow \\ K & \longleftarrow & [H,K] \end{array}$$

Alternatively, we have $[H, K] = G/(\ker(G \to H) \cap \ker(G \to K))$.

PROOF. This is indeed something self-explanatory, and elementary, with both the assertions coming from Theorem 4.22, and its proof. □

So long for the basics of the intersection and generation operations. The story is of course not over with the above results, because we still have as job, as apprentice or confirmed algebraists, to explore the obvious dual nature of these operations. And skipping some details here, that we will leave as an exercise, the situation is as follows:

FACT 4.24. *In the finite dimensional case, where the quotients $A \to B, C$ correspond to certain subalgebras $B^*, C^* \subset A^*$:*

(1) *The intersection and generation operations on the algebras B, C can be understood in terms of the dual algebras B^*, C^*.*
(2) *For the algebras of functions, and for group algebras too, all this is compatible with known formulae from group theory.*

As a last topic for this section, let us discuss now the connection with the notion of diagonal algebras, introduced earlier in this chapter. As explained there, associated to an affine Hopf algebra is a family of cocommutative diagonal algebras, as follows:

$$A^\Delta = \left\{ A_Q^\delta \middle| Q \in GL_N(F) \right\}$$

Generally speaking, we can expect this collection of diagonal algebras to encode the various algebraic and analytic properties of A. Here is a basic result on this subject:

THEOREM 4.25. *The following hold, over the complex numbers $F = \mathbb{C}$, both for the algebras $A = F(G)$ with $G \subset U_N$, and for the group algebras $A = F[H]$:*

(1) *Injectivity: the construction $A \to A^\Delta$ is injective, in the sense that $A \neq B$ implies $A_Q^\delta \neq B_Q^\delta$, for some $Q \in GL_N(F)$.*
(2) *Monotony: the construction $A \to A^\Delta$ is increasing, in the sense that passing to a quotient $A \to B$ decreases one of the diagonal algebras, $A_Q^\delta \neq B_Q^\delta$.*
(3) *Generation: any quantum group is generated by its tori, or, equivalently, any affine Hopf algebra A has the property $A = [A_Q^\delta | Q \in GL_N(F)]$.*

PROOF. We have two cases to be investigated, as follows:

(1) Assume first that we are in the group algebra case, $A = F(G)$. In order to prove the generation property we use the following formula, established before:

$$T_Q = G \cap Q^{-1}(F^*)^N Q$$

Now since any group element $U \in G$ is unitary, and so diagonalizable by basic linear algebra, we can write, for certain matrices $Q \in U_N$ and $D \in (F^*)^N$:

$$U = Q^{-1} D Q$$

But this shows that we have $U \in T_Q$, for this precise value of the spinning matrix $Q \in U_N$, used in the construction of the standard torus T_Q. Thus we have proved the generation property, and the injectivity and monotony properties follow from this.

(2) Regarding now the group algebras, here everything is trivial. Indeed, when these algebras are diagonally embedded we can take $Q = 1$, and when they are embedded by using a spinning matrix $Q \in GL_N(F)$, we can use precisely this matrix Q. □

Many other things can be said, as a continuation of the above, notably with some further verifications of the above conjectures, say in relation with product operations, and with some more specialized conjectures too. We will be back to this later in this book, when systematically discussing what happens over the complex numbers.

4d. Images and models

In order to further discuss now the generation operation $[\,,]$ introduced above, we will need the following construction, which is something of independent interest:

THEOREM 4.26. *Given a representation $\pi : A \to C$, with C being an associative algebra, there is a smallest Hopf algebra quotient $A \to B$ producing a factorization*

$$\pi : A \to B \to C$$

called Hopf image of π. More generally, given representations $\pi : A \to C_i$, with C_i being algebras, there is a smallest Hopf algebra quotient $A \to B$ producing factorizations

$$\pi_i : A \to B \to C_i$$

called joint Hopf image of the family of representations π_i.

PROOF. This is something quite obvious, obtained by dividing by a suitable ideal:

(1) Let I_π be sum of all Hopf ideals contained in $\ker(\pi)$. It is clear then that I_π is a Hopf ideal, and to be more precise, is the largest Hopf ideal contained in $\ker(\pi)$. Thus, we have the solution to our factorization problem, obtained as follows:

$$B = A/I_\pi$$

(2) As for the second assertion, regarding the Hopf image of an arbitrary family of representations $\pi_i : A \to C_i$, the proof here is similar, again by dividing by a suitable ideal, obtained as the sum of all Hopf ideals contained in all the kernels $\ker(\pi_i)$. □

The above construction might look quite trivial, but under some suitable extra assumptions, such as having the complex numbers as scalars, $F = \mathbb{C}$, a number of more subtle things can be said about it, and this even at the very general level, as follows:

(1) To start with, the Hopf image construction has a very simple description at the Tannakian level, namely "the Hom spaces are those in the model", and this can be taken as a definition for it. But more on this later, when talking Tannakian duality.

(2) As yet another approach, we can talk about idempotent states, and again, we have a simple description of the Hopf image construction, in such terms. But more on this later in this book, when talking Haar integration, and idempotent states.

Before going further, with some applications of the above construction to the computation of the $[\,,]$ operation, let us formulate the following definition, which is something of theoretical interest, that will appear on a regular basis, in what follows:

Definition 4.27. *We say that a representation $\pi : A \to C$ is inner faithful when there is no proper factorization of type*

$$\pi : A \to B \to C$$

that is, when its Hopf image is A itself.

As before with the notion of Hopf image, this might look like a trivial notion, but under a number of some suitable extra assumptions, such as having $F = \mathbb{C}$, some non-trivial things can be said too, including a simple Tannakian description of inner faithfulness, and an indempotent state formula as well. More on this, later in this book.

Getting now to the examples and illustrations, for the notions introduced above, these will come, as usual, from the case of the function algebras $A = F(G)$, and that of the group algebras $A = F[H]$. It is convenient to start with these latter algebras, $A = F[H]$, which are the most illustrating. Regarding them, we have the following result:

Theorem 4.28. *Given a matrix representation of a group algebra*

$$\pi : F[H] \to M_N(F)$$

coming by linearizing from a usual group representation

$$\rho : H \to GL_N(F)$$

the Hopf image factorization of π is obtained by taking the group image

$$\pi : F[H] \to F[\rho(H)] \to M_N(F)$$

and π is inner faithful when ρ is faithful, that is, when $H \subset GL_N(F)$.

Proof. This is something elementary and self-explanatory, with the first assertion coming from definitions, and with the second assertion coming from it. □

The above result is quite interesting, providing us with the key for understanding the Hopf image construction, and the related notion of inner faithfulness. For making things clear here, let us formulate our final conclusion a bit informally, as follows:

Conclusion 4.29. *When regarding the Hopf algebras as being of the form $A = F[H]$, with H being a quantum group:*

1. *The Hopf image construction produces the algebra $A' = F[H']$, with the quantum group H' being the image of H.*
2. *A representation of $A = F[H]$ is inner faithful precisely when the corresponding representation of H is faithful.*

With this discussed, let us get now to our other class of basic examples, the function algebras $A = F(G)$. Here the result is something quite simple as well, as follows:

THEOREM 4.30. *Given a diagonal representation of a function algebra*

$$\pi : F(G) \to M_N(F) \quad , \quad f \to \begin{pmatrix} f(g_1) & & \\ & \ddots & \\ & & f(g_K) \end{pmatrix}$$

coming from an arbitrary family of group elements, as follows,

$$g_1, \ldots, g_K \in G$$

the Hopf image factorization of π is obtained by taking the generated subgroup

$$\pi : F(G) \to F(< g_1, \ldots, g_K >) \to M_N(F)$$

and π is inner faithful when we have $G =< g_1, \ldots, g_K >$.

PROOF. This is again something elementary and self-explanatory, with the first assertion coming from definitions, and with the second assertion coming from it. □

So long for the notions of Hopf image, and inner faithfulness. There are of course some other examples too, and we will be back to this later. Also, as already mentioned, there is some further general theory to be developed too, under suitable extra assumptions, in relation with Tannakian duals and Haar integration, and we will come back to this later too. For the moment, what we have in the above, which is quite illustrating, will do.

Now by getting back to the generation operation $[\,,]$, as introduced before, we have the following result about it, which is something very useful, in practice:

THEOREM 4.31. *Assuming $A \to B, C$, the Hopf algebra $[B, C]$ is such that*

$$A \to [B, C] \to B, C$$

is the joint Hopf image of the following quotient maps:

$$A \to B, C$$

A similar result holds for an arbitrary family of quotients $A \to B_i$.

PROOF. In the particular case from the statement, the joint Hopf image appears as the smallest Hopf algebra quotient D producing factorizations as follows:

$$A \to D \to B, C$$

We conclude from this that we have $D = [B, C]$, as desired. As for the extension to the case of an arbitrary family of quotients $A \to B_i$, this is straightforward. □

As an application of the above Hopf image technology, let us discuss now matrix modeling questions. Let us start with something very basic, as follows:

Definition 4.32. *A matrix model for an affine Hopf algebra (A,u) is a morphism of associative algebras as follows, with T being a certain space:*

$$\pi : A \to M_K(F(T)) \quad , \quad u_{ij} \to U_{ij}$$

When this morphism π is an inclusion, we say that our model is faithful.

Obviously, this notion is potentially something quite useful. In practice now, we would like of course our matrix models to be faithful, and this in order for our computations inside the random matrix algebra $M_K(F(T))$ to be relevant, to our questions regarding A. In fact, this is why we chose above to use random matrix algebras $M_K(F(T))$ instead of plain matrix algebras $M_K(F)$, as to have more chances to have faithfulness.

But, the problem is that this situation is not always possible, due to a number of analytic reasons, the idea here being that the random matrix algebras $M_K(F(T))$ are quite "thin", from a certain functional analytic viewpoint, while the algebra A to be modeled might be "thick", from the same functional analytic viewpoint.

We will discuss more in detail such things later in this book, when talking about $F=\mathbb{C}$, and various analytic aspects. In the meantime, however, we certainly do have a problem, that is quite clear, and the point is that the notion of inner faithfulness from Definition 4.27 provides us with a potential solution to this problem, as follows:

Definition 4.33. *We say that a matrix model as above,*

$$\pi : A \to M_K(F(T)) \quad , \quad u_{ij} \to U_{ij}$$

is inner faithful when there is no Hopf algebra factorization as follows:

$$\pi : A \to B \to M_K(F(T)) \quad , \quad u_{ij} \to v_{ij} \to U_{ij}$$

That is, we can use our inner faithfulness notion, for the matrix models.

And the point now is that, with this notion in hand, we can model far more affine Hopf algebras than before, with the above-mentioned analytic obstructions dissapearing. In fact, there is no known obstruction on the algebras A than can be modeled as above.

Still in relation with the faithfulness problematics for models, let us record as well:

Theorem 4.34. *Given an arbitrary matrix model, as before,*

$$\pi : A \to M_K(F(T)) \quad , \quad u_{ij} \to U_{ij}$$

we can always factorize it via a smallest Hopf algebra, as follows,

$$\pi : A \to B \to M_K(F(T)) \quad , \quad u_{ij} \to v_{ij} \to U_{ij}$$

and the resulting factorized model $B \to M_K(F(T))$ is then inner faithful.

Proof. This is indeed something self-explanatory, coming by dividing by a suitable ideal, with the result itself being a particular case of Theorem 4.26. □

Summarizing, we have some interesting theory going on, for the matrix models. In practice now, in order to reach to something concrete, out of this, far more work is needed, and we will discuss this later in this book. In the meantime, let us formulate:

CONCLUSION 4.35. *In relation with the matrix models $\pi : A \to M_K(F(T))$:*

1. *The notion of inner faithful model is the good one, perfectly remembering A, and allowing us to model far more algebras A, than with usual faithfulness.*
2. *In general, we can also use the Hopf image technology in order to construct new Hopf algebras, by taking the Hopf image of an arbitrary model.*

And we will end this chapter, and the present opening Part I, with this. Good conclusions that we have here, waiting to be explored. We will be back to this, later.

4e. Exercises

We had a lot of interesting algebra in this chapter, and as exercises, we have:

EXERCISE 4.36. *Further study the projective versions.*

EXERCISE 4.37. *Further study the free complexifications.*

EXERCISE 4.38. *Work out some basic examples of intersections.*

EXERCISE 4.39. *Work out some basic examples of generations.*

EXERCISE 4.40. *Clarify the missing details for the Hopf image illustrations.*

EXERCISE 4.41. *Work out some basic examples of matrix models.*

As bonus exercise, figure out what happens to all the above when $F = \mathbb{C}$.

Part II

Compact groups

I'm going to Jackson
I'm gonna mess around
Yeah, I'm going to Jackson
Look out Jackson town

CHAPTER 5

Quantum groups

5a. Quantum spaces

Welcome to quantum groups. In this first part of the present book we discuss the construction and basic properties of the main quantum permutation and rotation groups.

The story here involves the foundational papers of Woronowicz [**99**], [**100**], from the end of the 80s, then the key paper of Wang [**92**], from the mid 90s, then my own papers from the late 90s and early 00s, and finally some more specialized papers from the mid and late 00s, including [**9**], [**12**], [**17**], containing a few fundamentals too.

In short, cavern man mathematics from about 20 years ago, but lots of things to be learned. We will provide here 100 pages on the subject, with a decent presentation of what is known about the main quantum groups, of fundamental type, coming in the form of theorems accompanied by short proofs. For further details on all this, you have my graduate textbook on quantum groups [**5**], along with the original papers cited above.

Getting started now, at the beginning of everything, we have:

Question 5.1 (Connes). *What is a quantum permutation group?*

This question is more tricky than it might seem. For solving it you need a good formalism of quantum groups, and there is a bewildering number of choices here, with most of these formalisms leading nowhere, in connection with the above question. So, we are into philosophy, and for truly getting started, we have to go back in time, with:

Question 5.2 (Heisenberg). *What is a quantum space?*

Regarding this latter question, there are as many answers as quantum physicists, starting with Heisenberg himself in the early 1920s, then Schrödinger and Dirac short after, with each coming with his own answer to the question. Not to forget Einstein, who labeled all these solutions as "nice, but probably fundamentally wrong".

In short, we are now into controversy, and a look at more modern physics does not help much, with the controversy basically growing instead of diminishing, over the time. So, in the lack of a good answer, let us take as starting point something nice and mathematical, rather agreed upon in the 1930s, coming from Dirac's work, namely:

Answer 5.3 (von Neumann). *A quantum space is the dual of an operator algebra.*

Fast forward now to the 90s and to Connes' question, this remains something non-trivial, even when knowing what a quantum space is, and this for a myriad technical reasons. You have to work a bit on that question, try all sorts of things which do not work, until you hit the good answer. With this good answer being as follows:

Answer 5.4 (Wang). *The quantum permutation group S_N^+ is the biggest compact quantum group acting on $\{1,\ldots,N\}$, by leaving the counting measure invariant.*

To be more precise, the idea is that $\{1,\ldots,N\}$ has all sorts of quantum permutations, and even when restricting the attention to the "correct" ones, namely those leaving invariant the counting measure, there is still an infinity of such quantum permutations, and the quantum group formed by this infinity of quantum permutations is compact.

This was for the story of the subject, very simplified, and as a final ingredient, two answers to two natural questions that you might have:

(1) Isn't the conclusion $|S_N^+|=\infty$ a bit too speculatory, not to say crazy? Certainly not, I would say, because in quantum mechanics particles do not have clear positions and speeds, and once you're deep into this viewpoint, "think quantum", a bit fuzzy about everything, why the set $\{1,\ldots,N\}$ not being allowed to have an infinity of quantum permutations, after all. So, no contradiction, philosophically speaking.

(2) Why was the theory of S_N^+ developed so late? Good question, and in answer, looking retrospectively, quantum groups and permutations should have been developed by von Neumann and Weyl, sometimes in the 1940s, perhaps with some help from Gelfand. But that never happened. As for the story after WW2, with mathematics, physics, and mankind in general: that was sex, drugs and rock and roll, forget about it.

Getting started now for good, we have the whole remainder of this chapter for understanding what Question 1.1 is about, and what its Answer 1.4 says. But before that, Question 1.2 and Answer 1.3 coming first. Leaving aside physics, we must first talk about operator algebras, and the starting definition here is as follows:

Definition 5.5. *A C^*-algebra is a complex algebra A, having a norm $||.||$ making it a Banach algebra, and an involution $*$, related to the norm by the formula*

$$||aa^*||=||a||^2$$

which must hold for any $a\in A$.

As a basic example, the algebra $M_N(\mathbb{C})$ of the complex $N\times N$ matrices is a C^*-algebra, with the usual matrix norm and involution of matrices, namely:

$$||M||=\sup_{||x||=1}||Mx|| \quad , \quad (M^*)_{ij}=\bar{M}_{ji}$$

More generally, any $*$-subalgebra $A \subset M_N(\mathbb{C})$ is automatically closed, and so is a C^*-algebra. In fact, in finite dimensions, the situation is as follows:

Proposition 5.6. *The finite dimensional C^*-algebras are exactly the algebras*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with norm $||(a_1, \ldots, a_k)|| = \sup_i ||a_i||$, and involution $(a_1, \ldots, a_k)^* = (a_1^*, \ldots, a_k^*)$.

Proof. In one sense this is clear. In the other sense, this comes by splitting the unit of our algebra A as a sum of central minimal projections, $1 = p_1 + \ldots + p_k$. Indeed, when doing so, each of the $*$-algebras $A_i = p_i A p_i$ follows to be a matrix algebra, $A_i \simeq M_{n_i}(\mathbb{C})$, and this gives the direct sum decomposition in the statement. □

In general now, a main theoretical result about C^*-algebras, due to Gelfand, Naimark and Segal, and called GNS representation theorem, is as follows:

Theorem 5.7. *Given a complex Hilbert space H, finite dimensional or not, the algebra $B(H)$ of linear operators $T : H \to H$ which are bounded, in the sense that*

$$||T|| = \sup_{||x||=1} ||Tx||$$

is finite, is a C^*-algebra, with the above norm, and with involution given by:

$$< Tx, y >=< x, T^*y >$$

More generally, and norm closed $*$-subalgebra of this full operator algebra

$$A \subset B(H)$$

is a C^*-algebra. Any C^*-algebra appears in this way, for a certain Hilbert space H.

Proof. There are several statements here, with the first ones being standard operator theory, and with the last one being the GNS theorem, the idea being as follows:

(1) First of all, the full operator algebra $B(H)$ is a Banach algebra. Indeed, given a Cauchy sequence $\{T_n\}$ inside $B(H)$, we can set $Tx = \lim_{n\to\infty} T_n x$, for any $x \in H$. It is then routine to check that we have $T \in B(H)$, and that $T_n \to T$ in norm.

(2) Regarding the involution, the point is that we must have $< Tx, y >=< x, T^*y >$, for a certain vector $T^*y \in H$. But this can serve as a definition for T^*, and the fact that T^* is indeed linear, and bounded, with the bound $||T^*|| = ||T||$, is routine. As for the formula $||TT^*|| = ||T||^2$, this is elementary as well, coming by double inequality.

(3) Finally, the fact that any C^*-algebra appears as $A \subset B(H)$, for a certain Hilbert space H, is advanced. The idea is that each $a \in A$ acts on A by multiplication, $T_a(b) = ab$. Thus, we are more or less led to the result, provided that we are able to convert our algebra A, regarded as a complex vector space, into a Hilbert space $H = L^2(A)$. But this latter conversion can be done, by taking some inspiration from abstract measure theory. □

As a third and last basic result about C^*-algebras, which will be of particular interest for us, we have the following well-known theorem of Gelfand:

THEOREM 5.8. *Given a compact space X, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a C^*-algebra, with norm and involution as follows:*

$$||f|| = \sup_{x\in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

This algebra is commutative, and any commutative C^*-algebra A is of this form, with $X = Spec(A)$ appearing as the space of Banach algebra characters $\chi : A \to \mathbb{C}$.

PROOF. Once again, there are several statements here, some of them being trivial, and some of them being advanced, the idea being as follows:

(1) First of all, the fact that $C(X)$ is indeed a Banach algebra is clear, because a uniform limit of continuous functions is continuous.

(2) Regarding now for the formula $||ff^*|| = ||f||^2$, this is something trivial for functions, because on both sides we obtain $\sup_{x\in X} |f(x)|^2$.

(3) Given a commutative C^*-algebra A, the character space $X = \{\chi : A \to \mathbb{C}\}$ is compact, and we have an evaluation morphism $ev : A \to C(X)$.

(4) The tricky point, which follows from basic spectral theory in Banach algebras, is to prove that ev is indeed isometric. This gives the last assertion. □

In what follows, we will be mainly using Definition 1.5 and Theorem 1.8, as general framework. To be more precise, in view of Theorem 1.8, let us formulate:

DEFINITION 5.9. *Given an arbitrary C^*-algebra A, we agree to write*

$$A = C(X)$$

and call the abstract space X a compact quantum space.

In other words, we can define the category of compact quantum spaces X as being the category of the C^*-algebras A, with the arrows reversed. A morphism $f : X \to Y$ corresponds by definition to a morphism $\Phi : C(Y) \to C(X)$, a product of spaces $X \times Y$ corresponds by definition to a product of algebras $C(X) \otimes C(Y)$, and so on.

All this is of course quite speculative, and as a first result regarding these compact quantum spaces, coming from Proposition 1.6, we have:

PROPOSITION 5.10. *The finite quantum spaces are exactly the disjoint unions of type*

$$X = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

where M_n is the finite quantum space given by $C(M_n) = M_n(\mathbb{C})$.

Proof. This is a reformulation of Proposition 1.6, by using the above philosophy. Indeed, for a compact quantum space X, coming from a C^*-algebra A via the formula $A = C(X)$, being finite can only mean that the following number is finite:

$$|X| = \dim_{\mathbb{C}} A < \infty$$

Thus, by using Proposition 1.6, we are led to the conclusion that we must have:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

But since direct sums of algebras A correspond to disjoint unions of quantum spaces X, via the correspondence $A = C(X)$, this leads to the conclusion in the statement. □

This was for the basic theory of C^*-algebras, the idea being that we have some basic operator theory results, that can be further learned from any standard book, such as Blackadar [31], and then we can talk about reformulations of these results in quantum space terms, by using Definition 1.9 and some basic common sense.

Finally, no discussion would be complete without a word about the von Neumann algebras. These are operator algebras of more advanced type, as follows:

Theorem 5.11. *For a $*$-algebra $A \subset B(H)$ the following conditions are equivalent, and if they are satisfied, we say that A is a von Neumann algebra:*

1. *A is closed with respect to the weak topology, making each $T \to Tx$ continuous.*
2. *A is equal to its algebraic bicommutant, $A = A''$, computed inside $B(H)$.*

As basic examples, we have the algebras $A = L^\infty(X)$, acting on $H = L^2(X)$. Such algebras are commutative, any any commutative von Neumann algebra is of this form.

Proof. There are several assertions here, the idea being as follows:

(1) The equivalence (1) $\iff$ (2) is the well-known bicommutant theorem of von Neumann, which can be proved by using an amplification trick, $H \to \mathbb{C}^N \otimes H$.

(2) Given a measured space X, we have indeed an emdedding $L^\infty(X) \subset B(L^2(X))$, with weakly closed image, given by $T_f : g \to fg$, as in the proof of the GNS theorem.

(3) Given a commutative von Neumann algebra $A \subset B(H)$ we can write $A =< T >$ with T being a normal operator, and the Spectral Theorem gives $A \simeq L^\infty(X)$. □

In the context of a C^*-algebra representation $A \subset B(H)$ we can consider the weak closure, or bicommutant $A'' \subset B(H)$, which is a von Neumann algebra. In the commutative case, $C(X) \subset B(L^2(X))$, the weak closure is $L^\infty(X)$. In general, we agree to write:

$$A'' = L^\infty(X)$$

For more on all this, basic theory of the C^*-algebras and von Neumann algebras, we refer to any standard operator algebra book, such as Blackadar [31].

5b. Quantum groups

We are ready now to introduce the compact quantum groups. The axioms here, due to Woronowicz [99], and slightly modified for our present purposes, are as follows:

Definition 5.12. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

define morphisms of C^*-algebras $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$ and $S : A \to A^{opp}$, called comultiplication, counit and antipode.

In this definition the tensor product needed for Δ can be any C^*-algebra tensor product. In order to get rid of redundancies, coming from this and from amenability issues, we will divide everything by an equivalence relation, as follows:

Definition 5.13. *We agree to identify two Woronowicz algebras, $(A, u) = (B, v)$, when we have an isomorphism of $*$-algebras*

$$< u_{ij} > \simeq < v_{ij} >$$

mapping standard coordinates to standard coordinates, $u_{ij} \to v_{ij}$.

We say that A is cocommutative when $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip. We have then the following key result, from [99], providing us with examples:

Proposition 5.14. *The following are Woronowicz algebras, which are commutative, respectively cocommutative:*

(1) *$C(G)$, with $G \subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi) = \left[(g, h) \to \varphi(gh)\right] \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S(\varphi) = \left[g \to \varphi(g^{-1})\right]$$

(2) *$C^*(\Gamma)$, with $F_N \to \Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. In both cases, we first have to exhibit a certain matrix u, and then prove that we have indeed a Woronowicz algebra. The constructions are as follows:

(1) For the first assertion, we can use the matrix $u = (u_{ij})$ formed by the standard matrix coordinates of G, which is by definition given by:

$$g = \begin{pmatrix} u_{11}(g) & \dots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \dots & u_{NN}(g) \end{pmatrix}$$

(2) For the second assertion, we can use the diagonal matrix formed by generators:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, regarding the last assertion, in the commutative case this follows from the Gelfand theorem, and in the cocommutative case, we will be back to this. □

In order to get now to quantum groups, we will need as well:

Proposition 5.15. *Assuming that $G \subset U_N$ is abelian, we have an identification of Woronowicz algebras $C(G) = C^*(\Gamma)$, with Γ being the Pontrjagin dual of G:*

$$\Gamma = \left\{ \chi : G \to \mathbb{T} \right\}$$

Conversely, assuming that $F_N \to \Gamma$ is abelian, we have an identification of Woronowicz algebras $C^*(\Gamma) = C(G)$, with G being the Pontrjagin dual of Γ:

$$G = \left\{ \chi : \Gamma \to \mathbb{T} \right\}$$

Thus, the Woronowicz algebras which are both commutative and cocommutative are exactly those of type $A = C(G) = C^*(\Gamma)$, with G, Γ being abelian, in Pontrjagin duality.

Proof. This follows from the Gelfand theorem applied to $C^*(\Gamma)$, and from the fact that the characters of a group algebra come from the characters of the group. □

In view of this result, and of the findings from Proposition 1.14 too, we have the following definition, complementing Definition 1.12 and Definition 1.13:

Definition 5.16. *Given a Woronowicz algebra, we write it as follows, and call G a compact quantum Lie group, and Γ a finitely generated discrete quantum group:*

$$A = C(G) = C^*(\Gamma)$$

Also, we say that G, Γ are dual to each other, and write $G = \widehat{\Gamma}, \Gamma = \widehat{G}$.

Let us discuss now some tools for studying the Woronowicz algebras, and the underlying quantum groups. First, we have the following result:

Proposition 5.17. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

(2) *S satisfies the antipode axiom, on the $*$-algebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

PROOF. As a first observation, the result holds in the commutative case, $A = C(G)$ with $G \subset U_N$. Indeed, here we know from Proposition 1.14 that Δ, ε, S appear as functional analytic transposes of the multiplication, unit and inverse maps m, u, i:

$$\Delta = m^t \quad , \quad \varepsilon = u^t \quad , \quad S = i^t$$

Thus, in this case, the various conditions in the statement on Δ, ε, S simply come by transposition from the group axioms satisfied by m, u, i, namely:

$$m(m \times id) = m(id \times m)$$

$$m(u \times id) = m(id \times u) = id$$

$$m(i \times id)\delta = m(id \times i)\delta = 1$$

Here $\delta(g) = (g, g)$. Observe also that the result holds as well in the cocommutative case, $A = C^*(\Gamma)$ with $F_N \to \Gamma$, trivially. In general now, the first axiom follows from:

$$(\Delta \otimes id)\Delta(u_{ij}) = (id \otimes \Delta)\Delta(u_{ij}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

As for the other axioms, the verifications here are similar. □

In order to reach to more advanced results, the idea will be that of doing representation theory. Following Woronowicz [99], let us start with the following definition:

DEFINITION 5.18. *Given (A, u), we call corepresentation of it any unitary matrix $v \in M_n(\mathcal{A})$, with $\mathcal{A} =< u_{ij} >$, satisfying the same conditions as u, namely:*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

We also say that v is a representation of the underlying compact quantum group G.

In the commutative case, $A = C(G)$ with $G \subset U_N$, we obtain in this way the finite dimensional unitary smooth representations $v : G \to U_n$, via the following formula:

$$v(g) = \begin{pmatrix} v_{11}(g) & \ldots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \ldots & v_{nn}(g) \end{pmatrix}$$

In the cocommutative case, $A = C^*(\Gamma)$ with $F_N \to \Gamma$, we will see in a moment that we obtain in this way the formal sums of elements of Γ, possibly rotated by a unitary. As a first result now regarding the corepresentations, we have:

PROPOSITION 5.19. *The corepresentations are subject to the following operations:*

1. *Making sums, $v + w = diag(v, w)$.*
2. *Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.*
3. *Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.*
4. *Rotating by a unitary, $v \to UvU^*$.*

PROOF. We first check the fact that the matrices in the statement are unitaries:

(1) The fact that $v+w$ is unitary is clear.

(2) Regarding now $v\otimes w$, this can be written in standard leg-numbering notation as $v\otimes w=v_{13}w_{23}$, and with this interpretation in mind, the unitarity is clear.

(3) In order to check that $\bar{v}$ is unitary, we can use the antipode. Indeed, by regarding the antipode as an antimultiplicative map $S:A\to A$, we have:

$$(\bar{v}v^t)_{ij}=\sum_k v_{ik}^*v_{jk}=\sum_k S(v_{kj}^*v_{ki})=S((v^*v)_{ji})=\delta_{ij}$$

$$(v^t\bar{v})_{ij}=\sum_k v_{ki}v_{kj}^*=\sum_k S(v_{jk}v_{ik}^*)=S((vv^*)_{ji})=\delta_{ij}$$

(4) Finally, the fact that UvU^* is unitary is clear. As for the verification of the comultiplicativity axioms, involving Δ,ε,S, this is routine, in all cases. □

As a consequence of the above result, we can formulate:

DEFINITION 5.20. *We denote by $u^{\otimes k}$, with $k=\circ\bullet\bullet\circ\ldots$ being a colored integer, the various tensor products between $u,\bar{u}$, indexed according to the rules*

$$u^{\otimes\emptyset}=1\quad,\quad u^{\otimes\circ}=u\quad,\quad u^{\otimes\bullet}=\bar{u}$$

and multiplicativity, $u^{\otimes kl}=u^{\otimes k}\otimes u^{\otimes l}$, and call them Peter-Weyl corepresentations.

Here are a few examples of such corepresentations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes\circ\circ}=u\otimes u\quad,\quad u^{\otimes\circ\bullet}=u\otimes\bar{u}$$

$$u^{\otimes\bullet\circ}=\bar{u}\otimes u\quad,\quad u^{\otimes\bullet\bullet}=\bar{u}\otimes\bar{u}$$

In order to do representation theory, we first need to know how to integrate over G. And we have here the following key result, due to Woronowicz [99]:

THEOREM 5.21. *Any Woronowicz algebra $A=C(G)$ has a unique Haar integration,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, by linearity we can assume that a is the coefficient of certain corepresentation, $a = (\tau \otimes id)v$. But in this case, an elementary computation gives the following formula, with P_φ being the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)v$:

$$\left(id \otimes \int_\varphi\right) v = P_\varphi$$

(2) Since $v\xi = \xi$ implies $[(id \otimes \varphi)v]\xi = \xi$, we have $P_\varphi \geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v) = \left\{\xi \in \mathbb{C}^n \Big| v\xi = \xi\right\}$$

The point now is that when $\varphi \in A^*$ is faithful, by using a standard positivity trick, we can prove that we have $P_\varphi = P$. Assume indeed $P_\varphi \xi = \xi$, and let us set:

$$a = \sum_i \left(\sum_j v_{ij}\xi_j - \xi_i\right)\left(\sum_k v_{ik}\xi_k - \xi_i\right)^*$$

A straightforward computation shows then that $\varphi(a) = 0$, and so $a = 0$, as desired.

(3) With this in hand, the left and right invariance of $\int_G = \int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

We can now develop a Peter-Weyl type theory for the corepresentations, in analogy with the theory from the classical case. We will need:

DEFINITION 5.22. *Given two corepresentations $v \in M_n(A), w \in M_m(A)$, we set*

$$Hom(v,w) = \left\{T \in M_{m\times n}(\mathbb{C}) \Big| Tv = wT\right\}$$

and we use the following conventions:

1. *We use the notations $Fix(v) = Hom(1,v)$, and $End(v) = Hom(v,v)$.*
2. *We write $v \sim w$ when $Hom(v,w)$ contains an invertible element.*
3. *We say that v is irreducible, and write $v \in Irr(G)$, when $End(v) = \mathbb{C}1$.*

In the classical case, where $A = C(G)$ with $G \subset U_N$ being a closed subgroup, we obtain in this way the usual notions regarding the representation intertwiners. Observe also that in the group dual case we have $g \sim h$ when $g = h$. Finally, observe that $v \sim w$ means that v, w are conjugated by an invertible matrix.

Here are now a few basic results, regarding the above linear spaces:

Proposition 5.23. *We have the following results:*

(1) $T\in Hom(u,v),S\in Hom(v,w)\implies ST\in Hom(u,w)$.
(2) $S\in Hom(u,v),T\in Hom(w,z)\implies S\otimes T\in Hom(u\otimes w,v\otimes z)$.
(3) $T\in Hom(v,w)\implies T^*\in Hom(w,v)$.

In other words, the Hom spaces form a tensor $*$-category.

Proof. The proofs are all elementary, as follows:

(1) Assume indeed that we have $Tu=vT$, $Sv=Ws$. We obtain, as desired:

$$STu=SvT=wST$$

(2) Assuming that we have $Su=vS$, $Tw=zT$, we obtain, as desired:

$$(S\otimes T)(u\otimes w)=(Su)_{13}(Tw)_{23}=(vS)_{13}(zT)_{23}=(v\otimes z)(S\otimes T)$$

(3) By conjugating, and then using the unitarity of v,w, we obtain:

$$\begin{aligned}Tv=wT\quad&\implies\quad v^*T^*=T^*w^*\\&\implies\quad vv^*T^*w=vT^*w^*w\\&\implies\quad T^*w=vT^*\end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3), the Hom spaces are linear spaces, and contain the units. □

Finally, in order to formulate the Peter-Weyl results, we will need as well:

Proposition 5.24. *The characters of the corepresentations, given by*

$$\chi_v=\sum_i v_{ii}$$

behave as follows, in respect to the various operations:

$$\chi_{v+w}=\chi_v+\chi_w\quad,\quad\chi_{v\otimes w}=\chi_v\chi_w\quad,\quad\chi_{\bar{v}}=\chi_v^*$$

In addition, given two equivalent corepresentations, $v\sim w$, we have $\chi_v=\chi_w$.

Proof. The three formulae in the statement are all clear from definitions. Regarding now the last assertion, assuming that we have $v=T^{-1}wT$, we obtain:

$$\chi_v=Tr(v)=Tr(T^{-1}wT)=Tr(w)=\chi_w$$

We conclude that $v\sim w$ implies $\chi_v=\chi_w$, as claimed. □

Consider the dense $*$-subalgebra $\mathcal{A}\subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$<a,b>=\int_G ab^*$$

With this convention, we have the following fundamental result, from [99]:

Theorem 5.25. *We have the following Peter-Weyl type results:*

(1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
(2) *Each irreducible corepresentation appears inside a certain* $u^{\otimes k}$.
(3) $\mathcal{A}=\bigoplus_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$*, the summands being pairwise orthogonal.*
(4) *The characters of irreducible corepresentations form an orthonormal system.*

Proof. All these results are from Woronowicz [99], the idea being as follows:

(1) Given a corepresentation $v\in M_n(A)$, we know from Proposition 1.23 that $End(v)$ is a finite dimensional C^*-algebra, and by using Proposition 1.6, we obtain:

$$End(v)=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

But this decomposition allows us to define subcorepresentations $v_i\subset v$, which are irreducible, so we obtain, as desired, a decomposition $v=v_1+\ldots+v_k$.

(2) To any corepresentation $v\in M_n(A)$ we associate its space of coefficients, given by $C(v)=span(v_{ij})$. The construction $v\to C(v)$ is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A}=\sum_{k\in\mathbb{N}*\mathbb{N}}C(u^{\otimes k})$$

Now given an arbitrary corepresentation $v\in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v)\subset\mathcal{A}$, and so we must have, for certain positive integers $k_1,\ldots,k_p$, an inclusion of vector spaces, as follows:

$$C(v)\subset C(u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p})$$

Thus we have $v\subset u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p}$, and by (1) we obtain the result.

(3) As a first observation, which follows from an elementary computation, for any two corepresentations v,w we have a Frobenius type isomorphism, as follows:

$$Hom(v,w)\simeq Fix(\bar{v}\otimes w)$$

Now assume $v\not\sim w$, and let us set $P_{ia,jb}=\int_G v_{ij}w_{ab}^*$. According to Theorem 1.21, the matrix P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(\bar{v},\bar{w})=\{0\}$$

Thus we have $P=0$, and so $C(v)\perp C(w)$, which gives the result.

(4) The fact that the characters form indeed an orthogonal system follows from (3). Regarding now the norm 1 assertion, consider the following integrals:

$$P_{ik,jl}=\int_G v_{ij}v_{kl}^*$$

We know from Theorem 1.21 that these integrals form the orthogonal projection onto $Fix(v \otimes \bar{v}) \simeq End(\bar{v}) = \mathbb{C}1$. By using this fact, we obtain the following formula:

$$\int_G \chi_v \chi_v^* = \sum_{ij} \int_G v_{ii} v_{jj}^* = \sum_i \frac{1}{N} = 1$$

Thus the characters have indeed norm 1, and we are done. □

Observe that in the cocommutative case, we obtain from (4) that our algebra must be of the form $A = C^*(\Gamma)$, for some discrete group Γ, as mentioned in Proposition 1.14. As another consequence of the above results, following Woronowicz [99], we have:

THEOREM 5.26. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $\Phi : A_{red} \to A_{full}$, by slicing the map $\widetilde{\Delta} : A_{red} \to A_{red} \otimes A_{full}$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some functional analysis. See [99]. □

With these results in hand, we can formulate, as a refinement of Definition 1.16:

DEFINITION 5.27. *Given a Woronowicz algebra A, we formally write as before*

$$A = C(G) = C^*(\Gamma)$$

and by GNS construction with respect to the Haar functional, we write as well

$$A'' = L^\infty(G) = L(\Gamma)$$

with G being a compact quantum group, and Γ being a discrete quantum group.

Now back to Theorem 1.26, as in the discrete group case, the most interesting criterion for amenability, leading to some interesting mathematics and physics, is the Kesten one, (4) there. This leads us into computing character laws:

Theorem 5.28. *Given a Woronowicz algebra (A,u), consider its main character:*

$$\chi=\sum_i u_{ii}$$

1. *The moments of χ are the numbers $M_k=\dim(Fix(u^{\otimes k}))$.*
2. *When $u\sim\bar{u}$ the law of χ is a real measure, supported by $\sigma(\chi)$.*
3. *The notion of coamenability of A depends only on $law(\chi)$.*

Proof. All this follows from the above results, the idea being as follows:

(1) This follows indeed from Peter-Weyl theory.

(2) When $u\sim\bar{u}$ we have $\chi=\chi^*$, which gives the result.

(3) This follows from Theorem 1.26 (4), and from (2) applied to $u+\bar{u}$. □

This was for the basic theory of compact and discrete quantum groups. For more on all this, we refer to Woronowicz [99] and related papers, or to the book [5].

5c. Quantum rotations

We know so far that the compact quantum groups include the usual compact Lie groups, $G\subset U_N$, and the abstract duals $G=\widehat{\Gamma}$ of the finitely generated groups $F_N\to\Gamma$. We can combine these examples by performing basic operations, as follows:

Proposition 5.29. *The class of Woronowicz algebras is stable under taking:*

1. *Tensor products, $A=A'\otimes A''$, with $u=u'+u''$. At the quantum group level we obtain usual products, $G=G'\times G''$ and $\Gamma=\Gamma'\times\Gamma''$.*
2. *Free products, $A=A'*A''$, with $u=u'+u''$. At the quantum group level we obtain dual free products $G=G'\,\hat{*}\,G''$ and free products $\Gamma=\Gamma'*\Gamma''$.*

Proof. Everything here is clear from definitions. In addition to this, let us mention as well that we have $\int_{A'\otimes A''}=\int_{A'}\otimes\int_{A''}$ and $\int_{A'*A''}=\int_{A'}*\int_{A''}$. Also, the corepresentations of the products can be explicitely computed. See Wang [92]. □

Here are some further basic operations, once again from Wang [92]:

Proposition 5.30. *The class of Woronowicz algebras is stable under taking:*

1. *Subalgebras $A'=<u'_{ij}>\subset A$, with u' being a corepresentation of A. At the quantum group level we obtain quotients $G\to G'$ and subgroups $\Gamma'\subset\Gamma$.*
2. *Quotients $A\to A'=A/I$, with I being a Hopf ideal, $\Delta(I)\subset A\otimes I+I\otimes A$. At the quantum group level we obtain subgroups $G'\subset G$ and quotients $\Gamma\to\Gamma'$.*

Proof. Once again, everything is clear, and we have as well some straightforward supplementary results, regarding integration and corepresentations. See [92]. □

Finally, here are two more operations, which are of key importance:

PROPOSITION 5.31. *The class of Woronowicz algebras is stable under taking:*

(1) *Projective versions, $PA =< w_{ia,jb} >\subset A$, where $w = u \otimes \bar{u}$. At the quantum group level we obtain projective versions, $G \to PG$ and $P\Gamma \subset \Gamma$.*
(2) *Free complexifications, $\widetilde{A} =< zu_{ij} >\subset C(\mathbb{T}) * A$. At the quantum group level we obtain free complexifications, denoted $\widetilde{G}$ and $\widetilde{\Gamma}$.*

PROOF. This is clear from the previous results. For details here, we refer to [92]. □

Once again following Wang [92] and related papers, let us discuss now a number of truly "new" quantum groups, obtained by liberating. We first have:

THEOREM 5.32. *The following universal algebras are Woronowicz algebras,*

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \bar{u}, u^t = u^{-1}\right)$$

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^* = u^{-1}, u^t = \bar{u}^{-1}\right)$$

so the underlying quantum spaces O_N^+, U_N^+ are compact quantum groups.

PROOF. This comes from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$(u^\Delta)_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij} = \delta_{ij} \quad , \quad (u^S)_{ij} = u_{ji}^*$$

Thus we can define Δ, ε, S by using the universal property of $C(O_N^+)$, $C(U_N^+)$. □

Now with this done, we can look for various intermediate subgroups $O_N \subset O_N^\times \subset O_N^+$ and $U_N \subset U_N^\times \subset U_N^+$. Following [17], a basic construction here is as follows:

THEOREM 5.33. *The following quotient algebras are Woronowicz algebras,*

$$C(O_N^*) = C(O_N^+)\Big/\left\langle abc = cba\Big|\forall a, b, c \in \{u_{ij}\}\right\rangle$$

$$C(U_N^*) = C(U_N^+)\Big/\left\langle abc = cba\Big|\forall a, b, c \in \{u_{ij}, u_{ij}^*\}\right\rangle$$

so the underlying quantum spaces O_N^*, U_N^* are compact quantum groups.

PROOF. This follows as in the proof of Theorem 1.32, because if the entries of u satisfy the half-commutation relations $abc = cba$, then so do the entries of $u^\Delta, u^\varepsilon, u^S$. □

Obviously, there are many more things that can be done here, with the above constructions being just the tip of the iceberg. But instead of discussing this, let us first verify that Theorem 1.32 and Theorem 1.33 provide us indeed with new quantum groups. For this purpose, we can use the notion of diagonal torus, which is as follows:

PROPOSITION 5.34. *Given a closed subgroup $G \subset U_N^+$, consider its diagonal torus, which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

PROOF. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

We can now distinguish between our various quantum groups, as follows:

THEOREM 5.35. *The diagonal tori of the basic unitary quantum groups, namely*

$$\begin{array}{ccccc}
U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
\end{array}$$

are the following discrete group duals,

$$\begin{array}{ccccc}
\widehat{\mathbb{Z}^N} & \longrightarrow & \widehat{\mathbb{Z}^{\circ N}} & \longrightarrow & \widehat{F_N} \\
\uparrow & & \uparrow & & \uparrow \\
\widehat{\mathbb{Z}_2^N} & \longrightarrow & \widehat{\mathbb{Z}_2^{\circ N}} & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
\end{array}$$

with $\circ$ standing for the half-classical product operation for groups.

PROOF. This is clear for U_N^+, where on the diagonal we obtain the biggest possible group dual, namely $\widehat{F_N}$. For the other quantum groups this follows by taking quotients, which correspond to taking quotients as well, at the level of the groups $\Lambda = \widehat{T}$. □

As a consequence of the above result, the quantum groups that we have are indeed distinct. There are many more things that can be said about these quantum groups, and about further versions of these quantum groups that can be constructed. More later.

5d. Quantum permutations

Eventually. Following Wang [92], let us discuss now the construction and basic properties of the quantum permutation group S_N^+. Let us first look at S_N. We have:

PROPOSITION 5.36. *Consider the symmetric group S_N, viewed as permutation group of the N coordinate axes of $\mathbb{R}^N$. The coordinate functions on $S_N \subset O_N$ are given by*

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

and the matrix $u = (u_{ij})$ that these functions form is magic, in the sense that its entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column.

PROOF. The action of S_N on the standard basis $e_1, \ldots, e_N \in \mathbb{R}^N$ being given by $\sigma : e_j \to e_{\sigma(j)}$, this gives the formula of u_{ij} in the statement. As for the fact that the matrix $u = (u_{ij})$ that these functions form is magic, this is clear. □

With a bit more effort, we obtain the following nice characterization of S_N:

THEOREM 5.37. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

PROOF. The universal algebra A in the statement being commutative, by the Gelfand theorem it must be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates form a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 entry on each row and each column, and so $X = S_N$. Thus we have proved the first assertion, and the second assertion is clear as well. □

Following now Wang [92], we can liberate S_N, as follows:

THEOREM 5.38. *The following universal C^*-algebra, with magic meaning as usual formed by projections ($p^2 = p^* = p$), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = \text{magic}\right)$$

is a Woronowicz algebra, with comultiplication, counit and antipode given by:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

Thus the space S_N^+ is a compact quantum group, called quantum permutation group.

PROOF. As a first observation, the universal C^*-algebra in the statement is indeed well-defined, because the conditions $p^2 = p^* = p$ satisfied by the coordinates give:

$$||u_{ij}|| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps Δ, ε, S given by the formulae in the statement. Consider the following matrices:

$$u^\Delta_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u^\varepsilon_{ij} = \delta_{ij} \quad , \quad u^S_{ij} = u_{ji}$$

Our claim is that, since u is magic, so are these three matrices. Indeed, regarding u^Δ, its entries are idempotents, as shown by the following computation:

$$(u^\Delta_{ij})^2 = \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj} = \sum_{kl} \delta_{kl}u_{ik} \otimes \delta_{kl}u_{kj} = u^\Delta_{ij}$$

These elements are self-adjoint as well, as shown by the following computation:

$$(u^\Delta_{ij})^* = \sum_k u^*_{ik} \otimes u^*_{kj} = \sum_k u_{ik} \otimes u_{kj} = u^\Delta_{ij}$$

The row and column sums for the matrix u^Δ can be computed as follows:

$$\sum_j u^\Delta_{ij} = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1$$

$$\sum_i u^\Delta_{ij} = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1$$

Thus, u^Δ is magic. Regarding now u^ε, u^S, these matrices are magic too, and this for obvious reasons. Thus, all our three matrices $u^\Delta, u^\varepsilon, u^S$ are magic, so we can define Δ, ε, S by the formulae in the statement, by using the universality property of $C(S_N^+)$. □

Our first task now is to make sure that Theorem 1.38 produces indeed new quantum groups, which do not collapse to S_N. Following Wang [**92**], we have:

THEOREM 5.39. *We have an embedding $S_N \subset S_N^+$, given at the algebra level by:*

$$u_{ij} \to \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

This is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. The fact that we have indeed an embedding as above follows from Theorem 1.37. Observe that in fact more is true, because Theorems 1.37 and 1.38 give:

$$C(S_N) = C(S_N^+) \Big/ \Big\langle ab = ba \Big\rangle$$

Thus, the inclusion $S_N \subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N=2$. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+=S_2$.

Case $N=3$. By using the same argument as in the $N=2$ case, and the symmetries of the problem, it is enough to check that u_{11},u_{22} commute. But this follows from:

$$\begin{aligned}u_{11}u_{22}&=u_{11}u_{22}(u_{11}+u_{12}+u_{13})\\&=u_{11}u_{22}u_{11}+u_{11}u_{22}u_{13}\\&=u_{11}u_{22}u_{11}+u_{11}(1-u_{21}-u_{23})u_{13}\\&=u_{11}u_{22}u_{11}\end{aligned}$$

Indeed, by applying the involution to this formula, we obtain that we have as well $u_{22}u_{11}=u_{11}u_{22}u_{11}$. Thus, we obtain $u_{11}u_{22}=u_{22}u_{11}$, as desired.

Case $N=4$. Consider the following matrix, with p,q being projections:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and we can choose $p,q\in B(H)$ as for the algebra $<p,q>$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N\geq 5$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well. □

The above result is quite surprising. How on Earth can the set $\{1,2,3,4\}$ have an infinity of quantum permutations, and will us be able to fully understand this, one day. But do not worry, the remainder of the present book will be here for that.

As a first observation, as a matter of doublechecking our findings, we are not wrong with our formalism, because as explained once again in [**92**], we have as well:

THEOREM 5.40. *The quantum permutation group S_N^+ acts on the set $X = \{1, \ldots, N\}$, the corresponding coaction map $\Phi : C(X) \to C(X) \otimes C(S_N^+)$ being given by:*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

In fact, S_N^+ is the biggest compact quantum group acting on X, by leaving the counting measure invariant, in the sense that $(tr \otimes id)\Phi = tr(.)1$, where $tr(e_i) = \frac{1}{N}, \forall i$.

PROOF. Our claim is that given a compact matrix quantum group G, the following formula defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u = (u_{ij})$ is a magic corepresentation of $C(G)$:

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

Indeed, let us first determine when Φ is multiplicative. We have:

$$\Phi(e_i)\Phi(e_k) = \sum_{jl} e_j e_l \otimes u_{ji} u_{lk} = \sum_j e_j \otimes u_{ji} u_{jk}$$

$$\Phi(e_i e_k) = \delta_{ik} \Phi(e_i) = \delta_{ik} \sum_j e_j \otimes u_{ji}$$

We conclude that the multiplicativity of Φ is equivalent to the following conditions:

$$u_{ji} u_{jk} = \delta_{ik} u_{ji} \quad , \quad \forall i, j, k$$

Similarly, Φ is unital when $\sum_i u_{ji} = 1$, $\forall j$. Finally, the fact that Φ is a $*$-morphism translates into $u_{ij} = u_{ij}^*$, $\forall i, j$. Summing up, in order for $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ to be a morphism of C^*-algebras, the elements u_{ij} must be projections, summing up to 1 on each row of u. Regarding now the preservation of the trace, observe that we have:

$$(tr \otimes id)\Phi(e_i) = \frac{1}{N} \sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements u_{ij} sum up to 1 on each of the columns of u. We conclude from this that $\Phi(e_i) = \sum_j e_j \otimes u_{ji}$ is a morphism of C^*-algebras preserving the trace precisely when u is magic, and since the coaction conditions on Φ are equivalent to the fact that u must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. □

As a technical comment here, the invariance of the counting measure is a key assumption in Theorem 1.40, in order to have an universal object S_N^+. That is, this condition is automatic for classical group actions, but not for quantum group actions, and when dropping it, there is no universal object of type S_N^+. This explains the main difficulty behind Question 1.1, and the credit for this discovery goes to Wang [**92**].

In order to study now S_N^+, we can use the technology that we have, which gives:

THEOREM 5.41. *The quantum groups S_N^+ have the following properties:*

(1) *We have $S_N^+\,\hat{*}\,S_M^+\subset S_{N+M}^+$, for any N,M.*
(2) *In particular, we have an embedding $\widehat{D_\infty}\subset S_4^+$.*
(3) *$S_4\subset S_4^+$ are distinguished by their spinned diagonal tori.*
(4) *If $\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}\to\Gamma$, with $N=\sum N_i$, then $\widehat{\Gamma}\subset S_N^+$.*
(5) *The quantum groups S_N^+ with $N\geq 5$ are not coamenable.*
(6) *The half-classical version $S_N^*=S_N^+\cap O_N^*$ collapses to S_N.*

PROOF. These results follow from what we have, the proofs being as follows:

(1) If we denote by u,v the fundamental corepresentations of $C(S_N^+),C(S_M^+)$, the fundamental corepresentation of $C(S_N^+\,\hat{*}\,S_M^+)$ is by definition:

$$w=\begin{pmatrix}u&0\\0&v\end{pmatrix}$$

But this matrix is magic, because both u,v are magic, and this gives the result.

(2) This result, which refines our $N=4$ trick from the proof of Theorem 1.39, follows from (1) with $N=M=2$. Indeed, we have the following computation:

$$\begin{aligned}S_2^+\,\hat{*}\,S_2^+&=S_2\,\hat{*}\,S_2=\mathbb Z_2\,\hat{*}\,\mathbb Z_2\\&\simeq\widehat{\mathbb Z_2}\,\hat{*}\,\widehat{\mathbb Z_2}=\widehat{\mathbb Z_2*\mathbb Z_2}\\&=\widehat{D_\infty}\end{aligned}$$

(3) Observe first that $S_4\subset S_4^+$ are not distinguished by their diagonal torus, which is $\{1\}$ for both of them. However, according to the Peter-Weyl theory applied to the group duals, the group dual $\widehat{D_\infty}\subset S_4^+$ from (2) must be a subgroup of the diagonal torus of (S_4^+,FuF^*), for a certain unitary $F\in U_4$, and this gives the result.

(4) This result, which generalizes (2), can be deduced as follows:

$$\begin{aligned}\widehat{\Gamma}&\subset\widehat{\mathbb Z_{N_1}*\ldots*\mathbb Z_{N_k}}=\widehat{\mathbb Z_{N_1}}\,\hat{*}\,\ldots\,\hat{*}\,\widehat{\mathbb Z_{N_k}}\\&\simeq\mathbb Z_{N_1}\,\hat{*}\,\ldots\,\hat{*}\,\mathbb Z_{N_k}\subset S_{N_1}\,\hat{*}\,\ldots\,\hat{*}\,S_{N_k}\\&\subset S_{N_1}^+\,\hat{*}\,\ldots\,\hat{*}\,S_{N_k}^+\subset S_N^+\end{aligned}$$

(5) This follows from (4), because at $N=5$ the dual of the group $\Gamma=\mathbb Z_2*\mathbb Z_3$, which is well-known not to be amenable, embeds into S_5^+. As for the general case, that of S_N^+ with $N\geq 5$, here the result follows by using the embedding $S_5^+\subset S_N^+$.

(6) We must prove that $S_N^*=S_N^+\cap O_N^*$ is classical. But here, we can use the fact that for a magic matrix, the entries on each row sum up to 1. Indeed, by making c vary over a full row of u, we obtain $abc=cba\implies ab=ba$, as desired. □

The above results are all quite interesting, notably with (2) providing us with a better understanding of why S_4^+ is infinite, and with (4) telling us that S_5^+ is not only infinite, but just huge. We have as well (6), suggesting that S_N^+ might be the only liberation of S_N. We will be back to these observations, with further results, in due time.

5e. Exercises

Exercises:

Exercise 5.42.

Exercise 5.43.

Exercise 5.44.

Exercise 5.45.

Exercise 5.46.

Exercise 5.47.

Exercise 5.48.

Exercise 5.49.

Bonus exercise.

CHAPTER 6

Diagrams, easiness

6a. Some philosophy

We have seen the definition and basic properties of S_N^+, and a number of more advanced results as well, such as the non-isomorphism of $S_N\subset S_N^+$ at $N\geq 4$, obtained by using suitable group duals $\widehat{\Gamma}\subset S_N^+$. It is possible to further build along these lines, but all this remains quite amateurish. For strong results, we must do representation theory.

So, let us first go back to the general closed subgroups $G\subset U_N^+$. We have seen in chapter 1 that such quantum groups have a Haar measure, and that by using this, a Peter-Weyl theory can be developed for them. However, all this is just a beginning, and many more things can be said, at the general level, which are all useful. We will present now this material, and go back afterwards to our problems regarding S_N^+.

Let us start with a claim, which is quite precise, and advanced, and which will stand as a guiding principle for this chapter, and in fact for the remainder of this book:

Claim 6.1. *Given a closed subgroup $G\subset_u U_N^+$, no matter what you want to do with it, of algebraic or analytic type, you must compute the following spaces:*

$$F_k=Fix(u^{\otimes k})$$

Moreover, for most questions, the computation of the dimensions $M_k=\dim F_k$, which are the moments of the main character $\chi=\sum_i u_{ii}$, will do.

This might look like a quite bold claim, so let us explain this. Assuming first that you are interested in doing representation theory for G, you will certainly run into the spaces F_k, via Peter-Weyl theory. In fact, Peter-Weyl tells you that the irreducible representations appear as $r\subset u^{\otimes k}$, so for finding them, you must compute the algebras $C_k=End(u^{\otimes k})$. But the knowledge of these algebras C_k is more or less the same thing as the knowledge of the spaces F_k, due to Frobenius duality, as follows:

Proposition 6.2. *Given a closed subgroup $G\subset_u U_N^+$, consider the following spaces:*

$$F_k=Fix(u^{\otimes k})\quad,\quad C_k=End(u^{\otimes k})\quad,\quad C_{kl}=Hom(u^{\otimes k},u^{\otimes l})$$

Then knowing the sequence $\{F_k\}$ is the same as knowing the double sequence $\{C_{kl}\}$, and in the case $1\in u$, this is the same as knowing the sequence $\{C_k\}$.

PROOF. In the particular case of the Peter-Weyl corepresentations, the Frobenius isomorphism $Hom(v,w) \simeq Fix(\bar{v} \otimes w)$, that we know from chapter 1, reads:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l}) = Fix(u^{\otimes \bar{k}l}) = F_{\bar{k}l}$$

But this gives the equivalence in the statement. Regarding now the last assertion, assuming $1 \in u$ we have $1 \in u^{\otimes k}$ for any colored integer k, and so:

$$F_k = Hom(1, u^{\otimes k}) \subset Hom(u^{\otimes k}, u^{\otimes k}) = C_k$$

Thus the spaces F_k can be identified inside the algebras C_k, and we are done. □

Summarizing, we have now good algebraic motivations for Claim 2.1. Before going further, however, let us point out that looking at Proposition 2.2 leads us a bit into a dillema, on which spaces are the best to use. And the traditional answer here is that the spaces C_{kl} are the best, due to Tannakian duality, which is as follows:

THEOREM 6.3. *The following operations are inverse to each other:*

(1) *The construction $G \to C$, which associates to a closed subgroup $G \subset_u U_N^+$ the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
(2) *The construction $C \to G$, associating to a tensor category C the closed subgroup $G \subset_u U_N^+$ coming from the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. This is something quite deep, going back to Woronowicz [100] in a slightly different form, and to Malacarne [71] in the simplified form above. The idea is that we have indeed a construction $G \to C_G$, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of finite dimensional Hilbert spaces, as follows:

$$(C_G)_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

We have as well a construction $C \to G_C$, obtained by setting:

$$C(G_C) = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C_{kl} \right\rangle$$

Regarding now the bijection claim, some elementary algebra shows that $C = C_{G_C}$ implies $G = G_{C_G}$, and that $C \subset C_{G_C}$ is automatic. Thus we are left with proving:

$$C_{G_C} \subset C$$

But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See Malacarne [71]. □

The above result is something quite abstract, yet powerful. We will see applications of it in a moment, in the form of Brauer theorems for U_N, O_N, S_N and U_N^+, O_N^+, S_N^+.

All this is very good, providing us with strong motivations for Claim 2.1. However, algebra is of course not everything, and we must comment now on analysis as well. As an analyst you would like to know how to integrate over G, and here, we have:

THEOREM 6.4. *The integration over $G \subset_u U_N^+$ is given by the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in D_k}\delta_\pi(i)\delta_\sigma(j)W_k(\pi,\sigma)$$

for any colored integer $k=e_1\ldots e_k$ and indices i,j, where D_k is a linear basis of $Fix(u^{\otimes k})$,

$$\delta_\pi(i)=<\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>$$

and $W_k=G_k^{-1}$, with $G_k(\pi,\sigma)=<\pi,\sigma>$.

PROOF. We know from chapter 1 that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(u^{\otimes k})=span(D_k)$$

Consider now the following linear map, with $D_k=\{\xi_k\}$ being as in the statement:

$$E(x)=\sum_{\pi\in D_k}<x,\xi_\pi>\xi_\pi$$

By a standard linear algebra computation, it follows that we have $P=WE$, where W is the inverse on $span(T_\pi|\pi\in D_k)$ of the restriction of E. But this restriction is the linear map given by G_k, and so W is the linear map given by W_k, and this gives the result. □

As a conclusion, regardless on whether you're an algebraist or an analyst, if you want to study $G\subset_u U_N$ you are led into the computation of the spaces $F_k=Fix(u^{\otimes k})$. However, the story is not over here, because you might say that you are a functional analyst, interested in the fine analytic properties of the dual $\Gamma=\widehat{G}$. But here, I would strike back with the following statement, based on the Kesten amenability criterion:

PROPOSITION 6.5. *Given a closed subgroup $G\subset_u U_N^+$, consider its main character:*

$$\chi=\sum_i u_{ii}$$

1. *The moments of χ are the numbers $M_k=\dim(Fix(u^{\otimes k}))$.*
2. *When $u\sim\bar{u}$ the law of χ is a real measure, supported by $\sigma(\chi)$.*
3. *The notion of amenability of $\Gamma=\widehat{G}$ depends only on $law(\chi)$.*

PROOF. This is something that we know from chapter 1, the idea being that (1) comes from Peter-Weyl theory, that (2) comes from $u\sim\bar{u}\implies\chi=\chi^*$, and that (3) comes from the Kesten amenability criterion, and from (2) applied to $u+\bar{u}$. □

Finally, you might argue that you are in fact a pure mathematician interested in the combinatorial beauty of the dual $\Gamma=\widehat{G}$. But I have an answer to this too, as follows, again urging you to look at the spaces $F_k=Fix(u^{\otimes k})$, before getting into Γ:

Proposition 6.6. *Consider a closed subgroup $G \subset_u U_N^+$, and assume, by enlarging if necessary u, that we have $1 \in u = \bar{u}$. The formula*

$$d(v,w) = \min\left\{k \in \mathbb{N}\Big| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

defines then a distance on $Irr(G)$, which coincides with the geodesic distance on the associated Cayley graph. Moreover, the moments of the main character,

$$\int_G \chi^k = \dim\left(Fix(u^{\otimes k})\right)$$

count the loops based at 1, having lenght k, on the corresponding Cayley graph.

Proof. Observe first the result holds indeed in the group dual case, where the Woronowicz algebra is $A = C^*(\Gamma)$, with $\Gamma =< S >$ being a finitely generated discrete group. In general, the fact that the lengths are finite follows from Peter-Weyl theory. The symmetry axiom is clear as well, and the triangle inequality is elementary to establish too. Finally, the last assertion, regarding the moments, is elementary too. □

As a conclusion, looks like I won the debate, with Claim 2.1 reigning over both the compact and discrete quantum group worlds, without opposition. Before getting further, let us record a result in relation with the second part of that claim, as follows:

Theorem 6.7. *Given a closed subgroup $G \subset_u U_N^+$, the law of its main character*

$$\chi = \sum_i u_{ii}$$

with respect to the Haar integration has the following properties:

1. *The moments of χ are the numbers $M_k = \dim(Fix(u^{\otimes k}))$.*
2. *M_k counts the lenght k loops at 1, on the Cayley graph of $\Gamma = \widehat{G}$.*
3. *$law(\chi)$ is the Kesten measure of the discrete quantum group $\Gamma = \widehat{G}$.*
4. *When $u \sim \bar{u}$ the law of χ is a usual measure, supported on $[-N, N]$.*
5. *$\Gamma = \widehat{G}$ is amenable precisely when $N \in supp(law(Re(\chi)))$.*
6. *Any inclusion $G \subset_u H \subset_v U_N^+$ must decrease the numbers M_k.*
7. *Such an inclusion is an isomorphism when $law(\chi_u) = law(\chi_v)$.*

Proof. All this is very standard, coming from the Peter-Weyl theory developed by Woronowicz in [**99**], and explained in chapter 1, the idea being as follows:

(1) This comes from the Peter-Weyl type theory, which tells us the number of fixed points of $v = u^{\otimes k}$ can be recovered by integrating the character $\chi_v = \chi_u^k$.

(2) This is something true, and well-known, for $G = \widehat{\Gamma}$ with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group. In general, the proof is quite similar.

(3) This is actually the definition of the Kesten measure, in the case $G = \widehat{\Gamma}$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group. In general, this follows from (2).

(4) The equivalence $u \sim \bar{u}$ translates into $\chi_u = \chi_u^*$, and this gives the first assertion. As for the support claim, this follows from $uu^* = 1 \implies ||u_{ii}|| \leq 1$, for any i.

(5) This is the Kesten amenability criterion, which can be established as in the group dual case, $G = \widehat{\Gamma}$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group.

(6) This is something elementary, which follows from (1), and from the fact that the inclusions of closed subgroups of U_N^+ decrease the spaces of fixed points.

(7) This follows by using (6), and the Peter-Weyl type theory, the idea being that if $G \subset H$ is not injective, then it must strictly decrease one of the spaces $Fix(u^{\otimes k})$. $\square$

As a conclusion to all this, somewhat improving Claim 2.1, given a closed subgroup $G \subset_u U_N^+$, regardless of our precise motivations, be that algebra, analysis or other, computing the law of $\chi = \sum_i u_{ii}$ is the "main problem" to be solved. Good to know.

6b. Diagrams, easiness

Let us discuss now the representation theory of S_N^+, and the computation of the law of the main character. Our main result here, which will be something quite conceptual, will be the fact that $S_N \subset S_N^+$ is a liberation of "easy quantum groups".

Looking at what has been said above, as a main tool, at the general level, we only have Tannakian duality. So, inspired by that, and following [**17**], let us formulate:

DEFINITION 6.8. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A collection of sets*

$$D = \bigsqcup_{k,l} D(k,l)$$

with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi, \sigma) \to [\pi\sigma]$.*
2. *Stability under the vertical concatenation, $(\pi, \sigma) \to [^\sigma_\pi]$.*
3. *Stability under the upside-down turning, $\pi \to \pi^*$.*
4. *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
5. *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

As a basic example, we have the category of all partitions P itself. Other basic examples are the category of pairings P_2, and the categories NC, NC_2 of noncrossing partitions, and pairings. We have as well the category $\mathcal{P}_2$ of pairings which are "matching", in the sense that they connect $\circ - \circ$, $\bullet - \bullet$ on the vertical, and $\circ - \bullet$ on the horizontal, and its subcategory $\mathcal{NC}_2 \subset \mathcal{P}_2$ consisting of the noncrossing matching pairings.

There are many other examples, and we will be back to this. Following [**17**], the relation with the Tannakian categories and duality comes from:

Proposition 6.9. *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

As for the composition and involution axioms, their proof is similar. □

In relation with quantum groups, we have the following result, from [**17**]:

Theorem 6.10. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and so a closed subgroup $G_N\subset_u U_N^+$.

Proof. Let call C_{kl} the spaces on the right. By using the axioms in Definition 2.8, and the categorical properties of the operation $\pi\to T_\pi$, from Proposition 2.9, we see that $C=(C_{kl})$ is a Tannakian category. Thus Theorem 2.3 applies, and gives the result. □

We can now formulate a key definition, as follows:

Definition 6.11. *A compact quantum group G_N is called easy when we have*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

for any colored integers k,l, for a certain category of partitions $D\subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool. In relation now with quantum permutation groups, and with the orthogonal and unitary quantum groups too, here is our main result, coming from [5], [17]:

THEOREM 6.12. *The basic quantum permutation and rotation groups,*

$$\begin{array}{ccccc} S_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N & \longrightarrow & U_N \end{array}$$

are all easy, the corresponding categories of partitions being as follows,

$$\begin{array}{ccccc} NC & \longleftarrow & NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P & \longleftarrow & P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

with 2 *standing for pairings, NC for noncrossing, and calligraphic for matching.*

PROOF. This is something quite fundamental, the proof being as follows:

(1) The quantum group U_N^+ is defined via the following relations:

$$u^* = u^{-1} \quad , \quad u^t = \bar{u}^{-1}$$

But, by doing some elementary computations, these relations tell us precisely that the following two operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi = \overset{\cap}{\circ\bullet} \ , \ \overset{\cap}{\bullet\circ}$$

Thus, the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =< \overset{\cap}{\circ\bullet} \ , \ \overset{\cap}{\bullet\circ} >= \mathcal{NC}_2$$

(2) The subgroup $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi = \begin{smallmatrix}\circ\\|\\\bullet\end{smallmatrix} \ , \ \begin{smallmatrix}\bullet\\|\\\circ\end{smallmatrix}$$

We conclude that the Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =< \mathcal{NC}_2, \begin{smallmatrix}\circ\\|\\\bullet\end{smallmatrix}, \begin{smallmatrix}\bullet\\|\\\circ\end{smallmatrix} >= NC_2$$

(3) The subgroup $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0 \quad , \quad [u_{ij}, \bar{u}_{kl}] = 0$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi = \slashoverback \, , \, \slashoverback$$

Thus the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D =< \mathcal{NC}_2, \slashoverback, \slashoverback >= \mathcal{P}_2$$

(4) In order to deal now with O_N, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us that O_N is indeed easy, coming from:

$$D =< NC_2, \mathcal{P}_2 >= P_2$$

(5) We know that the subgroup $S_N^+ \subset O_N^+$ appears as follows:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle u = \text{magic} \Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y \in P(2,1)$$

Given a corepresentation u, we have the following formulae:

$$(T_Y u^{\otimes 2})_{i,jk} = \sum_{lm} (T_Y)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

$$(u T_Y)_{i,jk} = \sum_l u_{il} (T_Y)_{l,jk} = \delta_{jk} u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle T_Y \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D =< Y >= NC$$

(6) Finally, in order to deal with S_N, we can use the following formula:

$$S_N = S_N^+ \cap O_N$$

At the categorical level, this tells us that S_N is indeed easy, coming from:

$$D =< NC, P_2 >= P$$

Thus, we are led to the conclusions in the statement. $\square$

The above result is something quite deep, and we will see in what follows countless applications of it. As a first such application, which is rather philosophical, we have:

THEOREM 6.13. *The constructions $G_N \to G_N^+$ with $G = U, O, S$ are easy quantum group liberations, in the sense that they come from the construction*

$$D \to D \cap NC$$

at the level of the associated categories of partitions.

PROOF. This is clear indeed from Theorem 2.12, and from the following trivial equalities, connecting the categories found there:

$$\mathcal{NC}_2 = \mathcal{P}_2 \cap NC \quad , \quad NC_2 = P_2 \cap NC \quad , \quad NC = P \cap NC$$

Thus, we are led to the conclusion in the statement. □

The above result is quite nice, because the various constructions $G_N \to G_N^+$ that we saw in chapter 1, although natural, were something quite ad-hoc. Now all this is no longer ad-hoc, and the next time that we will have to liberate a subgroup $G_N \subset U_N$, we know what the recipe is, namely check if G_N is easy, and if so, simply define $G_N^+ \subset U_N^+$ as being the easy quantum group coming from the category $D = D_G \cap NC$.

6c. Laws of characters

Let us discuss now some more advanced applications of Theorem 2.12, this time to the computation of the law of the main character, in the spirit of Claim 2.1. First, we have the following result, valid in the general easy quantum group setting:

PROPOSITION 6.14. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$, the moments of the main character are given by*

$$\int_{G_N} \chi^k = \dim\left(span\left(\xi_\pi\Big|\pi \in D(k)\right)\right)$$

where $D(k) = D(\emptyset, k)$, and with the notation $\xi_\pi = T_\pi$, for partitions $\pi \in D(k)$.

PROOF. According to the Peter-Weyl theory, and to the definition of easiness, the moments of the main character are given by the following formula:

$$\begin{aligned}\int_{G_N} \chi^k &= \int_{G_N} \chi_{u^{\otimes k}} \\ &= \dim\left(Fix(u^{\otimes k})\right) \\ &= \dim\left(span\left(\xi_\pi\Big|\pi \in D(k)\right)\right)\end{aligned}$$

Thus, we obtain the formula in the statement. □

With the above result in hand, you would probably say very nice, so in practice, this is just a matter of counting the partitions appearing in Theorem 2.12, and then recovering the measures having these numbers as moments. However, this is wrong, because such a computation would lead to a law of χ which is independent on $N \in \mathbb{N}$, and for the classical groups at least, S_N, O_N, U_N, we obviously cannot have such a result.

The mistake comes from the fact that the vectors ξ_π are not necessarily linearly independent. Let us record this finding, which will be of key importance for us:

Conclusion 6.15. *The vectors associated to the partitions $\pi \in P(k)$, namely*

$$\xi_\pi = \sum_{i_1 \dots i_k} \delta_\pi(i_1, \dots, i_k)\, e_{i_1} \otimes \dots \otimes e_{i_k}$$

are not linearly independent, with this making the main character moments for S_N,

$$\int_{S_N} \chi^k = \dim\left(span\left(\xi_\pi \Big| \pi \in P(k)\right)\right)$$

depend on $N \in \mathbb{N}$. Moreover, the same phenomenon happens for O_N, U_N.

All this suggests by doing some linear algebra for the vectors ξ_π, but this looks rather complicated, and let's keep that for later. What we can do right away, instead, is that of studying S_N with alternative, direct techniques. And here we have:

Theorem 6.16. *Consider the symmetric group S_N, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices.*

1. *The main character $\chi \in C(S_N)$, defined as usual as $\chi = \sum_i u_{ii}$, counts the number of fixed points, $\chi(\sigma) = \#\{i|\sigma(i) = i\}$.*
2. *The probability for a permutation $\sigma \in S_N$ to be a derangement, meaning to have no fixed points at all, becomes, with $N \to \infty$, equal to $1/e$.*
3. *The law of the main character $\chi \in C(S_N)$ becomes with $N \to \infty$ the Poisson law $p_1 = \frac{1}{e}\sum_k \delta_k/k!$, with respect to the counting measure.*

Proof. This is something very classical, the proof being as follows:

(1) We have indeed the following computation, which gives the result:

$$\chi(\sigma) = \sum_i u_{ii}(\sigma) = \sum_i \delta_{\sigma(i)i} = \#\left\{i\Big|\sigma(i) = i\right\}$$

(2) We use the inclusion-exclusion principle. Consider the following sets:

$$S_N^i = \left\{\sigma \in S_N \Big| \sigma(i) = i\right\}$$

The probability that we are interested in is then given by:

$$\begin{aligned}
P(\chi=0) &= \frac{1}{N!}\left(|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^i\cap S_N^j|-\sum_{i<j<k}|S_N^i\cap S_N^j\cap S_N^k|+\ldots\right)\\
&= \frac{1}{N!}\sum_{r=0}^N(-1)^r\sum_{i_1<\ldots<i_r}(N-r)!\\
&= \frac{1}{N!}\sum_{r=0}^N(-1)^r\binom{N}{r}(N-r)!\\
&= \sum_{r=0}^N\frac{(-1)^r}{r!}
\end{aligned}$$

Since we have here the expansion of $1/e$, this gives the result.

(3) This follows by generalizing the computation in (2). To be more precise, a similar application of the inclusion-exclusion principle gives the following formula:

$$\lim_{N\to\infty}P(\chi=k)=\frac{1}{k!e}$$

Thus, we obtain in the limit a Poisson law of parameter 1, as stated. $\square$

The above result is quite interesting, and tells us what to do next. As a first goal, we can try to recover (3) there by using Proposition 2.14, and easiness. Then, once this understood, we can try to look at S_N^+, and then at O_N,U_N and O_N^+,U_N^+ too, with the same objective, namely finding $N\to\infty$ results for the law of χ, using easiness.

So, back to Proposition 2.14 and Conclusion 2.15, and we have now to courageously attack the main problem, namely the linear independence question for the vectors ξ_π. This will be quite technical. Let us begin with some standard combinatorics:

Definition 6.17. *Let $P(k)$ be the set of partitions of $\{1,\ldots,k\}$, and $\pi,\sigma\in P(k)$.*

1. *We write $\pi\leq\sigma$ if each block of π is contained in a block of σ.*
2. *We let $\pi\vee\sigma\in P(k)$ be the partition obtained by superposing π,σ.*

Also, we denote by $|.|$ the number of blocks of the partitions $\pi\in P(k)$.

As an illustration here, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and we have:

$$||\leq\sqcap$$

Also, at $k=3$ we have $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!\!\mid\;,|\sqcap,\sqcap\!\sqcap\}$, and the order relation is as follows:

$$|||\ \leq\ \sqcap|\ ,\ \sqcap\!\!\!\!\!\mid\;,\ |\sqcap\ \leq\ \sqcap\!\sqcap$$

In relation with our linear independence questions, the idea will be that of using:

Proposition 6.18. *The Gram matrix of the vectors ξ_π is given by the formula*

$$< \xi_\pi, \xi_\sigma >= N^{|\pi\vee\sigma|}$$

where $\vee$ is the superposition operation, and $|.|$ is the number of blocks.

Proof. According to the formula of the vectors ξ_π, we have:

$$\begin{aligned} < \xi_\pi, \xi_\sigma > &= \sum_{i_1\ldots i_k} \delta_\pi(i_1,\ldots,i_k)\delta_\sigma(i_1,\ldots,i_k) \\ &= \sum_{i_1\ldots i_k} \delta_{\pi\vee\sigma}(i_1,\ldots,i_k) \\ &= N^{|\pi\vee\sigma|} \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to study the Gram matrix $G_k(\pi,\sigma) = N^{|\pi\vee\sigma|}$, and more specifically to compute its determinant, we will use several standard facts about the partitions. We have:

Definition 6.19. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi,\sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\sum_{\pi\leq\tau<\sigma}\mu(\pi,\tau) & \text{if } \pi < \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, for $P(2) = \{||, \sqcap\}$, we have by definition:

$$\mu(||, ||) = \mu(\sqcap, \sqcap) = 1$$

Also, $|| < \sqcap$, with no intermediate partition in between, so we obtain:

$$\mu(||, \sqcap) = -\mu(||, ||) = -1$$

Finally, we have $\sqcap \not\leq ||$, and so we have as well the following formula:

$$\mu(\sqcap, ||) = 0$$

Thus, as a conclusion, we have computed the Möbius matrix $M_2(\pi,\sigma) = \mu(\pi,\sigma)$ of the lattice $P(2) = \{||, \sqcap\}$, the formula being as follows:

$$M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which states that the following happens:

$$f(\sigma) = \sum_{\pi\leq\sigma} g(\pi) \quad \Longrightarrow \quad g(\sigma) = \sum_{\pi\leq\sigma}\mu(\pi,\sigma)f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

THEOREM 6.20. *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1 & \text{if } \pi\leq\sigma\\ 0 & \text{if } \pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.

PROOF. This is well-known, coming for instance from the fact that A_k is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 2.19. □

As an illustration, for $P(2)$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following key result:

PROPOSITION 6.21. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G_k=A_kL_k$, where

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1) & \text{if } \sigma\leq\pi\\ 0 & \text{otherwise}\end{cases}$$

and where A_k is the adjacency matrix of $P(k)$.

PROOF. We have the following computation, based on Proposition 2.18:

$$\begin{aligned}G_k(\pi,\sigma)&=N^{|\pi\vee\sigma|}\\&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According now to the definition of A_k,L_k, this formula reads:

$$\begin{aligned}G_k(\pi,\sigma)&=\sum_{\tau\geq\pi}L_k(\tau,\sigma)\\&=\sum_\tau A_k(\pi,\tau)L_k(\tau,\sigma)\\&=(A_kL_k)(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix}N^2&N\\ N&N\end{pmatrix}=\begin{pmatrix}1&1\\ 0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\ N&N\end{pmatrix}$$

We are led in this way to the following formula, due to Lindstöm [68]:

THEOREM 6.22. *The determinant of the Gram matrix G_k is given by*

$$\det(G_k)=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N<k$ we obtain 0.

PROOF. If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, A_k is upper triangular, and L_k is lower triangular. Thus, we have:

$$\begin{aligned}\det(G_k)&=\det(A_k)\det(L_k)\\ &=\det(L_k)\\ &=\prod_\pi L_k(\pi,\pi)\\ &=\prod_\pi N(N-1)\ldots(N-|\pi|+1)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Now back to easiness and laws of characters, we can formulate:

THEOREM 6.23. *For an easy quantum group $G=(G_N)$, coming from a category of partitions $D=(D(k,l))$, the asymptotic moments of the main character are given by*

$$\lim_{N\to\infty}\int_{G_N}\chi^k=|D(k)|$$

where $D(k)=D(\emptyset,k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the k-th term.

PROOF. We know from Proposition 2.14 that we have the following formula:

$$\int_{G_N}\chi^k=\dim\left(span\left(\xi_\pi\Big|\pi\in D(k)\right)\right)$$

Now since by Theorem 2.22 the vectors ξ_π are linearly independent with $N\geq k$, and in particular with $N\to\infty$, we obtain the formula in the statement. □

This is very nice, and as a first application, we can recover as promised the Poisson law result from Theorem 2.16, this time by using easiness, as follows:

THEOREM 6.24. *For the symmetric group S_N, the main character becomes Poisson*

$$\chi \sim p_1$$

in the $N \to \infty$ limit.

PROOF. As already mentioned, this is something that we already know, from Theorem 2.16. Alternatively, according to Theorem 2.23, we have the following formula:

$$\lim_{N\to\infty} \int_{S_N} \chi^k = |P(k)|$$

Now since a partition of $\{1, \ldots, k+1\}$ appears by choosing s neighbors for 1, among the k numbers available, and then partitioning the $k-s$ elements left, the numbers on the right $B_k = |P(k)|$, called Bell numbers, satisfy the following recurrence:

$$B_{k+1} = \sum_s \binom{k}{s} B_{k-s}$$

On the other hand, the moments M_k of the Poisson law $p_1 = \frac{1}{e}\sum_r \delta_r/r!$ are subject to the same recurrence formula, as shown by the following computation:

$$\begin{aligned} M_{k+1} &= \frac{1}{e}\sum_r \frac{(r+1)^k}{r!} \\ &= \frac{1}{e}\sum_r \frac{r^k}{r!}\left(1+\frac{1}{r}\right)^k \\ &= \frac{1}{e}\sum_r \frac{r^k}{r!}\sum_s \binom{k}{s} r^{-s} \\ &= \sum_s \binom{k}{s} \cdot \frac{1}{e}\sum_r \frac{r^{k-s}}{r!} \\ &= \sum_s \binom{k}{s} M_{k-s} \end{aligned}$$

As for the initial values, at $k = 1, 2$, these are $1, 2$, for both the Bell numbers B_k, and the Poisson moments M_k. Thus we have $B_k = M_k$, which gives the result. $\square$

6d. Free probability

Moving ahead, we have now to work out free analogues of Theorem 2.24 for the other easy quantum groups that we know. A bit of thinking at traces of unitary matrices suggests that for the groups O_N, U_N we should get the real and complex normal laws. As for O_N^+, U_N^+, S_N^+, we are a bit in the dark here, and we can only say that we can expect to have "free versions" of the real and complex normal laws, and of the Poisson law.

Long story short, the combinatorics ahead looks quite complicated, and we are in need of a crash course on probability. So, let us start with that, classical and free probability, and we will come back later to combinatorics and quantum groups. We first have:

DEFINITION 6.25. *Let A be a C^*-algebra, given with a trace $tr: A \to \mathbb{C}$.*

(1) *The elements $a \in A$ are called random variables.*
(2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
(3) *The law of such a variable is the functional $\mu : P \to tr(P(a))$.*

Here $k = \circ \bullet \bullet \circ \ldots$ is by definition a colored integer, and the corresponding powers a^k are defined by the following formulae, and multiplicativity:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial P, this is a noncommuting $*$-polynomial in one variable:

$$P \in \mathbb{C} < X, X^* >$$

Observe that the law is uniquely determined by the moments, because we have:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

Generally speaking, the above definition is something quite abstract, but there is no other way of doing things, at least at this level of generality. However, in certain special cases, the formalism simplifies, and we recover more familiar objects, as follows:

PROPOSITION 6.26. *Assuming that $a \in A$ is normal, $aa^* = a^*a$, its law corresponds to a probability measure on its spectrum $\sigma(a) \subset \mathbb{C}$, according to the following formula:*

$$tr(P(a)) = \int_{\sigma(a)} P(x) d\mu(x)$$

When the trace is faithful we have $supp(\mu) = \sigma(a)$. Also, in the particular case where the variable is self-adjoint, $a = a^*$, this law is a real probability measure.

PROOF. This is something very standard, coming from the Gelfand theorem, applied to the algebra $< a >$, which is commutative, and then the Riesz theorem. □

Following Voiculescu [90], we have the following two notions of independence:

DEFINITION 6.27. *Two subalgebras $A, B \subset C$ are called independent when*

$$tr(a) = tr(b) = 0 \implies tr(ab) = 0$$

holds for any $a \in A$ and $b \in B$, and free when

$$tr(a_i) = tr(b_i) = 0 \implies tr(a_1 b_1 a_2 b_2 \ldots) = 0$$

holds for any $a_i \in A$ and $b_i \in B$.

In short, we have here a straightforward extension of the usual notion of independence, in the framework of Definition 2.25, along with a quite natural free analogue of it. In order to understand what is going on, let us first discuss some basic models for independence and freeness. We have the following result, from [90], which clarifies things:

PROPOSITION 6.28. *Given two algebras (A, tr) and (B, tr), the following hold:*
(1) *A, B are independent inside their tensor product $A \otimes B$.*
(2) *A, B are free inside their free product $A * B$.*

PROOF. Both the assertions are clear from definitions, after some standard discussion regarding the tensor product and free product trace. See Voiculescu [90]. □

In relation with groups, we have the following result:

PROPOSITION 6.29. *We have the following results, valid for group algebras:*
(1) *$C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.*
(2) *$C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.*

PROOF. This follows from the general results in Proposition 2.28, along with the following two isomorphisms, which are both standard:

$$C^*(\Gamma \times \Lambda) = C^*(\Lambda) \otimes C^*(\Gamma) \quad , \quad C^*(\Gamma * \Lambda) = C^*(\Lambda) * C^*(\Gamma)$$

Alternatively, we can prove this directly, by using the fact that each algebra is spanned by the corresponding group elements, and checking the result on group elements. □

In order to study independence and freeness, our main tool will be:

THEOREM 6.30. *The convolution is linearized by the log of the Fourier transform,*

$$F_f(x) = E(e^{ixf})$$

and the free convolution is linearized by the R-transform, given by:

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

PROOF. In what regards the first assertion, if f, g are independent, we have indeed:

$$\begin{aligned}
F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixz} d(\mu_f * \mu_g)(z) \\
&= \int_{\mathbb{R}\times\mathbb{R}} e^{ix(z+t)} d\mu_f(z) d\mu_g(t) \\
&= \int_{\mathbb{R}} e^{ixz} d\mu_f(z) \int_{\mathbb{R}} e^{ixt} d\mu_g(t) \\
&= F_f(x) F_g(x)
\end{aligned}$$

As for the second assertion, here we need a good model for free convolution, and the best is to use the semigroup algebra of the free semigroup on two generators:

$$A = C^*(\mathbb{N} * \mathbb{N})$$

Indeed, we have some freeness in the semigroup setting, a bit in the same way as for the group algebras $C^*(\Gamma * \Lambda)$, from Proposition 2.29, and in addition to this fact, and to what happens in the group algebra case, the following two key things happen:

(1) The variables of type $S^*+f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and with $f \in \mathbb{C}[X]$ being a polynomial, model in moments all the distributions $\mu : \mathbb{C}[X] \to \mathbb{C}$. This is indeed something elementary, which can be checked via a direct algebraic computation.

(2) Given $f, g \in \mathbb{C}[X]$, the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N}*\mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$, are free, and their sum has the same law as $S^* + (f + g)(S)$. This follows indeed by using a 45° argument.

With this in hand, we can see that the operation $\mu \to f$ linearizes the free convolution. We are therefore left with a computation inside $C^*(\mathbb{N})$, whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ, as stated. See [90]. □

As a first result now, which is central and classical and free probability, we have:

THEOREM 6.31 (CLT). *Given self-adjoint variables $x_1, x_2, x_3, \ldots$ which are i.i.d./f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim g_t/\gamma_t$$

where g_t/γ_t are the normal and Wigner semicircle law of parameter t, given by:

$$g_t = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx \quad , \quad \gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

PROOF. This is routine, by using the Fourier transform and the R-transform. □

Next, we have the following complex version of the CLT:

THEOREM 6.32 (CCLT). *Given variables $x_1, x_2, x_3, \ldots$ which are i.i.d./f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim G_t/\Gamma_t$$

where G_t/Γ_t are the complex normal and Voiculescu circular law of parameter t, given by:

$$G_t = law\left(\frac{1}{\sqrt{2}}(a+ib)\right) \quad , \quad \Gamma_t = law\left(\frac{1}{\sqrt{2}}(\alpha+i\beta)\right)$$

where $a, b/\alpha, \beta$ are independent/free, each following the law g_t/γ_t.

Proof. This follows indeed from the CLT, by taking real and imaginary parts. □

Finally, we have the following discrete version of the CLT:

Theorem 6.33 (PLT). *The following Poisson limits converge, for any $t > 0$,*

$$p_t = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1\right)^{*n} \quad , \quad \pi_t = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1\right)^{\boxplus n}$$

the limiting measures being the Poisson law p_t, and the Marchenko-Pastur law π_t,

$$p_t = \frac{1}{e^t}\sum_{k=0}^\infty \frac{t^k\delta_k}{k!} \quad , \quad \pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

with at $t = 1$, the Marchenko-Pastur law being $\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$.

Proof. This is again routine, by using the Fourier and R-transform. □

This was for the basic classical and free probability. In relation now with combinatorics, we have the following result, which reminds easiness, and is of interest for us:

Theorem 6.34. *The moments of the various central limiting measures, namely*

$$\begin{array}{ccccc} \pi_t & - & \gamma_t & - & \Gamma_t \\ | & & | & & | \\ p_t & - & g_t & - & G_t \end{array}$$

are always given by the same formula, involving partitions, namely

$$M_k = \sum_{\pi\in D(k)} t^{|\pi|}$$

with the sets of partitions $D(k)$ in question being respectively

$$\begin{array}{ccccc} NC & \leftarrow & NC_2 & \leftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P & \leftarrow & P_2 & \leftarrow & \mathcal{P}_2 \end{array}$$

and with $|.|$ being the number of blocks.

Proof. This follows indeed from the various computations leading to Theorems 2.31, 2.32, 2.33, and details can be found in any free probability book. See [90]. □

It is possible to say more on this, following Rota in the classical case, Speicher in the free case, and Bercovici-Pata for the classical/free correspondence. We first have:

DEFINITION 6.35. *The cumulants of a self-adjoint variable $a\in A$ are given by*

$$\log F_a(\xi)=\sum_{n=1}^\infty k_n(a)\,\frac{(i\xi)^n}{n!}$$

and the free cumulants of the same variable $a\in A$ are given by:

$$R_a(\xi)=\sum_{n=1}^\infty \kappa_n(a)\xi^{n-1}$$

Moreover, we have extensions of these notions to the non-self-adjoint case.

In what follows we will only discuss the self-adjoint case, which is simpler, and illustrating. Since the classical and free cumulants are by definition certain linear combinations of the moments, we should have conversion formulae. The result here is as follows:

THEOREM 6.36. *The moments can be recaptured out of cumulants via*

$$M_n(a)=\sum_{\pi\in P(n)}k_\pi(a)\quad,\quad M_n(a)=\sum_{\pi\in NC(n)}\kappa_\pi(a)$$

with the convention that k_π,κ_π are defined by multiplicativity over blocks. Also,

$$k_n(a)=\sum_{\nu\in P(n)}\mu_P(\nu,1_n)M_\nu(a)\quad,\quad \kappa_n(a)=\sum_{\nu\in NC(n)}\mu_{NC}(\nu,1_n)M_\nu(a)$$

where μ_P,μ_{NC} are the Möbius functions of $P(n),NC(n)$.

PROOF. Here the first formulae follow from Definition 2.35, by doing some combinatorics, and the second formulae follow from them, via Möbius inversion. □

In relation with the various laws that we are interested in, we have:

PROPOSITION 6.37. *The classical and free cumulants are as follows:*

(1) *For $\mu=\delta_c$ both the classical and free cumulants are $c,0,0,\ldots$*
(2) *For $\mu=g_t/\gamma_t$ the classical/free cumulants are $0,t,0,0,\ldots$*
(3) *For $\mu=p_t/\pi_t$ the classical/free cumulants are $t,t,t,\ldots$*

PROOF. Here (1) is something trivial, and (2,3) can be deduced either directly, starting from the definition of the various laws involved, or by using Theorem 2.34. □

Following now Bercovici-Pata [18], let us formulate the following definition:

DEFINITION 6.38. *If the classical cumulants of η equal the free cumulants of μ,*

$$k_n(\eta)=\kappa_n(\mu)$$

we say that η is the classical version of μ, and that μ is the free version of η.

All this is quite interesting, and we have now a better understanding of Theorem 2.34, the point there being that on the vertical, we have measures in Bercovici-Pata bijection. Now back to quantum groups, we first have the following result, from [5]:

Theorem 6.39. *The asymptotic laws of characters for the basic quantum groups,*

$$\begin{array}{ccccc} S_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N & \longrightarrow & U_N \end{array}$$

are precisely the main laws in classical and free probability at $t = 1$.

Proof. This follows indeed from our various easiness considerations before, and from Theorem 2.34 applied at $t = 1$, which gives $M_k = |D(k)|$ in this case. □

More generally, again following [5], let us discuss now the computation for the truncated characters. These are variables constructed as follows:

Definition 6.40. *Associated to any Woronowicz algebra (A, u) are the variables*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

depending on a parameter $t \in (0, 1]$, called truncations of the main character.

In order to understand what these variables χ_t are about, let us first investigate the symmetric group S_N. We have here the following result:

Theorem 6.41. *For the symmetric group $S_N \subset O_N$, the truncated character*

$$\chi_t(g) = \sum_{i=1}^{[tN]} u_{ii}$$

becomes, with $N \to \infty$, a Poisson variable of parameter t.

Proof. This can be deduced via inclusion-exclusion, as in the proof of Theorem 2.16, but let us prove this via an alternative method, which is instructive as well. Our first claim is that the integrals over S_N are given by the following formula:

$$\int_{S_N} u_{i_1j_1} \ldots u_{i_kj_k} = \begin{cases} \frac{(N-|\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

Indeed, according to the definition of u_{ij}, the above integrals are given by:

$$\int_{S_N} u_{i_1j_1} \ldots u_{i_kj_k} = \frac{1}{N!} \# \left\{ \sigma \in S_N \Big| \sigma(j_1) = i_1, \ldots, \sigma(j_k) = i_k \right\}$$

But this proves our claim. Now with the above formula in hand, with S_{kb} being the Stirling numbers, counting the partitions in $P(k)$ having b blocks, we have:

$$\begin{aligned}\int_{S_N}\chi_t^k &= \sum_{i_1\ldots i_k=1}^{[tN]}\int_{S_N}u_{i_1i_1}\ldots u_{i_ki_k}\\ &= \sum_{\pi\in P(k)}\frac{[tN]!}{([tN]-|\pi|!)}\cdot\frac{(N-|\pi|!)}{N!}\\ &= \sum_{b=1}^{[tN]}\frac{[tN]!}{([tN]-b)!}\cdot\frac{(N-b)!}{N!}\cdot S_{kb}\end{aligned}$$

Thus with $N\to\infty$ the moments are $M_k\simeq\sum_{b=1}^k S_{kb}t^b$, which gives the result. □

Summarizing, we have nice results about S_N. In general, however, and in particular for O_N,U_N and S_N^+,O_N^+,U_N^+, there is no simple trick as for S_N, and we must use general integration methods, from [5], [36]. We have here the following formula:

THEOREM 6.42. *For an easy quantum group $G\subset_u U_N^+$, coming from a category of partitions $D=(D(k,l))$, we have the Weingarten integration formula*

$$\int_G u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where $D(k)=D(\emptyset,k)$, δ are usual Kronecker symbols, and $W_{kN}=G_{kN}^{-1}$, with

$$G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$$

where $|.|$ is the number of blocks.

PROOF. This follows from the general Weingarten formula from Theorem 2.4. Indeed, in the easy case we can take $D_k=D(k,k)$, and the Kronecker symbols are given by:

$$\delta_{\xi_\pi}(i)=<\xi_\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>=\delta_\pi(i_1,\ldots,i_k)$$

The Gram matrix being as well the correct one, we obtain the result. See [5]. □

With the above formula in hand, we can go back to the question of computing the laws of truncated characters. First, we have the following moment formula, from [5]:

PROPOSITION 6.43. *The moments of truncated characters are given by the formula*

$$\int_G(u_{11}+\ldots+u_{ss})^k=Tr(W_{kN}G_{ks})$$

where G_{kN} and $W_{kN}=G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

Proof. We have indeed the following computation:

$$\begin{aligned}\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int u_{i_1i_1}\ldots u_{i_ki_k}\\ &= \sum_{\pi,\sigma\in D(k)}W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\pi(i)\delta_\sigma(i)\\ &= \sum_{\pi,\sigma\in D(k)}W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\ &= Tr(W_{kN}G_{ks})\end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to process now the above formula, things are quite technical, and won't work well in general. We must impose here a uniformity condition, as follows:

Theorem 6.44. *For an easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, the following conditions are equivalent:*

1. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the embedding $U_{N-1}^+\subset U_N^+$ given by $u\to diag(u,1)$.*
2. *$G_{N-1}=G_N\cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+\subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

If these conditions are satisfied, we say that $G=(G_N)$ is uniform.

Proof. This is something very standard, the idea being as follows:

(1) $\iff$ (2) This equivalence is elementary, coming from the inclusion $S_N\subset G_N$, which makes everything S_N-invariant.

(1) $\iff$ (3) Given a closed subgroup $K\subset U_{N-1}^+$, with fundamental corepresentation u, consider the following $N\times N$ matrix:

$$v=\begin{pmatrix}u&0\\0&1\end{pmatrix}$$

Then for any $\pi\in P(k)$ a standard computation shows that we have:

$$\xi_\pi\in Fix(v^{\otimes k})\iff\xi_{\pi'}\in Fix(v^{\otimes k'}),\forall\pi'\in P(k'),\pi'\subset\pi$$

Now with this in hand, the result follows from Tannakian duality. □

By getting back now to the truncated characters, we have the following result:

Theorem 6.45. *For a uniform easy quantum group $G=(G_N)$, we have the formula*

$$\lim_{N\to\infty}\int_{G_N}\chi_t^k=\sum_{\pi\in D(k)}t^{|\pi|}$$

with $D\subset P$ being the associated category of partitions.

Proof. In the uniform case the Gram matrix, and so the Weingarten matrix too, are asymptotically diagonal, so the asymptotic moments are given by:

$$\int_{G_N}\chi_t^k=Tr(W_{kN}G_{k[tN]})\simeq\sum_{\pi\in D(k)}N^{-|\pi|}[tN]^{|\pi|}\simeq\sum_{\pi\in D(k)}t^{|\pi|}$$

Thus, we are led to the conclusion in the statement. See [5], [17]. □

We can now improve our quantum group results, as follows:

Theorem 6.46. *The asymptotic laws of truncated characters for the quantum groups*

$$\begin{array}{ccccc}
S_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
S_N & \longrightarrow & O_N & \longrightarrow & U_N
\end{array}$$

are precisely the main limiting laws in classical and free probability, namely:

$$\begin{array}{ccccc}
\pi_t & — & \gamma_t & — & \Gamma_t\\
| & & | & & |\\
p_t & — & g_t & — & G_t
\end{array}$$

Proof. This follows indeed from easiness, Theorem 2.34 and Theorem 2.45. □

6e. Exercises

Exercises:

Exercise 6.47.

Exercise 6.48.

Exercise 6.49.

Exercise 6.50.

Exercise 6.51.

Exercise 6.52.

Exercise 6.53.

Exercise 6.54.

Bonus exercise.

CHAPTER 7

Algebraic invariants

7a. Rotation groups

We have seen that the inclusion $S_N \subset S_N^+$, and its companion inclusions $O_N \subset O_N^+$ and $U_N \subset U_N^+$, are all liberations in the sense of easy quantum group theory, and that some representation theory consequences, in the $N \to \infty$ limit, can be derived from this. We discuss here the case where $N \in \mathbb{N}$ is fixed, which is more technical.

Let us first study the representations of O_N^+. We know that in the $N \to \infty$ limit we have $\chi \sim \gamma_1$, and as a first question, we would like to know how the irreducible representations of a "formal quantum group" should look like, when subject to the condition $\chi \sim \gamma_1$. And fortunately, the answer here is very simple, coming from SU_2:

THEOREM 7.1. *The group SU_2 is as follows:*

(1) *The main character is real, its odd moments vanish, and its even moments are the Catalan numbers:*

$$\int_{SU_2} \chi^{2k} = C_k$$

(2) *This main character follows the Wigner semicircle law, $\chi \sim \gamma_1$.*

(3) *The irreducible representations can be labelled by positive integers, r_k with $k \in \mathbb{N}$, and the fusion rules for these representations are:*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

(4) *The dimensions of these representations are* $\dim r_k = k + 1$.

PROOF. There are many possible proofs here, the idea being as follows:

(1,2) These statements are equivalent, and in order to prove them, a simple argument is by using the well-known isomorphism $SU_2 \simeq S^3_{\mathbb{R}}$, coming from:

$$SU_2 = \left\{ \begin{pmatrix} x+iy & z+it \\ -z+it & x-iy \end{pmatrix} \Big| x^2+y^2+z^2+t^2 = 1 \right\}$$

Indeed, in this picture the moments of $\chi = 2x$ can be computed via spherical coordinates and some calculus, and follow to be the Catalan numbers:

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

As for the formula $\chi\sim\gamma_1$, this follows from this, and is geometrically clear as well.

(3,4) Our claim is that we can construct, by recurrence on $k\in\mathbb N$, a sequence r_k of irreducible, self-adjoint and distinct representations of SU_2, satisfying:

$$r_0=1\quad,\quad r_1=u\quad,\quad r_{k-1}\otimes r_1=r_{k-2}+r_k$$

Indeed, assume that $r_0,\ldots,r_{k-1}$ are constructed, and let us construct r_k. We have:

$$r_{k-2}\otimes r_1=r_{k-3}+r_{k-1}$$

Thus $r_{k-1}\subset r_{k-2}\otimes r_1$, and since r_{k-2} is irreducible, by Frobenius we have:

$$r_{k-2}\subset r_{k-1}\otimes r_1$$

We conclude there exists a certain representation r_k such that:

$$r_{k-1}\otimes r_1=r_{k-2}+r_k$$

By recurrence, r_k is self-adjoint. Now observe that according to our recurrence formula, we can split $u^{\otimes k}$ as a sum of the following type, with positive coefficients:

$$u^{\otimes k}=c_kr_k+c_{k-2}r_{k-2}+c_{k-4}r_{k-4}+\ldots$$

We conclude by Peter-Weyl that we have an inequality as follows, with equality precisely when r_k is irreducible, and non-equivalent to the other summands r_i:

$$\sum_ic_i^2\leq\dim(End(u^{\otimes k}))$$

But by (1) the number on the right is C_k, and some straightforward combinatorics, based on the fusion rules, shows that the number on the left is C_k as well. Thus:

$$C_k=\sum_ic_i^2\leq\dim(End(u^{\otimes k}))=\int_{SU_2}\chi^{2k}=C_k$$

We conclude that we have equality in our estimate, so our representation r_k is irreducible, and non-equivalent to $r_{k-2},r_{k-4},\ldots$ Moreover, this representation r_k is not equivalent to $r_{k-1},r_{k-3},\ldots$ either, with this coming from $r_p\subset u^{\otimes p}$, and from:

$$\dim(Fix(u^{\otimes 2s+1}))=\int_{SU_2}\chi^{2s+1}=0$$

Thus, we have proved our claim. Now since each irreducible representation of SU_2 must appear in some tensor power $u^{\otimes k}$, and we know how to decompose each $u^{\otimes k}$ into sums of representations r_k, these representations r_k are all the irreducible representations of SU_2, and we are done with (3). As for the formula in (4), this is clear. $\square$

There are of course many other proofs for the above result, which are all instructive, and we recommend here any good book on geometry and physics. In what concerns us, the above will do, and we will be back to this later, with some further comments.

Getting back now to O_N^+, we know that in the $N\to\infty$ limit we have $\chi\sim\gamma_1$, so by the above when formally setting $N=\infty$, the fusion rules are the same as for SU_2. Miraculously, however, this happens in fact at any $N\geq 2$, the result being as follows:

THEOREM 7.2. *The quantum groups O_N^+ with $N\geq 2$ are as follows:*

(1) *The odd moments of the main character vanish, and the even moments are:*

$$\int_{O_N^+}\chi^{2k}=C_k$$

(2) *This main character follows the Wigner semicircle law, $\chi\sim\gamma_1$.*

(3) *The fusion rules for irreducible representations are as for SU_2, namely:*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+2}+\ldots+r_{k+l}$$

(4) *We have* $\dim r_k=(q^{k+1}-q^{-k-1})/(q-q^{-1})$, *with $q+q^{-1}=N$.*

PROOF. The idea is to skilfully recycle the proof of Theorem 3.1, as follows:

(1,2) These assertions are equivalent, and since we cannot prove them directly, we will simply say that these follow from the combinatorics in (3) below.

(3,4) As before, our claim is that we can construct, by recurrence on $k\in\mathbb{N}$, a sequence $r_0,r_1,r_2,\ldots$ of irreducible, self-adjoint and distinct representations of O_N^+, satisfying:

$$r_0=1\quad,\quad r_1=u\quad,\quad r_{k-1}\otimes r_1=r_{k-2}+r_k$$

In order to do so, we can use as before $r_{k-2}\otimes r_1=r_{k-3}+r_{k-1}$ and Frobenius, and we conclude there exists a certain representation r_k such that:

$$r_{k-1}\otimes r_1=r_{k-2}+r_k$$

As a first observation, r_k is self-adjoint, because its character is a certain polynomial with integer coefficients in χ, which is self-adjoint. In order to prove now that r_k is irreducible, and non-equivalent to $r_0,\ldots,r_{k-1}$, let us split as before $u^{\otimes k}$, as follows:

$$u^{\otimes k}=c_kr_k+c_{k-2}r_{k-2}+c_{k-4}r_{k-4}+\ldots$$

The point now is that we have the following equalities and inequalities:

$$C_k=\sum_ic_i^2\leq\dim(End(u^{\otimes k}))\leq|NC_2(k,k)|=C_k$$

Indeed, the equality at left is clear as before, then comes a standard inequality, then an inequality coming from easiness, then a standard equality. Thus, we have equality, so r_k is irreducible, and non-equivalent to $r_{k-2},r_{k-4},\ldots$ Moreover, r_k is not equivalent to $r_{k-1},r_{k-3},\ldots$ either, by using the same argument as for SU_2, and the end of the proof of (3) is exactly as for SU_2. As for (4), by recurrence we obtain, with $q+q^{-1}=N$:

$$\dim r_k=q^k+q^{k-2}+\ldots+q^{-k+2}+q^{-k}$$

But this gives the dimension formula in the statement, and we are done. □

The above result raises several interesting questions. For instance we would like to know if Theorem 3.1 can be unified with Theorem 3.2. Also, combinatorially speaking, we would like to have a better understanding of the "miracle" making Theorem 3.2 hold at any $N\geq 2$, instead of $N=\infty$ only. These questions will be answered in due time.

Regarding now the quantum group U_N^+, a similar result holds here, which is also elementary, using only algebraic techniques, based on easiness. Let us start with:

THEOREM 7.3. *We have isomorphisms as follows,*

$$U_N^+=\widetilde{O_N^+}\quad,\quad PO_N^+=PU_N^+$$

modulo the usual equivalence relation for compact quantum groups.

PROOF. The above isomorphisms both come from easiness, as follows:

(1) We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of U_N^+:

$$O_N^+\subset\widetilde{O_N^+}\subset U_N^+$$

We must prove that the embedding on the right is an isomorphism. In order to do so, let us denote by v,zv,u the fundamental representations of the above quantum groups. At the level of the associated Hom spaces we obtain reverse inclusions, as follows:

$$Hom(v^{\otimes k},v^{\otimes l})\supset Hom((zv)^{\otimes k},(zv)^{\otimes l})\supset Hom(u^{\otimes k},u^{\otimes l})$$

But the spaces on the left and on the right are known from chapter 2, the easiness result there stating that these are as follows:

$$span\left(T_\pi\Big|\pi\in NC_2(k,l)\right)\supset span\left(T_\pi\Big|\pi\in\mathcal{NC}_2(k,l)\right)$$

Regarding the spaces in the middle, these are obtained from those on the left by coloring, and we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O_N^+}\subset U_N^+$ is an isomorphism, modulo the usual equivalence relation.

(2) Regarding now the projective versions, the result here follows from:

$$PU_N^+=P\widetilde{O_N^+}=PO_N^+$$

Alternatively, with the notations in the proof of (1), we have:

$$Hom\left((v\otimes v)^k,(v\otimes v)^l\right)=span\left(T_\pi\Big|\pi\in NC_2((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

$$Hom\left((u\otimes\bar{u})^k,(u\otimes\bar{u})^l\right)=span\left(T_\pi\Big|\pi\in\mathcal{NC}_2((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusion $PO_N^+\subset PU_N^+$ preserves the corresponding Tannakian categories, and so must be an isomorphism. □

Getting now to the representations of U_N^+, the result here is as follows:

THEOREM 7.4. *The quantum groups U_N^+ with $N \geq 2$ are as follows:*

(1) *The moments of the main character count the matching pairings:*

$$\int_{U_N^+} \chi^k = |\mathcal{N}C_2(k)|$$

(2) *The main character follows the Voiculescu circular law of parameter* 1*:*

$$\chi \sim \Gamma_1$$

(3) *The irreducible representations are indexed by* $\mathbb{N} * \mathbb{N}$*, with as fusion rules:*

$$r_k \otimes r_l = \sum_{k=xy, l=\bar{y}z} r_{xz}$$

(4) *The corresponding dimensions* $\dim r_k$ *can be computed by recurrence.*

PROOF. There are several proofs here, the idea being as follows:

(1) The original proof, explained for instance in [5], is by construcing the representations r_k by recurrence, exactly as in the proof of Theorem 3.2, and then arguing, also as there, that the combinatorics found proves the first two assertions as well. In short, what we have is a "complex remake" of Theorem 3.2, which can be proved in a similar way.

(2) An alternative argument, discussed as well in [5], is by using Theorem 3.3. Indeed, the fusion rules for $U_N^+ = \widetilde{O_N^+}$ can be computed by using those of O_N^+, and we end up with the above "free complexification" of the Clebsch-Gordan rules. As for the first two assertions, these follow too from $U_N^+ = \widetilde{O_N^+}$, via standard free probability. □

As a conclusion, our results regarding O_N^+, U_N^+ show that the $N \to \infty$ convergence of the law of the main character to γ_1, Γ_1, known since chapter 2, is in fact stationary, starting with $N = 2$. And this is quite a miracle, for instance because for O_N, U_N, some elementary computations show that the same $N \to \infty$ convergence, this time to the normal laws g_1, G_1, is far from being stationary. Thus, it is tempting to formulate:

CONCLUSION 7.5. *The free world is simpler than the classical world.*

And please don't get me wrong, especially if you're new to the subject, having struggled with the free material explained so far in this book. What I'm saying here is that, once you're reasonably advanced, and familiar with freeness, and so you will be soon, a second look at what has been said so far in this book can only lead to the above conclusion.

More on this later, in connection with permutations and quantum permutations too. Finally, as an extra piece of evidence, we have the isomorphism $PO_N^+ = PU_N^+$ from Theorem 3.3, which is something quite intruiguing too, suggesting that the "free projective geometry is scalarless". We will be back to this later, with the answer that yes, free projective geometry is indeed scalarless, simpler than classical projective geometry.

7b. Clebsch-Gordan rules

We discuss now the representation theory of S_N^+ at $N\geq 4$. Let us begin our study exactly as for O_N^+. We know that in the $N\to\infty$ limit we have $\chi\sim\pi_1$, and as a first question, we would like to know how the irreducible representations of a "formal quantum group" should look like, when subject to the condition $\chi\sim\pi_1$. And fortunately, the answer here is very simple, involving this time the group SO_3:

THEOREM 7.6. *The group SO_3 is as follows:*

(1) *The moments of the main character are the Catalan numbers:*

$$\int_{SO_3}\chi^k=C_k$$

(2) *The main character follows the Marchenko-Pastur law of parameter 1:*

$$\chi\sim\pi_1$$

(3) *The fusion rules for irreducible representations are as follows:*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+1}+\ldots+r_{k+l}$$

(4) *The dimensions of these representations are* $\dim r_k=2k-1$.

PROOF. As before with SU_2, there are many possible proofs here, which are all instructive. Here is our take on the subject, in the spirit of our proof for SU_2:

(1,2) These statements are equivalent, and in order to prove them, a simple argument is by using the SU_2 result, and the double cover map $SU_2\to SO_3$. Indeed, let us recall from the proof for SU_2 that we have an isomorphism $SU_2\simeq S^3_{\mathbb R}$, coming from:

$$SU_2=\left\{\begin{pmatrix}x+iy&z+it\\-z+it&x-iy\end{pmatrix}\Big|x^2+y^2+z^2+t^2=1\right\}$$

The point now is that we have a double cover map $SU_2\to SO_3$, which gives the following formula for the generic elements of SO_3, called Euler-Rodrigues formula:

$$U=\begin{pmatrix}x^2+y^2-z^2-t^2&2(yz-xt)&2(xz+yt)\\2(xt+yz)&x^2+z^2-y^2-t^2&2(zt-xy)\\2(yt-xz)&2(xy+zt)&x^2+t^2-y^2-z^2\end{pmatrix}$$

It follows that the main character of SO_3 is given by the following formula:

$$\begin{aligned}\chi(U)&=Tr(U)+1\\&=3x^2-y^2-z^2-t^2+1\\&=4x^2\end{aligned}$$

On the other hand, we know from Theorem 3.1 and its proof that $2x\sim\gamma_1$. Now since we have $f\sim\gamma_1\implies f^2\sim\pi_1$, we obtain $\chi\sim\pi_1$, as desired.

(3,4) Our claim is that we can construct, by recurrence on $k \in \mathbb{N}$, a sequence r_k of irreducible, self-adjoint and distinct representations of SO_3, satisfying:

$$r_0 = 1 \quad , \quad r_1 = u - 1 \quad , \quad r_{k-1} \otimes r_1 = r_{k-2} + r_{k-1} + r_k$$

Indeed, assume that $r_0, \ldots, r_{k-1}$ are constructed, and let us construct r_k. The Frobenius trick from the proof for SU_2 will no longer work, as you can verify yourself, so we have to invoke (1). To be more precise, by integrating characters we obtain:

$$r_{k-1}, r_{k-2} \subset r_{k-1} \otimes r_1$$

Thus, there exists a representation r_k such that:

$$r_{k-1} \otimes r_1 = r_{k-2} + r_{k-1} + r_k$$

Once again by integrating characters, we conclude that r_k is irreducible, and non-equivalent to $r_1, \ldots, r_{k-1}$, and this proves our claim. Also, since any irreducible representation of SO_3 must appear in some tensor power of u, and we can decompose each $u^{\otimes k}$ into sums of representations r_p, we conclude that these representations r_p are all the irreducible representations of SO_3. Finally, the dimension formula is clear. □

Based on the above result, and on what we know about the relation between SU_2 and the quantum groups O_N^+ at $N \geq 2$, we can safely conjecture that the fusion rules for S_N^+ at $N \geq 4$ should be the same as for SO_3. However, a careful inspection of the proof of Theorem 3.6 shows that, when trying to extend it to S_4^+, a bit in the same way as the proof of Theorem 3.1 was extended to O_N^+, we run into a serious problem, namely:

PROBLEM 7.7. *Regarding S_N^+ with $N \geq 4$, we can't get away with the estimate*

$$\int_{S_N^+} \chi^k \leq C_k$$

because the Frobenius trick won't work. We need equality in this estimate.

To be more precise, the above estimate comes from easiness, and we have seen that for O_N^+ with $N \geq 2$, a similar easiness estimate, when coupled with the Frobenius trick, does the job. However, the proof of Theorem 3.6 makes it clear that no Frobenius trick is available, and so we need equality in the above estimate, as indicated.

So, how to prove the equality? The original argument, from [5], is something quick and advanced, saying that modulo some standard identifications, we are in need of the fact that the trace on the Temperley-Lieb algebra $TL_N(k) = span(NC_2(k,k))$ is faithful at index values $N \geq 4$, and with this being true by the results of Jones in [58]. However, while very quick, this remains something advanced, because the paper [58] itself is based on a good deal of von Neumann algebra theory, covering a whole book or so. And so, we don't want to get into this, at least at this stage of our presentation.

In short, we are a bit in trouble. But no worries, there should be a pedestrian way of solving our problem, because that is how reasonable mathematics is made, always available to pedestrians. Here is an idea for a solution, which is a no-brainer:

Solution 7.8. *We can get the needed equality at $N \geq 4$, namely*

$$\int_{S_N^+} \chi^k = C_k$$

by proving that the vectors $\{\xi_\pi|\pi \in NC(k)\}$ are linearly independent.

Indeed, this is something coming from easiness, and since this problem does not look that scary, let us try to solve it. As a starting point for our study, we have:

Proposition 7.9. *The following are linearly independent, at any $N \geq 2$:*

(1) *The linear maps $\left\{T_\pi \middle| \pi \in NC_2(k,l)\right\}$, with $k+l \in 2\mathbb{N}$.*
(2) *The vectors $\left\{\xi_\pi \middle| \pi \in NC_2(2k)\right\}$, with $k \in \mathbb{N}$.*
(3) *The linear maps $\left\{T_\pi \middle| \pi \in NC_2(k,k)\right\}$, with $k \in \mathbb{N}$.*

Proof. All this follows from the dimension equalities established in the proof of Theorem 3.2, because in all cases, the number of partitions is a Catalan number. □

In order to pass now to quantum permutations, we can use the following trick:

Proposition 7.10. *We have a bijection $NC(k) \simeq NC_2(2k)$, constructed by fattening and shrinking, as follows:*

(1) *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings too.*
(2) *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the above two operations are indeed inverse to each other is clear, by drawing pictures, and computing the corresponding compositions. □

At the level of the associated Gram matrices, the result is as follows:

Proposition 7.11. *The Gram matrices of $NC_2(2k) \simeq NC(k)$ are related by*

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1}G_{k,n^2}\Delta_{kn}^{-1})(\pi',\sigma')$$

where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}.

Proof. In the context of the bijection from Proposition 3.10, we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi\vee\sigma|} = n^{k+2|\pi'\vee\sigma'|-|\pi'|-|\sigma'|}$$

Thus, we are led to the formula in the statement. □

We can now formulate a "projective" version of Proposition 3.9, as follows:

PROPOSITION 7.12. *The following are linearly independent, for $N = n^2$ with $n \geq 2$:*
(1) *The linear maps $\{T_\pi | \pi \in NC(k,l)\}$, with $k,l \in 2\mathbb{N}$.*
(2) *The vectors $\{\xi_\pi | \pi \in NC(k)\}$, with $k \in \mathbb{N}$.*
(3) *The linear maps $\{T_\pi | \pi \in NC(k,k)\}$, with $k \in \mathbb{N}$.*

PROOF. This follows from the various linear independence results from Proposition 3.9, by using the Gram matrix formula from Proposition 3.11, along with the well-known fact that vectors are linearly independent when their Gram matrix is invertible. □

Good news, we can now discuss S_N^+ with $N = n^2$, $n \geq 2$, as follows:

THEOREM 7.13. *The quantum groups S_N^+ with $N = n^2$, $n \geq 2$ are as follows:*
(1) *The moments of the main character are the Catalan numbers:*

$$\int_{S_N^+} \chi^k = C_k$$

(2) *The main character follows the Marchenko-Pastur law, $\chi \sim \pi_1$.*
(3) *The fusion rules for irreducible representations are as for SO_3, namely:*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$$

(4) *We have $\dim r_k = (q^{k+1} - q^{-k})/(q-1)$, with $q + q^{-1} = N - 2$.*

PROOF. This is quite similar to the proof of Theorem 3.2, by using the linear independence result from Proposition 3.12 as main ingredient, as follows:

(1) We have the following computation, using Peter-Weyl, then the easiness property of S_N^+, then Proposition 3.12 (2), then Proposition 3.10, and the definition of C_k:

$$\int_{S_N^+} \chi^k = |NC(k)| = |NC_2(2k)| = C_k$$

(2) This is a reformulation of (1), using standard free probability theory.

(3) This is identical to the proof of Theorem 3.6 (3), based on (1).

(4) Finally, the dimension formula is clear by recurrence. □

All this is very nice, and although there is still some work, in order to reach to results for S_N^+ at any $N \geq 4$, let us just enjoy what we have. As a consequence, we have:

THEOREM 7.14. *The free quantum groups are as follows:*
(1) *U_N^+ is not coamenable at $N \geq 2$.*
(2) *O_N^+ is coamenable at $N = 2$, and not coamenable at $N \geq 3$.*
(3) *S_N^+ is coamenable at $N \leq 4$, and not coamenable at $N \geq 5$.*

PROOF. The various non-coamenability assertions are all clear, due to various examples of non-coamenable group dual subgroups $\widehat{\Gamma} \subset G$, coming from the theory in chapter 1. As for the amenability assertions, regarding O_2^+ and S_4^+, these come from Theorem 3.2 and Theorem 3.13, which show that the support of the spectral measure of χ is:

$$supp(\gamma_1) = [-2, 2] \quad , \quad supp(\pi_1) = [0, 4]$$

Thus the Kesten criterion from chapter 1, telling us that $G \subset O_N^+$ is coamenable precisely when $N \in supp(law(\chi))$, applies in both cases, and gives the result. $\square$

7c. Meander determinants

Let us discuss now the extension of Theorem 3.13, to all the quantum groups S_N^+ with $N \geq 4$. For this purpose we need an extension of the linear independence results from Proposition 3.12. This is something non-trivial, and the first thought goes to:

SPECULATION 7.15. *There should be a theory of deformed compact quantum groups, alowing us to talk about O_n^+ with $n \in [2, \infty)$, having the same fusion rules as SU_2, and therefore solving via partition shrinking our S_N^+ problems at any $N \geq 4$.*

This speculation is legit, and in what concerns the first part, generalities, that theory is indeed available, from the Woronowicz papers [99], [100]. Is it also possible to talk about deformations of O_N^+ in this setting, as explained in Wang's paper [92], with the new parameter $n \in [2, \infty)$ being of course not the dimension of the fundamental representation, but rather its "quantum dimension". And with this understood, all the rest is quite standard, and worked out in the quantum group literature. We refer to [5] for more about this, but we will not follow this path, which is too complicated.

As a second speculation now, which is something complicated too, but is far more conceptual, we have the idea, already mentioned before, of getting what we want via the trace on the Temperley-Lieb algebra $TL_N(k) = span(NC_2(k, k))$. We will not follow this path either, which is quite complicated too, but here is how this method works:

THEOREM 7.16. *Consider the Temperley-Lieb algebra of index $N \geq 4$, defined as*

$$TL_N(k) = span(NC_2(k, k))$$

with product given by the rule $\bigcirc = N$, when concatenating.

1. *We have a representation $i : TL_N(k) \to B((\mathbb{C}^N)^{\otimes k})$, given by $\pi \to T_\pi$.*
2. *$Tr(T_\pi) = N^{loops(<\pi>)}$, where $\pi \to <\pi>$ is the closing operation.*
3. *The linear form $\tau = Tr \circ i : TL_N(k) \to \mathbb{C}$ is a faithful positive trace.*
4. *The representation $i : TL_N(k) \to B((\mathbb{C}^N)^{\otimes k})$ is faithful.*

In particular, the vectors $\{\xi_\pi | \pi \in NC(k)\} \subset (\mathbb{C}^N)^{\otimes k}$ are linearly independent.

PROOF. All this is quite standard, but advanced, the idea being as follows:

(1) This is clear from the categorical properties of $\pi \to T_\pi$.

(2) This follows indeed from the following computation:

$$\begin{aligned} Tr(T_\pi) &= \sum_{i_1\ldots i_k} \delta_\pi\begin{pmatrix} i_1 \ldots i_k \\ i_1 \ldots i_k \end{pmatrix} \\ &= \#\left\{ i_1,\ldots,i_k \in \{1,\ldots,N\} \Big| \ker\begin{pmatrix} i_1 \ldots i_k \\ i_1 \ldots i_k \end{pmatrix} \geq \pi \right\} \\ &= N^{loops(<\pi>)} \end{aligned}$$

(3) The traciality of τ is clear from definitions. Regarding now the faithfulness, this is something well-known, and we refer here to Jones' paper [58].

(4) This follows from (3) above, via a standard positivity argument. As for the last assertion, this follows from (4), by fattening the partitions. □

We will be back to this later, when talking subfactors and planar algebras, with a closer look into Jones' paper [58]. In the meantime, however, Speculation 3.15 and Theorem 3.16 will not do, being too advanced, so we have to come up with something else, more pedestrian. And this can only be the computation of the Gram determinant.

We already know, from chapter 2, that for the group S_N the formula of the corresponding Gram matrix determinant, due to Lindstöm [68], is as follows:

THEOREM 7.17. *The determinant of the Gram matrix of S_N is given by*

$$\det(G_{kN}) = \prod_{\pi\in P(k)} \frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N < k$ we obtain 0.

PROOF. This is something that we know from chapter 2, the idea being that G_{kN} decomposes as a product of an upper triangular and lower triangular matrix. □

Although we will not need this here, let us discuss as well, for the sake of completness, the case of the orthogonal group O_N. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. The result is then as follows:

THEOREM 7.18. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN}) = \prod_{|\lambda|=k/2} f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda) = \prod_{(i,j)\in\lambda}(N+2j-i-1)$.

PROOF. This follows from the results of Zinn-Justin on the subject. Indeed, it is known from there that the Gram matrix is diagonalizable, as follows:

$$G_{kN}=\sum_{|\lambda|=k/2}f_N(\lambda)P_{2\lambda}$$

Here $1=\sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda})=f^{2\lambda}$, this gives the result. See [5]. □

For the free orthogonal and symmetric groups, the results, by Di Francesco [41], are substantially more complicated. Let us begin with some examples. We first have:

PROPOSITION 7.19. *The first Gram matrices and determinants for O_N^+ are*

$$\det\begin{pmatrix}N^2&N\\N&N^2\end{pmatrix}=N^2(N^2-1)$$

$$\det\begin{pmatrix}N^3&N^2&N^2&N^2&N\\N^2&N^3&N&N&N^2\\N^2&N&N^3&N&N^2\\N^2&N&N&N^3&N^2\\N&N^2&N^2&N^2&N^3\end{pmatrix}=N^5(N^2-1)^4(N^2-2)$$

with the matrices being written by using the lexicographic order on $NC_2(2k)$.

PROOF. The formula at $k=2$, where $NC_2(4)=\{\sqcap\sqcap,\Cap\}$, is clear. At $k=3$ however, things are tricky. We have $NC(3)=\{|||,\sqcap|,\sqcap\!\!\!|\,,|\sqcap,\sqcap\!\sqcap\}$, and the corresponding Gram matrix and its determinant are, according to Theorem 3.17:

$$\det\begin{pmatrix}N^3&N^2&N^2&N^2&N\\N^2&N^2&N&N&N\\N^2&N&N^2&N&N\\N^2&N&N&N^2&N\\N&N&N&N&N\end{pmatrix}=N^5(N-1)^4(N-2)$$

By using Proposition 3.11, the Gram determinant of $NC_2(6)$ is given by:

$$\begin{aligned}\det(G_{6N})&=\frac{1}{N^2\sqrt{N}}\times N^{10}(N^2-1)^4(N^2-2)\times\frac{1}{N^2\sqrt{N}}\\&=N^5(N^2-1)^4(N^2-2)\end{aligned}$$

Thus, we have obtained the formula in the statement. □

In general, such tricks won't work, because $NC(k)$ is strictly smaller than $P(k)$ at $k\geq 4$. However, following Di Francesco [41], we have the following result:

THEOREM 7.20. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb Z$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb Z$.

PROOF. This is something quite technical, obtained by using a decomposition as follows of the Gram matrix G_{kN}, with the matrix T_{kN} being lower triangular:

$$G_{kN}=T_{kN}T_{kN}^t$$

Thus, a bit as in the proof of the Lindstöm formula, we obtain the result, but the problem lies however in the construction of T_{kN}, which is non-trivial. See [41]. □

We refer to [5] for further details regarding the above result, including a short proof, based on the bipartite planar algebra combinatorics developed by Jones in [61]. Let us also mention that the Chebycheff polynomials have something to do with all this due to the fact that these are the orthogonal polynomials for the Wigner law. See [5].

Moving ahead now, regarding S_N^+, we have here the following formula, which is quite similar, obtained via shrinking, also from Di Francesco [41]:

THEOREM 7.21. *The determinant of the Gram matrix for S_N^+ is given by*

$$\det(G_{kN})=(\sqrt{N})^{a_k}\prod_{r=1}^{k}P_r(\sqrt{N})^{d_{kr}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb Z$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb Z$, and where $a_k=\sum_{\pi\in\mathcal P(k)}(2|\pi|-k)$.

PROOF. This follows indeed from Theorem 3.20, by using Proposition 3.11. □

Getting back now to our quantum permutation group questions, by using the above results we can produce a key technical ingredient, as follows:

PROPOSITION 7.22. *The following are linearly independent, for any $N\geq 4$:*

(1) *The linear maps $\left\{T_\pi\middle|\pi\in NC(k,l)\right\}$, with $k,l\in 2\mathbb{N}$.*
(2) *The vectors $\left\{\xi_\pi\middle|\pi\in NC(k)\right\}$, with $k\in\mathbb{N}$.*
(3) *The linear maps $\left\{T_\pi\middle|\pi\in NC(k,k)\right\}$, with $k\in\mathbb{N}$.*

PROOF. The statement is identical to Proposition 3.12, with the assumption $N=n^2$ lifted. As for the proof, this comes from the formula in Theorem 3.21. □

With this in hand, we have the following extension of Theorem 3.13:

THEOREM 7.23. *The quantum groups S_N^+ with $N\geq 4$ are as follows:*

(1) *The moments of the main character are the Catalan numbers:*
$$\int_{S_N^+}\chi^k=C_k$$
(2) *The main character follows the Marchenko-Pastur law, $\chi\sim\pi_1$.*
(3) *The fusion rules for irreducible representations are as for SO_3, namely:*
$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+1}+\ldots+r_{k+l}$$
(4) *We have $\dim r_k=(q^{k+1}-q^{-k})/(q-1)$, with $q+q^{-1}=N-2$.*

PROOF. This is identical to the proof of Theorem 3.13, by using this time the linear independence result from Proposition 3.22 as technical ingredient. □

So long for representations of S_N^+. All the above might seem quite complicated, but we repeat, up to some standard algebra, everything comes down to Proposition 3.22. And with some solid modern mathematical knowledge, be that operator algebras a la Jones, or deformed quantum groups a la Woronowicz, or meander determinants a la Di Francesco, the result there is in fact trivial. You can check here [5], [5], both short papers.

In what concerns us, we will be back to the similarity between S_N^+ and SO_3 on several occasions, with a number of further results on the subject, refining Theorem 3.23.

7d. Planar algebras

In the remainder of this chapter we keep developing some useful theory for U_N^+,O_N^+,S_N^+. We will present among others a result from [7], refining the Tannakian duality for the quantum permutation groups $G\subset S_N^+$, stating that these quantum groups are in correspondence with the subalgebras of Jones' spin planar algebra $P\subset\mathcal{S}_N$.

In order to get started, we need a lot of preliminaries, the lineup being von Neumann algebras, II_1 factors, subfactors, and finally planar algebras. We already met von Neumann algebras, in chapter 1. The advanced general theory regarding them is as follows:

THEOREM 7.24. *The von Neumann algebras $A\subset B(H)$ are as follows:*

(1) *Any such algebra decomposes as $A=\int_X A_xdx$, with X being the spectrum of the center, $Z(A)=L^\infty(X)$, and with the fibers A_x being factors, $Z(A_x)=\mathbb C$.*
(2) *The factors can be fully classified in terms of II_1 factors, which are those factors satisfying $\dim A=\infty$, and having a faithful trace $tr:A\to\mathbb C$.*
(3) *The II_1 factors enjoy the "continuous dimension geometry" property, in the sense that the traces of their projections can take any values in $[0,1]$.*
(4) *Among the II_1 factors, the smallest one is the Murray-von Neumann hyperfinite factor R, obtained as an inductive limit of matrix algebras.*

PROOF. This is something heavy, the idea being as follows:

(1) This is von Neumann's reduction theory theorem, which follows in finite dimensions from $A=M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$, and whose proof in general is quite technical.

(2) This comes from results of Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions.

(3) This is subtle functional analysis, with the rational traces being relatively easy to obtain, and with the irrational ones coming from limiting arguments.

(4) Once again, heavy results, by Murray-von Neumann and Connes, the idea being that any finite dimensional construction always leads to the same factor, called R. □

Let us discuss now subfactor theory, following Jones' fundamental paper [58]. Jones looked at the inclusions of II_1 factors $A\subset B$, called subfactors, which are quite natural objects in physics. Given such an inclusion, we can talk about its index:

DEFINITION 7.25. *The index of an inclusion of II_1 factors $A\subset B$ is the quantity*

$$[B:A]=\dim_AB\in[1,\infty]$$

constructed by using the Murray-von Neumann continuous dimension theory.

In order to explain Jones' result in [58], it is better to relabel our subfactor as $A_0\subset A_1$. We can construct the orthogonal projection $e_1:A_1\to A_0$, and set:

$$A_2=<A_1,e_1>$$

This remarkable procedure, called "basic construction", can be iterated, and we obtain in this way a whole tower of II_1 factors, as follows:

$$A_0\subset_{e_1}A_1\subset_{e_2}A_2\subset_{e_3}A_3\subset\ldots\ldots$$

Quite surprisingly, this construction leads to a link with the Temperley-Lieb algebra $TL_N=span(NC_2)$. The results can be summarized as follows:

THEOREM 7.26. *Let $A_0\subset A_1$ be an inclusion of* II_1 *factors.*

(1) *The sequence of projections $e_1,e_2,e_3,\ldots\in B(H)$ produces a representation of the Temperley-Lieb algebra of index $N=[A_1,A_0]$, as follows:*

$$TL_N\subset B(H)$$

(2) *The index $N=[A_1,A_0]$, which is a Murray-von Neumann continuous quantity $N\in[1,\infty]$, must satisfy the following condition:*

$$N\in\left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n\in\mathbb{N}\right\}\cup[4,\infty]$$

PROOF. This result, from [58], is something tricky, the idea being as follows:

(1) The idea here is that the functional analytic study of the basic construction leads to the conclusion that the sequence of projections $e_1,e_2,e_3,\ldots\in B(H)$ behaves algebrically, when rescaled, exactly as the sequence of diagrams $\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots\in TL_N$ given by:

$$\varepsilon_1={}^{\cup}_{\cap}\quad,\quad\varepsilon_2=|{}^{\cup}_{\cap}\quad,\quad\varepsilon_3=||{}^{\cup}_{\cap}\quad,\quad\ldots$$

But these diagrams generate TL_N, and so we have an embedding $TL_N\subset B(H)$, where H is the Hilbert space where our subfactor $A_0\subset A_1$ lives, as claimed.

(2) This is something quite surprising, which follows from (1), via some clever positivity considerations, involving the Perron-Frobenius theorem. In fact, the subfactors having index $N\in[1,4]$ can be classified by ADE diagrams, and the obstruction $N=4\cos^2(\frac{\pi}{n})$ comes from the fact that N must be the squared norm of such a graph. □

Quite remarkably, Theorem 3.26 is just the tip of the iceberg. One can prove indeed that the planar algebra structure of TL_N, taken in an intuitive sense, extends to a planar algebra structure on the sequence of relative commutants $P_k=A_0'\cap A_k$. In order to discuss this key result, due as well to Jones, from [60], and that we will need too, in connection with our quantum group problems, let us start with:

DEFINITION 7.27. *The planar algebras are defined as follows:*

(1) *A k-tangle, or k-box, is a rectangle in the plane, with $2k$ marked points on its boundary, containing r small boxes, each having $2k_i$ marked points, and with the $2k+\sum 2k_i$ marked points being connected by noncrossing strings.*
(2) *A planar algebra is a sequence of finite dimensional vector spaces $P=(P_k)$, together with linear maps $P_{k_1}\otimes\ldots\otimes P_{k_r}\to P_k$, one for each k-box, such that the gluing of boxes corresponds to the composition of linear maps.*

As basic example of a planar algebra, we have the Temperley-Lieb algebra TL_N. Indeed, putting $TL_N(k_i)$ diagrams into the small r boxes of a k-box clearly produces a $TL_N(k)$ diagram, so we have indeed a planar algebra, of somewhat "trivial" type.

In general, the planar algebras are more complicated than this, and we will be back later with some explicit examples. However, the idea is very simple, namely that "the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams". In relation now with subfactors, the result, which extends Theorem 3.26 (1), and which was found by Jones in [60], almost 20 years after [58], is as follows:

THEOREM 7.28. *Given a subfactor $A_0 \subset A_1$, the collection $P = (P_k)$ of linear spaces*

$$P_k = A_0' \cap A_k$$

has a planar algebra structure, extending the planar algebra structure of TL_N.

PROOF. As a first observation, since $e_1 : A_1 \to A_0$ commutes with A_0 we have $e_1 \in P_2'$. By translation we obtain $e_1, \ldots, e_{k-1} \in P_k$ for any k, and so:

$$TL_N \subset P$$

The point now is that the planar algebra structure of TL_N, obtained by composing diagrams, can be shown to extend into an abstract planar algebra structure of P. This is something quite technical, and we will not get into details here. See [60]. □

Getting back to quantum groups, all this machinery is interesting for us. We will need the construction of the tensor and spin planar algebras $\mathcal{T}_N, \mathcal{S}_N$. Let us start with:

DEFINITION 7.29. *The tensor planar algebra $\mathcal{T}_N$ is the sequence of vector spaces*

$$P_k = M_N(\mathbb{C})^{\otimes k}$$

with the multilinear maps $T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by the formula

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we are using here a construction which is very similar to the construction $\pi \to T_\pi$ that we used for easy quantum groups. We put the indices of the basic tensors on the marked points of the small boxes, in the obvious way, and the coefficients of the output tensor are then given by Kronecker symbols, exactly as in the easy case.

The fact that we have indeed a planar algebra, in the sense that the gluing of tangles corresponds to the composition of linear maps, as required by Definition 3.27, is something elementary, in the same spirit as the verification of the functoriality properties of the correspondence $\pi \to T_\pi$, discussed in chapter 2, and we refer here to Jones [60].

Let us discuss now a second planar algebra of the same type, which is important as well for various reasons, namely the spin planar algebra $\mathcal{S}_N$. This planar algebra appears somehow as the "square root" of the tensor planar algebra $\mathcal{T}_N$. Let us start with:

Definition 7.30. *We write the standard basis of $(\mathbb{C}^N)^{\otimes k}$ in $2\times k$ matrix form,*

$$e_{i_1\ldots i_k}=\begin{pmatrix}i_1&i_1&i_2&i_2&i_3&\ldots&\ldots\\ i_k&i_k&i_{k-1}&\ldots&\ldots&\ldots&\ldots\end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra $\mathcal{S}_N$, also from [60], as follows:

Definition 7.31. *The spin planar algebra $\mathcal{S}_N$ is the sequence of vector spaces*

$$P_k=(\mathbb{C}^N)^{\otimes k}$$

written as above, with the multiplinear maps $T_\pi:P_{k_1}\otimes\ldots\otimes P_{k_r}\to P_k$ being given by

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_r})=\sum_j\delta_\pi(i_1,\ldots,i_r:j)e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

Here are some illustrating examples for the spin planar algebra calculus:

(1) The identity 1_k is the (k,k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x,y)=1$ being the pairs of the form (x,x), this tangle 1_k acts by the identity:

$$1_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(2) The multiplication M_k is the (k,k,k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. It acts in the following way:

$$M_k\left(\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}\otimes\begin{pmatrix}l_1&\ldots&l_k\\ m_1&\ldots&m_k\end{pmatrix}\right)=\delta_{j_1m_1}\ldots\delta_{j_km_k}\begin{pmatrix}l_1&\ldots&l_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(3) The inclusion I_k is the $(k,k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x,y)=1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_l\begin{pmatrix}j_1&\ldots&j_k&l\\ i_1&\ldots&i_k&l\end{pmatrix}$$

(4) The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(5) The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting

from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1)=\sum_{ijl}\begin{pmatrix}i_1&\ldots&i_k&j&j\\ i_1&\ldots&i_k&l&l\end{pmatrix}$$

The elements $e_k=N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

(6) The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k=\begin{matrix}\Cap&|&|&|&\|\\ \|&&&&\|\\ \|&|&|&|&\Cup\end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2\ldots i_k})=e_{i_2\ldots i_ki_1}$$

There are many other interesting examples of k-tangles, but in view of our present purposes, we can actually stop here, due to the following fact:

THEOREM 7.32. *The multiplications, inclusions, expectations, Jones projections and rotations generate the set of all tangles, via the gluing operation.*

PROOF. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list. See [60]. □

Finally, in order for our discussion to be complete, we must talk as well about the $*$-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}^*=\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}$$

Getting back now to quantum groups, following [7], we have the following result:

THEOREM 7.33. *Given $G\subset S_N^+$, consider the tensor powers of the associated coaction map on $C(X)$, where $X=\{1,\ldots,N\}$, which are the folowing linear maps:*

$$\Phi^k:C(X^k)\to C(X^k)\otimes C(G)$$

$$e_{i_1\ldots i_k}\to\sum_{j_1\ldots j_k}e_{j_1\ldots j_k}\otimes u_{j_1i_1}\ldots u_{j_ki_k}$$

The fixed point spaces of these coactions, which are by definition the spaces

$$P_k=\left\{x\in C(X^k)\Big|\Phi^k(x)=1\otimes x\right\}$$

are given by $P_k=Fix(u^{\otimes k})$, and form a subalgebra of the spin planar algebra $\mathcal{S}_N$.

PROOF. Since the map Φ is a coaction, its tensor powers Φ^k are coactions too, and at the level of fixed point algebras we have the following formula:

$$P_k = Fix(u^{\otimes k})$$

In order to prove now the planar algebra assertion, we will use Theorem 3.32. Consider the rotation R_k. Rotating, then applying Φ^k, and rotating backwards by R_k^{-1} is the same as applying Φ^k, then rotating each k-fold product of coefficients of Φ. Thus the elements obtained by rotating, then applying Φ^k, or by applying Φ^k, then rotating, differ by a sum of Dirac masses tensored with commutators in $A = C(G)$:

$$\Phi^k R_k(x) - (R_k \otimes id)\Phi^k(x) \in C(X^k) \otimes [A, A]$$

Now let $\int_A$ be the Haar functional of A, and consider the conditional expectation onto the fixed point algebra P_k, which is given by the following formula:

$$\phi_k = \left(id \otimes \int_A\right) \Phi^k$$

Since $\int_A$ is a trace, it vanishes on commutators. Thus R_k commutes with ϕ_k:

$$\phi_k R_k = R_k \phi_k$$

The commutation relation $\phi_k T = T\phi_l$ holds in fact for any (l, k)-tangle T. These tangles are called annular, and the proof is by verification on generators of the annular category. In particular we obtain, for any annular tangle T:

$$\phi_k T \phi_l = T \phi_l$$

We conclude from this that the annular category is contained in the suboperad $\mathcal{P}' \subset \mathcal{P}$ of the planar operad consisting of tangles T satisfying the following condition, where $\phi = (\phi_k)$, and where $i(.)$ is the number of input boxes:

$$\phi T \phi^{\otimes i(T)} = T \phi^{\otimes i(T)}$$

On the other hand the multiplicativity of Φ^k gives $M_k \in \mathcal{P}'$. Now since the planar operad $\mathcal{P}$ is generated by multiplications and annular tangles, it follows that we have $\mathcal{P}' = P$. Thus for any tangle T the corresponding multilinear map between spaces $P_k(X)$ restricts to a multilinear map between spaces P_k. In other words, the action of the planar operad $\mathcal{P}$ restricts to P, and makes it a subalgebra of $\mathcal{S}_N$, as claimed. □

As a second result now, also from [7], completing our study, we have:

THEOREM 7.34. *We have a bijection between quantum permutation groups and subalgebras of the spin planar algebra,*

$$(G \subset S_N^+) \quad \longleftrightarrow \quad (Q \subset \mathcal{S}_N)$$

given in one sense by the construction in Theorem 3.33, and in the other sense by a suitable modification of Tannakian duality.

Proof. The idea is that this will follow by applying Tannakian duality to the annular category over Q. Let n, m be positive integers. To any element $T_{n+m} \in Q_{n+m}$ we associate a linear map $L_{nm}(T_{n+m}) : P_n(X) \to P_m(X)$ in the following way:

$$L_{nm}\begin{pmatrix} |\,|\,| \\ T_{n+m} \\ |\,|\,| \end{pmatrix} : \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \to \begin{pmatrix} |\ \ |\ \ \cap \\ \ \ T_{n+m}| \\ |\,|\ \ |\ | \\ a_n|\ \ |\ | \\ \cup\ \ |\ | \end{pmatrix}$$

That is, we consider the planar $(n, n+m, m)$-tangle having an small input n-box, a big input $n+m$-box and an output m-box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X) \otimes P_{n+m}(X) \to P_m(X)$$

If we put the element T_{n+m} in the big input box, we obtain in this way a certain linear map $P_n(X) \to P_m(X)$, that we call L_{nm}. With this convention, let us set:

$$Q_{nm} = \left\{ L_{nm}(T_{n+m}) : P_n(X) \to P_m(X) \Big| T_{n+m} \in Q_{n+m} \right\}$$

These spaces form a Tannakian category, so by [100] we obtain a Woronowicz algebra (A, u), such that the following equalities hold, for any m, n:

$$Hom(u^{\otimes m}, u^{\otimes n}) = Q_{mn}$$

We prove now that u is a magic unitary. We have $Hom(1, u^{\otimes 2}) = Q_{02} = Q_2$, so the unit of Q_2 must be a fixed vector of $u^{\otimes 2}$. But $u^{\otimes 2}$ acts on the unit of Q_2 as follows:

$$\begin{aligned} u^{\otimes 2}(1) &= u^{\otimes 2}\left(\sum_i \begin{pmatrix} i & i \\ i & i \end{pmatrix}\right) \\ &= \sum_{ikl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes u_{ki}u_{li} \\ &= \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (uu^t)_{kl} \end{aligned}$$

From $u^{\otimes 2}(1) = 1 \otimes 1$ ve get that uu^t is the identity matrix. Together with the unitarity of u, this gives the following formulae:

$$u^t = u^* = u^{-1}$$

Consider the Jones projection $E_1 \in Q_3$. After isotoping, $L_{21}(E_1)$ looks as follows:

$$L_{21}\begin{pmatrix} |\cup \\ |\cap \end{pmatrix} : \begin{pmatrix} |\ \ | \\ {\scriptstyle i}\ \ {\scriptstyle i} \\ {\scriptstyle j}\ \ {\scriptstyle j} \\ |\ \ | \end{pmatrix} \to \begin{pmatrix} | & \\ {\scriptstyle i}\ \ {\scriptstyle i} & \\ {\scriptstyle j}\ \ {\scriptstyle j} & \supset \\ | & \end{pmatrix} = \delta_{ij}\begin{pmatrix} | \\ {\scriptstyle i} \\ {\scriptstyle i} \\ | \end{pmatrix}$$

In other words, the linear map $M = L_{21}(E_1)$ is the multiplication $\delta_i \otimes \delta_j \to \delta_{ij}\delta_i$:

$$M \begin{pmatrix} i & i \\ j & j \end{pmatrix} = \delta_{ij} \begin{pmatrix} i \\ i \end{pmatrix}$$

In order to finish, consider the following element of $C(X) \otimes A$:

$$(M \otimes id)u^{\otimes 2} \left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) = \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes u_{ki}u_{kj}$$

Since $M \in Q_{21} = Hom(u^{\otimes 2}, u)$, this equals the following element of $C(X) \otimes A$:

$$u(M \otimes id) \left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) = \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes \delta_{ij}u_{ki}$$

Thus we have $u_{ki}u_{kj} = \delta_{ij}u_{ki}$ for any i, j, k, which shows that u is a magic unitary. Now if P is the planar algebra associated to u, we have $Hom(1, v^{\otimes n}) = P_n = Q_n$, as desired. As for the uniqueness, this is clear from the Peter-Weyl theory. □

All the above might seem a bit technical, but is worth learning, and for good reason, because it is extremely powerful. As an example of immediate application, if you agree with the bijection $G \leftrightarrow Q$ in Theorem 3.34, then $G = S_N^+$ itself, which is the biggest object on the left, must correspond to the smallest object on the right, namely $Q = TL_N$. Thus, more or less everything that we learned so far in this book is trivial.

Welcome to planar algebras. Try to master this technology. And once this understood, get to know some analysis too, which comes after. But it will be among our main purposes here to do so, getting you familiar with algebra, and with some analysis as well.

Back now to work, the results established above, regarding the subgroups $G \subset S_N^+$, have several generalizations, to the subgroups $G \subset O_N^+$ and $G \subset U_N^+$, as well as subfactor versions, going beyond the combinatorial level. At the algebraic level, we have:

THEOREM 7.35. *The following happen:*

(1) *The closed subgroups $G \subset O_N^+$ produce planar algebras $P \subset \mathcal{T}_N$, via the following formula, and any subalgebra $P \subset \mathcal{T}_N$ appears in this way:*
$$P_k = End(u^{\otimes k})$$
(2) *The closed subgroups $G \subset U_N^+$ produce planar algebras $P \subset \mathcal{T}_N$, via the following formula, and any subalgebra $P \subset \mathcal{T}_N$ appears in this way:*
$$P_k = End(\underbrace{u \otimes \bar{u} \otimes u \otimes \ldots}_{k\ terms})$$
(3) *In fact, the closed subgroups $G \subset PO_N^+ \simeq PU_N^+$ are in correspondence with the subalgebras $P \subset \mathcal{T}_N$, with $G \to P$ being given by $P_k = Fix(u^{\otimes k})$.*

PROOF. There is a long story with this result, whose origins go back to papers of mine written before the 1999 papers [5], [60], using Popa's standard lattice formalism, instead of the planar algebra one, and then to a number of papers written in the early 2000s, proving results which are more general. For the whole story, and a modern treatment of the subject, we refer to Tarrago-Wahl [87]. As in what regards the proof:

(1) This is similar to the proof of Theorem 3.33 and Theorem 3.34, ultimately coming from Woronowicz's Tannakian duality in [100]. Note however that the correspondence is not bijective, because the spaces P_k determine $PG\subset PO_N^+$, but not $G\subset O_N^+$ itself.

(2) This is an extension of (1), and the same comments apply. With the extra comment that the fact that the subgroups $PG\subset PO_N^+$ produce the same planar algebras as the subgroups $PG\subset PU_N^+$ should not be surprising, due to $PO_N^+=PU_N^+$.

(3) This is an extension of (2), and a further extension of (1), and is in fact the best result on the subject, due to the fact that we have there a true, bijective correspondence. As before, this ultimately comes from Woronowicz's Tannakian duality in [100].

(4) As a final comment, you might say that, now that we have (3) as ultimate result on the subject, why not saying a few words about the proof. In answer, (3) is in fact just the tip of the iceberg, so we prefer to discuss this later, once we'll see the whole iceberg. □

Finally, in relation with subfactors, the result here is as follows:

THEOREM 7.36. *The planar algebras coming the subgroups $G\subset S_N^+$ appear from fixed point subfactors, of the following type,*

$$A^G\subset(\mathbb C^N\otimes A)^G$$

and the planar algebras coming from the subgroups $G\subset PO_N^+=PU_N^+$ appear as well from fixed point subfactors, of the following type,

$$A^G\subset(M_N(\mathbb C)\otimes A)^G$$

with the action $G\curvearrowright A$ being assumed to be minimal, $(A^G)'\cap A=\mathbb C$.

PROOF. Again, there is a long story with this result, and besides needing some explanations, regarding the proof, all this is in need of some unification. We will be back to this in chapter 4, and in the meantime we refer to [5], [87] and related papers. □

Finally, let us mention that an important question, which is still open, is that of understanding whether the above subfactors can be taken to be hyperfinite, $A^G\simeq R$. This is related to the axiomatization of hyperfinite subfactors, another open question, which is of central importance in von Neumann algebras. We will be back to this.

7e. Exercises

Exercises:

EXERCISE 7.37.

EXERCISE 7.38.

EXERCISE 7.39.

EXERCISE 7.40.

EXERCISE 7.41.

EXERCISE 7.42.

EXERCISE 7.43.

EXERCISE 7.44.

Bonus exercise.

CHAPTER 8

Analytic aspects

8a. Matrix models

One potentially interesting method for the study of the closed subgroups $G \subset S_N^+$, that we have not tried yet, consists in modeling the standard coordinates $u_{ij} \in C(G)$ by concrete variables over some familiar C^*-algebra, $U_{ij} \in B$. Indeed, assuming that the model is faithful in some suitable sense, and that the variables U_{ij} are not too complicated, all questions about G would correspond in this way to routine questions inside B. We will discuss here such questions, which are quite interesting, first for the arbitrary closed subgroups $G \subset U_N^+$, and then for the quantum permutation groups $G \subset S_N^+$.

All this sounds good, mathematically speaking, and we will soon see that there are some potentially interesting connections with physics as well. Getting started now, we have a good idea, but we must first solve the following philosophical question:

Question 8.1. *What type of target algebras B shall we use for our matrix models $\pi : C(G) \to B$? We would like these to be simple enough, as for the computations inside them to be doable, but also general enough, as to model well our quantum groups.*

In answer, a good idea would be probably that of using random matrix algebras, $B = M_K(C(T))$, with $K \geq 1$ being an integer, and T being a compact space. Indeed, these algebras generalize the most familiar algebras that we know, namely the matrix ones $M_K(\mathbb{C})$, and the commutative ones $C(T)$, so they are definitely simple enough. As for their potential modeling power, my cat who knows some physics says okay.

In short, time to start our study, with the following definition:

Definition 8.2. *A matrix model for $G \subset U_N^+$ is a morphism of C^*-algebras*

$$\pi : C(G) \to M_K(C(T))$$

where $K \geq 1$ is an integer, and T is a compact space.

As a first comment, focusing on such models might look a bit restrictive, but we will soon discover that, with some know-how, we can do many things with such models. For the moment, let us develop some general theory. The main question to be solved is that of understanding the suitable faithfulness assumptions needed on π, as for the model to "remind" the quantum group. As we will see, this is something quite tricky.

The simplest situation is when π is faithful in the usual sense. Here π obviously reminds G. However, this is something quite restrictive, because in this case the algebra $C(G)$ must be quite small, admitting an embedding as follows:

$$\pi : C(G) \subset M_K(C(T))$$

Technically, this means that $C(G)$ must be of type I, as an operator algebra, and we will discuss this in a moment, with the comment that this is indeed something quite restrictive. However, there are many interesting examples here, and all this is worth a detailed look. First, we have the following result, providing us with basic examples:

Proposition 8.3. *The following closed subgroups $G \subset U_N^+$ have faithful models:*

(1) *The compact Lie groups $G \subset U_N$.*
(2) *The finite quantum groups $G \subset U_N^+$.*

In both cases, we can arrange for $\int_G$ to be restriction of the random matrix trace.

Proof. These assertions are all elementary, the proofs being as follows:

(1) This is clear, because we can simply use here the identity map:

$$id : C(G) \to M_1(C(G))$$

(2) Here we can use the left regular representation $\lambda : C(G) \to M_{|G|}(\mathbb{C})$. Indeed, let us endow the linear space $H = C(G)$ with the scalar product $< a, b >= \int_G ab^*$. We have then a representation of $*$-algebras, as follows:

$$\lambda : C(G) \to B(H) \quad , \quad a \to [b \to ab]$$

Now since we have $H \simeq \mathbb{C}^{|G|}$, we can view λ as a matrix model map, as above.

(3) Finally, our claim is that we can choose our model as for the following formula to hold, where $\int_T$ is the integration with respect to a given probability measure on T:

$$\int_G = \left(tr \otimes \int_T \right) \pi$$

But this is clear for the model in (1), by definition, and is clear as well for the model in (2), by using the basic properties of the left regular representation. □

In the above result, the last assertion is quite interesting, and suggests formulating the following definition, somewhat independently on the notion of faithfulness:

Definition 8.4. *A matrix model $\pi : C(G) \to M_K(C(T))$ is called stationary when*

$$\int_G = \left(tr \otimes \int_T \right) \pi$$

where $\int_T$ is the integration with respect to a given probability measure on T.

Here the term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit needed to be stationary, and this will be explained later. Yet another explanation comes from a certain relation with the lattice models, but this is something rather folklore, not axiomatized yet. We will be back to this.

We will see in a moment that stationarity implies faithfulness, so that stationarity can be regarded as being a useful, pragmatic version of faithfulness. But let us first discuss the examples. Besides those in Proposition 4.3, we can look at group duals. So, consider a discrete group Γ, and a model for the corresponding group algebra, as follows:

$$\pi : C^*(\Gamma) \to M_K(C(T))$$

Since a representation of a group algebra must come from a unitary representation of the group, such a matrix model must come from a representation as follows:

$$\rho : \Gamma \to C(T, U_K)$$

With this identification made, we have the following result:

PROPOSITION 8.5. *An matrix model $\rho : \Gamma \subset C(T, U_K)$ is stationary when:*

$$\int_T tr(g^x)dx = 0, \forall g \neq 1$$

Moreover, the examples include all abelian groups, and all finite groups.

PROOF. Consider indeed a group embedding $\rho : \Gamma \subset C(T, U_K)$, which produces by linearity a matrix model, as follows:

$$\pi : C^*(\Gamma) \to M_K(C(T))$$

It is enough to formulate the stationarity condition on the group elements $g \in C^*(\Gamma)$. Let us set $\rho(g) = (x \to g^x)$. With this notation, the stationarity condition reads:

$$\int_T tr(g^x)dx = \delta_{g,1}$$

Since this equality is trivially satisfied at $g = 1$, where by unitality of our representation we must have $g^x = 1$ for any $x \in T$, we are led to the condition in the statement. Regarding now the examples, these are both clear. More precisely:

(1) When Γ is abelian we can use the following trivial embedding:

$$\Gamma \subset C(\widehat{\Gamma}, U_1) \quad , \quad g \to [\chi \to \chi(g)]$$

(2) When Γ is finite we can use the left regular representation:

$$\Gamma \subset \mathcal{L}(\mathbb{C}\Gamma) \quad , \quad g \to [h \to gh]$$

Indeed, in both cases, the stationarity condition is trivially satisfied. □

In order to discuss now certain analytic aspects of the matrix models, let us go back to the von Neumann algebras, discussed in chapter 1, and in chapter 3. We recall from there that we have the following result, due to Murray-von Neumann and Connes:

THEOREM 8.6. *Given a von Neumann algebra $A\subset B(H)$, if we write its center as*

$$Z(A)=L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x having trivial center:

$$A=\int_X A_x\,dx$$

Moreover, the factors, $Z(A)=\mathbb C$, can be basically classified in terms of the II_1 *factors, which are those satisfying* $\dim A=\infty$*, and having a faithful trace $tr:A\to\mathbb C$.*

PROOF. This is something which is clear in finite dimensions, and in the commutative case too. In general, this is something heavy, the idea being as follows:

(1) The first assertion, regarding the decomposition into factors, is von Neumann's reduction theory main result, which is actually one of the heaviest results in fundamental mathematics, and whose proof uses advanced functional analysis techniques.

(2) The classification of factors, due to Murray-von Neumann and Connes, is again something heavy, the idea being that the II_1 factors are the "building blocks", with the other factors basically appearing from them via crossed product type constructions. □

Back now to matrix models, as a first general result, which is something which is not exactly trivial, and whose proof requires some functional analysis, we have:

THEOREM 8.7. *Assuming that a closed subgroup $G\subset U_N^+$ has a stationary model*

$$\pi:C(G)\to M_K(C(T))$$

it follows that G must be coamenable, and that the model is faithful. Moreover, π extends into an embedding of von Neumann algebras, as follows,

$$L^\infty(G)\subset M_K(L^\infty(T))$$

which commutes with the canonical integration functionals.

PROOF. Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi:C(G)\to C(G)_{red}\subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that π is faithful. With this assumption made, we have an embedding as follows:

$$C(G)\subset M_K(C(T))$$

By performing the GNS construction we obtain a better embedding, as follows:

$$L^\infty(G)\subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A=L^\infty(G)$. But this means that, when writing the center of this latter algebra as $Z(A)=L^\infty(X)$, the whole algebra decomposes over X, as an integral of type I factors:

$$L^\infty(G)=\int_X M_{K_x}(\mathbb C)\,dx$$

In particular, we can see from this that $C(G)\subset L^\infty(G)$ has a unique C^*-norm, and so G is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of π consists of the identity, and of an inclusion. □

In relation with the above, we have the following well-known result of Thoma:

THEOREM 8.8. *For a discrete group Γ, the following are equivalent:*

(1) *$C^*(\Gamma)$ is of type I, so that we have an embedding $\pi:C^*(\Gamma)\subset M_K(C(X))$, with X being a compact space.*
(2) *$C^*(\Gamma)$ has a stationary model of type $\pi:C^*(\Gamma)\to M_F(C(L))$, with F being a finite group, and L being a compact abelian group.*
(3) *Γ is virtually abelian, in the sense that we have an abelian subgroup $\Lambda\triangleleft\Gamma$ such that the quotient group $F=\Gamma/\Lambda$ is finite.*
(4) *Γ has an abelian subgroup $\Lambda\subset\Gamma$ whose index $K=[\Gamma:\Lambda]$ is finite.*

PROOF. There are several proofs for this fact, the idea being as follows:

(1) $\implies$ (4) This is the non-trivial implication, that we will not prove here. We refer instead to the literature, either Thoma's orignal paper, or books like those of Dixmier, mixing advanced group theory and advanced operator algebra theory.

(4) $\implies$ (3) We choose coset representatives $g_i\in\Gamma$, and we set:

$$\Lambda'=\bigcap_i g_i\Gamma g_i^{-1}$$

Then $\Lambda'\subset\Lambda$ has finite index, and we have $\Lambda'\triangleleft\Gamma$, as desired.

(3) $\implies$ (2) This follows by using the theory of induced representations. We can define a model $\pi:C^*(\Gamma)\to M_F(C(\widehat{\Lambda}))$ by setting:

$$\pi(g)(\chi)=Ind_\Lambda^\Gamma(\chi)(g)$$

Indeed, any character $\chi\in\widehat{\Lambda}$ is a 1-dimensional representation of Λ, and we can therefore consider the induced representation $Ind_\Lambda^\Gamma(\chi)$ of the group Γ. This representation is $|F|$-dimensional, and so maps the group elements $g\in\Gamma$ into order $|F|$ matrices $Ind_\Lambda^\Gamma(\chi)(g)$. Thus the above map π is well-defined, and the fact that it is a representation is clear as well. In order to check now the stationarity property of this representation, we can use the following well-known character formula, due to Frobenius:

$$Tr\left(Ind_\Lambda^\Gamma(\chi)(g)\right)=\sum_{x\in F}\delta_{x^{-1}gx\in\Lambda}\chi(x^{-1}gx)$$

By integrating with respect to $\chi\in\widehat{\Lambda}$, we deduce from this that we have:

$$\begin{aligned}\left(Tr\otimes\int_{\widehat{\Lambda}}\right)\pi(g) &= \sum_{x\in F}\delta_{x^{-1}gx\in\Lambda}\int_{\widehat{\Lambda}}\chi(x^{-1}gx)d\chi\\ &= \sum_{x\in F}\delta_{x^{-1}gx\in\Lambda}\delta_{g,1}\\ &= |F|\cdot\delta_{g,1}\end{aligned}$$

Now by dividing by $|F|$ we conclude that the model is stationary, as claimed.

(2) $\implies$ (1) This is the trivial implication, with the faithfulness of π following from the abstract functional analysis arguments from the proof of Theorem 4.7. □

We refer to [5] and related papers for more on all this, including for some partial extensions of Thoma's theorem, to the case of the discrete quantum groups.

Getting back now to Definition 4.2, more generally, we can model in that way the standard coordinates $x_i\in C(X)$ of various algebraic manifolds $X\subset S^{N-1}_{\mathbb{C},+}$. Indeed, these manifolds generalize the compact matrix quantum groups, which appear as:

$$G\subset U_N^+\subset S^{N^2-1}_{\mathbb{C},+}$$

Thus, we have many other interesting examples of such manifolds, such as the homogeneous spaces over our quantum groups. However, at this level of generality, not much general theory is available. It is elementary to show that, under the technical assumption $X^{class}\neq\emptyset$, there exists a universal $K\times K$ model for the algebra $C(X)$, which factorizes as follows, with $X^{(K)}\subset X$ being a certain algebraic submanifold:

$$\pi_K:C(X)\to C(X^{(K)})\subset M_K(C(T_K))$$

To be more precise, the universal $K\times K$ model space T_K appears by imposing to the complex $K\times K$ matrices the relations defining X, and the algebra $C(X^{(K)})$ is then by definition the image of π_K. In relation with this, we can set as well:

$$X^{(\infty)}=\bigcup_{K\in\mathbb{N}}X^{(K)}$$

We are led in this way to a filtration of X, as follows:

$$X^{class}=X^{(1)}\subset X^{(2)}\subset X^{(3)}\subset\ldots\ldots\subset X^{(\infty)}\subset X$$

It is possible to say a few non-trivial things about these manifolds $X^{(K)}$. In the compact quantum group case, however, that we are mainly interested in here, the matrix truncations $G^{(K)} \subset G$ are generically not quantum subgroups at $K \geq 2$, and so this theory is a priori not very useful, at least in its basic form presented above.

8b. Inner faithfulness

Let us discuss now the general, non-coamenable case, with the aim of finding a weaker notion of faithfulness, which still does the job, namely that of "reminding" the quantum group. The idea comes by looking at the group duals $G = \widehat{\Gamma}$. Consider indeed a general model for the associated group algebra, which can be written as follows:

$$\pi : C^*(\Gamma) \to M_K(C(T))$$

The point is that such a representation of the group algebra must come by linearization from a unitary group representation, as follows:

$$\rho : \Gamma \to C(T, U_K)$$

Now observe that when this group representation ρ is faithful, the representation π is in general not faithful, for instance because when $T = \{.\}$ its target algebra is finite dimensional. On the other hand, this representation "reminds" Γ, so can be used in order to fully understand Γ. Thus, we have an idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker.

This weaker notion, which will be of great interest for us, is called "inner faithfulness". The general theory here, from [10], starts with the following definition:

DEFINITION 8.9. *Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.*

(1) *The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization as follows:*

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

(2) *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

The above notions are quite tricky, and having them well understood will take us some time. As a first example, motivated by the above discussion, in the case where $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation, as follows:

$$\rho : \Gamma \to C(T, U_K)$$

Thus the minimal factorization in (1) is obtained by taking the image:

$$\rho : \Gamma \to \Lambda \subset C(T, U_K)$$

Thus, as a conclusion, in this case π is inner faithful precisely when we have:

$$\Gamma \subset C(T, U_K)$$

Dually now, given a compact Lie group G, and elements $g_1,\ldots,g_K\in G$, we have a diagonal representation $\pi:C(G)\to M_K(\mathbb{C})$, appearing as follows:

$$f\to\begin{pmatrix}f(g_1)&&\\&\ddots&\\&&f(g_K)\end{pmatrix}$$

The minimal factorization of this representation π, as in Definition 4.9 (1), is then via the algebra $C(H)$, with H being the following closed subgroup of G:

$$H=\overline{<g_1,\ldots,g_K>}$$

Thus, as a conclusion, π is inner faithful precisely when we have:

$$G=H$$

There are many other examples of inner faithful representations, which are however substantially more technically advanced, and we will discuss them later.

Back to general theory now, in the framework of Definition 4.9, the existence and uniqueness of the Hopf image come by dividing $C(G)$ by a suitable ideal, with this being something standard. Alternatively, in Tannakian terms, as explained in [**10**], we have:

Theorem 8.10. *Assuming $G\subset U_N^+$, with fundamental corepresentation $u=(u_{ij})$, the Hopf image of a model $\pi:C(G)\to M_K(C(T))$ comes from the Tannakian category*

$$C_{kl}=Hom(U^{\otimes k},U^{\otimes l})$$

where $U_{ij}=\pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

Proof. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(U^{\otimes k},U^{\otimes l})$$

More generally, we have such inclusions when replacing (G,u) with any pair producing a factorization of π. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions. On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H,v) given by:

$$Hom(v^{\otimes k},v^{\otimes l})=Hom(U^{\otimes k},U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of π, as claimed. □

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here, from [**5**], is as follows:

THEOREM 8.11. *Given an inner faithful model $\pi : C(G) \to M_K(C(T))$, we have*

$$\int_G = \lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^k \int_G^r$$

with the truncations of the integration on the right being given by

$$\int_G^r = (\varphi \circ \pi)^{*r}$$

with $\phi * \psi = (\phi \otimes \psi)\Delta$, and with $\varphi = tr \otimes \int_T$ being the random matrix trace.

PROOF. This is something quite tricky, the idea being as follows:

(1) As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in chapter 1. In fact, the above result holds more generally for any model $\pi : C(G) \to B$, with $\varphi \in B^*$ being a faithful trace.

(2) In order to prove now the result, we can proceed as in chapter 1. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that:

$$\int_G' = \int_G$$

It is enough to check this on the coefficients of the Peter-Weyl corepresentations, and if we let $v = u^{\otimes k}$ be one of these corepresentations, we must prove that we have:

$$\left(id \otimes \int_G'\right) v = \left(id \otimes \int_G\right) v$$

(3) In order to prove this, we already know, from the Haar measure theory from chapter 1, that the matrix on the right is the orthogonal projection onto $Fix(v)$:

$$\left(id \otimes \int_G\right) v = Proj\Big[Fix(v)\Big]$$

Regarding now the matrix on the left, the trick in [99] applied to the linear form $\varphi\pi$ tells us that this is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi\pi)v$:

$$\left(id \otimes \int_G'\right) v = Proj\Big[1 \in (id \otimes \varphi\pi)v\Big]$$

(4) Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have the following formula:

$$(id \otimes \varphi\pi)v = (id \otimes \varphi)V$$

Thus, we can apply the trick in [99], and we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 4.10, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired. □

In practice, Theorem 4.11 is something quite powerful. As an illustration, regarding the law of the main character, we obtain here the following result:

PROPOSITION 8.12. *Assume that $\pi : C(G) \to M_K(C(T))$ is inner faithful, let*

$$\mu = law(\chi)$$

and let μ^r be the law of χ with respect to $\int_G^r = (\varphi \circ \pi)^{*r}$, where $\varphi = tr \otimes \int_T$.

(1) *We have the following convergence formula, in moments:*

$$\mu = \lim_{k \to \infty} \frac{1}{k} \sum_{r=0}^{k} \mu^r$$

(2) *The moments of μ^r are the numbers $c_\varepsilon^r = Tr(T_\varepsilon^r)$, where:*

$$(T_\varepsilon)_{i_1 \ldots i_p, j_1 \ldots j_p} = \left(tr \otimes \int_T \right) (U_{i_1 j_1}^{\varepsilon_1} \ldots U_{i_p j_p}^{\varepsilon_p})$$

PROOF. These formulae are both elementary, by using the convergence result established in Theorem 4.11, the proof being as follows:

(1) This follows from the limiting formula in Theorem 4.11, by applying the linear forms there to the main character χ.

(2) This follows from the definitions of the measure μ^r and of the matrix T_e, by summing the entries of T_e over equal indices, $i_r = j_r$. □

Interestingly, the above results regarding inner faithfulness have applications as well to the notion of stationarity introduced before, clarifying among others the use of the word "stationary". To be more precise, in order to detect the stationary models, we have the following useful criterion, mixing linear algebra and analysis, from [10]:

THEOREM 8.13. *For a model $\pi : C(G) \to M_K(C(T))$, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and the Haar integration on it is:*

$$\psi = \left(tr \otimes \int_T \right) \pi$$

(2) *The linear form $\psi = (tr \otimes \int_T)\pi$ satisfies the idempotent state property:*

$$\psi * \psi = \psi$$

(3) *We have $T_e^2 = T_e$, $\forall p \in \mathbb{N}$, $\forall e \in \{1, *\}^p$, where:*

$$(T_e)_{i_1 \ldots i_p, j_1 \ldots j_p} = \left(tr \otimes \int_T \right) (U_{i_1 j_1}^{e_1} \ldots U_{i_p j_p}^{e_p})$$

If these conditions are satisfied, we say that π is stationary on its image.

Proof. Given a matrix model $\pi : C(G) \to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 4.9:

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

Now observe that (1,2,3) above depend only on the factorized representation:

$$\nu : C(H) \to M_K(C(T))$$

Thus, we can assume in practice that we have $G = H$, which means that we can assume that π is inner faithful. With this assumption made, the formula in Theorem 4.11 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent state equation:

$$\psi * \psi = \psi$$

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have $\psi^{*r} = \psi$ for any $r \in \mathbb{N}$, and Theorem 4.11 gives $\int_G = \psi$. By using now Theorem 4.7, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}) = (T_e^r)_{i_1\ldots i_p, j_1\ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_e and T_e^2 have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms ψ and $\psi * \psi$ coincide on any product of coefficients $u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. □

8c. Half-liberation

As a first illustration, we can apply the above criterion to certain models for O_N^*, U_N^*. We first have the following result, coming from the work in [5], [10]:

Proposition 8.14. *We have a matrix model as follows,*

$$C(O_N^*) \to M_2(C(U_N)) \quad , \quad u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

where v is the fundamental corepresentation of $C(U_N)$, as well as a model as follows,

$$C(U_N^*) \to M_2(C(U_N \times U_N)) \quad , \quad u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix}$$

where v, w are the fundamental corepresentations of the two copies of $C(U_N)$.

PROOF. It is routine to check that the matrices on the right are indeed biunitaries, and since the first matrix is also self-adjoint, we obtain in this way models as follows:

$$C(O_N^+)\to M_2(C(U_N))\quad,\quad C(U_N^+)\to M_2(C(U_N\times U_N))$$

Regarding now the half-commutation relations, this comes from something general, regarding the antidiagonal 2×2 matrices. Consider indeed matrices as follows:

$$X_i=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}$$

We have then the following computation:

$$X_iX_jX_k=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&x_j\\ y_j&0\end{pmatrix}\begin{pmatrix}0&x_k\\ y_k&0\end{pmatrix}=\begin{pmatrix}0&x_iy_jx_k\\ y_ix_jy_k&0\end{pmatrix}$$

Since this quantity is symmetric in i,k, we obtain from this:

$$X_iX_jX_k=X_kX_jX_i$$

Thus, the antidiagonal 2×2 matrices half-commute, and we conclude that our models for $C(O_N^+)$ and $C(U_N^+)$ constructed above factorize as in the statement. □

We can now formulate our first concrete modeling theorem, as follows:

THEOREM 8.15. *The above antidiagonal models, namely*

$$C(O_N^*)\to M_2(C(U_N))\quad,\quad C(U_N^*)\to M_2(C(U_N\times U_N))$$

are both stationary, and in particular they are faithful.

PROOF. Let us first discuss the case of O_N^*. We will use Theorem 4.13 (3). Since the fundamental representation is self-adjoint, the various matrices T_e with $e\in\{1,*\}^p$ are all equal. We denote this common matrix by T_p. We have, by definition:

$$(T_p)_{i_1\ldots i_p,j_1\ldots j_p}=\left(tr\otimes\int_H\right)\left[\begin{pmatrix}0&v_{i_1j_1}\\ \bar{v}_{i_1j_1}&0\end{pmatrix}\ldots\ldots\begin{pmatrix}0&v_{i_pj_p}\\ \bar{v}_{i_pj_p}&0\end{pmatrix}\right]$$

Since when multipliying an odd number of antidiagonal matrices we obtain an atidiagonal matrix, we have $T_p=0$ for p odd. Also, when p is even, we have:

$$\begin{aligned}(T_p)_{i_1\ldots i_p,j_1\ldots j_p}&=\left(tr\otimes\int_H\right)\begin{pmatrix}v_{i_1j_1}\ldots\bar{v}_{i_pj_p}&0\\ 0&\bar{v}_{i_1j_1}\ldots v_{i_pj_p}\end{pmatrix}\\ &=\frac{1}{2}\left(\int_H v_{i_1j_1}\ldots\bar{v}_{i_pj_p}+\int_H\bar{v}_{i_1j_1}\ldots v_{i_pj_p}\right)\\ &=\int_H Re(v_{i_1j_1}\ldots\bar{v}_{i_pj_p})\end{aligned}$$

We have $T_p^2=T_p=0$ when p is odd, so we are left with proving that for p even we have $T_p^2=T_p$. For this purpose, we use the following formula:

$$Re(x)Re(y)=\frac{1}{2}\left(Re(xy)+Re(x\bar{y})\right)$$

By using this identity for each of the terms which appear in the product, and multi-index notations in order to simplify the writing, we obtain:

$$\begin{aligned}(T_p^2)_{ij} &= \sum_{k_1\ldots k_p}(T_p)_{i_1\ldots i_p,k_1\ldots k_p}(T_p)_{k_1\ldots k_p,j_1\ldots j_p}\\ &= \int_H\int_H\sum_{k_1\ldots k_p}Re(v_{i_1k_1}\ldots\bar{v}_{i_pk_p})Re(w_{k_1j_1}\ldots\bar{w}_{k_pj_p})dvdw\\ &= \frac{1}{2}\int_H\int_H\sum_{k_1\ldots k_p}Re(v_{i_1k_1}w_{k_1j_1}\ldots\bar{v}_{i_pk_p}\bar{w}_{k_pj_p})+Re(v_{i_1k_1}\bar{w}_{k_1j_1}\ldots\bar{v}_{i_pk_p}w_{k_pj_p})dvdw\\ &= \frac{1}{2}\int_H\int_H Re((vw)_{i_1j_1}\ldots(\bar{v}\bar{w})_{i_pj_p})+Re((v\bar{w})_{i_1j_1}\ldots(\bar{v}w)_{i_pj_p})dvdw\end{aligned}$$

Now since $vw\in H$ is uniformly distributed when $v,w\in H$ are uniformly distributed, the quantity on the left integrates up to $(T_p)_{ij}$. Also, since H is conjugation-stable, $\bar{w}\in H$ is uniformly distributed when $w\in H$ is uniformly distributed, so the quantity on the right integrates up to the same quantity, namely $(T_p)_{ij}$. Thus, we have:

$$(T_p^2)_{ij}=\frac{1}{2}\Big((T_p)_{ij}+(T_p)_{ij}\Big)=(T_p)_{ij}$$

Summarizing, we have obtained that for any p, we have $T_p^2=T_p$. Thus Theorem 4.13 applies, and shows that our model is stationary, as claimed. As for the proof of the stationarity for the model for U_N^*, this is similar. See [16]. □

As a second illustration, regarding H_N^*,K_N^*, we have:

THEOREM 8.16. *We have a stationary matrix model as follows,*

$$C(H_N^*)\to M_2(C(K_N))\quad,\quad u_{ij}\to\begin{pmatrix}0&v_{ij}\\ \bar{v}_{ij}&0\end{pmatrix}$$

where v is the fundamental corepresentation of $C(K_N)$, as well as a stationary model

$$C(K_N^*)\to M_2(C(K_N\times K_N))\quad,\quad u_{ij}\to\begin{pmatrix}0&v_{ij}\\ w_{ij}&0\end{pmatrix}$$

where v,w are the fundamental corepresentations of the two copies of $C(K_N)$.

PROOF. This follows by adapting the proof of Proposition 4.14 and Theorem 4.15, by adding there the H_N^+,K_N^+ relations. All this is in fact part of a more general phenomenon, concerning half-liberation in general, and we refer here to [5], [10]. □

As a consequence of this, we can now work out the discrete group case:

PROPOSITION 8.17. *Any reflection group $\Gamma =< g_1,\ldots,g_N >$ which is half-abelian, in the sense that its standard generators half-commute,*

$$g_ig_jg_k = g_kg_jg_i$$

has an algebraic stationary model, with $K = 2$.

PROOF. This follows from Theorem 4.15. To be more precise, in the non-abelian case, the results in [5] show that $\widehat{\Gamma} \subset O_N^*$ must come from a group dual $\widehat{\Lambda} \subset U_N$, via the construction there, and with $\Lambda =< h_1,\ldots,h_N >$, the corresponding model is:

$$\Gamma \subset C(\widehat{\Lambda}, U_2) \quad , \quad g_i \to \left[\chi \to \begin{pmatrix} 0 & \chi(h_i) \\ \bar{\chi}(h_i) & 0 \end{pmatrix}\right]$$

As for the abelian case, the result here follows from Proposition 4.5. □

More generally now, we have the following result, from [5]:

PROPOSITION 8.18. *If L is a compact group, having a N-dimensional unitary corepresentation v, and an order K automorphism $\sigma : L \to L$, we have a matrix model*

$$\pi : C(U_N^*) \to M_K(C(L)) \quad , \quad u_{ij} \to \tau[v_{ij}^{(1)},\ldots,v_{ij}^{(K)}]$$

where $v^{(i)}(g) = v(\sigma^i(g))$, and where $\tau[x_1,\ldots,x_K]$ is obtained by filling the standard K-cycle $\tau \in M_K(0,1)$ with the elements $x_1,\ldots,x_K$. We call such models "cyclic".

PROOF. The matrices $U_{ij} = \tau[v_{ij}^{(1)},\ldots,v_{ij}^{(K)}]$ in the statement appear by definition as follows, with the convention that all the blank spaces denote 0 entries:

$$U_{ij} = \begin{pmatrix} & & & v_{ij}^{(1)} \\ v_{ij}^{(2)} & & & \\ & \ddots & & \\ & & v_{ij}^{(K)} & \end{pmatrix}$$

The matrix $U = (U_{ij})$ is then unitary, and so is $\bar{U} = (U_{ij}^*)$. Thus, if we denote by $w = (w_{ij})$ the fundamental corepresentation of $C(U_N^+)$, we have a model as follows:

$$\rho : C(U_N^+) \to M_K(C(L)) \quad , \quad w_{ij} \to U_{ij}$$

Now observe that the matrices $U_{ij}U_{kl}^*, U_{ij}^*U_{kl}$ are all diagonal, so in particular, they commute. Thus the above morphism ρ factorizes through $C(U_N^*)$, as claimed. □

In relation with the above models, we have the following result, also from [5]:

THEOREM 8.19. *Any cyclic model in the above sense,*

$$\pi : C(U_N^*) \to M_K(C(L))$$

is stationary on its image, with the corresponding closed subgroup $[L] \subset U_N^*$, given by

$$Im(\pi) = C([L])$$

being the quotient $L \rtimes \mathbb{Z}_K \to [L]$ having as coordinates the variables $u_{ij} = v_{ij} \otimes \tau$.

PROOF. Assuming that (L, σ) are as in Proposition 4.18, we have an action $\mathbb{Z}_K \curvearrowright L$, and we can therefore consider the following short exact sequence:

$$1 \to \mathbb{Z}_K \to L \rtimes \mathbb{Z}_K \to L \to 1$$

By performing a Thoma type construction we obtain a model as follows, where $x^{(i)} = \tilde{\sigma}^i(x)$, with $\tilde{\sigma} : C(L) \to C(L)$ being the automorphism induced by $\sigma : L \to L$:

$$\rho : C(L \rtimes \mathbb{Z}_K) \subset M_K(C(L)) \quad , \quad x \otimes \tau^i \to \tau^i[x^{(1)}, \ldots, x^{(K)}]$$

Consider now the quotient quantum group $L \rtimes \mathbb{Z}_K \to [L]$ having as coordinates the variables $u_{ij} = v_{ij} \otimes \tau$. We have then a injective morphism, as follows:

$$\nu : C([L]) \subset C(L \rtimes \mathbb{Z}_K) \quad , \quad u_{ij} \to v_{ij} \otimes \tau$$

By composing the above two embeddings, we obtain an embedding as follows:

$$\rho\nu : C([L]) \subset M_K(C(L)) \quad , \quad u_{ij} \to \tau[v_{ij}^{(1)}, \ldots, v_{ij}^{(K)}]$$

Now since ρ is stationary, and since ν commutes with the Haar funtionals as well, it follows that this morphism $\rho\nu$ is stationary, and this finishes the proof. □

As an illustration, we can now recover the following result, from [5]:

PROPOSITION 8.20. *For any non-classical $G \subset O_N^*$ we have a stationary model*

$$\pi : C(G) \to M_2(C(L)) \quad , \quad u_{ij} = \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

where $L \subset U_N$, with coordinates denoted v_{ij}, is the lift of $PG \subset PO_N^* = PU_N$.

PROOF. Assume first that $L \subset U_N$ is self-conjugate, in the sense that $g \in L \implies \bar{g} \in L$. If we consider the order 2 automorphism of $C(L)$ induced by $g_{ij} \to \bar{g}_{ij}$, we can apply Theorem 4.19, and we obtain a stationary model, as follows:

$$\pi : C([L]) \subset M_2(C(L)) \quad , \quad u_{ij} \otimes 1 = \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

The point now is that, as explained in [5], any non-classical subgroup $G \subset O_N^*$ must appear as $G = [L]$, for a certain self-conjugate subgroup $L \subset U_N$. Moreover, since we have $PG = P[L]$, it follows that $L \subset U_N$ is the lift of $PG \subset PO_N^* = PU_N$, as claimed. □

In the unitary case now, and with the matrix size $K\in\mathbb N$ being arbitrary, we recall from [5], [10] and related papers that U_N^* has a certain "arithmetic version" $U_{N,K}^*\subset U_N^*$, obtained by imposing some natural length $2K$ relations on the standard coordinates. As basic examples, at $K=1$ we have $U_{N,1}^*=U_N$, the defining relations being $ab=ba$ with $a,b\in\{u_{ij},u_{ij}^*\}$, and at $K=2$ we have $U_{N,2}^*=U_N^{**}$, with the latter quantum group appearing via the relations $ab\cdot cd=cd\cdot ab$, for any $a,b,c,d\in\{u_{ij},u_{ij}^*\}$.

With this convention, we have the following result, also from [5]:

THEOREM 8.21. *For any subgroup $G\subset U_{N,K}^*$ which is K-symmetric, in the sense that $u_{ij}\to e^{2\pi i/K}u_{ij}$ defines an automorphism of $C(G)$, we have a stationary model*

$$\pi:C(G)\to M_K(C(L))\quad,\quad u_{ij}\to\tau[v_{ij}^{(1)},\ldots,v_{ij}^{(K)}]$$

with $L\subset U_N^K$ being a closed subgroup which is symmetric, in the sense that it is stable under the cyclic action $\mathbb Z_K\curvearrowright U_N^K$.

PROOF. This follows from what we have, as follows:

(1) Assuming that $L\subset U_N^K$ is symmetric in the above sense, we have representations $v^{(i)}:L\subset U_N^K\to U_N^{(i)}$ for any i, and the cyclic action $\mathbb Z_K\curvearrowright U_N^K$ restricts into an order K automorphism $\sigma:L\to L$. Thus we can apply Theorem 4.19, and we obtain a certain closed subgroup $[L]\subset U_{N,K}^*$, having a stationary model as in the statement.

(2) Conversely now, assuming that $G\subset U_{N,K}^*$ is K-symmetric, the main result in [10] applies, and shows that we must have $C(G)\subset C(L)\rtimes\mathbb Z_K$, for a certain closed subgroup $L\subset U_N^K$ which is symmetric. But this shows that we have $G=[L]$, and we are done. □

We refer to [5], [10] and related papers for more on the above.

8d. Group duals

Let us discuss now the group dual case, where we have a closed subgroup $\widehat\Gamma\subset S_N^+$, with Γ being a discrete group. Following [5], we use the following construction:

PROPOSITION 8.22. *The following happen:*

1. *Given integers $K_1,\ldots,K_M$ satisfying $K_1+\ldots+K_M=N$, the dual of any quotient group $\mathbb Z_{K_1}*\ldots*\mathbb Z_{K_M}\to\Gamma$ appears as a closed subgroup $\widehat\Gamma\subset S_N^+$.*
2. *By refining if necessary the partition $N=K_1+\ldots+K_M$, we can always assume that the M morphisms $\mathbb Z_{K_i}\to\Gamma$ are all injective.*
3. *Assuming that the partition $N=K_1+\ldots+K_M$ is refined, as above, this partition is precisely the one describing the orbit structure of $\widehat\Gamma\subset S_N^+$.*

PROOF. The idea for (1) is that we have embeddings $\widehat{\mathbb{Z}}_{K_i} \simeq \mathbb{Z}_{K_i} \subset S_{K_i} \subset S_{K_i}^+$, and by performing a free product construction, we obtain an embedding as follows:

$$\widehat{\Gamma} \subset \mathbb{Z}_{K_1} \widehat{* \ldots *} \mathbb{Z}_{K_M} \subset S_N^+$$

To be more precise, the magic unitary that we get is as follows, where $F_i = \frac{1}{\sqrt{K_i}}(w_i^{ab})_{ab}$ with $w_i = e^{2\pi i/K_i}$, and $V_i = (g_i^a)_a$, with g_i being the standard generator of $\mathbb{Z}_{K_i}$:

$$u = diag(u_i) \quad , \quad u_i = \frac{1}{\sqrt{K_i}}\begin{pmatrix} (F_iV_i)_0 & \ldots & (F_iV_i)_{K_i-1} \\ (F_iV_i)_{K_i-1} & \ldots & (F_iV_i)_{K_i-2} \\ \vdots & \vdots & \vdots \\ (F_iV_i)_1 & \ldots & (F_iV_i)_0 \end{pmatrix}$$

Regarding (2,3), the idea here is that the orbit structure of any $\widehat{\Gamma} \subset S_N^+$ produces a partition $N = K_1 + \ldots + K_M$, and then a quotient map $\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M} \to \Gamma$. □

Following the material from the previous chapters, we will be mainly interested in what follows in the quasi-transitive case. Let us start with the following definition:

DEFINITION 8.23. *Given a subgroup $G \subset S_N^+$, a random matrix model of type*

$$\pi : C(G) \to M_K(C(T))$$

is called quasi-flat when the fibers $P_{ij}^x = \pi(u_{ij})(x)$ all have rank ≤ 1.

We will explore more in detail this notion later. Now with this convention made, and getting back to the group duals, we have the following result, from [5]:

PROPOSITION 8.24. *The quasi-transitive group duals $\widehat{\Gamma} \subset S_N^+$, with orbits having K elements, appearing as above, have the following properties:*

1. *These come from the quotients $\mathbb{Z}_K^{*M} \to \Gamma$, having the property that the corresponding M morphisms $\mathbb{Z}_K^{(i)} \subset \mathbb{Z}_K^{*M} \to \Gamma$ are all injective.*
2. *For such a quotient, a matrix model $\pi : C^*(\Gamma) \to M_K(\mathbb{C})$ is quasi-flat if and only if it is stationary on each subalgebra $C^*(\mathbb{Z}_K^{(i)}) \subset C^*(\Gamma)$.*

PROOF. The first assertion follows from Proposition 4.23. Regarding the second assertion, consider a matrix model $\pi : C^*(\Gamma) \to M_K(\mathbb{C})$, mapping $g_i \to U_i$, where g_i is the standard generator of $\mathbb{Z}_K^{(i)}$. With notations from the proof of Proposition 4.23, the images of the nonzero standard coordinates on $\widehat{\Gamma} \subset S_N^+$ are mapped as follows:

$$\pi : \frac{1}{\sqrt{K}}(FV_i)_c \to \frac{1}{\sqrt{K}}(FW_i)_c$$

Here $V_i = (g_i^a)_a$, $W_i = (U_i^a)_a$, and $F = \frac{1}{\sqrt{K}}(w^{ab})_{ab}$ with $w = e^{2\pi i/K}$. With this formula in hand, the flatness condition on π simply states that we must have:

$$Tr((FW_i)_c) = \sqrt{K} \quad , \quad \forall i, \forall c$$

In terms of the trace vectors $T_i = (Tr(U_i^a))_a$ this condition becomes $FT_i = \sqrt{K}\xi$, where $\xi \in \mathbb{C}^K$ is the all-one vector. Thus we must have $T_i = \sqrt{K}F^*\xi$, which reads:

$$\begin{pmatrix} Tr(1) \\ Tr(U_i) \\ \vdots \\ Tr(U_i^{K-1}) \end{pmatrix} = \sqrt{K}F^* \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} K \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad , \quad \forall i$$

In other words, we have reached to the conclusion that π is flat precisely when its restrictions to each subalgebra $C^*(\mathbb{Z}_K^{(i)}) \subset C^*(\Gamma)$ are stationary, as claimed. $\square$

We would like to end our study with a purely group-theoretical formulation of these results, and of some related questions, that we believe of interest. Let us start with:

Definition 8.25. *A discrete group Γ is called uniform when:*

1. *Γ is finitely generated, $\Gamma =< g_1, \ldots, g_M >$.*
2. *The generators $g_1, \ldots, g_M$ have common order $K < \infty$.*
3. *Γ appears as an intermediate quotient $\mathbb{Z}_K^{*M} \to \Gamma \to \mathbb{Z}_K^M$.*
4. *We have an action $S_M \curvearrowright \Gamma$, given by $\sigma(g_i) = g_{\sigma(i)}$.*

Here the conditions (1-3) basically come from [22], via Proposition 4.24 (1), and together with some extra considerations from [5], which prevent us from using groups of type $\Gamma = (\mathbb{Z}_K * \mathbb{Z}_K) \times \mathbb{Z}_K$, we are led to the condition (4) as well.

Observe that some of the above conditions are technically redundant, with (4) implying that the generators $g_1, \ldots, g_M$ have common order, as stated in (2), and also with (3) implying that the group is finitely generated, with the generators having finite order. We have as well the following notion, which is once again group-theoretical:

Definition 8.26. *If a discrete group Γ is uniform, as above, a unitary representation $\rho : \Gamma \to U_K$ is called quasi-flat when the eigenvalues of each*

$$U_i = \rho(g_i) \in U_K$$

are uniformly distributed.

To be more precise, assuming that $\Gamma =< g_1, \ldots, g_M >$ with $ord(g_i) = K$ is as in Definition 4.25, any unitary representation $\rho : \Gamma \to U_K$ is uniquely determined by the images $U_i = \rho(g_i) \in U_K$ of the standard generators. Now since each of these unitaries satisfies $U_i^K = 1$, its eigenvalues must be among the K-th roots of unity, and our quasi-flatness assumption states that each eigenvalue must appear with multiplicity 1.

With these notions in hand, we have the following result:

THEOREM 8.27. *If* $\Gamma =< g_1, \ldots, g_M >$ *is uniform, with* $ord(g_i) = K$*, a model*

$$\pi : C^*(\Gamma) \to M_K(C(X))$$

is quasi-flat precisely when the associated unitary representation

$$\rho : \Gamma \to C(X, U_K)$$

has quasi-flat fibers, in the sense of Definition 4.26.

PROOF. According to Proposition 4.24 (2), the model is quasi-flat precisely when the following compositions are all stationary:

$$\pi_i : C^*(\mathbb{Z}_K^{(i)}) \subset C^*(\Gamma) \to M_K(C(X))$$

On the other hand, as already observed in the proof of Proposition 4.24, a matrix model $\rho : C^*(\mathbb{Z}_K) \to M_K(C(X))$ is stationary precisely when the unitary $U = \rho(g)$, where g is the standard generator of $\mathbb{Z}_K$, satisfies the following condition:

$$\begin{pmatrix} tr(1) \\ tr(U) \\ \vdots \\ tr(U^{K-1}) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus, such a model is stationary precisely when the eigenvalues of U are uniformly distributed, over the K-th roots of unity. We conclude that π is quasi-flat precisely when the eigenvalues of each $U_i = \rho(g_i)$ are uniformly distributed, as in Definition 4.26. □

We are interested now in the matrix models for the discrete group algebras, which are stationary. We use a lift of the quasi-flat models, in the following sense:

PROPOSITION 8.28. *The affine lift of the universal quasi-flat model for* $C^*(\mathbb{Z}_K^{*M})$*,*

$$\pi : C^*(\mathbb{Z}_K^{*M}) \to M_K(C(U_K^M))$$

is given on the canonical generator g_i *of the* i*-th factor by*

$$\pi(g_i)(U^1, \ldots, U^M) = \sum_j w^j P_{U_j^i}$$

where U_j^i *is the* j*-th column of* U^i *and* P_ξ *denotes the orthogonal projection onto* $\mathbb{C}\xi$*.*

PROOF. There is indeed a canonical quotient map $U_K \to E_K$, obtained by parametrizing the orthonormal bases of $\mathbb{C}^K$ by the unitary group U_K, and this gives the result. □

We know that the maximal group dual subgroups $\widehat{\Gamma} \subset S_N^+$ are the free products of type $\mathbb{Z}_{K_1} * \ldots * \mathbb{Z}_{K_M}$ with $K_1 + \ldots + K_M = N$. In the quasi-transitive case, where by definition $K_1 = \ldots = K_M = K$ with $K|N$, we have the following result, from [5]:

THEOREM 8.29. *The universal quasi-flat model for the group*

$$\Gamma = \mathbb{Z}_K^{*M}$$

is inner faithful.

PROOF. It is enough to prove that the affine lift of the universal model in the statement is inner faithful, and this is indeed something very standard. □

8e. Exercises

Exercises:

EXERCISE 8.30.

EXERCISE 8.31.

EXERCISE 8.32.

EXERCISE 8.33.

EXERCISE 8.34.

EXERCISE 8.35.

EXERCISE 8.36.

EXERCISE 8.37.

Bonus exercise.

Part III

Discrete groups

If it hadn't been for Cotton-Eye Joe
I'd been married long time ago
Where did you come from, where did you go
Where did you come from, Cotton-Eye Joe

CHAPTER 9

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.

CHAPTER 10

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.

Exercise 11.4.

Exercise 11.5.

Exercise 11.6.

Exercise 11.7.

Exercise 11.8.

Bonus exercise.

CHAPTER 12

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Fourier analysis

Come on let's twist again
Like we did last Summer
Yeah, let's twist again
Like we did last year

CHAPTER 13

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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