Basic complex analysis

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Abstract. This is an introduction to complex analysis, with all needed preliminaries included. We first have a look at $\mathbb{C}$ and $\mathbb{C}^N$, mostly algebraic and geometric, notably with basic results regarding the complex linear maps and polynomials. Then we develop the standard theory of holomorphic functions $f : X \to \mathbb{C}$ with $X \subset \mathbb{C}$, with the Cauchy formula, its versions, and its basic applications. We then go on a discussion in higher dimensions, featuring harmonic functions, Fourier analysis and Riemann surfaces. Finally, we get back to one variable, with various approximation results, analytic continuation and its applications, and an introduction to complex dynamics.

Preface

What are the complex numbers good for? Solving $x^2 = -1$ you would say, but this remains something quite abstract, do we really need solutions to this equation, in our everyday life. In answer now, complex numbers can be good for many things:

(1) To start with, any polynomial $P \in \mathbb{C}[X]$ has a root, and in fact a full collection of $\deg P$ roots, when counted with multiplicities. This is something truly fundamental, generalizing $i^2 = -1$, which all of the sudden takes a high philosophical dimension.

(2) As an application, any matrix $A \in M_N(\mathbb{R})$, and in fact any matrix $A \in M_N(\mathbb{C})$, has N eigenvalues, when counted with multiplicities. Thus, there are far more chances for a usual matrix $A \in M_N(\mathbb{R})$, regarded as $A \in M_N(\mathbb{C})$, to be diagonalizable.

(3) As an illustration here, the rotation of angle $t \in [0, 2\pi)$ in the plane has obviously no eigenvalues, unless $t = 0, \pi$. In the complex setting, however, some magic appears, with $e^{\pm it}$ being eigenvalues, and with the rotation being diagonalizable.

(4) As another illustration, the all-one matrix $\mathbb{I}_N \in M_N(\mathbb{R})$ can be certainly diagonalized, but with some pain, the problem coming from uniformly solving $x_1 + \ldots + x_N = 0$. Over the complex numbers, this equation is elegantly solved by the roots of unity.

(5) But, do we really care about matrices and diagonalization? Sure we do, because the derivative of any function $f : \mathbb{R}^N \to \mathbb{R}^N$ is such a matrix, $f'(x) \in M_N(\mathbb{R})$. Also, the second derivative of any function $f : \mathbb{R}^N \to \mathbb{R}$ is such a matrix too, $f''(x) \in M_N(\mathbb{R})$.

(6) So, no escape from matrices and diagonalization, no matter what we want to do, algebra, geometry or analysis. And getting back to theory, as a key result, the matrices $A \in M_N(\mathbb{C})$ are in fact generically diagonalizable. Which is very good news.

As you can see, the complex numbers potentially solve many basic problems from mathematics. At the advanced level the applications abound too, with operator theory, Fourier analysis and many other things using complex numbers. As for truly applied mathematics, in relation with physics, here the need for complex numbers is even more drastic, with the understanding of all sorts of waves requiring them. And there is even more, because quantum mechanics, and so this whole world, naturally lives over $\mathbb{C}$.

This book is an introduction to this, complex numbers and analysis, with all the needed preliminaries included. The book is organized in 4 parts, as follows:

Part I - we first have a look at $\mathbb{C}$ and $\mathbb{C}^N$, mostly algebraic and geometric, notably with basic results regarding the complex linear maps and polynomials.

Part II - here we develop the standard theory of holomorphic functions, $f : X \to \mathbb{C}$ with $X \subset \mathbb{C}$, with the Cauchy formula, its versions, and its basic applications.

Part III - we go here on a discussion in higher dimensions, featuring harmonic functions, Fourier analysis and applications, and Riemann surfaces.

Part IV - here we get back to one variable, with various approximation results, analytic continuation and its applications, and an introduction to complex dynamics.

Many thanks to my colleagues, present and former, and in particular to Euler, for his formula $e^{it} = \cos t + i \sin t$. Thanks as well to my cats, they compute complex lengths, areas and volumes as I breathe, hope one day they will fully teach me this.

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CHAPTER 1

Complex numbers

1a. Complex numbers

You certainly know how to deal with the functions $f:\mathbb{R}\to\mathbb{R}$, by computing derivatives and integrals, and with this being good knowledge, many questions in mathematics and science having formulations in terms of such functions $f:\mathbb{R}\to\mathbb{R}$.

You probably know about multivariable functions $f:\mathbb{R}^N\to\mathbb{R}^M$ too, which are essential for formulating more complicated mathematics and science questions. Again, the main tools here are the derivatives and integrals, and in most cases, everything reduces to a mix of one-variable functions, and linear algebra. Indeed, associated to $f:\mathbb{R}^N\to\mathbb{R}^M$ is the $M\times N$ matrix formed its one-variable components $f_{ij}:\mathbb{R}\to\mathbb{R}$ at a given $x\in\mathbb{R}^N$, which does not fully determine f there, but which captures its essentials.

All this is very nice, and even looks like the end of the story, the instructions being:

(1) Better learn how to model your problems in terms of functions $f:\mathbb{R}^N\to\mathbb{R}^M$.

(2) Better learn how to deal with the one-variable functions, such as $f_{ij}:\mathbb{R}\to\mathbb{R}$.

(3) Better learn how to deal with the real matrices, including with $A=(f_{ij}'(x))$.

Well, this was for the theory. In practice, mathematics has not said its final word, and there is a further twist to this, involving the complex numbers. With the idea being that the above, and many other things, are better done by upgrading from $\mathbb{R}$ to $\mathbb{C}$.

This book will be here for telling you this, how to master the complex numbers, and their applications to analysis, up to the point of even forgetting about the real numbers themselves, the complex numbers being just so good, for everything. In short, powerful and mysterious technology that we will learn here, and expect to become soon a "mutant". Like professional mathematicians, or physicists, and even like cats, small or big.

Getting started now, with the complex numbers, their definition is something quite crazy and algebraic, as follows, that will take us some time to understand:

DEFINITION 1.1. *The complex numbers are variables of the form*

$$x = a + ib$$

with $a, b \in \mathbb{R}$, which add in the obvious way, and multiply according to the following rule:

$$i^2 = -1$$

Each real number can be regarded as a complex number, $a = a + i \cdot 0$.

In other words, we consider variables as above, without bothering for the moment with their precise meaning. Now consider two such complex numbers:

$$x = a + ib \quad , \quad y = c + id$$

The formula for the sum is then the obvious one, as follows:

$$x + y = (a + c) + i(b + d)$$

As for the formula of the product, by using the rule $i^2 = -1$, we obtain:

$$\begin{aligned} xy &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Thus, the complex numbers as introduced above are well-defined. The multiplication formula is of course quite tricky, and hard to memorize, but we will see later some alternative ways, which are more conceptual, for performing the multiplication.

The advantage of using the complex numbers comes from the fact that the equation $x^2 = 1$ has now a solution, $x = i$. In fact, this equation has two solutions, namely:

$$x = \pm i$$

This is of course very good news. More generally, we have the following result, regarding the arbitrary degree 2 equations, with real coefficients:

THEOREM 1.2. *The complex solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of negative real numbers being defined as

$$\sqrt{-m} = \pm i\sqrt{m}$$

and with the square root of positive real numbers being the usual one.

PROOF. We can write our equation in the following way:

$$\begin{aligned}
ax^2+bx+c=0 &\iff x^2+\frac{b}{a}x+\frac{c}{a}=0\\
&\iff \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0\\
&\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\\
&\iff x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We will see later that any degree 2 complex equation has solutions as well, and that more generally, any polynomial equation, real or complex, has solutions. Moving ahead now, we can represent the complex numbers in the plane, in the following way:

PROPOSITION 1.3. *The complex numbers, written as usual*

$$x=a+ib$$

can be represented in the plane, according to the following identification:

$$x=\begin{pmatrix}a\\b\end{pmatrix}$$

With this convention, the sum of complex numbers is the usual sum of vectors.

PROOF. Consider indeed two arbitrary complex numbers:

$$x=a+ib\quad,\quad y=c+id$$

Their sum is then by definition the following complex number:

$$x+y=(a+c)+i(b+d)$$

Now let us represent x,y in the plane, as in the statement:

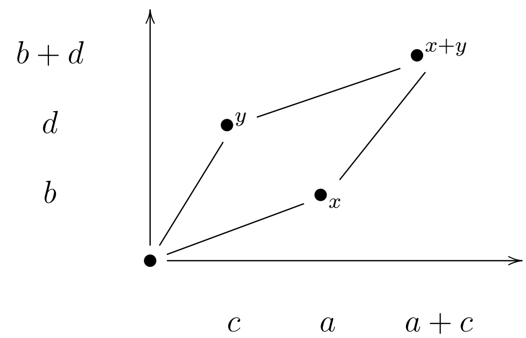
$$x=\begin{pmatrix}a\\b\end{pmatrix}\quad,\quad y=\begin{pmatrix}c\\d\end{pmatrix}$$

In this picture, their sum is given by the following formula:

$$x+y=\begin{pmatrix}a+c\\b+d\end{pmatrix}$$

But this is indeed the vector corresponding to $x+y$, so we are done. □

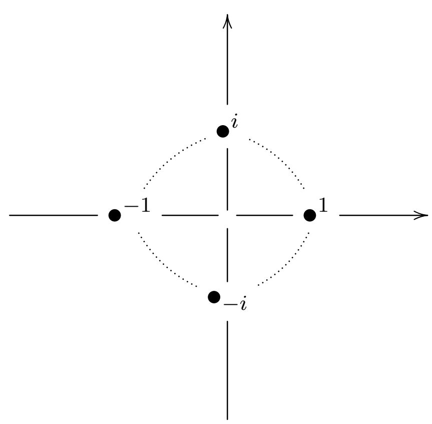
Here we have assumed that you are a bit familiar with vector calculus. If not, no problem, the idea is simply that vectors add by forming a parallelogram, as follows:



Observe that in our geometric picture from Proposition 1.3, the real numbers correspond to the numbers on the Ox axis. As for the purely imaginary numbers, these lie on the Oy axis, with the number i itself being given by the following formula:

$$i=\begin{pmatrix}0\\1\end{pmatrix}$$

As an illustration for this, let us record now a basic picture, with some key complex numbers, namely $1,i,-1,-i$, represented according to our conventions:



You might perhaps wonder why I chose to draw that circle, connecting the numbers $1,i,-1,-i$, which does not look very useful. More on this in a moment, the idea being that this circle can be extremely useful. In fact, coming in advance, some advice:

Advice 1.4. *When drawing complex numbers, always begin with the coordinate axes Ox,Oy, and with a copy of the unit circle.*

And more on this later in this chapter, when discussing more advanced mathematics involving the complex numbers, in relation with trigonometry.

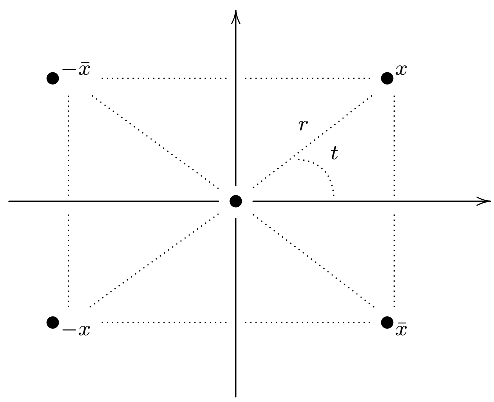
As a last general topic regarding the complex numbers, let us discuss conjugation. This is something quite tricky, complex number specific, as follows:

DEFINITION 1.5. *The complex conjugate of $x = a + ib$ is the following number,*

$$\bar{x} = a - ib$$

obtained by making a reflection with respect to the Ox axis.

As before with other such operations on complex numbers, a quick picture says it all. Here is the picture, with the numbers $x, \bar{x}, -x, -\bar{x}$ being all represented:



Observe that the conjugate of a real number $x \in \mathbb{R}$ is the number itself, $x = \bar{x}$. In fact, the equation $x = \bar{x}$ characterizes the real numbers, among the complex numbers. At the level of non-trivial examples now, we have the following formula:

$$\bar{i} = -i$$

There are many things that can be said about the conjugation of the complex numbers, and here is a summary of the basic such things that can be said:

THEOREM 1.6. *The conjugation operation $x \to \bar{x}$ has the following properties:*

(1) *$x = \bar{x}$ precisely when x is real.*
(2) *$x = -\bar{x}$ precisely when x is purely imaginary.*
(3) *$x\bar{x} = |x|^2$, with $|x|$ being the usual vector length.*
(4) *We have the formula $\overline{xy} = \bar{x}\bar{y}$, for any $x, y \in \mathbb{C}$.*
(5) *The solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$ are conjugate.*

PROOF. These results are all elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something clear too, because with $x = a + ib$ our equation $x = -\bar{x}$ reads $a + ib = -a + ib$, and so $a = 0$, which amounts in saying that x is purely imaginary.

(3) This is a key formula, which can be proved as follows, with $x = a + ib$:

$$\begin{aligned} x\bar{x} &= (a+ib)(a-ib) \\ &= a^2 + b^2 \\ &= |x|^2 \end{aligned}$$

(4) This is something quite magic, which can be proved as follows:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd)-i(ad+bc) \\ &= (a-ib)(c-id) \end{aligned}$$

(5) This comes from the formula of the solutions, that we know from Theorem 1.2, but we can deduce this as well directly, without computations. Indeed, by using our assumption that the coefficients are real, $a, b, c \in \mathbb{R}$, we have:

$$\begin{aligned} ax^2 + bx + c = 0 &\implies \overline{ax^2 + bx + c} = 0 \\ &\implies \bar{a}\bar{x}^2 + \bar{b}\bar{x} + \bar{c} = 0 \\ &\implies a\bar{x}^2 + b\bar{x} + c = 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Getting now to more advanced aspects, we have the following result:

THEOREM 1.7. *Any complex number $x = a + ib$ has two square roots, given by*

$$\sqrt{x} = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} \pm i\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$$

with the signs being identical when $b > 0$, and opposite when $b < 0$.

PROOF. This is something quite routine, the idea being as follows:

(1) With $x = a + ib$ as in the statement, and $\sqrt{x} = c + id$, our equation is:

$$(c+id)^2 = a + ib$$

In terms of the real and imaginary parts, we have two equations, as follows:

$$c^2 - d^2 = a \quad , \quad 2cd = b$$

(2) Let us first compute the number $u = c^2$. The equation for it is as follows:

$$u - \frac{b^2}{4u} = a$$

Thus, the number $u = c^2$ satisfies the following degree 2 equation:

$$u^2 - au - \frac{b^2}{4} = 0$$

But this latter equation has a unique positive solution, given by:

$$u=\frac{a+\sqrt{a^2+b^2}}{2}$$

Thus, we are led to the formula of $c=\pm\sqrt{u}$ in the statement.

(3) Similarly, let us compute now $v=d^2$. The equation for it is as follows:

$$\frac{b^2}{4v}-v=a$$

Thus, the number $v=d^2$ satisfies the following degree 2 equation:

$$v^2+av-\frac{b^2}{4}=0$$

But this latter equation has a unique positive solution, given by:

$$v=\frac{-a+\sqrt{a^2+b^2}}{2}$$

Thus, we are led to the formula of $d=\pm\sqrt{v}$ in the statement, and this gives the result, with the last assertion regarding signs being clear, coming from $2cd=b$.

(4) Time now for a doublecheck, for sleeping well at night. Given $a,b\in\mathbb{R}$, consider the following numbers $c,d\in\mathbb{R}$, with the sign on the right being that of b:

$$c=\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}\quad,\quad d=\pm\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$$

We have then $(c+id)^2=(c^2-d^2)+2icd$, whose real part is given by:

$$\begin{aligned}c^2-d^2&=\frac{a+\sqrt{a^2+b^2}}{2}-\frac{-a+\sqrt{a^2+b^2}}{2}\\&=\frac{a}{2}+\frac{a}{2}\\&=a\end{aligned}$$

As for the imaginary part, this can be computed as follows:

$$\begin{aligned}2cd&=\pm2\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}\cdot\frac{-a+\sqrt{a^2+b^2}}{2}}\\&=\pm2\sqrt{\frac{-a^2+a^2+b^2}{4}}\\&=\pm|b|\\&=b\end{aligned}$$

Thus we have indeed $(c+id)^2=a+ib$, as claimed. □

Good news, we can now formulate a general result in degree 2, as follows:

THEOREM 1.8. *The complex solutions of* $ax^2 + bx + c = 0$ *with* $a, b, c \in \mathbb{C}$ *are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of $b^2 - 4ac = p + iq$ *being extracted as above, namely*

$$\sqrt{p + iq} = \pm\sqrt{\frac{p + \sqrt{p^2 + q^2}}{2}} \pm i\sqrt{\frac{-p + \sqrt{p^2 + q^2}}{2}}$$

with the signs being identical when $q > 0$, *and opposite when* $q < 0$.

PROOF. This follows indeed from our old degree 2 computation, from the proof of Theorem 1.2, with the square roots being extracted as in Theorem 1.7. □

We will see in a moment a better method for extracting the square roots of complex numbers, and for formulating our final degree 2 result. Also, we will see later in this chapter that a similar result holds in fact for any $P \in \mathbb{C}[X]$, with each such polynomial having exactly $\deg P$ complex roots, when counted with multiplicities.

1b. Polar coordinates

We have so far a quite good understanding of their complex numbers, and their addition. In order to understand now the multiplication operation, we must do something more complicated, namely using polar coordinates. Let us start with:

DEFINITION 1.9. *The complex numbers* $x = a + ib$ *can be written in polar coordinates,*

$$x = r(\cos t + i \sin t)$$

with the connecting formulae being as follows,

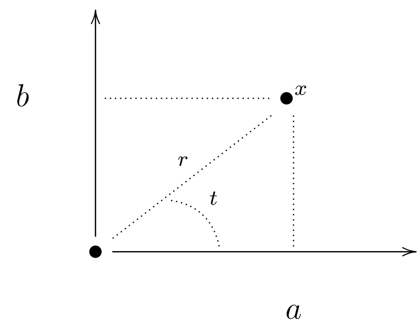
$$a = r \cos t \quad , \quad b = r \sin t$$

and in the other sense being as follows,

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

and with r, t *being called modulus, and argument.*

There is a clear relation here with the vector notation from Proposition 1.3, because r is the length of the vector, and t is the angle made by the vector with the Ox axis. To be more precise, the picture for what is going on in Definition 1.9 is as follows:



As a basic example here, the number i takes the following form:

$$i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$$

The point now is that in polar coordinates, the multiplication formula for the complex numbers, which was so far something quite opaque, takes a very simple form:

THEOREM 1.10. *Two complex numbers written in polar coordinates,*

$$x = r(\cos s + i\sin s) \quad , \quad y = p(\cos t + i\sin t)$$

multiply according to the following formula:

$$xy = rp(\cos(s+t) + i\sin(s+t))$$

In other words, the moduli multiply, and the arguments sum up.

PROOF. This follows from the following formulae, that we know well:

$$\cos(s+t) = \cos s\cos t - \sin s\sin t$$

$$\sin(s+t) = \cos s\sin t + \sin s\cos t$$

Indeed, we can assume that we have $r = p = 1$, by dividing everything by these numbers. Now with this assumption made, we have the following computation:

$$\begin{aligned} xy &= (\cos s + i\sin s)(\cos t + i\sin t) \\ &= (\cos s\cos t - \sin s\sin t) + i(\cos s\sin t + \sin s\cos t) \\ &= \cos(s+t) + i\sin(s+t) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result, which is based on some non-trivial trigonometry, is quite powerful. As a basic application of it, we can now compute powers, as follows:

THEOREM 1.11. *The powers of a complex number, written in polar form,*

$$x = r(\cos t + i \sin t)$$

are given by the following formula, valid for any exponent $k \in \mathbb{N}$:

$$x^k = r^k(\cos kt + i \sin kt)$$

Moreover, this formula holds in fact for any $k \in \mathbb{Z}$, and even for any $k \in \mathbb{Q}$.

PROOF. Given a complex number x, written in polar form as above, and an exponent $k \in \mathbb{N}$, we have indeed the following computation, with k terms everywhere:

$$\begin{aligned} x^k &= x \ldots x \\ &= r(\cos t + i \sin t) \ldots r(\cos t + i \sin t) \\ &= r^k([\cos(t + \ldots + t) + i \sin(t + \ldots + t)) \\ &= r^k(\cos kt + i \sin kt) \end{aligned}$$

Thus, we are done with the case $k \in \mathbb{N}$. Regarding now the generalization to the case $k \in \mathbb{Z}$, it is enough here to do the verification for $k = -1$, where the formula is:

$$x^{-1} = r^{-1}(\cos(-t) + i \sin(-t))$$

But this number x^{-1} is indeed the inverse of x, as shown by:

$$\begin{aligned} xx^{-1} &= r(\cos t + i \sin t) \cdot r^{-1}(\cos(-t) + i \sin(-t)) \\ &= \cos(t - t) + i \sin(t - t) \\ &= \cos 0 + i \sin 0 \\ &= 1 \end{aligned}$$

Finally, regarding the generalization to the case $k \in \mathbb{Q}$, it is enough to do the verification for exponents of type $k = 1/n$, with $n \in \mathbb{N}$. The claim here is that:

$$x^{1/n} = r^{1/n} \left[\cos\left(\frac{t}{n}\right) + i \sin\left(\frac{t}{n}\right)\right]$$

In order to prove this, let us compute the n-th power of this number. We can use the power formula for the exponent $n \in \mathbb{N}$, that we already established, and we obtain:

$$\begin{aligned} (x^{1/n})^n &= (r^{1/n})^n \left[\cos\left(n \cdot \frac{t}{n}\right) + i \sin\left(n \cdot \frac{t}{n}\right)\right] \\ &= r(\cos t + i \sin t) \\ &= x \end{aligned}$$

Thus, we have indeed a n-th root of x, and our proof is now complete. $\square$

We should mention that there is a bit of ambiguity in the above, in the case of the exponents $k \in \mathbb{Q}$, due to the fact that the square roots, and the higher roots as well, can take multiple values, in the complex number setting. We will be back to this.

As a basic application of Theorem 1.11, we have the following result:

THEOREM 1.12. *Each complex number, written in polar form,*

$$x = r(\cos t + i \sin t)$$

has two square roots, given by the following formula:

$$\sqrt{x} = \pm\sqrt{r}\left[\cos\left(\frac{t}{2}\right) + i \sin\left(\frac{t}{2}\right)\right]$$

When $x \in \mathbb{R}$, these roots are $\pm\sqrt{x}$ for $x > 0$, and $\pm i\sqrt{-x}$ for $x < 0$.

PROOF. The first assertion is clear indeed from the general formula in Theorem 1.11, at $k = 1/2$. As for its particular cases with $x \in \mathbb{R}$, these are clear from it. □

As a comment here, for $x > 0$ we are very used to call the usual $\sqrt{x}$ square root of x. However, for $x < 0$, or more generally for $x \in \mathbb{C} - \mathbb{R}_+$, there is less interest in choosing one of the possible $\sqrt{x}$ and calling it "the" square root of x, because all this is based on our convention that i comes up, instead of down, which is something rather arbitrary. Actually, clocks turning clockwise, i should be rather coming down. All this is a matter of taste, and in any case, for our math, the best is to keep some ambiguity, as above.

Good news, we can improve now our general result in degree 2, as follows:

THEOREM 1.13. *The complex solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{C}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of $b^2 - 4ac = r(\cos t + i \sin t)$ being extracted as above, namely

$$\sqrt{r(\cos t + i \sin t)} = \pm\sqrt{r}\left[\cos\left(\frac{t}{2}\right) + i \sin\left(\frac{t}{2}\right)\right]$$

and with this generalizing the usual real formula, where $t = 0$.

PROOF. This follows indeed from our old degree 2 computation, from the proof of Theorem 1.2, with the square roots being extracted as in Theorem 1.12. □

As yet another interesting application of the complex numbers in polar form, which is something really beautiful, we have the roots of unity, which are as follows:

THEOREM 1.14. *The equation $x^N=1$ has N complex solutions, namely*

$$\left\{w^k\Big|k=0,1,\ldots,N-1\right\}\quad,\quad w=\cos\left(\frac{2\pi}{N}\right)+i\sin\left(\frac{2\pi}{N}\right)$$

which are called roots of unity of order N.

PROOF. This follows as well from the general multiplication formula from Theorem 1.10. Indeed, with $x=r(\cos t+i\sin t)$ our equation $x^N=1$ reads:

$$r^N(\cos Nt+i\sin Nt)=1$$

Thus $r=1$, and $t\in[0,2\pi)$ must be a multiple of $2\pi/N$, as stated. □

The roots of unity are very useful variables, and have many interesting properties. As a first application, we can now solve the ambiguity questions related to the extraction of N-th roots, coming from Theorem 1.11, the statement here being as follows:

THEOREM 1.15. *Any $x=r(\cos t+i\sin t)$ has N roots of order N, namely*

$$y=r^{1/N}\left(\cos\left(\frac{t}{N}\right)+i\sin\left(\frac{t}{N}\right)\right)$$

multiplied by the N roots of unity of order N.

PROOF. We must solve the equation $z^N=x$, over the complex numbers. Since the number y in the statement clearly satisfies $y^N=x$, our equation is equivalent to:

$$z^N=y^N$$

We conclude from this that the solutions z appear by multiplying y by the solutions of $t^N=1$, which are the N-th roots of unity, as claimed. □

As already mentioned before, the above results are not the end of the story, and we will see later that a similar result holds in fact for any $P\in\mathbb{C}[X]$, with each such polynomial having exactly $\deg P$ complex roots, when counted with multiplicities.

1c. Euler formula

In order to advance, in our understanding of the complex numbers, we need to know a bit about the complex functions $f:\mathbb{C}\to\mathbb{C}$. Normally this is the main topic of this book, to be systematically investigated starting with chapter 4 below. In the meantime, however, here are some very basics, that we will need in what follows:

DEFINITION 1.16. *A complex function $f:\mathbb{C}\to\mathbb{C}$, or more generally $f:X\to\mathbb{C}$, with $X\subset\mathbb{C}$ being a subset, is called continuous when, for any $x_n,x\in X$*

$$x_n\to x\implies f(x_n)\to f(x)$$

where the convergence of the sequences of complex numbers, $x_n\to x$, means by definition that for n big enough, the quantity $|x_n-x|$ becomes arbitrarily small.

Observe that in real coordinates, $x=(a,b)$, the distances appearing in the definition of the convergence $x_n\to x$ are given by the following formula:

$$|x_n-x|=\sqrt{(a_n-a)^2+(b_n-b)^2}$$

Thus $x_n\to x$ in the complex sense means that $(a_n,b_n)\to(a,b)$ in the usual, intuitive sense, with respect to the usual distance in the plane $\mathbb{R}^2$. As a consequence, a function $f:\mathbb{C}\to\mathbb{C}$ is continuous precisely when it is continuous, in an intuitive sense, when regarded as function $f:\mathbb{R}^2\to\mathbb{R}^2$. But more on this, later in this book.

At the level of key examples of complex functions, we have the exponential:

THEOREM 1.17. *We can exponentiate the complex numbers, according to the formula*

$$e^x=\sum_{k=0}^\infty\frac{x^k}{k!}$$

and the function $x\to e^x$ is continuous, and satisfies $e^{x+y}=e^xe^y$.

PROOF. We must first prove that the series converges. But this follows from:

$$\begin{aligned}|e^x|&=\left|\sum_{k=0}^\infty\frac{x^k}{k!}\right|\\&\leq\sum_{k=0}^\infty\left|\frac{x^k}{k!}\right|\\&=\sum_{k=0}^\infty\frac{|x|^k}{k!}\\&=e^{|x|}<\infty\end{aligned}$$

Regarding the formula $e^{x+y}=e^xe^y$, this follows too as in the real case, as follows:

$$\begin{aligned}e^{x+y}&=\sum_{k=0}^\infty\frac{(x+y)^k}{k!}\\&=\sum_{k=0}^\infty\sum_{s=0}^k\binom{k}{s}\cdot\frac{x^sy^{k-s}}{k!}\\&=\sum_{k=0}^\infty\sum_{s=0}^k\frac{x^sy^{k-s}}{s!(k-s)!}\\&=e^xe^y\end{aligned}$$

Finally, the continuity of $x \to e^x$ comes at $x = 0$ from the following computation:

$$\begin{aligned} |e^t - 1| &= \left|\sum_{k=1}^\infty \frac{t^k}{k!}\right| \\ &\leq \sum_{k=1}^\infty \left|\frac{t^k}{k!}\right| \\ &= \sum_{k=1}^\infty \frac{|t|^k}{k!} \\ &= e^{|t|} - 1 \end{aligned}$$

As for the continuity of $x \to e^x$ in general, this can be deduced now as follows:

$$\begin{aligned} \lim_{t\to 0} e^{x+t} &= \lim_{t\to 0} e^x e^t \\ &= e^x \lim_{t\to 0} e^t \\ &= e^x \cdot 1 \\ &= e^x \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

We will be back to more complex functions later, starting with chapter 4, notably by adding complex analogues of $\sin, \cos, \log$ to the above complex analogue of exp.

Getting back now to the basics, the complex numbers themselves, we would like to discuss now the final and most convenient writing for these numbers. This ultimate writing, due to Euler, is an analytic variation on the polar writing, as follows:

$$x = re^{it}$$

For this purpose, recall from Theorem 1.17 that the complex exponentials are subject to the formula $e^{x+y} = e^x e^y$, as in the real case. As a consequence of this, we have:

PROPOSITION 1.18. *The exponential of complex numbers is given by*

$$e^{s+it} = e^s e^{it}$$

with e^s being a usual real exponential, and with e^{it}, in need to be computed.

PROOF. This is indeed something self-explanatory, coming from $e^{x+y} = e^x e^y$, and with the somewhat non-standard notation $x = s + it$ being something needed later. □

Now let us get to the remaining problem, namely computation of e^{it} with $t \in \mathbb{R}$. Here are a few elementary observations, regarding the operation $t \to e^{it}$:

PROPOSITION 1.19. *For $t \in \mathbb{R}$ the number e^{it} belongs to the unit circle,*

$$e^{it} \in \mathbb{T}$$

and the operation $t \to e^{it}$ is subject to the following formulae,

$$e^{i(s+t)} = e^{is}e^{it} \quad , \quad e^{i0} = 1 \quad , \quad (e^{it})^{-1} = e^{it}$$

telling us $t \to e^{it}$ is a group morphism $\mathbb{R} \to \mathbb{T}$.

PROOF. There are several things going on here, the idea being as follows:

(1) To start with, we have the following formula, valid for any $x \in \mathbb{C}$:

$$e^{\bar{x}} = \sum_{k=0}^{\infty} \frac{\bar{x}^k}{k!} = \overline{\sum_{k=0}^{\infty} \frac{x^k}{k!}} = \overline{e^x}$$

We have as well the following computation, again valid for any $x \in \mathbb{C}$:

$$e^x e^{-x} = e^{x-x} = e^0 = 1 \implies (e^x)^{-1} = e^{-x}$$

(2) But with these two formulae in hand, we can prove the first assertion. Indeed, the first formula, applied with $x = it$, with $t \in \mathbb{R}$, gives the following equality:

$$e^{-it} = \overline{e^{it}}$$

As for the second formula above, again applied with $x = it$, this gives:

$$(e^{it})^{-1} = e^{it}$$

We conclude that the complex number $z = e^{it}$ has the following property:

$$z^{-1} = \bar{z}$$

But this is exactly the equation of the unit circle $\mathbb{T}$, as desired.

(3) Regarding now the various formulae in the statement, for the operation $t \to e^{it}$, these are all trivial, coming from the multiplicativity formula $e^{x+y} = e^x e^y$.

(4) As for the final conclusion, this is something quite intuitive, telling us that $t \to e^{it}$ transforms the additive structure of $\mathbb{R}$ into the multiplicative structure of $\mathbb{T}$. □

What is next? Well, we will have to trick a bit, and we are led in this way to the following fundamental result of Euler, regarding the complex exponential:

THEOREM 1.20. *We have the following formula,*

$$e^{it} = \cos t + i \sin t$$

valid for any $t \in \mathbb{R}$.

PROOF. We have several possible proofs here, all instructive, as follows:

(1) Intuitive proof. We know from Proposition 1.19 that $t \to e^{it}$ is a group morphism $\mathbb{R} \to \mathbb{T}$. But in view of this, barring any pathologies, this operation can only appear by "wrapping". That is, we must have a formula as follows, for a certain $\alpha \in \mathbb{R}$:

$$e^{it} = \cos(\alpha t) + i\sin(\alpha t)$$

In order now to find the parameter $\alpha \in \mathbb{R}$, let us look at what happens around $t = 0$. As a first observation, at $t = 0$ precisely, our formula is as follows, true:

$$e^0 = \cos 0 + i \sin 0$$

The point now is that, around $t = 0$, we have the following elementary estimate, simply obtained by truncating the series defining the exponential:

$$e^{it} \simeq 1 + it$$

On the other hand, in what regards sin and cos, we can use here the fact, which is something clear on pictures, that we have $\sin t \simeq t$ for $t \simeq 0$, plus the fact, which follows from this and from Pythagoras, $\sin^2 + \cos^2 = 1$, that we have as well $\cos t \simeq 1 - t^2/2$, for $t \simeq 0$. We conclude that we have the following estimate, again around $t = 0$:

$$\cos(\alpha t) + i\sin(\alpha t) \simeq 1 + i\alpha t$$

Thus $\alpha = 1$, and we are led to the Euler formula, namely:

$$e^{it} = \cos t + i \sin t$$

(2) Analysis proof. Consider the following function $f : \mathbb{R} \to \mathbb{C}$:

$$f(t) = \frac{\cos t + i \sin t}{e^{it}}$$

We want to prove that f is constant, with the constant in question being $f(0) = 1$. In order to do so, the idea will be that of proving that the derivative of f vanishes:

$$f' = 0$$

So, calculus. To start with, we can rewrite the formula of f, as follows:

$$f(t) = e^{-it}(\cos t + i \sin t)$$

In order to differentiate this expression, which is a product, we can use the Leibnitz formula for products, which is something elementary, as follows:

$$(gh)' = g'h + gh'$$

In what regards now the components, for the exponential we have:

$$\begin{aligned}(e^x)' &= \left(\sum_{k=0}^\infty \frac{x^k}{k!}\right)' \\ &= \sum_{k=0}^\infty \frac{kx^{k-1}}{k!} \\ &= e^x\end{aligned}$$

Regarding now sin, the computation here goes as follows, by using the standard estimates for sin and cos around $t=0$, discussed in the context of (1):

$$\begin{aligned}(\sin x)' &= \lim_{t\to 0}\frac{\sin(x+t)-\sin x}{t} \\ &= \lim_{t\to 0}\frac{\sin x\cos t+\cos x\sin t-\sin x}{t} \\ &= \lim_{t\to 0}\sin x\cdot\frac{\cos t-1}{t}+\cos x\cdot\frac{\sin t}{t} \\ &= \cos x\end{aligned}$$

Finally, regarding cos, the computation here is similar, as follows:

$$\begin{aligned}(\cos x)' &= \lim_{t\to 0}\frac{\cos(x+t)-\cos x}{t} \\ &= \lim_{t\to 0}\frac{\cos x\cos t-\sin x\sin t-\cos x}{t} \\ &= \lim_{t\to 0}\cos x\cdot\frac{\cos t-1}{t}-\sin x\cdot\frac{\sin t}{t} \\ &= -\sin x\end{aligned}$$

By putting now everything together, we have the following computation:

$$\begin{aligned}f'(t) &= (e^{-it}(\cos t+i\sin t))' \\ &= -ie^{-it}(\cos t+i\sin t)+e^{-it}(-\sin t+i\cos t) \\ &= e^{-it}(-i\cos t+\sin t)+e^{-it}(-\sin t+i\cos t) \\ &= 0\end{aligned}$$

We conclude that $f:\mathbb{R}\to\mathbb{C}$ is constant, equal to $f(0)=1$, as desired.

(3) Physics proof. We have indeed the following elegant computation, with the end making use of the well-known formulae for the Taylor series of cos and sin:

$$\begin{aligned}
e^{it} &= \sum_{k=0}^\infty \frac{(it)^k}{k!} \\
&= \sum_{k=2l} \frac{(it)^k}{k!} + \sum_{k=2l+1} \frac{(it)^k}{k!} \\
&= \sum_{l=0}^\infty (-1)^l \frac{t^{2l}}{(2l)!} + i\sum_{l=0}^\infty (-1)^l \frac{t^{2l+1}}{(2l+1)!} \\
&= \cos t + i \sin t
\end{aligned}$$

And I will leave it to you to figure out what is wrong with this, and what the fix is. □

Now back to our $x = re^{it}$ objectives, with the above theory in hand we can indeed use from now on this notation, the complete statement being as follows:

THEOREM 1.21. *The complex numbers $x = a + ib$ can be written in polar coordinates,*

$$x = re^{it}$$

with the connecting formulae being

$$a = r\cos t \quad , \quad b = r \sin t$$

and in the other sense being

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

and with r, t being called modulus, and argument.

PROOF. This is a reformulation of our previous Definition 1.9, by using the formula $e^{it} = \cos t + i \sin t$ from Theorem 1.20, and multiplying everything by r. □

With this in hand, we can now go back to the basics, namely the addition and multiplication of the complex numbers. We have the following result:

THEOREM 1.22. *In polar coordinates, the complex numbers multiply as*

$$re^{is} \cdot pe^{it} = rp\, e^{i(s+t)}$$

with the arguments s, t being taken modulo 2π.

PROOF. This is something that we already know, from Theorem 1.10, reformulated by using the notations from Theorem 1.21. Observe that this follows as well directly, from the fact that we have $e^{a+b} = e^a e^b$, that we know from Theorem 1.17. □

We can investigate as well more complicated operations, as follows:

THEOREM 1.23. *We have the following operations on the complex numbers, written in polar form, as above:*

(1) *Inversion:* $(re^{it})^{-1} = r^{-1}e^{-it}$.
(2) *Square roots:* $\sqrt{re^{it}} = \pm\sqrt{r}e^{it/2}$.
(3) *Powers:* $(re^{it})^a = r^a e^{ita}$.
(4) *Conjugation:* $\overline{re^{it}} = re^{-it}$.

PROOF. This is something that we already know, from Theorem 1.11, but we can now discuss all this, from a more conceptual viewpoint, the idea being as follows:

(1) We have indeed the following computation, using Theorem 1.22:

$$\begin{aligned}(re^{it})(r^{-1}e^{-it}) &= rr^{-1} \cdot e^{i(t-t)} \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

(2) Once again by using Theorem 1.22, we have:

$$(\pm\sqrt{r}e^{it/2})^2 = (\sqrt{r})^2 e^{i(t/2+t/2)} = re^{it}$$

(3) Given an arbitrary number $a \in \mathbb{R}$, we can define, as stated:

$$(re^{it})^a = r^a e^{ita}$$

Due to Theorem 1.22, this operation $x \to x^a$ is indeed the correct one.

(4) This comes from the fact, that we know from Theorem 1.6, that the conjugation operation $x \to \bar{x}$ keeps the modulus, and switches the sign of the argument. □

Summarizing, the exponential writing of the complex numbers $x = re^{it}$ is good for pretty much everything, except for the summing operation, $(x, y) \to x + y$.

1d. Polynomials

Getting now to polynomials, we can upgrade our degree 2 result, as follows:

THEOREM 1.24. *The complex solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{C}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of $b^2 - 4ac = re^{it}$ being extracted as above, namely

$$\sqrt{re^{it}} = \pm\sqrt{r}e^{it/2}$$

and with this generalizing the usual real formula, where $t = 0$.

PROOF. This is a bit tiring, I know, with this being the third consecutive upgrade of Theorem 1.2, that we started with. But, this is the final upgrade, I promise. □

We can upgrade as well our theory of roots of unity, as follows:

THEOREM 1.25. *The equation $x^N = 1$ has N complex solutions, namely*

$$\left\{w^k\Big|k = 0, 1, \ldots, N-1\right\} \quad , \quad w = e^{2\pi i/N}$$

which are called roots of unity of order N.

PROOF. This follows from the general multiplication formula for complex numbers from Theorem 1.22. Indeed, with $x = re^{it}$ our equation reads:

$$r^N e^{itN} = 1$$

Thus $r = 1$, and $t \in [0, 2\pi)$ must be a multiple of $2\pi/N$, as stated. □

As an illustration here, the roots of unity of small order, along with some of their basic properties, which are very useful for computations, are as follows:

$\underline{N=1}$. Here the unique root of unity is 1.

$\underline{N=2}$. Here we have two roots of unity, namely 1 and -1.

$\underline{N=3}$. Here we have 1, then $w = e^{2\pi i/3}$, and then $w^2 = \bar{w} = e^{4\pi i/3}$.

$\underline{N=4}$. Here the roots of unity, read as usual counterclockwise, are $1, i, -1, -i$.

$\underline{N=5}$. Here, with $w = e^{2\pi i/5}$, the roots of unity are $1, w, w^2, w^3, w^4$.

$\underline{N=6}$. Here a useful alternative writing is $\{\pm 1, \pm w, \pm w^2\}$, with $w = e^{2\pi i/3}$.

$\underline{N=7}$. Here, with $w = e^{2\pi i/7}$, the roots of unity are $1, w, w^2, w^3, w^4, w^5, w^6$.

$\underline{N=8}$. Here the roots of unity, read as usual counterclockwise, are the numbers $1, w, i, iw, -1, -w, -i, -iw$, with $w = e^{\pi i/4}$, which is also given by $w = (1+i)/\sqrt{2}$.

The roots of unity appear in connection with many interesting questions, and there are many useful formulae relating them, which are good to know. Here is a basic such formula, which is very beautiful, to be used many times in what follows:

PROPOSITION 1.26. *The roots of unity, $\{w^k\}$ with $w = e^{2\pi i/N}$, have the property*

$$\sum_{k=0}^{N-1} (w^k)^s = N\delta_{N|s}$$

for any exponent $s \in \mathbb{N}$, where on the right we have a Kronecker symbol.

PROOF. The numbers in the statement, when written more conveniently as $(w^s)^k$ with $k = 0, \ldots, N-1$, form a certain regular polygon in the plane P_s. Thus, if we denote by C_s the barycenter of this polygon, we have the following formula:

$$\frac{1}{N}\sum_{k=0}^{N-1} w^{ks} = C_s$$

Now observe that in the case $N\not| s$ our polygon P_s is non-degenerate, circling around the unit circle, and having center $C_s = 0$. As for the case $N|s$, here the polygon is degenerate, lying at 1, and having center $C_s = 1$. Thus, we have the following formula:

$$C_s = \delta_{N|s}$$

Thus, we obtain the formula in the statement. □

As an interesting philosophical fact, regarding the roots of unity, and the complex numbers in general, we can now solve the following equation, in a "uniform" way:

$$x_1 + \ldots + x_N = 0$$

With this being not a joke. Frankly, can you find some nice-looking family of real numbers $x_1, \ldots, x_N$ satisfying $x_1 + \ldots + x_N = 0$? Certainly not. But with complex numbers we have now our answer, the sum of the N-th roots of unity being zero.

As a third and last general result now, regarding the polynomials, which is something very general, generalizing both Theorem 1.24 and Theorem 1.25, we have:

THEOREM 1.27. *Any polynomial $P \in \mathbb{C}[X]$ decomposes as*

$$P = c(X - r_1) \ldots (X - r_N)$$

with $c \in \mathbb{C}$ and with $r_1, \ldots, r_N \in \mathbb{C}$.

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, the problem is that of proving that our polynomial has at least one root, because afterwards we can proceed by recurrence, in the obvious way.

(2) We prove this fact, that P has at least one root, by contradiction. So, assume that P has no roots, and pick a number $z \in \mathbb{C}$ where $|P|$ attains its minimum:

$$|P(z)| = \min_{x \in \mathbb{C}} |P(x)| > 0$$

(3) Since $Q(t) = P(z + t) - P(z)$ is a polynomial which vanishes at $t = 0$, this polynomial must be of the form ct^k + higher terms, with $c \neq 0$, and with $k \geq 1$ being an integer. We obtain from this that, with $t \in \mathbb{C}$ small, we have the following estimate:

$$P(z + t) \simeq P(z) + ct^k$$

Now let us write $t = rw$, with $r > 0$ small, and with $|w| = 1$. Our estimate becomes:

$$P(z + rw) \simeq P(z) + cr^k w^k$$

(4) But, recall that we assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$\begin{aligned} |P(z + rw)| &\simeq |P(z) + cr^k w^k| \\ &= |P(z)|(1 - |c|r^k) \end{aligned}$$

(5) Now by choosing $r > 0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(z + rw)| < |P(z)|$$

(6) But this contradicts our definition of $z \in \mathbb{C}$, as a point where $|P|$ attains its minimum. Thus P has a root, and by recurrence it has N roots, as stated. $\square$

All this is very nice, and we will see many applications later. As a word of warning, however, we should mention that the above result remains something quite theoretical. Indeed, the proof is by contradiction, and there is no way of recycling the material there into something explicit, that can be used for effectively computing the roots.

1e. Exercises

Welcome to the complex numbers, and as exercises about them, we have:

EXERCISE 1.28. *What happens to $\mathbb{C}$ when representing it upside-down?*

EXERCISE 1.29. *What about drawing $\mathbb{R}$ from right to left? Or doing both?*

EXERCISE 1.30. *Can you prove $\sum_n x^n = 1/(1-x)$, geometrically, for $|x| < 1$?*

EXERCISE 1.31. *Find and solve basic geometry problems, using complex numbers.*

EXERCISE 1.32. *Study more $e^{it} = \cos t + i \sin t$, including Taylor series discussion.*

EXERCISE 1.33. *By the way, do not forget to learn as well about $E = mc^2$.*

EXERCISE 1.34. *Meditate on how to define $\sin, \cos, \log$, as complex functions.*

EXERCISE 1.35. *Study the real and complex solutions of $x_1 + \ldots + x_N = 0$.*

As bonus exercise, review if needed your basic trigonometry knowledge.

CHAPTER 2

Plane geometry

2a. Plane geometry

We have seen so far that complex numbers are kings, when it comes to solve equations of type $P(x) = 0$, with P being a polynomial. What is next? Many things, based on this, especially in relation with linear algebra, and with multivariable calculus too, where such equations $P(x) = 0$ appear naturally, via the Jacobian and Hessian matrices, and their characteristic polynomials. We will discuss this starting from chapter 3 below.

Before that, however, we have unfinished philosophical business, with the complex numbers themselves. Here is the question that we would like to solve:

Question 2.1. *What happens to the very first area of modern mathematics, namely plane geometry, as developed by Thales and others, when using complex numbers?*

So, this will be what we will be talking about, in this chapter, and take this as a relaxing discussion, before getting to more complicated things and analysis, later. And with this being something useful too, if you intend to do afterwards, as your daily job, any sort of geometry, or pure mathematics, or theoretical physics, where geometry rules too, please be sure that knowing the answer to Question 2.1 will help you a lot.

Turning now to the answer of Question 2.1, there are many theorems in plane geometry, and tricky exercises too, and generally speaking, all these have 3 solutions, as follows:

(1) First we have the old-style solution, essentially based on the Thales theorem, and all sorts of tricks. With this being usually quite hard to beat, cannot really compete with 2000 years of knowledge and improvements, when it comes to doing such things.

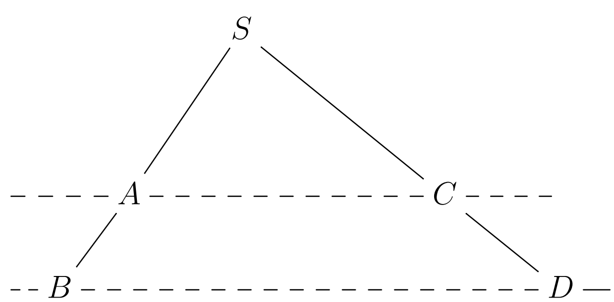
(2) Then we have the solution using real coordinates, label everything as $(a, b) \in \mathbb{R}^2$, and compute. This can be sometimes more elegant than the old-style solution, with the existence of the barycenter of triangles being a good illustration here.

(3) Alternatively, we can use complex numbers, by labeling everything as $x \in \mathbb{C}$, and computing. And with this being often better than using real coordinates, with for instance the equation of the circles being something very simple, $|x - c| = r$.

So, this is the situation, with at the end of the day, a choice between (1) and (3), and with (2) being somehow obsolete. We will discuss this in what follows, with a review of the plane geometry basics, by using (1) and (3), and comparing these methods.

Getting started now, at the beginning of everything, we have the Thales theorem:

THEOREM 2.2 (Thales). *Proportions are kept, along parallel lines. That is, given a configuration as follows, consisting of two parallel lines, and of two extra lines,*



the following equality holds:

$$\frac{SA}{SB}=\frac{SC}{SD}$$

Moreover, the converse of this holds too, in the sense that, in the context of a picture as above, if this equality is satisfied, then the lines AC and BD must be parallel.

PROOF. This can be proved indeed, by using our various methods. □

Coming next, we have a useful technical result, as follows:

PROPOSITION 2.3. *We have a duality between points and lines, obtained by fixing a circle in the plane, say of center O and radius $r>0$, and doing the following,*

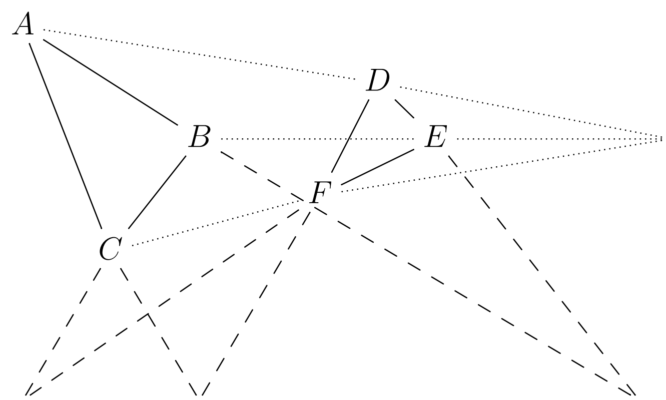
(1) *Given a point P, construct Q on the line OP, as to have $OP\cdot OQ=r^2$,*
(2) *Draw the perpendicular at Q on the line OQ. This is the dual line p,*

and this duality $P\leftrightarrow p$ transforms collinear points into concurrent lines.

PROOF. Again, this can be proved indeed, by using our various methods. □

Coming next, we have the Desargues theorem, which is as follows:

THEOREM 2.4 (Desargues). *Two triangles are in perspective centrally,*



if and only if they are in perspective axially.

PROOF. This can be proved indeed by using our various methods, and with Proposition 2.3 being of great help, in relation with the old-style proof. □

Next, we have the following useful technical result:

PROPOSITION 2.5. *We can talk about the cross ratio of four collinear points A, B, C, D, as being the following quantity, signed according to our usual sign conventions,*

$$(A, B, C, D) = \frac{AC \cdot BD}{BC \cdot AD}$$

and with this notion in hand, points in central perspective have the same cross ratio:

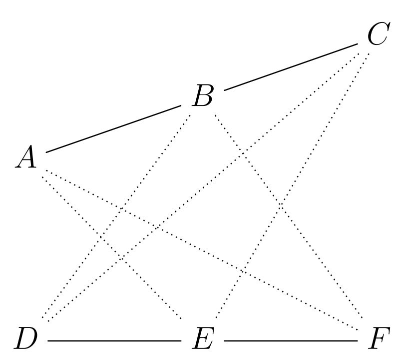
$$(A, B, C, D) = (A', B', C', D')$$

Moreover, the converse of this fact holds too.

PROOF. Again, this can be proved indeed, by using our various methods. □

Coming next, we have the Pappus theorem, which is as follows:

THEOREM 2.6 (Pappus). *Given a configuration as follows,*



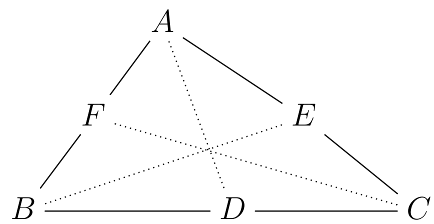
the three middle points are collinear.

PROOF. This can be proved indeed by using our various methods, and with Proposition 2.5 being of great help, in relation with the old-style proof. ☐

2b. Triangles, centers

Getting now to the core of plane geometry, triangles and their centers, we first have the barycenter theorem, which drastically simplifies with coordinates, as follows:

THEOREM 2.7 (Barycenter). *Given a triangle ABC, its medians cross,*



at a point called barycenter, lying at $1/3 - 2/3$ on each median.

PROOF. Let us call $A, B, C \in \mathbb{C}$ the coordinates of the vertices A, B, C, and consider the average $P = (A + B + C)/3$. We have then:

$$P = \frac{1}{3} \cdot A + \frac{2}{3} \cdot \frac{B + C}{2}$$

Thus P lies on the median emanating from A, and a similar argument shows that P lies as well on the medians emanating from B, C. Thus, we have our barycenter. ☐

Getting now to the other centers of a triangle, we have here:

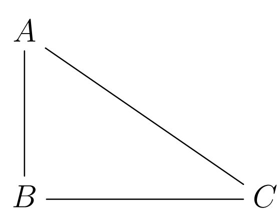
THEOREM 2.8. *Given a triangle ABC, the following happen:*

(1) *The angle bisectors cross, at a point called incenter.*
(2) *The perpendicular bisectors cross, at a point called circumcenter.*
(3) *The altitudes cross, at a point called orthocenter.*

PROOF. Again, such things can be proved with coordinates, and patience. We will leave some of the calculations here as an instructive exercise for you, reader. ☐

Coming next, we have the theorem of Pythagoras:

THEOREM 2.9 (Pythagoras). *In a right triangle ABC,*

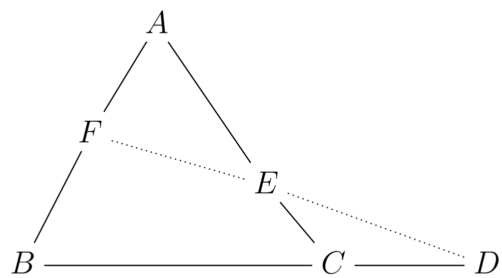


we have $AB^2 + BC^2 = AC^2$.

Proof. Again, this is clear with coordinates. □

Next, we have the following key result, due to Menelaus:

Theorem 2.10 (Menelaus). *In a configuration of the following type, with a triangle ABC cut by a line DEF,*



we have the following formula, with all segments being taken oriented:

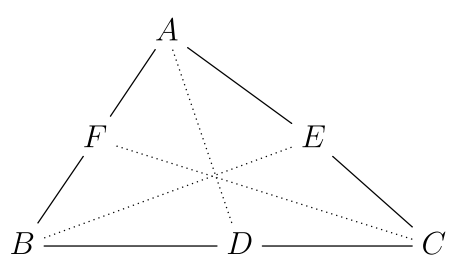
$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=-1$$

Moreover, the converse holds, with this formula guaranteeing that D,E,F are colinear.

Proof. Again, this can be proved with coordinates. □

Next, we have the following remarkable result, due to Ceva:

Theorem 2.11 (Ceva). *In a configuration of the following type, with a triangle ABC containing inner lines AD,BE,CF which cross,*



we have the following formula:

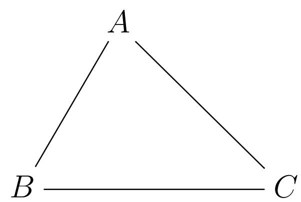
$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

Moreover, the converse holds, with this formula guaranteeing that AD,BE,CF cross.

Proof. This traditionally follows from Menelaus, applied 3 times, and can be proved as well directly, with coordinates. □

At a more advanced level now, we have the following key result:

Theorem 2.12. *Besides the* 4 *main centers of a triangle, discussed in the above, many more remarkable points can be associated to a triangle* ABC*,*

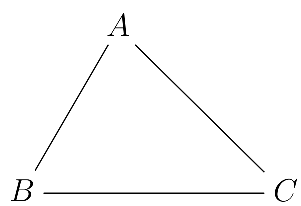


and most of these lie on a line, called Euler line of ABC*. In particular, the barycenter* G*, the circumcenter* O *and the orthocenter* H *lie on this line, and* $GH = 2GO$*.*

Proof. Again, this can be proved with coordinates. □

Along the same lines, advanced plane geometry, we have as well the following result:

Theorem 2.13. *Associated to any triangle* ABC*,*



we have a nine-point circle, passing through the following points:

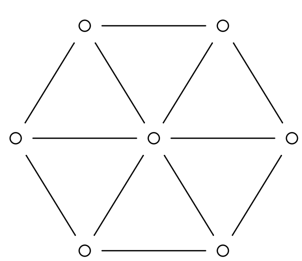
(1) *The midpoints of each side.*
(2) *The feet of each altitude.*
(3) *The midpoint of each segment vertex - orthocenter.*

Moreover, the center of this circle lies on the Euler line, midway between H *and* O*.*

Proof. Again, this can be proved with coordinates. □

Coming next, at a more complicated level, we have the following result:

Theorem 2.14 (Pascal). *Given a hexagon lying on a circle*

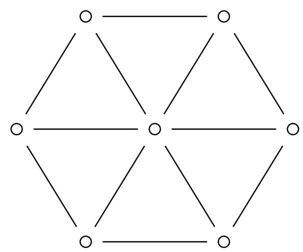


the pairs of opposite sides intersect in points which are collinear.

PROOF. This can be proved indeed, with some tricks. Observe the similarity with Pappus. We will see in fact, later in this chapter, when talking about conics, that the Pascal theorem generalizes to the case of conics, and with this fully generalizing Pappus. □

And here is now, at a truly advanced level, a quite scary theorem:

THEOREM 2.15 (Brianchon). *Given a hexagon circumscribed around on a circle*



the main diagonals intersect.

PROOF. This is nearly impossible to prove, with bare hands, and ask around kids preparing for Math Olympiads, they will witness for that. But, this follows by duality from Pascal. As before with Pascal, we will see later that this extends to conics. □

And with this, good news, done, and we are now experts in algebraic geometry.

2c. Ellipses, conics

Looking up, to the sky, the first thing that you see is the Sun, seemingly moving around the Earth on a circle, but a more careful study reveals that this circle is rather a deformed circle, called ellipse. As for the other stars and planets, these have all sort of weird trajectories, but a more careful study reveals that, with due attention to what the best "center" is, replacing our Earth, the trajectories are often ellipses:

(1) Indeed, this applies to all the planets in our Solar System, which move around the biggest object in the system, which is by far the Sun, on ellipses.

(2) The same trick applies to the trajectories of various distant stars, the rule being always the same, "small moves around big, on an ellipse".

(3) However, there are counterexamples too, such as asteroids reaching our Solar system, but then traveling outwards, never to be seen again.

Summarizing, modulo some annoying asteroids that we will leave for later, we are led in this way to ellipses, and their mathematics. And good news, a full theory of ellipses is available, and this since the ancient Greeks, whose main findings were as follows:

Theorem 2.16. *The ellipses, taken centered at the origin 0, and squarely oriented with respect to Oxy, can be defined in 4 possible ways, as follows:*

(1) *As the curves given by an equation as follows, with $a, b > 0$:*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

(2) *Or given by an equation as follows, with $q > 0$, $p = -q$, and $l \in (0, 2q)$:*

$$d(z, p) + d(z, q) = l$$

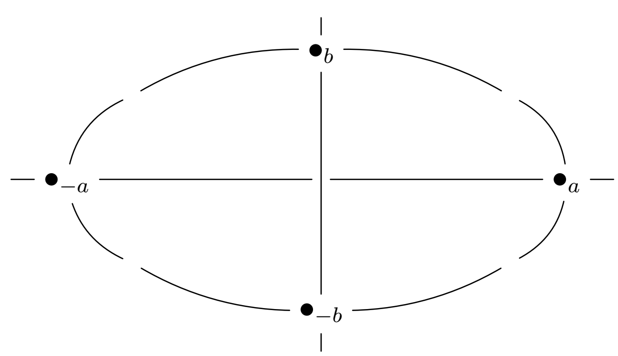
(3) *As the curves appearing when drawing a circle, from various perspectives:*

$$\bigcirc \quad \to \quad ?$$

(4) *As the closed non-degenerate curves appearing by cutting a cone with a plane.*

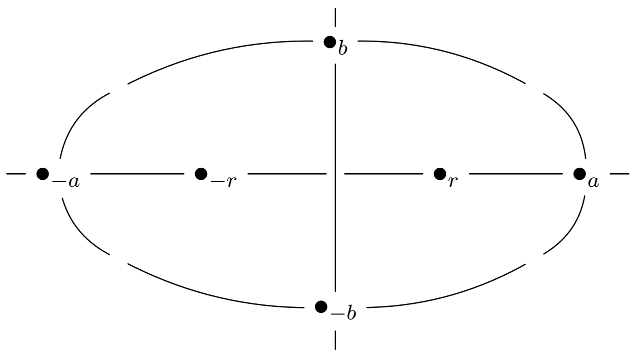
Proof. This might look a bit confusing, and you might say, what exactly is to be proved here. Good point, and in answer, what is to be proved is that the above constructions (1-4) give rise to the same class of curves. And this can be done as follows:

(1) To start with, let us draw a picture from what comes out of (1), which will be our main definition for the ellipses, in what follows. Here that is, making it clear what the parameters $a, b > 0$ stand for, with $2a \times 2b$ being the gift box size for our ellipse:



(2) Let us prove now that such an ellipse has two focal points, as stated in (2). We must look for a number $r > 0$, and a number $l > 0$, such that our ellipse appears as

$d(z,p)+d(z,q)=l$, with $p=(0,-r)$ and $q=(0,r)$, according to the following picture:



(3) Let us first compute these numbers $r,l>0$. Assuming that our result holds indeed as stated, by taking $z=(0,a)$, we see that the length l is:

$$l=(a-r)+(a+r)=2a$$

As for the parameter r, by taking $z=(b,0)$, we conclude that we must have:

$$2\sqrt{b^2+r^2}=2a\implies r=\sqrt{a^2-b^2}$$

(4) With these observations made, let us prove now the result. Given $l,r>0$, and setting $p=(0,-r)$ and $q=(0,r)$, we have the following computation, with $z=(x,y)$:

$$\begin{aligned}
& d(z,p)+d(z,q)=l\\
\iff\ & \sqrt{(x+r)^2+y^2}+\sqrt{(x-r)^2+y^2}=l\\
\iff\ & \sqrt{(x+r)^2+y^2}=l-\sqrt{(x-r)^2+y^2}\\
\iff\ & (x+r)^2+y^2=(x-r)^2+y^2+l^2-2l\sqrt{(x-r)^2+y^2}\\
\iff\ & 2l\sqrt{(x-r)^2+y^2}=l^2-4xr\\
\iff\ & 4l^2(x^2+r^2-2xr+y^2)=l^4+16x^2r^2-8l^2xr\\
\iff\ & 4l^2x^2+4l^2r^2+4l^2y^2=l^4+16x^2r^2\\
\iff\ & (4x^2-l^2)(4r^2-l^2)=4l^2y^2
\end{aligned}$$

(5) Now observe that we can further process the equation that we found as follows:

$$\begin{aligned}(4x^2-l^2)(4r^2-l^2)=4l^2y^2 &\iff \frac{4x^2-l^2}{l^2}=\frac{4y^2}{4r^2-l^2}\\ &\iff \frac{4x^2-l^2}{l^2}=\frac{y^2}{r^2-l^2/4}\\ &\iff \left(\frac{x}{2l}\right)^2-1=\left(\frac{y}{\sqrt{r^2-l^2/4}}\right)^2\\ &\iff \left(\frac{x}{2l}\right)^2+\left(\frac{y}{\sqrt{r^2-l^2/4}}\right)^2=1\end{aligned}$$

(6) Thus, our result holds indeed, and with the numbers $l, r > 0$ appearing, and no surprise here, via the formulae $l = 2a$ and $r = \sqrt{a^2 - b^2}$, found in (3) above.

(7) Getting back to our theorem, we have two other assertions there at the end, (3,4). But, thinking a bit, these assertions are equivalent, and (4) can be established by doing some 3D computations, that we will leave here as an instructive exercise, for you. □

All this is very nice, but before getting into physics, let us settle as well the question of wandering asteroids. These can travel on parabolas and hyperbolas, so what we need as mathematics is a unified theory of ellipses, parabolas and hyperbolas.

And fortunately, this theory exists too, also since the ancient Greeks. The basics of this theory, called theory of conics, can be summarized as follows:

THEOREM 2.17. *The conics, which are the algebraic curves of degree* 2 *in the plane,*

$$C = \left\{(x,y) \in \mathbb{R}^2 \middle| P(x,y) = 0\right\}$$

with $\deg P \leq 2$, *appear modulo degeneration by cutting a* 2*-sided cone with a plane, and can be classified into ellipses, parabolas and hyperbolas.*

PROOF. This follows by further building on Theorem 2.16, as follows:

(1) Let us first classify the conics up to non-degenerate linear transformations of the plane, which are by definition transformations as follows, with $\det A \neq 0$:

$$\begin{pmatrix}x\\y\end{pmatrix} \to A\begin{pmatrix}x\\y\end{pmatrix}$$

Our claim is that as solutions we have the circles, parabolas, hyperbolas, along with some degenerate solutions, namely $\emptyset$, points, lines, pairs of lines, $\mathbb{R}^2$.

(2) As a first remark, it looks like we forgot precisely the ellipses, but via linear transformations these become circles, so things fine. As a second remark, all our claimed solutions can appear. Indeed, the circles, parabolas, hyperbolas can appear as follows:

$$x^2+y^2=1 \quad , \quad x^2=y \quad , \quad xy=1$$

As for $\emptyset$, points, lines, pairs of lines, $\mathbb{R}^2$, these can appear too, as follows, and with our polynomial P chosen, whenever possible, to be of degree exactly 2:

$$x^2=-1 \quad , \quad x^2+y^2=0 \quad , \quad x^2=0 \quad , \quad xy=0 \quad , \quad 0=0$$

Observe here that, when dealing with these degenerate cases, assuming $\deg P = 2$ instead of $\deg P \leq 2$ would only rule out $\mathbb{R}^2$ itself, which is not worth it.

(3) Getting now to the proof of our claim in (1), classification up to linear transformations, consider an arbitrary conic, written as follows, with $a,b,c,d,e,f \in \mathbb{R}$:

$$ax^2+by^2+cxy+dx+ey+f=0$$

Assume first $a \neq 0$. By making a square out of ax^2, up to a linear transformation in (x,y), we can get rid of the term cxy, and we are left with:

$$ax^2+by^2+dx+ey+f=0$$

In the case $b \neq 0$ we can make two obvious squares, and again up to a linear transformation in (x,y), we are left with an equation as follows:

$$x^2 \pm y^2 = k$$

In the case of positive sign, $x^2+y^2=k$, the solutions are the circle, when $k \geq 0$, the point, when $k=0$, and $\emptyset$, when $k<0$. As for the case of negative sign, $x^2-y^2=k$, which reads $(x-y)(x+y)=k$, here once again by linearity our equation becomes $xy=l$, which is a hyperbola when $l \neq 0$, and two lines when $l=0$.

(4) In the case $b \neq 0$ the study is similar, with the same solutions, so we are left with the case $a=b=0$. Here our conic is as follows, with $c,d,e,f \in \mathbb{R}$:

$$cxy+dx+ey+f=0$$

If $c \neq 0$, by linearity our equation becomes $xy=l$, which produces a hyperbola or two lines, as explained before. As for the remaining case, $c=0$, here our equation is:

$$dx+ey+f=0$$

But this is generically the equation of a line, unless we are in the case $d=e=0$, where our equation is $f=0$, having as solutions $\emptyset$ when $f \neq 0$, and $\mathbb{R}^2$ when $f=0$.

(5) Thus, done with the classification, up to linear transformations as in (1). But this classification leads to the classification in general too, by applying now linear transformations to the solutions that we found. So, done with this, and very good.

(6) It remains to discuss the cone cutting. By suitably choosing our coordinate axes (x,y,z), we can assume that our cone is given by an equation as follows, with $k>0$:

$$x^2+y^2=kz^2$$

In order to prove the result, we must in principle intersect this cone with an arbitrary plane, which has an equation as follows, with $(a,b,c)\neq(0,0,0)$:

$$ax+by+cz=d$$

(7) However, before getting into computations, observe that what we want to find is a certain degree 2 equation in the above plane, for the intersection. Thus, it is convenient to change the coordinates, as for our plane to be given by the following equation:

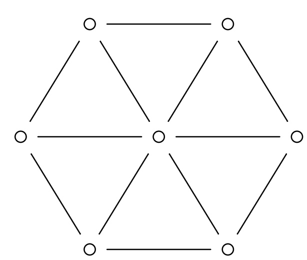
$$z=0$$

(8) But with this done, what we have to do is to see how the cone equation $x^2+y^2=kz^2$ changes, under this change of coordinates, and then set $z=0$, as to get the (x,y) equation of the intersection. But this leads, via some thinking or computations, to the conclusion that the cone equation $x^2+y^2=kz^2$ becomes in this way a degree 2 equation in (x,y), which can be arbitrary, and so to the final conclusion in the statement. □

In practice now, we already know many things about ellipses, from the beginning of this chapter. Similar things can be said about parabolas and hyperbolas.

Getting now to more advanced plane geometry, as a first application of our conic technology, we can prove the following statement, announced earlier:

Theorem 2.18 (Pascal). *Given a hexagon lying on a conic*

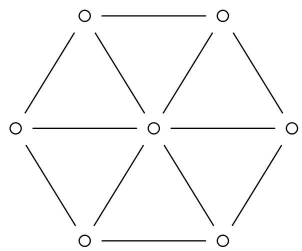


the pairs of opposite sides intersect in points which are collinear.

Proof. This can be proved indeed, by using coordinates, and with this being a good exercise, but the simplest in fact is to argue, as did Pascal himself, back in the days, that this follows from the result that we already know about the circles, by projecting. Observe also, among others, that our statement generalizes the Desargues theorem. □

Still following the material from before, we have as well the following result:

Theorem 2.19 (Brianchon). *Given a hexagon circumscribed around on a conic*



the main diagonals intersect.

Proof. This can be proved as well, similarly, by using coordinates, or by projecting. Observe among others that our statement generalizes the Pappus theorem. □

We can go back now to physics and gravity, and we have the following result:

Theorem 2.20. *Planets and other celestial bodies move around the Sun on conics,*

$$C=\left\{(x,y)\in\mathbb R^2\Big|P(x,y)=0\right\}$$

with $P\in\mathbb R[x,y]$ being of degree 2, which can be ellipses, parabolas or hyperbolas.

Proof. This is something quite long, due to Kepler and Newton, which actually requires a bit of knowledge of calculus and equations, the idea being as follows:

(1) According to observations and calculations performed over the centuries, since the ancient times, and first formalized by Newton, following some groundbreaking work of Kepler, the force of attraction between two bodies of masses M,m is given by:

$$||F||=G\cdot\frac{Mm}{d^2}$$

Here d is the distance between the two bodies, and $G\simeq 6.674\times 10^{-11}$ is a constant. Now assuming that M is fixed at $0\in\mathbb R^3$, the force exterted on m positioned at $x\in\mathbb R^3$, regarded as a vector $F\in\mathbb R^3$, is given by the following formula:

$$\begin{aligned}F&=-||F||\cdot\frac{x}{||x||}\\&=-\frac{GMm}{||x||^2}\cdot\frac{x}{||x||}\\&=-\frac{GMmx}{||x||^3}\end{aligned}$$

But $F=ma=m\ddot{x}$, with $a=\ddot{x}$ being the acceleration, second derivative of the position, so the equation of motion of m, assuming that M is fixed at 0, is:

$$\ddot{x}=-\frac{GMx}{||x||^3}$$

(2) Obviously, the problem happens in 2 dimensions, and you can even find, as an exercise, a formal proof of that, based on the above equation. Now here the most convenient is to use standard x,y coordinates, and denote our point as $z=(x,y)$. With this change made, and by setting $K=GM$, the equation of motion becomes:

$$\ddot{z}=-\frac{Kz}{||z||^3}$$

In other words, in terms of the coordinates x,y, the equations are:

$$\ddot{x}=-\frac{Kx}{(x^2+y^2)^{3/2}}\quad,\quad\ddot{y}=-\frac{Ky}{(x^2+y^2)^{3/2}}$$

(3) Let us begin with a simple particular case, that of the circular solutions. To be more precise, we are interested in solutions of the following type:

$$x=r\cos\alpha t\quad,\quad y=r\sin\alpha t$$

In this case we have $||z||=r$, so our equation of motion becomes:

$$\ddot{z}=-\frac{Kz}{r^3}$$

On the other hand, differentiating x,y leads to the following formula:

$$\ddot{z}=(\ddot{x},\ddot{y})=-\alpha^2(x,y)=-\alpha^2z$$

Thus, we have a circular solution when the parameters r,α satisfy:

$$r^3\alpha^2=K$$

(4) In the general case now, the problem can be solved via some calculus. Let us write indeed our vector $z=(x,y)$ in polar coordinates, as follows:

$$x=r\cos\theta\quad,\quad y=r\sin\theta$$

We have then $||z||=r$, and our equation of motion becomes, as in (3):

$$\ddot{z}=-\frac{Kz}{r^3}$$

Let us differentiate now x,y. By using the standard calculus rules, we have:

$$\dot{x}=\dot{r}\cos\theta-r\sin\theta\cdot\dot{\theta}$$

$$\dot{y}=\dot{r}\sin\theta+r\cos\theta\cdot\dot{\theta}$$

Differentiating one more time gives the following formulae:

$$\ddot{x}=\ddot{r}\cos\theta-2\dot{r}\sin\theta\cdot\dot{\theta}-r\cos\theta\cdot\dot{\theta}^2-r\sin\theta\cdot\ddot{\theta}$$

$$\ddot{y}=\ddot{r}\sin\theta+2\dot{r}\cos\theta\cdot\dot{\theta}-r\sin\theta\cdot\dot{\theta}^2+r\cos\theta\cdot\ddot{\theta}$$

Consider now the following two quantities, appearing as coefficients in the above:

$$a=\ddot{r}-r\dot{\theta}^2\quad,\quad b=2\dot{r}\dot{\theta}+r\ddot{\theta}$$

In terms of these quantities, our second derivative formulae read:

$$\ddot{x} = a\cos\theta - b\sin\theta$$

$$\ddot{y} = a\sin\theta + b\cos\theta$$

(5) We can now solve the equation of motion from (4). Indeed, with the formulae that we found for $\ddot{x}, \ddot{y}$, our equation of motion takes the following form:

$$a\cos\theta - b\sin\theta = -\frac{K}{r^2}\cos\theta$$

$$a\sin\theta + b\cos\theta = -\frac{K}{r^2}\sin\theta$$

But these two formulae can be written in the following way:

$$\left(a + \frac{K}{r^2}\right)\cos\theta = b\sin\theta$$

$$\left(a + \frac{K}{r^2}\right)\sin\theta = -b\cos\theta$$

By making now the product, and assuming that we are in a non-degenerate case, where the angle θ varies indeed, we obtain by positivity that we must have:

$$a + \frac{K}{r^2} = b = 0$$

(6) We are almost there. Let us first examine the second equation, $b = 0$. Remembering who b is, from (4), this equation can be solved as follows:

$$\begin{aligned} b = 0 \quad &\Longleftrightarrow \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \\ &\Longleftrightarrow \quad \frac{\ddot{\theta}}{\dot{\theta}} = -2\frac{\dot{r}}{r} \\ &\Longleftrightarrow \quad (\log\dot{\theta})' = (-2\log r)' \\ &\Longleftrightarrow \quad \log\dot{\theta} = -2\log r + c \\ &\Longleftrightarrow \quad \dot{\theta} = \frac{\lambda}{r^2} \end{aligned}$$

As for the first equation the we found, namely $a + K/r^2 = 0$, remembering from (4) that a was by definition given by $a = \ddot{r} - r\dot{\theta}^2$, this equation now becomes:

$$\ddot{r} - \frac{\lambda^2}{r^3} + \frac{K}{r^2} = 0$$

(7) As a conclusion to all this, in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, our equations of motion are as follows, with λ being a constant, not depending on t:

$$\ddot{r} = \frac{\lambda^2}{r^3} - \frac{K}{r^2} \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

Even better now, by writing $K = \lambda^2/c$, these equations read:

$$\ddot{r} = \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right) \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

(8) As an illustration, let us quickly work out the case of a circular motion, where r is constant. Here $\ddot{r} = 0$, so the first equation gives $c = r$. Also we have $\dot{\theta} = \alpha$, with:

$$\alpha = \frac{\lambda}{r^2}$$

Assuming $\theta = 0$ at $t = 0$, from $\dot{\theta} = \alpha$ we obtain $\theta = \alpha t$, and so, as in (3) above:

$$x = r\cos\alpha t \quad , \quad y = r\sin\alpha t$$

Observe also that the condition found in (3) is indeed satisfied:

$$r^3\alpha^2 = \frac{\lambda^2}{r} = \frac{\lambda^2}{c} = K$$

(9) Back to the general case now, our claim is that we have the following formula, for the distance $r = r(t)$ as function of the angle $\theta = \theta(t)$, for some $\varepsilon, \delta \in \mathbb{R}$:

$$r = \frac{c}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

Let us first check that this formula works indeed. With r being as above, and by using our second equation found before, $\dot{\theta} = \lambda/r^2$, we have the following computation:

$$\begin{aligned}\dot{r} &= \frac{c(\varepsilon\sin\theta - \delta\cos\theta)\dot{\theta}}{(1 + \varepsilon\cos\theta + \delta\sin\theta)^2}\\ &= \frac{\lambda c(\varepsilon\sin\theta - \delta\cos\theta)}{r^2(1 + \varepsilon\cos\theta + \delta\sin\theta)^2}\\ &= \frac{\lambda(\varepsilon\sin\theta - \delta\cos\theta)}{c}\end{aligned}$$

Thus, the second derivative of the above function r is given, as desired, by:

$$\begin{aligned}\ddot{r} &= \frac{\lambda(\varepsilon\cos\theta + \delta\sin\theta)\dot{\theta}}{c}\\ &= \frac{\lambda^2(\varepsilon\cos\theta + \delta\sin\theta)}{r^2 c}\\ &= \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right)\end{aligned}$$

(10) The above check was something quite informal, and now we must prove that our formula is indeed the correct one. For this purpose, we use a trick. Let us write:

$$r(t) = \frac{1}{f(\theta(t))}$$

Abbreviated, and by always reminding that f takes $\theta=\theta(t)$ as variable, this reads:

$$r=\frac{1}{f}$$

With the convention that dots mean as usual derivatives with respect to t, and that the primes will denote derivatives with respect to $\theta=\theta(t)$, we have:

$$\dot{r}=-\frac{f'\dot{\theta}}{f^2}=-\frac{f'}{f^2}\cdot\frac{\lambda}{r^2}=-\lambda f'$$

By differentiating one more time with respect to t, we obtain:

$$\ddot{r}=-\lambda f''\dot{\theta}=-\lambda f''\cdot\frac{\lambda}{r^2}=-\frac{\lambda^2}{r^2}f''$$

On the other hand, our equation for $\ddot{r}$ found in (7) reads:

$$\ddot{r}=\frac{\lambda^2}{r^2}\left(\frac{1}{r}-\frac{1}{c}\right)=\frac{\lambda^2}{r^2}\left(f-\frac{1}{c}\right)$$

Thus, in terms of $f=1/r$ as above, our equation for $\ddot{r}$ simply reads:

$$f''+f=\frac{1}{c}$$

But this latter equation is elementary to solve. Indeed, both functions $\cos t,\sin t$ satisfy g" $+g=0$, so any linear combination of them satisfies as well this equation. But the solutions of $f''+f=1/c$ being those of $g''+g=0$ shifted by $1/c$, we obtain:

$$f=\frac{1+\varepsilon\cos\theta+\delta\sin\theta}{c}$$

Now by inverting, we obtain the formula announced in (9), namely:

$$r=\frac{c}{1+\varepsilon\cos\theta+\delta\sin\theta}$$

(11) But this leads to the conclusion that the trajectory is a conic. Indeed, in terms of the parameter θ, the formulae of the coordinates are:

$$x=\frac{c\cos\theta}{1+\varepsilon\cos\theta+\delta\sin\theta}$$

$$y=\frac{c\sin\theta}{1+\varepsilon\cos\theta+\delta\sin\theta}$$

But these are precisely the equations of conics in polar coordinates.

(12) To be more precise, in order to find the precise equation of the conic, observe that the two functions x,y that we found above satisfy the following formula:

$$\begin{aligned} x^2+y^2 &= \frac{c^2(\cos^2\theta+\sin^2\theta)}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \\ &= \frac{c^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \end{aligned}$$

On the other hand, these two functions satisfy as well the following formula:

$$\begin{aligned} (\varepsilon x+\delta y-c)^2 &= \frac{c^2\big(\varepsilon\cos\theta+\delta\sin\theta-(1+\varepsilon\cos\theta+\delta\sin\theta)\big)^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \\ &= \frac{c^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \end{aligned}$$

We conclude that our coordinates x,y satisfy the following equation:

$$x^2+y^2=(\varepsilon x+\delta y-c)^2$$

But what we have here is an equation of a conic, as claimed. □

2d. Higher dimensions

Speaking now algebra, it is quite remarkable that both $\mathbb{R}$ and $\mathbb{R}^2$ have field structures. This fails for $\mathbb{R}^3$, but then for $\mathbb{R}^4$ something can be done, as follows:

Theorem 2.21. *In contrast with $\mathbb{R}$, and with $\mathbb{R}^2=\mathbb{C}$, which are fields:*

(1) *The vector space $\mathbb{R}^3$ does not have a multiplication, making it a field.*
(2) *For $\mathbb{R}^4$ however, something can be done, of rather physics flavor.*

Proof. This is obviously something informal, the idea being as follows:

(1) Let us first examine the field structures on $\mathbb{R}^N$, with $N\in\mathbb{N}$ arbitrary. A first idea, which is very natural, is that any multiplication on $\mathbb{R}^N$ must come by linearity from a multiplication on the unit sphere $S^{N-1}_\mathbb{R}\subset\mathbb{R}^N$. That is, once we know how to multiply the norm one vectors $x,y\in S^{N-1}_\mathbb{R}$, we can set, by linearity:

$$(\lambda x)*(\mu y)=(\lambda\mu)(x*y)$$

At the level of examples, this is certainly what happens at $N=1,2$, where the corresponding unit spheres are as follows, and with the multiplication on $\mathbb{R}^N$ itself appearing as above, from the obvious multiplication on these unit spheres, by linearity:

$$S^0_\mathbb{R}=\{-1,1\}\quad,\quad S^1_\mathbb{R}=\mathbb{T}$$

(2) In practice now, such ideas require first proving that $||x||=||y||=1$ implies $||x*y||=1$, with $||x||=\sqrt{\sum x_i^2}$ being the usual norm, and while not exactly obvious, this can be done indeed. As another remark, getting back now to $N=1,2$, while the

possible multiplication on $S^0_\mathbb{R}=\{-1,1\}$ is unique, $(-1)^2=1$, in what regards the possible multiplications on $S^1_\mathbb{R}=\mathbb{T}$ things are more complicated, of topology flavor. So, as conclusion, it is pretty much clear that all this leads us into geometry, and topology.

(3) Moving now to $\mathbb{R}^3$, you would say that the vector product $x\times y$ does the job, but this is wrong, because $x\sim y$ implies $x\times y=0$, so definitely wrong way. However, thinking well, a multiplication on $\mathbb{R}^3$ would induce a multiplication on the unit sphere $S^2_\mathbb{R}\subset\mathbb{R}^3$, as explained above, and the point is that there is a topological obstruction to this. However, this obstruction is a bit difficult to explain, involving some advanced mathematics, and our result at $N=3$ being negative anyway, we will not further insist on this.

(4) Getting now to $\mathbb{R}^4$, as a good surprise here, the unit sphere $S^3_\mathbb{R}\subset\mathbb{R}^4$ is naturally a group, $S^3_\mathbb{R}=SU_2$. Indeed, solving $U^*=U^{-1}$ under the assumption $\det U=1$ gives:

$$SU_2=\left\{\begin{pmatrix}a&b\\-\bar{b}&\bar{a}\end{pmatrix}\middle||a|^2+|b|^2=1\right\}$$

Here we use complex numbers, and in real number notation, the result is:

$$SU_2=\left\{\begin{pmatrix}x+iy&z+it\\-z+it&x-iy\end{pmatrix}\middle|x^2+y^2+z^2+t^2=1\right\}$$

(5) But this is obviously good news, we more or less solved our problem, and it remains to work out the details. So, consider the following matrices:

$$c_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad,\quad c_2=\begin{pmatrix}i&0\\0&-i\end{pmatrix}\quad,\quad c_3=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad,\quad c_4=\begin{pmatrix}0&i\\i&0\end{pmatrix}$$

In terms of these matrices, which by the way are called Pauli spin matrices, discovered by Pauli in relation with quantum mechanics, but let us not get into this here, maybe later, towards the end of the present book, our result above reads:

$$SU_2=\left\{c_1x+c_2y+c_3z+c_4t\middle|x^2+y^2+z^2+t^2=1\right\}$$

In order to figure out how the resulting multiplication on $\mathbb{R}^4$ looks like, we must first multiply the Pauli matrices. Their products are given by the following formulae:

$$c_2^2=c_3^2=c_4^2=-1$$

$$c_2c_3=-c_3c_2=c_4$$

$$c_3c_4=-c_4c_3=c_2$$

$$c_4c_2=-c_2c_4=c_3$$

Thus, we are led in this way to a multiplication on $\mathbb{R}^4$, as stated.

(6) Alternatively, we have the following real matrices, multiplying quite similarly:

$$1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\quad,\quad i=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$j=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\quad,\quad k=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

Thus, one way or another, we are led to the conclusion in the statement.

(7) Finally, there is as well a purely algebraic approach to this, using formal numbers $1,i,j,k$, called quaternions, with $1,i$ being the $1,i$ that we know, and with j,k being constructed similarly, a bit like i was, formally via $i^2=-1$, when introducing $\mathbb{C}$. To be more precise, the multiplication rules for i,j,k, found by Hamilton, are as follows:

$$i^2=j^2=k^2=ijk=-1$$

Observe that these are precisely the multiplication rules for the Pauli matrices, from (5) above. Thus, we are again led to the conclusion in the statement. □

2e. Exercises

Exciting chapter that we had here, and as exercises about this, we have:

Exercise 2.22. *Work out the details in the proof of Desargues and Pappus.*

Exercise 2.23. *Work out the details in the proof of Menelaus and Ceva.*

Exercise 2.24. *Do the same for the incenter, circumcenter and orthocenter.*

Exercise 2.25. *Work out the details for the Euler line, and nine-point circle.*

Exercise 2.26. *Work out the details in the proof of Pascal and Brianchon.*

Exercise 2.27. *Learn more about conics, and various formulae for them.*

Exercise 2.28. *Solve the two-body problem in* 1 *dimension, for a free fall.*

Exercise 2.29. *Clarify what we said at the end, regarding quaternions and* $\mathbb{R}^4$.

As bonus exercise, find and start reading an old-style algebraic geometry book.

CHAPTER 3

Linear algebra

3a. Linear algebra

We know from chapter 1 that any polynomial $P \in \mathbb{R}[X]$, or more generally $P \in \mathbb{C}[X]$, has $\deg P$ roots, when counted with multiplicities. We discuss here the applications of this phenomenon to questions from linear algebra, and multivariable calculus.

You surely know well linear algebra, but always good to recall this. We first have:

THEOREM 3.1. *The linear maps $T : \mathbb{C}^N \to \mathbb{C}^N$ are in correspondence with the square matrices $A \in M_N(\mathbb{C})$, with the linear map associated to such a matrix being*

$$Tx = Ax$$

and with the matrix associated to a linear map being $A_{ij} =< Te_j, e_i >$.

PROOF. The first assertion is clear, because a linear map $T : \mathbb{C}^N \to \mathbb{C}^N$ must send a vector $x \in \mathbb{C}^N$ to a certain vector $Tx \in \mathbb{C}^N$, all whose components are linear combinations of the components of x. Thus, we can write, for certain complex numbers $A_{ij} \in \mathbb{C}$:

$$T\begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + \ldots + A_{1N}x_N \\ \vdots \\ \vdots \\ A_{N1}x_1 + \ldots + A_{NN}x_N \end{pmatrix}$$

Now the parameters $A_{ij} \in \mathbb{C}$ can be regarded as being the entries of a square matrix $A \in M_N(\mathbb{C})$, and with the usual convention for matrix multiplication, we have:

$$Tx = Ax$$

Regarding the second assertion, with $Tx = Ax$ as above, if we denote by $e_1, \ldots, e_N$ the standard basis of $\mathbb{C}^N$, then we have the following formula:

$$Te_j = \begin{pmatrix} A_{1j} \\ \vdots \\ \vdots \\ A_{Nj} \end{pmatrix}$$

But this gives the second formula, $< Te_j, e_i >= A_{ij}$, as desired. □

Our claim now is that, no matter what we want to do with T or A, of advanced type, we will run at some point into their adjoints T^* and A^*, constructed as follows:

THEOREM 3.2. *The adjoint operator $T^* : \mathbb{C}^N \to \mathbb{C}^N$, which is given by*

$$< Tx, y >=< x, T^*y >$$

corresponds to the adjoint matrix $A^* \in M_N(\mathbb{C})$, given by

$$(A^*)_{ij} = \bar{A}_{ji}$$

via the correspondence between linear maps and matrices constructed above.

PROOF. Given a linear map $T : \mathbb{C}^N \to \mathbb{C}^N$, fix $y \in \mathbb{C}^N$, and consider the linear form $\varphi(x) =< Tx, y >$. This form must be as follows, for a certain vector $T^*y \in \mathbb{C}^N$:

$$\varphi(x) =< x, T^*y >$$

Thus, we have constructed a map $y \to T^*y$ as in the statement, which is obviously linear, and that we can call T^*. Now by taking the vectors $x, y \in \mathbb{C}^N$ to be elements of the standard basis of $\mathbb{C}^N$, our defining formula for T^* reads:

$$< Te_i, e_j >=< e_i, T^*e_j >$$

By reversing the scalar product on the right, this formula can be written as:

$$< T^*e_j, e_i >= \overline{< Te_i, e_j >}$$

But this means that the matrix of T^* is given by $(A^*)_{ij} = \bar{A}_{ji}$, as desired. □

Getting back to our claim, the adjoints $*$ are indeed ubiquitous, as shown by:

THEOREM 3.3. *The following happen:*

(1) *$T(x) = Ux$ with $U \in M_N(\mathbb{C})$ is an isometry precisely when $U^* = U^{-1}$.*
(2) *$T(x) = Px$ with $P \in M_N(\mathbb{C})$ is a projection precisely when $P^2 = P^* = P$.*

PROOF. Let us first recall that the lengths, or norms, of the vectors $x \in \mathbb{C}^N$ can be recovered from the knowledge of the scalar products, as follows:

$$||x|| = \sqrt{< x, x >}$$

Conversely, we can recover the scalar products out of norms, by using the following difficult to remember formula, called complex polarization identity:

$$4 < x, y >= ||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2$$

Finally, we will use Theorem 3.2, and more specifically the following formula coming from there, valid for any matrix $A \in M_N(\mathbb{C})$ and any two vectors $x, y \in \mathbb{C}^N$:

$$< Ax, y >=< x, A^*y >$$

(1) Given a matrix $U \in M_N(\mathbb{C})$, we have indeed the following equivalences, with the first one coming from the polarization identity, and the other ones being clear:

$$\begin{aligned} ||Ux|| = ||x|| &\iff < Ux, Uy > = < x, y > \\ &\iff < x, U^*Uy > = < x, y > \\ &\iff U^*Uy = y \\ &\iff U^*U = 1 \\ &\iff U^* = U^{-1} \end{aligned}$$

(2) Given a matrix $P \in M_N(\mathbb{C})$, in order for $x \to Px$ to be an oblique projection, we must have $P^2 = P$. Now observe that this projection is orthogonal when:

$$\begin{aligned} < Px - x, Py > = 0 &\iff < P^*Px - P^*x, y > = 0 \\ &\iff P^*Px - P^*x = 0 \\ &\iff P^*P - P^* = 0 \\ &\iff P^*P = P^* \end{aligned}$$

The point now is that by conjugating the last formula, we obtain $P^*P = P$. Thus we must have $P = P^*$, and this gives the result. □

Summarizing, the linear operators come in pairs T, T^*, and the associated matrices come as well in pairs A, A^*. This is something quite interesting, philosophically speaking, and we will keep this in mind, and come back to it later, on numerous occasions.

3b. Diagonalization

Let us discuss now the diagonalization question for the linear maps and matrices. The basic diagonalization theory, formulated in terms of matrices, is as follows:

PROPOSITION 3.4. *A vector $v \in \mathbb{C}^N$ is called eigenvector of $A \in M_N(\mathbb{C})$, with corresponding eigenvalue λ, when A multiplies by λ in the direction of v:*

$$Av = \lambda v$$

In the case where $\mathbb{C}^N$ has a basis $v_1, \ldots, v_N$ formed by eigenvectors of A, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_N$, in this new basis A becomes diagonal, as follows:

$$A \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

Equivalently, if we denote by $D = diag(\lambda_1, \ldots, \lambda_N)$ the above diagonal matrix, and by $P = [v_1 \ldots v_N]$ the square matrix formed by the eigenvectors of A, we have:

$$A = PDP^{-1}$$

In this case we say that the matrix A is diagonalizable.

PROOF. This is something which is clear, the idea being as follows:

(1) The first assertion is clear, because the matrix which multiplies each basis element v_i by a number λ_i is precisely the diagonal matrix $D = diag(\lambda_1, \ldots, \lambda_N)$.

(2) The second assertion follows from the first one, by changing the basis. We can prove this by a direct computation as well, because we have $Pe_i = v_i$, and so:

$$\begin{aligned} PDP^{-1}v_i &= PDe_i \\ &= P\lambda_i e_i \\ &= \lambda_i Pe_i \\ &= \lambda_i v_i \end{aligned}$$

Thus, the matrices A and PDP^{-1} coincide, as stated. □

Let us recall as well that the basic example of a non diagonalizable matrix, over the complex numbers as above, is the following matrix:

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Indeed, we have $J\binom{x}{y} = \binom{y}{0}$, so the eigenvectors are the vectors of type $\binom{x}{0}$, all with eigenvalue 0. Thus, we have not enough eigenvectors for constructing a basis of $\mathbb{C}^2$.

In general, in order to study the diagonalization problem, the idea is that the eigenvectors can be grouped into linear spaces, called eigenspaces, as follows:

THEOREM 3.5. *Let $A \in M_N(\mathbb{C})$, and for any eigenvalue $\lambda \in \mathbb{C}$ define the corresponding eigenspace as being the vector space formed by the corresponding eigenvectors:*

$$E_\lambda = \left\{ v \in \mathbb{C}^N \Big| Av = \lambda v \right\}$$

These eigenspaces E_λ are then in a direct sum position, in the sense that given vectors $v_1 \in E_{\lambda_1}, \ldots, v_k \in E_{\lambda_k}$ corresponding to different eigenvalues $\lambda_1, \ldots, \lambda_k$, we have:

$$\sum_i c_i v_i = 0 \implies c_i = 0$$

In particular we have the following estimate, with sum over all the eigenvalues,

$$\sum_\lambda \dim(E_\lambda) \leq N$$

and our matrix is diagonalizable precisely when we have equality.

PROOF. We prove the first assertion by recurrence on $k \in \mathbb{N}$. Assume by contradiction that we have a formula as follows, with the scalars $c_1, \ldots, c_k$ being not all zero:

$$c_1 v_1 + \ldots + c_k v_k = 0$$

By dividing by one of these scalars, we can assume that our formula is:

$$v_k = c_1v_1 + \ldots + c_{k-1}v_{k-1}$$

Now let us apply A to this vector. On the left we obtain:

$$Av_k = \lambda_k v_k = \lambda_k c_1 v_1 + \ldots + \lambda_k c_{k-1} v_{k-1}$$

On the right we obtain something different, as follows:

$$\begin{aligned} A(c_1v_1 + \ldots + c_{k-1}v_{k-1}) &= c_1Av_1 + \ldots + c_{k-1}Av_{k-1} \\ &= c_1\lambda_1 v_1 + \ldots + c_{k-1}\lambda_{k-1}v_{k-1} \end{aligned}$$

We conclude from this that the following equality must hold:

$$\lambda_k c_1 v_1 + \ldots + \lambda_k c_{k-1} v_{k-1} = c_1\lambda_1 v_1 + \ldots + c_{k-1}\lambda_{k-1} v_{k-1}$$

On the other hand, we know by recurrence that the vectors $v_1, \ldots, v_{k-1}$ must be linearly independent. Thus, the coefficients must be equal, at right and at left:

$$\lambda_k c_1 = c_1\lambda_1$$

$$\vdots$$

$$\lambda_k c_{k-1} = c_{k-1}\lambda_{k-1}$$

Now since at least one of the numbers c_i must be nonzero, from $\lambda_k c_i = c_i\lambda_i$ we obtain $\lambda_k = \lambda_i$, which is a contradiction. Thus our proof by recurrence of the first assertion is complete. As for the second assertion, this follows from the first one. □

In order to reach now to more advanced results, we can use the characteristic polynomial, which appears via the following fundamental result:

THEOREM 3.6. *Given a matrix $A \in M_N(\mathbb{C})$, consider its characteristic polynomial:*

$$P(x) = \det(A - x1_N)$$

The eigenvalues of A are then the roots of P. Also, we have the inequality

$$\dim(E_\lambda) \leq m_\lambda$$

where m_λ is the multiplicity of λ, as root of P.

PROOF. The first assertion follows from the following computation, using the fact that a linear map is bijective when the determinant of the associated matrix is nonzero:

$$\begin{aligned} \exists v, Av = \lambda v &\iff \exists v, (A - \lambda 1_N)v = 0 \\ &\iff \det(A - \lambda 1_N) = 0 \end{aligned}$$

Regarding now the second assertion, given an eigenvalue λ of our matrix A, consider the dimension $d_\lambda = \dim(E_\lambda)$ of the corresponding eigenspace. By changing the basis of

$\mathbb{C}^N$, as for the eigenspace E_λ to be spanned by the first d_λ basis elements, our matrix becomes as follows, with B being a certain smaller matrix:

$$A \sim \begin{pmatrix}\lambda 1_{d_\lambda} & 0\\ 0 & B\end{pmatrix}$$

We conclude that the characteristic polynomial of A is of the following form:

$$P_A = P_{\lambda 1_{d_\lambda}} P_B = (\lambda - x)^{d_\lambda} P_B$$

Thus the multiplicity m_λ of our eigenvalue λ, as a root of P, satisfies $m_\lambda \geq d_\lambda$, and this leads to the conclusion in the statement. □

Now recall that we are over $\mathbb{C}$. By using this, we obtain the following result:

THEOREM 3.7. *Given a matrix $A \in M_N(\mathbb{C})$, consider its characteristic polynomial*

$$P(X) = \det(A - X1_N)$$

then factorize this polynomial, by computing the complex roots, with multiplicities,

$$P(X) = (-1)^N (X - \lambda_1)^{n_1} \ldots (X - \lambda_k)^{n_k}$$

and finally compute the corresponding eigenspaces, for each eigenvalue found:

$$E_i = \left\{ v \in \mathbb{C}^N \middle| Av = \lambda_i v \right\}$$

The dimensions of these eigenspaces satisfy then the following inequalities,

$$\dim(E_i) \leq n_i$$

and A is diagonalizable precisely when we have equality for any i.

PROOF. This follows by combining Theorems 3.5 and 3.6. Indeed, by summing the inequalities $\dim(E_\lambda) \leq m_\lambda$ from Theorem 3.6, we obtain an inequality as follows:

$$\sum_\lambda \dim(E_\lambda) \leq \sum_\lambda m_\lambda \leq N$$

On the other hand, we know from Theorem 3.5 that our matrix is diagonalizable when we have global equality. Thus, we are led to the conclusion in the statement. □

In practice, diagonalizing a matrix remains something quite complicated. Let us record a useful algorithmic version of the above result, as follows:

THEOREM 3.8. *The square matrices $A \in M_N(\mathbb{C})$ can be diagonalized as follows:*

(1) *Compute the characteristic polynomial.*
(2) *Factorize the characteristic polynomial.*
(3) *Compute the eigenvectors, for each eigenvalue found.*
(4) *If there are no N eigenvectors, A is not diagonalizable.*
(5) *Otherwise, A is diagonalizable, $A = PDP^{-1}$.*

PROOF. This is an informal reformulation of Theorem 3.7, with (4) referring to the total number of linearly independent eigenvectors found in (3), and with $A = PDP^{-1}$ in (5) being the usual diagonalization formula, with P, D being as before. □

As an illustration for all this, which is a must-know computation, we have:

THEOREM 3.9. *The rotation of angle $t \in \mathbb{R}$ in the plane diagonalizes as:*

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}e^{-it} & 0\\ 0 & e^{it}\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}$$

Over the reals this is impossible, unless $t = 0, \pi$, where the rotation is diagonal.

PROOF. Observe first that, as indicated, unlike we are in the case $t = 0, \pi$, where our rotation is $\pm 1_2$, our rotation is a "true" rotation, having no eigenvectors in the plane. Fortunately the complex numbers come to the rescue, via the following computation:

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}\cos t - i\sin t\\ i\cos t + \sin t\end{pmatrix} = e^{-it}\begin{pmatrix}1\\ i\end{pmatrix}$$

We have as well a second complex eigenvector, coming from:

$$\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix} = \begin{pmatrix}\cos t + i\sin t\\ -i\cos t + \sin t\end{pmatrix} = e^{it}\begin{pmatrix}1\\ -i\end{pmatrix}$$

Thus, we are led to the conclusion in the statement. □

As another basic illustration, we have the following result:

THEOREM 3.10. *The all-one matrix diagonalizes as follows,*

$$\begin{pmatrix}1 & \ldots & 1\\ \vdots & & \vdots\\ 1 & \ldots & 1\end{pmatrix} = \frac{1}{N}F_N\begin{pmatrix}N &&&\\ & 0 &&\\ && \ddots &\\ &&& 0\end{pmatrix}F_N^*$$

with $F_N = (w^{ij})_{ij}$ with $w = e^{2\pi i/N}$ being the Fourier matrix.

PROOF. The all-one matrix being N times the projection on the all-one vector, the diagonal form is the one in the statement. In order to find now the explicit diagonalization formula, with passage matrix and its inverse, we must solve the following equation:

$$x_1 + \ldots + x_N = 0$$

And this is not an easy task, if we want a nice basis for the space of solutions. Fortunately, the complex numbers come to the rescue, via the following formula:

$$\sum_{k=0}^{N-1} w^{ks} = N\delta_{N|s}$$

Thus, we are led to the conclusion in the statement. □

Getting back now to general theory, we have the following advanced result:

THEOREM 3.11. *The following happen, regarding the matrices $A \in M_N(\mathbb{C})$:*

(1) *The self-adjoint matrices, $A = A^*$, are diagonalizable.*
(2) *The unitary matrices, $A^* = A^{-1}$, are diagonalizable too.*
(3) *In fact, the normal matrices, $AA^* = A^*A$, are diagonalizable.*

Moreover, the commuting families of normal matrices are jointly diagonalizable.

PROOF. This is something more advanced, the idea being as follows:

(1) This generalizes the fact, that you probably know from basic linear algebra, that any symmetric matrix $A \in M_N(\mathbb{R})$ is diagonalizable. And with the proof being similar.

(2) This is certainly something complex number specific, for instance in view of Theorem 3.9. However, the proof is quite routine, coming as a variation of (1).

(3) This is a joint generalization of (1) and (2), the matrices there being normal, and for the proof here, we refer to any advanced linear algebra book.

(4) As for the last assertion, again we refer here to any advanced linear algebra book. Alternatively, you can consult any introductory operator theory book. □

As a last general linear algebra result, that you should know too, we have:

THEOREM 3.12. *Given a matrix $A \in M_N(\mathbb{C})$, the following happen:*

(1) *A^*A being positive, we can extract its square root $|A| = \sqrt{A^*A}$.*
(2) *When A is invertible, we have $A = U|A|$, with U being a unitary.*
(3) *In general, we still have $A = U|A|$, with U being a partial isometry.*

PROOF. Again, this is something more advanced, the idea being as follows:

(1) The matrix A^*A being self-adjoint, and with positive eigenvalues, with this coming from $< A^*Ax, x >= ||Ax||^2$, our claim follows from Theorem 3.11 (1), by diagonalizing this matrix, and then taking the square roots of all eigenvalues.

(2) According to our definition of the modulus, namely $|A| = \sqrt{A^*A}$, we have:

$$\begin{aligned} < |A|x, |A|y > &= < x, |A|^2 y > \\ &= < x, A^*Ay > \\ &= < Ax, Ay > \end{aligned}$$

We conclude that the following linear application is well-defined, and isometric:

$$U : Im|A| \to Im(A) \quad , \quad |A|x \to Ax$$

But with A assumed to be invertible, U is a unitary, and $A = U|A|$, as desired.

(3) Getting back to the linear isometric map U constructed in (2), we can extend this map into a partial isometry $U:\mathbb{C}^N\to\mathbb{C}^N$, in a straightforward way, by setting:

$$Ux=0\quad,\quad\forall x\in Im|A|^\perp$$

And the point is that, with this convention, we have again $A=U|A|$, as desired. □

3c. Density tricks

Back to Theorem 3.7 and generalities, at the level of basic examples of diagonalizable matrices, we have the following result, providing us with the "generic" examples:

THEOREM 3.13. *For a matrix $A\in M_N(\mathbb{C})$ the following conditions are equivalent,*

(1) *The eigenvalues are different, $\lambda_i\neq\lambda_j$,*
(2) *The characteristic polynomial P has simple roots,*
(3) *The characteristic polynomial satisfies $(P,P')=1$,*
(4) *The resultant of P,P' is nonzero, $R(P,P')\neq 0$,*
(5) *The discriminant of P is nonzero, $\Delta(P)\neq 0$,*

and in this case, the matrix is diagonalizable.

PROOF. The last assertion holds indeed, due to Theorem 3.7. As for the equivalences in the statement, these are all standard, the idea for their proofs, along with some more theory, needed for using in practice the present result, being as follows:

(1) $\iff$ (2) This follows from Theorem 3.7.

(2) $\iff$ (3) This is standard, the double roots of P being roots of P'.

(3) $\iff$ (4) The idea here is that associated to any two polynomials P,Q is their resultant $R(P,Q)$, which checks whether P,Q have a common root. Let us write:

$$P=c(X-a_1)\ldots(X-a_k)$$

$$Q=d(X-b_1)\ldots(X-b_l)$$

We can define then the resultant as being the following quantity:

$$R(P,Q)=c^ld^k\prod_{ij}(a_i-b_j)$$

The point now, that we will explain as well, is that this is a polynomial in the coefficients of P,Q, with integer coefficients. Indeed, this can be checked as follows:

– We can expand the formula of $R(P,Q)$, and in what regards $a_1,\ldots,a_k$, which are the roots of P, we obtain in this way certain symmetric functions in these variables, which will be therefore polynomials in the coefficients of P, with integer coefficients.

– We can then look what happens with respect to the remaining variables $b_1,\ldots,b_l$, which are the roots of Q. Once again what we have here are certain symmetric functions, and so polynomials in the coefficients of Q, with integer coefficients.

– Thus, we are led to the above conclusion, that $R(P,Q)$ is a polynomial in the coefficients of P,Q, with integer coefficients, and with the remark that the c^ld^k factor is there for these latter coefficients to be indeed integers, instead of rationals.

Alternatively, let us write our two polynomials in usual form, as follows:

$$P = p_kX^k + \ldots + p_1X + p_0$$

$$Q = q_lX^l + \ldots + q_1X + q_0$$

The corresponding resultant appears then as the determinant of an associated matrix, having size $k+l$, and having 0 coefficients at the blank spaces, as follows:

$$R(P,Q) = \begin{vmatrix} p_k & & & q_l & & \\ \vdots & \ddots & & \vdots & \ddots & \\ p_0 & & p_k & q_0 & & q_l \\ & \ddots & \vdots & & \ddots & \vdots \\ & & p_0 & & & q_0 \end{vmatrix}$$

(4) $\iff$ (5) Once again this is something standard, the idea here being that the discriminant $\Delta(P)$ of a polynomial $P \in \mathbb{C}[X]$ is, modulo scalars, the resultant $R(P,P')$. To be more precise, let us write our polynomial as follows:

$$P(X) = cX^N + dX^{N-1} + \ldots$$

Its discriminant is then defined as being the following quantity:

$$\Delta(P) = \frac{(-1)^{\binom{N}{2}}}{c}R(P,P')$$

This is a polynomial in the coefficients of P, with integer coefficients, with the division by c being indeed possible, under $\mathbb{Z}$, and with the sign being there for various reasons, including the compatibility with some well-known formulae, at small values of N. □

All the above might seem a bit complicated, so as an illustration, let us work out an example. Consider the case of a polynomial of degree 2, and a polynomial of degree 1:

$$P = ax^2 + bx + c \quad , \quad Q = dx + e$$

In order to compute the resultant, let us factorize our polynomials:

$$P = a(x-p)(x-q) \quad , \quad Q = d(x-r)$$

The resultant can be then computed as follows, by using the two-step method:

$$\begin{aligned} R(P,Q) &= ad^2(p-r)(q-r) \\ &= ad^2(pq-(p+q)r+r^2) \\ &= cd^2 + bd^2r + ad^2r^2 \\ &= cd^2 - bde + ae^2 \end{aligned}$$

Observe that $R(P,Q)=0$ corresponds indeed to the fact that P,Q have a common root. Indeed, the root of Q is $r=-e/d$, and we have:

$$P(r)=\frac{ae^2}{d^2}-\frac{be}{d}+c=\frac{R(P,Q)}{d^2}$$

We can recover as well the resultant as a determinant, as follows:

$$R(P,Q)=\begin{vmatrix}a&d&0\\b&e&d\\c&0&e\end{vmatrix}=ae^2-bde+cd^2$$

Finally, in what regards the discriminant, let us see what happens in degree 2. Here we must compute the resultant of the following two polynomials:

$$P=aX^2+bX+c\quad,\quad P'=2aX+b$$

The resultant is then given by the following formula:

$$\begin{aligned}R(P,P')&=ab^2-b(2a)b+c(2a)^2\\&=4a^2c-ab^2\\&=-a(b^2-4ac)\end{aligned}$$

Now by doing the discriminant normalizations, we obtain, as we should:

$$\Delta(P)=b^2-4ac$$

As already mentioned, one can prove that the matrices having distinct eigenvalues are "generic", so that Theorem 3.13 basically captures the whole situation. We have in fact the following collection of density results, which are quite advanced:

THEOREM 3.14. *The following happen, inside $M_N(\mathbb{C})$:*

(1) *The invertible matrices are dense.*
(2) *The matrices having distinct eigenvalues are dense.*
(3) *The diagonalizable matrices are dense.*

PROOF. These are quite advanced results, which can be proved as follows:

(1) This is clear, intuitively speaking, because the invertible matrices are given by the condition $\det A\neq 0$. Thus, the set formed by these matrices appears as the complement of the hypersurface $\det A=0$, and so must be dense inside $M_N(\mathbb{C})$, as claimed.

(2) Here we can use a similar argument, this time by saying that the set formed by the matrices having distinct eigenvalues appears as the complement of the hypersurface given by $\Delta(P_A)=0$, and so must be dense inside $M_N(\mathbb{C})$, as claimed.

(3) This follows from (2), via the fact that the matrices having distinct eigenvalues are diagonalizable, that we know from Theorem 3.13. There are of course some other proofs as well, for instance by putting the matrix in Jordan form. $\square$

As an application of the above results, and of our methods in general, we have:

THEOREM 3.15. *The following happen:*

(1) *We have $P_{AB}=P_{BA}$, for any two matrices $A,B\in M_N(\mathbb{C})$.*
(2) *AB,BA have the same eigenvalues, with the same multiplicities.*
(3) *If A has eigenvalues $\lambda_1,\ldots,\lambda_N$, then $f(A)$ has eigenvalues $f(\lambda_1),\ldots,f(\lambda_N)$.*

PROOF. These results can be deduced by using Theorem 3.14, as follows:

(1) It follows from definitions that the characteristic polynomial of a matrix is invariant under conjugation, in the sense that we have the following formula:

$$P_C=P_{ACA^{-1}}$$

Now observe that, when assuming that A is invertible, we have:

$$AB=A(BA)A^{-1}$$

Thus, we have the result when A is invertible. By using now Theorem 3.14 (1), we conclude that this formula holds for any matrix A, by continuity.

(2) This is a reformulation of (1), via the fact that P encodes the eigenvalues, with multiplicities, which is hard to prove with bare hands.

(3) This is something quite informal, clear for the diagonal matrices D, then for the diagonalizable matrices PDP^{-1}, and finally for all matrices, by using Theorem 3.14 (3), provided that f has suitable regularity properties. We will be back to this. □

Let us go back now to the main problem raised by the diagonalization procedure, namely the computation of the roots of characteristic polynomials. We have here:

THEOREM 3.16. *The complex eigenvalues of a matrix $A\in M_N(\mathbb{C})$, counted with multiplicities, have the following properties:*

(1) *Their sum is the trace.*
(2) *Their product is the determinant.*

PROOF. Consider indeed the characteristic polynomial P of the matrix:

$$\begin{aligned}P(X)&=\det(A-X1_N)\\&=(-1)^NX^N+(-1)^{N-1}Tr(A)X^{N-1}+\ldots+\det(A)\end{aligned}$$

We can factorize this polynomial, by using its N complex roots, and we obtain:

$$\begin{aligned}P(X)&=(-1)^N(X-\lambda_1)\ldots(X-\lambda_N)\\&=(-1)^NX^N+(-1)^{N-1}\left(\sum_i\lambda_i\right)X^{N-1}+\ldots+\prod_i\lambda_i\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the intermediate terms, we have here:

THEOREM 3.17. *Assume that $A\in M_N(\mathbb C)$ has eigenvalues $\lambda_1,\ldots,\lambda_N\in\mathbb C$, counted with multiplicities. The basic symmetric functions of these eigenvalues, namely*

$$c_k=\sum_{i_1<\ldots<i_k}\lambda_{i_1}\ldots\lambda_{i_k}$$

are then given by the fact that the characteristic polynomial of the matrix is:

$$P(X)=(-1)^N\sum_{k=0}^N(-1)^kc_kX^k$$

Moreover, all symmetric functions of the eigenvalues, such as the sums of powers

$$d_s=\lambda_1^s+\ldots+\lambda_N^s$$

appear as polynomials in these characteristic polynomial coefficients c_k.

PROOF. These results can be proved by doing some algebra, as follows:

(1) Consider indeed the characteristic polynomial P of the matrix, factorized by using its N complex roots, taken with multiplicities. By expanding, we obtain:

$$\begin{aligned}P(X)&=(-1)^N(X-\lambda_1)\ldots(X-\lambda_N)\\&=(-1)^NX^N+(-1)^{N-1}\left(\sum_i\lambda_i\right)X^{N-1}+\ldots+\prod_i\lambda_i\\&=(-1)^NX^N+(-1)^{N-1}c_1X^{N-1}+\ldots+(-1)^0c_N\\&=(-1)^N\left(X^N-c_1X^{N-1}+\ldots+(-1)^Nc_N\right)\end{aligned}$$

With the convention $c_0=1$, we are led to the conclusion in the statement.

(2) This is something standard, coming by doing some abstract algebra. Working out the formulae for the sums of powers $d_s=\sum_i\lambda_i^s$, at small values of the exponent $s\in\mathbb N$, is an excellent exercise, which shows how to proceed in general, by recurrence. □

3d. Jacobians, Hessians

Getting now to analysis, and applications of the above, let us discuss the differentiability in several variables. At order 1, the situation is quite simple, as follows:

THEOREM 3.18. *The derivative of a function $f:\mathbb R^N\to\mathbb R^M$, making the formula*

$$f(x+t)\simeq f(x)+f'(x)t$$

work, must be the matrix of partial derivatives at x, namely

$$f'(x)=\left(\frac{df_i}{dx_j}(x)\right)_{ij}\in M_{M\times N}(\mathbb R)$$

acting on the vectors $t\in\mathbb R^N$ by usual multiplication.

PROOF. As a first observation, the formula in the statement makes sense indeed, as an equality, or rather approximation, of vectors in $\mathbb{R}^M$, as follows:

$$f\begin{pmatrix}x_1+t_1\\ \vdots\\ x_N+t_N\end{pmatrix}\simeq f\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}+\begin{pmatrix}\frac{df_1}{dx_1}(x)&\ldots&\frac{df_1}{dx_N}(x)\\ \vdots&&\vdots\\ \frac{df_M}{dx_1}(x)&\ldots&\frac{df_M}{dx_N}(x)\end{pmatrix}\begin{pmatrix}t_1\\ \vdots\\ t_N\end{pmatrix}$$

In order to prove now this formula, we can proceed by recurrence, as follows:

(1) First of all, at $N=M=1$ what we have is a usual 1-variable function $f:\mathbb{R}\to\mathbb{R}$, and the formula in the statement is something that we know well, namely:

$$f(x+t)\simeq f(x)+f'(x)t$$

(2) Let us discuss now the case $N=2,M=1$. Here what we have is a function $f:\mathbb{R}^2\to\mathbb{R}$, and by using twice the basic approximation result from (1), we obtain:

$$\begin{aligned}f\begin{pmatrix}x_1+t_1\\ x_2+t_2\end{pmatrix}&\simeq f\begin{pmatrix}x_1+t_1\\ x_2\end{pmatrix}+\frac{df}{dx_2}(x)t_2\\ &\simeq f\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\frac{df}{dx_1}(x)t_1+\frac{df}{dx_2}(x)t_2\\ &= f\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}\frac{df}{dx_1}(x)&\frac{df}{dx_2}(x)\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix}\end{aligned}$$

(3) More generally, we can deal in this way with the case $N\in\mathbb{N},M=1$, by recurrence. But this gives the result in the general case $N,M\in\mathbb{N}$ too. Indeed, let us write:

$$f=\begin{pmatrix}f_1\\ \vdots\\ f_M\end{pmatrix}$$

We can apply our result to each of the components $f_i:\mathbb{R}^N\to\mathbb{R}$, and we get:

$$f_i\begin{pmatrix}x_1+t_1\\ \vdots\\ x_N+t_N\end{pmatrix}\simeq f_i\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}+\begin{pmatrix}\frac{df_i}{dx_1}(x)&\ldots&\frac{df_i}{dx_N}(x)\end{pmatrix}\begin{pmatrix}t_1\\ \vdots\\ t_N\end{pmatrix}$$

But this is precisely what we want, at the level of the global map $f:\mathbb{R}^N\to\mathbb{R}^M$. □

As a technical complement to the above result, we have:

THEOREM 3.19. *For a function $f:X\to\mathbb{R}^M$, with $X\subset\mathbb{R}^N$, the following conditions are equivalent, and in this case we say that f is continuously differentiable:*

1. *f is differentiable, and the map $x\to f'(x)$ is continuous.*
2. *f has partial derivatives, which are continuous with respect to $x\in X$.*

If these conditions are satisfied, $f'(x)$ is the matrix fomed by the partial derivatives at x.

Proof. We already know, from Theorem 3.18, that the last assertion holds. Regarding now the proof of the equivalence, this goes as follows:

(1) $\implies$ (2) Assuming that f is differentiable, we know from Theorem 3.18 that $f'(x)$ is the matrix fomed by the partial derivatives at x. Thus, for any $x,y\in X$:

$$\frac{df_i}{dx_j}(x)-\frac{df_i}{dx_j}(y)=f'(x)_{ij}-f'(y)_{ij}$$

By applying now the absolute value, we obtain from this the following estimate:

$$\begin{aligned}\left|\frac{df_i}{dx_j}(x)-\frac{df_i}{dx_j}(y)\right| &= |f'(x)_{ij}-f'(y)_{ij}|\\ &= |(f'(x)-f'(y))_{ij}|\\ &\leq ||f'(x)-f'(y)||\end{aligned}$$

But this gives the result, because if the map $x\to f'(x)$ is assumed to be continuous, then the partial derivatives follow to be continuous with respect to $x\in X$.

(2) $\implies$ (1) This is something more technical. For simplicity, let us assume $M=1$, the proof in general being similar. Given $x\in X$ and $\varepsilon>0$, let us pick $r>0$ such that the ball $B=B_x(r)$ belongs to X, and such that the following happens, over B:

$$\left|\frac{df}{dx_j}(x)-\frac{df}{dx_j}(y)\right|<\frac{\varepsilon}{N}$$

Our claim is that, with this choice made, we have the following estimate, for any $t\in\mathbb{R}^N$ satisfying $||t||<r$, with A being the vector of partial derivatives at x:

$$|f(x+t)-f(x)-At|\leq\varepsilon||t||$$

In order to prove this claim, the idea will be that of suitably applying the mean value theorem, over the N directions of $\mathbb{R}^N$. Indeed, consider the following vectors:

$$t^{(k)}=\begin{pmatrix}t_1\\ \vdots\\ t_k\\ 0\\ \vdots\\ 0\end{pmatrix}$$

In terms of these vectors, we have the following formula:

$$f(x+t)-f(x)=\sum_{j=1}^N f(x+t^{(j)})-f(x+t^{(j-1)})$$

Also, the mean value theorem gives a formula as follows, with $s_j \in [0,1]$:

$$f(x+t^{(j)}) - f(x+t^{(j-1)}) = \frac{df}{dx_j}(x + s_j t^{(j)} + (1-s_j)t^{(j-1)}) \cdot t_j$$

But, according to our assumption on $r > 0$ from the beginning, the derivative on the right differs from $\frac{df}{dx_j}(x)$ by something which is smaller than ε/N:

$$\left|\frac{df}{dx_j}(x + s_j t^{(j)} + (1-s_j)t^{(j-1)}) - \frac{df}{dx_j}(x)\right| < \frac{\varepsilon}{N}$$

Now by putting everything together, we obtain the following estimate:

$$\begin{aligned}
|f(x+t) - f(x) - At| &= \left|\sum_{j=1}^N f(x+t^{(j)}) - f(x+t^{(j-1)}) - \frac{df}{dx_j}(x) \cdot t_j\right| \\
&\leq \sum_{j=1}^N \left|f(x+t^{(j)}) - f(x+t^{(j-1)}) - \frac{df}{dx_j}(x) \cdot t_j\right| \\
&= \sum_{j=1}^N \left|\frac{df}{dx_j}(x + s_j t^{(j)} + (1-s_j)t^{(j-1)}) \cdot t_j - \frac{df}{dx_j}(x) \cdot t_j\right| \\
&= \sum_{j=1}^N \left|\frac{df}{dx_j}(x + s_j t^{(j)} + (1-s_j)t^{(j-1)}) - \frac{df}{dx_j}(x)\right| \cdot |t_j| \\
&\leq \sum_{j=1}^N \frac{\varepsilon}{N} \cdot |t_j| \\
&\leq \varepsilon ||t||
\end{aligned}$$

Thus we have proved our claim, and this gives the result. □

Moving on, with this done, our first task will be that of extending to several variables our basic results from one-variable calculus. As a standard result here, we have:

Theorem 3.20. *We have the chain derivative formula*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

as an equality of matrices.

Proof. This is something standard in one variable, and in several variables the proof is similar, by using the abstract notion of derivative coming from Theorem 3.18. To be more precise, consider a composition of functions, as follows:

$$f : \mathbb{R}^N \to \mathbb{R}^M \quad , \quad g : \mathbb{R}^K \to \mathbb{R}^N \quad , \quad f \circ g : \mathbb{R}^K \to \mathbb{R}^M$$

According to Theorem 3.18, the derivatives of these functions are certain linear maps, corresponding to certain rectangular matrices, as follows:

$$f'(g(x)) \in M_{M\times N}(\mathbb{R}) \quad , \quad g'(x) \in M_{N\times K}(\mathbb{R}) \qquad (f\circ g)'(x) \in M_{M\times K}(\mathbb{R})$$

Thus, our formula makes sense indeed. As for proof, this comes from:

$$\begin{aligned}(f\circ g)(x+t) &= f(g(x+t))\\ &\simeq f(g(x)+g'(x)t)\\ &\simeq f(g(x))+f'(g(x))g'(x)t\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In what regards the change of variables, the result here is as follows:

THEOREM 3.21. *Given a transformation $\varphi = (\varphi_1, \ldots, \varphi_N)$, we have*

$$\int_E f(x)dx = \int_{\varphi^{-1}(E)} f(\varphi(t))|J_\varphi(t)|dt$$

with the J_φ quantity, called Jacobian, being given by

$$J_\varphi(t) = \det\left[\left(\frac{d\varphi_i}{dx_j}(x)\right)_{ij}\right]$$

and with this generalizing the usual formula from one variable calculus.

PROOF. This is something quite tricky, the idea being as follows:

(1) Observe first that the above formula generalizes indeed the change of variable formula in 1 dimension, the point here being that the absolute value on the derivative appears as to compensate for the lack of explicit bounds for the integral.

(2) As a second observation, we can assume if we want, by linearity, that we are dealing with the constant function $f = 1$. For this function, our formula reads:

$$vol(E) = \int_{\varphi^{-1}(E)} |J_\varphi(t)|dt$$

In terms of $D = \varphi^{-1}(E)$, this amounts in proving that we have:

$$vol(\varphi(D)) = \int_D |J_\varphi(t)|dt$$

And here, as a first remark, our formula is clear for the linear maps φ, by using the definition of the determinant of real matrices, as a signed volume.

(3) However, the extension of this to the case of non-linear maps φ is something non-trivial, so we will not follow this path. In order to prove now the result, as stated,

our first claim is that the validity of the theorem is stable under taking compositions of transformations φ. In order to prove this claim, consider a composition, as follows:

$$\varphi : E \to F \quad , \quad \psi : D \to E \quad , \quad \varphi \circ \psi : D \to F$$

Assuming that the theorem holds for φ, ψ, we deduce that we have, as desired:

$$\begin{aligned}\int_F f(x)dx &= \int_E f(\varphi(s))|J_\varphi(s)|ds \\ &= \int_D f(\varphi \circ \psi(t))|J_\varphi(\psi(t))| \cdot |J_\psi(t)|dt \\ &= \int_D f(\varphi \circ \psi(t))|J_{\varphi\circ\psi}(t)|dt\end{aligned}$$

(4) Next, as a key ingredient, let us examine the case where we are in $N = 2$ dimensions, and our transformation φ has one of the following special forms:

$$\varphi(x, y) = (\psi(x, y), y) \quad , \quad \varphi(x, y) = (x, \psi(x, y))$$

By symmetry, it is enough to deal with the first case. Here the Jacobian is $d\psi/dx$, and by replacing if needed $\psi \to -\psi$, we can assume that this Jacobian is positive, $d\psi/dx > 0$. Now by assuming as before that $D = \varphi^{-1}(E)$ is a rectangle, $D = [a, b] \times [c, d]$, we can prove our formula by using the change of variables in 1 dimension, as follows:

$$\begin{aligned}\int_E f(s)ds &= \int_{\varphi(D)} f(x, y)dxdy \\ &= \int_c^d \int_{\psi(a,y)}^{\psi(b,y)} f(x, y)dxdy \\ &= \int_c^d \int_a^b f(\psi(x, y), y)\frac{d\psi}{dx}\,dxdy \\ &= \int_D f(\varphi(t))J_\varphi(t)dt\end{aligned}$$

(5) But with this, we can now prove the theorem, in $N = 2$ dimensions. Indeed, given a transformation $\varphi = (\varphi_1, \varphi_2)$, consider the following two transformations:

$$\phi(x, y) = (\varphi_1(x, y), y) \quad , \quad \psi(x, y) = (x, \varphi_2 \circ \phi^{-1}(x, y))$$

We have then $\varphi = \psi \circ \phi$, and by using (4) for ψ, ϕ, which are of the special form there, and then (3) for composing, we conclude that the theorem holds indeed for φ, as desired. Thus, theorem proved in $N = 2$ dimensions, and the extension of the above proof to arbitrary N dimensions is straightforward, that we will leave here as an exercise. □

Moving on, we can talk as well about higher derivatives, simply by performing the operation of taking derivatives recursively. As a first result here, we have:

Theorem 3.22. *The double derivatives of a function $f:\mathbb{R}^2\to\mathbb{R}$ satisfy*

$$\frac{d^2f}{dxdy}=\frac{d^2f}{dydx}$$

called Clairaut formula.

Proof. This is something very standard, the idea being as follows:

(1) Before pulling out a formal proof, as an intuitive justification for our formula, let us consider a product of power functions, $f(z)=x^py^q$. We have then:

$$\frac{d^2f}{dxdy}=\frac{d}{dx}\left(\frac{dx^py^q}{dy}\right)=\frac{d}{dx}\left(qx^py^{q-1}\right)=pqx^{p-1}y^{q-1}$$

$$\frac{d^2f}{dydx}=\frac{d}{dy}\left(\frac{dx^py^q}{dx}\right)=\frac{d}{dy}\left(px^{p-1}y^q\right)=pqx^{p-1}y^{q-1}$$

Next, let us consider a linear combination of power functions, $f(z)=\sum_{pq}c_{pq}x^py^q$, which can be finite or not. We have then, by using the above computation:

$$\frac{d^2f}{dxdy}=\frac{d^2f}{dydx}=\sum_{pq}c_{pq}pqx^{p-1}y^{q-1}$$

Thus, we can see that our commutation formula for derivatives holds indeed, and this due to the fact that the functions in x and y commute. Of course, this does not prove our formula, in general. But exercise for you, to have this idea fully working.

(2) Getting now to more standard techniques, given a point in the plane, $z=(a,b)$, consider the following functions, depending on $h,k\in\mathbb{R}$ small:

$$u(h,k)=f(a+h,b+k)-f(a+h,b)$$

$$v(h,k)=f(a+h,b+k)-f(a,b+k)$$

$$w(h,k)=f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)$$

By the mean value theorem, for $h,k\neq0$ we can find $\alpha,\beta\in\mathbb{R}$ such that:

$$\begin{aligned}
w(h,k)&=u(h,k)-u(0,k)\\
&=h\cdot\frac{d}{dx}u(\alpha h,k)\\
&=h\left(\frac{d}{dx}f(a+\alpha h,b+k)-\frac{d}{dx}f(a+\alpha h,b)\right)\\
&=hk\cdot\frac{d}{dy}\cdot\frac{d}{dx}f(a+\alpha h,b+\beta k)
\end{aligned}$$

Similarly, again for $h,k\neq 0$, we can find $\gamma,\delta\in\mathbb R$ such that:

$$\begin{aligned} w(h,k) &= v(h,k)-v(h,0)\\ &= k\cdot\frac{d}{dy}\,v(h,\delta k)\\ &= k\left(\frac{d}{dy}\,f(a+h,b+\delta k)-\frac{d}{dy}\,f(a,b+\delta k)\right)\\ &= hk\cdot\frac{d}{dx}\cdot\frac{d}{dy}\,f(a+\gamma h,b+\delta k)\end{aligned}$$

Now by dividing everything by $hk\neq 0$, we conclude from this that the following equality holds, with the numbers $\alpha,\beta,\gamma,\delta\in\mathbb R$ being found as above:

$$\frac{d}{dy}\cdot\frac{d}{dx}\,f(a+\alpha h,b+\beta k)=\frac{d}{dx}\cdot\frac{d}{dy}\,f(a+\gamma h,b+\delta k)$$

But with $h,k\to 0$ we get from this the Clairaut formula, at $z=(a,b)$, as desired. □

In arbitrary dimensions now, we have the following result:

THEOREM 3.23. *Given $f:\mathbb R^N\to\mathbb R$, we can talk about its higher derivatives,*

$$\frac{d^kf}{dx_{i_1}\ldots dx_{i_k}}=\frac{d}{dx_{i_1}}\cdots\frac{d}{dx_{i_k}}(f)$$

provided that these derivatives exist indeed. Moreover, due to the Clairaut formula,

$$\frac{d^2f}{dx_idx_j}=\frac{d^2f}{dx_jdx_i}$$

the order in which these higher derivatives are computed is irrelevant.

PROOF. There are several things going on here, the idea being as follows:

(1) First of all, we can talk about the quantities in the statement, with the remark however that at each step of our recursion, the corresponding partial derivative can exist of not. We will say in what follows that our function is k times differentiable if the quantities in the statement exist at any $l\leq k$, and smooth, if this works with $k=\infty$.

(2) Regarding now the second assertion, this is something more tricky. Let us first recall from Theorem 3.22 that the second derivatives of a twice differentiable function of two variables $f:\mathbb R^2\to\mathbb R$ are subject to the Clairaut formula, namely:

$$\frac{d^2f}{dxdy}=\frac{d^2f}{dydx}$$

(3) But this result clearly extends to our function $f:\mathbb R^N\to\mathbb R$, simply by ignoring the unneeded variables, so we have the Clairaut formula in general, also called Schwarz

formula, which is the one in the statement, namely:

$$\frac{d^2f}{dx_idx_j}=\frac{d^2f}{dx_jdx_i}$$

(4) Now observe that this tells us that the order in which the higher derivatives are computed is irrelevant. That is, we can permute the order of our partial derivative computations, and a standard way of doing this is by differentiating first with respect to x_1, as many times as needed, then with respect to x_2, and so on. Thus, the collection of partial derivatives can be written, in a more convenient form, as follows:

$$\frac{d^kf}{dx_1^{k_1}\ldots dx_N^{k_N}}=\frac{d^{k_1}}{dx_1^{k_1}}\cdots\frac{d^{k_N}}{dx_N^{k_N}}(f)$$

(5) To be more precise, here $k\in\mathbb{N}$ is as usual the global order of our derivatives, the exponents $k_1,\ldots,k_N\in\mathbb{N}$ are subject to the condition $k_1+\ldots+k_N=k$, and the operations on the right are the familiar one-variable higher derivative operations.

(6) This being said, for certain tricky questions it is more convenient not to order the indices, or rather to order them according to what order best fits our computation, so what we have in the statement is the good formula, and (4-5) are mere remarks.

(7) And with the remark too that for trivial questions, what we have in the statement is the good formula, simply because there are less indices to be written, when compared to what we have to write when using the ordering procedure in (4-5) above. □

All this is very nice, and as an illustration, let us work out in detail the case $k=2$. Here things are quite special, and we can formulate the following definition:

DEFINITION 3.24. *Given a twice differentiable function $f:\mathbb{R}^N\to\mathbb{R}$, we set*

$$f''(x)=\left(\frac{d^2f}{dx_idx_j}\right)_{ij}$$

which is a symmetric matrix, called Hessian matrix of f at the point $x\in\mathbb{R}^N$.

To be more precise, we know that when $f:\mathbb{R}^N\to\mathbb{R}$ is twice differentiable, its order $k=2$ partial derivatives are the numbers in the statement. Now since these numbers naturally form a $N\times N$ matrix, the temptation is high to call this matrix $f''(x)$, and so we will do. And finally, we know from Clairaut that this matrix is symmetric:

$$f''(x)_{ij}=f''(x)_{ji}$$

Observe that at $N=1$ this is compatible with the usual definition of the second derivative f'', and this because in this case, the 1×1 matrix from Definition 3.24 is:

$$f''(x)=(f''(x))\in M_{1\times1}(\mathbb{R})$$

As a word of warning, however, never use Definition 3.24 for functions $f:\mathbb{R}^N\to\mathbb{R}^M$, where the second derivative can only be something more complicated. Also, never attempt either to do something similar at $k=3$ or higher, for functions $f:\mathbb{R}^N\to\mathbb{R}$ with $N>1$, because again, that beast has too many indices, for being a true, honest matrix.

Back now to business, analysis, with these notions, we have the following result:

THEOREM 3.25. *Given a twice differentiable function $f:\mathbb{R}^N\to\mathbb{R}$, we have*

$$f(x+t)\simeq f(x)+f'(x)t+\frac{<f''(x)t,t>}{2}$$

where $f''(x)\in M_N(\mathbb{R})$ stands as usual for the Hessian matrix.

PROOF. This is something more tricky, the idea being as follows:

(1) As a first observation, at $N=1$ the Hessian matrix as constructed in Definition 3.24 is the 1×1 matrix having as entry the second derivative $f''(x)$, and the formula in the statement is something that we know well from basic calculus, namely:

$$f(x+t)\simeq f(x)+f'(x)t+\frac{f''(x)t^2}{2}$$

(2) In general now, this is in fact something which does not need a new proof, because it follows from the one-variable formula above, applied to the restriction of f to the following segment in $\mathbb{R}^N$, which can be regarded as being a one-variable interval:

$$I=[x,x+t]$$

To be more precise, let $y\in\mathbb{R}^N$, and consider the following function, with $r\in\mathbb{R}$:

$$g(r)=f(x+ry)$$

We know from (1) that the Taylor formula for g, at the point $r=0$, reads:

$$g(r)\simeq g(0)+g'(0)r+\frac{g''(0)r^2}{2}$$

And our claim is that, with $t=ry$, this is precisely the formula in the statement.

(3) So, let us see if our claim is correct. By using the chain rule, we have the following formula, with on the right, as usual, a row vector multiplied by a column vector:

$$g'(r)=f'(x+ry)\cdot y$$

By using again the chain rule, we can compute the second derivative as well:

$$\begin{aligned} g''(r) &= (f'(x+ry)\cdot y)' \\ &= \left(\sum_i \frac{df}{dx_i}(x+ry)\cdot y_i\right)' \\ &= \sum_i\sum_j \frac{d^2f}{dx_idx_j}(x+ry)\cdot\frac{d(x+ry)_j}{dr}\cdot y_i \\ &= \sum_i\sum_j \frac{d^2f}{dx_idx_j}(x+ry)\cdot y_iy_j \\ &= < f''(x+ry)y,y > \end{aligned}$$

(4) Time now to conclude. We know that we have $g(r)=f(x+ry)$, and according to our various computations above, we have the following formulae:

$$g(0)=f(x)\quad,\quad g'(0)=f'(x)\quad,\quad g''(0)=< f''(x)y,y >$$

Buit with this data in hand, the usual Taylor formula for our one variable function g, at order 2, at the point $r=0$, takes the following form, with $t=ry$:

$$\begin{aligned} f(x+ry) &\simeq f(x)+f'(x)ry+\frac{< f''(x)y,y > r^2}{2} \\ &= f(x)+f'(x)t+\frac{< f''(x)t,t >}{2} \end{aligned}$$

Thus, we have obtained the formula in the statement.

(5) Finally, for completness, let us record as well a more numeric formulation of what we found. According to our usual rules for matrix calculus, what we found is:

$$f(x+t)\simeq f(x)+\sum_{i=1}^N\frac{df}{dx_i}\,t_i+\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N\frac{d^2f}{dx_idx_j}\,t_it_j$$

Observe that, since the Hessian matrix $f''(x)$ is symmetric, most of the terms on the right will appear in pairs, making it clear what the $1/2$ is there for, namely avoiding redundancies. However, this is only true for the off-diagonal terms, so instead of further messing up our numeric formula above, we will just leave it like this. □

As in the one variable case, the Taylor formula is useful for computing the local extrema of the function. Indeed, let us first look at the order 1 formula, namely:

$$f(x+t)\simeq f(x)+f'(x)t$$

It is clear then, exactly as in the one-variable case, that in order to have a local extremum, we must have $f'(x)=0$. Next, assuming that this holds, let us look at the order 2 Taylor formula, which in the case $f'(x)=0$ takes the following form:

$$f(x+t)\simeq f(x)+\frac{<f''(x)t,t>}{2}$$

We conclude from this, again as in the one-variable case, that when $f''(x)>0$, with this meaning that the symmetric matrix $f''(x)\in M_N(\mathbb{R})$ must be strictly positive, we have a local minimum, and that when $f''(x)<0$, we have a local maximum.

At higher order now, things become more complicated, as follows:

THEOREM 3.26. *Given an order k differentiable function $f:\mathbb{R}^N\to\mathbb{R}$, we have*

$$f(x+t)\simeq f(x)+f'(x)t+\frac{<f''(x)t,t>}{2}+\ldots$$

and this helps in identifying the local extrema, when $f'(x)=0$ and $f''(x)=0$.

PROOF. The study here is very similar to that at $k=2$, from the proof of Theorem 3.25, with everything coming from the usual Taylor formula, applied on:

$$I=[x,x+t]$$

Thus, it is pretty much clear that we are led to the conclusion in the statement. We will leave some study here as an instructive exercise. □

3e. Exercises

This was a standard linear algebra chapter, and as exercises on this, we have:

EXERCISE 3.27. *Prove the complex polarization identity.*

EXERCISE 3.28. *Find and prove the parallelogram identity too.*

EXERCISE 3.29. *Read the proof of diagonalization of self-adjoints, $A=A^*$.*

EXERCISE 3.30. *Then do the same for the unitary matrices, $A^*=A^{-1}$.*

EXERCISE 3.31. *Then do the same for the normal matrices, $AA^*=A^*A$.*

EXERCISE 3.32. *Learn more about the positive matrices, and their properties.*

EXERCISE 3.33. *Learn more things about the resultant and discriminant.*

EXERCISE 3.34. *Learn as well the Cardano formulae, in degree 3 and 4.*

As bonus exercise, review if needed your multivariable calculus knowledge.

CHAPTER 4

Complex functions

4a. Complex functions

Time to get into the real thing, and main topic of this book, study of the complex functions $f : \mathbb{C} \to \mathbb{C}$. We will see that many results about the real functions $f : \mathbb{R} \to \mathbb{R}$ extend to the complex case, but there will be quite a number of new phenomena too.

We will need, in order to get started, the following basic definition:

DEFINITION 4.1. *The distance between two complex numbers is given by:*

$$d(x, y) = |x - y|$$

With this, we can talk about convergence, by saying that $x_n \to x$ when $d(x_n, x) \to 0$.

Here the fact that $d(x, y) = |x-y|$ is indeed the usual distance in the plane is clear for $y = 0$, because we have $d(x, 0) = |x|$, by definition of the modulus $|x|$. As for the general case, $y \in \mathbb{C}$, this comes from the fact that the distance in the plane is given by:

$$d(x, y) = d(x - y, 0) = |x - y|$$

Observe that in real coordinates, the distance formula is quite complicated, namely:

$$\begin{aligned} d(a + ib, c + id) &= |(a + ib) - (c + id)| \\ &= |(a - c) + i(b - d)| \\ &= \sqrt{(a - c)^2 + (b - d)^2} \end{aligned}$$

However, for most computations, we will not need this formula, and we can get away with the various tricks regarding complex numbers that we know. As a first result now, regarding $\mathbb{C}$ and its distance, that we will need in what follows, we have:

PROPOSITION 4.2. *The complex plane $\mathbb{C}$ is complete, in the sense that any Cauchy sequence converges.*

PROOF. Consider indeed a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$. If we write $x_n = a_n + ib_n$ for any $n \in \mathbb{N}$, then we have the following estimates:

$$|a_n - a_m| \leq \sqrt{(a_n - a_m)^2 + (b_n - b_m)^2} = |x_n - x_m|$$

$$|b_n - b_m| \leq \sqrt{(a_n - a_m)^2 + (b_n - b_m)^2} = |x_n - x_m|$$

Thus both the sequences $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ are Cauchy, and since we know that $\mathbb{R}$ itself is complete, we can consider the limits of these sequences:

$$a_n \to a \quad , \quad b_n \to b$$

With $x = a + ib$, our claim is that $x_n \to x$. Indeed, we have:

$$\begin{aligned} |x_n - x| &= \sqrt{(a_n - a)^2 + (b_n - b)^2} \\ &\leq |a_n - a| + |b_n - b| \end{aligned}$$

It follows that we have $x_n \to x$, as claimed, and this gives the result. □

Talking complex functions now, we have here the following definition:

Definition 4.3. *A complex function $f : \mathbb{C} \to \mathbb{C}$, or more generally $f : X \to \mathbb{C}$, with $X \subset \mathbb{C}$ being a subset, is called continuous when, for any $x_n, x \in X$:*

$$x_n \to x \implies f(x_n) \to f(x)$$

Also, we can talk about pointwise convergence of functions, $f_n \to f$, and about uniform convergence too, $f_n \to_u f$, exactly as for the real functions.

Observe that, since $x_n \to x$ in the complex sense means that $(a_n, b_n) \to (a, b)$ in the usual, real plane sense, a function $f : \mathbb{C} \to \mathbb{C}$ is continuous precisely when it is continuous when regarded as real function, $f : \mathbb{R}^2 \to \mathbb{R}^2$. But more on this later in this book.

At the level of examples now, we first have the polynomials, $P \in \mathbb{C}[X]$. We already met polynomials in chapter 1, so let us recall from there that we have:

Theorem 4.4. *Each polynomial $P \in \mathbb{C}[X]$ can be regarded as a function*

$$P : \mathbb{C} \to \mathbb{C}$$

which is continuous. Moreover, we have the factorization formula

$$P(x) = a(x - r_1) \ldots (x - r_n)$$

with $a \in \mathbb{C}$, and with the numbers $r_1, \ldots, r_n \in \mathbb{C}$ being the roots of P.

Proof. This is something that we know from chapter 1. Assume indeed that P has no roots, and pick a number $x \in \mathbb{C}$ where $|P|$ attains its minimum:

$$|P(x)| = \min_{z\in\mathbb{C}} |P(z)| > 0$$

Since $Q(t) = P(x + t) - P(x)$ vanishes at $t = 0$, we have an estimate as follows:

$$P(x + t) \simeq P(x) + ct^k$$

On the other hand, since $P(x) \neq 0$, we can choose our variable $t \simeq 0$ such that ct^k points in the opposite direction to that of $P(x)$, and we obtain in this way:

$$|P(x + t)| < |P(x)|$$

Thus, contradiction, P has a root, and by recurrence it has n roots, as stated. □

Next in line, we have the rational functions, which are defined as follows:

THEOREM 4.5. *The quotients of complex polynomials $f = P/Q$ are called rational funtions. When written in reduced form, with P, Q prime to each other,*

$$f = \frac{P}{Q}$$

is well-defined and continuous outside the zeroes $P_f \subset \mathbb{C}$ of Q, called poles of f:

$$f : \mathbb{C} - P_f \to \mathbb{C}$$

In addition, the rational functions, regarded as algebraic expressions, are stable under summing, making products and taking inverses.

PROOF. There are several things going on here, the idea being as follows:

(1) First of all, we can surely talk about quotients of polynomials, $f = P/Q$, regarded as abstract algebraic expressions. Also, the last assertion is clear, because we can indeed perform sums, products, and take inverses, by using the following formulae:

$$\frac{P}{Q} + \frac{R}{S} = \frac{PS + QR}{QS} \quad , \quad \frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS} \quad , \quad \left(\frac{P}{Q}\right)^{-1} = \frac{Q}{P}$$

(2) The question is now, given a rational function f, can we regard it as a complex function? In general, we cannot say that we have $f : \mathbb{C} \to \mathbb{C}$, for instance because $f(x) = x^{-1}$ is not defined at $x = 0$. More generally, assuming $f = P/Q$ with $P, Q \in \mathbb{C}$, we cannot talk about $f(x)$ when x is a root of Q, unless of course we are in the special situation where x is a root of P too, and we can simplify the fraction.

(3) In view of this discussion, in order to solve our question, we must avoid the situation where the polynomials P, Q have common roots. But this can be done by writing our rational function f in reduced form, as follows, with $P, Q \in \mathbb{C}[X]$ prime to each other:

$$f = \frac{P}{Q}$$

(4) Now with this convention made, it is clear that f is well-defined, and continuous too, outside of the zeroes of f. Now since these zeroes can be obviously recovered from the knowledge of f itself, as being the points where "f explodes", we can call them poles of f, and so we have a function $f : \mathbb{C} - P_f \to \mathbb{C}$, as in the statement. □

As a comment here, the term "pole" does not come from the Poles who invented this, but rather from the fact that, when trying to draw the graph of f, or rather imagine that graph, which takes place in $2+2 = 4$ real dimensions, we are faced with some sort of tent, which is suspended by infinite poles, which lie, guess where, at the poles of f.

Getting back now to Theorem 4.5, as stated, that is obviously a mixture of algebra and analysis. So, let us first further clarify the algebra part. We know that the rational functions are stable under summing, making products and taking inverses, and this makes the link with the following notion, from number theory and abstract algebra:

Definition 4.6. *A field is a set F with a sum operation $+$ and a product operation $\times$, subject to the following conditions:*

(1) $a+b=b+a$, $a+(b+c)=(a+b)+c$, *there exists* $0\in F$ *such that* $a+0=0$, *and any* $a\in F$ *has an inverse* $-a\in F$, *satisfying* $a+(-a)=0$.
(2) $ab=ba$, $a(bc)=(ab)c$, *there exists* $1\in F$ *such that* $a1=a$, *and any* $a\neq 0$ *has a multiplicative inverse* $a^{-1}\in F$, *satisfying* $aa^{-1}=1$.
(3) *The sum and product are compatible via* $a(b+c)=ab+ac$.

As basic examples of fields, we have the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. Some further examples of fields of numbers, which are more specialized, and useful in number theory, can be constructed as well. In view of this, it is useful to think of any field F as being a "field of numbers", and this because the elements $a,b,c,\ldots\in F$ behave under the operations $+$ and $\times$ exactly as the usual numbers do.

In what regards the various spaces of functions, such as the polynomials $\mathbb{C}[X]$, or the continuous functions $C(\mathbb{R})$, these certainly have sum and product operations $+$ and $\times$, but are in general not fields, because they do not satisfy the following field axiom:

$$f\neq 0\implies \exists f^{-1}$$

However, and here comes our point, Theorem 4.5 tells us that the rational functions form a field. This is quite interesting, and opposite to the general spirit of analysis and function spaces, which are in general not fields. Let us record this finding, as follows:

Definition 4.7. *We denote by $\mathbb{C}(X)$ the field of rational functions*

$$f=\frac{P}{Q}\quad,\quad P,Q\in\mathbb{C}[X]$$

with the usual sum and product operations $+$ and $\times$ for the rational functions.

To be more precise, this is some sort of reformulation of Theorem 4.5, or rather of the algebraic content of Theorem 4.5, telling us that the rational functions form indeed a field. And to the question, how can a theorem suddenly become a definition, the answer is that this is quite commonplace in mathematics, and especially in algebra.

Back now to analysis, let us point out that, contrary to what the above might suggest, everything does not always extend trivally from the real to the complex case.

For instance, we have the following result, regarding the basic geometric series:

Proposition 4.8. *We have the following formula, valid for any $|x|<1$,*

$$\frac{1}{1-x}=1+x+x^2+\ldots$$

but, for $x\in\mathbb{C}-\mathbb{R}$, the geometric meaning of this formula is quite unclear.

Proof. Here the formula in the statement holds indeed, by multiplying and cancelling terms, and with the convergence being justified by the following estimate:

$$\left|\sum_{n=0}^\infty x^n\right|\leq\sum_{n=0}^\infty|x|^n=\frac{1}{1-|x|}$$

As for the last assertion, this is something quite informal, the idea being as follows:

(1) To start with, at the value $x=1/2$ our formula is clear, by cutting the interval $[0,2]$ into half, and so on, which gives the following formula, as desired:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=2$$

(2) More generally, for $x\in(-1,1)$ the meaning of the formula in the statement is something quite intuitive, geometrically speaking, by using a similar argument.

(3) However, when x is complex, and not real, we are led into a kind of mysterious spiral there, and the only case where the formula is "obvious", geometrically speaking, is that when $x=rw$, with $r\in[0,1)$, and with w being a root of unity.

(4) To be more precise here, assume that we have a number $w\in\mathbb{C}$ satisfying $w^N=1$, for some $N\in\mathbb{N}$. We have then the following formula, for our infinite sum:

$$\begin{aligned}1+rw+r^2w^2+\ldots&=(1+rw+\ldots+r^{N-1}w^{N-1})\\&+(r^N+r^{N+1}w\ldots+r^{2N-1}w^{N-1})\\&+(r^{2N}+r^{2N+1}w\ldots+r^{3N-1}w^{N-1})\\&+\ldots\end{aligned}$$

(5) Thus, by grouping the terms with the same argument, our infinite sum is:

$$\begin{aligned}1+rw+r^2w^2+\ldots&=(1+r^N+r^{2N}+\ldots)\\&+(r+r^{N+1}+r^{2N+1}+\ldots)w\\&+\ldots\\&+(r^{N-1}+r^{2N-1}+r^{3N-1}+\ldots)w^{N-1}\end{aligned}$$

(6) But the sums of each ray can be computed with the real formula for geometric series, that we know and understand well, and with an extra bit of algebra, we get:

$$\begin{aligned}1+rw+r^2w^2+\ldots &= \frac{1}{1-r^N}+\frac{rw}{1-r^N}+\ldots+\frac{r^{N-1}w^{N-1}}{1-r^N}\\ &= \frac{1}{1-r^N}\left(1+rw+\ldots+r^{N-1}w^{N-1}\right)\\ &= \frac{1}{1-r^N}\cdot\frac{1-r^N}{1-rw}\\ &= \frac{1}{1-rw}\end{aligned}$$

(7) Summarizing, as claimed above, the geometric series formula can be understood, in a purely geometric way, for variables of type $x=rw$, with $r\in[0,1)$, and with w being a root of unity. In general, however, this formula tells us that the numbers on a certain infinite spiral sum up to a certain number, which remains something quite mysterious. □

Getting back now to less mysterious mathematics, the main tool for dealing with the continuous functions $f:\mathbb{R}\to\mathbb{R}$ was the intermediate value theorem. In the complex setting, that of the functions $f:\mathbb{C}\to\mathbb{C}$, we do not have such a theorem, at least in its basic formulation, because there is no order relation for the complex numbers, or things like complex intervals. However, the intermediate value theorem in its advanced formulation, that with connected sets, extends. In order to discuss this, let us first formulate:

Proposition 4.9. *We can talk about open and closed sets in $\mathbb{C}$, and about compact and connected sets too, in the obvious way, and the following happen:*

(1) *Open disks are open, closed disks are closed.*
(2) *Union of open sets is open, intersection of closed sets is closed.*
(3) *Finite intersection of open sets is open, finite union of closed sets is closed.*
(4) *The open sets are exactly the complements of closed sets.*
(5) *Compact, as defined with covers, is the same as being bounded and closed.*
(6) *Connected means by definition to be impossible to split in 2 parts.*

Proof. This is obviously something quick and informal, the idea being as follows:

(1) This is something which is obvious.

(2) Again, something clear, and a good exercise for you.

(3) Another good exercise, and think at some related counterexamples too.

(4) Exercise too, provided that you didn't use this as definition, for open or closed.

(5) This is the only non-trivial thing, in all this, and exercise for you, again.

(6) This is the definition of the connected sets. What can we say about these? □

Getting now to the functions, we have the following result about them:

THEOREM 4.10. *Assuming that $f:\mathbb{C}\to\mathbb{C}$ is continuous:*

(1) *If O is open, then $f^{-1}(O)$ is open.*
(2) *If C is closed, then $f^{-1}(C)$ is closed.*
(3) *If K is compact, then $f(K)$ is compact.*
(4) *If E is connected, then $f(E)$ is connected.*

PROOF. This is indeed something very standard, with (1,2) coming from definitions, and with their converses holding too, and with (3,4) coming too from definitions. □

Getting now to what we really wanted to say about complex functions, intermediate value theorem, this is (4) in the above, so let us have this highlighted, as follows:

THEOREM 4.11 (Intermediate values). *Assuming that a function*

$$f:X\to\mathbb{C}$$

with $X\subset\mathbb{C}$ is continuous, if the domain X is connected, so is its image $f(X)$.

PROOF. We have already stated this in Theorem 4.10 (4), but let us see now how the detailed proof goes as well. Assume by contradiction that $f(X)$ is not connected, which in practice means that we can find two disjoint open sets $A,B\in\mathbb{C}$ such that:

$$f(X)\subset A\sqcup B$$

By taking inverse images, we obtain from this a disjoint union as follows:

$$X\subset f^{-1}(A)\sqcup f^{-1}(B)$$

Now since inverse image of an open set is open, wich this being something which is clear from definitions, both the above sets $f^{-1}(A)$ and $f^{-1}(B)$ are open. Thus we have managed to split X into two parts, contradicting its connectivity, as desired. □

4b. Sin, cos, exp, log

Getting now to more complicated functions, such as sin, cos, exp, log, again many things extend well from real to complex, the basic theory here being as follows:

THEOREM 4.12. *The functions* $\sin,\cos,\exp,\log$ *have complex extensions, given by*

$$\sin x=\sum_{l=0}^\infty(-1)^l\frac{x^{2l+1}}{(2l+1)!}\quad,\quad\cos x=\sum_{l=0}^\infty(-1)^l\frac{x^{2l}}{(2l)!}$$

$$e^x=\sum_{k=0}^\infty\frac{x^k}{k!}\quad,\quad\log(1+x)=\sum_{k=1}^\infty(-1)^{k+1}\frac{x^k}{k}$$

with $|x|<1$ needed for $\log$*, which are continuous over their domain, and satisfy the formulae $e^{x+y}=e^xe^y$ and $e^{ix}=\cos x+i\sin x$.*

Proof. This is a mixture of trivial and non-trivial results, as follows:

(1) We already know about e^x from chapter 1, the idea being that the convergence of the series, and then the continuity of e^x, come from the following estimate:

$$|e^x|\leq\sum_{k=0}^\infty\frac{|x|^k}{k!}=e^{|x|}<\infty$$

(2) Regarding $\sin x$, we can formally define it over the whole $\mathbb{C}$ by the formula in the statement, with the convergence being guaranteed by the following estimate:

$$|\sin x|\leq\sum_{l=0}^\infty\frac{|x|^{2l+1}}{(2l+1)!}\leq\sum_{k=0}^\infty\frac{|x|^k}{k!}=e^{|x|}$$

(3) The same goes for $\cos x$, with the needed estimate here being as follows:

$$|\cos x|\leq\sum_{l=0}^\infty\frac{|x|^{2l}}{(2l)!}\leq\sum_{k=0}^\infty\frac{|x|^k}{k!}=e^{|x|}$$

(4) Regarding now the formulae satisfied by $\sin,\cos,\exp$, we already know from chapter 1 that the exponential has the following property, exactly as in the real case:

$$e^{x+y}=e^xe^y$$

We also have the following formula, connecting our functions $\sin,\cos,\exp$:

$$\begin{aligned}e^{ix}&=\sum_{k=0}^\infty\frac{(ix)^k}{k!}\\&=\sum_{k=2l}\frac{(ix)^k}{k!}+\sum_{k=2l+1}\frac{(ix)^k}{k!}\\&=\sum_{l=0}^\infty(-1)^l\frac{x^{2l}}{(2l)!}+i\sum_{l=0}^\infty(-1)^l\frac{x^{2l+1}}{(2l+1)!}\\&=\cos x+i\sin x\end{aligned}$$

(5) Finally, still in relation with $\sin,\cos,\exp$, it remains to justify the fact that the complex functions $\sin,\cos:\mathbb{C}\to\mathbb{C}$ constructed above extend the usual real functions $\sin,\cos:\mathbb{R}\to\mathbb{R}$. But this comes from the Euler formula, from chapter 1, namely:

$$e^{it}=\cos t+i\sin t$$

Indeed, in this formula $t\in\mathbb{R}$ is real, and $\sin,\cos:\mathbb{R}\to\mathbb{R}$ are the usual trigonometric functions. On the other hand, we know from (4) that we have $e^{it}=\cos t+i\sin t$, with $\sin,\cos:\mathbb{C}\to\mathbb{C}$ being this time constructed as in (2,3). We therefore conclude that the restrictions $\sin,\cos:\mathbb{R}\to\mathbb{C}$ coming from (2,3) are the usual ones, as desired.

(6) In order to discuss now the complex logarithm function log, let us first study some more the complex exponential function exp. By using $e^{x+y} = e^x e^y$ we obtain $e^x \neq 0$ for any $x \in \mathbb{C}$, so the complex exponential function is as follows:

$$\exp : \mathbb{C} \to \mathbb{C} - \{0\}$$

Now since we have $e^{x+iy} = e^x e^{iy}$ for $x, y \in \mathbb{R}$, with e^x being surjective onto $(0, \infty)$, and with e^{iy} being surjective onto the unit circle $\mathbb{T}$, we deduce that $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ is surjective. Also, again by using $e^{x+iy} = e^x e^{iy}$, we deduce that we have:

$$e^x = e^y \iff x - y \in 2\pi i\mathbb{Z}$$

(7) With these ingredients in hand, we can now talk about log. Indeed, let us fix a horizontal strip in the complex plane, having width 2π:

$$S = \left\{ x + iy \middle| x \in \mathbb{R}, y \in [a, a + 2\pi) \right\}$$

We know from the above that the restriction map $\exp : S \to \mathbb{C} - \{0\}$ is bijective, so we can define log as to be the inverse of this map:

$$\log = \exp^{-1} : \mathbb{C} - \{0\} \to S$$

(8) In practice now, the best is to choose for instance $a = 0$, or $a = -\pi$, as to have the whole real line included in our strip, $\mathbb{R} \subset S$. In this case on $\mathbb{R}_+$ we recover the usual logarithm, while on $\mathbb{R}_-$ we obtain complex values, as for instance $\log(-1) = \pi i$ in the case $a = 0$, or $\log(-1) = -\pi i$ in the case $a = -\pi$, coming from $e^{\pi i} = -1$.

(9) Finally, assuming $|x| < 1$, we can consider the following series, which converges:

$$f(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

We have then $e^{f(x)} = 1 + x$, and so $f(x) = \log(1 + x)$, when $1 + x \in S$. □

As an interesting consequence of the above result, which is of practical interest, we have the following useful method, for remembering the basic math formulae:

METHOD 4.13. *Knowing* $e^x = \sum_k x^k/k!$ *and* $e^{ix} = \cos x + i \sin x$ *gives you*

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

right away, in case you forgot these formulae, as well as

$$\sin x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+1}}{(2l + 1)!} \quad , \quad \cos x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l}}{(2l)!}$$

again, right away, in case you forgot these formulae.

To be more precise, assume that we forgot everything trigonometry, which is something that can happen to everyone, in the real life, but still know the formulae $e^x = \sum_k x^k/k!$ and $e^{ix} = \cos x + i\sin x$. Then, we can recover the formulae for sums, as follows:

$$\begin{aligned} e^{i(x+y)} = e^{ix}e^{iy} \quad &\implies \quad \cos(x+y) + i\sin(x+y) = (\cos x + i\sin x)(\cos y + i\sin y) \\ &\implies \quad \begin{cases} \cos(x+y) = \cos x\cos y - \sin x\sin y \\ \sin(x+y) = \sin x\cos y + \cos x\sin y \end{cases} \end{aligned}$$

And isn't this smart. Also, and even more impressively, we can recover the Taylor formulae for $\sin, \cos$, which are certainly quite difficult to memorize, as follows:

$$\begin{aligned} e^{ix} = \sum_k \frac{(ix)^k}{k!} \quad &\implies \quad \cos x + i\sin x = \sum_k \frac{(ix)^k}{k!} \\ &\implies \quad \begin{cases} \cos x = \sum_{l=0}^\infty (-1)^l \frac{x^{2l}}{(2l)!} \\ \sin x = \sum_{l=0}^\infty (-1)^l \frac{x^{2l+1}}{(2l+1)!} \end{cases} \end{aligned}$$

Finally, in what regards log, there is a trick here too, which is partial, namely:

$$\begin{aligned} \log(\exp x) = x \quad &\implies \quad \log\left(1 + x + \frac{x^2}{2} + \ldots\right) = x \\ &\implies \quad \log(1+y) = y - \frac{y^2}{2} + \ldots \end{aligned}$$

To be more precise, $\log(1+y) \simeq y$ is clear, and with a bit more work, that we will leave here as an instructive exercise, you can recover $\log(1+y) = y - y^2/2$ too. Of course, the higher terms can be recovered too, with some work involved, at each step.

Back to theory, we can talk as well about powers, in the following way:

FACT 4.14. *Under suitable assumptions, we can talk about x^y with $x, y \in \mathbb{C}$, and in particular about the complex functions a^x and x^a, with $a \in \mathbb{C}$.*

To be more precise, in what regards x^y, we already know from basic calculus that things are quite tricky, even in the real case. In the complex case the same problems appear, along with some more, but these questions can be solved by using the above theory of $\exp, \log$. To be more precise, in order to solve the first question, we can set:

$$x^y = e^{y\log x}$$

We will be back to these functions later, when we will have more tools for studying them. In fact, all of a sudden, we are now into quite complicated mathematics, and we cannot really deal with such questions, with bare hands. More later.

4c. Hyperbolic functions

Moving ahead, Theorem 4.12 leads us into the question on whether the other formulae that we know about $\sin,\cos$, such as the values of these functions on sums $x+y$, or on doubles $2x$, extend to the complex setting. Things are quite tricky here, and in relation with this, we have the following result, which is something of general interest:

THEOREM 4.15. *The following functions, called hyperbolic sine and cosine,*

$$\sinh x=\frac{e^x-e^{-x}}{2}\quad,\quad\cosh x=\frac{e^x+e^{-x}}{2}$$

are subject to the following formulae:

(1) $e^x=\cosh x+\sinh x$.
(2) $\sinh(ix)=i\sin x$, $\cosh(ix)=\cos x$, *for* $x\in\mathbb{R}$.
(3) $\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y$.
(4) $\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$.
(5) $\sinh x=\sum_l\frac{x^{2l+1}}{(2l+1)!}$, $\cosh x=\sum_l\frac{x^{2l}}{(2l)!}$.

PROOF. The formula (1) follows from definitions. As for (2), this follows from:

$$\sinh(ix)=\frac{e^{ix}-e^{-ix}}{2}=\frac{\cos x+i\sin x}{2}-\frac{\cos x-i\sin x}{2}=i\sin x$$

$$\cosh(ix)=\frac{e^{ix}+e^{-ix}}{2}=\frac{\cos x+i\sin x}{2}+\frac{\cos x-i\sin x}{2}=\cos x$$

Regarding now (3,4), observe first that the formula $e^{x+y}=e^x+e^y$ reads:

$$\cosh(x+y)+\sinh(x+y)=(\cosh x+\sinh x)(\cosh y+\sinh y)$$

Thus, we have some good explanation for (3,4), and in practice, these formulae can be checked by direct computation, as follows:

$$\frac{e^{x+y}-e^{-x-y}}{2}=\frac{e^x-e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x+e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$$

$$\frac{e^{x+y}+e^{-x-y}}{2}=\frac{e^x+e^{-x}}{2}\cdot\frac{e^y+e^{-y}}{2}+\frac{e^x-e^{-x}}{2}\cdot\frac{e^y-e^{-y}}{2}$$

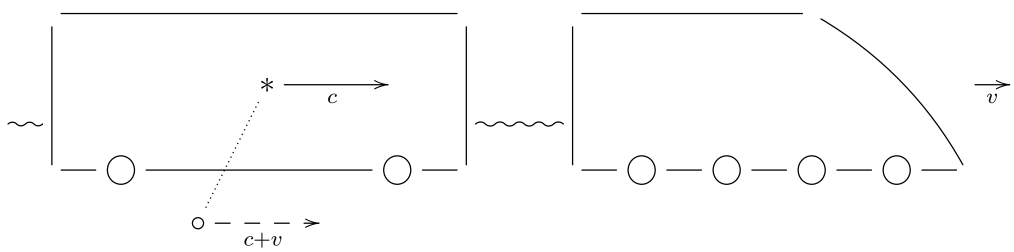
Finally, (5) is clear from the definition of sinh, cosh, and from $e^x=\sum_k\frac{x^k}{k!}$. $\square$

In relation with this, ready for some physics? Based on experiments by Fizeau and others, and on some theory by Maxwell and Lorentz too, Einstein came upon:

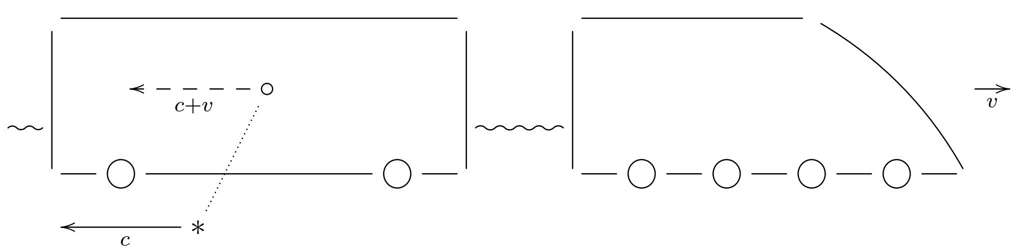
FACT 4.16 (Einstein principles). *The following happen:*

(1) *Light travels in vacuum at a finite speed,* $c<\infty$.
(2) *This speed* c *is the same for all inertial observers.*
(3) *In non-vacuum, the light speed is lower,* $v<c$.
(4) *Nothing can travel faster than light,* $v\not>c$.

The point now is that, obviously, something is wrong here. Indeed, assuming for instance that we have a train, running in vacuum at speed $v > 0$, and someone on board lights a flashlight $*$ towards the locomotive, then an observer $\circ$ on the ground will see the light traveling at speed $c + v > c$, which is a contradiction:



Equivalently, with the same train running, in vacuum at speed $v > 0$, if the observer on the ground lights a flashlight $*$ towards the back of the train, then viewed from the train, that light will travel at speed $c + v > c$, which is a contradiction again:



Summarizing, Fact 4.16 implies $c + v = c$, so contradicts classical mechanics, which therefore needs a fix. By dividing all speeds by c, as to have $c = 1$, and by restricting the attention to the 1D case, to start with, we are led to the following puzzle:

PUZZLE 4.17. *How to define speed addition on the space of* 1D *speeds, which is*

$$I = [-1, 1]$$

with our $c = 1$ convention, as to have $1 + c = 1$, as required by physics?

In view of our geometric knowledge so far, a natural idea here would be that of wrapping $[-1, 1]$ into a circle, and then stereographically projecting on $\mathbb{R}$. Indeed, we can then "import" to $[-, 1, 1]$ the usual addition on $\mathbb{R}$, via the inverse of this map.

So, let us see where all this leads us. First, the formula of our map is as follows:

PROPOSITION 4.18. *The map wrapping $[-1, 1]$ into the unit circle, and then stereographically projecting on $\mathbb{R}$ is given by the formula*

$$\varphi(u) = \tan\left(\frac{\pi u}{2}\right)$$

with the convention that our wrapping is the most straightforward one, making correspond $\pm 1 \to i$, with negatives on the left, and positives on the right.

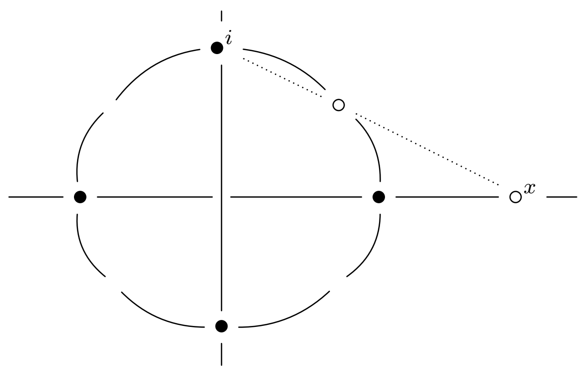
PROOF. Regarding the wrapping, as indicated, this is given by:

$$u \to e^{it} \quad , \quad t = \pi u - \frac{\pi}{2}$$

Indeed, this correspondence wraps $[-1,1]$ as above, the basic instances of our correspondence being as follows, and with everything being fine modulo 2π:

$$-1 \to \frac{\pi}{2} \quad , \quad -\frac{1}{2} \to -\pi \quad , \quad 0 \to -\frac{\pi}{2} \quad , \quad \frac{1}{2} \to 0 \quad , \quad 1 \to \frac{\pi}{2}$$

Regarding now the stereographic projection, the picture here is as follows:



Thus, by Thales, the formula of the stereographic projection is as follows:

$$\frac{\cos t}{x} = \frac{1 - \sin t}{1} \implies x = \frac{\cos t}{1 - \sin t}$$

Now if we compose our wrapping operation above with the stereographic projection, what we get is, via the above Thales formula, and some trigonometry:

$$\begin{aligned} x &= \frac{\cos t}{1 - \sin t} \\ &= \frac{\cos\left(\pi u - \frac{\pi}{2}\right)}{1 - \sin\left(\pi u - \frac{\pi}{2}\right)} \\ &= \frac{\cos\left(\frac{\pi}{2} - \pi u\right)}{1 + \sin\left(\frac{\pi}{2} - \pi u\right)} \\ &= \frac{\sin(\pi u)}{1 + \cos(\pi u)} \\ &= \frac{2\sin\left(\frac{\pi u}{2}\right)\cos\left(\frac{\pi u}{2}\right)}{2\cos^2\left(\frac{\pi u}{2}\right)} \\ &= \tan\left(\frac{\pi u}{2}\right) \end{aligned}$$

Thus, we are led to the conclusion in the statement. $\square$

The above result is very nice, but when it comes to physics, things do not work, for instance because of the wrong slope of the function $\varphi(u) = \tan\left(\frac{\pi u}{2}\right)$ at the origin, which makes our summing on $[-1,1]$ not compatible with the Galileo addition, at low speeds.

So, what to do? Obviously, trash Proposition 4.18, and start all over again. Getting back now to Puzzle 4.17, this has in fact a simpler solution, based this time on algebra, and which in addition is the good, physically correct solution, as follows:

THEOREM 4.19. *If we sum the speeds according to the Einstein formula*

$$u +_e v = \frac{u+v}{1+uv}$$

then the Galileo formula still holds, approximately, for low speeds

$$u +_e v \simeq u + v$$

and if we have $u = 1$ or $v = 1$, the resulting sum is $u +_e v = 1$.

PROOF. All this is self-explanatory, and clear from definitions, and with the Einstein formula of $u +_e v$ itself being just an obvious solution to Puzzle 4.17, provided that, importantly, we know 0 geometry, and rely on very basic algebra only. □

So, very nice, problem solved, at least in 1D. But, shall we give up with geometry, and the stereographic projection? Certainly not, let us try to recycle that material. In order to do this, let us recall that the usual trigonometric functions are given by:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad , \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad , \quad \tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$

The point now is that, and you might know this from calculus, the above functions have some natural "hyperbolic" or "imaginary" analogues, constructed as follows:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^{x} + e^{-x}}{2} \quad , \quad \tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

But the function on the right, tanh, starts reminding the formula of Einstein addition, from Theorem 4.19. So, we have our idea, and we are led to the following result:

THEOREM 4.20. *The Einstein speed summation in 1D is given by*

$$\tanh x +_e \tanh y = \tanh(x+y)$$

with $\tanh : [-\infty, \infty] \to [-1, 1]$ *being the hyperbolic tangent function.*

PROOF. This follows by putting together our various formulae above, but it is perhaps better, for clarity, to prove this directly. Our claim is that we have:

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

But this can be checked via direct computation, from the definitions, as follows:

$$\begin{aligned}
&\frac{\tanh x+\tanh y}{1+\tanh x\tanh y}\\
=\ &\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}+\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)\Big/\left(1+\frac{e^x-e^{-x}}{e^x+e^{-x}}\cdot\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)\\
=\ &\frac{(e^x-e^{-x})(e^y+e^{-y})+(e^x+e^{-x})(e^y-e^{-y})}{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y+e^{-y})}\\
=\ &\frac{2(e^{x+y}-e^{-x-y})}{2(e^{x+y}+e^{-x-y})}\\
=\ &\tanh(x+y)
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Very nice all this, hope you agree. As a conclusion, passing from the Riemann stereographic projection sum to the Einstein summation basically amounts in replacing:

$$\tan\to\tanh$$

Let us formulate as well this finding more philosophically, as follows:

CONCLUSION 4.21. *The Einstein speed summation in 1D is the imaginary analogue of the summation on $[-1,1]$ obtained via Riemann's stereographic projection.*

Which looks quite deep, and we will stop here. More on this later in this book, when discussing curved spacetime, in full generality, and with more advanced tools.

4d. Gamma, zeta, eta

We would like talk now about some more specialized complex functions, which are, among others, of great use in arithmetic, called gamma, zeta and eta.

Let us start our discussion with a general-purpose question, namely finding the volume of the unit sphere in $\mathbb{R}^N$. In order to do our computation here, we will need:

THEOREM 4.22 (Wallis). *We have the following formulae,*

$$\int_0^{\pi/2}\cos^n t\,dt=\int_0^{\pi/2}\sin^n t\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(n)}\frac{n!!}{(n+1)!!}$$

where $\varepsilon(n)=1$ if n is even, and $\varepsilon(n)=0$ if n is odd, and where

$$m!!=(m-1)(m-3)(m-5)\ldots$$

with the product ending at 2 if m is odd, and ending at 1 if m is even.

PROOF. Let us first compute the integral on the left in the statement:

$$I_n=\int_0^{\pi/2}\cos^nt\,dt$$

We do this by partial integration. We have the following formula:

$$\begin{aligned}(\cos^nt\sin t)'&=n\cos^{n-1}t(-\sin t)\sin t+\cos^nt\cos t\\&=n\cos^{n+1}t-n\cos^{n-1}t+\cos^{n+1}t\\&=(n+1)\cos^{n+1}t-n\cos^{n-1}t\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain the following formula:

$$(n+1)I_{n+1}=nI_{n-1}$$

Thus we can compute I_n by recurrence, and we obtain:

$$\begin{aligned}I_n&=\frac{n-1}{n}\,I_{n-2}\\&=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\,I_{n-4}\\&=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\,I_{n-6}\\&\ \ \vdots\\&=\frac{n!!}{(n+1)!!}\,I_{1-\varepsilon(n)}\end{aligned}$$

But $I_0=\frac{\pi}{2}$ and $I_1=1$, so we get the result. As for the second formula, this follows from the first one, with $t=\frac{\pi}{2}-s$. Thus, we have proved both formulae in the statement. □

We can now compute the volumes of the N-dimensional spheres, as follows:

THEOREM 4.23. *The volume of the unit sphere in $\mathbb{R}^N$ is given by*

$$V=\left(\frac{\pi}{2}\right)^{[N/2]}\frac{2^N}{(N+1)!!}$$

with our usual convention $N!!=(N-1)(N-3)(N-5)\ldots$

PROOF. If we denote by V_N the volume of the unit sphere in $\mathbb{R}^N$, we have:

$$\begin{aligned}
V_N &= \int_{-1}^1 (1-x^2)^{(N-1)/2} dx \cdot V_{N-1} \\
&= 2V_{N-1}\int_0^1 (1-x^2)^{(N-1)/2} dx \\
&= 2V_{N-1}\int_0^{\pi/2} (1-\sin^2 t)^{(N-1)/2} \cos t dt \\
&= 2V_{N-1}\int_0^{\pi/2} \cos^{N-1} t \cos t dt \\
&= 2V_{N-1}\int_0^{\pi/2} \cos^N t dt
\end{aligned}$$

Now by recurrence, using the Wallis formula from Theorem 4.22, we obtain:

$$\begin{aligned}
V_N &= 2^N \int_0^{\pi/2} \cos^N t dt \int_0^{\pi/2} \cos^{N-1} t dt \ldots\ldots \int_0^{\pi/2} \cos t dt \\
&= 2^N \left(\frac{\pi}{2}\right)^{\varepsilon(N)+\varepsilon(N-1)+\ldots+\varepsilon(1)} \frac{N!!}{(N+1)!!} \cdot \frac{(N-1)!!}{N!!} \cdots \frac{1!!}{2!!} \\
&= \left(\frac{\pi}{2}\right)^{\varepsilon(N)+\varepsilon(N-1)+\ldots+\varepsilon(1)} \frac{2^N}{(N+1)!!} \\
&= \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N+1)!!}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us record as well the asymptotics, obtained via Stirling, as follows:

THEOREM 4.24. *The volume of the unit sphere in $\mathbb{R}^N$ is given by*

$$V \simeq \left(\frac{2\pi e}{N}\right)^{N/2} \frac{1}{\sqrt{\pi N}}$$

in the $N \to \infty$ limit.

PROOF. This is something very standard, the idea being as follows:

(1) We use the exact formula found in Theorem 4.23, namely:

$$V = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{2^N}{(N+1)!!}$$

(2) But the double factorials can be estimated by using the Stirling formula. Indeed, in the case where $N=2K$ is even, we have the following computation:

$$\begin{aligned}(N+1)!! &= 2^K K!\\ &\simeq \left(\frac{2K}{e}\right)^K \sqrt{2\pi K}\\ &= \left(\frac{N}{e}\right)^{N/2} \sqrt{\pi N}\end{aligned}$$

As for the case where $N=2K-1$ is odd, here the estimate goes as follows:

$$\begin{aligned}(N+1)!! &= \frac{(2K)!}{2^K K!}\\ &\simeq \frac{1}{2^K}\left(\frac{2K}{e}\right)^{2K}\sqrt{4\pi K}\left(\frac{e}{K}\right)^K \frac{1}{\sqrt{2\pi K}}\\ &= \left(\frac{2K}{e}\right)^K \sqrt{2}\\ &= \left(\frac{N+1}{e}\right)^{(N+1)/2}\sqrt{2}\\ &= \left(\frac{N}{e}\right)^{N/2}\left(\frac{N+1}{N}\right)^{N/2}\sqrt{\frac{N+1}{e}}\cdot\sqrt{2}\\ &\simeq \left(\frac{N}{e}\right)^{N/2}\sqrt{e}\cdot\sqrt{\frac{N}{e}}\cdot\sqrt{2}\\ &= \left(\frac{N}{e}\right)^{N/2}\sqrt{2N}\end{aligned}$$

(3) Now back to the spheres, when N is even, the estimate goes as follows:

$$\begin{aligned}V &= \left(\frac{\pi}{2}\right)^{N/2}\frac{2^N}{(N+1)!!}\\ &\simeq \left(\frac{\pi}{2}\right)^{N/2} 2^N \left(\frac{e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}}\\ &= \left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}}\end{aligned}$$

As for the case where N is odd, here the estimate goes as follows:

$$\begin{aligned} V &= \left(\frac{\pi}{2}\right)^{(N-1)/2}\frac{2^N}{(N+1)!!} \\ &\simeq \left(\frac{\pi}{2}\right)^{(N-1)/2} 2^N\left(\frac{e}{N}\right)^{N/2}\frac{1}{\sqrt{2N}} \\ &= \sqrt{\frac{2}{\pi}}\left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{2N}} \\ &= \left(\frac{2\pi e}{N}\right)^{N/2}\frac{1}{\sqrt{\pi N}} \end{aligned}$$

Thus, we are led to the uniform formula in the statement. □

The above results are quite interesting, and raise the following question:

QUESTION 4.25. *How to unify the theory of factorials $N!$ and double factorials $N!!$, with $N\in\mathbb{N}$? Also, how to have a continuous theory, using $N\in\mathbb{R}$, or $N\in\mathbb{C}$?*

Which might sound a bit crazy as a question, but before formulating such criticisms, please listen to the following result, which provides an answer to our crazy question:

THEOREM 4.26. *We can talk about the gamma function*

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx$$

extending the usual factorial of integers, $\Gamma(s)=(s-1)!$.

PROOF. The integral converges indeed, and by partial integration we have:

$$\begin{aligned} \Gamma(s+1) &= \int_0^\infty x^s e^{-x}\,dx \\ &= \int_0^\infty s x^{s-1}e^{-x}\,dx \\ &= s\Gamma(s) \end{aligned}$$

Regarding now the case $s\in\mathbb{N}$, for the initial value $s=1$ we have:

$$\Gamma(1)=\int_0^\infty e^{-x}dx=1$$

Thus, for $s\in\mathbb{N}$ we have indeed $\Gamma(s)=(s-1)!$, as claimed. □

Many interesting things can be said about the gamma function, notably with a computation at the half-integers, and with this being related to geometry, as follows:

THEOREM 4.27. *The gamma function is given at half-integers by*

$$\Gamma(n)=(n-1)!\quad,\quad\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!!}{2^n}\sqrt{\pi}$$

and we have the following formula for it, with $c=\sqrt{2},\sqrt{\pi}$ *for* N *even, odd:*

$$\Gamma\left(\frac{N}{2}\right)=\frac{(N-1)!!}{2^{(N-1)/2}}\,c$$

Moreover, the volume of the unit sphere in $\mathbb{R}^N$*, which is given by*

$$V=\left(\frac{\pi}{2}\right)^{[N/2]}\frac{2^N}{(N+1)!!}$$

can be expressed in terms of these values of the gamma function.

PROOF. There are several things going on here, the idea being as follows:

(1) We already know, from Theorem 4.26, that for $n\in\mathbb{N}$ we have:

$$\Gamma(n)=(n-1)!$$

(2) Regarding now the half-integers, we first have the following computation, using at the end the Gauss integral, which comes by squaring and using polar coordinates:

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)&=\int_0^\infty x^{-1/2}e^{-x}\,dx\\&=\int_0^\infty y^{-1}e^{-y^2}2y\,dy\\&=2\int_0^\infty e^{-y^2}dy\\&=\sqrt{\pi}\end{aligned}$$

Next, by using $\Gamma(s+1)=s\Gamma(s)$, we have the following computations:

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)=\frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)=\frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\,\Gamma\left(\frac{5}{2}\right)=\frac{15}{8}\sqrt{\pi}$$

$$\vdots$$

Thus, we can solve the problem by recurrence, and we obtain in this way, with our usual convention $N!! = (N-1)(N-3)(N-5)\ldots$ for the double factorials:

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1.3.5\ldots(2n-1)}{2^n}\sqrt{\pi}=\frac{(2n)!!}{2^n}\sqrt{\pi}$$

(3) Regarding now the unification of the formulae that we have in (1) and (2), let us first rewrite the formula that we just found in (2), in the following way:

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{(2n)!!}{2^n}\sqrt{\pi}$$

Which looks good and in final form, no questions about this, so for a unification we are left with refurbishing the nice formula found in (1), in the following way:

$$\begin{aligned}\Gamma\left(\frac{2n}{2}\right) &= \Gamma(n)\\ &= (n-1)!\\ &= \frac{2.4.6\ldots(2n-2)}{2^{n-1}}\\ &= \frac{(2n-1)!!}{2^{n-1}}\\ &= \frac{(2n-1)!!}{2^{n-1/2}}\sqrt{2}\end{aligned}$$

And with this, job done, we are led to the uniform formula in the statement.

(4) Finally, regarding the spheres, the assertion, which is something quite traditional, is self-explanatory. However, doing so will only result in complicating our formulae, so we will not do this. For more on this, we refer to Theorem 4.23 and Theorem 4.24. □

Time now for some arithmetic. Let us start with the following basic result:

PROPOSITION 4.28. *We can talk about the Riemann zeta function*

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

at any $s\in\mathbb{C}$ with $Re(z)>1$.

PROOF. We have indeed the following computation, with $s = r + it$ with $r > 1$:

$$\begin{aligned}|\zeta(s)| &\leq \sum_{n=1}^{\infty}\frac{1}{|n^s|}\\ &= \sum_{n=1}^{\infty}\frac{1}{n^r}\\ &< 1+\int_1^{\infty}\frac{1}{x^r}\,dx\\ &= 1+\frac{1}{r-1}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a first result regarding the zeta function, making a remarkable connection with primes and arithmetic, we can write it as an Euler product, as follows:

THEOREM 4.29. *We have the following formula, with product over the primes,*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

valid for any exponent $s \in \mathbb{C}$ with $Re(s) > 1$.

PROOF. We have the following computation, with everything converging:

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s}\\ &= \prod_p\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\ldots\right)\\ &= \prod_p\left(1-\frac{1}{p^s}\right)^{-1}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

But, how to study zeta? The answer comes via the gamma function, as follows:

THEOREM 4.30. *We have the following formula,*

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}\,dx$$

valid for any $s \in \mathbb{C}$ with $Re(s) > 1$.

PROOF. We have indeed the following computation:

$$\begin{aligned}
\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx &= \int_0^\infty \frac{x^{s-1}}{e^x}\cdot\frac{1}{1-e^{-x}}\,dx\\
&= \int_0^\infty x^{s-1}(e^{-x}+e^{-2x}+e^{-3x}+\ldots)\\
&= \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\,dx\\
&= \sum_{n=1}^\infty \int_0^\infty \left(\frac{y}{n}\right)^{s-1} e^{-y}\,\frac{dy}{n}\\
&= \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty y^{s-1}e^{-y}\,dy\\
&= \zeta(s)\Gamma(s)
\end{aligned}$$

Thus, we are led to the formula in the statement. □

At a more advanced level now, we can try to compute particular values of ζ. Things are quite tricky here, and we have the following result, based on Theorem 4.30:

THEOREM 4.31. *We have the following formula, for the even integers $s=2k$,*

$$\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}$$

with B_n being the Bernoulli numbers, which in practice gives the formulae

$$\zeta(2)=\frac{\pi^2}{6}\quad,\quad\zeta(4)=\frac{\pi^4}{90}\quad,\quad\zeta(6)=\frac{\pi^6}{945}\quad,\quad\zeta(8)=\frac{\pi^8}{9450}\quad,\quad\ldots$$

generalizing the formula $\zeta(2)=\pi^2/6$ of Euler, solving the Basel problem.

PROOF. As already mentioned, all this comes from the formula from Theorem 4.30, by doing some standard computations, that we will leave here as an instructive exercise. □

Let us get now into the true unknown, $Re(s)<1$, with our first objective being that of understanding what happens in the strip $0<Re(s)<1$. We have here:

THEOREM 4.32. *We have the following formula,*

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^\infty\frac{(-1)^{n+1}}{n^s}$$

which can stand as definition for ζ, in the strip $0<Re(s)<1$.

PROOF. This is something elementary, known since Dirichlet and Euler, but of key importance, and with many consequences, the idea being as follows:

(1) To start with, we can define indeed the Dirichlet eta function η as being the signed version of the Riemann zeta function ζ, according to the following formula:

$$\eta(s)=\sum_{n=1}^\infty\frac{(-1)^{n+1}}{n^s}$$

Observe that this function converges indeed in the strip $0<Re(s)<1$.

(2) We must now connect ζ and η, at $Re(s)>1$, and this can be done as follows:

$$\begin{aligned}\zeta(s)+\eta(s)&=\sum_{n=1}^\infty\frac{1}{n^s}+\sum_{n=1}^\infty\frac{(-1)^{n+1}}{n^s}\\&=2\sum_{k=1}^\infty\frac{1}{(2k)^s}\\&=2^{1-s}\sum_{k=1}^\infty\frac{1}{k^s}\\&=2^{1-s}\zeta(s)\end{aligned}$$

(3) But this intuitively leads to the conclusion in the statement. In practice, all this needs a bit more discussion, and complex analysis knowledge. We will be back to this. □

4e. Exercises

This was a particularly beautiful and concrete chapter, and as exercises, we have:

EXERCISE 4.33. *Work out the convergence basics for the functions $f:\mathbb C\to\mathbb C$.*

EXERCISE 4.34. *Work out as well the more advanced theory, using topology.*

EXERCISE 4.35. *Further investigate the spiral problem, $\sum_n x^n=1/(1-x)$.*

EXERCISE 4.36. *Further investigate as well the complex logarithm, and powers.*

EXERCISE 4.37. *Learn more about the hyperbolic functions and geometry.*

EXERCISE 4.38. *Learn more, from Einstein [28], about the space we live in.*

EXERCISE 4.39. *Review, if needed, your knowledge of the Stirling formula.*

EXERCISE 4.40. *Learn as well some more complicated Wallis formulae.*

As bonus exercise, and no surprise here, learn more about gamma, zeta, eta.

Part II

Cauchy formula

CHAPTER 5

Cauchy formula

5a. Differentiation

Welcome to complex analysis, which is the study of the functions $f:\mathbb C\to\mathbb C$, or more generally $f:X\to\mathbb C$, with $X\subset\mathbb C$. And expect of course, as it was the case with the complex numbers themselves, a lot of mysterious phenomena, appearing here.

Let us first study the differentiability of the functions $f:\mathbb C\to\mathbb C$. Things here will be quite tricky, but to start with, we have a straightforward definition, as follows:

Definition 5.1. *We say that a function $f:X\to\mathbb C$ is differentiable in the complex sense when the following limit is defined for any $x\in X$:*

$$f'(x)=\lim_{t\to0}\frac{f(x+t)-f(x)}{t}$$

In this case, we also say that f is holomorphic, and we write $f\in H(X)$.

As basic examples, we have the power functions $f(x)=x^n$. Indeed, the derivative of such a power function can be computed exactly as in the real case, and we get:

$$\begin{aligned}(x^n)' &= \lim_{t\to0}\frac{(x+t)^n-x^n}{t}\\ &= \lim_{t\to0}\frac{nx^{n-1}t+\binom{n}{2}x^{n-2}t^2+\ldots+t^n}{t}\\ &= \lim_{t\to0}\frac{nx^{n-1}t}{t}\\ &= nx^{n-1}\end{aligned}$$

We will see later more computations of this type, similar to those from the real case. To summarize, our definition of differentiability seems to work nicely, so let us start developing some theory. The general results from the real case extend well, as follows:

Theorem 5.2. *We have the following results:*

(1) $(f+g)'=f'+g'$.
(2) $(\lambda f)'=\lambda f'$.
(3) $(fg)'=f'g+fg'$.
(4) $(f\circ g)'=f'(g)g'$.

PROOF. These formulae are all clear from definitions, as in the real case:

(1) This follows indeed from definitions, the computation being as follows:

$$\begin{aligned}(f+g)'(x) &= \lim_{t\to 0}\frac{(f+g)(x+t)-(f+g)(x)}{t}\\ &= \lim_{t\to 0}\left(\frac{f(x+t)-f(x)}{t}+\frac{g(x+t)-g(x)}{t}\right)\\ &= \lim_{t\to 0}\frac{f(x+t)-f(x)}{t}+\lim_{t\to 0}\frac{g(x+t)-g(x)}{t}\\ &= f'(x)+g'(x)\end{aligned}$$

(2) This follows again from definitions, the computation being as follows:

$$\begin{aligned}(\lambda f)'(x) &= \lim_{t\to 0}\frac{(\lambda f)(x+t)-(\lambda f)(x)}{t}\\ &= \lambda\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}\\ &= \lambda f'(x)\end{aligned}$$

(3) This follows from definitions too, the computation, by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative, being as follows:

$$\begin{aligned}(fg)(x+t) &= f(x+t)g(x+t)\\ &\simeq (f(x)+f'(x)t)(g(x)+g'(x)t)\\ &\simeq f(x)g(x)+(f'(x)g(x)+f(x)g'(x))t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of t, namely:

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x)$$

(4) Regarding compositions, the computation here is as follows, again by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative:

$$\begin{aligned}(f\circ g)(x+t) &= f(g(x+t))\\ &\simeq f(g(x)+g'(x)t)\\ &\simeq f(g(x))+f'(g(x))g'(x)t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of t, namely:

$$(f\circ g)'(x)=f'(g(x))g'(x)$$

Thus, we are led to the conclusions in the statement. □

As a consequence of Theorem 5.2 (1,2), any poynomial $P\in\mathbb C[X]$ is differentiable, with its derivative being given by the same formula as in the real case, namely:

$$P(x)=\sum_{k=0}^nc_kx^k\implies P'(x)=\sum_{k=1}^nkc_kx^{k-1}$$

More generally, any rational function $f\in\mathbb C(X)$ is differentiable on its domain, that is, outsides its poles, because if we write $f=P/Q$ with $P,Q\in\mathbb C[X]$, we have:

$$f'=\left(\frac{P}{Q}\right)'=\frac{P'Q-PQ'}{Q^2}$$

Let us record these conclusions in a statement, as follows:

PROPOSITION 5.3. *The following happen:*

(1) *Any polynomial $P\in\mathbb C[X]$ is holomorphic, and in fact infinitely differentiable in the complex sense, with all its derivatives being polynomials.*
(2) *Any rational function $f\in\mathbb C(X)$ is holomorphic, and in fact infinitely differentiable, with all its derivatives being rational functions.*

PROOF. This follows indeed from the above discussion. □

Let us look now into more complicated complex functions that we know. And here, surprise, things are quite tricky, the result being as follows:

THEOREM 5.4. *The following happen:*

(1) $\sin,\cos,\exp,\log$ *are holomorphic, and in fact are infinitely differentiable, with their derivatives being given by the same formulae as in the real case.*
(2) *However, functions like $\bar{x}$ or $|x|$ are not holomorphic, and this because the limit defining $f'(x)$ depends on the way we choose $t\to0$.*

PROOF. There are several things going on here, the idea being as follows:

(1) Here the first assertion is standard, because our functions $\sin,\cos,\exp,\log$ have Taylor series that we know, and the derivative can be therefore computed by using the same rule as in the real case, similar to the one for polynomials, namely:

$$f(x)=\sum_{k=0}^\infty c_kx^k\implies f'(x)=\sum_{k=1}^\infty kc_kx^{k-1}$$

(2) Regarding now the function $f(x)=\bar{x}$, the point here is that we have:

$$\frac{f(x+t)-f(x)}{t}=\frac{\bar{x}+\bar{t}-\bar{x}}{t}=\frac{\bar{t}}{t}$$

But this limit does not converge with $t\to0$, for instance because with $t\in\mathbb R$ we obtain 1 as limit, while with $t\in i\mathbb R$ we obtain -1 as limit. In fact, with $t=rw$ with $|w|=1$

fixed and $r\in\mathbb{R}$, $r\to 0$, we can obtain as limit any number on the unit circle:

$$\lim_{r\to 0}\frac{f(x+rw)-f(x)}{rw}=\lim_{r\to 0}\frac{r\bar{w}}{rw}=\bar{w}^2$$

(3) The situation for the function $f(x)=|x|$ is similar. To be more precise, we have:

$$\frac{f(x+rw)-f(x)}{rw}=\frac{|x+rw|-|x|}{r}\cdot\bar{w}$$

Thus with $|w|=1$ fixed and $r\to 0$ we obtain a certain multiple of $\bar{w}$, with the multiplication factor being computed as follows:

$$\begin{aligned}\frac{|x+rw|-|x|}{r} &= \frac{|x+rw|^2-|x|^2}{(|x+rw|+|x|)r}\\ &\simeq \frac{xr\bar{w}+\bar{x}rw}{2|x|r}\\ &= Re\left(\frac{x\bar{w}}{|x|}\right)\end{aligned}$$

Now by making w vary on the unit circle, as in (2) above, we can obtain in this way limits pointing in all possible directions, so our limit does not converge, as stated. □

The above result is quite surprising, because we are so used, from the real case, to the notion of differentiability to correspond to some form of "smoothness at first order" of the function. Nevermind. Moving ahead, based on the above computations, let us formulate the following definition, coming as a complement to Definition 5.1:

DEFINITION 5.5. *A function $f:X\to\mathbb{C}$ is called differentiable:*

(1) *In the real sense, if the following two limits converge, for any $x\in X$:*

$$f_1'(x)=\lim_{t\in\mathbb{R}\to 0}\frac{f(x+t)-f(x)}{t}\quad,\quad f_i'(x)=\lim_{t\in i\mathbb{R}\to 0}\frac{f(x+t)-f(x)}{t}$$

(2) *In a radial sense, if the following limit converges, for any $x\in X$, and $w\in\mathbb{T}$:*

$$f_w'(x)=\lim_{t\in w\mathbb{R}\to 0}\frac{f(x+t)-f(x)}{t}$$

(3) *In the complex sense, if the following limit converges, for any $x\in X$:*

$$f'(x)=\lim_{t\to 0}\frac{f(x+t)-f(x)}{t}$$

If f is differentiable in the complex sense, we also say that f is holomorphic.

We can see now more clearly what is going on. We have (3) $\implies$ (2) $\implies$ (1) in general, and most of the functions that we know, namely the polynomials, the rational functions, and $\sin,\cos,\exp,\log$, satisfy (3). As for the functions $\bar{x},|x|$, these do not satisfy

(3), and do not satisfy (2) either, but they satisfy however (1). It is possible to say more about all this, and we will certainly come back to this topic, later in this book.

Still in relation with the differentiability basics, let us formulate as well:

DEFINITION 5.6. *The Cauchy-Riemann operators are*

$$\partial=\frac{1}{2}\left(\frac{d}{dx}-i\frac{d}{dy}\right)\quad,\quad\bar{\partial}=\frac{1}{2}\left(\frac{d}{dx}+i\frac{d}{dy}\right)$$

where $\frac{d}{dx}$ *and* $\frac{d}{dy}$ *are the usual partial derivatives for complex functions.*

There are many things that can be said about the Cauchy-Riemann operators $\partial,\bar{\partial}$, the idea being that in many contexts, these are better to use than the usual partial derivatives $\frac{d}{dx},\frac{d}{dy}$, and with this being a bit like the usage of the variables $z,\bar{z}$, instead of the decomposition $z=a+ib$, for many questions regarding the complex numbers.

At the general level, the main properties of $\partial,\bar{\partial}$ can be summarized as follows:

THEOREM 5.7. *Assume that* $f:X\to\mathbb{C}$ *is differentiable in the real sense.*

(1) f *is holomorphic precisely when* $\bar{\partial}f=0$.
(2) *In this case, its derivative is* $f'=\partial f$.

PROOF. We can assume by linearity that we are dealing with differentiability questions at 0. Since our function $f:X\to\mathbb{C}$ is differentiable in the real sense, we have a formula as follows, with $z=x+iy$, and with $a,b\in\mathbb{C}$ being the partial derivatives at 0:

$$f(z)=ax+by+o(z)$$

Now observe that we can write this formula in the following way:

$$\begin{aligned}f(z)&=a\cdot\frac{z+\bar{z}}{2}+b\cdot\frac{z-\bar{z}}{2i}+o(z)\\&=a\cdot\frac{z+\bar{z}}{2}+b\cdot\frac{i\bar{z}-iz}{2}+o(z)\\&=\frac{a-ib}{2}\cdot z+\frac{a+ib}{2}\cdot\bar{z}+o(z)\end{aligned}$$

Now by dividing by z, we obtain from this the following formula:

$$\begin{aligned}\frac{f(z)}{z}&=\frac{a-ib}{2}+\frac{a+ib}{2}\cdot\frac{\bar{z}}{z}+o(1)\\&=\partial f(0)+\bar{\partial}f(0)\cdot\frac{\bar{z}}{z}+o(1)\end{aligned}$$

But this gives the result, because in order for $f'(0)$ to exist, as the $z\to 0$ limit of the above quantity, the coefficient of $\bar{z}/z$, which does not converge, must vanish. □

5b. Analytic functions

We have seen that all the basic examples of holomorphic functions that we have are infinitely differentiable, and this raises the question of finding a function such that f' exists, while f'' does not exist. Quite surprisingly, we will see that such functions do not exist. In order to get into this latter phenomenon, let us start with:

THEOREM 5.8. *Each power series $f(x)=\sum_n c_nx^n$ has a radius of convergence*

$$R\in[0,\infty]$$

which is such that f converges for $|x|<R$, and diverges for $|x|>R$. We have:

$$R=\frac{1}{C}\quad,\quad C=\limsup_{n\to\infty}\sqrt[n]{|c_n|}$$

Also, in the case $|x|=R$ the function f can either converge, or diverge.

PROOF. This follows from the Cauchy criterion for series, from basic real analysis, which says that a series $\sum_n x_n$ converges if $c<1$, and diverges if $c>1$, where:

$$c=\limsup_{n\to\infty}\sqrt[n]{|x_n|}$$

Indeed, with $x_n=|c_nx^n|$ we obtain that the convergence radius $R\in[0,\infty]$ exists, and is given by the formula in the statement. Finally, for the examples and counterexamples at the end, when $|x|=R$, the simplest here is to use $f(x)=\sum_n x^n$, where $R=1$. □

Back now to our questions regarding derivatives, we have:

THEOREM 5.9. *Assuming that a function $f:X\to\mathbb{C}$ is analytic, in the sense that it is a series, around each point $x\in X$,*

$$f(x+t)=\sum_{n=0}^\infty c_nt^n$$

it follows that f is infinitely differentiable, in the complex sense. In particular, f' exists, and so f is holomorphic in our sense.

PROOF. Assuming that f is analytic, as in the statement, we have:

$$f'(x+t)=\sum_{n=1}^\infty nc_nt^{n-1}$$

Moreover, the radius of convergence is the same, as shown by the following computation, using the Cauchy formula for the convergence radius, and $\sqrt[n]{n}\to1$:

$$\frac{1}{R'}=\limsup_{n\to\infty}\sqrt[n]{|nc_n|}=\limsup_{n\to\infty}\sqrt[n]{|c_n|}=\frac{1}{R}$$

Thus f' exists and is analytic, on the same domain, and this gives the result. □

Our goal in what follows will be that of proving that any holomorphic function is analytic. This is something quite subtle, which cannot be proved with bare hands, and requires lots of preliminaries. Getting to these preliminaries now, our claim is that a lot of useful knowledge, in order to deal with the holomorphic functions, can be gained by further studying the analytic functions, and even the usual polynomials $P \in \mathbb{C}[X]$.

So, let us further study the polynomials $P \in \mathbb{C}[X]$, and other analytic functions. We already know from chapter 1 that in the polynomial case, $P \in \mathbb{C}[X]$, some interesting things happen, because any such polynomial has a root, and even $\deg(P)$ roots, after a recurrence. Keeping looking at polynomials, with the same methods, we are led to:

THEOREM 5.10. *Any polynomial $P \in \mathbb{C}[X]$ satisfies the maximum principle, in the sense that given a domain D, with boundary γ, we have:*

$$\exists x \in \gamma \quad , \quad |P(x)| = \max_{y \in D} |P(y)|$$

That is, the maximum of $|P|$ over a domain is attained on its boundary.

PROOF. In order to prove this, we can split D into connected components, and then assume that D is connected. Moreover, we can assume that D has no holes, and so is homeomorphic to a disk, and even assume that D itself is a disk. But with this assumption made, the result follows from by contradiction, by using the same arguments as in the proof of the existence of a root, from chapter 1. To be more precise, assume $\deg P \geq 1$, and that the maximum of $|P|$ is attained at the center of a disk $D = D(z, r)$:

$$|P(z)| = \max_{x \in D} |P(x)|$$

We can write then $P(z + t) \simeq P(z) + ct^k$ with $c \neq 0$, for t small, and by suitably choosing the argument of t on the unit circle we conclude, exactly as in chapter 1, that the function $|P|$ cannot have a local maximum at z, as stated. ☐

A good explanation for the fact that the maximum principle holds for polynomials $P \in \mathbb{C}[X]$ could be that the values of such a polynomial inside a disk can be recovered from its values on the boundary. And fortunately, this is indeed the case, and we have:

THEOREM 5.11. *Given a polynomial $P \in \mathbb{C}[X]$, and a disk D, with boundary γ, we have the following formulae, with the integrations being the normalized, mass 1 ones:*

(1) *P satisfies the plain mean value formula $P(x) = \int_D P(y)dy$.*
(2) *P satisfies the boundary mean value formula $P(x) = \int_\gamma P(y)dy$.*

PROOF. As a first observation, the two mean value formulae in the statement are equivalent, by restricting the attention to disks D, having as boundaries circles γ, and using annuli and polar coordinates for the proof of the equivalence. As for the formulae themselves, these can be checked by direct computation for a disk D, with the formulation in (2) being the most convenient. Indeed, for a monomial $P(x)=x^n$ we have:

$$\begin{aligned}\int_\gamma y^n dy &= \frac{1}{2\pi}\int_0^{2\pi}(x+re^{it})^n dt\\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^n\binom{n}{k}x^k(re^{it})^{n-k}dt\\ &= \sum_{k=0}^n\binom{n}{k}x^kr^{n-k}\frac{1}{2\pi}\int_0^{2\pi}e^{i(n-k)t}dt\\ &= \sum_{k=0}^n\binom{n}{k}x^kr^{n-k}\delta_{kn}\\ &= x^n\end{aligned}$$

Here we have used the following key identity, valid for any exponent $m\in\mathbb{Z}$:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}e^{imt}dt &= \frac{1}{2\pi}\int_0^{2\pi}\cos(mt)+i\sin(mt)dt\\ &= \delta_{m0}+i\cdot 0\\ &= \delta_{m0}\end{aligned}$$

Thus, we have the result for monomials, and the general case follows by linearity. □

All the above is very nice, but we can in fact do even better, with a more powerful integration formula. Let us start with some preliminaries. We first have:

PROPOSITION 5.12. *We can integrate functions f over curves γ by setting*

$$\int_\gamma f(x)dx=\int_a^b f(\gamma(t))\gamma'(t)dt$$

with this quantity being independent on the parametrization $\gamma:[a,b]\to\mathbb{C}$.

PROOF. We must prove that the quantity in the statement is independent on the parametrization. In other words, we must prove that if we pick two different parametrizations $\gamma,\eta:[a,b]\to\mathbb{C}$ of our curve, then we have the following formula:

$$\int_a^b f(\gamma(t))\gamma'(t)dt=\int_a^b f(\eta(t))\eta'(t)dt$$

But for this purpose, let us write $\gamma=\eta\phi$, with $\phi:[a,b]\to[a,b]$ being a certain function, that we can assume to be bijective, via an elementary cut-and-paste argument.

By using the chain rule for derivatives, and the change of variable formula, we have:

$$\begin{aligned}\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b f(\eta\phi(t))(\eta\phi)'(t)dt\\ &= \int_a^b f(\eta\phi(t))\eta'(\phi(t))\phi'(t)dt\\ &= \int_a^b f(\eta(t))\eta'(t)dt\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The main properties of the above integration method are as follows:

PROPOSITION 5.13. *We have the following formula, for a union of paths:*

$$\int_{\gamma\cup\eta} f(x)dx = \int_\gamma f(x)dx + \int_\eta f(x)dx$$

Also, when reversing the path, the integral changes its sign.

PROOF. Here the first assertion is clear from definitions, and the second assertion comes from the change of variable formula, by using Proposition 5.12. □

Now by getting back to polynomials, we have the following result:

THEOREM 5.14. *Any polynomial $P\in\mathbb C[X]$ satisfies the Cauchy formula*

$$P(x)=\frac{1}{2\pi i}\int_\gamma\frac{P(y)}{y-x}\,dy$$

with the integration over γ being constructed as above.

PROOF. This follows by using abstract arguments and computations similar to those in the proof of Theorem 5.11. Indeed, by linearity we can assume $P(x)=x^n$. Also, by using a cut-and-paste argument, we can assume that we are on a circle:

$$\gamma:[0,2\pi]\to\mathbb C\quad,\quad\gamma(t)=x+re^{it}$$

By using now the computation from the proof of Theorem 5.11, we obtain:

$$\begin{aligned}\int_\gamma\frac{y^n}{y-x}\,dy &= \int_0^{2\pi}\frac{(x+re^{it})^n}{re^{it}}\,rie^{it}dt\\ &= i\int_0^{2\pi}(x+re^{it})^n dt\\ &= i\cdot 2\pi x^n\end{aligned}$$

Thus, we are led to the formula in the statement. □

5c. Cauchy formula

All the above is quite interesting, and obviously, we are now into serious mathematics. Importantly, Theorem 5.10, Theorem 5.11 and Theorem 5.14 provide us with a path for proving the converse of Theorem 5.9. Indeed, if we manage to prove the Cauchy formula for any holomorphic function $f : X \to \mathbb{C}$, then it will follow that our function is in fact analytic, and so infinitely differentiable. So, let us start with the following result:

THEOREM 5.15. *The Cauchy formula, namely*

$$f(x) = \frac{1}{2\pi i}\int_\gamma \frac{f(y)}{y-x}\,dy$$

holds for any holomorphic function $f : X \to \mathbb{C}$.

PROOF. This is something standard, which can be proved as follows:

(1) Our first claim is that given $f \in H(X)$, with $f' \in C(X)$, the integral of f' vanishes on any path. Indeed, by using the change of variable formula, we have:

$$\begin{aligned}\int_\gamma f'(x)dx &= \int_a^b f'(\gamma(t))\gamma'(t)dt\\ &= f(\gamma(b)) - f(\gamma(a))\\ &= 0\end{aligned}$$

(2) Our second claim is that given $f \in H(X)$ and a triangle $\Delta \subset X$, we have:

$$\int_\Delta f(x)dx = 0$$

Indeed, let us call $\Delta = ABC$ our triangle. Now consider the midpoints A', B', C' of the edges BC, CA, AB, and then consider the following smaller triangles:

$$\Delta_1 = AC'B' \quad , \quad \Delta_2 = BA'C' \quad , \quad \Delta_3 = CB'A' \quad , \quad \Delta_4 = A'B'C'$$

These smaller triangles partition then Δ, and due to our above conventions for the vertex ordering, which produce cancellations when integrating over them, we have:

$$\int_\Delta f(x)dx = \sum_{i=1}^4 \int_{\Delta_i} f(x)dx$$

Thus we can pick, among the triangles Δ_i, a triangle $\Delta^{(1)}$ such that:

$$\left|\int_\Delta f(x)dx\right| \leq 4\left|\int_{\Delta^{(1)}} f(x)dx\right|$$

In fact, by repeating the procedure, we obtain triangles $\Delta^{(n)}$ such that:

$$\left|\int_\Delta f(x)dx\right| \leq 4^n\left|\int_{\Delta^{(n)}} f(x)dx\right|$$

(3) Now let z be the limiting point of these triangles $\Delta^{(n)}$, and fix $\varepsilon > 0$. By using the fact that the functions $1, x$ integrate over paths up to 0, coming from (1), we obtain the following estimate, with $n \in \mathbb{N}$ being big enough, and L being the perimeter of Δ:

$$\begin{aligned}\left|\int_{\Delta^{(n)}} f(x)dx\right| &= \left|\int_{\Delta^{(n)}} f(x) - f(z) - f'(z)(x-z)dx\right|\\ &\leq \int_{\Delta^{(n)}} |f(x) - f(z) - f'(z)(x-z)|\, dx\\ &\leq \int_{\Delta^{(n)}} \varepsilon|x-z|dx\\ &\leq \varepsilon(2^{-n}L)^2\end{aligned}$$

Now by combining this with the estimate in (2), this proves our claim.

(4) The rest is quite routine. First, we can pass from triangles to boundaries of convex sets, in a straightforward way, with the same conclusion as in (2), namely:

$$\int_\gamma f(x)dx = 0$$

Getting back to what we want to prove, namely the Cauchy formula for an arbitrary holomorphic function $f \in H(X)$, let $x \in X$, and consider the following function:

$$g(y) = \begin{cases}\frac{f(y)-f(x)}{y-x} & (y \neq x)\\ f'(x) & (y = x)\end{cases}$$

Now assuming that γ encloses a convex set, we can apply what we found, namely vanishing of the integral, to this function g, and we obtain the Cauchy formula for f.

(5) Finally, the extension to general curves is standard, and standard as well is the discussion of what exactly happens at x, in the above proof. See Rudin [**79**]. □

As a main application of the Cauchy formula, we have:

THEOREM 5.16. *The following conditions are equivalent, for a function $f : X \to \mathbb{C}$:*
(1) *f is holomorphic.*
(2) *f is infinitely differentiable.*
(3) *f is analytic.*
(4) *The Cauchy formula holds for f.*

PROOF. This is routine from what we have, the idea being as follows:

(1) $\Longrightarrow$ (4) is non-trivial, but we know this from Theorem 5.15.

(4) $\Longrightarrow$ (3) is something trivial, because we can expand the series in the Cauchy formula, and we conclude that our function is indeed analytic.

(3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are both elementary, known from Theorem 5.9. □

As another application of the Cauchy formula, we have:

THEOREM 5.17. *Any holomorphic function $f : X \to \mathbb{C}$ satisfies the maximum principle, in the sense that given a domain D, with boundary γ, we have:*

$$\exists x \in \gamma \quad , \quad |f(x)| = \max_{y \in D} |f(y)|$$

That is, the maximum of $|f|$ over a domain is attained on its boundary.

PROOF. This follows indeed from the Cauchy formula. Observe that the converse is not true, for instance because functions like $\bar{x}$ satisfy too the maximum principle. We will be back to this later in this book, when talking about harmonic functions. □

As before with polynomials, a good explanation for the fact that the maximum principle holds could be that the values of our function inside a disk can be recovered from its values on the boundary. And fortunately, this is indeed the case, and we have:

THEOREM 5.18. *Given an holomorphic function $f : X \to \mathbb{C}$, and a disk D, with boundary γ, the following happen:*

(1) *f satisfies the plain mean value formula $f(x) = \int_D f(y)dy$.*
(2) *f satisfies the boundary mean value formula $f(x) = \int_\gamma f(y)dy$.*

PROOF. As usual, this follows from the Cauchy formula, with of course some care in passing from integrals constructed as in Proposition 5.12 to integrals viewed as averages, which are those that we refer to, in the present statement. □

Finally, as yet another application of the Cauchy formula, which is something nice-looking and conceptual, we have the following statement, called Liouville theorem:

THEOREM 5.19. *An entire, bounded holomorphic function*

$$f : \mathbb{C} \to \mathbb{C} \quad , \quad |f| \leq M$$

must be constant. In particular, knowing $f \to 0$ with $z \to \infty$ gives $f = 0$.

PROOF. This follows as usual from the Cauchy formula, namely:

$$f(x) = \frac{1}{2\pi i} \int_\gamma \frac{f(y)}{y - x} \, dy$$

Alternatively, we can view this as a consequence of Theorem 5.18, because given two points $x \neq y$, we can view the values of f at these points as averages over big disks centered at these points, say $D = D_x(R)$ and $E = D_y(R)$, with $R >> 0$:

$$f(x) = \int_D f(z)dz \quad , \quad f(y) = \int_E f(z)dz$$

Indeed, the point is that when the radius goes to ∞, these averages tend to be equal, and so we have $f(x) \simeq f(y)$, which gives $f(x) = f(y)$ in the limit. □

5d. Stieltjes inversion

We would like to end this chapter with an interesting and useful application of the complex functions to basic probability theory, which is as follows:

THEOREM 5.20. *The density of a real probability measure μ can be recaptured from the sequence of moments $\{M_k\}_{k\geq 0}$ via the Stieltjes inversion formula*

$$d\mu(x)=\lim_{t\searrow 0}-\frac{1}{\pi}\,Im\left(G(x+it)\right)\cdot dx$$

where the function on the right, given in terms of moments by

$$G(\xi)=\xi^{-1}+M_1\xi^{-2}+M_2\xi^{-3}+\ldots$$

is the Cauchy transform of the measure μ.

PROOF. The Cauchy transform of our measure μ is given by:

$$\begin{aligned}
G(\xi) &= \xi^{-1}\sum_{k=0}^{\infty}M_k\xi^{-k}\\
&= \int_{\mathbb{R}}\frac{\xi^{-1}}{1-\xi^{-1}y}\,d\mu(y)\\
&= \int_{\mathbb{R}}\frac{1}{\xi-y}\,d\mu(y)
\end{aligned}$$

Now with $\xi=x+it$, we obtain the following formula:

$$\begin{aligned}
Im(G(x+it)) &= \int_{\mathbb{R}}Im\left(\frac{1}{x-y+it}\right)d\mu(y)\\
&= \int_{\mathbb{R}}\frac{1}{2i}\left(\frac{1}{x-y+it}-\frac{1}{x-y-it}\right)d\mu(y)\\
&= -\int_{\mathbb{R}}\frac{t}{(x-y)^2+t^2}\,d\mu(y)
\end{aligned}$$

By integrating over $[a,b]$ we obtain, with the change of variables $x=y+tz$:

$$\begin{aligned}
\int_a^b Im(G(x+it))dx &= -\int_{\mathbb{R}}\int_a^b\frac{t}{(x-y)^2+t^2}\,dx\,d\mu(y)\\
&= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{t}{(tz)^2+t^2}\,t\,dz\,d\mu(y)\\
&= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{1}{1+z^2}\,dz\,d\mu(y)\\
&= -\int_{\mathbb{R}}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)d\mu(y)
\end{aligned}$$

Now observe that with $t \searrow 0$ we have:

$$\lim_{t\searrow 0}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)=\begin{cases}\frac{\pi}{2}-\frac{\pi}{2}=0 & (y<a)\\ \frac{\pi}{2}-0=\frac{\pi}{2} & (y=a)\\ \frac{\pi}{2}-(-\frac{\pi}{2})=\pi & (a<y<b)\\ 0-(-\frac{\pi}{2})=\frac{\pi}{2} & (y=b)\\ -\frac{\pi}{2}-(-\frac{\pi}{2})=0 & (y>b)\end{cases}$$

We therefore obtain the following formula:

$$\lim_{t\searrow 0}\int_a^b Im(G(x+it))dx=-\pi\left(\mu(a,b)+\frac{\mu(a)+\mu(b)}{2}\right)$$

Thus, we are led to the conclusion in the statement. □

Before getting further, let us mention that the above result does not fully solve the moment problem, because we still have the question of understanding when a sequence of numbers $M_1, M_2, M_3, \ldots$ can be the moments of a measure μ. We have here:

THEOREM 5.21. *A sequence of numbers $M_0, M_1, M_2, M_3, \ldots \in \mathbb{R}$, with $M_0 = 1$, is the series of moments of a real probability measure μ precisely when:*

$$\begin{vmatrix}M_0\end{vmatrix}\geq 0 \quad , \quad \begin{vmatrix}M_0 & M_1\\ M_1 & M_2\end{vmatrix}\geq 0 \quad , \quad \begin{vmatrix}M_0 & M_1 & M_2\\ M_1 & M_2 & M_3\\ M_2 & M_3 & M_4\end{vmatrix}\geq 0 \quad , \quad \ldots$$

That is, the associated Hankel determinants must be all positive.

PROOF. This is something a bit more advanced, the idea being as follows:

(1) As a first observation, the positivity conditions in the statement tell us that the following associated linear forms must be positive:

$$\sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}\geq 0$$

(2) But this is something very classical, in one sense the result being elementary, coming from the following computation, which shows that we have positivity indeed:

$$\int_{\mathbb{R}}\left|\sum_{i=1}^n c_ix^i\right|^2 d\mu(x)=\int_{\mathbb{R}}\sum_{i,j=1}^n c_i\bar{c}_jx^{i+j}d\mu(x)=\sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}$$

(3) As for the other sense, here the result comes once again from the above formula, this time via some standard functional analysis, that we will leave as an exercise. □

Getting back now to more concrete things, the point is that we have:

FACT 5.22. *Given a graph X, with distinguished vertex $*$, we can talk about the probability measure μ having as k-th moment the number of length k loops based at $*$:*

$$M_k = \left\{ * - i_1 - i_2 - \ldots - i_k = * \right\}$$

As basic examples, for the graph $\mathbb{N}$ the moments must be the Catalan numbers C_k, and for the graph $\mathbb{Z}$, the moments must be the central binomial coefficients D_k.

To be more precise, the first assertion, regarding the existence and uniqueness of μ, follows from a basic linear algebra computation, by diagonalizing the adjacency matrix of X. As for the examples, we will leave them as an instructive exercise.

Needless to say, counting loops on graphs, as in Fact 5.22, is something important in applied mathematics, and physics. So, back to our business now, motivated by all this, as a basic application of the Stieltjes formula, let us solve the moment problem for the Catalan numbers C_k, and for the central binomial coefficients D_k. We first have:

THEOREM 5.23. *The real measure having as even moments the Catalan numbers, $C_k = \frac{1}{k+1}\binom{2k}{k}$, and having all odd moments 0 is the measure*

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

called Wigner semicircle law on $[-2,2]$.

PROOF. In order to apply the inversion formula, our starting point will be the well-known formula for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^{\infty} C_k z^k = \frac{1-\sqrt{1-4z}}{2z}$$

By using this formula with $z = \xi^{-2}$, we obtain the following formula:

$$\begin{aligned} G(\xi) &= \xi^{-1}\sum_{k=0}^{\infty} C_k \xi^{-2k} \\ &= \xi^{-1} \cdot \frac{1-\sqrt{1-4\xi^{-2}}}{2\xi^{-2}} \\ &= \frac{\xi}{2}\left(1-\sqrt{1-4\xi^{-2}}\right) \\ &= \frac{\xi}{2} - \frac{1}{2}\sqrt{\xi^2-4} \end{aligned}$$

Now let us apply Theorem 5.20. The study here goes as follows:

(1) According to the general philosophy of the Stieltjes formula, the first term, namely $\xi/2$, which is “trivial”, will not contribute to the density.

(2) As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{\xi^2-4}$ will be replaced by the "dual" square root $\sqrt{4-x^2}\,dx$, and that we have to multiply everything by $-1/\pi$.

(3) As a conclusion, by Stieltjes inversion we obtain the following density:

$$d\mu(x)=-\frac{1}{\pi}\cdot-\frac{1}{2}\sqrt{4-x^2}\,dx=\frac{1}{2\pi}\sqrt{4-x^2}dx$$

Thus, we have obtained the mesure in the statement, and we are done. □

We have the following version of the above result:

THEOREM 5.24. *The real measure having as sequence of moments the Catalan numbers, $C_k=\frac{1}{k+1}\binom{2k}{k}$, is the measure*

$$\pi_1=\frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

called Marchenko-Pastur law on $[0,4]$.

PROOF. As before, we use the standard formula for the generating series of the Catalan numbers. With $z=\xi^{-1}$ in that formula, we obtain the following formula:

$$\begin{aligned}G(\xi)&=\xi^{-1}\sum_{k=0}^\infty C_k\xi^{-k}\\&=\xi^{-1}\cdot\frac{1-\sqrt{1-4\xi^{-1}}}{2\xi^{-1}}\\&=\frac{1}{2}\left(1-\sqrt{1-4\xi^{-1}}\right)\\&=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\xi^{-1}}\end{aligned}$$

With this in hand, let us apply now the Stieltjes inversion formula, from Theorem 5.20. We obtain, a bit as before in Theorem 5.23, the following density:

$$d\mu(x)=-\frac{1}{\pi}\cdot-\frac{1}{2}\sqrt{4x^{-1}-1}\,dx=\frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

Thus, we are led to the conclusion in the statement. □

Regarding now the central binomial coefficients, we have here:

THEOREM 5.25. *The real probability measure having as moments the central binomial coefficients, $D_k=\binom{2k}{k}$, is the measure*

$$\alpha_1=\frac{1}{\pi\sqrt{x(4-x)}}\,dx$$

called arcsine law on $[0,4]$.

PROOF. We have the following computation, using some standard formulae:

$$\begin{aligned}
G(\xi) &= \xi^{-1}\sum_{k=0}^\infty D_k\xi^{-k}\\
&= \frac{1}{\xi}\sum_{k=0}^\infty D_k\left(-\frac{t}{4}\right)^k\\
&= \frac{1}{\xi}\cdot\frac{1}{\sqrt{1-4/\xi}}\\
&= \frac{1}{\sqrt{\xi(\xi-4)}}
\end{aligned}$$

But this gives the density in the statement, via Theorem 5.20. □

Finally, we have the following version of the above result:

THEOREM 5.26. *The real probability measure having as moments the middle binomial coefficients,* $E_k=\binom{k}{[k/2]}$, *is the following law on* $[-2,2]$,

$$\sigma_1=\frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\,dx$$

called modified the arcsine law on $[-2,2]$.

PROOF. In terms of the central binomial coefficients D_k, we have:

$$E_{2k}=\binom{2k}{k}=\frac{(2k)!}{k!k!}=D_k$$

$$E_{2k-1}=\binom{2k-1}{k}=\frac{(2k-1)!}{k!(k-1)!}=\frac{D_k}{2}$$

Standard calculus based on the Taylor formula for $(1+t)^{-1/2}$ gives:

$$\frac{1}{2x}\left(\sqrt{\frac{1+2x}{1-2x}}-1\right)=\sum_{k=0}^\infty E_kx^k$$

With $x=\xi^{-1}$ we obtain the following formula for the Cauchy transform:

$$\begin{aligned} G(\xi) &= \xi^{-1}\sum_{k=0}^\infty E_k\xi^{-k} \\ &= \frac{1}{\xi}\left(\sqrt{\frac{1+2/\xi}{1-2/\xi}}-1\right) \\ &= \frac{1}{\xi}\left(\sqrt{\frac{\xi+2}{\xi-2}}-1\right) \end{aligned}$$

By Stieltjes inversion we obtain the density in the statement. □

All this is very nice, and we are obviously building here, as this book goes by, some good knowledge in probability theory. We will be back to this later.

5e. Exercises

This was a standard introduction to complex functions, and as exercises, we have:

Exercise 5.27. *Learn some more about partial, and radial derivatives.*

Exercise 5.28. *Learn as well more about the Cauchy-Riemann operators.*

Exercise 5.29. *Review if needed the convergence radius basics, from real analysis.*

Exercise 5.30. *Learn more, say from physicists, about integrating over curves.*

Exercise 5.31. *Fill in the details at the end, in the proof of the Cauchy formula.*

Exercise 5.32. *Work out some alternative proofs for the Liouville theorem.*

Exercise 5.33. *Learn more about the moment problem, and Hankel determinants.*

Exercise 5.34. *Learn also more about the Catalan numbers, and their properties.*

As bonus exercise, have a look at harmonic functions too. More on these later.

CHAPTER 6

Residue formula

6a. Rational functions

Rational functions.

6b. Meromorphic functions

Meromorphic functions.

6c. Residue formula

Residue formula.

6d. Basic applications

Basic applications.

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Bonus exercise.

CHAPTER 7

Analytic continuation

7a. Regular points

Regular points.

7b. Analytic continuation

Analytic continuation.

7c. Monodromy theorem

Monodromy theorem.

7d. Picard theorem

Picard theorem.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

Some applications

8a. Back to zeta

Time to review the material from the end of chapter 4, regarding the zeta function, and further build on that. We recall that ζ is defined for $Re(s) > 1$ according to:

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

To be more precise, the convergence comes as follows, with $s = r + it$ with $r > 1$:

$$\begin{aligned} |\zeta(s)| &\leq \sum_{n=1}^\infty \frac{1}{|n^s|} \\ &= \sum_{n=1}^\infty \frac{1}{n^r} \\ &< 1 + \int_1^\infty \frac{1}{x^r}\,dx \\ &= 1 + \frac{1}{r-1} \end{aligned}$$

Let us recall as well that, in relation with prime numbers, and with arithmetic in general, we can write zeta as an Euler product, as follows:

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^\infty \frac{1}{n^s} \\ &= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right) \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \end{aligned}$$

Coming next, we have the following useful formula for the zeta function:

THEOREM 8.1. *The inverse of the zeta function is given by*

$$\frac{1}{\zeta(s)}=\sum_{n=1}^\infty\frac{\mu(n)}{n^s}$$

with μ being the Möbius function of the integers, given by the formula

$$\mu(n)=\begin{cases}(-1)^k&\text{if }n=p_1\ldots p_k\\0&\text{if }n\text{ is not square}-\text{free}\end{cases}$$

with this being valid for any exponent $s\in\mathbb C$ with $Re(s)>1$.

PROOF. We have the following computation, with everything converging:

$$\begin{aligned}\frac{1}{\zeta(s)}&=\prod_p\left(1-\frac{1}{p^s}\right)\\&=\sum_{k=0}^\infty(-1)^k\prod_{p_1\ldots p_k}\frac{1}{p_1^s\ldots p_k^s}\\&=\sum_{n=1}^\infty\frac{\mu(n)}{n^s}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Along the same lines, as another elementary result about zeta, we have:

THEOREM 8.2. *The square of the zeta function is given by*

$$\zeta^2(s)=\sum_{n=1}^\infty\frac{\tau(n)}{n^s}$$

with $\tau(n)$ being the number of divisors of n, for any $s\in\mathbb C$ with $Re(s)>1$.

PROOF. We have the following computation, with everything converging:

$$\zeta(s)^2=\sum_{k=1}^\infty\sum_{l=1}^\infty\frac{1}{(kl)^s}=\sum_{n=1}^\infty\frac{\tau(n)}{n^s}$$

Thus, we are led to the conclusion in the statement. □

There are some further formulae of the same type, again involving the zeta function, and the above function $\tau(n)$, counting the number of divisors of n. First, we have:

$$\frac{\zeta^3(s)}{\zeta(2s)}=\sum_{n=1}^\infty\frac{\tau(n^2)}{n^s}$$

Along the same lines, we have as well the following formula:

$$\frac{\zeta^4(s)}{\zeta(2s)}=\sum_{n=1}^\infty\frac{\tau(n)^2}{n^s}$$

Next, let us recall from chapter 4 that we have the following key formula:

THEOREM 8.3. *We have the following formula,*

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx$$

valid for any $s\in\mathbb{C}$ with $Re(s)>1$.

PROOF. We have indeed the following computation:

$$\begin{aligned}
\int_0^\infty\frac{x^{s-1}}{e^x-1}\,dx
&=\int_0^\infty\frac{x^{s-1}}{e^x}\cdot\frac{1}{1-e^{-x}}\,dx\\
&=\int_0^\infty x^{s-1}(e^{-x}+e^{-2x}+e^{-3x}+\ldots)\\
&=\sum_{n=1}^\infty\int_0^\infty x^{s-1}e^{-nx}\,dx\\
&=\sum_{n=1}^\infty\int_0^\infty\left(\frac{y}{n}\right)^{s-1}e^{-y}\,\frac{dy}{n}\\
&=\sum_{n=1}^\infty\frac{1}{n^s}\int_0^\infty y^{s-1}e^{-y}\,dy\\
&=\zeta(s)\Gamma(s)
\end{aligned}$$

Thus, we are led to the formula in the statement. □

Finally, again at the general level, but in relation this time with the values of zeta at the integers, let us record as well the following result, a bit of physics flavor:

THEOREM 8.4. *We have the following formula,*

$$\zeta(s)=\int_0^1\ldots\int_0^1\frac{dx_1\ldots dx_s}{1-x_1\ldots x_s}$$

valid for any $s\in\mathbb{N}$, $s\geq 2$.

PROOF. This follows as usual from some calculus, the idea being as follows:

(1) At $s=2$ we have indeed the following computation, using Theorem 8.3:

$$\begin{aligned}\int_0^1\int_0^1\frac{1}{1-xy}\,dxdy &= \int_0^1\left[-\frac{\log(1-xy)}{y}\right]_0^1 dy\\ &= -\int_0^1\frac{\log(1-y)}{y}\,dy\\ &= -\int_0^\infty\frac{\log(e^{-t})}{1-e^{-t}}\,e^{-t}dt\\ &= \int_0^\infty\frac{t}{e^t-1}\,dt\\ &= \zeta(2)\Gamma(2)\\ &= \zeta(2)\end{aligned}$$

(2) In the general case, $s\in\mathbb{N}$, the best is to start with the following formula:

$$\frac{1}{1-x_1\ldots x_s}=\sum_{n=0}^\infty(x_1\ldots x_s)^n$$

Thus, the integral in the statement is given by the following formula:

$$\int_0^1\ldots\int_0^1\frac{dx_1\ldots dx_s}{1-x_1\ldots x_s}=\sum_{n=0}^\infty\int_0^1\ldots\int_0^1(x_1\ldots x_s)^n dx_1\ldots dx_s$$

But this eventually leads to the formula in the statement, after some standard calculus computations, that we will leave here as an instructive exercise.

(3) Before leaving, let us see as well, out of mathematical curiosity, what happens at the exponent $s=1$. Here the integral in the statement is:

$$\begin{aligned}\int_0^1\frac{1}{1-x}\,dx &= [-\log(1-x)]_0^1\\ &= -\log(1-1)+\log(1-0)\\ &= \infty+0\\ &= \zeta(1)\end{aligned}$$

Not a big deal, you would say, but as an interesting remark, since $\log(1-x)\simeq -x$, we are led to the conclusion that ζ, when suitably extended by analytic continuation, should have a simple pole at $s=1$, with residue 1. We will be back to this, in a moment. □

Summarizing, we have ζ up and working in the complex domain $Re(s)>1$.

8b. Special values

At a more advanced level now, we can try to compute particular values of ζ. Things are quite tricky here, and we first have the following result, briefly discussed before:

THEOREM 8.5. *We have the following formula, for the even integers $s=2k$,*

$$\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}$$

with B_n being the Bernoulli numbers, which in practice gives the formulae

$$\zeta(2)=\frac{\pi^2}{6}\quad,\quad\zeta(4)=\frac{\pi^4}{90}\quad,\quad\zeta(6)=\frac{\pi^6}{945}\quad,\quad\zeta(8)=\frac{\pi^8}{9450}\quad,\quad\ldots$$

generalizing the formula $\zeta(2)=\pi^2/6$ of Euler, solving the Basel problem.

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, at $s=2$ the Euler computation, from before, was as follows:

$$\begin{aligned}\frac{\sin x}{x}&=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\\&=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\\&=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots\\&=1-\frac{1}{\pi^2}\sum_{n=1}^\infty\frac{1}{n^2}x^2+\ldots\end{aligned}$$

It is possible to use the same idea for dealing with $\zeta(2k)$ with $k\in\mathbb{N}$, but this is quite complicated, and in addition the above method of Euler needs some justification, that we have not really provided before, so in short, not a path to be followed.

(2) Instead, we have the following luminous computation, using Theorem 8.3:

$$\begin{aligned}\zeta(2k)&=\frac{1}{\Gamma(2k)}\int_0^\infty\frac{x^{2k-1}}{e^x-1}\,dx\\&=\frac{1}{(2k-1)!}\int_0^\infty\frac{x^{2k-1}}{e^x-1}\,dx\\&=\frac{1}{(2k-1)!}\int_0^\infty\frac{(2\pi t)^{2k-1}}{e^{2\pi t}-1}\,2\pi dt\\&=\frac{(2\pi)^{2k}}{(2k-1)!}\int_0^\infty\frac{t^{2k-1}}{e^{2\pi t}-1}\,dt\end{aligned}$$

(3) But, we recognize on the right the integral giving rise to the even Bernoulli numbers, with one of the many definitions of these numbers being as follows:

$$B_{2k}=4k(-1)^{k+1}\int_0^\infty\frac{t^{2k-1}}{e^{2\pi t}-1}\,dt$$

Thus, we can finish our computation of the values $\zeta(2k)$ as follows:

$$\begin{aligned}\zeta(2k) &= \frac{(2\pi)^{2k}}{(2k-1)!}\cdot(-1)^{k+1}\frac{B_{2k}}{4k}\\ &= (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}\end{aligned}$$

(4) Regarding now the Bernoulli numbers, there is a long story here. At the beginning, we have the following formula of Bernoulli, standing as a definition for them:

$$\sum_{k=0}^{n-1}k^m=\frac{1}{m+1}\sum_{k=0}^m B_k n^{m+1-k}$$

This leads to the following recurrence relation, which computes them:

$$B_m=-\frac{1}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k}B_k$$

In practice, we can see that the odd Bernoulli numbers all vanish, except for the first one, $B_1=-1/2$, and that the even Bernoulli numbers are as follows:

$$\frac{1}{6}\ ,\quad -\frac{1}{30}\ ,\quad \frac{1}{42}\ ,\quad -\frac{1}{30}\ ,\quad \frac{5}{66}\ ,\quad -\frac{691}{2730}\ ,\quad \frac{7}{6}\ ,\quad \ldots$$

(5) For analytic purposes, the Bernoulli numbers are best viewed as follows, with this coming from the fact that the coefficients satisfy the above recurrence relation:

$$\begin{aligned}\frac{x}{e^x-1} &= \sum_{n=0}^\infty B_n\frac{x^n}{n!}\\ &= 1-\frac{1}{2}x+\frac{1}{6}\cdot\frac{x^2}{2!}-\frac{1}{30}\cdot\frac{x^4}{4!}+\frac{1}{42}\cdot\frac{x^6}{6!}-\frac{1}{30}\cdot\frac{x^8}{8!}+\ldots\end{aligned}$$

Observe that all this is related as well to the hyperbolic functions, via:

$$\frac{x}{2}\left(\coth\frac{x}{2}-1\right)=\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}$$

The point now is that, in relation with our zeta business, the above analytic formulae give, after some calculus, the formula that we used in (3), namely:

$$B_{2k}=4k(-1)^{k+1}\int_0^\infty\frac{t^{2k-1}}{e^{2\pi t}-1}\,dt$$

(6) Finally, no discussion about the Bernoulli numbers would be complete without mentioning the Euler-Maclaurin formula, involving them, which is as follows:

$$\begin{aligned}\sum_{k=0}^{n-1} f(x) &\simeq \int_0^n f(x)dx - \frac{1}{2}(f(n)-f(0)) \\ &+\frac{1}{6}\cdot\frac{f'(n)-f'(0)}{2!} - \frac{1}{30}\cdot\frac{f^{(3)}(n)-f^{(3)}(0)}{4!} \\ &+\frac{1}{42}\cdot\frac{f^{(5)}(n)-f^{(5)}(0)}{6!} - \frac{1}{30}\cdot\frac{f^{(7)}(n)-f^{(7)}(0)}{8!}+\ldots\end{aligned}$$

(7) And there is more coming from the complex extension of the zeta function, by analytic continuation, that we will discuss later. An an announcement here, the values of zeta at the negative integers $0,-1,-2,-3,\ldots$ will not be ∞, but rather given by:

$$\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}$$

Alternatively, we have the following formula for the Bernoulli numbers:

$$B_n = (-1)^{n-1}n\zeta(1-n)$$

(8) In any case, we are led to the various conclusions in the statement, both theoretical and numeric. And exercise for you of course to learn more about the Bernoulli numbers, and beware of the freakish notations used by mathematicians there. □

As a more digest form of Theorem 8.5, let us record as well:

Theorem 8.6. *The generating function of the numbers $\zeta(2k)$ with $k\in\mathbb{N}$ is*

$$\sum_{k=0}^\infty \zeta(2k)x^{2k} = -\frac{\pi x}{2}\cot(\pi x)$$

and with this generalizing the formula involving Bernoulli numbers.

Proof. This is something tricky, again, the idea being as follows:

(1) A version of the recurrence formula for Bernoulli numbers is as follows:

$$B_{2n} = -\frac{1}{n+1/2}\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}$$

Now observe that this formula can be written in the following way:

$$\frac{B_{2n}}{(2n)!} = -\frac{1}{n+1/2}\sum_{k=1}^{n-1}\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2n-2k}}{(2n-2k)!}$$

In view of Theorem 8.5, we obtain the following formula, valid at any $n > 1$:

$$\zeta(2n) = \frac{1}{n+1/2}\sum_{k=1}^{n-1}\zeta(2k)\zeta(2n-2k)$$

(2) But this allows the computation of the series in the statement, by squaring that series. Indeed, consider the following modified version of that series:

$$f(x) = 2\sum_{k=0}^{\infty}\zeta(2k)\left(\frac{x}{\pi}\right)^{2k}$$

By squaring, and using the recurrence formula for the numbers $\zeta(2n)$ found in (1), with some care at the values $n = 0, 1$, not covered by that formula, we obtain:

$$f^2 + f + x^2 = xf'$$

(3) But this is precisely the functional equation satisfied by $g(x) = -x\cot x$. Indeed, by using the well-known formula $\cot' = -\cot^2 - 1$, we have:

$$\begin{aligned} xg' &= x(-\cot x - x\cot' x) \\ &= x(-\cot x + x\cot^2 x + x) \\ &= g + g^2 + x^2 \end{aligned}$$

(4) We conclude that we have $f = g$, which reads:

$$2\sum_{k=0}^{\infty}\zeta(2k)\left(\frac{x}{\pi}\right)^{2k} = -x\cot x$$

Now by replacing $x \to \pi x$, we obtain the formula in the statement. □

Regarding now the values $\zeta(2k+1)$ with $k \in \mathbb{N}$, the story here is more complicated, with the first such number being the Apéry constant, given by:

$$\zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

There has been a lot of work on this number, by Apéry and others, and on the higher $\xi(2k+1)$ values as well. We will be back to this, later in this chapter.

Many other things can be said about ζ and its special values, as a continuation of the above, and you can check here any advanced number theory book. In what concerns us, we will rather head towards the remaining domain $Re(s) \leq 1$, using complex analysis.

8c. Riemann formula

Quite remarkably, with a bit of complex analysis, we can have the zeta function working in the whole complex plane, via analytic continuation. However, analytic continuation being Devil's business, we will explain this slowly, by gradually going from the analytic right half-plane $Re(s) > 1$, that we understand well, to other parts of $\mathbb{C}$.

Getting started, let us first see what happens at $s = 1$. Here we have:

PROPOSITION 8.7. *We have the following formula,*

$$\lim_{s\to 1}(s-1)\zeta(s) = 1$$

showing that the complex zeta has a simple pole at $s = 1$, with residue 1.

PROOF. We have the following computation, using $\Gamma(1) = 1$:

$$\begin{aligned}\lim_{s\to 1}(s-1)\zeta(s) &= \lim_{s\to 1}(s-1)\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx \\ &= \lim_{t\to 0}\int_0^\infty \frac{tx^t}{e^x-1}\,dx \\ &= 1\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a more advanced result now, on the same topic, we have:

THEOREM 8.8. *We have the following formula,*

$$\lim_{s\to 1}\left|\zeta(s) - \frac{1}{s-1}\right| = \gamma$$

with γ being the Euler-Mascheroni constant.

PROOF. This is something more advanced, the idea being as follows:

(1) The Euler-Mascheroni constant is related to the zeta function by:

$$\gamma = \sum_{n=2}^\infty (-1)^n \frac{\zeta(n)}{n}$$

(2) On the other hand, we have we well the following formula:

$$\gamma = \lim_{s\to 1^+}\sum_{n=1}^\infty \frac{1}{n^s} - \frac{1}{s^n}$$

But in terms of the zeta function, this latter formula simply reads:

$$\gamma = \lim_{s\to 1^+}\zeta(s) - \frac{1}{s-1}$$

(3) Thus, we are led to the formula in the statement. Note that we have as well:

$$\gamma = \lim_{s\to 0} \frac{\zeta(1+s)+\zeta(1-s)}{2}$$

Indeed, this follows from the formula in the statement. □

Leaving aside now $s=1$, let us focus on the other points, $s=1+it$ with $t\neq 0$, of the boundary line $Re(s)=1$, between known and unknown. We have here:

THEOREM 8.9. *The Riemann zeta function, namely*

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

converges at any $s = 1+it$ *with* $t\neq 0$.

PROOF. We have the following computation, to start with:

$$\begin{aligned}\zeta(1+it) &= \sum_{n=1}^\infty \frac{1}{n^{1+it}}\\ &= \sum_{n=1}^\infty \frac{1}{ne^{it\log n}}\\ &= \sum_{n=1}^\infty \frac{e^{-it\log n}}{n}\\ &= \sum_{n=1}^\infty \frac{\cos(t\log n) - i\sin(t\log n)}{n}\end{aligned}$$

And then, the convergence at $t\neq 0$ can be proved via some calculus. □

Let us get now into the true unknown, $Re(s)<1$, with our first objective being that of understanding what happens in the strip $0<Re(s)<1$. We first have here:

PROPOSITION 8.10. *Unlike the standard Riemann series, which diverges,*

$$\zeta(1) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots = \infty$$

the signed version of this series, called standard Dirichlet series, converges,

$$\eta(1) = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots < \infty$$

and we can even compute its value, $\eta(1) = \log 2$.

PROOF. Here the convergence of the series $\eta(1)$ can be proved in a variety of ways, for instance by grouping terms and comparing to $\zeta(2) < \infty$:

$$\eta(1) = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \ldots < \zeta(2) < \infty$$

As for the exact formula of $\eta(1)$, this follows from the Taylor formula for log:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots$$

Indeed, by plugging in $x = 1$, we obtain the formula in the statement. □

Thus, we have our idea, "forcing" zeta to converge in the strip $0 < Re(s) < 1$, by adding signs, and then recovering zeta, or rather its analytic continuation, in this same strip, by removing the signs. This leads to the following remarkable result:

THEOREM 8.11. *We have the following formula,*

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}$$

which can stand as definition for ζ, in the strip $0 < Re(s) < 1$.

PROOF. This is something elementary, known since Dirichlet and Euler, but of key importance, and with many consequences, the idea being as follows:

(1) To start with, we can define the Dirichlet function η as being the signed version of ζ, exactly as we did in Proposition 8.10 at $s = 1$, as follows:

$$\eta(s) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}$$

Observe that this function converges indeed in the strip $0 < Re(s) < 1$.

(2) We must now connect ζ and η, at $Re(s) > 1$, and this can be done as follows:

$$\begin{aligned}
\zeta(s) + \eta(s) &= \sum_{n=1}^\infty \frac{1}{n^s} + \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s} \\
&= 2\sum_{k=1}^\infty \frac{1}{(2k)^s} \\
&= 2^{1-s} \sum_{k=1}^\infty \frac{1}{k^s} \\
&= 2^{1-s}\zeta(s)
\end{aligned}$$

(3) But this gives the following formula, valid at any exponent $s \in \mathbb{C}$ satisfying $Re(s) > 1$, and which is the formula in the statement:

$$\zeta(s) = \frac{1}{1-2^{1-s}}\,\eta(s)$$

(4) In order now to conclude, we can invoke the theory of analytic continuation. Indeed, what we have in the statement is a formula for ζ in the right half-plane, $Re(s) > 0$, which is analytic, and more specifically meromorphic, with a single pole, at $s = 1$, and which coincides with the usual formula of ζ on the usual domain of definition, $Re(s) > 1$. But, in this situation, the theory of analytic continuation tells us that we can redefine ζ all over the right half-plane, $Re(s) > 0$, by the formula in the statement, and with this extension being unique, as per the general properties of the meromorphic functions.

(5) Finally, observe that our present result proves Theorem 8.9 as well. Thinking retrospectively, we were in need there precisely of a Dirichlet type idea. □

Getting now to the left half-plane, $Re(s) < 0$, many methods are available here, and with the main one, due to Riemann himself, which is something quite tough, but unavoidable for understanding the zeta function as a whole, being as follows:

Theorem 8.12. *We have the following formula of Riemann, relating the values of zeta at s and $1-s$,*

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

which holds on the strip $0 < Re(s) < 1$, and can serve as definition for zeta in the left half-plane, $Re(s) < 0$, by analytic continuation.

Proof. This is something subtle, with even understanding the statement being non-trivial business, and with the proof being complicated too, the idea being as follows:

(1) To start with, let us check our formula for mistakes. With $Re(s) > 1$ our formula tells us that the familiar $\zeta(s)$ can be expressed in terms of some virtual number $\zeta(1-s)$, which remains to be defined later, and normally no problem with this.

(2) However, looking more carefully, there might be a problem coming from the sine, which vanishes at $s = 2k$ with $k \in \mathbb{N}$. But, the point is that $\Gamma(1-s)$ has a pole at $s = 2k$, compensating for this vanishing of the sine. So, as a conclusion here, not only we avoided the contradictory $\zeta(2k) = 0$, but also know that, later when it will come to discuss $\zeta(1-2k)$, that will be a usual complex number, with no need for a pole there.

(3) Conversely now, let us plug in numbers with $Re(s) < 0$, so that $Re(1-s) > 1$. Here what our formula tells us is that the familiar $\zeta(1-s)$, when multiplied by the quantities in the statement, produces a candidate $\zeta(s)$ for the analytic continuation in the left half-plane $Re(s) < 0$. So, very good, no contradiction whatsoever here, and in addition this tells us, confirming the finding in (2), that zeta will have no poles at $Re(s) < 0$.

(4) Now let us have a look at the strip $0 < Re(s) < 1$. Here our function ζ is already existent, thanks to Theorem 8.11, and we have something to prove, namely that the Riemann formula in the statement holds indeed, in this strip $0 < Re(s) < 1$.

(5) But this is something that can be proved indeed, via some non-trivial calculus, done by Riemann a long time ago, and which has been barely simplified, since. In order to get started, we use the following formula for the gamma function:

$$\Gamma\left(\frac{s}{2}\right) = n^s \pi^{\frac{s}{2}} \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx$$

(6) Thus, we are led to the following formula for the zeta function:

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{\frac{s}{2}} \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x} dx \\ &= \pi^{\frac{s}{2}} \int_0^\infty x^{\frac{s}{2}-1} \sum_{n=1}^\infty e^{-n^2\pi x} dx\end{aligned}$$

(7) Now let us call Ψ the function appearing on the right, namely:

$$\Psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x} dx$$

With this convention, the formula that we found can be written as follows:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \Psi(x) dx$$

(8) Now let us have a look at the function Ψ. By Poisson summation we obtain:

$$\sum_{n=-\infty}^\infty e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^\infty e^{-\frac{n^2\pi}{x}}$$

We conclude that our function Ψ satisfies the following equation:

$$2\Psi(x) + 1 = \frac{1}{\sqrt{x}}\left(2\Psi\left(\frac{1}{x}\right) + 1\right)$$

(9) With this equation in hand, let us go back to the formula for zeta in (7). We can further process that formula, in the following way:

$$\begin{aligned}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}\Psi(x)dx \\ &= \int_0^1 x^{\frac{s}{2}-1}\Psi(x)dx+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)dx \\ &= \int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\Psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{2}}-\frac{1}{2}\right)dx+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)dx \\ &= \frac{1}{s-1}+\frac{1}{s}+\int_0^1 x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)dx+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)dx\end{aligned}$$

(10) We conclude from this that we have the following formula:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty\left(x^{-\frac{s+1}{2}}+x^{\frac{s}{2}-1}\right)\Psi(x)dx$$

Now since the expression on the right is invariant under $s\to 1-s$, we obtain:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

But with this we are done, because this latter formula is equivalent to the Riemann symmetry formula in the statement, namely:

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

(11) Next, there is some discussion at the border of the strip too, with the formula relating the values at $Re(s)=1$, all finite except for a pole at $s=1$, to the values at $Re(s)=0$, which all follow to be finite, thanks to the mechanism explained in (2).

(12) Now with this done, we can take the formula in the statement as a definition for zeta in the left half-plane, $Re(s)<0$, and with the general theory of analytic continuation telling us, a bit like before, at the end of the proof of Theorem 15.18, that this continuation is unique, thanks to the general properties of the meromorphic functions.

(13) So, this was for the idea of the proof, and in practice, there are of course many details still in need to be checked, and we will leave this as an instructive exercise. □

Observe that, in what regards the Riemann formula itself, this remains a key symmetry formula of our newly defined zeta function, as a meromorphic function over $\mathbb{C}$.

All the above starts to be a bit heavy, and as a summary of all this, we have:

THEOREM 8.13. *We can talk about the Riemann zeta, as a meromorphic function $\zeta : \mathbb{C} \to \mathbb{C}$, with a single pole, at $s = 1$ with residue 1. At $Re(s) > 1$ we have*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and more generally at $Re(s) > 0$ we have the following formula:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

Also, the values of zeta at any s and $1 - s$ are related by the Riemann formula

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

with Γ being as usual the gamma function.

PROOF. This is a summary of our various findings from Theorems 8.11 and 8.12 and their proofs, and with the thing to be always kept in mind, when dealing with all this, being that the formula at $Re(s) > 0$ generalizes indeed the formula at $Re(s) > 1$, thanks to a trivial computation, explained in the proof of Theorem 8.11. □

Getting back now to the Riemann formula from Theorem 8.12, passed the technical difficulties for establishing it, this is something very beautiful and useful, with a lot of symmetry in it, making it clear that the strip $0 < Re(s) < 1$ is what matters, and that the vertical axis $Re(s) = 1/2$ is where interesting things should happen.

As a consequence of the Riemann formula, we have the following version of it:

THEOREM 8.14. *We have the following version of the Riemann formula,*

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

symmetric in $s, 1 - s$, which is in fact equivalent to it.

PROOF. The above formula is indeed equivalent to the one in Theorem 8.12, and is in fact what comes out from computations, when proving Theorem 8.12. □

In practice, the quantity in Theorem 8.14 is best normalized as follows:

THEOREM 8.15. *The following function, called ξ function,*

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

satisfies $\xi(s) = \xi(1 - s)$.

PROOF. Again, the above Riemann formula is equivalent to the previous ones, with the function ξ being what is used in computations, when proving Theorem 8.12. □

We have zeta up and working in the full complex plane $\mathbb{C}$, as a meromorphic function with a single pole at 1, and this gives rise to many interesting questions. In particular, getting now to the zeroes of zeta, as a consequence of Theorem 8.12, we have:

THEOREM 8.16. *We have the following formula, for any integer $k \geq 1$,*

$$\zeta(-2k) = 0$$

with these being called the "trivial zeroes" of ζ.

PROOF. We recall that the Riemann symmetry formula from Theorem 8.12 is as follows, valid all over the complex plane, as an equality of meromorphic functions:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

By plugging in the value $s = -2k$, with $k \geq 1$ integer, we obtain:

$$\begin{aligned}\zeta(-2k) &= 2^{-2k}\pi^{-2k-1}\sin(k\pi)\Gamma(1+2k)\xi(1+2k)\\ &= 0\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

8d. Prime distribution

Getting now to prime numbers and their distribution, let us start with:

DEFINITION 8.17. *The modified Chebycheff and von Mangoldt functions are*

$$\psi(x) = \sum_{p^k \leq x} \log p \quad , \quad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

related by the formulae $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and $\Lambda(n) = \psi(n) - \psi(n-)$.

You might of course ask, why using two functions instead of one. Good point, and in answer, we will see a bit later that, in the context of certain delicate questions, the Chebycheff function and the von Mangoldt function are not exactly the same thing. In relation with the Prime Number Theorem, that we want to prove, we have:

PROPOSITION 8.18. *We have the following equivalence,*

$$\pi(x) \sim \frac{x}{\log x} \iff \psi(x) \sim x$$

with the condition on the left being the Prime Number Theorem.

PROOF. This is something elementary, coming from two estimates, as follows:

(1) In one sense, we have the following basic estimate:

$$\begin{aligned}\psi(x) &= \sum_{p^k\leq x}\log p\\ &= \sum_{p\leq x}\log p\left[\frac{\log x}{\log p}\right]\\ &\leq \sum_{p\leq x}\log x\\ &= \pi(x)\log x\end{aligned}$$

(2) In the other sense, we have the following estimate, valid for any $\varepsilon > 0$:

$$\begin{aligned}\psi(x) &= \sum_{p^k\leq x}\log p\\ &\geq \sum_{x^{1-\varepsilon}\leq p\leq x}\log p\\ &\geq \sum_{x^{1-\varepsilon}\leq p\leq x}(1-\varepsilon)\log x\\ &= (1-\varepsilon)(\pi(x)+O(x^{1-\varepsilon}))\log x\end{aligned}$$

Thus, we are led to the equivalence in the statement. □

In order to estimate now the Chebycheff function ψ, we would need an analytic formula for it. However, finding such a formula is not obvious with bare hands, so let us examine instead the same question for the von Mangoldt function Λ. And good news, we have:

PROPOSITION 8.19. *The von Mangoldt function satisfies*

$$\sum_{n=1}^\infty\frac{\Lambda(n)}{n^s} = -(\log\zeta(s))'$$

with ζ being the Riemann zeta function.

PROOF. We use the Euler product formula for zeta, namely:

$$\zeta(s) = \prod_p\left(1-\frac{1}{p^s}\right)^{-1}$$

By taking the logarithm, we obtain from this the following formula:

$$\log\zeta(s) = -\sum_p\log\left(1-\frac{1}{p^s}\right)$$

Now by differentiating, we obtain the following formula:

$$\begin{aligned}
(\log\zeta(s))' &= -\sum_p\left(1-\frac{1}{p^s}\right)^{-1}\frac{d(1-p^{-s})}{ds}\\
&= \sum_p\left(1-\frac{1}{p^s}\right)^{-1}\frac{dp^{-s}}{ds}\\
&= -\sum_p\left(1-\frac{1}{p^s}\right)^{-1}p^{-s}\log p\\
&= -\sum_p\frac{p^s}{p^s-1}\cdot\frac{1}{p^s}\log p\\
&= -\sum_p\frac{\log p}{p^s-1}
\end{aligned}$$

On the other hand, the sum on the left in the statement is given by:

$$\begin{aligned}
\sum_{n=1}^\infty\frac{\Lambda(n)}{n^s} &= \sum_{n=p^k}\frac{\log p}{n^s}\\
&= \sum_p\log p\sum_{k=1}^\infty\frac{1}{p^{ks}}\\
&= \sum_p\log p\cdot\frac{1}{p^s}\left(1-\frac{1}{p^s}\right)^{-1}\\
&= \sum_p\frac{\log p}{p^s-1}
\end{aligned}$$

Thus, we are led to the equality in the statement. □

Now let us turn to the second part of our plan, namely reformulating the formula for Λ that we found in terms of ψ. This is something more delicate, leading to:

THEOREM 8.20. *The modified Chebycheff function is given by*

$$\psi(x)=x-\log(2\pi)-\sum_{\zeta(s)=0}\frac{x^s}{s}$$

for $x\notin\mathbb Z$, with the sum being over all the zeroes of zeta.

PROOF. This follows via some complex analysis and tricks, as follows:

(1) To start with, we know from Definition 8.17 that the functions ψ and Λ are related by the following conversion formulae, which are both trivial:

$$\psi(x)=\sum_{n\leq x}\Lambda(n)\quad,\quad\Lambda(n)=\psi(n)-\psi(n-)$$

The problem now is to use these conversion formulae, in order to reformulate in terms of ψ the formula for Λ that we found in Proposition 8.19, namely:

$$\sum_{n=1}^\infty\frac{\Lambda(n)}{n^s}=-(\log\zeta(s))'$$

(2) As a first step, we have the following computation, with at the beginning the $n=1$ term ignored, and at the end, the $n=1$ term added, because these vanish anyway:

$$\begin{aligned}\sum_{n=1}^\infty\frac{\Lambda(n)}{n^s}&=\sum_{n=2}^\infty\frac{\psi(n)-\psi(n-)}{n^s}\\&=\sum_{n=2}^\infty\frac{\psi(n)-\psi(n-1)}{n^s}\\&=\sum_{n=1}^\infty\psi(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)\end{aligned}$$

(3) Thus, we have the following equation, in terms of the function ψ:

$$\sum_{n=1}^\infty\psi(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)=-(\log\zeta(s))'$$

(4) The problem is now, how to fine-tune this, into something truly analytical, involving the function $\psi(x)$ with real argument, $x>1$. For this purpose, it is convenient to further modify the Chebycheff step function ψ, by making it continuous, as follows:

$$\varphi(x)=\int_1^x\psi(t)dt$$

(5) Observe that this latter function can be expressed in terms of Λ, as follows:

$$\varphi(x)=\sum_{n\leq x}(x-n)\Lambda(n)$$

Also, as another remark, in relation with Proposition 8.18, we have:

$$\psi(x)\sim x\iff\varphi(x)\sim\frac{x^2}{2}$$

Thus, we can normally do everything with φ instead of ψ. However, for our purposes here, φ will be a secondary object, with our main function remaining ψ.

(6) The point now is that we have the following formula, as a contour integral, with $r>1$, coming via some manipulations involving the Cauchy formula:

$$\frac{\varphi(x)}{x^2}=\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}\sum_{n=1}^\infty\psi(n)\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right)ds$$

(7) We recognize on the right the sum from (3), and by plugging that in, we get:

$$\begin{aligned}\frac{\varphi(x)}{x^2}&=-\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}(\log\zeta(s))'ds\\&=-\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}\cdot\frac{\zeta'(s)}{\zeta(s)}\,ds\end{aligned}$$

(8) Now since the function $\zeta'(s)/\zeta(s)$ has a simple pole at 1, with residue -1, we can separate the contribution of that pole, and we get, again with $r>1$:

$$\frac{\varphi(x)}{x^2}=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}\frac{x^{s-1}}{s(s+1)}\left(\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}\right)ds$$

(9) In order to simplify notation, let us introduce the following function:

$$f(s)=\frac{1}{s(s+1)}\left(\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}\right)$$

In terms of this function, the formula that we found above reads:

$$\begin{aligned}\frac{\varphi(x)}{x^2}&=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{1}{2\pi i}\int_{r-\infty i}^{r+\infty i}x^{s-1}f(x)\,ds\\&=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{1}{2\pi}\int_{-\infty}^{\infty}x^{r+it-1}f(r+it)\,dt\\&=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{x^{r-1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt\end{aligned}$$

(10) Thus, getting back now to the usual Chebycheff function ψ, we have:

$$\frac{1}{x^2}\int_1^x\psi(t)dt=\frac{1}{2}\left(1-\frac{1}{x}\right)^2-\frac{x^{r-1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt$$

By multiplying both sides by x^2, we have the following formula:

$$\int_1^x\psi(t)dt=\frac{(x-1)^2}{2}-\frac{x^{r+1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt$$

(11) Now by taking the derivative with respect to x, this formula gives:

$$\begin{aligned}\psi(x) &= \frac{d}{dx}\left[\frac{(x-1)^2}{2}-\frac{x^{r+1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt\right]\\ &= x-1+\frac{d}{dx}\left[\frac{x^{r+1}}{2\pi}\int_{-\infty}^{\infty}e^{it\log x}f(r+it)\,dt\right]\end{aligned}$$

(12) The point now is that, by computing the derivative on the right, we get:

$$\psi(x)=x-\log(2\pi)-\sum_{\zeta(s)=0}\frac{x^s}{s}$$

Thus, we are led to the conclusion in the statement. □

Now remember from Proposition 8.18 that what we want to do is to estimate ψ, with the following estimate, proving the Prime Number theorem, being our goal:

$$\psi(x)\sim x$$

Looking at the formula in Theorem 8.20, the x is already there, $\log(2\pi)$ does not matter, and what is left to prove that the sum over zeroes of ζ does not matter either:

$$\sum_{\zeta(s)=0}\frac{x^s}{s}=o(x)$$

In what regards the trivial zeroes, things are easily settled here, as follows:

PROPOSITION 8.21. *The contribution to the modified Chebycheff function ψ of the trivial zeroes of zeta, namely $-2,-4,-6,\ldots$, is given by*

$$\sum_{k=1}^{\infty}\frac{x^{-2k}}{2k}=-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)$$

and this quantity vanishes in the $x\to\infty$ limit.

PROOF. We have indeed the following computation:

$$\sum_{k=1}^{\infty}\frac{x^{-2k}}{2k}=\sum_{k=1}^{\infty}\frac{1}{2kx^{2k}}=-\log\left(1-\frac{1}{x^2}\right)$$

Thus, we are led to the conclusion in the statement. □

Regarding now the non-trivial zeroes of zeta, we know from our study before that these lie inside the strip $0\le Re(s)\le 1$, and as a first observation, we have:

Proposition 8.22. *The contribution to the modified Chebycheff function ψ of the non-trivial zeroes of zeta lying in the strip $0 \leq Re(s) < 1$ satisfies*

$$\sum_{\zeta(s)=0} \frac{x^s}{s} = o(x)$$

so we are left with studying the zeroes on the line $Re(s) = 1$.

Proof. This is something quite self-explanatory, with some care needed however when summing all the $o(x)$ quantities associated to the zeroes in question. As for the final conclusion, this comes by combining our finding with Proposition 8.21. □

We are now getting to the core of the proof, with the key ingredient being:

Theorem 8.23. *The Riemann zeta function has no zero on the line*

$$Re(s) = 1$$

and no zero on the line $Re(s) = 0$ either.

Proof. This is something quite tricky, the idea being as follows:

(1) To start with, the $Re(s) = 0$ result, that we will not need here for our current purposes, in view of Proposition 8.22, but which of course has great theoretical interest, follows from the $Re(s) = 1$ result, via the Riemann reflection formula.

(2) In order to study now the zeta function on the line $Re(s) = 1$, we use the Euler product formula for this function, namely:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

By taking the logarithm, we obtain from this the following formula:

$$\begin{aligned} \log \zeta(s) &= -\sum_p \log \left(1 - \frac{1}{p^s}\right) \\ &= \sum_p \sum_{k=0}^{\infty} \frac{1}{kp^{ks}} \end{aligned}$$

(3) Now with $s=r+it$ as usual, this formula reads:

$$\begin{aligned}\log\zeta(s) &= \sum_p\sum_{k=0}^\infty\frac{1}{kp^{k(r+it)}}\\ &= \sum_p\sum_{k=0}^\infty\frac{p^{-kit}}{kp^{kr}}\\ &= \sum_p\sum_{k=0}^\infty\frac{e^{-kit\log p}}{kp^{kr}}\\ &= \sum_p\sum_{k=0}^\infty\frac{\cos(kt\log p)-i\sin(kt\log p)}{kp^{kr}}\end{aligned}$$

(4) Now remember the following formula, for the complex exponentials:

$$|e^z|^2=e^z\cdot\overline{e^z}=e^ze^{\bar{z}}=e^{z+\bar{z}}=e^{2Re(z)}$$

Thus we have $|e^z|=e^{Re(z)}$, and by using this with $z=\log\zeta(s)$, we get:

$$\begin{aligned}|\zeta(s)| &= |\exp(\log\zeta(s))|\\ &= \exp(Re(\log\zeta(s)))\\ &= \exp\left(\sum_p\sum_{k=0}^\infty\frac{\cos(kt\log p)}{kp^{kr}}\right)\end{aligned}$$

(5) In order to get an estimate, we use the following formula, valid for any $\alpha\in\mathbb{R}$:

$$\begin{aligned}2(1+\cos\alpha)^2 &= 2+4\cos\alpha+2\cos^2\alpha\\ &= 3+4\cos\alpha+\cos(2\alpha)\end{aligned}$$

Indeed, by using this, we obtain from the formula in (4) the following estimate:

$$\begin{aligned}|\zeta(r)^3\zeta(r+it)^4\zeta(r+2it)| &= \exp\left(\sum_p\sum_{k=0}^\infty\frac{3+4\cos(kt\log p)+\cos(2kt\log p)}{kp^{kr}}\right)\\ &= \exp\left(\sum_p\sum_{k=0}^\infty\frac{2(1+\cos(kt\log p))^2}{kp^{kr}}\right)\\ &\geq 1\end{aligned}$$

(6) But with this, we can now finish. Assume indeed by contradiction $\zeta(1+it)=0$, for some $t\neq 0$, and let us look at the following quantity, in the $r\to 1^+$ limit:

$$K=\zeta(r)^3\zeta(r+it)^4\zeta(r+2it)$$

What happens then in the $r \to 1^+$ limit is that we have $\zeta(r)^3 \to \infty$ with triple pole behavior, $\zeta(r+it)^4 \to 0$ with quadruple zero behavior, and $\zeta(r+2it) \to \zeta(2it)$ with analytic behavior. But since $3 < 4$ the quadruple zero will kill the triple pole, and so:

$$\lim_{r \to 1^+} K = 0$$

But this contradicts the estimate found in (5), and so our theorem is proved. □

By putting now everything together, we obtain:

Theorem 8.24 (Prime Number Theorem). *We have*

$$\pi(x) \sim \frac{\log x}{x}$$

in the $x \to \infty$ limit.

Proof. This follows by putting everything together, as follows:

(1) We know from Proposition 8.18 that $\pi(x) \sim x/\log x$ is equivalent to $\psi(x) \sim x$.

(2) We have in Theorem 8.20 a formula for $\psi(x)$, in terms of the zeroes of zeta.

(3) Most of these zeroes are taken care of by Propositions 8.21 and 8.22.

(4) As for the remaining zeroes, there are none, as shown by Theorem 8.23. □

8e. Exercises

This was a beautiful but quite technical chapter, and as exercises, we have:

Exercise 8.25. *Learn more about the Möbius function, and its properties.*

Exercise 8.26. *Establish the formulae for $\zeta^3(s)/\zeta(2s)$ and $\zeta^4(s)/\zeta(2s)$.*

Exercise 8.27. *Establish the multiple integral formula for $\zeta(s)$, with $s \in \mathbb{N}$.*

Exercise 8.28. *Learn more about Bernoulli numbers, and their properties.*

Exercise 8.29. *Learn about Dirichlet eta functions, and their properties.*

Exercise 8.30. *Learn also about the work of Hasse and others, regarding ζ.*

Exercise 8.31. *Read about Mertens theorems, and Chebycheff estimates.*

Exercise 8.32. *Learn also the Selberg proof of the Prime Number theorem.*

As bonus exercise, and no surprise here, solve the Riemann Hypothesis.

Part III

Harmonic functions

CHAPTER 9

Harmonic functions

9a. Laplace operator

With the Cauchy formula proved, and applied, done with complex analysis, you might say? You must be kidding, that was just the tip of the iceberg, and many things remain to be discussed. We will have a look at them, in the remainder of this book.

In order to reach to a continuation of what we know, let us formulate:

Definition 9.1. *The Laplace operator in 2 dimensions is:*

$$\Delta f = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2}$$

A function $f : \mathbb{R}^2 \to \mathbb{C}$ satisfying $\Delta f = 0$ will be called harmonic.

Here the Laplace operator is something very standard, coming from virtually all branches of physics, and its presence here remains to be explained, yes I know. So, let us try to find the functions $f : \mathbb{R}^2 \to \mathbb{C}$ which are harmonic. And here, as a good surprise, we have an interesting link with the holomorphic functions:

Theorem 9.2. *Any holomorphic function $f : \mathbb{C} \to \mathbb{C}$, when regarded as function*

$$f : \mathbb{R}^2 \to \mathbb{C}$$

is harmonic. Moreover, the conjugates $\bar{f}$ of holomorphic functions are harmonic too.

Proof. The first assertion follows from the following computation, for the power functions $f(z) = z^n$, with the usual notation $z = x + iy$:

$$\begin{aligned}
\Delta z^n &= \frac{d^2 z^n}{dx^2} + \frac{d^2 z^n}{dy^2} \\
&= \frac{d(nz^{n-1})}{dx} + \frac{d(inz^{n-1})}{dy} \\
&= n(n-1)z^{n-2} - n(n-1)z^{n-2} \\
&= 0
\end{aligned}$$

As for the second assertion, this follows from $\Delta \bar{f} = \overline{\Delta f}$, which is clear from definitions, and which shows that if f is harmonic, then so is its conjugate $\bar{f}$. $\square$

All this is quite interesting, and the idea in what follows will be that of developing the theory of harmonic functions, as a generalization of the theory that we know for the holomorphic functions, but covering as well functions of type $\bar{z}$.

As a first goal, in order to understand the harmonic functions, we can try to find the homogeneous polynomials $P \in \mathbb{R}[x,y]$ which are harmonic. In order to do so, the most convenient is to use the variable $z = x + iy$, and think of these polynomials as being homogeneous polynomials $P \in \mathbb{R}[z,\bar{z}]$. With this convention, the result is as follows:

THEOREM 9.3. *The degree n homogeneous polynomials $P \in \mathbb{R}[x,y]$ which are harmonic are precisely the linear combinations of*

$$P = z^n \quad , \quad P = \bar{z}^n$$

with the usual identification $z = x + iy$.

PROOF. As explained above, any homogeneous polynomial $P \in \mathbb{R}[x,y]$ can be regarded as an homogeneous polynomial $P \in \mathbb{R}[z,\bar{z}]$, with the change of variables $z = x+iy$, and in this picture, the degree n homogeneous polynomials are as follows:

$$P(z) = \sum_{k+l=n} c_{kl} z^k \bar{z}^l$$

In oder to solve now the Laplace equation $\Delta P = 0$, we must compute the quantities $\Delta(z^k \bar{z}^l)$, for any k,l. But the computation here is routine. We first have the following formula, with the derivatives being computed with respect to the variable x:

$$\begin{aligned} \frac{d(z^k \bar{z}^l)}{dx} &= (z^k)' \bar{z}^l + z^k (\bar{z}^l)' \\ &= k z^{k-1} \bar{z}^l + l z^k \bar{z}^{l-1} \end{aligned}$$

By taking one more time the derivative with respect to x, we obtain:

$$\begin{aligned} \frac{d^2(z^k \bar{z}^l)}{dx^2} &= k(z^{k-1}\bar{z}^l)' + l(z^k \bar{z}^{l-1})' \\ &= k\left[(z^{k-1})'\bar{z}^l + z^{k-1}(\bar{z}^l)'\right] + l\left[(z^k)'\bar{z}^{l-1} + z^k(\bar{z}^{l-1})'\right] \\ &= k\left[(k-1)z^{k-2}\bar{z}^l + l z^{k-1}\bar{z}^{l-1}\right] + l\left[k z^{k-1}\bar{z}^{l-1} + (l-1)z^k \bar{z}^{l-2}\right] \\ &= k(k-1)z^{k-2}\bar{z}^l + 2kl z^{k-1}\bar{z}^{l-1} + l(l-1)z^k \bar{z}^{l-2} \end{aligned}$$

With respect to the variable y, the computations are similar, but some $\pm i$ factors appear, due to $z' = i$ and $\bar{z}' = -i$, coming from $z = x + iy$. We first have:

$$\begin{aligned} \frac{d(z^k \bar{z}^l)}{dy} &= (z^k)' \bar{z}^l + z^k (\bar{z}^l)' \\ &= ik z^{k-1} \bar{z}^l - il z^k \bar{z}^{l-1} \end{aligned}$$

By taking one more time the derivative with respect to y, we obtain:

$$\begin{aligned}
\frac{d^2(z^k\bar{z}^l)}{dy^2} &= ik(z^{k-1}\bar{z}^l)' - il(z^k\bar{z}^{l-1})' \\
&= ik\left[(z^{k-1})'\bar{z}^l + z^{k-1}(\bar{z}^l)'\right] - il\left[(z^k)'\bar{z}^{l-1} + z^k(\bar{z}^{l-1})'\right] \\
&= ik\left[i(k-1)z^{k-2}\bar{z}^l - ilz^{k-1}\bar{z}^{l-1}\right] - il\left[ikz^{k-1}\bar{z}^{l-1} - i(l-1)z^k\bar{z}^{l-2}\right] \\
&= -k(k-1)z^{k-2}\bar{z}^l + 2klz^{k-1}\bar{z}^{l-1} - l(l-1)z^k\bar{z}^{l-2}
\end{aligned}$$

We can now sum the formulae that we found, and we obtain:

$$\begin{aligned}
\Delta(z^k\bar{z}^l) &= \frac{d^2(z^k\bar{z}^l)}{dx^2} + \frac{d^2(z^k\bar{z}^l)}{dy^2} \\
&= k(k-1)z^{k-2}\bar{z}^l + 2klz^{k-1}\bar{z}^{l-1} + l(l-1)z^k\bar{z}^{l-2} \\
&\quad -k(k-1)z^{k-2}\bar{z}^l + 2klz^{k-1}\bar{z}^{l-1} - l(l-1)z^k\bar{z}^{l-2} \\
&= 4klz^{k-1}\bar{z}^{l-1}
\end{aligned}$$

In other words, we have reached to the following nice formula:

$$f = z^k\bar{z}^l \implies \Delta f = \frac{4klf}{|z|^2}$$

Now let us get back to our homogeneous polynomial P, written as follows:

$$P(z) = \sum_{k+l=n} c_{kl}z^k\bar{z}^l$$

By using the above formula, it follows that the Laplacian of P is given by:

$$\Delta P(z) = \frac{4}{|z|^2}\sum_{k+l=n} klc_{kl}z^k\bar{z}^l$$

We conclude that the Laplace equation for P takes the following form:

$$\begin{aligned}
\Delta P = 0 &\iff klc_{kl} = 0, \forall k, l \\
&\iff [k, l \neq 0 \implies c_{kl} = 0] \\
&\iff P = c_{n0}z^n + c_{0n}\bar{z}^n
\end{aligned}$$

Thus, we are led to the conclusion in the statement. And with the observation that the real formulation of the final result is something quite complicated, and so, for one more time, the use of the complex variable $z = x + iy$ is something very useful. □

As a conclusion to what we have so far, we know that the holomorphic functions, and so their real and imaginary parts too, are harmonic. That is, if f is holomorphic, then the following function is harmonic, for any values of the parameters $\alpha, \beta \in \mathbb{C}$:

$$f_{\alpha\beta} = \alpha Re(f) + \beta Im(f)$$

Observe that this result covers all the examples that we have so far, for instance with the function $\bar{z}$, that we know to be harmonic, appearing as follows:

$$\bar{z} = Re(z) - iIm(z)$$

Moreover, Theorem 9.3 seems to suggest that the converse of this is true. We will see later that this is something which happens indeed, at least locally.

At the theoretical level now, recall from chapter 5 the following definition:

DEFINITION 9.4. *The Cauchy-Riemann operators are*

$$\partial = \frac{1}{2}\left(\frac{d}{dx} - i\frac{d}{dy}\right) \quad , \quad \bar{\partial} = \frac{1}{2}\left(\frac{d}{dx} + i\frac{d}{dy}\right)$$

where $\frac{d}{dx}$ *and* $\frac{d}{dy}$ *are the usual partial derivatives for complex functions.*

There are many things that can be said about these Cauchy-Riemann operators $\partial, \bar{\partial}$, the idea being that in many contexts, these are better to use than the usual partial derivatives $\frac{d}{dx}, \frac{d}{dy}$, and with this being a bit like the usage of the variables $z, \bar{z}$, instead of the decomposition $z = x + iy$, for many questions regarding the complex numbers. At the general level, the main properties of $\partial, \bar{\partial}$ can be summarized as follows:

THEOREM 9.5. *Assume that $f : X \to \mathbb{C}$ is differentiable in the real sense.*

1. *f is holomorphic precisely when $\bar{\partial} f = 0$.*
2. *In this case, its derivative is $f' = \partial f$.*
3. *The Laplace operator is given by $\Delta = 4\partial\bar{\partial}$.*
4. *f is harmonic precisely when $\partial\bar{\partial} f = 0$.*

PROOF. We already know the first two assertions from chapter 5, with these coming, at the point 0, from the following formula, which itself comes from definitions:

$$\frac{f(z)}{z} = \partial f(0) + \bar{\partial} f(0) \cdot \frac{\bar{z}}{z} + o(1)$$

Indeed, this gives the first two assertions, because in order for the derivative $f'(0)$ to exist, appearing as the $z \to 0$ limit of the above quantity, the coefficient of $\bar{z}/z$, which does not converge, must vanish. Regarding now the third assertion, this follows from:

$$\begin{aligned}\Delta &= \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \\ &= \left(\frac{d}{dx} - i\frac{d}{dy}\right)\left(\frac{d}{dx} + i\frac{d}{dy}\right) \\ &= 4\partial\bar{\partial}\end{aligned}$$

As for the last assertion, this is clear from this latter formula of Δ. □

With this discussed, as a next objective, and getting back to more concrete things, let us try to find the harmonic functions which are radial, in the following sense:

$$f(z)=\varphi(|z|)$$

However, things are quite tricky here, involving a blowup phenomenon at the dimension value $N=2$, which is precisely the one that we are interested in. In view of this phenomenon, it makes sense to move now to arbitrary $N\in\mathbb N$ dimensions. So, let us introduce the Laplace operator, acting on the functions $f:\mathbb R^N\to\mathbb C$, as follows:

$$\Delta f=\sum_{i=1}^N\frac{d^2f}{dx_i^2}$$

As before in 2 dimensions, we say that a function $f:\mathbb R^N\to\mathbb C$ is harmonic when $\Delta f=0$. With these conventions, the result about radial harmonics is as follows:

THEOREM 9.6. *The fundamental radial solutions of $\Delta f=0$ are*

$$f(x)=\begin{cases}||x||^{2-N}&(N\neq2)\\ \log||x||&(N=2)\end{cases}$$

with the log *at $N=2$ basically coming from* $\log'=1/x$.

PROOF. Consider indeed a radial function, defined outside the origin $x=0$. This function can be written as follows, with $\varphi:(0,\infty)\to\mathbb C$ being a certain function:

$$f:\mathbb R^N-\{0\}\to\mathbb C\quad,\quad f(x)=\varphi(||x||)$$

Our first goal will be that of reformulating the Laplace equation $\Delta f=0$ in terms of the one-variable function $\varphi:(0,\infty)\to\mathbb C$. For this purpose, observe that we have:

$$\begin{aligned}\frac{d||x||}{dx_i}&=\frac{d\sqrt{\sum_{i=1}^Nx_i^2}}{dx_i}\\&=\frac{1}{2}\cdot\frac{1}{\sqrt{\sum_{i=1}^Nx_i^2}}\cdot\frac{d\left(\sum_{i=1}^Nx_i^2\right)}{dx_i}\\&=\frac{1}{2}\cdot\frac{1}{||x||}\cdot2x_i\\&=\frac{x_i}{||x||}\end{aligned}$$

By using this formula, we have the following computation:

$$\begin{aligned}\frac{df}{dx_i} &= \frac{d\varphi(||x||)}{dx_i}\\ &= \varphi'(||x||)\cdot\frac{d||x||}{dx_i}\\ &= \varphi'(||x||)\cdot\frac{x_i}{||x||}\end{aligned}$$

By differentiating one more time, we obtain the following formula:

$$\begin{aligned}\frac{d^2f}{dx_i^2} &= \frac{d}{dx_i}\left(\varphi'(||x||)\cdot\frac{x_i}{||x||}\right)\\ &= \frac{d\varphi'(||x||)}{dx_i}\cdot\frac{x_i}{||x||}+\varphi'(||x||)\cdot\frac{d}{dx_i}\left(\frac{x_i}{||x||}\right)\\ &= \left(\varphi''(||x||)\cdot\frac{x_i}{||x||}\right)\cdot\frac{x_i}{||x||}+\varphi'(||x||)\cdot\frac{||x||-x_i\cdot x_i/||x||}{||x||^2}\\ &= \varphi''(||x||)\cdot\frac{x_i^2}{||x||^2}+\varphi'(||x||)\cdot\frac{||x||^2-x_i^2}{||x||^3}\end{aligned}$$

Now by summing over $i\in\{1,\ldots,N\}$, this gives the following formula:

$$\begin{aligned}\Delta f &= \sum_{i=1}^N\varphi''(||x||)\cdot\frac{x_i^2}{||x||^2}+\sum_{i=1}^N\varphi'(||x||)\cdot\frac{||x||^2-x_i^2}{||x||^3}\\ &= \varphi''(||x||)\cdot\frac{||x||^2}{||x||^2}+\varphi'(||x||)\cdot\frac{(N-1)||x||^2}{||x||^3}\\ &= \varphi''(||x||)+\varphi'(||x||)\cdot\frac{N-1}{||x||}\end{aligned}$$

Thus, with $r=||x||$, the Laplace equation $\Delta f=0$ can be reformulated as follows:

$$\varphi''(r)+\frac{(N-1)\varphi'(r)}{r}=0$$

Equivalently, the equation that we want to solve is as follows:

$$r\varphi''+(N-1)\varphi'=0$$

Now observe that we have the following formula:

$$\begin{aligned}(r^{N-1}\varphi')' &= (N-1)r^{N-2}\varphi'+r^{N-1}\varphi''\\ &= r^{N-2}((N-1)\varphi'+r\varphi'')\end{aligned}$$

Thus, the equation to be solved can be simply written as follows:

$$(r^{N-1}\varphi')'=0$$

We conclude that $r^{N-1}\varphi'$ must be a constant K, and so, that we must have:

$$\varphi' = Kr^{1-N}$$

But the fundamental solutions of this latter equation are as follows:

$$\varphi(r) = \begin{cases} r^{2-N} & (N \neq 2) \\ \log r & (N = 2) \end{cases}$$

Thus, we are led to the conclusion in the statement. □

9b. Harmonic functions

Back now to the general theory of harmonic functions, the passage to arbitrary $N \in \mathbb{N}$ dimensions, that we made in Theorem 9.6, and that we will adopt, proves to be something fruitful, allowing us to see many things obscured by various $N = 2$ phenomena.

Among others, in analogy with the usual theory of the holomorphic functions, that we know well since chapter 5, we can now formulate the following statement:

THEOREM 9.7. *The harmonic functions in N dimensions obey to the same general principles as the holomorphic functions, namely:*

1. *The plain mean value formula.*
2. *The boundary mean value formula.*
3. *The maximum modulus principle.*
4. *The Liouville theorem.*

PROOF. This is something quite straightforward, the idea being as follows:

(1) Regarding the plain mean value formula, here the statement is that given an harmonic function $f : X \to \mathbb{C}$, and a ball B, the following happens:

$$f(x) = \int_B f(y)dy$$

In order to prove this formula, we can assume that our ball B is centered at 0, say of radius $r > 0$. Now if we denote by χ_r the characteristic function of this ball, nomalized as to integrate up to 1, we want to prove that we have the following formula:

$$f = f * \chi_r$$

To be more precise, here $*$ is the standard convolution operation, given by:

$$(f * g)(x) = \int_{\mathbb{R}^N} f(x-y)g(y)dy$$

For proving the above formula, let us pick a number $0 < s < r$, and a solution w of the following equation, supported on B, which can be constructed explicitly:

$$\Delta w = \chi_r - \chi_s$$

By using the basic properties of the convolution operation $*$, we have:

$$\begin{aligned} f*\chi_r - f*\chi_s &= f*(\chi_r-\chi_s)\\ &= f*\Delta w\\ &= \Delta f*w\\ &= 0 \end{aligned}$$

Thus $f*\chi_r=f*\chi_s$, and by letting now $s\to 0$, we get $f*\chi_r=f$, as desired.

(2) Regarding the boundary mean value formula, here the statement is that given an harmonic function $f:X\to\mathbb{C}$, and a ball B, with boundary γ, the following happens:

$$f(x)=\int_\gamma f(y)dy$$

But this follows as a consequence of the plain mean value formula in (1), with our two mean value formulae, the one there and the one here, being in fact equivalent, by using annuli and radial integration for the proof of the equivalence, in the obvious way.

(3) Regarding the maximum modulus principle, the statement here is that any holomorphic function $f:X\to\mathbb{C}$ has the property that the maximum of $|f|$ over a domain is attained on its boundary. That is, given a domain D, with boundary γ, we have:

$$\exists x\in\gamma\quad,\quad |f(x)|=\max_{y\in D}|f(y)|$$

But this is something which follows again from the mean value formula in (1), first for the balls, and then in general, by using a standard division argument.

(4) Finally, regarding the Liouville theorem, the statement here is that an entire, bounded harmonic function must be constant:

$$f:\mathbb{R}^N\to\mathbb{C}\quad,\quad \Delta f=0\quad,\quad |f|\leq M\quad\implies\quad f=\text{constant}$$

As a slightly weaker statement, again called Liouville theorem, we have the fact that an entire harmonic function which vanishes at ∞ must vanish globally:

$$f:\mathbb{R}^N\to\mathbb{C}\quad,\quad \Delta f=0\quad,\quad \lim_{x\to\infty}f(x)=0\quad\implies\quad f=0$$

But can view these as a consequence of the mean value formula in (1), because given two points $x\neq y$, we can view the values of f at these points as averages over big balls centered at these points, say $B=B_x(R)$ and $C=B_y(R)$, with $R>>0$:

$$f(x)=\int_B f(z)dz\quad,\quad f(y)=\int_C f(z)dz$$

Indeed, the point is that when the radius goes to ∞, these averages tend to be equal, and so we have $f(x)\simeq f(y)$, which gives $f(x)=f(y)$ in the limit, as desired. $\square$

So long for harmonic functions in arbitrary N dimensions. Getting back now to 2 dimensions, as a useful complement to what we have in Theorem 9.8, we have:

THEOREM 9.8. *The real harmonic functions $f : X \to \mathbb{R}$ with $X \subset \mathbb{C}$ are locally the real parts of the holomorphic functions $g : X \to \mathbb{C}$.*

PROOF. This is something that we announced before, and which is a bit more technical, the idea being that this can be indeed established by using the technology from Theorem 9.8, at $N = 2$. We refer for instance to Rudin [79] for the proof of this. □

9c. Waves and heat

In order to get more insight into the Laplace operator Δ, and the related notion of harmonic function, the way to be followed is physics. To start with, in 1 dimension, where $\Delta f = f''$, we have the following observation, which sounds a bit like physics:

PROPOSITION 9.9. *Intuitively, the second derivative of a function $f : \mathbb{R} \to \mathbb{R}$,*

$$f''(x) \in \mathbb{R}$$

computes how much different is $f(x)$, compared to the average of $f(y)$, with $y \simeq x$.

PROOF. This is obviously something a bit heuristic, but which is good to know. Let us write the Taylor formula for f at order 2 as such, and with $\to -t$ too:

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

$$f(x-t) \simeq f(x) - f'(x)t + \frac{f''(x)}{2}\,t^2$$

By making the average, we obtain from this the following formula:

$$\frac{f(x+t)+f(x-t)}{2} \simeq f(x) + \frac{f''(x)}{2}\,t^2$$

Thus, thinking a bit, we are led to the conclusion in the statement. It is of course possible to say more here, but we will not really need all this, in what follows. □

In arbitrary N dimensions, the situation is similar, as follows:

PROPOSITION 9.10. *Intuitively, the Laplacian of $f : \mathbb{R}^N \to \mathbb{R}$, given by*

$$\Delta f = \sum_{i=1}^N \frac{d^2 f}{dx_i^2}$$

computes how much different is $f(x)$, compared to the average of $f(y)$, with $y \simeq x$.

Proof. As before with Proposition 9.9, this is something a bit heuristic, but good to know. Let us write the Taylor formula for f at order 2 as such, and with $t \to -t$ too:

$$f(x+t) \simeq f(x) + f'(x)t + \frac{< f''(x)t, t >}{2}$$
$$f(x-t) \simeq f(x) - f'(x)t + \frac{< f''(x)t, t >}{2}$$

By making the average, we obtain the following formula:

$$\frac{f(x+t)+f(x-t)}{2} \simeq f(x) + \frac{< f''(x)t, t >}{2}$$

Thus, thinking a bit, we are led to the conclusion in the statement. It is of course possible to say more here, but we will not really need all the details, in what follows. □

With this discussed, and getting now into true physics, we first have the following key result, dealing with the most important equation of them all, the wave one:

Theorem 9.11. *The wave equation in $\mathbb{R}^N$ is*

$$\ddot{\varphi} = v^2 \Delta \varphi$$

where $v > 0$ is the propagation speed.

Proof. The equation in the statement is of course what comes out of physics experiments. However, allowing us a bit of imagination, and trust in this imagination, we can mathematically "prove" this equation, by discretizing, as follows:

(1) Let us first consider the 1D case. In order to understand the propagation of waves, we will model $\mathbb{R}$ as a network of balls, with springs between them, as follows:

$$\cdots \times\times\times \bullet \times\times\times \bullet \times\times\times \bullet \times\times\times \bullet \times\times\times \bullet \times\times\times \cdots$$

Now let us send an impulse, and see how the balls will be moving. For this purpose, we zoom on one ball. The situation here is as follows, l being the spring length:

$$\cdots\cdots \bullet_{\varphi(x-l)} \times\times\times \bullet_{\varphi(x)} \times\times\times \bullet_{\varphi(x+l)} \cdots\cdots$$

We have two forces acting at x. First is the Newton motion force, mass times acceleration, which is as follows, with m being the mass of each ball:

$$F_n = m \cdot \ddot{\varphi}(x)$$

And second is the Hooke force, displacement of the spring, times spring constant. Since we have two springs at x, this is as follows, k being the spring constant:

$$\begin{aligned} F_h &= F_h^r - F_h^l \\ &= k(\varphi(x+l) - \varphi(x)) - k(\varphi(x) - \varphi(x-l)) \\ &= k(\varphi(x+l) - 2\varphi(x) + \varphi(x-l)) \end{aligned}$$

We conclude that the equation of motion, in our model, is as follows:

$$m \cdot \ddot{\varphi}(x) = k(\varphi(x+l) - 2\varphi(x) + \varphi(x-l))$$

(2) Now let us take the limit of our model, as to reach to continuum. For this purpose we will assume that our system consists of $B >> 0$ balls, having a total mass M, and spanning a total distance L. Thus, our previous infinitesimal parameters are as follows, with K being the spring constant of the total system, which is of course lower than k:

$$m = \frac{M}{B} \quad , \quad k = KB \quad , \quad l = \frac{L}{B}$$

With these changes, our equation of motion found in (1) reads:

$$\ddot{\varphi}(x) = \frac{KB^2}{M}(\varphi(x+l) - 2\varphi(x) + \varphi(x-l))$$

Now observe that this equation can be written, more conveniently, as follows:

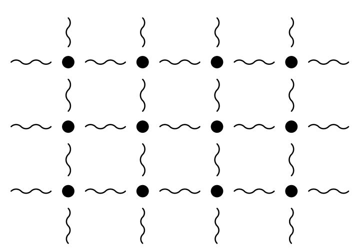
$$\ddot{\varphi}(x) = \frac{KL^2}{M} \cdot \frac{\varphi(x+l) - 2\varphi(x) + \varphi(x-l)}{l^2}$$

With $N \to \infty$, and therefore $l \to 0$, we obtain in this way:

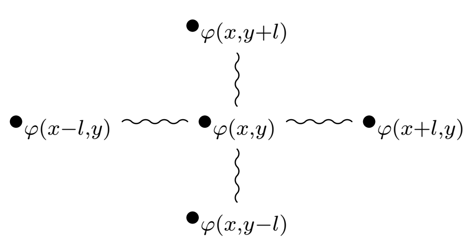
$$\ddot{\varphi}(x) = \frac{KL^2}{M} \cdot \frac{d^2\varphi}{dx^2}(x)$$

We are therefore led to the wave equation in the statement, which is $\ddot{\varphi} = v^2\varphi''$ in our present $N = 1$ dimensional case, the propagation speed being $v = \sqrt{K/M} \cdot L$.

(3) In 2 dimensions now, the same argument carries on. Indeed, we can use here a lattice model as follows, with all the edges standing for small springs:



As before in one dimension, we send an impulse, and we zoom on one ball. The situation here is as follows, with l being the spring length:



We have two forces acting at (x, y). First is the Newton motion force, mass times acceleration, which is as follows, with m being the mass of each ball:

$$F_n = m \cdot \ddot{\varphi}(x, y)$$

And second is the Hooke force, displacement of the spring, times spring constant. Since we have four springs at (x, y), this is as follows, k being the spring constant:

$$\begin{aligned} F_h &= F_h^r - F_h^l + F_h^u - F_h^d \\ &= k(\varphi(x+l, y) - \varphi(x, y)) - k(\varphi(x, y) - \varphi(x-l, y)) \\ &+ k(\varphi(x, y+l) - \varphi(x, y)) - k(\varphi(x, y) - \varphi(x, y-l)) \\ &= k(\varphi(x+l, y) - 2\varphi(x, y) + \varphi(x-l, y)) \\ &+ k(\varphi(x, y+l) - 2\varphi(x, y) + \varphi(x, y-l)) \end{aligned}$$

We conclude that the equation of motion, in our model, is as follows:

$$\begin{aligned} m \cdot \ddot{\varphi}(x, y) &= k(\varphi(x+l, y) - 2\varphi(x, y) + \varphi(x-l, y)) \\ &+ k(\varphi(x, y+l) - 2\varphi(x, y) + \varphi(x, y-l)) \end{aligned}$$

(4) Now let us take the limit of our model, as to reach to continuum. For this purpose we will assume that our system consists of $B^2 >> 0$ balls, having a total mass M, and spanning a total area L^2. Thus, our previous infinitesimal parameters are as follows, with K being the spring constant of the total system, taken to be equal to k:

$$m = \frac{M}{B^2} \quad , \quad k = K \quad , \quad l = \frac{L}{B}$$

With these changes, our equation of motion found in (3) reads:

$$\begin{aligned} \ddot{\varphi}(x, y) &= \frac{KB^2}{M}(\varphi(x+l, y) - 2\varphi(x, y) + \varphi(x-l, y)) \\ &+ \frac{KB^2}{M}(\varphi(x, y+l) - 2\varphi(x, y) + \varphi(x, y-l)) \end{aligned}$$

Now observe that this equation can be written, more conveniently, as follows:

$$\begin{aligned} \ddot{\varphi}(x, y) &= \frac{KL^2}{M} \times \frac{\varphi(x+l, y) - 2\varphi(x, y) + \varphi(x-l, y)}{l^2} \\ &+ \frac{KL^2}{M} \times \frac{\varphi(x, y+l) - 2\varphi(x, y) + \varphi(x, y-l)}{l^2} \end{aligned}$$

With $N \to \infty$, and therefore $l \to 0$, we obtain in this way:

$$\ddot{\varphi}(x, y) = \frac{KL^2}{M}\left(\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2}\right)(x, y)$$

Thus, we are led in this way to the following wave equation in two dimensions, with $v=\sqrt{K/M}\cdot L$ being the propagation speed of our wave:

$$\ddot{\varphi}(x,y)=v^2\left(\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}\right)(x,y)$$

But we recognize at right the Laplace operator, and we are done. As before in 1D, there is of course some discussion to be made here, arguing that our spring model in (3) is indeed the correct one. But do not worry, experiments confirm our findings.

(5) In 3 dimensions now, which is the case of the main interest, corresponding to our real-life world, the same argument carries over, and the wave equation is as follows:

$$\ddot{\varphi}(x,y,z)=v^2\left(\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}\right)(x,y,z)$$

(6) Finally, the same argument, namely a lattice model, carries on in arbitrary N dimensions, and the wave equation here is as follows:

$$\ddot{\varphi}(x_1,\ldots,x_N)=v^2\sum_{i=1}^N\frac{d^2\varphi}{dx_i^2}(x_1,\ldots,x_N)$$

Thus, we are led to the conclusion in the statement. □

In relation now with the next important phenomenon in physics, heat diffusion, the equation here is quite similar to the one for the waves, as follows:

THEOREM 9.12. *Heat diffusion in $\mathbb{R}^N$ is described by the heat equation*

$$\dot{\varphi}=\alpha\Delta\varphi$$

where $\alpha>0$ is the thermal diffusivity of the medium, and Δ is the Laplace operator.

PROOF. The study here is quite similar to the study of waves, as follows:

(1) To start with, as an intuitive explanation for the equation, since the second derivative φ'' in one dimension, or the quantity $\Delta\varphi$ in general, computes the average value of a function φ around a point, minus the value of φ at that point, the heat equation as formulated above tells us that the rate of change $\dot{\varphi}$ of the temperature of the material at any given point must be proportional, with proportionality factor $\alpha>0$, to the average difference of temperature between that given point and the surrounding material.

(2) The heat equation as formulated above is of course something approximative, and several improvements can be made to it, first by incorporating a term accounting for heat radiation, and then doing several fine-tunings, depending on the material involved. But more on this later, for the moment let us focus on the heat equation above.

(3) In relation with our modeling questions, we can recover this equation a bit as we did for the wave equation before, by using a basic lattice model. Indeed, let us first

assume, for simplifying, that we are in the one-dimensional case, $N = 1$. Here our model looks as follows, with distance $l > 0$ between neighbors:

$$—— \circ_{x-l} \overset{l}{——} \circ_x \overset{l}{——} \circ_{x+l} ——$$

In order to model heat diffusion, we have to implement the intuitive mechanism explained above, namely "the rate of change of the temperature of the material at any given point must be proportional, with proportionality factor $\alpha > 0$, to the average difference of temperature between that given point and the surrounding material".

(4) In practice, this leads to a condition as follows, expressing the change of the temperature φ, over a small period of time $\delta > 0$:

$$\varphi(x, t + \delta) = \varphi(x, t) + \frac{\alpha\delta}{l^2} \sum_{x \sim y} [\varphi(y, t) - \varphi(x, t)]$$

To be more precise, we have made several assumptions here, as follows:

– General heat diffusion assumption: the change of temperature at any given point x is proportional to the average over neighbors, $y \sim x$, of the differences $\varphi(y, t) - \varphi(x, t)$ between the temperatures at x, and at these neighbors y.

– Infinitesimal time and length conditions: in our model, the change of temperature at a given point x is proportional to small period of time involved, $\delta > 0$, and is inverse proportional to the square of the distance between neighbors, l^2.

(5) Regarding these latter assumptions, the one regarding the proportionality with the time elapsed $\delta > 0$ is something quite natural, physically speaking, and mathematically speaking too, because we can rewrite our equation as follows, making it clear that we have here an equation regarding the rate of change of temperature at x:

$$\frac{\varphi(x, t + \delta) - \varphi(x, t)}{\delta} = \frac{\alpha}{l^2} \sum_{x \sim y} [\varphi(y, t) - \varphi(x, t)]$$

As for the second assumption that we made above, namely inverse proportionality with l^2, this can be justified on physical grounds too, but again, perhaps the best is to do the math, which will show right away where this proportionality comes from.

(6) So, let us do the math. In the context of our 1D model the neighbors of x are the points $x \pm l$, and so the equation that we wrote above takes the following form:

$$\frac{\varphi(x, t + \delta) - \varphi(x, t)}{\delta} = \frac{\alpha}{l^2} \Big[(\varphi(x + l, t) - \varphi(x, t)) + (\varphi(x - l, t) - \varphi(x, t)) \Big]$$

Now observe that we can write this equation as follows:

$$\frac{\varphi(x, t + \delta) - \varphi(x, t)}{\delta} = \alpha \cdot \frac{\varphi(x + l, t) - 2\varphi(x, t) + \varphi(x - l, t)}{l^2}$$

(7) As it was the case with the wave equation before, we recognize on the right the usual approximation of the second derivative, coming from calculus. Thus, when taking the continuous limit of our model, $l \to 0$, we obtain the following equation:

$$\frac{\varphi(x,t+\delta)-\varphi(x,t)}{\delta} = \alpha \cdot \varphi''(x,t)$$

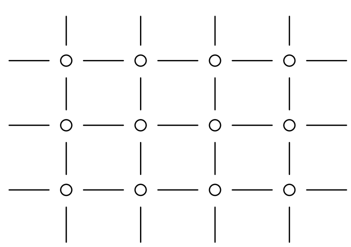
Now with $t \to 0$, we are led in this way to the heat equation, namely:

$$\dot{\varphi}(x,t) = \alpha \cdot \varphi''(x,t)$$

Summarizing, we are done with the 1D case, with our proof being quite similar to the one for the wave equation, from the previous section.

(8) In practice now, there are of course still a few details to be discussed, in relation with all this, for instance at the end, in relation with the precise order of the limiting operations $l \to 0$ and $\delta \to 0$ to be performed, but these remain minor aspects, because our equation makes it clear, right from the beginning, that time and space are separated, and so that there is no serious issue with all this. And so, fully done with 1D.

(9) With this done, let us discuss now 2 dimensions. Here, as before for the waves, we can use a lattice model as follows, with all lengths being $l > 0$, for simplifying:



(10) We have to implement now the physical heat diffusion mechanism, namely "the rate of change of the temperature of the material at any given point must be proportional, with proportionality factor $\alpha > 0$, to the average difference of temperature between that given point and the surrounding material". In practice, this leads to a condition as follows, expressing the change of the temperature φ, over a small period of time $\delta > 0$:

$$\varphi(x,y,t+\delta) = \varphi(x,y,t) + \frac{\alpha\delta}{l^2} \sum_{(x,y)\sim(u,v)} [\varphi(u,v,t) - \varphi(x,y,t)]$$

In fact, we can rewrite our equation as follows, making it clear that we have here an equation regarding the rate of change of temperature at x:

$$\frac{\varphi(x,y,t+\delta)-\varphi(x,y,t)}{\delta} = \frac{\alpha}{l^2} \sum_{(x,y)\sim(u,v)} [\varphi(u,v,t) - \varphi(x,y,t)]$$

(11) So, let us do the math. In the context of our 2D model the neighbors of x are the points $(x \pm l, y \pm l)$, so the equation above takes the following form:

$$\begin{aligned} & \frac{\varphi(x,y,t+\delta)-\varphi(x,y,t)}{\delta} \\ = & \frac{\alpha}{l^2}\Big[(\varphi(x+l,y,t)-\varphi(x,y,t))+(\varphi(x-l,y,t)-\varphi(x,y,t))\Big] \\ + & \frac{\alpha}{l^2}\Big[(\varphi(x,y+l,t)-\varphi(x,y,t))+(\varphi(x,y-l,t)-\varphi(x,y,t))\Big] \end{aligned}$$

Now observe that we can write this equation as follows:

$$\begin{aligned} \frac{\varphi(x,y,t+\delta)-\varphi(x,y,t)}{\delta} &= \alpha\cdot\frac{\varphi(x+l,y,t)-2\varphi(x,y,t)+\varphi(x-l,y,t)}{l^2} \\ &+ \alpha\cdot\frac{\varphi(x,y+l,t)-2\varphi(x,y,t)+\varphi(x,y-l,t)}{l^2} \end{aligned}$$

(12) As it was the case when modeling the wave equation before, we recognize on the right the usual approximation of the second derivative, coming from calculus. Thus, when taking the continuous limit of our model, $l \to 0$, we obtain the following equation:

$$\frac{\varphi(x,y,t+\delta)-\varphi(x,y,t)}{\delta} = \alpha\left(\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}\right)(x,y,t)$$

Now with $t \to 0$, we are led in this way to the heat equation, namely:

$$\dot{\varphi}(x,y,t) = \alpha\cdot\Delta\varphi(x,y,t)$$

Finally, in arbitrary N dimensions the same argument carries over, namely a straightforward lattice model, and gives the heat equation, as formulated in the statement. $\square$

9d. Some computations

We have seen in Theorem 9.11 that the mechanical waves propagate according to the wave equation, which is as follows, with Δ being the Laplace operator:

$$\ddot{\varphi} = v^2\Delta\varphi$$

Let us do now some math. With a bit of work, we can in fact fully solve the 1D wave equation. In order to explain this, we will need a standard calculus result, as follows:

PROPOSITION 9.13. *The derivative of a function of type*

$$\varphi(x) = \int_{g(x)}^{h(x)} f(s)ds$$

is given by the formula $\varphi'(x) = f(h(x))h'(x) - f(g(x))g'(x)$.

PROOF. Consider a primitive of the function that we integrate, $F' = f$. We have:

$$\begin{aligned}\varphi(x) &= \int_{g(x)}^{h(x)} f(s)ds \\ &= \int_{g(x)}^{h(x)} F'(s)ds \\ &= F(h(x)) - F(g(x))\end{aligned}$$

By using now the chain rule for derivatives, we obtain from this:

$$\begin{aligned}\varphi'(x) &= F'(h(x))h'(x) - F'(g(x))g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Now back to the 1D waves, the general result here, due to d'Alembert, along with a little more, in relation with our lattice models above, is as follows:

THEOREM 9.14. *The solution of the 1D wave equation with initial value conditions $\varphi(x,0) = f(x)$ and $\dot{\varphi}(x,0) = g(x)$ is given by the d'Alembert formula, namely:*

$$\varphi(x,t) = \frac{f(x-vt) + f(x+vt)}{2} + \frac{1}{2v}\int_{x-vt}^{x+vt} g(s)ds$$

In the context of our previous lattice model discretizations, what happens is more or less that the above d'Alembert integral gets computed via Riemann sums.

PROOF. There are several things going on here, the idea being as follows:

(1) Let us first check that the d'Alembert solution is indeed a solution of the wave equation $\ddot{\varphi} = v^2\varphi''$. The first time derivative is computed as follows:

$$\dot{\varphi}(x,t) = \frac{-vf'(x-vt) + vf'(x+vt)}{2} + \frac{1}{2v}(vg(x+vt) + vg(x-vt))$$

The second time derivative is computed as follows:

$$\ddot{\varphi}(x,t) = \frac{v^2f''(x-vt) + v^2f(x+vt)}{2} + \frac{vg'(x+vt) - vg'(x-vt)}{2}$$

Regarding now space derivatives, the first one is computed as follows:

$$\varphi'(x,t) = \frac{f'(x-vt) + f'(x+vt)}{2} + \frac{1}{2v}(g'(x+vt) - g'(x-vt))$$

As for the second space derivative, this is computed as follows:

$$\varphi''(x,t) = \frac{f''(x-vt) + f''(x+vt)}{2} + \frac{g''(x+vt) - g''(x-vt)}{2v}$$

Thus we have indeed $\ddot{\varphi} = v^2\varphi''$. As for the initial conditions, $\varphi(x,0) = f(x)$ is clear from our definition of φ, and $\dot{\varphi}(x,0) = g(x)$ is clear from our above formula of $\dot{\varphi}$.

(2) Conversely now, we must show that our solution is unique, but instead of going here into abstract arguments, we will simply solve our equation, which among others will doublecheck the computations in (1). Let us make the following change of variables:

$$\xi = x - vt \quad , \quad \eta = x + vt$$

With this change of variables, which is quite tricky, mixing space and time variables, our wave equation $\ddot{\varphi} = v^2\varphi''$ reformulates in a very simple way, as follows:

$$\frac{d^2\varphi}{d\xi d\eta} = 0$$

But this latter equation tells us that our new ξ, η variables get separated, and we conclude from this that the solution must be of the following special form:

$$\varphi(x,t) = F(\xi) + G(\eta) = F(x - vt) + G(x + vt)$$

Now by taking into account the intial conditions $\varphi(x,0) = f(x)$ and $\dot{\varphi}(x,0) = g(x)$, and then integrating, we are led to the d'Alembert formula in the statement.

(3) In regards now with our discretization questions, by using a 1D lattice model with balls and springs as before, what happens to all the above is more or less that the above d'Alembert integral gets computed via Riemann sums, in our model, as stated. □

Getting now to arbitrary N dimensions, a natural idea here is that of reformulating the wave equation in spherical coordinates. In order to to do, we will need:

Theorem 9.15. *The Laplace operator in spherical coordinates is*

$$\Delta = \frac{1}{r^2} \cdot \frac{d}{dr}\left(r^2 \cdot \frac{d}{dr}\right) + \frac{1}{r^2 \sin s} \cdot \frac{d}{ds}\left(\sin s \cdot \frac{d}{ds}\right) + \frac{1}{r^2 \sin^2 s} \cdot \frac{d^2}{dt^2}$$

with our standard conventions for these coordinates, in 3D.

Proof. There are several proofs here, a short, elementary one being as follows:

(1) Let us first see how Δ behaves under a change of coordinates $\{x_i\} \to \{y_i\}$, in arbitrary N dimensions. Our starting point is the chain rule for derivatives:

$$\frac{d}{dx_i} = \sum_j \frac{d}{dy_j} \cdot \frac{dy_j}{dx_i}$$

By using this rule, then Leibnitz for products, then again this rule, we obtain:

$$\begin{aligned}\frac{d^2f}{dx_i^2} &= \sum_j \frac{d}{dx_i}\left(\frac{df}{dy_j}\cdot\frac{dy_j}{dx_i}\right)\\ &= \sum_j \frac{d}{dx_i}\left(\frac{df}{dy_j}\right)\cdot\frac{dy_j}{dx_i}+\frac{df}{dy_j}\cdot\frac{d}{dx_i}\left(\frac{dy_j}{dx_i}\right)\\ &= \sum_j\left(\sum_k\frac{d}{dy_k}\cdot\frac{dy_k}{dx_i}\right)\left(\frac{df}{dy_j}\right)\cdot\frac{dy_j}{dx_i}+\frac{df}{dy_j}\cdot\frac{d^2y_j}{dx_i^2}\\ &= \sum_{jk}\frac{d^2f}{dy_kdy_j}\cdot\frac{dy_k}{dx_i}\cdot\frac{dy_j}{dx_i}+\sum_j\frac{df}{dy_j}\cdot\frac{d^2y_j}{dx_i^2}\end{aligned}$$

(2) Now by summing over i, we obtain the following formula, with A being the derivative of $x \to y$, that is to say, the matrix of partial derivatives dy_i/dx_j:

$$\begin{aligned}\Delta f &= \sum_{ijk}\frac{d^2f}{dy_kdy_j}\cdot\frac{dy_k}{dx_i}\cdot\frac{dy_j}{dx_i}+\sum_{ij}\frac{df}{dy_j}\cdot\frac{d^2y_j}{dx_i^2}\\ &= \sum_{ijk}A_{ki}A_{ji}\frac{d^2f}{dy_kdy_j}+\sum_{ij}\frac{d^2y_j}{dx_i^2}\cdot\frac{df}{dy_j}\\ &= \sum_{jk}(AA^t)_{jk}\frac{d^2f}{dy_kdy_j}+\sum_j\Delta(y_j)\frac{df}{dy_j}\end{aligned}$$

(3) So, this will be the formula that we will need. Observe that this formula can be further compacted as follows, with all the notations being self-explanatory:

$$\Delta f = Tr(AA^tH_y(f)) + < \Delta(y), \nabla_y(f) >$$

(4) Getting now to spherical coordinates, $(x, y, z) \to (r, s, t)$, the derivative of the inverse, obtained by differentiating x, y, z with respect to r, s, t, is given by:

$$A^{-1} = \begin{pmatrix}\cos s & -r\sin s & 0\\ \sin s\cos t & r\cos s\cos t & -r\sin s\sin t\\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t\end{pmatrix}$$

The product $(A^{-1})^tA^{-1}$ of the transpose of this matrix with itself is then:

$$\begin{pmatrix}\cos s & \sin s\cos t & \sin s\sin t\\ -r\sin s & r\cos s\cos t & r\cos s\sin t\\ 0 & -r\sin s\sin t & r\sin s\cos t\end{pmatrix}\begin{pmatrix}\cos s & -r\sin s & 0\\ \sin s\cos t & r\cos s\cos t & -r\sin s\sin t\\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t\end{pmatrix}$$

But everything simplifies here, and we have the following remarkable formula, which by the way is something very useful, worth to be memorized:

$$(A^{-1})^t A^{-1} = \begin{pmatrix}1&0&0\\0&r^2&0\\0&0&r^2\sin^2 s\end{pmatrix}$$

Now by inverting, we obtain the following formula, in relation with the above:

$$AA^t = \begin{pmatrix}1&0&0\\0&1/r^2&0\\0&0&1/(r^2\sin^2 s)\end{pmatrix}$$

(5) Let us compute now the Laplacian of r, s, t. We first have the following formula, that we will use many times in what follows, and is worth to be memorized:

$$\begin{aligned}\frac{dr}{dx} &= \frac{d}{dx}\sqrt{x^2+y^2+z^2}\\ &= \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+y^2+z^2}}\\ &= \frac{x}{r}\end{aligned}$$

Of course the same computation works for y, z too, and we therefore have:

$$\frac{dr}{dx}=\frac{x}{r}\quad,\quad\frac{dr}{dy}=\frac{y}{r}\quad,\quad\frac{dr}{dz}=\frac{z}{r}$$

(6) By using the above formulae, twice, we can compute the Laplacian of r:

$$\begin{aligned}\Delta(r) &= \Delta\left(\sqrt{x^2+y^2+z^2}\right)\\ &= \frac{d}{dx}\left(\frac{x}{r}\right)+\frac{d}{dy}\left(\frac{y}{r}\right)+\frac{d}{dz}\left(\frac{z}{r}\right)\\ &= \frac{r^2-x^2}{r^3}+\frac{r^2-y^2}{r^3}+\frac{r^2-z^2}{r^3}\\ &= \frac{2}{r}\end{aligned}$$

(7) In what regards now s, the computation here goes as follows:

$$\begin{aligned}
\Delta(s) &= \Delta\left(\arccos\left(\frac{x}{r}\right)\right)\\
&= \frac{d}{dx}\left(-\frac{\sqrt{r^2-x^2}}{r^2}\right)+\frac{d}{dy}\left(\frac{xy}{r^2\sqrt{r^2-x^2}}\right)+\frac{d}{dz}\left(\frac{xz}{r^2\sqrt{r^2-x^2}}\right)\\
&= \frac{2x\sqrt{r^2-x^2}}{r^4}+\frac{r^2(z^2-2y^2)+2x^2y^2}{r^4\sqrt{r^2-x^2}}+\frac{r^2(y^2-2z^2)+2x^2z^2}{r^4\sqrt{r^2-x^2}}\\
&= \frac{2x\sqrt{r^2-x^2}}{r^4}+\frac{x(2x^2-r^2)}{r^4\sqrt{r^2-x^2}}\\
&= \frac{x}{r^2\sqrt{r^2-x^2}}\\
&= \frac{\cos s}{r^2\sin s}
\end{aligned}$$

(8) Finally, in what regards t, the computation here goes as follows:

$$\begin{aligned}
\Delta(t) &= \Delta\left(\arctan\left(\frac{z}{y}\right)\right)\\
&= \frac{d}{dx}(0)+\frac{d}{dy}\left(-\frac{z}{y^2+z^2}\right)+\frac{d}{dz}\left(\frac{y}{y^2+z^2}\right)\\
&= 0-\frac{2yz}{(y^2+z^2)^2}+\frac{2yz}{(y^2+z^2)^2}\\
&= 0
\end{aligned}$$

(9) We can now plug the data from (4) and (6,7,8) in the general formula that we found in (2) above, and we obtain in this way:

$$\begin{aligned}
\Delta f &= \frac{d^2f}{dr^2}+\frac{1}{r^2}\cdot\frac{d^2f}{ds^2}+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2f}{dt^2}+\frac{2}{r}\cdot\frac{df}{dr}+\frac{\cos s}{r^2\sin s}\cdot\frac{df}{ds}\\
&= \frac{2}{r}\cdot\frac{df}{dr}+\frac{d^2f}{dr^2}+\frac{\cos s}{r^2\sin s}\cdot\frac{df}{ds}+\frac{1}{r^2}\cdot\frac{d^2f}{ds^2}+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2f}{dt^2}\\
&= \frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{df}{dr}\right)+\frac{1}{r^2\sin s}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{df}{ds}\right)+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2f}{dt^2}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

The point now is that, with the above formula for Δ in hand, the wave equation can be reformulated in spherical coordinates, and the same goes for the heat equation. Many things can be said here, and we will be back to this in chapter 10.

9e. Exercises

This was a rather mathematical physics chapter, and as exercises on this, we have:

EXERCISE 9.16. *Learn more about the Cauchy-Riemann operators.*

EXERCISE 9.17. *Clarify the details, in the proof of the mean value formula.*

EXERCISE 9.18. *Fill in the details, in the proof of the Liouville theorem too.*

EXERCISE 9.19. *And do the same for* $f = Re(g)$ *theorem, in two dimensions.*

EXERCISE 9.20. *Learn more about waves, via classical mechanics, and elasticity.*

EXERCISE 9.21. *Learn also about the various corrections to the heat equation.*

EXERCISE 9.22. *Work out the discretization of the d'Alembert integral.*

EXERCISE 9.23. *Learn also more about the heat equation, in one dimension.*

As bonus exercise, read about electrostatics too, and the Laplace equation there.

CHAPTER 10

Fourier analysis

10a. Function spaces

In this chapter we go back to the functions of one real variable, $f:\mathbb R\to\mathbb R$ or $f:\mathbb R\to\mathbb C$, with some applications, by using our complex function technology. We will be mainly interested in constructing the Fourier transform, which is an operation $f\to\widehat{f}$ on such functions, which can solve various questions.

Before doing that, however, let us study the spaces that the functions $f:\mathbb R\to\mathbb C$ can form. Let us start with some well-known and useful inequalities, as follows:

THEOREM 10.1. *Given two functions $f,g:\mathbb R\to\mathbb C$ and an exponent $p\geq 1$, we have*

$$\left(\int_{\mathbb R}|f+g|^p\right)^{1/p}\leq\left(\int_{\mathbb R}|f|^p\right)^{1/p}+\left(\int_{\mathbb R}|g|^p\right)^{1/p}$$

called Minkowski inequality. Also, assuming that $p,q\geq 1$ satisfy $1/p+1/q=1$, we have

$$\int_{\mathbb R}|fg|\leq\left(\int_{\mathbb R}|f|^p\right)^{1/p}\left(\int_{\mathbb R}|g|^q\right)^{1/q}$$

called Hölder inequality. These inequalities hold as well for ∞ values of the exponents.

PROOF. All this is very standard, the idea being as follows:

(1) As a first observation, at $p=2$, which is a special exponent, we have $q=2$ as well, and the Minkowski and Hölder inequalities are as follows:

$$\left(\int_{\mathbb R}|f+g|^2\right)^{1/2}\leq\left(\int_{\mathbb R}|f|^2\right)^{1/2}+\left(\int_{\mathbb R}|g|^2\right)^{1/2}$$

$$\int_{\mathbb R}|fg|\leq\left(\int_{\mathbb R}|f|^2\right)^{1/2}\left(\int_{\mathbb R}|g|^2\right)^{1/2}$$

But the proof of the Hölder inequality, called Cauchy-Schwarz inequality in this case, is something elementary, coming from the fact that $I(t)=\int_{\mathbb R}|f+twg|^2$ with $|w|=1$ is a positive degree 2 polynomial in $t\in\mathbb R$, and so its discriminant must be negative. As for the Minkowski inequality, this follows from this, by taking squares and simplifying.

(2) In general, let us first prove Hölder, in the case of finite exponents, $p,q\in(1,\infty)$. By linearity we can assume that f,g are normalized, in the following way:

$$\int_\mathbb{R}|f|^p=\int_\mathbb{R}|g|^q=1$$

In this case, we want to prove that we have $\int_\mathbb{R}|fg|\leq 1$. And for this purpose, we can use the Young inequality, which gives, for any $x\in\mathbb{R}$:

$$|f(x)g(x)|\leq\frac{|f(x)|^p}{p}+\frac{|g(x)|^q}{q}$$

By integrating now over $x\in\mathbb{R}$, we obtain from this, as desired:

$$\int_\mathbb{R}|fg|\leq\int_\mathbb{R}\frac{|f|^p}{p}+\frac{|g|^q}{q}=\frac{1}{p}+\frac{1}{q}=1$$

Let us prove now Minkowski, again in the finite exponent case, $p\in(1,\infty)$. We have the following estimate, using the Hölder inequality, and the conjugate exponent:

$$\begin{aligned}
\int_\mathbb{R}|f+g|^p &= \int_\mathbb{R}|f+g|\cdot|f+g|^{p-1}\\
&\leq \int_\mathbb{R}|f|\cdot|f+g|^{p-1}+\int_\mathbb{R}|g|\cdot|f+g|^{p-1}\\
&\leq \left(\int_\mathbb{R}|f|^p\right)^{1/p}\left(\int_\mathbb{R}|f+g|^{(p-1)q}\right)^{1/q}\\
&+ \left(\int_\mathbb{R}|g|^p\right)^{1/p}\left(\int_\mathbb{R}|f+g|^{(p-1)q}\right)^{1/q}\\
&= \left[\left(\int_\mathbb{R}|f|^p\right)^{1/p}+\left(\int_\mathbb{R}|g|^p\right)^{1/p}\right]\left(\int_\mathbb{R}|f+g|^p\right)^{1-1/p}
\end{aligned}$$

Thus, we are led to the Minkowski inequality in the statement.

(3) Finally, with the convention that $(\int_\mathbb{R}|f|^p)^{1/p}$ takes as value at $p=\infty$ the essential supremum of f, the Minkowski inequality holds as well at $p=\infty$, trivially:

$$\sup\left|f+g\right|\leq\sup\left|f\right|+\sup\left|g\right|$$

The same goes for the Hölder inequality at $p=\infty,q=1$, which is simply:

$$\int_\mathbb{R}|fg|\leq\sup\left|f\right|\times\int_\mathbb{R}|g|$$

And finally, the same goes for the Hölder inequality at $p=1,q=\infty$. □

As a consequence of the above results, we can formulate:

THEOREM 10.2. *Given an interval $I \subset \mathbb{R}$ and an exponent $p \geq 1$, the following space, with the convention that functions are identified up to equality almost everywhere,*

$$L^p(I) = \left\{ f : I \to \mathbb{C} \Big| \int_I |f(x)|^p dx < \infty \right\}$$

is a vector space, and the following quantity,

$$||f||_p = \left(\int_I |f(x)|^p \right)^{1/p}$$

is a norm on it, in the sense that it satisfies the usual conditions for a vector space norm. Moreover, $L^p(I)$ is complete with respect to the distance $d(f,g) = ||f-g||_p$.

PROOF. This basically follows from Theorem 10.1, the idea being as follows:

(1) Again, let us first see what happens at $p = 2$. Here everything is standard from what we have in Theorem 10.1, and with the remark that the space $L^2(I)$ that we obtain is more than just a normed vector space, because we have as well a scalar product, related to the norm by the formula $||f||_2 = \sqrt{< f, f >}$, constructed as follows:

$$< f, g >= \int_I f(x) \overline{g(x)} \, dx$$

(2) In the general case now, where $p \geq 1$ is still finite, but arbitrary, the proof is similar, basically coming from the Minkowski inequality from Theorem 10.1.

(3) Finally, the extension at $p = \infty$ is clear too, coming from definitions, and with the various conventions made at the end of the proof of Theorem 10.1. □

Going ahead now with our study of functions $f : \mathbb{R} \to \mathbb{C}$, let us define an interesting operation on such functions, called convolution, which is useful for many purposes:

DEFINITION 10.3. *The convolution of two functions $f, g : \mathbb{R} \to \mathbb{C}$ is the function*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$$

provided that the function $y \to f(x-y)g(y)$ is indeed integrable, for any x.

There are many reasons for introducing this operation, that we will gradually discover, in what follows. As a basic example, let us take $g = \chi_{[0,1]}$. We have then:

$$(f * g)(x) = \int_0^1 f(x-y)dy$$

Thus, with this choice of g, the operation $f \to f * g$ has some sort of "regularizing effect", that can be useful for many purposes. We will be back to this, later.

At the level of the axiomatics, we have the following basic result:

THEOREM 10.4. *The convolution operation is well-defined on the space*

$$C_c(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) \Big| supp(f) = \text{ compact} \right\}$$

of continuous functions $f : \mathbb{R} \to \mathbb{C}$ having compact support.

PROOF. We have several things to be proved, the idea being as follows:

(1) First we must show that given two functions $f, g \in C_c(\mathbb{R})$, their convolution $f * g$ is well-defined, as a function $f * g : \mathbb{R} \to \mathbb{C}$. But this follows from the following estimate, where l denotes the length of the compact subsets of $\mathbb{R}$:

$$\begin{aligned}
\int_{\mathbb{R}} |f(x-y)g(y)|dy &= \int_{supp(g)} |f(x-y)g(y)|dy \\
&\leq \max(g) \int_{supp(g)} |f(x-y)|dy \\
&\leq \max(g) \cdot l(supp(g)) \cdot \max(f) \\
&< \infty
\end{aligned}$$

(2) Next, we must show that the function $f * g : \mathbb{R} \to \mathbb{C}$ that we constructed is indeed continuous. But this follows from the following estimate, where K_f is the constant of uniform continuity for the function $f \in C_c(\mathbb{R})$:

$$\begin{aligned}
|(f*g)(x+\varepsilon) - (f*g)(x)| &= \left| \int_{\mathbb{R}} f(x+\varepsilon-y)g(y)dy - \int_{\mathbb{R}} f(x-y)g(y)dy \right| \\
&= \left| \int_{\mathbb{R}} \left(f(x+\varepsilon-y) - f(x-y)\right) g(y)dy \right| \\
&\leq \int_{\mathbb{R}} |f(x+\varepsilon-y) - f(x-y)| \cdot |g(y)|dy \\
&\leq K_f \cdot \varepsilon \cdot \int_{\mathbb{R}} |g|
\end{aligned}$$

(3) Finally, we must show that the function $f * g \in C(\mathbb{R})$ that we constructed has indeed compact support. For this purpose, our claim is that we have:

$$supp(f+g) \subset supp(f) + supp(g)$$

In order to prove this claim, observe that we have, by definition of $f * g$:

$$\begin{aligned}
(f*g)(x) &= \int_{\mathbb{R}} f(x-y)g(y)dy \\
&= \int_{supp(g)} f(x-y)g(y)dy
\end{aligned}$$

But this latter quantity being 0 for $x \notin supp(f) + supp(g)$, this gives the result. $\square$

Here are now a few remarkable properties of the convolution operation:

PROPOSITION 10.5. *The following hold, for the functions in* $C_c(\mathbb{R})$*:*

(1) $f * g = g * f$.
(2) $f * (g * h) = (f * g) * h$.
(3) $f * (\lambda g + \mu h) = \lambda f * g + \mu f * h$.

PROOF. These formulae are all elementary, the idea being as follows:

(1) This follows from the following computation, with $y = x - t$:

$$\begin{aligned}(f * g)(x) &= \int_{\mathbb{R}} f(x-y)g(y)dy \\ &= \int_{\mathbb{R}} f(t)g(x-t)dt \\ &= \int_{\mathbb{R}} g(x-t)f(t)dt \\ &= (g * f)(x)\end{aligned}$$

(2) This is clear from definitions.

(3) Once again, this is clear from definitions. □

In relation with derivatives, and with the "regularizing effect" of the convolution operation mentioned after Definition 10.3, we have the following result:

THEOREM 10.6. *Given two functions* $f, g \in C_c(\mathbb{R})$*, assuming that* g *is differentiable, then so is* $f * g$*, with derivative given by the following formula:*

$$(f * g)' = f * g'$$

More generally, given $f, g \in C_c(\mathbb{R})$*, and assuming that* g *is* k *times differentiable, then so is* $f * g$*, with* k*-th derivative given by* $(f * g)^{(k)} = f * g^{(k)}$*.*

PROOF. In what regards the first assertion, with $y = x - t$, then $t = x - y$, we get:

$$\begin{aligned}(f * g)'(x) &= \frac{d}{dx}\int_{\mathbb{R}} f(x-y)g(y)dy \\ &= \frac{d}{dx}\int_{\mathbb{R}} f(t)g(x-t)dt \\ &= \int_{\mathbb{R}} f(t)g'(x-t)dt \\ &= \int_{\mathbb{R}} f(x-y)g'(y)dy \\ &= (f * g')(x)\end{aligned}$$

As for the second assertion, this follows form the first one, by recurrence. □

Finally, getting beyond the compactly supported continuous functions, we have the following result, which is of particular theoretical importance:

THEOREM 10.7. *The convolution operation is well-defined on $L^1(\mathbb{R})$, and we have:*

$$||f * g||_1 \leq ||f||_1||g||_1$$

Thus, if $f \in L^1(\mathbb{R})$ and $g \in C_c^k(\mathbb{R})$, then $f * g$ is well-defined, and $f * g \in C_c^k(\mathbb{R})$.

PROOF. In what regards the first assertion, this follows from the following computation, involving an intuitive manipulation on the double integrals, called Fubini theorem, that we will use as such here, and that we will fully clarify later on, in this book:

$$\begin{aligned}\int_{\mathbb{R}}|(f*g)(x)|dx &\leq \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)g(y)|dydx\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-y)g(y)|dxdy\\ &= \int_{\mathbb{R}}|f(x)|dx\int_{\mathbb{R}}|g(y)|dy\end{aligned}$$

As for the second assertion, this follows from the first one, and from Theorem 10.6. □

Summarizing, we have now some good knowledge of the various spaces that the functions $f : \mathbb{R} \to \mathbb{C}$ can form, and we have as well an interesting regularization operation $f \to f * g$ on such functions, that can be used for various purposes.

10b. Fourier transform

We discuss here the construction and main properties of the Fourier transform, which is the main tool in analysis, and even in mathematics in general. We first have:

DEFINITION 10.8. *Given $f \in L^1(\mathbb{R})$, we define a function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{ix\xi} f(x)dx$$

and call it Fourier transform of f.

As a first observation, even if f is a real function, $\widehat{f}$ is a complex function, which is not necessarily real. Also, $\widehat{f}$ is obviously well-defined, because $f \in L^1(\mathbb{R})$ and $|e^{ix\xi}| = 1$. Also, the condition $f \in L^1(\mathbb{R})$ is basically needed for construcing $\widehat{f}$, because:

$$\widehat{f}(0) = \int_{\mathbb{R}} f(x)dx$$

Generally speaking, the Fourier transform is there for helping with various computations, with the above formula $\widehat{f}(0) = \int f$ being something quite illustrating. Here are some basic properties of the Fourier transform, all providing some good motivations:

PROPOSITION 10.9. *The Fourier transform has the following properties:*

(1) *Linearity:* $\widehat{f+g}=\widehat{f}+\widehat{g}$, $\widehat{\lambda f}=\lambda\widehat{f}$.
(2) *Regularity:* $\widehat{f}$ *is continuous and bounded.*
(3) *If* f *is even then* $\widehat{f}$ *is even.*
(4) *If* f *is odd then* $\widehat{f}$ *is odd.*

PROOF. These results are all elementary, as follows:

(1) The additivity formula is clear from definitions, as follows:

$$\begin{aligned}\widehat{f+g}(\xi) &= \int_{\mathbb{R}} e^{ix\xi}(f+g)(x)dx\\ &= \int_{\mathbb{R}} e^{ix\xi}f(x)dx+\int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= \widehat{f}(\xi)+\widehat{g}(\xi)\end{aligned}$$

As for the formula $\widehat{\lambda f}=\lambda\widehat{f}$, this is clear as well.

(2) The continuity of $\widehat{f}$ follows indeed from:

$$\begin{aligned}|\widehat{f}(\xi+\varepsilon)-\widehat{f}(\xi)| &\leq \int_{\mathbb{R}}\left|(e^{ix(\xi+\varepsilon)}-e^{ix\xi})f(x)\right|dx\\ &= \int_{\mathbb{R}}\left|e^{ix\xi}(e^{ix\varepsilon}-1)f(x)\right|dx\\ &\leq |e^{ix\varepsilon}-1|\int_{\mathbb{R}}|f|\end{aligned}$$

As for the boundedness of $\widehat{f}$, this is clear as well.

(3) This follows from the following computation, assuming that f is even:

$$\begin{aligned}\widehat{f}(-\xi) &= \int_{\mathbb{R}} e^{-ix\xi}f(x)dx\\ &= \int_{\mathbb{R}} e^{ix\xi}f(-x)dx\\ &= \int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= \widehat{f}(\xi)\end{aligned}$$

(4) The proof here is similar to the proof of (3), by changing some signs. □

We will be back to more theory in a moment, but let us explore now the examples. Here are some basic computations of Fourier transforms:

Proposition 10.10. *We have the following Fourier transform formulae,*

$$f=\chi_{[-a,a]}\implies\widehat{f}(\xi)=\frac{2\sin(a\xi)}{\xi}$$

$$f=e^{-ax}\chi_{[0,\infty]}(x)\implies\widehat{f}(\xi)=\frac{1}{a-i\xi}$$

$$f=e^{ax}\chi_{[-\infty,0]}(x)\implies\widehat{f}(\xi)=\frac{1}{a+i\xi}$$

$$f=e^{-a|x|}\implies\widehat{f}(\xi)=\frac{2a}{a^2+\xi^2}$$

$$f=sgn(x)e^{-a|x|}\implies\widehat{f}(\xi)=\frac{2i\xi}{a^2+\xi^2}$$

valid for any number $a>0$.

Proof. All this follows from some calculus, as follows:

(1) In what regards first formula, assuming $f=\chi_{[-a,a]}$, we have, by using the fact that $\sin(x\xi)$ is an odd function, whose integral vanishes on centered intervals:

$$\begin{aligned}\widehat{f}(\xi)&=\int_{-a}^{a}e^{ix\xi}dx\\&=\int_{-a}^{a}\cos(x\xi)dx+i\int_{-a}^{a}\sin(x\xi)dx\\&=\int_{-a}^{a}\cos(x\xi)dx\\&=\left[\frac{\sin(x\xi)}{\xi}\right]_{-a}^{a}\\&=\frac{2\sin(a\xi)}{\xi}\end{aligned}$$

(2) With $f(x)=e^{-ax}\chi_{[0,\infty]}(x)$, the computation goes as follows:

$$\begin{aligned}\widehat{f}(\xi)&=\int_{0}^{\infty}e^{ix\xi-ax}dx\\&=\int_{0}^{\infty}e^{(i\xi-a)x}dx\\&=\left[\frac{e^{(i\xi-a)x}}{i\xi-a}\right]_{0}^{\infty}\\&=\frac{1}{a-i\xi}\end{aligned}$$

(3) Regarding the third formula, this follows from the second one, by using the following fact, generalizing the parity computations from Proposition 10.9:

$$F(x) = f(-x) \implies \widehat{F}(\xi) = \widehat{f}(-\xi)$$

(4) The last 2 formulae follow from what we have, by making sums and differences, and the linearity properties of the Fourier transform, from Proposition 10.9. □

We will see many other examples, in what follows. Getting back now to theory, we have the following result, adding to the various general properties in Proposition 10.9, and providing more motivations for the Fourier transform:

PROPOSITION 10.11. *Given $f, g \in L^1(\mathbb{R})$ we have $\widehat{f}g, f\widehat{g} \in L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} f(\xi)\widehat{g}(\xi)d\xi = \int_{\mathbb{R}} \widehat{f}(x)g(x)dx$$

called "exchange of hat" formula.

PROOF. Regarding the fact that we have indeed $\widehat{f}g, f\widehat{g} \in L^1(\mathbb{R})$, this is actually a bit non-trivial, but we will be back to this later. Assuming this, we have:

$$\int_{\mathbb{R}} f(\xi)\widehat{g}(\xi)d\xi = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\xi)e^{ix\xi}g(x)dxd\xi$$

On the other hand, we have as well the following formula:

$$\int_{\mathbb{R}} \widehat{f}(x)g(x)dx = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi}f(x)g(\xi)dxd\xi$$

Thus, with $x \leftrightarrow \xi$, we are led to the formula in the statement. □

As an important result, showing the power of the Fourier transform, this transforms the derivative into something very simple, namely a multiplication by the variable:

THEOREM 10.12. *Given $f : \mathbb{R} \to \mathbb{C}$ such that $f, f' \in L^1(\mathbb{R})$, we have:*

$$\widehat{f'}(\xi) = -i\xi\widehat{f}(\xi)$$

More generally, assuming $f, f', f'', \ldots, f^{(n)} \in L^1(\mathbb{R})$, we have

$$\widehat{f^{(k)}}(\xi) = (-i\xi)^k\widehat{f}(\xi)$$

for any $k = 1, 2, \ldots, n$.

PROOF. These results follow by doing a partial integration, as follows:

(1) Assuming that $f:\mathbb{R}\to\mathbb{C}$ has compact support, we have indeed:

$$\begin{aligned}\widehat{f'}(\xi) &= \int_{\mathbb{R}} e^{ix\xi}f'(x)dx\\ &= -\int_{\mathbb{R}} i\xi e^{ix\xi}f(x)dx\\ &= -i\xi\int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= -i\xi\widehat{f}(\xi)\end{aligned}$$

(2) Regarding the higher derivatives, the formula here follows by recurrence. □

Importantly, we have a converse statement as well, as follows:

THEOREM 10.13. *Assuming that $f\in L^1(\mathbb{R})$ is such that $F(x)=xf(x)$ belongs to $L^1(\mathbb{R})$ too, the function $\widehat{f}$ is differentiable, with derivative given by:*

$$(\widehat{f})'(\xi)=i\widehat{F}(\xi)$$

More generally, if $F_k(x)=x^kf(x)$ belongs to $L^1(\mathbb{R})$, for $k=0,1,\ldots,n$, we have

$$(\widehat{f})^{(k)}(\xi)=i^k\widehat{F_k}(\xi)$$

for any $k=1,2,\ldots,n$.

PROOF. These results are both elementary, as follows:

(1) Regarding the first assertion, the computation here is as follows:

$$\begin{aligned}(\widehat{f})'(\xi) &= \frac{d}{d\xi}\int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= \int_{\mathbb{R}} ixe^{ix\xi}f(x)dx\\ &= i\int_{\mathbb{R}} e^{ix\xi}xf(x)dx\\ &= i\widehat{F}(\xi)\end{aligned}$$

(2) As for the second assertion, this follows from the first one, by recurrence. □

As a conclusion to all this, we are on a good way with our theory, and we have:

CONCLUSION 10.14. *Modulo normalization factors, the Fourier transform converts the derivatives into multiplications by the variable, and vice versa.*

And isn't this interesting, because isn't computing derivatives a difficult task. Here is now another useful result, of the same type, this time regarding convolutions:

THEOREM 10.15. *Assuming $f,g\in L^1(\mathbb R)$, the following happens:*

$$\widehat{f*g}=\widehat{f}\cdot\widehat{g}$$

Moreover, under suitable assumptions, the formula $\widehat{fg}=\widehat{f}*\widehat{g}$ holds too.

PROOF. This is something quite subtle, the idea being as follows:

(1) Regarding the first assertion, this is something elementary, as follows:

$$\begin{aligned}
\widehat{f*g}(\xi) &= \int_{\mathbb R} e^{ix\xi}(f*g)(x)dx\\
&= \int_{\mathbb R}\int_{\mathbb R} e^{ix\xi}f(x-y)g(y)dxdy\\
&= \int_{\mathbb R} e^{iy\xi}\left(\int_{\mathbb R} e^{i(x-y)\xi}f(x-y)dx\right)g(y)dy\\
&= \int_{\mathbb R} e^{iy\xi}\left(\int_{\mathbb R} e^{it\xi}f(t)dt\right)g(y)dy\\
&= \int_{\mathbb R} e^{iy\xi}\widehat{f}(\xi)g(y)dy\\
&= \widehat{f}(\xi)\widehat{g}(\xi)
\end{aligned}$$

(2) As for the second assertion, this is something more tricky, and we will be back to it later. In the meantime, here is however some sort of proof, not very honest:

$$\begin{aligned}
(\widehat{f}*\widehat{g})(\xi) &= \int_{\mathbb R}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\\
&= \int_{\mathbb R}\int_{\mathbb R}\int_{\mathbb R} e^{ix(\xi-\eta)}f(x)e^{iy\eta}g(y)dxdyd\eta\\
&= \int_{\mathbb R}\int_{\mathbb R}\int_{\mathbb R} e^{ix\eta}e^{i(y-x)\eta}f(x)g(y)dxdyd\eta\\
&= \int_{\mathbb R} e^{ix\eta}f(x)g(x)dx\\
&= \widehat{fg}(\eta)
\end{aligned}$$

To be more precise, the point here is that we can pass from the triple to the single integral by arguing that "we must have $x=y$". We will be back to this later. □

As an updated conclusion to all this, we have, modulo a few bugs, to be fixed:

CONCLUSION 10.16. *The Fourier transform converts the derivatives into multiplications by the variable, and convolutions into products, and vice versa.*

We will see applications of this later, after developing some more general theory. So, let us develop now more theory for the Fourier transform. We first have:

THEOREM 10.17. *Given $f \in L^1(\mathbb{R})$, its Fourier transform satisfies*

$$\lim_{\xi \to \pm\infty} \widehat{f}(\xi) = 0$$

called Riemann-Lebesgue property of $\widehat{f}$.

PROOF. This is something quite technical, as follows:

(1) Given a function $f : \mathbb{R} \to \mathbb{C}$ and a number $y \in \mathbb{R}$, let us set:

$$f_y(x) = f(x - y)$$

Our claim is then that if $f \in L^p(\mathbb{R})$, then the following function is uniformly continuous, with respect to the usual p-norm on the right:

$$\mathbb{R} \to L^p(\mathbb{R}) \quad , \quad y \to f_y$$

(2) In order to prove this, fix $\varepsilon > 0$. Since $f \in L^p(\mathbb{R})$, we can find a function of type $g : [-K, K] \to \mathbb{C}$ which is continuous, such that:

$$||f - g||_p < \varepsilon$$

Now since g is uniformly continuous, we can find $\delta \in (0, K)$ such that:

$$|s - t| < \delta \implies |g(s) - g(t)| < (3K)^{-1/p}\varepsilon$$

But this shows that we have the following estimate:

$$\begin{aligned} ||g_s - g_t||_p &= \left(\int_{\mathbb{R}} \left|g(x - s) - g(x - t)\right|^p dx\right)^{1/p} \\ &< \left[(3K)^{-1}\varepsilon^p(2k + \delta)\right]^{1/p} \\ &< \varepsilon \end{aligned}$$

By using now the formula $||f||_p = ||f_s||_p$, which is clear, we obtain:

$$\begin{aligned} ||f_s - f_t||_p &\leq ||f_s - g_s||_p + ||g_s - g_t||_p + ||g_t - f_t||_p \\ &< \varepsilon + \varepsilon + \varepsilon \\ &= 3\varepsilon \end{aligned}$$

But this being true for any $|s - t| < \delta$, we have proved our claim.

(3) Let us prove now the Riemann-Lebesgue property of $\widehat{f}$, as formulated in the statement. By using $e^{\pi i}=-1$, and the change of variables $x\to x-\pi/\xi$, we have:

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= -\int_{\mathbb{R}} e^{ix\xi}e^{\pi i}f(x)dx\\ &= -\int_{\mathbb{R}} e^{i\xi(x+\pi/\xi)}f(x)dx\\ &= -\int_{\mathbb{R}} e^{ix\xi}f\left(x-\frac{\pi}{\xi}\right)dx\end{aligned}$$

On the other hand, we have as well the following formula:

$$\widehat{f}(\xi)=\int_{\mathbb{R}} e^{ix\xi}f(x)dx$$

Thus by summing, we obtain the following formula:

$$2\widehat{f}(\xi)=\int_{\mathbb{R}} e^{ix\xi}\left(f(x)-f\left(x-\frac{\pi}{\xi}\right)\right)dx$$

But this gives the following estimate, with notations from (1):

$$2|\widehat{f}(\xi)|\leq ||f-f_{\pi/\xi}||_1$$

Since by (1) this goes to 0 with $\xi\to\pm\infty$, this gives the result. □

Quite remarkably, and as a main result now regarding Fourier transforms, a function $f:\mathbb{R}\to\mathbb{C}$ can be recovered from its Fourier transform $\widehat{f}:\mathbb{R}\to\mathbb{C}$, as follows:

Theorem 10.18. *Assuming $f,\widehat{f}\in L^1(\mathbb{R})$, we have*

$$f(x)=\int_{\mathbb{R}} e^{-ix\xi}\widehat{f}(\xi)d\xi$$

almost everywhere, called Fourier inversion formula.

Proof. This is something quite tricky, due to the fact that a direct attempt by double integration fails. Consider the following function, depending on a parameter $\lambda>0$:

$$\varphi_\lambda(x)=\int_{\mathbb{R}} e^{-ix\xi-\lambda|\xi|}d\xi$$

We have then the following computation:

$$\begin{aligned}(f*\varphi_\lambda)(x) &= \int_{\mathbb{R}} f(x-y)\varphi_\lambda(y)dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)e^{-iy\xi-\lambda|\xi|}d\xi dy\\ &= \int_{\mathbb{R}} e^{-\lambda|\xi|}\left(\int_{\mathbb{R}} f(x-y)e^{-iy\xi}dy\right)d\xi\\ &= \int_{\mathbb{R}} e^{-\lambda|\xi|}e^{-ix\xi}\widehat{f}(\xi)d\xi\end{aligned}$$

By letting now $\lambda\to 0$, we obtain from this the following formula:

$$\lim_{\lambda\to 0}(f*\varphi_\lambda)(x)=\int_{\mathbb{R}} e^{-ix\xi}\widehat{f}(\xi)d\xi$$

On the other hand, by using Theorem 10.17 we obtain that, almost everywhere:

$$\lim_{\lambda\to 0}(f*\varphi_\lambda)(x)=f(x)$$

Thus, we are led to the conclusion in the statement. □

10c. Distributions

Let us discuss now a related topic, the mathematical distributions. These are something quite smart, and as an advertisement for what we will be doing, we have:

ADVERTISEMENT 10.19. *With a suitable theory of distributions, covering both the functions and the Dirac masses, the basic step function, namely*

$$H(x)=\begin{cases}0 & (x\leq 0)\\ 1 & (x>0)\end{cases}$$

is differentiable when viewed as distribution, with derivative $H'=\delta_0$.

And isn't this crazy, hope you agree with me. Getting started now, there is a price to pay for doing such things, namely formulating a technical definition, as follows:

DEFINITION 10.20. *A distribution on an open interval* $I\subset\mathbb{R}$ *is a functional*

$$\varphi: C_c^\infty(I)\to\mathbb{C}$$

such that for any $K\subset I$ *compact, there exist* $n\in\mathbb{N}$ *and* $c>0$ *such that*

$$|\varphi(f)|\leq c||f||_{C^n(K)}$$

for any $f\in C_c^\infty(I)$ *having support in* K*, where* $||f||_{C^n(K)}=\sup_{x\in K}\sum_{i=0}^n|f^{(i)}(x)|$.

Obviously, this is something quite technical, but believe me, every little thing in the above is there for a reason, in order to the whole theory to work fine.

At the level of main examples of distributions, we have the integration functionals associated to the measures, and in particular to the measures having a density. In view of this, we can consider any function $f \in L^1(I)$, viewed as density, as being a distribution. Other basic examples include the Dirac masses δ_x at the points $x \in I$.

Regarding the general theory of distributions, that is quite similar to the theory of functions. Algebrically, the distributions form a vector space, and are subject to a number of supplementary operations too, such as dilations, translations and so on, and multiplication by functions too. Analytically, we can talk about convergence of distributions, $\varphi_n \to \varphi$, and about their support too, $supp(\varphi) \subset I$, in a quite straightforward way.

Getting now to what we wanted to do, derivatives, we have here:

THEOREM 10.21. *We can talk about the derivatives of distributions, given by*

$$\varphi'(f) = -\varphi(f')$$

and with this notion in hand, the following happen:

1. *When φ is a usual differentiable function, φ' is the usual derivative.*
2. *For the basic step function we have $H' = \delta_0$, as previously advertised.*
3. *In fact, for a function $\varphi = g$ with jumps at $\{x_i\}$, we have $\varphi' = g' + \sum_i J_g(x_i)\delta_{x_i}$.*

PROOF. The first assertion, which by the way explains the need for the $-$ sign, follows from the Leibnitz rule for derivatives. Regarding the second assertion, this follows from:

$$\begin{aligned} H'(f) &= -H(f') \\ &= -\int_0^\infty f'(x)dx \\ &= -f(\infty) + f(0) \\ &= f(0) \\ &= \delta_0(f) \end{aligned}$$

As for the third assertion, which generalizes (1,2), we will leave this as an exercise. □

10d. Independence, limits

With the above theory in hand, let us go back to probability. Our claim is that things are very interesting here, with the real-life notion of independence corresponding to our mathematical notion of convolution, and with the Fourier transform being something very useful in probability, in order to understand the independence. Let us start with:

Definition 10.22. *Two variables $f,g\in L^\infty(X)$ are called independent when*

$$E(f^kg^l)=E(f^k)\,E(g^l)$$

happens, for any $k,l\in\mathbb N$.

This definition hides some non-trivial things. Indeed, by linearity, we would like to have a formula as follows, valid for any polynomials $P,Q\in\mathbb R[X]$:

$$E[P(f)Q(g)]=E[P(f)]\,E[Q(g)]$$

By a continuity argument, it is enough to have this formula for the characteristic functions χ_I,χ_J of the arbitrary measurable sets of real numbers $I,J\subset\mathbb R$:

$$E[\chi_I(f)\chi_J(g)]=E[\chi_I(f)]\,E[\chi_J(g)]$$

Thus, we are led to the usual definition of independence, namely:

$$P(f\in I,g\in J)=P(f\in I)\,P(g\in J)$$

All this might seem a bit abstract, but in practice, the idea is of course that f,g must be independent, in an intuitive, real-life sense. As a first result now, we have:

Theorem 10.23. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$\mu_{f+g}=\mu_f*\mu_g$$

where $*$ is the convolution of real probability measures.

Proof. We have the following computation, using the independence of f,g:

$$\begin{aligned}M_k(f+g)&=E((f+g)^k)\\&=\sum_r\binom{k}{r}E(f^rg^{k-r})\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\int_{\mathbb R}x^kd(\mu_f*\mu_g)(x)&=\int_{\mathbb R\times\mathbb R}(x+y)^kd\mu_f(x)d\mu_g(y)\\&=\sum_r\binom{k}{r}\int_{\mathbb R}x^rd\mu_f(x)\int_{\mathbb R}y^{k-r}d\mu_g(y)\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

Thus μ_{f+g} and $\mu_f*\mu_g$ have the same moments, and so they coincide, as claimed. □

Here is now a second result on independence, which is something more advanced:

THEOREM 10.24. *Assuming that $f, g \in L^\infty(X)$ are independent, we have*

$$F_{f+g} = F_f F_g$$

where $F_f(x) = E(e^{ixf})$ is the Fourier transform.

PROOF. We have the following computation, by using Theorem 10.23:

$$\begin{aligned}
F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixz} d\mu_{f+g}(z) \\
&= \int_{\mathbb{R}} e^{ixz} d(\mu_f * \mu_g)(z) \\
&= \int_{\mathbb{R}\times\mathbb{R}} e^{ix(z+t)} d\mu_f(z) d\mu_g(t) \\
&= \int_{\mathbb{R}} e^{ixz} d\mu_f(z) \int_{\mathbb{R}} e^{ixt} d\mu_g(t) \\
&= F_f(x) F_g(x)
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As an application, we can now establish the Central Limit Theorem, as follows:

THEOREM 10.25 (CLT). *Given random variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i \sim g_t$$

where g_t is the Gaussian law of parameter t, having as density $\frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy$.

PROOF. In terms of moments, the Fourier transform is given by:

$$\begin{aligned}
F_f(x) &= E\left(\sum_{k=0}^{\infty} \frac{(ixf)^k}{k!}\right) \\
&= \sum_{k=0}^{\infty} \frac{(ix)^k E(f^k)}{k!} \\
&= \sum_{k=0}^{\infty} \frac{i^k M_k(f)}{k!} x^k
\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}
F(x) &= \left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n \\
&= \left[1-\frac{tx^2}{2n}+O(n^{-2})\right]^n \\
&\simeq \left[1-\frac{tx^2}{2n}\right]^n \\
&\simeq e^{-tx^2/2}
\end{aligned}$$

On the other hand, the Fourier transform of g_t is given by:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-y^2/2t+ixy}dy \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2}dy \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-z^2-tx^2/2}\sqrt{2t}dz \\
&= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\int_{\mathbb{R}} e^{-z^2}dz \\
&= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\cdot\sqrt{\pi} \\
&= e^{-tx^2/2}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

10e. Exercises

This was a key analytic chapter, and as exercises here, we have:

EXERCISE 10.26. *Revise if needed the Young inequality, coming from Jensen.*

EXERCISE 10.27. *Learn more about L^p spaces, notably with $(L^p)^* = L^q$.*

EXERCISE 10.28. *Learn about the Fourier transform over the space L^2.*

EXERCISE 10.29. *Learn also about Fourier over the Schwarz space $\mathcal{S}$.*

EXERCISE 10.30. *Learn also about Fourier over a locally compact group G.*

EXERCISE 10.31. *Clarify what we said above, in relation with distributions.*

EXERCISE 10.32. *Learn more about the CLT, and related limiting theorems.*

EXERCISE 10.33. *What does all the above tell us, about waves and heat?*

As bonus exercise, start learning some systematic functional analysis.

CHAPTER 11

Conformal maps

11a. Conformal maps

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.

CHAPTER 12

Riemann surfaces

12a. Riemann surfaces

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Advanced aspects

CHAPTER 13

Rational approximation

13a. Rational approximation

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

Polynomial approximation

14a. Polynomial approximation

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.

CHAPTER 15

Algebraic curves

15a. Algebraic curves

We would like to discuss, in the remainder of this book, some interesting phenomena appearing in the complex plane $\mathbb{C}$, of rather geometric nature. Let us start with:

DEFINITION 15.1. *An algebraic curve in $\mathbb{R}^2$ is the vanishing set*

$$C=\left\{(x,y)\in\mathbb{R}^2\Big|P(x,y)=0\right\}$$

of a polynomial $P\in\mathbb{R}[X,Y]$ of arbitrary degree.

We already know well the algebraic curves in degree 2, which are the conics, and a first problem is, what results from what we learned about conics have a chance to be relevant to the arbitrary algebraic curves. And normally none, because the ellipses, parabolas and hyperbolas are obviously very particular curves, having very particular properties.

Let us record however a useful statement here, as follows:

PROPOSITION 15.2. *The conics can be written in cartesian, polar, parametric or complex coordinates, with the equations for the unit circle being*

$$x^2+y^2=1\quad,\quad r=1\quad,\quad x=\cos t\,,\,y=\sin t\quad,\quad |z|=1$$

and with the equations for ellipses, parabolas and hyperbolas being similar.

PROOF. The equations for the circle are clear, those for ellipses can be found in the above, and we will leave as an exercise those for parabolas and hyperbolas. □

As a true answer to our question now, coming this time from a very modest conic, namely $xy=0$, that we dismissed in the above as being "degenerate", we have:

THEOREM 15.3. *The following happen, for curves C defined by polynomials P:*

1. *In degree $d=2$, curves can have singularities, such as $xy=0$ at $(0,0)$.*
2. *In general, assuming $P=P_1\ldots P_k$, we have $C=C_1\cup\ldots\ldots\cup C_k$.*
3. *A union of curves $C_i\cup C_j$ is generically non-smooth, unless disjoint.*
4. *Due to this, we say that C is non-degenerate when P is irreducible.*

PROOF. All this is self-explanatory, the details being as follows:

(1) This is something obvious, just the story of two lines crossing.

(2) This comes from the following trivial fact, with the notation $z=(x,y)$:

$$P_1\ldots P_k(z)=0 \iff P_1(z)=0,\ \text{or } P_2(z)=0,\ \ldots\ ,\ \text{or } P_k(z)=0$$

(3) This is something very intuitive, and it actually takes a bit of time to imagine a situation where $C_1\cap C_2\neq\emptyset$, $C_1\not\subset C_2$, $C_2\not\subset C_1$, but $C_1\cup C_2$ is smooth. In practice now, "generically" has of course a mathematical meaning, in relation with probability, and our assertion does say something mathematical, that we are supposed to prove. But, we will not insist on this, and leave this as an instructive exercise, precise formulation of the claim, and its proof, in the case you are familiar with probability theory.

(4) This is just a definition, based on the above, that we will use in what follows. □

With degree 1 and 2 investigated, and our conclusions recorded, let us get now to degree 3, see what new phenomena appear here. And here, to start with, we have the following remarkable curve, well-known from calculus, because 0 is not a maximum or minimum of the function $x\to y$, despite the derivative vanishing there:

$$x^3=y$$

Also, in relation with set theory and logic, and with the foundations of mathematics in general, we have the following curve, which looks like the empyset $\emptyset$:

$$(x-y)(x^2+y^2-1)=0$$

But, it is not about counterexamples to calculus, or about logic, that we want to talk about here. As a first truly remarkable degree 3 curve, or cubic, we have the cusp:

PROPOSITION 15.4. *The standard cusp, which is the cubic given by*

$$x^3=y^2$$

has a singularity at $(0,0)$*, with only* 1 *tangent line at that singularity.*

PROOF. The two branches of the cusp are indeed both tangent to Ox, because:

$$y'=\pm\frac{3}{2}\sqrt{x}\implies y'(0)=0$$

Observe also that what happens for the cusp is different from what happens for $xy=0$, precisely because we have 1 line tangent at the singularity, instead of 2. □

As a second remarkable cubic, which gets the crown, and the right to have a Theorem about it, we have the Tschirnhausen curve, which is as follows:

THEOREM 15.5. *The Tschirnhausen cubic, given by the following equation,*

$$x^3=x^2-3y^2$$

makes the dream of $xy=0$ *come true, by self-intersecting, and being non-degenerate.*

PROOF. This is something self-explanatory, by drawing a picture, but there are several other interesting things that can be said about this curve, and the family of curves containing it, depending on a parameter, and up to basic transformations, as follows:

(1) Let us start with the curve written in polar coordinates as follows:

$$r\cos^3\left(\frac{\theta}{3}\right)=a$$

With $t=\tan(\theta/3)$, the equations of the coordinates are as follows:

$$x=a(1-3t^2)\quad,\quad y=at(3-t^2)$$

Now by eliminating t, we reach to the following equation:

$$(a-x)(8a+x)^2=27ay^2$$

(2) By translating horizontally by $8a$, and changing signs of variables, we have:

$$x=3a(3-t^2)\quad,\quad y=at(3-t^2)$$

Now by eliminating t, we reach to the following equation:

$$x^3=9a(x^2-3y^2)$$

But with $a=1/9$ this is precisely the equation in the statement. □

In degree 4 now, quartics, we have enough dimensions for "improving" the cusp and the Tschirnhausen curve. First we have the cardioid, which is as follows:

PROPOSITION 15.6. *The cardioid, which is a quartic, given in polar coordinates by*

$$2r=a(1-\cos\theta)$$

makes the dream of $x^3=y^2$ come true, by being a closed curve, with a cusp.

PROOF. As before with the Tschirnhausen curve, this is something self-explanatory, by drawing a picture, but there are several things that must be said, as follows:

(1) The cardioid appears by definition by rolling a circle of radius $c>0$ around another circle of same radius $c>0$. With θ being the rolling angle, we have:

$$x=2c(1-\cos\theta)\cos\theta$$

$$y=2c(1-\cos\theta)\sin\theta$$

(2) Thus, in polar coordinates we get the equation in the statement, with $a=4c$:

$$r=2c(1-\cos\theta)$$

(3) Finally, in cartesian coordinates, the equation is as follows:

$$(x^2+y^2)^2+4cx(x^2+y^2)=4c^2y^2$$

Thus, what we have is indeed a degree 4 curve, as claimed. □

Still in degree 4, the crown gets to the Bernoulli lemniscate, which is as follows:

THEOREM 15.7. *The Bernoulli lemniscate, a quartic, which is given by*

$$r^2 = a^2 \cos 2\theta$$

makes the dream of $x^3 = x^2 - 3y^2$ *come true, by being closed, and self-intersecting.*

PROOF. As usual, this is something self-explanatory, by drawing a picture, which looks like ∞, but there are several other things that must be said, as follows:

(1) In cartesian coordinates, the equation is as follows, with $a^2 = 2c^2$:

$$(x^2 + y^2)^2 = c^2(x^2 - y^2)$$

(2) Also, we have the following nice complex reformulation of this equation:

$$|z + c| \cdot |z - c| = c^2$$

Thus, we are led to the conclusions in in the statement. □

In degree 5, in the lack of any spectacular quintic, let us record:

THEOREM 15.8. *Unlike in degree* $3, 4$*, where equations can be solved, by the Cardano formula, in degree* 5 *this generically does not happen, an example being*

$$x^5 - x - 1 = 0$$

having Galois group S_5*, not solvable. Geometrically, this tells us that the intersection of the quintic* $y = x^5 - x - 1$ *with the line* $y = 0$ *cannot be computed.*

PROOF. Obviously off-topic, but with no good quintic available, and still a few more minutes before the bell ringing, I had to improvise a bit, and tell you about this:

(1) As indicated, the degree 3 equations can be solved a bit like the degree 2 ones, but with the formula, due to Cardano, being more complicated. With some square making tricks, which are non-trivial either, the Cardano formula applies to degree 4 as well.

(2) In degree 5 or higher, none of this is possible. Long story here, the idea being that in order for $P = 0$ to be solvable, the group $Gal(P)$ must be solvable, in the sense of group theory. But, unlike S_3, S_4 which are solvable, S_5 and higher are not solvable. □

Back now to our usual business, in degree 6, sextics, we first have here:

PROPOSITION 15.9. *The trefoil sextic, or Kiepert curve, which is given by*

$$r^3 = a^3 \cos 3\theta$$

looks like a trefoil, closed curve, with a triple self-intersection.

PROOF. As before, drawing a picture is mandatory. With $z = re^{i\theta}$ we have:

$$\begin{aligned}
r^3 = a^3\cos 3\theta \quad &\Longleftrightarrow \quad r^3\cos 3\theta = \left(\frac{r^2}{a}\right)^3\\
&\Longleftrightarrow \quad z^3 + \bar{z}^3 = 2\left(\frac{z\bar{z}}{a}\right)^3\\
&\Longleftrightarrow \quad (x+iy)^3 + (x-iy)^3 = 2\left(\frac{x^2+y^2}{a}\right)^3\\
&\Longleftrightarrow \quad x^3 - 3xy^2 = \left(\frac{x^2+y^2}{a}\right)^3\\
&\Longleftrightarrow \quad (x^2+y^2)^3 = a^3(x^3 - 3xy^2)
\end{aligned}$$

Thus, we have indeed a sextic, as claimed. □

We also have in degree 6 the most beautiful of curves them all, the Cayley sextic:

THEOREM 15.10. *The Cayley sextic, given in polar coordinates by*

$$r = a\cos^3\left(\frac{\theta}{3}\right)$$

makes the dream of everyone come true, by looking like a self-intersecting heart.

PROOF. As before, picture mandatory. With $z = re^{i\theta}$ and $u = z^{1/3}$ we have:

$$\begin{aligned}
r = a\cos^3\left(\frac{\theta}{3}\right) \quad &\Longleftrightarrow \quad ar\cos^3\left(\frac{\theta}{3}\right) = r^2\\
&\Longleftrightarrow \quad a\left(\frac{u+\bar{u}}{2}\right)^3 = r^2\\
&\Longleftrightarrow \quad a(u^3 + \bar{u}^3 + 3u\bar{u}(u+\bar{u})) = 8r^2\\
&\Longleftrightarrow \quad 3au\bar{u}\cdot\frac{u+\bar{u}}{2} = 4r^2 - ax\\
&\Longleftrightarrow \quad 27a^3r^6\cdot\frac{r^2}{a} = (4r^2 - ax)^3\\
&\Longleftrightarrow \quad 27a^2(x^2+y^2)^2 = (4x^2 + 4y^2 - ax)^3
\end{aligned}$$

Thus, we have indeed a sextic, as claimed. □

And we will stop here our study of the remarkable small degree curves. Of course, far more can be said, about the above curves, and about some related curves too.

Quite remarkably, most of the above curves are sinusoidal spirals, in the following sense, and with actually the term "sinusoidal spiral" being a bit unfortunate:

THEOREM 15.11. *The sinusoidal spirals, which are as follows,*

$$r^n = a^n \cos n\theta$$

with $a \neq 0$ and $n \in \mathbb{Q} - \{0\}$, include the following curves:

(1) $n = -1$ *line.*
(2) $n = 1$ *circle,* $n = -1/2$ *parabola,* $n = -2$ *hyperbola.*
(3) $n = -3$ *Humbert cubic,* $n = -1/3$ *Tschirnhausen curve.*
(4) $n = 1/2$ *cardioid,* $n = 2$ *Bernoulli lemniscate.*
(5) $n = 3$ *Kiepert trefoil,* $n = 1/3$ *Cayley sextic.*

PROOF. We first have to prove that the sinusoidal spirals are indeed algebraic curves. But this is best done by using the complex coordinate $z = re^{i\theta}$, as follows:

$$\begin{aligned} r^n = a^n \cos n\theta \quad &\iff \quad r^n \cos n\theta = \left(\frac{r^2}{a}\right)^n \\ &\iff \quad z^n + \bar{z}^n = 2\left(\frac{z\bar{z}}{a}\right)^n \\ &\iff \quad (x+iy)^n + (x-iy)^n = 2\left(\frac{x^2+y^2}{a}\right)^n \end{aligned}$$

As a first observation now, in the case $n \in \mathbb{N}$ we can simply use the binomial formula, and we get an algebraic equation of degree $2n$, as follows:

$$\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} = \left(\frac{x^2+y^2}{a}\right)^n$$

In general, things are a bit more complicated, as shown for instance by our computation for the Cayley sextic. However, the same idea as there applies, and we are led in this way to the equation of an algebraic curve, as claimed. Regarding now the examples:

(1) At $n = -1$ the equation is as follows, producing a line:

$$r\cos\theta = a \iff x = a$$

(2) At $n = 1$ the equation is as follows, producing a circle:

$$r = a\cos\theta \iff r^2 = ax \iff x^2 + y^2 = ax$$

(3) At $n = -1/2$ the equation is as follows, producing a parabola:

$$a = r\cos^2(\theta/2) \iff r + x = 2a \iff y^2 = 4a(a-x)$$

(4) At $n = -2$ the equation is as follows, producing a hyperbola:

$$a^2 = r\cos^2 2\theta \iff a^2 = 2x^2 - r^2 \iff (x+y)(x-y) = a^2$$

(5) At $n=-3$ the equation is as follows, producing a curve with 3 components, which looks like some sort of "trivalent hyperbola", called Humbert cubic:

$$r^3\cos 3\theta = a^3 \iff z^3+\bar{z}^3=2a^3 \iff x^3-3xy^2=a^3$$

(6) As for the other curves, this follows from our various formulae above. □

Let us study now more in detail the sinusoidal spirals. We first have:

PROPOSITION 15.12. *The sinusoidal spirals, which with $z=x+iy$ are*

$$z^n+\bar{z}^n=2\left(\frac{z\bar{z}}{a}\right)^n$$

with $a\neq 0$ and $n\in\mathbb{Q}-\{0\}$, are as follows:

(1) *With $n=-m$, $m\in\mathbb{N}$, the equation is $z^m+\bar{z}^m=2a^m$, degree m.*
(2) *With $n=m$, $m\in\mathbb{N}$, the equation is $z^m+\bar{z}^m=2(z\bar{z}/a)^m$, degree $2m$.*
(3) *With $n=-1/m$, $m\in\mathbb{N}$, the equation is $(z^{1/m}+\bar{z}^{1/m})^m=2^m a$.*
(4) *With $n=1/m$, $m\in\mathbb{N}$, the equation is $(z^{1/m}+\bar{z}^{1/m})^m=2^m z\bar{z}/a$.*

PROOF. This is something self-explanatory, the details being as follows:

(1) With $n=-m$ and $m\in\mathbb{N}$ as in the statement, the equation is, as claimed:

$$z^{-m}+\bar{z}^{-m}=2\left(\frac{z\bar{z}}{a}\right)^{-m} \iff z^m+\bar{z}^m=2a^m$$

(2) This is an empty statement, just a matter of using the new variable $m=n$.

(3) With $n=-1/m$ and $m\in\mathbb{N}$ as in the statement, the equation is, as claimed:

$$\begin{aligned}z^{-1/m}+\bar{z}^{-1/m}=2\left(\frac{z\bar{z}}{a}\right)^{-1/m} &\iff z^{1/m}+\bar{z}^{1/m}=2a^{1/m}\\ &\iff (z^{1/m}+\bar{z}^{1/m})^m=2^m a\end{aligned}$$

(4) With $n=1/m$ and $m\in\mathbb{N}$ as in the statement, the equation is, as claimed:

$$z^{1/m}+\bar{z}^{1/m}=2\left(\frac{z\bar{z}}{a}\right)^{1/m} \iff (z^{1/m}+\bar{z}^{1/m})^m=2^m\cdot\frac{z\bar{z}}{a}$$

Thus, we are led to the conclusions in the statement. □

Observe that in the fractionary cases, $n=\pm 1/m$, the equations in the above statement are not polynomial in x,y, unless at very small values of m. To be more precise:

(1) In the case $n=-1/m$, we certainly have at $m=1,2,3$ the $d=1$ line, $d=2$ parabola, and $d=3$ Tschirnhausen curve, but at $m=4$ things change, with the equation $(z^{1/4}+\bar{z}^{1/4})^4=16a$ being no longer polynomial in x,y, and requiring a further square operation to make it polynomial, and therefore leading to a curve of degree $d=8$.

(2) As for the case $n = 1/m$, this is more complicated, with the data that we have at $m = 1, 2, 3$, namely the $d = 2$ circle, $d = 3$ cardioid, and $d = 6$ Cayley sextic, being not very good, and with things getting even more complicated at $m = 4$ and higher.

In short, things quite complicated, and the general case, $n = \pm p/q$ with $p, q \in \mathbb{N}$, is certainly even more complicated. Instead of insisting on this, let us focus now on the simplest sinusoidal spirals that we have, namely those with $n = \pm m$, with $m \in \mathbb{N}$.

The point indeed is that the sinusoidal spirals with $n \in \mathbb{N}$ are also part of another remarkable family of plane algebraic curves, going back to Cassini, as follows:

THEOREM 15.13. *The polynomial lemniscates, which are as follows,*

$$|P(z)| = b^n$$

with $P \in \mathbb{C}[X]$ having n distinct roots, and $b > 0$, include the following curves:

1. *The sinusoidal spirals with $n \in \mathbb{N}$, including the $n = 1$ circle, $n = 2$ Bernoulli lemniscate, and $n = 3$ Kiepert trefoil.*
2. *The Cassini ovals, which are the quartics given by $|z + c| \cdot |z - c| = b^2$, covering too the Bernoulli lemniscate, appearing at $b = c$.*

PROOF. This is something quite self-explanatory, the details being as follows:

(1) Regarding the sinusoidal spirals with $n \in \mathbb{N}$, their equation is, with $a^n = 2c^n$:

$$\begin{aligned} z^n + \bar{z}^n = 2\left(\frac{z\bar{z}}{a}\right)^n &\iff c^n(z^n + \bar{z}^n) = (z\bar{z})^n \\ &\iff (z^n - c^n)(\bar{z}^n - c^n) = c^{2n} \\ &\iff |z^n - c^n| = c^n \end{aligned}$$

(2) Regarding the Cassini ovals, these correspond to the case where the polynomial $P \in \mathbb{C}[X]$ has degree 2, and we already know from the above that these cover the Bernoulli lemniscate. In general, the equation for the Cassini ovals is:

$$\begin{aligned} |z + c| \cdot |z - c| = b^2 &\iff |z^2 - c^2| = b^2 \\ &\iff (z^2 - c^2)(\bar{z}^2 - c^2) = b^4 \\ &\iff (z\bar{z})^2 - c^2(z^2 + \bar{z}^2) + c^4 = b^4 \\ &\iff (x^2 + y^2)^2 - c^2(x^2 - y^2) + c^4 = b^4 \\ &\iff (x^2 + y^2)^2 = c^2(x^2 - y^2) + b^4 - c^4 \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The polynomial lemniscates can be geometrically understood as follows:

THEOREM 15.14. *The equation $|P(z)| = b$ defining the polynomial lemniscates can be written as follows, in terms of the roots $c_1, \ldots, c_n$ of the polynomial P,*

$$\sqrt[n]{\prod_{k=1}^{n} |z - c_i|} = b$$

telling us that the geometric mean of the distances from z to the vertices of the polygon formed by $c_1, \ldots, c_n$ must be the constant $b > 0$.

PROOF. This is something self-explanatory, and as an illustration, let us work out the case of sinusoidal spirals with $n \in \mathbb{N}$. Here with $w = e^{2\pi i/n}$ we have:

$$z^n - c^n = \prod_{k=1}^{n} (z - cw^k)$$

Thus, the sinusoidal spiral equation reformulates as follows:

$$\begin{aligned} |z^n - c^n| = c^n \quad &\iff \quad \prod_{k=1}^{n} |z - cw^k| = c^n \\ &\iff \quad \sqrt[n]{\prod_{k=1}^{n} |z - cw^k|} = c \end{aligned}$$

Thus, for a sinusoidal spiral with positive integer parameter, the geometric mean of the distances to the vertices of a regular polygon must equal the radius of the polygon. □

Regarding now the sinusoidal spirals with $n \in -\mathbb{N}$, these are too part of another remarkable family of plane algebraic curves, constructed as follows:

THEOREM 15.15. *Given points in the plane $c_1, \ldots, c_n \in \mathbb{C}$ and a number $d \in \mathbb{R}$, construct the associated stelloid as being the set of points $z \in \mathbb{C}$ verifying*

$$\frac{1}{n} \sum_{k=1}^{n} \alpha_v (z - c_i) = d$$

with α_v denoting the angle with respect to a direction v. Then the stelloid is an algebraic curve, not depending on v, and at the level of examples we have the sinusoidal spirals with $n \in -\mathbb{N}$, including the $n = -1$ line, $n = -2$ hyperbola, and $n = -3$ Humbert cubic.

PROOF. All this is quite self-explanatory, and we will leave the verification of the various generalities regarding the stelloids, as well as the verification of the relation with the sinusoidal spirals with $n \in -\mathbb{N}$, as an instructive exercise. As a bonus exercise, try understanding the precise relation between stelloids, and polynomial lemniscates. □

So long for plane algebraic curves. Needless to say, all the above is old-style, first class mathematics, having countless applications. For instance when doing classical mechanics or electrodynamics, you will certainly meet polynomial lemniscates and stelloids, when looking at the field lines. Also, the image of any circle passing though 0 by $z \to z^2$ is a cardioid, and the famous Mandelbrot set is organized around such a cardioid.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.16.

Exercise 15.17.

Exercise 15.18.

Exercise 15.19.

Exercise 15.20.

Exercise 15.21.

Exercise 15.22.

Exercise 15.23.

Bonus exercise.

CHAPTER 16

Complex dynamics

16a. Complex dynamics

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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