AN INVITATION TO VON NEUMANN ALGEBRAS

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Abstract. This is an introduction to the von Neumann algebras $A \subset B(H)$. Following John von Neumann, we insist on the case where the algebra has a trace $tr : A \to \mathbb{C}$. In this case we have $A = L^\infty(X)$, with X being a noncommutative measured space. We discuss the basics, then more advanced aspects, following Connes, Jones, Voiculescu.

Contents

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Introduction

Let H be a complex Hilbert space. A linear operator $T:H\to H$ is called bounded when the following quantity, which is a norm, is bounded:

$$||T||=\sup_{||x||=1}||Tx||$$

The bounded operators $T:H\to H$ form a Banach algebra, denoted $B(H)$. In finite dimensions, where $H=\mathbb C^N$, what we have here is the matrix algebra $M_N(\mathbb C)$.

In infinite dimensions it is known that H has an orthonormal basis $\{e_i\}_{i\in I}$, and so we have an embedding of complex algebras, as follows:

$$B(H)\subset M_I(\mathbb C)$$

To be more precise, $B(H)$ consists of the matrices $M\in M_I(\mathbb C)$ subject to the condition that the associated linear operator $T:H\to H$ is well-defined, and satisfies:

$$||T||<\infty$$

This is certainly nice, but remains something quite theoretical. Indeed, the typical Hilbert spaces are those of type $H=L^2(X)$, having quite complicated bases.

In analogy with the usual matrices, each bounded operator $T:H\to H$ has an adjoint operator $T^*:H\to H$, given by the following formula:

$$<Tx,y>=<x,T^*y>$$

Indeed, since $f(x)=<Tx,y>$ is linear, we must have $f(x)=<x,T^*y>$, for a certain vector $T^*y\in H$. The map $y\to T^*y$ is then linear, and bounded as well, because:

$$||T||=||T^*||$$

Generally speaking, the original operator $T:H\to H$ and its adjoint $T^*:H\to H$ can be thought of as being "twin brothers". They are indeed related by a lot of interesting mathematics, and in particular, we have the following formula:

$$||TT^*||=||T||^2$$

Once again, all this is very nice. The operator algebra $B(H)$ is therefore a Banach $*$-algebra, with the norm and involution being related by the above formulae.

A von Neumann algebra is a $*$-algebra of operators $A \subset B(H)$ which is closed under the weak topology, making each of the following maps continuous:

$$T \to Tx$$

As a basic example, given an operator $T \in B(H)$, we can take A to be the weak closure of the algebra generated by T, and its adjoint T^*. This algebra "encodes" T.

These algebras were introduced by John von Neumann about 100 years ago, with motivation coming from quantum mechanics. His first result, which is something very remarkable, is the bicommutant theorem. Given a subalgebra $A \subset B(H)$, let us set:

$$A' = \left\{ T \in B(H) \Big| TS = ST, \forall S \in A \right\}$$

We obtain in this way an algebra, called commutant of A. The point now is that if we repeat this procedure, we obtain a second algebra A'', and we obviously have:

$$A \subset A''$$

The bicommutant theorem states that for a von Neumann algebra we have $A = A''$, and that, conversely, this condition characterizes the von Neumann algebras.

As a basic application of this theorem, given a subset $F \subset B(H)$, we can construct its commutant as above, and we obtain in this way a von Neumann algebra:

$$A = F'$$

Observe that any von Neumann algebra appears in this way, as a commutant. Indeed, given a von Neumann algebra A, we can take $F = A'$, and we have $F' = A$.

A second key theorem of von Neumann deals with the commutative case. Given a measured space X, consider the corresponding Hilbert space of L^2 functions:

$$H = L^2(X)$$

Consider as well the complex algebra of L^∞ functions on X:

$$A = L^\infty(X)$$

This latter algebra A is then a von Neumann algebra of operators on H, with the L^∞ functions acting by multiplication on the L^2 functions, as follows:

$$A \subset B(H)$$

$$f \to (g \to fg)$$

The above-mentioned second theorem of von Neumann states that, up to a certain multiplicity, any commutative von Neumann algebra appears in this way.

Summarizing, up to some technicalities, the commutative von Neumann algebras are precisely those of the following form, with X being a measured space:

$$A = L^\infty(X)$$

This is something extremely interesting, and starting from this point, several different things can be done with the von Neumann algebras, as follows:

I. Classification. This is what von Neumann did, the idea being very simple, namely that of writing the center, which is a commutative algebra, in the above form:

$$Z(A) = L^\infty(X)$$

With this result in hand, it is possible then to prove that the whole algebra A decomposes as an integral of von Neumann algebras, as follows:

$$A = \int_X A_x dx$$

The fibers A_x are von Neumann algebras with trivial center, called "factors". Murray-von Neumann, and then Connes, divided these factors into several classes:

$$\mathrm{I_N}, \mathrm{I_\infty}$$

$$\mathrm{II_1}, \mathrm{II_\infty}$$

$$\mathrm{III_0}, \mathrm{III_\lambda}, \mathrm{III_1}$$

The type I factors are the algebras $B(H)$, indexed by $N = \dim H$. The "interesting" factors are those of type $\mathrm{II_1}$, which are infinite dimensional, and have a trace:

$$tr : A \to \mathbb{C}$$

The factors of type $\mathrm{II_\infty}$ are tensor products of $\mathrm{II_1}$ factors with $B(H)$. As for the type III factors, these appear as well from the $\mathrm{II_1}$ ones, via crossed product constructions.

II. Geometry. This is something a bit more speculative, the idea being once again very simple. Motivated by the above result, given an arbitrary von Neumann algebra A, we can write it as follows, with X being a "noncommutative measured space":

$$A = L^\infty(X)$$

In order to construct now examples of such noncommutative spaces X, and study them, we can take some inspiration from the classical case. Here most of the interesting measured spaces are actually smooth manifolds, namely the compact Lie groups $G \subset U_N$, or the homogeneous spaces over such compact Lie groups, which can be written as:

$$X = G/H$$

Thus, what we have to do is to extend this theory, and do "noncommutative geometry". This can be done indeed, and as first, and very basic examples of von Neumann algebras that we can obtain in this way, we have the group von Neumann algebras:

$$L(\Gamma) = L^\infty(\widehat{\Gamma})$$

To be more precise, here Γ is a discrete group, and $L(\Gamma)$ is the weak closure of the group algebra $\mathbb{C}[\Gamma]$. In the abelian case, we obtain the usual algebra $L^\infty(\widehat{\Gamma})$.

III. Probability. This is once again something speculative, the idea being once again very simple. As before, given an arbitrary von Neumann algebra A, let us write it as follows, with X being a "noncommutative measured space":

$$A = L^\infty(X)$$

The problem now is whether X can be thought of as being a "noncommutative probability space", and generally speaking, the answer here is yes, provided that our von Neumann algebra A comes along with an integration functional:

$$\int_X : A \to \mathbb{C}$$

To be more precise, this integration functional must be positive, in an appropriate sense, and unital. Once we have this, we can study A via various "noncommutative probability" techniques. Very successful here was the free probability theory of Voiculescu.

IV. Subfactors. This is something more recent, and quite advanced. The idea is that given an inclusion of von Neumann algebras $A_0 \subset A_1$, which must be of a certain special type, namely II_1 factors, we can consider the expectation $e_1 : A_1 \to A_0$, inside the ambient algebra $B(H)$, and then construct the following algebra:

$$A_2 =< A_1, e_1 >$$

Jones discovered that this construction, called "basic construction", is a kind of mirroring procedure, with $A_1 \subset A_2$ appearing to be somehow opposite to $A_0 \subset A_1$. This construction can be iterated, and we obtain a whole tower of algebras, as follows:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

The point now is that the sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ behaves exactly as the sequence of Temperley-Lieb diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \in TL_N$, given by:

$$\varepsilon_1 = {}^{\cup}_{\cap} \quad , \quad \varepsilon_2 = |{}^{\cup}_{\cap} \quad , \quad \varepsilon_3 = ||{}^{\cup}_{\cap} \quad , \quad \ldots\ldots$$

This is very surprising, and one can go well beyond this, for instance with a result stating that the following commutants have a planar algebra structure:

$$P_k = A_0' \cap A_k$$

To be more precise, the Jones projections $e_1, e_2, e_3, \ldots$ belong to these commutants, and the result states that the diagrammatic structure of the Temperley-Lieb algebra $TL =< e_1, e_2, e_3, \ldots >$ extends to a planar algebra structure of $P = (P_k)$.

Summarizing, many interesting things can be done with the von Neumann algebras. Needless to say, all the above is related, because classification leads into group duals and other noncommutative spaces, and vice versa, and these latter spaces are best studied by using probability techniques, and vice versa, and free probability is needed for classification, and vice versa, and finally subfactor theory is of course related to all this.

One common interesting feature of all the above is that everything best works when the von Neumann algebra has a trace $tr : A \to \mathbb{C}$, satisfying:

$$tr(ab) = tr(ba)$$

Indeed, in what regards the classification, the tracial case is what von Neumann was mostly interested in, and most of the interesting results and work still lie here.

In what regards quantum groups and other noncommutative manifolds, here the trace condition corresponds to the following group theory condition:

$$(g^{-1})^{-1} = g$$

This condition is well-known to be necessary, in order to develop advanced quantum group theory, in the operator algebra framework.

In what regards probability, here the trace condition is once again very welcome, with formulae of the following type being technically quite unavoidable:

$$\mathbb{E}(XY) = \mathbb{E}(X^{1/2}YX^{1/2})$$

This is particularly true for Voiculescu's free probability theory, where most of the interesting combinatorics which has been developed uses such formulae.

Finally, regarding the Jones subfactor theory, here the basic objects are towers of II_1 factors $A_0 \subset A_1 \subset A_2 \subset \ldots$ which do have a trace, by definition:

$$tr : A_k \to \mathbb{C}$$

There are of course some extensions of subfactor theory to the type III factors as well, but the bulk of the interesting theory and results remains in the type II_1 case.

Summarizing, the notion of von Neumann algebra $A \subset B(H)$ is quite broad, and the assumption that the algebra has a trace $tr : A \to \mathbb{C}$ appears to be a good one.

Getting even more philosophical here, from a mathematical viewpoint the problem is what von Neumann algebras are to be investigated, and what to do with them:

(1) All the above suggests that the algebras to look at are those having traces, and what is to be done with them is a mixture of "noncommutative" mathematics.

(2) From a more purist point of view, the algebras to be studied are the II_1 factors, and to be done with them is some "highly noncommutative" mathematics.

(3) At the purely feline level, the algebra to be studied is the Murray-von Neumann hyperfinite II_1 factor R, and the problem is that of classifying its subfactors.

The present book is an introduction to all this, basic von Neumann algebra theory, often focusing on the tracial case. The organization is as follows:

(1) In 1-4 we discuss the basic theory of von Neumann algebras. Everything here is accessible with a minimal knowledge of real and complex analysis.

(2) In 5-8 we discuss noncommutative geometry and probability aspects. Once again, this is an introduction to the subject, with full preliminaries given.

(3) In 9-12 we get into technical functional analysis, and we discuss more advanced aspects, and notably the theory of II_1 factors.

(4) In 13-16 we discuss the basics of subfactor theory, and its relation with noncommutative geometry and probability.

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1. Bounded operators

Hilbert spaces.
Bounded operators.
Spectral theory.
C^*-algebras.
Finite dimensions.
Gelfand.
GNS.

2. Von Neumann algebras

Weak topology.
Ultraweak too.
We call strong topology the one given by the norm.
Von Neumann algebras.
Bicommutant.
Basic examples.
Commutative case.
Traces.
Basic examples.

3. Factors, examples

Factors.
Types I, II, III.
Some theory.
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4. Reduction theory

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Finite von Neumann algebras.
Here the proof is simpler.

5. Quantum groups

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Center, ICC.
Quantum groups.
Integration, trace.
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6. Algebraic manifolds

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NCG.

7. Integration theory

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Gram, Weingarten.
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11. Classification questions

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12. Hyperfiniteness

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Murray-von Neumann.
Connes.
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Fuss-Catalan, other.

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Ocneanu, Popa.
More Popa.
And Jones.
Heavy stuff.

16. Small and big index

Small index.
Index 6 problems.
Big index.
Large N limit.

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