Random classical mechanics

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Abstract. This is an introduction to classical mechanics, and more specifically to the gravitational N-body problem, viewed from a probabilistic viewpoint, in the $N >> 0$ regime. Our aim is to describe the various available techniques, and to distinguish, probabilistically speaking, between order and chaos. At the level of applications, we insist on various configurations met in the context of celestial mechanics, from thin interstellar dust, up to macroscopic phenomena such as galaxies, and clusters of galaxies.

Preface

What is the difference between order and chaos? Not very clear, and this is rather a question for God who created this world. As humans, we can however make a few observations, as to have a bit of an idea of what that divine "order" really means.

The simplest such observations, that we all made since childhood, concerm the life surrounding us. Cat eats mouse, mouse eats grains, grains eat minerals from the soil. Obviously, matter tends to organize into an hierarchic way, and this is where the beauty and order of this world come from. Technically, all this can be fully understood too, via organic chemistry, and then a lot of biology, culminating with Darwin.

The same phenomenon, matter organizing into an orderly and hierarchic way, can be observed as well at smaller scales. Quarks manage to team up and form elementary particles, then elementary particles team up and form atoms, and then atoms combine and form molecules. And afterwards, the biggest and most advanced molecules, which are the organic ones, team up and form cells, which cells organize into life.

Things are not over here, with the same phenomenon, matter organizing into an orderly and hierarchic way, being observable as well at bigger scales. Here the unit is typically some sort of big chunk of matter, and the organization which results from this, via the laws of physics, is into Solar systems, galaxies, clusters of galaxies, and so on.

In this book we will be interested in these latter phenomena. The point indeed is that, while seemingly being a bit far away from us, and from our potential scientific knowledge and tools, these latter phenomena are in fact the simplest. To be more precise, passed the formation and functioning of stars, which is certainly a difficult topic, involving all sorts of physics, things up in the skies are basically governed by gravity. Thus, what we need in order to understand these phenomena is just good old classical mechanics.

With the remark, however, that what we need is a probabilistic-flavored version of classical mechanics. Indeed, in order to understand how matter organizes in a hierarchic way, under the effect of gravity, we certainly have to deal with the gravitational N-body problem, but viewed from a probabilistic viewpoint, in the $N >> 0$ regime.

So, this will be the plan for the book, first reviewing all sorts of basic questions and methods from classical mechanics and dynamical systems, and fluid mechanics too, and with a touch of statistical mechanics as well, with emphasis on the difference between mechanical order and chaos, probabilistically speaking, and then moving ahead with an exploration of various questions from celestial mechanics, armed with these tools, starting with basic phenomena which are relentlessly at work, concerning interstellar dust, up to truly macroscopic phenomena such as galaxies, and clusters of galaxies.

The book contains for the most very basic material, based of course on standard classical mechanics, but we will recall the classical mechanics basics. Mathematically, we will assume some familiarity with basic calculus, and probability theory, but again, we will recall the basics here, whenever needed. Finally, in what regards the main problems to be solved, these will be a bit more advanced, at the level of classical mechanics from the early and mid 20th century, but again, we will explain all these techniques.

I would like to thank my colleagues in Cergy, with everyone doing some sort of quantum there, the discussions in the coffee room usually revolve around classical and celestial mechanics, and other peaceful themes, and I learned many things in this way. Many thanks to Vladimir Arnold for his wonderful books, and for his talks too. I had the pleasure and privilege of attending one of them in the early 90s, as an undergraduate student, and boy, that was some talk, that I will always have in mind.

Finally, many thanks go to my cats. We will be talking about order and chaos in this book, with what happens in the skies being our main source of inspiration. But, no really need to go up there for observing things and drawing conclusions: extreme order and chaos can be an everyday matter, happening at home, right in front of your eyes.

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Order and chaos

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Video killed the radio star
In my mind and in my car
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CHAPTER 1

Basic mechanics

1a. Pendulum, clocks

I bet that, perhaps a bit scared by the complexity of mechanical watches, you don't have a Rolex. For basic timekeeping and fashion matters, an average Casio will do. And for a few dozen bucks more, they even have fancy models, with several dials and everything, a bit like the Daytona. Add to this some cheap tools for replacing the battery, which cannot be properly done with bare hands, and you're ready to go.

This being said, in this book we will be mainly interested in mechanical order and chaos, and the best starting point for all this are mechanical watches. What can be more basic and orderly than a swinging pendulum keeping the time, of course skilfully adjusted as to really work, and with this being the secret of mechanical watches.

1b. Friction, resonance

Mathematics of friction and resonance, illustrated with various devices.

1c. Planets and comets

You surely know a bit about gravity, including the findings of Kepler and Newton, but always good to remember this. The result here, which is the pride of mathematics, physics, and human knowledge in general, is the following famous theorem:

THEOREM 1.1 (Kepler, Newton). *The following happen:*

(1) *Planets and other celestial bodies move around the Sun on conics, that is, curves given by* $P(x,y)=0$*, with* $P\in\mathbb{R}[x,y]$ *being of degree* 2.
(2) *The conics are the curves which appear by cutting a 2-sided cone with a plane, and can be classified into ellipses, parabolas and hyperbolas.*

PROOF. The idea here is that (1) is something tough, due to Kepler and Newton, and (2), due to the ancient Greeks, is elementary. Let us mention too that the above statement is a bit informal, with the 3 viewpoints on the conics, coming from gravity, cutting cones and classification, agreeing in the non-degenerate case, $\deg P=2$, modulo some normalizations, and with some disagreements in the degenerate case, $\deg P\leq 1$. But the complete statement, including a full discussion of the normalizations and of the degenerate cases, being too long, we have preferred to formulate things as above, and for more we refer to the comments at the end of the proof. Getting started now:

(1) According to observations and calculations performed over the centuries, since the ancient times, and first formalized by Newton, following some groundbreaking work of Kepler, the force of attraction between two bodies of masses M, m is given by:

$$||F|| = G \cdot \frac{Mm}{d^2}$$

Here d is the distance between the two bodies, and $G \simeq 6.674 \times 10^{-11}$ is a constant. Now assuming that M is fixed at $0 \in \mathbb{R}^3$, the force extered on m positioned at $x \in \mathbb{R}^3$, regarded as a vector $F \in \mathbb{R}^3$, is given by the following formula:

$$F = -||F|| \cdot \frac{x}{||x||} = -\frac{GMm}{||x||^2} \cdot \frac{x}{||x||} = -\frac{GMmx}{||x||^3}$$

But $F = ma = m\ddot{x}$, with $a = \ddot{x}$ being the acceleration, second derivative of the position, so the equation of motion of m, assuming that M is fixed at 0, is:

$$\ddot{x} = -\frac{GMx}{||x||^3}$$

Obviously, the problem happens in 2 dimensions, and you can even find, as an exercise, a formal proof of that, based on the above equation, if your really want to. Now here the most convenient is to use standard x, y coordinates, and denote our point as $z = (x, y)$. With this change made, and by setting $K = GM$, the equation of motion becomes:

$$\ddot{z} = -\frac{Kz}{||z||^3}$$

(2) The idea now is that the problem can be solved via some calculus. Let us write indeed our vector $z = (x, y)$ in polar coordinates, as follows:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

We have then $||z|| = r$, and our equation of motion becomes:

$$\ddot{z} = -\frac{Kz}{r^3}$$

Let us differentiate now x, y. By using the standard calculus rules, we have:

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta}$$

Differentiating one more time gives the following formulae:

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\sin\theta \cdot \dot{\theta} - r\cos\theta \cdot \dot{\theta}^2 - r\sin\theta \cdot \ddot{\theta}$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\cos\theta \cdot \dot{\theta} - r\sin\theta \cdot \dot{\theta}^2 + r\cos\theta \cdot \ddot{\theta}$$

Consider now the following two quantities, appearing as coefficients in the above:

$$a = \ddot{r} - r\dot{\theta}^2 \quad , \quad b = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

In terms of these quantities, our second derivative formulae read:

$$\ddot{x} = a\cos\theta - b\sin\theta$$

$$\ddot{y} = a\sin\theta + b\cos\theta$$

(3) We can now solve the equation of motion from (1). Indeed, with the formulae that we found for $\ddot{x}, \ddot{y}$, our equation of motion takes the following form:

$$a\cos\theta - b\sin\theta = -\frac{K}{r^2}\cos\theta$$

$$a\sin\theta + b\cos\theta = -\frac{K}{r^2}\sin\theta$$

But these two formulae can be written in the following way:

$$\left(a + \frac{K}{r^2}\right)\cos\theta = b\sin\theta$$

$$\left(a + \frac{K}{r^2}\right)\sin\theta = -b\cos\theta$$

By making now the product, and assuming that we are in a non-degenerate case, where the angle θ varies indeed, we obtain by positivity that we must have:

$$a + \frac{K}{r^2} = b = 0$$

(4) Let us first examine the second equation, $b = 0$. This can be solved as follows:

$$\begin{aligned}
b = 0 \quad &\iff \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \\
&\iff \quad \frac{\ddot{\theta}}{\dot{\theta}} = -2\frac{\dot{r}}{r} \\
&\iff \quad (\log\dot{\theta})' = (-2\log r)' \\
&\iff \quad \log\dot{\theta} = -2\log r + c \\
&\iff \quad \dot{\theta} = \frac{\lambda}{r^2}
\end{aligned}$$

As for the first equation the we found, namely $a + K/r^2 = 0$, this becomes:

$$\ddot{r} - \frac{\lambda^2}{r^3} + \frac{K}{r^2} = 0$$

As a conclusion to all this, in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, our equations of motion are as follows, with λ being a constant, not depending on t:

$$\ddot{r} = \frac{\lambda^2}{r^3} - \frac{K}{r^2} \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

Even better now, let us introduce a new constant K, as follows:

$$K = \frac{\lambda^2}{c}$$

With this convention, our equations above simply read:

$$\ddot{r} = \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right) \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

(5) In order to study the first equation, we use a trick. Let us write:

$$r(t) = \frac{1}{f(\theta(t))}$$

Abbreviated, and by reminding that f takes $\theta = \theta(t)$ as variable, this reads:

$$r = \frac{1}{f}$$

With the convention that dots mean as usual derivatives with respect to t, and that the primes will denote derivatives with respect to $\theta = \theta(t)$, we have:

$$\dot{r} = -\frac{f'\dot{\theta}}{f^2} = -\frac{f'}{f^2} \cdot \frac{\lambda}{r^2} = -\lambda f'$$

By differentiating one more time with respect to t, we obtain:

$$\ddot{r} = -\lambda f''\dot{\theta} = -\lambda f'' \cdot \frac{\lambda}{r^2} = -\frac{\lambda^2}{r^2} f''$$

On the other hand, our equation for $\ddot{r}$ found in (4) above reads:

$$\ddot{r} = \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right) = \frac{\lambda^2}{r^2}\left(f - \frac{1}{c}\right)$$

Thus, in terms of $f = 1/r$ as above, our equation for $\ddot{r}$ simply reads:

$$f'' + f = \frac{1}{c}$$

But this latter equation is elementary to solve. Indeed, both functions $\cos t, \sin t$ satisfy $g" + g = 0$, so any linear combination of them satisfies as well this equation. But the solutions of $f'' + f = 1/c$ being those of $g'' + g = 0$ shifted by $1/c$, we obtain:

$$f = \frac{1 + \varepsilon\cos\theta + \delta\sin\theta}{c}$$

Now by inverting, we obtain the following formula:

$$r = \frac{c}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

(6) But this leads to the conclusion that the trajectory is a conic. Indeed, in terms of the parameter θ, the formulae of the coordinates are:

$$x=\frac{c\cos\theta}{1+\varepsilon\cos\theta+\delta\sin\theta}$$

$$y=\frac{c\sin\theta}{1+\varepsilon\cos\theta+\delta\sin\theta}$$

Now observe that these two functions x,y satisfy the following formula:

$$\begin{aligned}x^2+y^2&=\frac{c^2(\cos^2\theta+\sin^2\theta)}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2}\\&=\frac{c^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2}\end{aligned}$$

On the other hand, these two functions satisfy as well the following formula:

$$\begin{aligned}(\varepsilon x+\delta y-c)^2&=\frac{c^2\big(\varepsilon\cos\theta+\delta\sin\theta-(1+\varepsilon\cos\theta+\delta\sin\theta)\big)^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2}\\&=\frac{c^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2}\end{aligned}$$

We conclude that our coordinates x,y satisfy the following equation:

$$x^2+y^2=(\varepsilon x+\delta y-c)^2$$

But what we have here is an equation of a conic, and this ends the proof of the first assertion of the theorem. Which is not bad at all, because you probably know now more classical mechanics than the average nerd, and even math professor.

(7) Before getting into the mathematics of conics, a bit more physics. Astronomy and Kepler tell us that for planets the trajectory should be an ellipsis, and this can be deduced from what we have, the missing piece of math, which is elementary, being that of proving that a bounded non-degenerate conic must be an ellipsis. However, this will follow as well from the classification results below, so we will stop physics here.

(8) The classification of the conics, going back to the ancient Greeks, is standard. Consider indeed one of these conics:

$$C=\left\{(x,y)\in\mathbb R^2\Big|P(x,y)=0,\deg P\leq 2\right\}$$

By doing some suitable manipulations on the degree 2 polynomial $P\in\mathbb R[x,y]$, up to affine transformations of the curve, we can have this curve written in some simple, "standard" form, with standard depending a bit on you, matter of taste. But this standard form can only lead to the 3 cases in the statement, namely ellipses, parabolas and hyperbolas, up to degeneration, with the degenerate cases being the lines, double lines, points, empty set, and $\mathbb R^2$ itself, basically appearing when $\deg P\leq 1$.

(9) The fact that the conics appear by cutting a 2-sided cone with a plane is also elementary, and also known since the ancient Greeks. A first proof is by doing some abstract algebra, and verifying that the cut must be indeed a curve of degree 2. A second proof is by computing the cut in the various cases that might appear, depending on the angle of the plane with respect to the cone, with this leading to the curves found in (8), namely ellipses, parabolas and hyperbolas, up to degeneration.

(10) Summarizing, we are done with what was announced in the theorem, but there is still some discussion to be made, in relation with degeneration. With respect to what was found in (8), when cutting cones with a plane, if you want to get exactly the same list of conics, you have to allow the degenerate cone, of angle 180°, which in practice means a plane, in order to have as examples the empty set, and $\mathbb{R}^2$ itself.

(11) Also in relation to what was found in (8), what comes out of gravity basically agrees, namely ellipses, parabolas and hyperbolas. However, at the level of degenerate examples, these are different, consisting of the point, and then of the segment, which corresponds to the object m falling into the object M on a perfectly straight line.

(12) Finally, there is as well a discussion concerning normalization, because in the Kepler problem we assumed M to be fixed at 0. However, when changing coordinates via a translation, we can obtain in this way all the ellipses, parabolas and hyperbolas. □

Still with me, I hope, after all these computations. For further applications, here is a sort of "best of" the formulae found in the proof of Theorem 1.1:

THEOREM 1.2 (Kepler, Newton). *In the context of a 2-body problem, with M fixed at 0, and m starting its movement from Ox, the equation of motion of m, namely*

$$\ddot{z} = -\frac{Kz}{||z||^3}$$

with $K = GM$, and $z = (x, y)$, becomes in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$,

$$\ddot{r} = \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right) \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

for some $\lambda, c \in \mathbb{R}$, related by $\lambda^2 = Kc$. The value of r in terms of θ is given by

$$r = \frac{c}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

for some $\varepsilon, \delta \in \mathbb{R}$. At the level of the affine coordinates x, y, this means

$$x = \frac{c\cos\theta}{1 + \varepsilon\cos\theta + \delta\sin\theta} \quad , \quad y = \frac{c\sin\theta}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

with $\theta = \theta(t)$ being subject to $\dot{\theta} = \lambda^2/r$, as above. Finally, we have

$$x^2 + y^2 = (\varepsilon x + \delta y - c)^2$$

which is a degree 2 equation, and so the resulting trajectory is a conic.

PROOF. As already mentioned, this is a sort of "best of" the formulae found in the proof of Theorem 1.1. And in the hope of course that we have not forgotten anything. Finally, let us mention that the simplest illustration for this is the circular motion, and for details on this, not included in the above, we refer to the proof of Theorem 1.1. □

As a first question, we would like to understand how the various parameters appearing above, namely $\lambda, c, \varepsilon, \delta$, which via some basic math can only tell us more about the shape of the orbit, appear from the initial data. The formulae here are as follows:

PROPOSITION 1.3. *In the context of Theorem 1.2, and in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, the initial data is as follows, with $R = r_0$:*

$$r_0 = \frac{c}{1+\varepsilon} \quad , \quad \theta_0 = 0$$

$$\dot{r}_0 = -\frac{\delta\sqrt{K}}{\sqrt{c}} \quad , \quad \dot{\theta}_0 = \frac{\sqrt{Kc}}{R^2}$$

$$\ddot{r}_0 = \frac{\varepsilon K}{R^2} \quad , \quad \ddot{\theta}_0 = \frac{4\delta K}{R^2}$$

The corresponding formulae for the affine coordinates x, y can be deduced from this. Also, the various motion parameters c, ε, δ and $\lambda = \sqrt{Kc}$ can be recovered from this data.

PROOF. We have several assertions here, the idea being as follows:

(1) As mentioned in Theorem 1.2, the object m begins its movement on Ox. Thus we have $\theta_0 = 0$, and from this we get the formula of r_0 in the statement.

(2) Regarding the initial speed now, the formula of $\dot{\theta}_0$ follows from:

$$\dot{\theta} = \frac{\lambda}{r^2} = \frac{\sqrt{Kc}}{r^2}$$

Also, in what concerns the radial speed, the formula of $\dot{r}_0$ follows from:

$$\begin{aligned}
\dot{r} &= \frac{c(\varepsilon\sin\theta - \delta\cos\theta)\dot{\theta}}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \\
&= \frac{c(\varepsilon\sin\theta - \delta\cos\theta)}{c^2/r^2} \cdot \frac{\sqrt{Kc}}{r^2} \\
&= \frac{\sqrt{K}(\varepsilon\sin\theta - \delta\cos\theta)}{\sqrt{c}}
\end{aligned}$$

(3) Regarding now the initial acceleration, by using $\dot{\theta} = \sqrt{Kc}/r^2$ we find:

$$\ddot{\theta} = -2\sqrt{Kc} \cdot \frac{2r\dot{r}}{r^3} = -\frac{4\sqrt{Kc} \cdot \dot{r}}{r^2}$$

In particular at $t=0$ we obtain the formula in the statement, namely:

$$\ddot{\theta}_0=-\frac{4\sqrt{Kc}\cdot\dot{r}_0}{R^2}=\frac{4\sqrt{Kc}}{R^2}\cdot\frac{\delta\sqrt{K}}{\sqrt{c}}=\frac{4\delta K}{R^2}$$

(4) Also regarding acceleration, with $\lambda=\sqrt{Kc}$ our main motion formula reads:

$$\ddot{r}=\frac{Kc}{r^2}\left(\frac{1}{r}-\frac{1}{c}\right)$$

In particular at $t=0$ we obtain the formula in the statement, namely:

$$\ddot{r}_0=\frac{Kc}{R^2}\left(\frac{1}{R}-\frac{1}{c}\right)=\frac{Kc}{R^2}\cdot\frac{\varepsilon}{c}=\frac{\varepsilon K}{R^2}$$

(5) Finally, the last assertion is clear, and since the formulae look better anyway in polar coordinates than in affine coordinates, we will not get into details here. □

With the above formulae in hand, which are a precious complement to Theorem 1.2, we can do some reverse engineering at the level of parameters, and work out how various inital speeds and accelerations lead to various types of conics.

1d. The Solar system

The Solar system: Sun, planets, satellites, comets, asteroids, dust. There is an amazing number of interesting phenomena that can be observed here, and what we will be doing here will be a constant source of inspiration, for the remainder of this book.

1e. Exercises

Exercises.

CHAPTER 2

Lagrange points

2a. Conservative forces

We have seen so far the foundations of classical mechanics, with emphasis on gravity. Our goal here is to complete this discussion with an introduction to more advanced topics, belonging both to theoretical physics and applied physics, as follows:

(1) On one hand, we would like to further build on the general theory developed so far, following the work of Lagrange and Hamilton. Besides being very useful for all sorts of purposes, and particularly for us later on, when doing quantum mechanics, their formulation of classical mechanics is especially useful when dealing with 3-body problems, with in practice the bodies being usually the Earth, the Sun, and a satellite.

(2) On the other hand, we would like to systematically discuss certain concrete questions coming from engineering, in relation with rockets, ballistics and satellites. We will discuss such questions with and without drag, and at the theoretical level, this will lead us into a number of interesting versions and generalizations of classical mechanics, involving drag or friction, and also into the general topic of fluid mechanics.

Summarizing, many things to be done. As usual, we will be quite brief, and our standard references will be the classical mechanics books [**2**], [**32**], [**36**], [**53**], [**56**], [**88**]. For fluid mechanics, a standard reference is Batchelor [**11**], and Arnold-Khesin [**5**] for more advanced aspects. Finally, for differences between solids and fluids, a subtle question that we will get into as well, standard reads here are Ashcroft-Mermin [**7**], Chaikin-Lubensky [**15**], Goodstein [**37**], Harrison [**44**], but only a quick look at them, because we will discuss this later in this book too, more in detail, in Part III and Part IV below.

Getting started now, we first need to clarify a bit the things discussed at the end of chapter 1. The discussion here, involving some non-trivial mathematics, will start erring on the upper undergraduate side of things, and a good, concise reference here is the opening chapter of Goldman's classical graduate textbook [**36**]. As for the detailed mathematics needed, this can be found in any multivariable calculus book.

In order to further clarify our concept of "conservative force", which generalizes in an efficient and elegant way the usual gravity, the best is to start by talking about work.

This is certainly something that we already discussed, when talking exercices and the need to solve them, but here is now the exact, physical definition of work:

DEFINITION 2.1. *The work done by a force $F = F(x)$ for moving a particle from point $p \in \mathbb{R}^3$ to point $q \in \mathbb{R}^3$ via a given path $\gamma : p \to q$ is the following quantity:*

$$W(\gamma) = \int_\gamma < F(x), dx >$$

We say that F is conservative if this work quantity $W(\gamma)$ does not depend on the chosen path $\gamma : p \to q$, and in this case we denote this quantity by $W(p,q)$.

We will see in a moment that this definition is compatible with our previous definition for the conservative forces. As a first comment now, assume that we have two paths $\gamma : p \to q$ and $\delta : p \to q$. We can then consider the path $\circ : p \to p$ obtained by going along $\gamma : p \to q$, and then along δ reversed, $\delta^{-1} : q \to p$, and we have:

$$W(\circ) = W(\gamma) - W(\delta)$$

Thus F is conservative precisely when, for any loop $\circ : p \to p$, we have:

$$W(\circ) = 0$$

Intuitively, this means that F is some sort of "clean", ideal force, with no dirty things like friction involved. As we will soon see, gravity is such a clean force, with a simple example coming from throwing a rock up in the sky. That rock will travel on a loop $p \to q \to p$, and will come back here to p unchanged, save for the fact that its speed vector is reversed. Thus, and assuming now that work has something to do with energy, which is intuitive, there has been no overall work of gravity on this loop, $W(\circ) = 0$.

An even better example, avoiding any reference to energy, is the movement of the Earth around the Sun. Every year that passes the Earth makes a loop, and wih the Sun obviously not even noticing that, so the yearly work done by the Sun is $W(\circ) = 0$.

As a counterexample now, friction is not conservative. I would definitely prefer to make loops with my lawn mower in my garden, and say to myself, for motivation, that I'm doing $-W(\circ) = 0$, rather than taking that thing up to the North Pole, and back.

As a first result now, regarding the conservative forces, we have:

THEOREM 2.2. *The work done by a conservative force F on a mass m object is*

$$W(p,q) = T(q) - T(p)$$

with $T = m||v||^2/2$ standing as usual for the kinetic energy of the object.

Proof. Assuming that F is conservative, and acts via the usual formula $F = ma$ on our object of mass m, we have the following computation, as desired:

$$\begin{aligned}
W(p,q) &= \int_p^q < F(x), dx > \\
&= m\int_p^q < a(x), dx > \\
&= m\int_p^q \left\langle \frac{dv(x)}{dt}, v(x)dt \right\rangle \\
&= \frac{m}{2}\int_p^q \frac{d < v(x), v(x) >}{dt}\, dt \\
&= \frac{m}{2}\int_p^q \frac{d||v(x)||^2}{dt}\, dt \\
&= \frac{m}{2}\left(||v(q)||^2 - ||v(p)||^2\right) \\
&= T(q) - T(p)
\end{aligned}$$

Here we have used in the middle the fact that the time derivative of a scalar product of functions $< v, w >$ consists of two terms, which are equal when $v = w$. □

Next, we have the following result, which uses some more advanced mathematics:

Theorem 2.3. *A force F is conservative precisely when it is of the form*

$$F = -\nabla V$$

for a certain function V, and in this case the work done by it is given by:

$$W(p,q) = V(p) - V(q)$$

Also, the gravitation force is conservative, coming from $V = -Km/||x||$.

Proof. This is something quite tricky, the idea being as follows:

(1) In one sense, assume that F is conservative. Since the work $W(p,q) = W(\gamma)$ is independent of the chosen path $\gamma : p \to q$, we can find a function V such that:

$$W(p,q) = V(p) - V(q)$$

Observe that this function V is well-defined up to an additive constant. Now with this formula in hand, we further obtain, as desired:

$$\begin{aligned}
W(p,q) = V(p) - V(q) &\implies < F, dx >= -dV \\
&\implies < F, x_i >= -\frac{dV}{dx_i} \\
&\implies F = -\nabla V
\end{aligned}$$

(2) In the other sense now, assuming $F=-\nabla V$, we have the following computation, valid for any loop $\circ:p\to p$, which shows that F is indeed conservative:

$$\begin{aligned} W(\circ) &= -\int_\circ \nabla V \\ &= 0 \end{aligned}$$

More generally, regarding the work done by such a force $F=-\nabla V$, along a path $\gamma:p\to q$, which is independent on this path γ, this is given by:

$$\begin{aligned} W(p,q) &= -\int_p^q \nabla V \\ &= V(p)-V(q) \end{aligned}$$

(3) Finally, the last assertion, regarding the gravitation, this is something that we know from chapter 1, coming via a quick gradient computation, done there. □

We can put now everything together, and we have the following result, which makes the link with the various conservation energy results from chapter 1, and to be more precise generalizes them, and fully clarifies the situation:

THEOREM 2.4. *Given a conservative force F, appearing as follows, with V being uniquely determined up to an additive constant,*

$$F=-\nabla V$$

the movements of a particle under F preserve the total energy, given by

$$E=T+V$$

with $T=m||v||^2/2$ being the kinetic energy, and with V being called potential energy.

PROOF. This is something that we already know from chapter 1, established there by using a computation using the chain rule for derivatives, the idea being as follows:

$$\dot{T}=<F,v>=-<\nabla V,v>=-\dot{V}$$

However, we can now provide an alternative proof for this fact, based on the theory developed above, and more specifically on Theorem 2.2 and Theorem 2.3, which give:

$$W(p,q)=T(q)-T(p)$$

$$W(p,q)=V(p)-V(q)$$

Indeed, we obtain from these equalities the following formula:

$$T(p)+V(p)=T(q)+V(q)$$

Thus, the total energy $E=T+V$ is conserved, as claimed. □

As a side remark here, observe that Theorem 2.4 completely closes the discussion about conservation of energy, at least linguistically, our conclusion being:

CONCLUSION 2.5. *Conservative forces conserve energy.*

So, this will be the general principle to remember. This being said, don't leave a fish out in the sun, it will not be conserved by gravity. Instead, use a refrigerator.

This was for the basic theory of conservative forces, and for more on all this, we refer as usual to [2], [32], [36], [53], [56], [88]. We will be back to such general forces, and to more theory about them later on, when talking about magnetism.

2b. Lagrange and Hamilton

Back now to gravity, and to the various questions that we would like to solve here, left open in the previous chapters, the point is that by using the above energy theory, we can reformulate the whole classical mechanics formalism and equations, in a far more efficient way. Discussing this, following Lagrange and Hamilton, will be our next task.

Things are quite tricky here, involving some sort of unexpected discovery, of the type that you can stumble upon when looking at various formulae coming from physics, with a good mathematical background. So, let us begin with mathematics. The "good mathematical background" that we will need, and that Lagrange needed too, for his discovery, is the following theorem, due to guess who, Euler and Lagrange himself:

THEOREM 2.6. *Given a function $f:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$, the integral*

$$I=\int_{x_0}^{x_1}f(u,\dot{u})dx$$

is stationary, in the sense that it is left unchanged by small variations of $u=u(x)$, which vanish at the endpoints x_0,x_1, precisely when $u=u(x)$ satisfies the equations

$$\frac{df}{du_i}=\frac{d}{dx}\left(\frac{df}{d\dot{u}_i}\right)$$

called Euler-Lagrange equations.

PROOF. Let us just work out the case $N=1$, the general case being similar. Consider a small variation $\Delta u(x)$, which vanishes at the endpoints x_0,x_1, as required above:

$$\Delta u(x_0)=\Delta u(x_1)=0$$

The corresponding variation of $f(u,\dot{u})$, at first order, is then given by:

$$\Delta f=\frac{df}{du}\Delta u+\frac{df}{d\dot{u}}\Delta\dot{u}$$

Thus the corresponding variation of the integral in the statement is given by:

$$\begin{aligned}
\Delta I &= \int_{x_0}^{x_1} \frac{df}{du}\,\Delta u\,dx + \int_{x_0}^{x_1} \frac{df}{d\dot{u}}\,\Delta\dot{u}\,dx \\
&= \int_{x_0}^{x_1} \frac{df}{du}\,\Delta u\,dx + \int_{x_0}^{x_1} \frac{df}{d\dot{u}} \cdot \frac{d(\Delta u)}{dx}\,dx \\
&= \int_{x_0}^{x_1} \frac{df}{du}\,\Delta u\,dx + \left[\frac{df}{d\dot{u}}\,\Delta u\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{df}{d\dot{u}}\right)\Delta u\,dx \\
&= \int_{x_0}^{x_1} \frac{df}{du}\,\Delta u\,dx - \int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{df}{d\dot{u}}\right)\Delta u\,dx \\
&= \int_{x_0}^{x_1} \left(\frac{df}{du} - \frac{d}{dx}\left(\frac{df}{d\dot{u}}\right)\right)\Delta u\,dx
\end{aligned}$$

We conclude that I is stationary precisely when the following equation is satisfied:

$$\frac{df}{du} = \frac{d}{dx}\left(\frac{df}{d\dot{u}}\right)$$

But this is the Euler-Lagrange equation in the statement, as desired. □

The point now with the above is that, when looking at the usual motion equations of mechanics, but written in a somewhat bizarre way, we will get precisely into the Euler-Lagrange equations. So, remember our struggle from chapter 1 with the gravitational potential, and more specifically with $E = T + V$ vs $E = T - V$. We had to ask at that time cat for help, and he said that $E = T + V$ is the good formula, but that $E = T - V$ looks like something quite interesting too. So, following now cat, let us formulate:

Definition 2.7. *In the context of a conservative force F acting, the quantity*

$$L = T - V$$

is called Lagrangian of the system.

This is something quite tricky, but we will get more familiar with it when working out explicit examples, and with the reminder of course that we have already played a bit with $T - V$, in the simplest case, that of a 1D free fall, in chapter 1. In relation now with Theorem 2.6, the connection is very simple, called Hamilton principle, as follows:

Theorem 2.8. *The following integral, called action integral, is stationary,*

$$I = \int_{t_0}^{t_1} L\,dt$$

and the corresponding Euler-Lagrange equations are precisely the equations of motion.

PROOF. According to Definition 2.7, the Lagrangian is given by the following formula, with V being the potential associated to our conservative force, via $F = -\nabla V$:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

Thus, we are in the general framework of Theorem 2.6, with the function u there being played by the coordinates x, y, z. Now let us pick one of these coordinates, $s = x, y, z$, and compute the derivatives of L with respect to $s, \dot{s}$. By using $F = -\nabla V$ we have:

$$\frac{dL}{ds} = -\frac{dV}{ds} = F_s$$

Also since the potential V is time-independent, we have:

$$\frac{dL}{d\dot{s}} = m\dot{s} = ma_s$$

Now consider the equation of motion, under the influence of the force F:

$$F = ma$$

This is a vector equation, with 3 components, and according to the above formulae its 3 components can be written as follows, in terms of the Lagrangian L:

$$\frac{dL}{ds} = \frac{d}{dt}\left(\frac{dL}{d\dot{s}}\right)$$

But these are precisely the Euler-Lagrange equations for the stationarity of the action integral $I = \int L$, and we are therefore led to the conclusions in the statement. □

The point now with the above result is that it leads right away into another result, which this time is something fundamental and powerful, as we will soon discover:

THEOREM 2.9. *The Euler-Lagrange equations for the action integral*

$$I = \int_{t_0}^{t_1} L\, dt$$

hold in any system of coordinates (q_1, q_2, q_3), *and are as follows:*

$$\frac{dL}{dq} = \frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right)$$

These latter equations are called the Lagrange equations of motion.

PROOF. We know from Theorem 2.8 that the action integral is stationary, with respect to the standard coordinates (x, y, z). But this shows that the action integral is stationary with respect to any system of coordinates (q_1, q_2, q_3), and so the corresponding Euler-Lagrange equations, which are the equations in the statement, hold indeed. □

All this might seem a bit complicated, but we will see examples in what follows. The idea every time will be the same, namely thinking a bit, than picking up a suitable system of coordinates (q_1, q_2, q_3) for our problem, and then instead of doing all sorts of computations for reformulating the equations of motion in terms of these coordinates (q_1, q_2, q_3), simply writing the Lagrange equations, which are there for that.

As a basic illustration here, the Newton solution to the Kepler problem, discussed in chapter 1, was using polar coordinates, or rather cylindrical coordinates with z not mattering, to be fully correct. But the computations there can be heavily simplified by starting directly with the Lagrange equations in cylindrical coordinates.

Moving ahead now, let us discuss some further interesting manipulations on the Lagrangian and the Lagrange equations, due to Hamilton. Let us start with:

Definition 2.10. *Given a system of coordinates $q_1, \ldots, q_n$, the quantities*

$$p_i = \frac{dL}{d\dot{q}_i}$$

are called generalized momenta. In terms of them, the Lagrange equations read

$$\frac{dL}{dq_i} = \dot{p}_i$$

analogously to the usual motion equations $F = \dot{p}$.

What are these new variables good for? Let us recall from Definition 2.7 that the Lagrangian was by definition a function as follows:

$$L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

The point now, which is something quite subtle, and useful for all sorts of purposes, in classical mechanics, and especially in its versions and generalizations, is that we can get rid of the derivatives $\dot{q}_1, \ldots, \dot{q}_n$, which are variables in the Lagrange formulation of mechanics, by replacing them by the generalized momenta $p_1, \ldots, p_n$ constructed above. In order to do so we need a clever new quantity H, replacing L, and we have here:

Definition 2.11. *With $q_1, \ldots, q_n$ and $p_1, \ldots, p_n$ being as above, the quantity*

$$H(q, p) = \sum_i p_i \dot{q}_i - L$$

is called Hamiltonian of the system.

As we will soon see, for many simple systems H is in fact the total energy. Before that, however, let us explain how H replaces L, as a quantity which encapsulates as well what is going on, namely the equations of motion. The result here is as follows:

Theorem 2.12. *The Hamiltonian $H = H(q,p)$ is subject to the equations*

$$\frac{dH}{dp_i} = \dot{q}_i \quad , \quad \frac{dH}{dq_i} = -\dot{p}_i$$

called Hamilton equations, which are equivalent to the usual equations of motion.

Proof. As a first observation, this statement reminds right away Theorem 2.9, namely the Lagrange formulation of mechanics, who was claiming the same type of thing. However, there are some differences. On one hand, the new variables $q_1,\ldots,q_n$ and $p_1,\ldots,p_n$ are certainly a bit more abstract than the old ones $q_1,\ldots,q_n$ and $\dot{q}_1,\ldots,\dot{q}_n$. On the other hand, the new equations look great. Regarding now the proof, everything follows from the definition of the variables p_i and from the Lagrange equations, namely:

$$p_i = \frac{dL}{d\dot{q}_i} \quad , \quad \frac{dL}{dq_i} = \dot{p}_i$$

(1) By using the definition of the variables p_i, we obtain the following formula:

$$\begin{aligned}
\frac{dH}{dp_i} &= \frac{d\left(\sum_j p_j\dot{q}_j - L\right)}{dp_i} \\
&= \dot{q}_i + \sum_j p_j\frac{d\dot{q}_j}{dp_i} - \sum_j \frac{dL}{d\dot{q}_j}\cdot\frac{d\dot{q}_j}{dp_i} \\
&= \dot{q}_i + \sum_j p_j\frac{d\dot{q}_j}{dp_i} - \sum_j p_j\frac{d\dot{q}_j}{dp_i} \\
&= \dot{q}_i
\end{aligned}$$

(2) By using the Lagrange equations, we obtain the following formula:

$$\begin{aligned}
\frac{dH}{dq_i} &= \frac{d\left(\sum_j p_j\dot{q}_j - L\right)}{dq_i} \\
&= -\frac{dL}{dq_i} + \sum_j p_j\frac{d\dot{q}_j}{dq_i} - \sum_j \frac{dL}{d\dot{q}_j}\cdot\frac{d\dot{q}_j}{dq_i} \\
&= -\dot{p}_i + \sum_j p_j\frac{d\dot{q}_j}{dq_i} - \sum_j p_j\frac{d\dot{q}_j}{dq_i} \\
&= -\dot{p}_i
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

There are many other things that can be said about the Lagrangian L and the Hamiltonian H, and we will see some examples and general theory, in what follows.

2c. The N body problem

We have now in our bag all the standard tools of the classical mechanician. There are all sorts of problems that we can solve with them, and as usual here we refer to our standard books [2], [32], [36], [53], [56], [88]. In what follows we will focus on the generalizations of the Kepler 2-body problem, which is more or less the only serious problem that we have solved, so far. These generalizations follow into several classes:

(1) 2-body problem with round bodies.
(2) 2-body problem with atmospheric drag.
(3) 2-body problem with one of the bodies rotating.
(4) 3-body problem with 2 bodies being fixed.
(5) 3-body problem with 1 big, distant body.
(6) 3-body problem with 1 body being tiny.
(7) Combinations of the above, and more.

All this screams for help, looks like we are in a complete jungle. And the problem is that (1-7) above are all serious questions, related to problems in the real life. Just throw a rock in front of you, and you're instantly into (1,2,3), taken altogether. Throw something more complicated, like a rocket with a satellite, and you'll have a taste of (4,5,6) too.

Can mathematics save us? We know so far how to solve the Kepler 2-body problem, and looking at the above list (1-7), and thinking underlying math, suggests that question (4) might be the easiest. Indeed, we should normally be able to replace our 2 fixed bodies with a single one, then solve the problem, and have our first 3-body theorem.

In order to discuss this, let us start with the following general notion:

Definition 2.13. *Associated to a system of bodies $M_1,\ldots,M_k$, located at positions $c_1,\ldots,c_k\in\mathbb R^3$ is their center of mass, located at the following position:*

$$c=\frac{\sum_ic_iM_i}{\sum_iM_i}$$

A single body of mass $\sum_iM_i$ located there, at the center of mass, and with $M_1,\ldots,M_k$ being erased, will be called average of the system formed by $M_1,\ldots,M_k$.

You are certainly a bit familiar with this, and we will work out the math in a moment. However, before doing that, let us see if this can help in connection with problem (4) above. When looking in real life for two fixed bodies M_1,M_2 the first thing which comes to mind is a dumbell. Which is however something quite small, so let us formulate:

Definition 2.14. *The Devil's dumbell is a system of two fixed objects*

$$\bullet_{M_1}\,\text{——}_d\text{—}\,\bullet_{M_2}$$

which can have arbitrary characteristics $M_1>0$, $M_2>0$, $d>0$.

As already mentioned, the simplest example is a usual dumbell, which however at the galactic level is subject to $M_i \simeq 0$ and $d \simeq 0$. Bigger examples exist in distant galaxies, with the giants there training with specially designed dumbells, M_i being roughly the mass of an average planet, and d being accordingly large. As for the fully versatile dumbell in Definition 2.14, that is a creation of the Devil, for annoying us physicists.

Getting back now to question (4), we would like to understand the motion of an object of mass m around a dumbell of parameters M_1, M_2, d, as in Definition 2.14, by using the notion of center of mass of that dumbell, constructed according to Definition 2.13:

$$\circ_m$$

$$\bullet_{M_1} \text{——} \star_c \text{——} \bullet_{M_2}$$

And the preliminary observations here are quite grim, as follows:

Observations 2.15. *In the context of a body of mass m moving around a dumbell of parameters M_1, M_2, d, the following happen:*

(1) *When m is at distance $x >> 0$ from the dumbell, the gravitation force acting on it is $F_1 + F_2 \simeq F_c$, so m should travel on some sort of approximate conic.*
(2) *However, when m is at distance $x \simeq 0$ from the dumbell, the physics and trajectory have nothing to do with the center of mass, $F_1 + F_2 \not\simeq F_c$.*

Here in what regards forces, (1) is something quite obvious and intuitive, and we will do the math in a moment, with a proof of $F_1 + F_2 \simeq F_c$, and a study of the correction term as well. As for (2), this is something obvious and intuitive too, because if you place m on the line joining M_1, M_2, bad things will happen, and the only possibility where c can be of help is when m was initially placed precisely on c, and with this, point on a line being at a prescribed location, being an event happening with probability 0.

As for the trajectory claim in (1), that is something sort of intuitive, but not really, because as you might know from observations with pendulums, balls rolling on various surfaces, and so on, involving equilibrium and non-equilibrium, there is no guarantee that the solution of a perturbed problem is a perturbation of the initial solution.

Getting back now to our list (1-7), it looks like the simplest question there, (4), while suggesting a quick solution using the center of mass, is something quite difficult. So what to do. As usual in such difficult situations, we will ask the cat. And cat says:

Advice 2.16. *Start with some basic math of the N body systems, but beware here of the center of mass, and of what physicists claim they can do with it.*

What cat says here is quite frightening. So not only we're into difficult questions, but in addition no one, including our standard mechanics books, can help us. Are all the formulae there, regarding conservation of momentum, angular momentum, energy and so on, all obtained by using an elegant use of the center of mass, correct or not?

To be seen. So let us start with the beginning, basic mathematics of the center of mass, as constructed in Definition 2.13. To be kept in mind first is:

PROPOSITION 2.17. *The center of mass is not a center of gravity, in the sense that the gravity there is not necessarily* 0. *For instance the center of mass of a dumbell is*

$$\bullet_{M_1}\underbrace{\qquad\qquad}_{\frac{M_2 d}{M_1+M_2}}\star_{cm}\underbrace{\qquad\qquad\qquad}_{\frac{M_1 d}{M_1+M_2}}\circ_{M_2}$$

while the center of gravity, which is the unique point where the gravity is 0*, is:*

$$\bullet_{M_1}\underbrace{\qquad\qquad\qquad}_{\frac{\sqrt{M_1}d}{\sqrt{M_1}+\sqrt{M_2}}}\star_{cg}\quad\underbrace{\qquad\qquad}_{\frac{\sqrt{M_2}d}{\sqrt{M_1}+\sqrt{M_2}}}\circ_{M_2}$$

The systems of $k\geq 3$ bodies might have several centers of gravity, usually uncomputable.

PROOF. There are several assertions here, the idea being as follows:

(1) Regarding the dumbell, pictured above with $M_1>M_2$, the formula for the center of mass is clear from definitions. Regarding now the center of gravity, the formula there can be found by doing the math, and it works, because the acceleration there is:

$$a=-\frac{GM_1}{\left(\frac{\sqrt{M_1}d}{\sqrt{M_1}+\sqrt{M_2}}\right)^2}+\frac{GM_2}{\left(\frac{\sqrt{M_2}d}{\sqrt{M_1}+\sqrt{M_2}}\right)^2}=0$$

(2) Getting now to systems $M_1,\ldots,M_k$ with $k\geq 3$, things here are quite complicated. With c_i being the position of M_i, a center of gravity $x\in\mathbb R^3$ must satisfy:

$$\sum_i\frac{M_i(x-c_i)}{||x-c_i||^3}=0$$

Equivalently, we are looking for solutions of the equation $\nabla V=0$, where:

$$V=-\sum_i\frac{GmM_i}{||x-c_i||}$$

(3) Let us first examine the simplest case, that of 3 bodies on a line, at distinct positions. Here, by obvious reasons, we have 2 centers of gravity, as follows:

$$\bullet_{M_1}-\star_{x_1}-\bullet_{M_2}-\star_{x_2}-\bullet_{M_3}$$

More generally, again by obvious reasons, a system of aligned bodies $M_1,\ldots,M_k$ has $k-1$ centers of gravity, one in between each pair of consecutive bodies.

(4) In the 2D case now, and with $k\geq 3$, we are looking the the center of gravity of a triangle, with vertices weighted by masses M_1,M_2,M_3. The simplest possible situation is that of an equilateral triangle, with equal masses M,M,M at its vertices, and it is quite clear here that we will have 3 solutions, 1 lying on each of the 3 symmetry axes.

(5) This is of course quite bad news, because we have now 3 solutions, instead of the 2 ones found in one dimension, in (3). And for the disaster to be complete, let us attempt now to compute these solutions. Best is to use here complex numbers, with our triangle being $1, w, w^2$ in the complex plane, with $w = e^{2\pi i/3}$. We will only look for the solution on the Ox axis, say $r \in \mathbb{R}$, the other solutions being wr, w^2r. The equation is:

$$\frac{r-1}{|r-1|^3} + \frac{r-w}{|r-w|^3} + \frac{r-w^2}{|r-w^2|^3} = 0$$

By simplifying at left and using $1 + w + w^2 = 0$ for the right terms, this reads:

$$\frac{2r+1}{|r-w|^3} = \frac{1}{(1-r)^2}$$

(6) And that is pretty much it, for computing r we must raise this to the square, which leads us into a degree 6 equation, and we will not do this. As a conclusion, things are hard for the equilateral triangle equally weighted, and can only be harder in general.

(7) As a final comment, however, and forgetting perhaps about exact numerics, this is a geometry problem. Indeed, the equations in (2) suggest looking at $\mathbb{R}^3$ with a "hole" at each M_i, and the problem is that of understanding which points are in equilibrium. This is some sort of an Einstein idea, and we will be back to this later. □

Moving ahead, and looking for an easier question, let us still examine the gravity of a rigid object, formed by fixed bodies $M_1, \ldots, M_k$, but at a distance. We have here:

THEOREM 2.18. *Consider a rigid object, consisting of fixed bodies $M_1, \ldots, M_k$, located at positions $c_1, \ldots, c_k \in \mathbb{R}^3$. The corresponding gravitation force, $F = -\nabla V$ with*

$$V = -\sum_i \frac{GmM_i}{||x - c_i||}$$

can be approximated by the force coming from the center of mass, $F_c = -\nabla V$ with

$$V_c = -\frac{Gm\sum_i M_i}{||x - c||}$$

at order zero, when $x >> c_i$. The correction term can be computed as well.

PROOF. We have several assertions here, the idea being as follows:

(1) The first assertion, $F \simeq F_c$ when $x >> c_i$, is something clear, and with this not even needing c to be the center of mass. Indeed, with V, V_c as above, we have:

$$V = -\sum_i \frac{GmM_i}{||x - c_i||} \simeq -\sum_i \frac{GmM_i}{||x - c||} = V_c$$

(2) Regarding now the correction term, the error to be estimated is:

$$\begin{aligned} V - V_c &= -\sum_i \frac{GmM_i}{||x - c_i||} + \frac{Gm\sum_i M_i}{||x - c||} \\ &= \sum_i GmM_i \left(\frac{1}{||x - c||} - \frac{1}{||x - c_i||} \right) \end{aligned}$$

(3) By translation we may assume $c = 0$. In order to evaluate the difference of inverses on the right, we use the following trick, valid for any two vectors $x >> d$:

$$\begin{aligned} \frac{1}{||x||} - \frac{1}{||x - d||} &= \frac{||x - d|| - ||x||}{||x|| \cdot ||x - d||} \\ &= \frac{||x - d||^2 - ||x||^2}{||x|| \cdot ||x - d|| \cdot (||x|| + ||x - d||)} \\ &= \frac{||d||^2 - 2 < x, d >}{||x|| \cdot ||x - d|| \cdot (||x|| + ||x - d||)} \\ &\simeq -\frac{< x, d >}{||x||^3} \end{aligned}$$

To be more precise, in the last step we have neglected upstairs the order 0 term $||d||^2$, and downstairs we have approximated all norms appearing there by $||x||$.

(4) Getting back now to the formula in (2), assuming $c = 0$ by translation, as already mentioned above, and by using the trick in (3), we obtain:

$$\begin{aligned} V - V_c &= \sum_i GmM_i \left(\frac{1}{||x||} - \frac{1}{||x - c_i||} \right) \\ &\simeq -\sum_i GmM_i \frac{< x, c_i >}{||x||^3} \\ &= -\frac{Gm}{||x||^3} \left\langle x, \sum_i M_i c_i \right\rangle \end{aligned}$$

(5) Let us think now at the meaning of the above formula. Normally this is the formula of the first order correction, but when $c = 0$ is the center of mass, this means precisely that we have $\sum_i M_i c_i = 0$, so this correction that we computed vanishes. So, in short, on one hand good news, we are on the good way, with the center of mass c being the ideal location for the origin of the approximating potential V_c, but on the other hand, bad news, precisely due to this fact, we must fine-tune our tricks from (3) above, get a better inequality there, that we can use in order to compute the nonzero correction at c.

(6) So here we go again with estimates. Getting back to the end of the computation in (3), we need there an estimate for $||x-d||$, and this can be found as follows:

$$\begin{aligned} ||x-d|| &= \sqrt{||x||^2+||d||^2-2<x,d>} \\ &\simeq \sqrt{||x||^2-2<x,d>} \\ &\simeq ||x||-<x,d> \end{aligned}$$

With this in hand, we can improve our master estimate in (3), as follows:

$$\begin{aligned} \frac{1}{||x||}-\frac{1}{||x-d||} &= \frac{||d||^2-2<x,d>}{||x||\cdot||x-d||\cdot(||x||+||x-d||)} \\ &\simeq \frac{||d||^2-2<x,d>}{||x||\cdot(||x||-<x,d>)\cdot(2||x||-<x,d>)} \\ &\simeq \frac{||d||^2-2<x,d>}{||x||\cdot(2||x||^2-3||x||<x,d>)} \\ &= \frac{||d||^2-2<x,d>}{||x||^2\cdot(2||x||-3<x,d>)} \end{aligned}$$

Now if we denote by α the angle between x and d, this formula becomes:

$$\begin{aligned} \frac{1}{||x||}-\frac{1}{||x-d||} &\simeq \frac{||d||^2-2||x||\cdot||d||\cdot\cos\alpha}{||x||^2\cdot(2||x||-3||x||\cdot||d||\cdot\cos\alpha)} \\ &= \frac{||d||}{||x||^3}\cdot\frac{||d||-2||x||\cdot\cos\alpha}{2-3||d||\cdot\cos\alpha} \end{aligned}$$

(7) Thus, we can improve the estimate found in (4), with the conclusion that at the center of mass, taken to be at the origin, $c=0$, the error is as follows, with α_i being the angles between our body m, and the components $M_1,\ldots,M_k$ of the rigid body:

$$\begin{aligned} V-V_c &= \sum_i GmM_i\left(\frac{1}{||x||}-\frac{1}{||x-c_i||}\right) \\ &\simeq \sum_i\frac{||c_i||}{||x||^3}\cdot\frac{||c_i||-2||x||\cdot\cos\alpha_i}{2-3||c_i||\cdot\cos\alpha_i} \end{aligned}$$

(8) Assuming in addition that we are in a generic position, where $\alpha_i\neq\pi/2$ for any i, the upper terms can be further estimated, and we obtain in this way:

$$\begin{aligned} V-V_c &\simeq -\sum_i\frac{||c_i||}{||x||^3}\cdot\frac{2||x||\cdot\cos\alpha_i}{2-3||c_i||\cdot\cos\alpha_i} \\ &= -\sum_i\frac{||c_i||}{||x||^2}\cdot\frac{2\cos\alpha_i}{2-3||c_i||\cdot\cos\alpha_i} \end{aligned}$$

Summarizing, we have proved our result, and with a few bonus conclusions, namely that the center of mass c is indeed the ideal location for the approximate potential V_c, and that the computation of the error term there ultimately involves the angles $\alpha_1,\ldots,\alpha_k$ between our body m, and the components $M_1,\ldots,M_k$ of our rigid body. □

Before going ahead and leaving this subject, let us mention that an interesting generalization of the above comes when considering a "true" rigid body, made of matter arranged according to a certain density function ρ inside it. We will not go into details here, and instead let us just formulate a basic statement, as follows:

THEOREM 2.19. *Consider a rigid body, made of matter arranged according to a certain density function ρ inside it. Its gravitational force is then $F=-\nabla V$ with*

$$V=-\int\frac{Gm\rho(z)}{||x-z||}\,dz$$

and can be approximated by the force coming from the center of mass, $F_c=-\nabla V$ with

$$V_c=-\frac{Gm\int\rho(z)dz}{\left|\left|x-\int u\rho(u)du\right|\right|}\,dz$$

at order zero, when m is far away. The correction term can be computed as well.

PROOF. Here the formulae in the statement, which are perfectly similar to those in Theorem 2.18, can be obtained via the usual philosophy "replace sums by integrals". Observe in particular the formula of the center of mass, producing V_c, namely:

$$c=\int u\rho(u)du$$

As for the last assertion, this can only hold too, by proceeding as in the proof of Theorem 2.18, and replacing everywhere at the end the sums by integrals. □

The above results, Theorem 2.18 and Theorem 2.19, are both quite interesting, and suggest a whole string of further questions, and potential generalizations. What happens in the context of Theorem 2.18 when the constituents $M_1,\ldots,M_k$ are allowed to move a bit, say by being confined by an external force? Then, what happens when the constituents $M_1,\ldots,M_k$ are allowed to freely move? Also, what about Theorem 2.19, if we allow there some kind of fluid movement inside the body? And so on. These are obviously all difficult questions, of general N-body problem type, so perhaps time to stop here.

Moving ahead now, and still in connection with Advice 2.16, let us examine now various conservation questions. The simplest problematics here is most likely that of the angular momentum, but as we will soon discover, things are quite tricky here.

As a first question, we know from chapter 1 that when M_2 moves around M_1, positioned at 0, its angular momentum J_2 is constant. By symmetry, if we regard M_1 moving around

M_2, fixed at 0, its angular momentum J_1 will be constant too. Thus in both cases $J = J_1 + J_2$ is constant, which raises the question whether J is constant or not when computed at other points of $\mathbb{R}^3$. And here, we first have the following result:

PROPOSITION 2.20. *In the context of the 2-body problem, the following happen:*

1. *J is conserved when assuming that M_1 or M_2 is fixed at 0.*
2. *More generally, J is conserved at any $\lambda_1 M_1 + \lambda_2 M_2$, with $\lambda_1 + \lambda_2 = 1$.*
3. *In particular, J is conserved when computed at the center of mass.*
4. *However, J is not conserved when assuming that M_1 or M_2 is fixed at $d \neq 0$.*

PROOF. We have several assertions here, the idea being as follows:

(1) This is something that we know well, from chapter 1, as explained above.

(2) Assume first, as in (1), that M_1 is fixed at 0, and that M_2 moves around it, with position vector $x \in \mathbb{R}^3$. Given parameters λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 = 1$, let us set:

$$y = \lambda_1 \cdot 0 + \lambda_2 \cdot x = \lambda_2 x$$

We make now the convention that at $t = 0$ this point was the origin, with coordinate axes parallel to our original coordinate axes, and that at any $t > 0$ this is still the origin, with the directions of the coordinate axes being unchanged. Thus, we have a new frame, and the coordinates of our objects M_1, M_2 with respect to this new frame are:

$$z_1 = -\lambda_1 x \quad , \quad z_2 = \lambda_2 x$$

But in this new frame the momenta of M_1, M_2 are both proportional, by factors $\lambda_1^2 M_2/M_1$ and λ_2^2, to the original momentum of M_2, computed in (1), which was constant. Thus both these momenta J_1, J_2 are constant, and so is their sum $J = J_1 + J_2$.

(3) This follows from (2), with $\lambda_1 = M_2/(M_1 + M_2)$ and $\lambda_2 = M_1/(M_1 + M_2)$.

(4) Assuming that M_1 is fixed at a given point $d \neq 0$, and that M_2 travels around it, with position vector $d + x$, with x being as in (1), we still have $J_1 = 0$, and so:

$$\begin{aligned}
J &= J_2 \\
&= (d + x) \times p \\
&= d \times p + x \times p \\
&= d \times p + \text{constant} \\
&\neq \text{constant}
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a conclusion, the conservation of angular momentum depends on the "quality" of the frame that we are using. In a good frame, as in (1,2,3) above, the momentum will be conserved, while in a bad frame, as in (4), the momentum will be not conserved.

Generally speaking, we will be talking frames later, when discussing relativity. In connection with our questions here we will be quite brief, and in order to keep moving, let us formulate the following informal definition, that will do:

Definition 2.21. *An inertial frame is a frame where all basic formulae, namely*

$$||F||=\frac{Gm_1m_2}{||x_1-x_2||^2}\quad,\quad F=ma\quad,\quad a=\dot{v}\quad,\quad v=\dot{x}\quad,\quad F_{12}=-F_{21}$$

hold, with the last formula standing for Newton's action-reaction principle.

To be more precise here, the first 4 formulae are something that we have been heavily using, so far in this book. As for the last formula, also called Newton's third law, this expresses the fact that when an object 1 acts on an object 2, say via gravity, with force F_{12}, then object 2 acts as well on object 1, with force $F_{21}=-F_{12}$.

As already mentioned, we will discuss all this later, more in detail. In relation with our present considerations, we have the following basic examples:

Proposition 2.22. *In the context of the 2-body problem, the frames of type*

$$\lambda_1M_1+\lambda_2M_2$$

constructed above are all non-inertial, including the center of mass frame.

Proof. Since our definition of an inertial frame was something quite informal, so will be this proof. We want to check whether the forces between M_1,M_2 satisfy:

$$F_{12}=-F_{21}=\frac{GM_1M_2(x_1-x_2)}{||x_1-x_2||^3}$$

(1) In the case of the frame centered at M_1, the formula $F_{12}=-F_{21}$ certainly does not hold, because the acceleration of M_1 is in this case $\ddot{0}=0$, and so no force acting upon it, at least from our calculus viewpoint. The same holds for the frame centered at M_2.

(2) In general now, where we have parameters λ_1,λ_2 satisfying $\lambda_1+\lambda_2=0$, as in Proposition 2.20, as explained there, the positions of M_1,M_2 are:

$$z_1=-\lambda_1x\quad,\quad z_2=\lambda_2x$$

Thus the forces acting upon M_1,M_2, computed according to calculus, are:

$$F_{21}=-M_1\lambda_1\ddot{x}\quad,\quad F_{12}=-M_2\lambda_2\ddot{x}$$

Thus, in order to have $F_{12}=-F_{21}$, the parameters λ_1,λ_2 satisfy $M_1\lambda_1=M_2\lambda_2$. But these are exactly the parameters of the center of mass.

(3) But the center of mass frame is not inertial either, because due to the fact that we performed a dilation, the magnitude of $F_{12}=-F_{21}$ is not the correct one. □

Now back to momentum, we have the following extension of Proposition 2.20 (3):

THEOREM 2.23. *In an inertial frame, the total angular momentum*

$$J=\sum_i x_i\times p_i$$

of a system of bodies $M_1,\ldots,M_k$ is conserved.

PROOF. Our inertial frame assumption tells us that we can use at will all formulae in Definition 2.21, and by using them, and notably $F_{12}=-F_{21}$, we obtain:

$$\begin{aligned}\dot{J} &= \sum_i x_i\times\sum_{j\neq i}F_{ji}\\ &= \sum_{i<j}x_i\times F_{ji}+x_j\times F_{ij}\\ &= \sum_{i<j}x_i\times F_{ji}-x_j\times F_{ji}\\ &= \sum_{i<j}(x_i-x_j)\times F_{ji}\\ &= 0\end{aligned}$$

Now since we have $\dot{J}=0$, the angular momentum J is conserved, as claimed. □

Moving ahead now, our next problem will concern the conservation of energy. Here things are a bit similar with angular momentum, but more can be said, as follows:

THEOREM 2.24. *With a suitable potential formalism, the total energy*

$$E=\sum_i T_i+V_i$$

of a system of bodies $M_1,\ldots,M_k$ is conserved. Also, the individual energy

$$E'=T'+\sum_i V_i'$$

of an extra body m added is conserved as well, again with a suitable formalism.

PROOF. There are several questions here, the idea being as follows:

(1) In what regards $T=\sum_i T_i$ we have, exactly as in the 2-body problem:

$$\begin{aligned}\dot{T} &= \sum_{i\neq j}<v_i,F_{ji}>\\ &= \sum_{i\neq j}<v_i,-\nabla V_{ji}>\\ &= -\sum_{i\neq j}\dot{V}_{ji}\end{aligned}$$

(2) With this in hand, we can group pairs of terms, as in the proof of Theorem 2.23, and we are led to the conclusion in the statement, and with the remark however that all the potentials appearing there are now time-dependent.

(3) In what regards now the second assertion, this is not exactly something of the same nature as the first assertion, becase assuming that by some kind of miracle we would have a theory where all the bodies conserve their energy, the total energy of the system would be trivially conserved too, just by summing, and this does not look normal. So, getting now to the second assertion as formulated, we have, by computing as in (1) above:

$$\dot{T}' = -\sum_i \dot{V}_i'$$

(4) Thus, we are led to the conclusion in the statement, with the problem however that all the potentials appearing there are now time-dependent. □

2d. Lagrange points

With the above questions discussed, let us go back now to our to-do list (1-7), from the beginning of this section. We have all sorts of difficult questions there, and we will focus first on (6), namely 3-body problem with one of the bodies being tiny.

Things here are quite complicated, technically speaking. Let us start with:

Definition 2.25. *The 3-body problem is the general gravitational problem for three bodies M_1, M_2, M_3, with the corresponding equations of motion being as follows:*

$$\ddot{x}_1 = -\frac{GM_2(x_1 - x_2)}{||x_1 - x_2||^3} - \frac{GM_3(x_1 - x_3)}{||x_1 - x_3||^3}$$

$$\ddot{x}_2 = -\frac{GM_3(x_2 - x_3)}{||x_2 - x_3||^3} - \frac{GM_1(x_2 - x_1)}{||x_2 - x_1||^3}$$

$$\ddot{x}_3 = -\frac{GM_1(x_3 - x_1)}{||x_3 - x_1||^3} - \frac{GM_2(x_3 - x_2)}{||x_3 - x_2||^3}$$

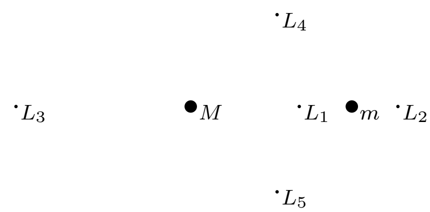
The planar 3-body problem is this problem, in a plane. The restricted 3-body problem is also this problem, in the situation $M_1, M_2 >> M_3$, and usually considered in a plane.

The general 3-body problem has all sorts of weird solutions, and cannot be solved via exact mathematics. You would probably say no problem, just give it to a computer, but guess what, the computer cannot solve that either. The problem with computers is that, no matter how big and powerful they are, they can still only operate with a finite amount of data, and at a certain speed, and what happens, somehow, is that the 3 equations in Definition 2.25 can produce after some time $t > 0$ all sorts of bizarre phenomena, going in all senses, and managing all this is impossible, even for a powerful computer.

The best here is to go on internet, and look up some animations. Just by looking at them, and how bizarre they can be, you will quickly realize that such things are beyond what you, what your professors, and even what a powerful computer, can do.

This being said, and in connection now with our satellite problem that we have in mind, question (6) on our to-do list, some mathematics is however possible, in certain simple cases, and we have the following remarkable result, due to Lagrange:

Theorem 2.26. *The restricted* 3*-body problem has* 5 *distinguished solutions,*



called Lagrange points L1-L5, whose positions with respect to M, m *are as above.*

Proof. This is certainly something quite complicated, using all sorts of advanced mathematics, further building on what has been developed in the above, and we won't get into details here. Instead, let us describe at least how each L1-L5 functions:

(1) L1 is the most intuitive solution, placed in between M and m, and closer to m, at that precise point where the gravitation of M equals in magnitude the gravitation of m. The math here for finding the distances is very simple, but recall however that m is supposed to move around M, on an ellipsis. Thus, the picture is that an object ε placed at L1 moves as well around M, by staying aligned with m, on a smaller ellipsis.

(2) One problem with L1 comes from the fact that it is unstable, in the sense that an object ε placed around there, but not exactly there, will not stay there. Indeed, was ε to be placed a tiny little bit towards M, it will start slowly moving towards M, and gone it will be. And the same can happen in the other sense, with m being able to capture it too, since L1 is some sort of no man's land between the gravities of M, m, which look equal from there. Thus, L1 looks like some sort of fake solution to the problem.

(3) However, all this is useful in practice, because with just a little bit of homemade acceleration, from time to time, in order to correct the trajectory and keep it on L1, a satellite can be placed there at L1, and will stay there. In fact, most of the scientific satellites are placed there at L1, first because its proximity to Earth, but also because there is no dirt like asteroids trapped there, due to the fact that L1 is unstable.

(4) Getting to L2 now, the functioning mechanism here is different, crucially relying on the fact that m, and so L2 too, does move around M, on an elliptic orbit. Indeed, what looks impossible in 1D, namely an object ε placed behind m not to be attracted by both M, m, and start going towards them, is now possible in the context of the 2D elliptical

movement, with the normal movement of ε with respect to M being slightly altered by the presence of m, which in practice tends to pull ε away from M, m, and with the precise distance being that where equilibrium is achieved, in all this. As for L1, this point L2 is not far from Earth, and unstable, making it usable for satellites.

(5) In what regards now L3, yet another functioning mechanism going on here, which is this time something very simple, namely an object ε placed there at L3 will simply travel around M, in a standard elliptic way, basically on the same orbit as m, slightly adjusted as to take into account the gravity of m too. Again, this is an unstable point, suitable for satellites, but who would go up there to install one.

(6) Finally, regarding L4 and L5, these are two extra solutions, discovered later by Lagrange, located at the positions where they form, along with M, m, equilateral triangles. An object ε say placed at L4 will stay there, or rather travel on an elliptical orbit around M passing through L4, keeping its L4 relative position with respect to M, m, due to the fact that, due to the geometry of the equilateral triangle $\varepsilon - m - M$, the extra pull from m keeps it in tune with m, on that smaller orbit. And the same goes for L5.

(7) These two last points L4, L5 are stable, provided that the masses M, m of the two big objects satisfy $M/m > 24.96$, which is the case for instance for the Sun-Earth system, and for the Earth-Moon system too. However, the stability makes them unsuitable for satellite use, due to the tons of space garbage accumulated there, over the years. □

So long for the N-body problem, and for the Lagrange points. Obviously there is some very interesting mathematics and physics going on here, which is relevant for anything in relation with the Solar System, be that natural or human-made. In fact, speaking astronomy, the Sun-Earth-Moon system is already something that you can spend your whole life of scientist on, because due to pure 3-body gravitation, and then also to various imperfections in the shape, density, and many other parameters of these objects, things in this system are continuously evolving, over the passing years, and with the very long term predictions on what will really happen being quite complicated to make.

Getting back now to our to-do list from the beginning of the previous section, let us focus now on question (2) there, namely the 2-body problem with atmospheric drag. This is something not discussed yet, and of crucial real-life importance, and for everything in relation with engineering, and which can bring us far, deep into fluid mechanics.

Mathematically, we have to go back to the Kepler 2-body problem, with the aim of doing some more study here, in the parabolic trajectory case, which is traditionally the field of ballistics. A well regulated militia being necessary to the security of a free state, we have the following result, that you might find of interest:

Proposition 2.27. *Ballistics.*

PROOF. This follows indeed by doing some computations. In what regards the notion of escape velocity, the conclusions here might seem quite surprising. □

Speaking arms, war and related topics, we have as well the following result:

PROPOSITION 2.28. *Rockets, again.*

PROOF. We already know about rockets from chapter 1. In order to beat now gravity, we can just write down the rocket equations, coming from the conservation of the momentum principle, and do some computations in relation with gravitation. □

In the elliptic trajectory case, which is the most interesting, mathematically speaking, and for various peaceful applications too, such as scientific satellites, we have:

PROPOSITION 2.29. *Satellites.*

PROOF. This follows again by doing some computations. □

Still in the elliptic trajectory case, at a more advanced level now, allowing the launch of satellites far away from the dust, radiation and other mess on Earth, we have:

PROPOSITION 2.30. *Outer space satellites.*

PROOF. This is something that we already know, from our study of Lagrange points, performed in the above. □

Moving ahead now towards a different problematics, and in practice by getting back to, guess what, the good old Kepler 2-body problem, most of the questions studied above, regarding ballistics and rockets, are subject in the real life to atmospheric drag. Thus, we must discuss now the needed corrections. We first have here the following result:

PROPOSITION 2.31. *Ballistics, with drag.*

PROOF. This follows indeed by doing some computations. In what regards the notion of escape velocity, the conclusions here might seem quite surprising. □

Regarding now rockets, we have here the following result:

PROPOSITION 2.32. *Rockets, with drag.*

PROOF. This follows again by doing some computations. □

There are of course many other things that can be said on all the above, and about versions of the various systems considered above, with all this being very useful for engineering, or any other concrete application of mechanics, and we refer here as usual to our go-to mechanics books, namely [**2**], [**32**], [**36**], [**53**], [**56**], [**88**].

Then, what about forgetting about objects moving through fluids, and investigating the fluids themselves? There are many interesting things that can be said here:

Fact 2.33. *Fluid mechanics equations.*

Actually things here are quite mixed, with some of these equations being facts, and some other being mathematical theorems. The mathematics however needed for axiomatization is quite complicated, that of the so-called diffeomorphism groups.

Finally, we have seen the basic physics of the solid-fluid and fluid-fluid interactions. For our discussion to be complete, we should talk as well about solid-solid interactions.

2e. Exercises

Exercises.

CHAPTER 3

State space

3a.

3b.

3c.

3d.

3e. Exercises

CHAPTER 4

Stability loss

4a.

4b.

4c.

4d.

4e. Exercises

Part II

Fluids, granularity

Gotta make a move to a town
That's right for me
Town to keep me moving
Keep me grooving with some energy

CHAPTER 5

5a.

5b.

5c.

5d.

5e. Exercises

CHAPTER 6

6a.

6b.

6c.

6d.

6e. Exercises

CHAPTER 7

7a.

7b.

7c.

7d.

7e. Exercises

CHAPTER 8

8a.

8b.

8c.

8d.

8e. Exercises

Part III

Galaxies, clustering

Join me for a ride
Speed up the music
Join me for a ride
Maximum overdrive

CHAPTER 9

9a.

9b.

9c.

9d.

9e. Exercises

CHAPTER 10

10a.

10b.

10c.

10d.

10e. Exercises

CHAPTER 11

11a.

11b.

11c.

11d.

11e. Exercises

CHAPTER 12

12a.

12b.

12c.

12d.

12e. Exercises

Part IV

Interstellar dust

I got the poison
I got the remedy
I got the pulsating
Rhythmical remedy

CHAPTER 13

13a.

13b.

13c.

13d.

13e. Exercises

CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

CHAPTER 15

15a.

15b.

15c.

15d.

15e. Exercises

CHAPTER 16

16a.

16b.

16c.

16d.

16e. Exercises

Bibliography

[1] V.I. Arnold, Ordinary differential equations, Springer (1973).

[2] V.I. Arnold, Mathematical methods of classical mechanics, Springer (1974).

[3] V.I. Arnold, Lectures on partial differential equations, Springer (1997).

[4] V.I. Arnold, Catastrophe theory, Springer (1974).

[5] V.I. Arnold and B.A. Khesin, Topological methods in hydrodynamics, Springer (1998).

[6] V.I. Arnold, V.V. Kozlov and A.I. Neishtadt, Mathematical aspects of classical and celestial mechanics, Springer (2006).

[7] N.W. Ashcroft and N.D. Mermin, Solid state physics, Saunders College Publ. (1976).

[8] M.F. Atiyah, The geometry and physics of knots, Cambridge Univ. Press (1990).

[9] T. Banica, Calculus and applications (2023).

[10] T. Banica, Introduction to modern physics (2023).

[11] G.K. Batchelor, An introduction to fluid dynamics, Cambridge Univ. Press. (1967).

[12] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[13] S.J. Blundell and K.M. Blundell, Concepts in thermal physics, Oxford Univ. Press (2006).

[14] S.M. Carroll, Spacetime and geometry, Cambridge Univ. Press (2004).

[15] P.M. Chaikin and T.C. Lubensky, Principles of condensed matter physics, Cambridge Univ. Press (1995).

[16] A.R. Choudhuri, Astrophysics for physicists, Cambridge Univ. Press (2012).

[17] D.D. Clayton, Principles of stellar evolution and nucleosynthesis, Univ. of Chicago Press (1968).

[18] A. Connes, Noncommutative geometry, Academic Press (1994).

[19] W.N. Cottingham and D.A. Greenwood, An introduction to the standard model of particle physics, Cambridge Univ. Press (2012).

[20] P.A. Davidson, Introduction to magnetohydrodynamics, Cambridge Univ. Press (2001).

[21] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.

[22] A.P. French, Special relativity, Taylor and Francis (1968).

[23] P.A.M. Dirac, Principles of quantum mechanics, Oxford Univ. Press (1930).

[24] M.P. do Carmo, Differential geometry of curves and surfaces, Dover (1976).

[25] M.P. do Carmo, Riemannian geometry, Birkhäuser (1992).

[26] S. Dodelson, Modern cosmology, Academic Press (2003).

[27] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).

[28] A. Einstein, Relativity: the special and the general theory, Dover (1916).

[29] L.C. Evans, Partial differential equations, AMS (1998).

[30] W. Feller, An introduction to probability theory and its applications, Wiley (1950).

[31] E. Fermi, Thermodynamics, Dover (1937).

[32] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics I: mainly mechanics, radiation and heat, Caltech (1963).

[33] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics II: mainly electromagnetism and matter, Caltech (1964).

[34] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics III: quantum mechanics, Caltech (1966).

[35] S. Galtier, Introduction to modern magnetohydrodynamics, Cambridge Univ. Press (2016).

[36] H. Goldstein, C. Safko and J. Poole, Classical mechanics, Addison-Wesley (1980).

[37] D.L. Goodstein, States of matter, Dover (1975).

[38] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, Cambridge Univ. Press (2012).

[39] W. Greiner, L. Neise and H. Stöcker, Thermodynamics and statistical mechanics, Springer (2012).

[40] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).

[41] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).

[42] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).

[43] D.J. Griffiths, Revolutions in twentieth-century physics, Cambridge Univ. Press (2012).

[44] J. Harris, Algebraic geometry, Springer (1992).

[45] W.A. Harrison, Solid state theory, Dover (1970).

[46] M.P. Hobson, G.P. Efstathiou and A.N. Lasenby, General relativity, Cambridge Univ. Press (2006).

[47] K. Huang, Introduction to statistical physics, CRC Press (2001).

[48] K. Huang, Quantum field theory, Wiley (1998).

[49] K. Huang, Quarks, leptons and gauge fields, World Scientific (1982).

[50] K. Huang, Fundamental forces of nature, World Scientific (2007).

[51] V.F.R. Jones, Subfactors and knots, AMS (1991).

[52] L.P. Kadanoff, Statistical physics: statics, dynamics and renormalization, World Scientific (2000).

[53] T. Kibble and F.H. Berkshire, Classical mechanics, Imperial College Press (1966).

[54] M. Kumar, Quantum: Einstein, Bohr, and the great debate about the nature of reality, Norton (2009).

[55] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).

[56] L.D. Landau and E.M. Lifshitz, Mechanics, Pergamon Press (1960).

[57] L.D. Landau and E.M. Lifshitz, The classical theory of fields, Addison-Wesley (1951).

[58] L.D. Landau and E.M. Lifshitz, Quantum mechanics: non-relativistic theory, Pergamon Press (1959).

[59] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, Quantum electrodynamics, Butterworth-Heinemann (1982).

[60] P. Lax, Linear algebra and its applications, Wiley (2007).

[61] P. Lax, Functional analysis, Wiley (2002).
[62] J.M. Lee, Introduction to topological manifolds, Springer (2011).
[63] J.M. Lee, Introduction to smooth manifolds, Springer (2012).
[64] J.M. Lee, Introduction to Riemannian manifolds, Springer (2019).
[65] M.L. Mehta, Random matrices, Elsevier (2004).
[66] R.K. Pathria and and P.D. Beale, Statistical mechanics, Elsevier (1972).
[67] P. Petersen, Linear algebra, Springer (2012).
[68] P. Petersen, Riemannian geometry, Springer (2006).
[69] W. Rudin, Principles of mathematical analysis, McGraw-Hill (1964).
[70] W. Rudin, Real and complex analysis, McGraw-Hill (1966).
[71] B.M. Peterson and B. Ryden, Foundations of astrophysics, Cambridge Univ. Press (2010).
[72] B. Ryden, Introduction to cosmology, Cambridge Univ. Press (2002).
[73] B. Ryden and R.W. Pogge, Interstellar and intergalactic medium, Cambridge Univ. Press (2021).
[74] D.V. Schroeder, An introduction to thermal physics, Oxford Univ. Press (1999).
[75] B. Schutz, A first course in general relativity, Cambridge Univ. Press (2009).
[76] J. Schwinger, Einstein's legacy: the unity of space and time, Dover (1986).
[77] J. Schwinger, L.L. DeRaad Jr., K.A. Milton and W.Y. Tsai, Classical electrodynamics, CRC Press (1998).
[78] J. Schwinger and B.H. Englert, Quantum mechanics: symbolism of atomic measurements, Springer (2001).
[79] I.R. Shafarevich, Basic algebraic geometry, Springer (1974).
[80] R. Shankar, Fundamentals of physics I: mechanics, relativity, and thermodynamics, Yale Univ. Press (2014).
[81] R. Shankar, Fundamentals of physics II: electromagnetism, optics, and quantum mechanics, Yale Univ. Press (2016).
[82] R. Shankar, Principles of quantum mechanics, Springer (1980).
[83] R. Shankar, Quantum field theory and condensed matter: an introduction, Cambridge Univ. Press (2017).
[84] J.H. Smith, Introduction to special relativity, Dover (1965).
[85] A.M. Steane, Thermodynamics, Oxford Univ. Press (2016).
[86] A.M. Steane, Relativity made relatively easy I, Oxford Univ. Press (2012).
[87] A.M. Steane, Relativity made relatively easy II, Oxford Univ. Press (2021).
[88] J.R. Taylor, Classical mechanics, Univ. Science Books (2003).
[89] N.H. Temperley and E.H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem, *Proc. Roy. Soc. London* **322** (1971), 251–280.
[90] M. Tsamparlis, Special relativity, Springer (2010).
[91] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).
[92] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).

[93] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).

[94] S. Weinberg, Lectures on astrophysics, Cambridge Univ. Press (2019).

[95] S. Weinberg, Cosmology, Oxford Univ. Press (2008).

[96] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).

[97] H. Weyl, The classical groups: their invariants and representations, Princeton Univ. Press (1939).

[98] H. Weyl, Space, time, matter, Princeton Univ. Press (1918).

[99] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–399.

[100] B. Zwiebach, A first course in string theory, Cambridge Univ. Press (2004).