World of colors

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ABSTRACT. This is an introduction to light and colors, with due scientific explanations, and kept as elementary as possible. We first discuss in some detail the physics of light, namely mechanisms of light creation, and properties of light. Then we go on a long discussion regarding optics and colors, then the mathematics of colors, then the classification of colors, and then various practical aspects, including a number of engineering matters. We end with a discussion regarding sound, and smell, taste and touch.

Preface

Which of your senses would you happily dispose of, if gently asked? After all, there are so many people living with this or that missing sense, and claiming that life is just wonderful. Add to this the cohorts of organ donors, not necessarily related to senses, that's true, or people doing all sorts of dangerous sports, just for fun, again not caring much about losing this or that part of their body. Not to forget professional soldiers, and other mercenaries. And high-risk jobs, too. And many more, there is certainly a long list that can be made here. In view of all this, the question seems to makes sense.

In fact, this is a question that you might already aware of, just browse the social media, and you will certainly stumble upon some tricky questions, of type how much money would be one of your fingers worth, or worse, how much money would be one of your friend's fingers worth. Usually this is the type of sickening question that you run away from, quickly, not enough moderation on these websites, you would say. But, after all, why not, as scientists at least, we should have an answer to everything, right.

In answer, you might be too young, so listen here to the old man, writing these lines. Getting old is not funny at all, and although this is traditionally a well-kept secret, by the billions of people getting older, you should know about it. So, health is what you have, enjoy it, and to anyone inquiring about it, the answer should be an AR-15.

So, happily enjoying our sight, smell, hearing, taste and touch, but, coming as a continuation of the above, after all, which of these senses is the most useful?

Probably sight, I would say, and with this being in tune with the main findings of math and physics, where light and colors are the alpha and omega of everything. We are consumers of color, this is how most of our knowledge builds up. Followed, again from a math and physics perspective, by hearing, and this because sounds have many things in common with light and color. As for touch, and smell and taste, these are respectively a bit too rudimentary, and too advanced, that is, basic mechanics, and then chemistry.

The present book is an introduction to light and colors, with due scientific explanations, and kept as elementary as possible. We discuss as well sound, as a ramification of our main discussion about colors. Smell, taste and touch will be not forgotten either.

PREFACE

The book is organized in 4 parts, with Part I dealing with the generalities of basic physics and light, then Part II and Part III going deep into colors, their meaning, mathematics, classification, applications, and many more, and finally with Part IV going into the other senses, with a discussion mixing basic chemistry and basic biology.

Many thanks go to my math and physics colleagues, and all sorts of other friends too, there is so much to learn about colors, with everyone having their say, usually interesting things, regardless of their scientific or literary background. Thanks as well to my cats, learning from them every day too, and doing some progresses with smell.

Cergy, January 2025 Teo Banica

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Part I

Physics, light

Hot and cold emotion Confusing my brain I could not decide Between pleasure and pain

CHAPTER 1

Electromagnetism

1a. Charges, field lines

What is light? Good question, and as you certainly know, it took mankind a lot of time, basically up to the very modern times, last century or so, in order to figure out some sort of answer to this. And with the present answer as we know it being, as you might also be aware of, something rather awkward and temporary, and this due to various difficulties, which still persist, in quantum mechanics and related disciplines.

But, where does the problem with light come from? Well, passed the difficulties with observing the speed and other properties of light, which are something quite outstanding, basically requiring modern machinery, or sometimes very complicated tricks, in relation with astronomy and such, the main problem comes from the possible origins of light, which are all quite complicated to understand. As a summary here, we have:

FACT 1.1. Light observed by us humans can come from:

- (1) The Sun. With this being as old as our Planet Earth, but go understand what happens inside the Sun, that is no easy business, even by modern standards.
- (2) The fire. Again, with this being as old as mankind, but go understand what exactly happens inside a fire, that is again no easy business, for us scientists.
- (3) More modern sources, such as a usual light bulb switched on. Not to forget more advanced methods either, such as detonating an atomic bomb.

And with this, we can clearly see where the problem with understanding light comes from. It is just that our ancestors were stuck for ages with (1) and (2), with both being no-go directions, and with no possibility to escape from this. As simple as that.

As modern humans, however, we can trick, because we have (3). And here, leaving atomic bombs and other advanced forms of light production aside, we are left with switching on a light bulb, which is a quite simple operation. So, here will be our plan:

PLAN 1.2. In order to understand light, we will be guided by the light bulb:

- (1) We will first talk about charges, and electromagnetism.
- (2) Then about accelerating charges, and electromagnetic radiation, a.k.a. light.
- (3) Followed by explanations regarding the bulb, and with a look into fire too.
- (4) Finally, we will discuss more fearsome phenomena, such as Solar light.

Which sounds quite good, so let us go for this, the Light Bulb way, and as a last philosophical comment on this, now that we are about to start, with our modern approach, never underestimate of course what our ancestors did, without the said bulb. In fact, and more on this later, when talking more in detail about light and its properties, what they did as scientific effort for understanding light, without access to our modern electricity and bulbs, is truly remarkable, and in fact shame on us, modern scientists, for not having been able to come up with much more, and this same subject, light, in view of all the modern tools that we have, namely electricity, light bulbs, and many more.

Getting started now, what is electricity? Very good question, and do not expect a simple answer to it. However, the answer to it is not that terribly complicated either, this is definitely something that can be learned, it is just that this will take some time. Let us start with something very basic, at least by our modern scientific standards, as follows:

FACT 1.3. Each piece of matter has a charge $q \in \mathbb{R}$, which is normally neutral, q = 0, but that we can make positive or negative, by using various methods. We say that responsible for the charge is the amount of electrons present, as follows:

- (1) When the matter lacks electrons, the charge is positive, q > 0.
- (2) When there are more electrons than needed, the charge is negative, q < 0.

Moving ahead now, as our first result, due to Coulomb, and that will come as a physics fact instead of a mathematics theorem, because, well, I must admit that what we have in Fact 1.3 is indeed more than borderline, as axiomatics for a theory, we have:

FACT 1.4 (Coulomb law). Any pair of charges $q_1, q_2 \in \mathbb{R}$ is subject to a force as follows, which is attractive if $q_1q_2 < 0$ and repulsive if $q_1q_2 > 0$,

$$||F|| = K \cdot \frac{|q_1q_2|}{d^2}$$

where d > 0 is the distance between the charges, and K > 0 is a certain constant.

Observe the amazing similarity with the Newton law for gravity. However, as we will discover soon, passed a few simple facts, things will be far more complicated here.

As in the gravity case, the force F appearing above is understood to be parallel to the vector $x_2 - x_1 \in \mathbb{R}^3$ joining as $x_1 \to x_2$ the locations $x_1, x_2 \in \mathbb{R}^3$ of our charges, and by taking into account the attraction/repulsion rules above, we have:

PROPOSITION 1.5. The Coulomb force of q_1 at x_1 acting on q_2 at x_2 is

$$F = K \cdot \frac{q_1 q_2 (x_2 - x_1)}{||x_2 - x_1||^3}$$

with K > 0 being the Coulomb constant, as above.

PROOF. We have indeed the following computation:

$$F = sgn(q_1q_2) \cdot ||F|| \cdot \frac{x_2 - x_1}{||x_2 - x_1||}$$

= $sgn(q_1q_2) \cdot K \cdot \frac{|q_1q_2|}{||x_2 - x_1||^2} \cdot \frac{x_2 - x_1}{||x_2 - x_1||}$
= $K \cdot \frac{q_1q_2(x_2 - x_1)}{||x_2 - x_1||^3}$

Thus, we are led to the formula in the statement.

All the above looks quite encouraging, and it is tempting to try now to develop some Newton type theory for electricity, inspired from gravity, with some mathematics, ellipses and everything. And, why not with charges replaced by single electrons -, and their antiparticles protons +, as a matter of further clarifying our axiomatics and formalism.

Fortunately cat, who is still present, quickly intervenes, and says:

CAT 1.6. Don't even think about Newton type stuff, when charges move they produce magnetism, what you have in Fact 1.4 is just part of the story. And don't think either about talking about single electrons, Fact 1.4 is something of statistical nature.

Which sounds quite frightening, if I understand well what we have to do is to build a theory of electrostatics, based on Fact 1.4, then further upgrade that into a theory of electromagnetism, and then, well, get beyond statistics, with some precise laws for the movements of the electrons, position, speed and everything, and why not about their shape too, are these round, or perhaps looking like strings, or other weird shapes.

Nevermind. We won't be scared by this, and slowly develop all the needed theory, but as a remark here for the math reader, coming as a continuation of the discussion started after Fact 1.3, sure I will keep my promise to come back with axioms, in due time, but I'm afraid that this will be probably towards the end of the book. Deal, I hope.

So, forgetting now about high hopes and abstractions, and getting back, with due modesty, to what we have in Fact 1.4, let us further explore the physics there, matter of having it perfectly understood. In relation with the value of the constant K appearing there, called Coulomb constant, things are a bit tricky, as follows:

FACT 1.7. The Coulomb constant K is given by the formula

 $K = 8.987\ 551\ 7923(14) \times 10^9$

in standard units, with the charges being measured in coulombs C, given by

 $1C \simeq 6.241\ 509 \times 10^{18} \, e$

where e is the elementary charge, namely minus that of an electron.

There are in fact several interesting things going on here. First, at the end you would say why not simply saying that e is the charge of the proton +, but the thing is that the proton + and the electron - do not have in fact the same exact charge, with sign switched, and the electron was preferred, as always, over the proton for formulating things.

Which takes us into the question of why the charge of the electron is -, instead of +. And there is a long story here, involving debates among the 18th century greats, and with a little bit of confusion being involved too, because the electrons - are attracted by positive charges q > 0, and so observed around these positive charges q > 0, which might lead to the idea that they might have themselves a positive charge +, contributing to q > 0. Benjamin Franklin is generally credited for the - convention.

Things were later restored in the early 20th century, with the atomic theory of Bohr and others, where electrons – spin around a proton and neutron core q > 0, and with this picture, including the signs, looking like something very reasonable.

Passed all this, another peculiarity of Fact 1.7 comes in relation with the definition of the coulomb, which is in fact given by definition by an exact formula, namely:

$$1C = \frac{5 \times 10^{18}}{0.801\ 088\ 317} \ e$$

This in practice gives the following more precise formula for the coulomb, which shows that a charge of 1C is something fractionary, that cannot be realized in real life:

$$C = 6241 \ 509 \ 074 \ 460 \ 762 \ 607.776 \ e$$

The problem comes from the following alternative definition of the coulomb, in terms of the ampere, which is something more complicated, that we will talk about later:

$$1C = 1A \cdot 1s$$

Hang on, we are not done yet. Adding to the confusion, the Coulomb constant is usually denoted K, but also k, or most often k_e , but in fact the most often is written in the following form, with ε_0 being the so-called permittivity of free space:

$$K = \frac{1}{4\pi\varepsilon_0}$$

And the story is not over here, because ε_0 itself is given by the following formula, with μ_0 being the magnetic permeability of free space, and c being the speed of light:

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}$$

And we are surely still not done, because all the above discussion assumes that the other units that are used are standard, namely meter and second, and this is not always standard, due to the about 50 orders of magnitude physics has to deal with.

In any case, let us end this interesting discussion about units with something concrete, useful, and very illustrating, in relation with gravity, as follows:

THEOREM 1.8. The electrical repulsion between two electrons is about

$$R = 10^{42}$$

times bigger than their gravitational attraction.

PROOF. Consider indeed two electrons, having masses m, m and charges -e, -e. The magnitudes of the electric repulsion F_e and gravity attraction F_g are given by:

$$||F_e|| = \frac{Ke^2}{d^2}$$
 , $||F_g|| = \frac{Gm^2}{d^2}$

Thus the ratio of forces R that we want to measure is given by:

$$R = \frac{||F_e||}{||F_g||} = \frac{Ke^2}{Gm^2}$$

Regarding now the data, this is as follows, with m at rest, and in standard units, namely meters and seconds, also kilograms, and including now coulombs too:

$$K = 8.897 \times 10^9$$
 , $G = 6.674 \times 10^{-11}$
 $e = 1.602 \times 10^{-19}$, $m = 9.109 \times 10^{-31}$

We obtain the following approximation for the ratio R considered above:

$$R = \frac{8.897 \times 1.602^2}{6.674 \times 9.109^2} \times \frac{10^9 \times 10^{-38}}{10^{-11} \times 10^{-62}}$$

= (4.123 × 10⁻²) × 10⁴⁴
 $\simeq 10^{42}$

Thus, we are led to the conclusion in the statement.

Let us develop now the basic mathematics for electrostatics. We first have:

DEFINITION 1.9. Given charges $q_1, \ldots, q_k \in \mathbb{R}$ located at positions $x_1, \ldots, x_k \in \mathbb{R}^3$, we define their electric field to be the vector function

$$E(x) = K \sum_{i} \frac{q_i(x - x_i)}{||x - x_i||^3}$$

so that their force applied to a charge $Q \in \mathbb{R}$ positioned at $x \in \mathbb{R}^3$ is given by F = QE.

More generally, we will be interested in electric fields of various non-discrete configurations of charges, such as curves, surfaces and solid bodies. So, let us go ahead with:

DEFINITION 1.10. The electric field of a charge configuration $L \subset \mathbb{R}^3$, with charge density function $\rho: L \to \mathbb{R}$, is the vector function

$$E(x) = K \int_{L} \frac{\rho(z)(x-z)}{||x-z||^3} dz$$

so that the force of L applied to a charge Q positioned at x is given by F = QE.

It is most convenient now to forget about the charges, and focus on the corresponding electric fields E. These fields are by definition vector functions $E : \mathbb{R}^3 \to \mathbb{R}^3$, with the convention that they take $\pm \infty$ values at the places where the charges are located, and intuitively, are best represented by their field lines, constructed as follows:

DEFINITION 1.11. The field lines of $E : \mathbb{R}^3 \to \mathbb{R}^3$ are the oriented curves

$$\gamma \subset \mathbb{R}^3$$

pointing at every point $x \in \mathbb{R}^3$ at the direction of the field, $E(x) \in \mathbb{R}^3$.

As a basic example here, for one charge the field lines are the half-lines emanating from its position, oriented according to the sign of the charge:

$\overline{\}$	\uparrow	\nearrow	\searrow	\downarrow	\checkmark
\leftarrow	\oplus	\rightarrow	\rightarrow	\ominus	\leftarrow
\checkmark	\downarrow	\searrow	\nearrow	\uparrow	K

For two charges now, if these are of opposite signs, + and -, you get a picture that you are very familiar with, namely that of the field lines of a bar magnet:

\nearrow	\nearrow	\rightarrow	\rightarrow	\rightarrow	\rightarrow	\searrow	\searrow
$\overline{\ }$	\uparrow	\nearrow	\rightarrow	\rightarrow	\searrow	\downarrow	\swarrow
\leftarrow	\oplus	\rightarrow	\rightarrow	\rightarrow	\rightarrow	\ominus	\leftarrow
\swarrow	\downarrow	\searrow	\rightarrow	\rightarrow	\nearrow	\uparrow	$\overline{\}$
\searrow	\searrow	\rightarrow	\rightarrow	\rightarrow	\rightarrow	\nearrow	\nearrow

If the charges are +, + or -, -, you get something of similar type, but repulsive this time, with the field lines emanating from the charges being no longer shared:



The field lines obviously do not encapsulate the whole information about the field, with the direction of each vector $E(x) \in \mathbb{R}^3$ being there, but with the magnitude $||E(x)|| \ge 0$ of this vector missing. However, in practice, when drawing, when picking up uniformly radially spaced field lines around each charge, and with the number of these lines being proportional to the magnitude of the charge, and then completing the picture, the density

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of the field lines around each point $x \in \mathbb{R}$ will give you then the magnitude $||E(x)|| \ge 0$ of the field there, up to a scalar. With this being, of course, very practical.

Let us summarize these observations as a mathematical result, follows:

THEOREM 1.12. Given an electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$, the knowledge of its field lines is the same as the knowledge of the composition

$$nE: \mathbb{R}^3 \to \mathbb{R}^3 \to S$$

where $S \subset \mathbb{R}^3$ is the unit sphere, and $n : \mathbb{R}^3 \to S$ is the rescaling map, namely:

$$n(x) = \frac{x}{||x||}$$

However, in practice, when the field lines are accurately drawn, the density of the field lines gives you the magnitude of the field, up to a scalar.

PROOF. This follows indeed from the above discussion. It is possible to be a bit more mathematical here, but we will not really need this, in what follows. \Box

1b. Flux, Gauss law

Let us introduce now a key definition, as follows:

DEFINITION 1.13. The flux of an electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$ through a surface $S \subset \mathbb{R}^3$, assumed to be oriented, is the quantity

$$\Phi_E(S) = \int_S \langle E(x), n(x) \rangle dx$$

with n(x) being unit vectors orthogonal to S, following the orientation of S. Intuitively, the flux measures the signed number of field lines crossing S.

Here by orientation of S we mean precisely the choice of unit vectors n(x) as above, orthogonal to S, which must vary continuously with x. For instance a sphere has two possible orientations, one with all these vectors n(x) pointing inside, and one with all these vectors n(x) pointing outside. More generally, any surface has locally two possible orientations, so if it is connected, it has two possible orientations. In what follows the convention is that the closed surfaces are oriented with each n(x) pointing outside.

As a first illustration, let us do a basic computation, as follows:

PROPOSITION 1.14. For a point charge $q \in \mathbb{R}$ at the center of a sphere S,

$$\Phi_E(S) = \frac{q}{\varepsilon_0}$$

where the constant is $\varepsilon_0 = 1/(4\pi K)$, independently of the radius of S.

PROOF. Assuming that S has radius r, we have the following computation:

$$\Phi_E(S) = \int_S \langle E(x), n(x) \rangle dx$$

=
$$\int_S \left\langle \frac{Kqx}{r^3}, \frac{x}{r} \right\rangle dx$$

=
$$\int_S \frac{Kq}{r^2} dx$$

=
$$\frac{Kq}{r^2} \times 4\pi r^2$$

=
$$4\pi Kq$$

Thus with $\varepsilon_0 = 1/(4\pi K)$ as above, we obtain the result.

More generally now, we have the following result:

THEOREM 1.15. The flux of a field E through a sphere S is given by

$$\Phi_E(S) = \frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed by S, and $\varepsilon_0 = 1/(4\pi K)$.

PROOF. This can be done in several steps, as follows:

(1) Before jumping into computations, let us do some manipulations. First, by discretizing the problem, we can assume that we are dealing with a system of point charges. Moreover, by additivity, we can assume that we are dealing with a single charge. And if we denote by $q \in \mathbb{R}$ this charge, located at $v \in \mathbb{R}^3$, we want to prove that we have the following formula, where $B \subset \mathbb{R}^3$ denotes the ball enclosed by S:

$$\Phi_E(S) = \frac{q}{\varepsilon_0} \,\delta_{v \in B}$$

(2) By linearity we can assume that we are dealing with the unit sphere S. Moreover, by rotating we can assume that our charge q lies on the Ox axis, that is, that we have v = (r, 0, 0) with $r \ge 0, r \ne 1$. The formula that we want to prove becomes:

$$\Phi_E(S) = \frac{q}{\varepsilon_0} \,\delta_{r<1}$$

(3) Let us start now the computation. With u = (x, y, z), we have:

$$\begin{split} \Phi_E(S) &= \int_S \langle E(u), u \rangle du \\ &= \int_S \left\langle \frac{Kq(u-v)}{||u-v||^3}, u \right\rangle du \\ &= Kq \int_S \frac{\langle u-v, u \rangle}{||u-v||^3} du \\ &= Kq \int_S \frac{1-\langle v, u \rangle}{||u-v||^3} du \\ &= Kq \int_S \frac{1-rx}{(1-2xr+r^2)^{3/2}} du \end{split}$$

(4) In order to compute the above integral, we can use spherical coordinates for the unit sphere S. Our integral from (3) becomes in this way:

$$\Phi_E(S) = Kq \int_S \frac{1 - rx}{(1 - 2xr + r^2)^{3/2}} du$$

= $Kq \int_0^{2\pi} \int_0^{\pi} \frac{1 - r\cos s}{(1 - 2r\cos s + r^2)^{3/2}} \cdot \sin s \, ds \, dt$
= $2\pi Kq \int_0^{\pi} \frac{(1 - r\cos s)\sin s}{(1 - 2r\cos s + r^2)^{3/2}} \, ds$
= $\frac{q}{2\varepsilon_0} \int_0^{\pi} \frac{(1 - r\cos s)\sin s}{(1 - 2r\cos s + r^2)^{3/2}} \, ds$

(5) The point now is that the integral on the right can be computed with the change of variables $x = \cos s$. Indeed, we have $dx = -\sin s \, ds$, and we obtain:

$$\int_{0}^{\pi} \frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^{2})^{3/2}} ds = \int_{-1}^{1} \frac{1-rx}{(1-2rx+r^{2})^{3/2}} dx$$
$$= \left[\frac{x-r}{\sqrt{1-2rx+r^{2}}}\right]_{-1}^{1}$$
$$= \frac{1-r}{\sqrt{1-2r+r^{2}}} - \frac{-1-r}{\sqrt{1+2r+r^{2}}}$$
$$= \frac{1-r}{|1-r|} + 1$$
$$= 2\delta_{r<1}$$

Thus, we are led to the formula in the statement.

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As a comment here, at r = 1, which is normally avoided by our problematics, the integral I_r computed in (5) above converges too, and can be evaluated as follows:

$$I_1 = \left[\frac{x-1}{\sqrt{2-2x}}\right]_{-1}^1 = \left[-\sqrt{\frac{1-x}{2}}\right]_{-1}^1 = 1$$

Thus, we have the correct middle step between the 0, 2 values of the integral I_r , and getting back now to the flux, at r = 1 we formally have $\Phi_E(S) = q/(2\varepsilon_0)$, which again is the correct middle step between the $0, q/\varepsilon_0$ values of the flux.

Even more generally now, we have the following result, due to Gauss:

THEOREM 1.16 (Gauss law). The flux of a field E through a surface S is given by

$$\Phi_E(S) = \frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed by S, and $\varepsilon_0 = 1/(4\pi K)$.

PROOF. This basically follows from Theorem 1.15, or even from Proposition 1.14, by adding to the results there a number of new ingredients, as follows:

(1) Our first claim is that given a closed surface S, with no charges inside, the flux through it of any choice of external charges vanishes:

$$\Phi_E(S) = 0$$

This claim is indeed supported by the intuitive interpretation of the flux, as corresponding to the signed number of field lines crossing S. Indeed, any field line entering as + must exit somewhere as -, and vice versa, so when summing we get 0.

(2) In practice now, in order to prove this rigorously, there are several ways. A first argument, which is quite elementary, is the one used by Feynman in [34], based on the fact that, due to $F \sim 1/d^2$, local deformations of S will leave invariant the flux, and so in the end we are left with a rotationally invariant surface, where the result is clear.

(3) A second argument, which basically uses the same idea, but is perhaps a bit more robust, is by redoing the computations in the proof of Theorem 1.15, by assuming this time that the integration takes place on an arbitrary surface as follows:

$$S_{\lambda} = \left\{ \lambda(u) u \middle| u \in S \right\}$$

To be more precise, here $\lambda : S \to (0, \infty)$ is a certain function, defining the surface, whose derivatives will appear both in the construction of the normal vectors n(x) with $x = \lambda(u)u$, and in the Jacobian of the change of variables $x \to u$, and in the end, when integrating over S as in the proof of Theorem 1.15, this function λ dissapears.

(4) A third argument, used by basically all electrodynamics books at the graduate level, and by some undergraduate books too, is by using heavy calculus, namely partial integration in 3D, and we will discuss this later, more in detail, a bit later.

(5) A fourth argument is by following the idea in (1), namely carefully axiomatizing the field lines, and their relation with the field, and then obtaining $\Phi_E(S) = 0$ by using the in-and-out trick in (1), as explained for instance by Griffiths in [44].

(6) To summarize, we are led to the conclusion that given a closed surface S, with no charges inside, the flux through it of any choice of external charges vanishes:

$$\Phi_E(S) = 0$$

(7) The point now is that, with this and Proposition 1.14 in hand, we can finish by using a standard math trick. Let us assume indeed, by discretizing, that our system of charges is discrete, consisting of enclosed charges $q_1, \ldots, q_k \in \mathbb{R}$, and an exterior total charge Q_{ext} . We can surround each of q_1, \ldots, q_k by small disjoint spheres U_1, \ldots, U_k , chosen such that their interiors do not touch S, and we have:

$$\Phi_E(S) = \Phi_E(S - \cup U_i) + \Phi_E(\cup U_i)$$

= $0 + \Phi_E(\cup U_i)$
= $\sum_i \Phi_E(U_i)$
= $\sum_i \frac{q_i}{\varepsilon_0}$
= $\frac{Q_{enc}}{\varepsilon_0}$

(8) To be more precise, in the above the union $\cup U_i$ is a usual disjoint union, and the flux is of course additive over components. As for the difference $S - \cup U_i$, this is by definition the disjoint union of S with the disjoint union $\cup (-U_i)$, with each $-U_i$ standing for U_i with orientation reversed, and since this difference has no enclosed charges, the flux through it vanishes by (6). Finally, the end makes use of Proposition 1.14.

We have the following point of view on the Gauss formula, more conceptual:

THEOREM 1.17 (Gauss). Given an electric potential E, its divergence is given by

$$<\nabla, E>=rac{
ho}{arepsilon_0}$$

where ρ denotes as usual the charge distribution. Also, we have

$$\nabla \times E = 0$$

meaning that the curl of E vanishes.

PROOF. The first formula, called Gauss law in differential form, follows from:

$$\int_{B} \langle \nabla, E \rangle = \int_{S} \langle E(x), n(x) \rangle dx$$
$$= \Phi_{E}(S)$$
$$= \frac{Q_{enc}}{\varepsilon_{0}}$$
$$= \int_{B} \frac{\rho}{\varepsilon_{0}}$$

As a side remark, the Gauss law in differential form can be established as well directly, with the computation, involving a Dirac mass, being as follows:

$$\langle \nabla, E \rangle (x) = \left\langle \nabla, K \int_{\mathbb{R}^3} \frac{\rho(z)(x-z)}{||x-z||^3} dz \right\rangle$$

$$= K \int_{\mathbb{R}^3} \left\langle \nabla, \frac{x-z}{||x-z||^3} \right\rangle \rho(z) dz$$

$$= K \int_{\mathbb{R}^3} 4\pi \delta_x \cdot \rho(z) dz$$

$$= 4\pi K \int_{\mathbb{R}^3} \delta_x \rho(z) dz$$

$$= \frac{\rho(x)}{\varepsilon_0}$$

Regarding the curl, by discretizing and linearity we can assume that we are dealing with a single charge q, positioned at 0. We have, by using spherical coordinates r, s, t:

$$\begin{aligned} \int_{a}^{b} \langle E(x), dx \rangle &= \int_{a}^{b} \left\langle \frac{Kqx}{||x||^{3}}, dx \right\rangle \\ &= \int_{a}^{b} \left\langle \frac{Kq}{r^{2}} \cdot \frac{x}{||x||}, dx \right\rangle \\ &= \int_{a}^{b} \frac{Kq}{r^{2}} dr \\ &= \left[-\frac{Kq}{r} \right]_{a}^{b} \\ &= Kq \left(\frac{1}{r_{a}} - \frac{1}{r_{b}} \right) \end{aligned}$$

In particular the integral of E over any closed loop vanishes, and by using now the Stokes theorem, we conclude that the curl of E vanishes, as stated.

1C. MAGNETIC FIELDS

1c. Magnetic fields

Just by feeding a light bulb with a battery, and looking at the cables, and playing a bit with them, we are led to the following interesting conclusion:

FACT 1.18. Parallel electric currents in opposite directions repel, and parallel electric currents in the same direction attract.

We can in fact say even more, by further playing with the cables, armed this time with a compass. The conclusion is that each cable produces some kind of "magnetic field" around it, which interestingly, is not oriented in the direction of the current, but is rather orthogonal to it, given by the right-hand rule, as follows:

FACT 1.19 (Right-hand rule). An electric current produces a magnetic field B which is orthogonal to it, whose direction is given by the right-hand rule,



namely wrap your right hand around the cable, with the thumb pointing towards the direction of the current, and the movement of your wrist will give you the direction of B.

This is something even more interesting than Fact 1.18. Indeed, not only moving charges produce something new, that we'll have to investigate, but they know well about 3D, and more specifically about orientation there, left and right, even if living in 1D.

And isn't this amazing. Let us summarize this discussion with:

FACT 1.20. Charges are smart, they know about 3D, and about left and right.

With this discussed, let us go ahead and investigate the charge smartness, and more specifically the magnetic fields discovered above. In order to evaluate the properties of the magnetic fields B coming from electric currents, the simplest way is that of making them act on exterior charges Q. And we have here the following formula:

FACT 1.21 (Lorentz force law). The magnetic force on a charge Q, moving with velocity v in a magnetic field B, is as follows, with \times being a vector product:

$$F_m = (v \times B)Q$$

In the presence of both electric and magnetic fields, the total force on Q is

$$F = (E + v \times B)Q$$

where E is the electric field.

Here the occurrence of the vector product \times is not surprising, due to the fact that the right-hand rule appears both in Fact 1.19, and in the definition of \times . In fact, the Lorentz force law is just a fancy mathematical reformulation of Fact 1.19, telling us that, once the magnetic fields *B* duly axiomatized, and with this being a remaining big problem, their action on exterior charges *Q* will be proportional to the charge, $F_m \sim Q$, and with the orientation and magnitude coming from the 3D of the right-hand rule in Fact 1.19.

As an interesting application of the Lorentz force law, we have:

THEOREM 1.22. Magnetic forces do not work.

PROOF. This might seem quite surprising, but the math is there, as follows:

$$dW_m = \langle F_m, dx \rangle$$

= $\langle (v \times B)Q, v dt \rangle$
= $Q \langle v \times B, v \rangle dt$
= 0

Thus, we are led to the conclusion in the statement.

Moving ahead now, let us talk axiomatization of electric currents, including units. We have here the following definition, clarifying our previous discussion about coulombs:

DEFINITION 1.23. The electric currents I are measured in amperes, given by:

$$1A = 1C/s$$

As a consequence, the coulomb is given by $1C = 1A \times 1s$.

With this notion in hand, let us keep building the math and physics of magnetism. So, assume that we are dealing with an electric current I, producing a magnetic field B. In this context, the Lorentz force law from Fact 1.21 takes the following form:

$$F_m = \int (dx \times B)I$$

The current being typically constant along the wire, this reads:

$$F_m = I \int dx \times B$$

We can deduce from this the following result:

THEOREM 1.24. The volume current density J satisfies

$$<\nabla, J>=-\dot{\rho}$$

called continuity equation.

PROOF. We have indeed the following computation, for any surface S enclosing a volume V, based on the Lorentz force law, and on the overall chage conservation:

$$\int_{V} \langle \nabla, J \rangle = \int_{S} \langle J, n(x) \rangle dx$$
$$= -\frac{d}{dt} \int_{V} \rho$$
$$= -\int_{V} \dot{\rho}$$

Thus, we are led to the conclusion in the statement.

Moving ahead now, let us formulate the following definition:

DEFINITION 1.25. The realm of magnetostatics is that of the steady currents,

$$\dot{\rho} = 0 \quad , \quad J = 0$$

in analogy with electrostatics, dealing with fixed charges.

As a first observation, for steady currents the continuity equation reads:

$$\langle \nabla, J \rangle = 0$$

We have here a bit of analogy between electrostatics and magnetostatics, and with this in mind, let us look for equations for the magnetic field B. We have:

FACT 1.26 (Biot-Savart law). The magnetic field of a steady line current is given by

$$B = \frac{\mu_0}{4\pi} \int \frac{I \times x}{||x||^3}$$

where μ_0 is a certain constant, called the magnetic permeability of free space.

This law not only gives us all we need, for studying steady currents, and we will talk about this in a moment, with math and everything, but also makes an amazing link with the Coulomb force law, due to the following fact, which is also part of it:

FACT 1.27 (Biot-Savart, continued). The electric permittivity of free space ε_0 and the magnetic permeability of free space μ_0 are related by the formula

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}$$

where c is as usual the speed of light.

This is something truly remarkable, and very deep, that will have numerous consequences, in what follows, be that for investigating phenomena like radiation, or for making the link with Einstein's relativity theory, both crucially involving c.

But, first of all, this is certainly an invitation to rediscuss units and constants, as a continuation of our previous discussion on this topic. In what regards the units, we won't be impressed by the ampere, and keep using the coulomb, as a main unit:

CONVENTIONS 1.28. We keep using standard units, namely meters, kilograms, seconds, along with the coulomb, defined by the following exact formula

$$1C = \frac{5 \times 10^{18}}{0.801 \ 088 \ 317} \ e$$

with e being minus the charge of the electron, which in practice means:

$$1C \simeq 6.241 \times 10^{18} e$$

We will also use the ampere, defined as 1A = 1C/s, for measuring currents.

In what regards constants, however, time to do some cleanup. We have been boycotting for some time already the Coulomb constant K, and using instead $\varepsilon_0 = 1/(4\pi K)$, due to the ubiquitous 4π factor, first appearing as the area of the unit sphere, $A = 4\pi$, in the computation for the Gauss law for the unit sphere. Together with Fact 1.27, this suggests using the numbers ε_0, μ_0 as our new constants, by always keeping in mind $\varepsilon_0\mu_0 = 1/c^2$, and by having of course c as constant too, and we are led in this way into:

CONVENTIONS 1.29. We use from now on as constants the electric permittivity of free space ε_0 and the magnetic permeability of free space μ_0 , given by

$$\varepsilon_0 = 8.854 \ 187 \ 8128(13) \times 10^{-12}$$

 $\mu_0 = 1.256 \ 637 \ 062 \ 12(19) \times 10^{-6}$

as well as the speed of light, given by the following exact formula,

 $c = 299\,792\,458$

which are related by $\varepsilon_0 \mu_0 = 1/c^2$, and with the Coulomb constant being $K = 1/(4\pi\varepsilon_0)$.

Observe in passing that we are not messing up our figures, which can be quite often the case in this type of situation, because according to our data, and by truncating instead of rounding, as busy theoretical physicists usually do, we have:

 $\varepsilon_0 \mu_0 c^2 = 8.854 \times 1.256 \times 2.997^2 \times 10^{16-12-6} = 0.998$

Getting back now to theory and math, the Biot-Savart law has as consequence:

THEOREM 1.30. We have the following formula:

 $\langle \nabla, B \rangle = 0$

That is, the divergence of the magnetic field vanishes.

PROOF. We recall that the Biot-Savart law tells us that the magnetic field B of a steady line current I is given by the following formula:

$$B = \frac{\mu_0}{4\pi} \int \frac{I \times x}{||x||^3}$$

By applying the divergence operator to this formula, we obtain:

$$\langle \nabla, B \rangle = \frac{\mu_0}{4\pi} \int \left\langle \nabla, \frac{I \times x}{||x||^3} \right\rangle$$

$$= \frac{\mu_0}{4\pi} \int \left\langle \nabla \times J, \frac{x}{||x||^3} \right\rangle - \left\langle \nabla \times \frac{x}{||x||^3}, J \right\rangle$$

$$= \frac{\mu_0}{4\pi} \int \left\langle 0, \frac{x}{||x||^3} \right\rangle - \left\langle 0, J \right\rangle$$

$$= 0$$

Thus, we are led to the conclusion in the statement.

Regarding now the curl, we have here a similar result, as follows:

THEOREM 1.31 (Ampère law). We have the following formula,

$$\nabla \times B = \mu_0 J$$

computing the curl of the magnetic field.

PROOF. Again, we use the Biot-Savart law, telling us that the magnetic field B of a steady line current I is given by the following formula:

$$B = \frac{\mu_0}{4\pi} \int \frac{I \times x}{||x||^3}$$

-

By applying the curl operator to this formula, we obtain:

$$\nabla \times B = \frac{\mu_0}{4\pi} \int \nabla \times \frac{I \times x}{||x||^3}$$
$$= \frac{\mu_0}{4\pi} \int \left\langle \nabla, \frac{x}{||x||^3} \right\rangle J - \langle \nabla, J \rangle \frac{x}{||x||^3}$$
$$= \frac{\mu_0}{4\pi} \int 4\pi \delta_x \cdot J - \frac{\mu_0}{4\pi} \cdot 0$$
$$= \mu_0 \int \delta_x \cdot J$$
$$= \mu_0 J$$

Thus, we are led to the conclusion in the statement.

As a conclusion to all this, the equations of magnetostatics are as follows:

THEOREM 1.32. The equations of magnetostatics are

 $\langle \nabla, B \rangle = 0$, $\nabla \times B = \mu_0 J$

with the second equation being the Ampère law.

PROOF. This follows indeed from the above discussion, and more specifically from Theorem 1.30 and Theorem 1.31, which both follow from the Biot-Savart law. \Box

Observe the obvious analogy with the Gauss equations of electrostatics, namely:

$$< \nabla, E >= rac{
ho}{arepsilon_0} \quad , \quad \nabla \times E = 0$$

As a conclusion to all this, looks like someone has played here with basic 3D math, vectors, products and so on, and messed them up, as for electrostatics to become magnetostatics, and vice versa. More on this later, when talking about unification.

1d. Maxwell equations

Quite remarkably, and at the origin of all modern theory of electromagnetism, and of any type of modern electrical engineering too, we have:

FACT 1.33 (Faraday laws). The following happen:

- (1) Moving a wire loop γ through a magnetic field B produces a current through γ .
- (2) Keeping γ fixed, but changing the strength of B, produces too current through γ .

In order to understand what is going on here, let us start with the simplest electric loop that we know, namely a battery feeding a light bulb:

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Here the star stands for the fact that we don't really know what happens inside the battery, typically a complicated chemical process. Nor we will actually worry about the bulb, let us simply assume that this bulb does not exist at all. We will be interested in the force driving the current around the loop, and we have here:

PROPOSITION 1.34. When writing the force driving the current through a loop γ as

$$F = F_{\star} + F_e$$

with F_{\star} coming from the source, and F_{e} coming from the loop, the quantity

$$\mathcal{E} = \int_{\gamma} < F(x), dx >$$

called electromotive force, or emf of the loop, is simply obtained by integrating F_{\star} .

PROOF. We have indeed the following computation, based on the fact that F_e being an electrostatic force, its integral over the loop vanishes:

$$\mathcal{E} = \int_{\gamma} \langle F(x), dx \rangle$$

=
$$\int_{\gamma} \langle F_{\star}(x), dx \rangle + \int_{\gamma} \langle F_{e}(x), dx \rangle$$

=
$$\int_{\gamma} \langle F_{\star}(x), dx \rangle + 0$$

=
$$\int_{\gamma} \langle F_{\star}(x), dx \rangle$$

Thus, we have our result, and with the remark of course that the emf $\mathcal{E} \in \mathbb{R}$ is not really a force, but this is the standard terminology, and we will use it.

In relation now with the Faraday principles from Fact 1.33, these can be fine-tuned, and reformulated in terms of the emf, in the following way:

FACT 1.35 (Faraday). The emf of a loop γ moving through a magnetic field B is

$$\mathcal{E} = -\dot{\Phi}$$

where Φ is the flux of the field B through the loop γ , given by:

$$\Phi = \int_{\gamma} < B(x), dx >$$

As for the emf of a fixed loop γ in a changing magnetic field B, this is

$$\mathcal{E} = -\int_{\gamma} < \dot{B}(x), dx >$$

which by Stokes is equivalent to the Faraday law $\Delta \times E = -\dot{B}$.

All the above is very useful in electromechanics, for construcing electric motors. Getting back now to theory, the above considerations lead to the following conclusion:

FACT 1.36 (Faraday). In the context of moving chages, the electrostatics law

$$\nabla \times E = 0$$

must be replaced by the following equation,

$$\nabla \times E = -\dot{B}$$

called Faraday law.

Along the same lines, and following now Maxwell, there is a correction as well to be made to the main law of magnetostatics, namely the Ampère law, as follows:

FACT 1.37 (Maxwell). In the context of moving chages, the Ampère law

 $\nabla \times B = \mu_0 J$

must be replaced by the following equation,

$$\nabla \times B = \mu_0 (J + \varepsilon_0 E)$$

called Ampère law with Maxwell correction term.

Now by putting everything together, and perhaps after doublecheking as well, with all sorts of experiments, that the remaining electrostatics and magnetostatics laws, that we have not modified, work indeed fine in the dynamic setting, we obtain:

THEOREM 1.38 (Maxwell). Electrodynamics is governed by the formulae

$$\langle \nabla, E \rangle = \frac{\rho}{\varepsilon_0} \quad , \quad \langle \nabla, B \rangle = 0$$

 $\nabla \times E = -\dot{B} \quad , \quad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \dot{E}$

called Maxwell equations.

PROOF. This follows indeed from the above, the details being as follows:

(1) The first equation is the Gauss law, that we know well.

- (2) The second equation is something anonymous, that we know well too.
- (3) The third equation is a previously anonymous law, modified into Faraday's law.

(4) And the fourth equation is the Ampère law, as modified by Maxwell.

The Maxwell equations are in fact not the end of everything, because in the context of the 2-body problem, they must be replaced by quantum mechanics. More later.

1e. Exercises

Exercises:

EXERCISE 1.39.

EXERCISE 1.40.

EXERCISE 1.41.

EXERCISE 1.42.

EXERCISE 1.43.

EXERCISE 1.44.

EXERCISE 1.45.

EXERCISE 1.46.

Bonus exercise.

CHAPTER 2

Waves, light

2a. Wave equation

Time now to talk about light, and other waves. Things are quite tricky here, and coming a bit in advance, the truth about waves is as follows:

FACT 2.1. Waves can be of many types, and basically fall into two classes:

- (1) Mechanical waves, such as the usual water waves, but also the sound waves, or the seismic waves. In all these cases, the wave propagates mechanically, via a certain medium, which can be solid, liquid or gaseous.
- (2) Electromagnetic waves, coming via a more complicated mechanism, namely an accelerating charge in the context of electromagnetism. These are the radio waves, microwaves, IR, visible light, UV, X-rays and γ-rays.

Quite remarkably, the behavior of all the above waves is basically described by the same wave equation, which looks as follows, and details on this in a moment:

$$\ddot{\varphi} = v^2 \Delta \varphi$$

Getting started for good now, some mathematics first. What is the Laplace operator Δ , appearing in the above? The answer here is very simple, coming from:

PRINCIPLE 2.2. The second derivative of a function $\varphi : \mathbb{R}^N \to \mathbb{R}$, making the formula

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + \frac{\langle \varphi''(x)h, h \rangle}{2}$$

work, is its Hessian matrix $\varphi''(x) \in M_N(\mathbb{R})$, given by the following formula:

$$\varphi''(x) = \left(\frac{d^2\varphi}{dx_i dx_j}\right)_{ij}$$

However, when needing a number, as second derivative, the trace of $\varphi''(x)$, denoted

$$\Delta \varphi = \sum_{i=1}^{N} \frac{d^2 \varphi}{dx_i^2}$$

and called Laplacian of φ , usually does the job.

2. WAVES, LIGHT

So, let us try to understand this principle. In one variable, given a function $\varphi : \mathbb{R} \to \mathbb{R}$, the first job is that of finding a quantity $\varphi'(x) \in \mathbb{R}$ making the following formula work:

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h$$

But here, there are not so many choices, and the solution is that of defining the number $\varphi'(x) \in \mathbb{R}$ by the following formula, provided that the limit converges indeed:

$$\varphi'(x) = \lim_{h \to 0} \frac{\varphi(x+h) - \varphi(x)}{h}$$

Still in one variable, we can talk as well about second derivatives, as follows:

THEOREM 2.3. The second derivative of a function $\varphi : \mathbb{R} \to \mathbb{R}$, making the formula

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + \frac{\varphi''(x)h^2}{2}$$

work, is the derivative φ'' of the derivative $\varphi' : \mathbb{R} \to \mathbb{R}$.

PROOF. Assume indeed that φ is twice differentiable at x, and let us try to construct an approximation of φ around x by a quadratic function, as follows:

$$\varphi(x+h) \simeq a + bh + ch^2$$

We must have $a = \varphi(x)$, and we also know that $b = \varphi'(x)$ is the correct choice for the coefficient of h. Thus, our approximation must be as follows:

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + ch^2$$

In order to find the correct choice for $c \in \mathbb{R}$, observe that the function $\psi(h) = \varphi(x+h)$ matches with $P(h) = \varphi(x) + \varphi'(x)h + ch^2$ in what regards the value at h = 0, and also in what regards the value of the derivative at h = 0. Thus, the correct choice of $c \in \mathbb{R}$ should be the one making match the second derivatives at h = 0, and this gives:

$$c = \frac{\varphi''(x)}{2}$$

Thus, we are led to the formula in the statement. Now if we denote by $\psi(h) \simeq P(h)$ that formula, to be proved, then we have, by using L'Hôpital's rule:

$$\frac{\psi(h) - P(h)}{h^2} \simeq \frac{\psi'(h) - P'(h)}{2h}$$
$$\simeq \frac{\psi''(h) - P''(h)}{2}$$
$$= \frac{\varphi''(h) - \varphi''(h)}{2}$$
$$= 0$$

Thus, we are led to the conclusion in the statement.

2A. WAVE EQUATION

Many other things can be said, as a continuation of the above, and you surely know all this. Before leaving the subject, however, let us record the following statement, which is something a bit heuristic, which will play an important role, in what follows:

PROPOSITION 2.4. Intuitively speaking, the second derivative $\varphi''(x) \in \mathbb{R}$ computes how much different is $\varphi(x)$, compared to the average of $\varphi(y)$, with $y \simeq x$.

PROOF. As already mentioned, this is something a bit heuristic, but which is good to know. Let us write the formula in Theorem 2.3, as such, and with $h \to -h$ too:

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + \frac{\varphi''(x)}{2}h^2$$
$$\varphi(x-h) \simeq \varphi(x) - \varphi'(x)h + \frac{\varphi''(x)}{2}h^2$$

By making the average, we obtain the following formula:

$$\frac{\varphi(x+h) + \varphi(x-h)}{2} = \varphi(x) + \frac{\varphi''(x)}{2}h^2$$

Thus, thinking a bit, we are led to the conclusion in the statement.

Moving now to several variables, $N \geq 2$, as a first job, given a function $\varphi : \mathbb{R}^N \to \mathbb{R}$, we would like to find a quantity $\varphi'(x)$ making the following formula work:

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h$$

But here, again there are not so many choices, and the solution is that of defining $\varphi'(x)$ as being the row vector formed by the partial derivatives at x:

$$\varphi'(x) = \left(rac{d\varphi}{dx_1} \quad \dots \quad rac{d\varphi}{dx_N}
ight)$$

Moving now to second derivatives, the result here, generalizing Theorem 2.3, is:

THEOREM 2.5. The second derivative of a function $\varphi : \mathbb{R}^N \to \mathbb{R}$, making the formula

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + \frac{\langle \varphi''(x)h, h \rangle}{2}$$

work, is its Hessian matrix $\varphi''(x) \in M_N(\mathbb{R})$, given by the following formula:

$$\varphi''(x) = \left(\frac{d^2\varphi}{dx_i dx_j}\right)_{ij}$$

Moreover, this Hessian matrix is symmetric, $\varphi''(x)_{ij} = \varphi'(x)_{ji}$.

2. WAVES, LIGHT

PROOF. There are several things going on here, the idea being as follows:

(1) As a first observation, at N = 1 the Hessian matrix constructed above is simply the 1×1 matrix having as entry the second derivative $\varphi''(x)$, and the formula in the statement is something that we know well from Theorem 2.3, namely:

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + \frac{\varphi''(x)h^2}{2}$$

(2) At N = 2 now, we obviously need to differentiate φ twice, and the point is that we come in this way upon the following formula, called Clairaut formula:

$$\frac{d^2\varphi}{dxdy} = \frac{d^2\varphi}{dydx}$$

But, is this formula correct or not? As an intuitive justification for it, let us consider a product of power functions, $\varphi(z) = x^p y^q$. We have then our formula, due to:

$$\frac{d^2\varphi}{dxdy} = \frac{d}{dx} \left(\frac{dx^p y^q}{dy}\right) = \frac{d}{dx} \left(qx^p y^{q-1}\right) = pqx^{p-1}y^{q-1}$$
$$\frac{d^2\varphi}{dydx} = \frac{d}{dy} \left(\frac{dx^p y^q}{dx}\right) = \frac{d}{dy} \left(px^{p-1}y^q\right) = pqx^{p-1}y^{q-1}$$

Next, let us consider a linear combination of power functions, $\varphi(z) = \sum_{pq} c_{pq} x^p y^q$, which can be finite or not. We have then, by using the above computation:

$$\frac{d^2\varphi}{dxdy} = \frac{d^2\varphi}{dydx} = \sum_{pq} c_{pq} pq x^{p-1} y^{q-1}$$

Thus, we can see that our commutation formula for derivatives holds indeed, due to the fact that the functions in x, y commute. And exercise for you, to have this idea fully working, or to look up the standard proof of Clairaut, using the mean value theorem.

(3) Moving now to N = 3 and higher, we can use here the Clairaut formula with respect to any pair of coordinates, which gives the Schwarz formula, namely:

$$\frac{d^2\varphi}{dx_i dx_i} = \frac{d^2\varphi}{dx_i dx_i}$$

Thus, the second derivative, or Hessian matrix, is symmetric, as claimed.

(4) Getting now to the main topic, namely approximation formula in the statement, in arbitrary N dimensions, this is in fact something which does not need a new proof, because it follows from the one-variable formula in (1), applied to the restriction of φ to the following segment in \mathbb{R}^N , which can be regarded as being a one-variable interval:

$$I = [x, x+h]$$

To be more precise, let $y \in \mathbb{R}^N$, and consider the following function, with $r \in \mathbb{R}$:

$$f(r) = \varphi(x + ry)$$

We know from (1) that the Taylor formula for f, at the point r = 0, reads:

$$f(r) \simeq f(0) + f'(0)r + \frac{f''(0)r^2}{2}$$

And our claim is that, with h = ry, this is precisely the formula in the statement.

(5) So, let us see if our claim is correct. By using the chain rule, we have the following formula, with on the right, as usual, a row vector multiplied by a column vector:

$$f'(r) = \varphi'(x + ry) \cdot y$$

By using again the chain rule, we can compute the second derivative as well:

$$f''(r) = (\varphi'(x+ry) \cdot y)'$$

= $\left(\sum_{i} \frac{d\varphi}{dx_{i}}(x+ry) \cdot y_{i}\right)'$
= $\sum_{i} \sum_{j} \frac{d^{2}\varphi}{dx_{i}dx_{j}}(x+ry) \cdot \frac{d(x+ry)_{j}}{dr} \cdot y_{i}$
= $\sum_{i} \sum_{j} \frac{d^{2}\varphi}{dx_{i}dx_{j}}(x+ry) \cdot y_{i}y_{j}$
= $< \varphi''(x+ry)y, y >$

(6) Time now to conclude. We know that we have $f(r) = \varphi(x + ry)$, and according to our various computations above, we have the following formulae:

$$f(0) = \varphi(x)$$
 , $f'(0) = \varphi'(x)$, $f''(0) = \langle \varphi''(x)y, y \rangle$

Buit with this data in hand, the usual Taylor formula for our one variable function f, at order 2, at the point r = 0, takes the following form, with h = ry:

$$\varphi(x+ry) \simeq \varphi(x) + \varphi'(x)ry + \frac{\langle \varphi''(x)y, y \rangle r^2}{2}$$
$$= \varphi(x) + \varphi'(x)t + \frac{\langle \varphi''(x)h, h \rangle}{2}$$

Thus, we have obtained the formula in the statement.

Getting back now to what we wanted to do, namely understand Principle 2.2, it remains to talk about the Laplace operator Δ . Things are quite tricky here, basically requiring some physics that we still need to develop, but as something mathematical to start with, we have the following higher dimensional analogue of Proposition 2.4:

PROPOSITION 2.6. Intuitively, the following quantity, called Laplacian of φ ,

$$\Delta \varphi = \sum_{i=1}^{N} \frac{d^2 \varphi}{dx_i^2}$$

measures how much different is $\varphi(x)$, compared to the average of $\varphi(y)$, with $y \simeq x$.

PROOF. As before with Proposition 2.4, this is something a bit heuristic, but good to know. Let us write the formula in Theorem 2.5, as such, and with $h \to -h$ too:

$$\varphi(x+h) \simeq \varphi(x) + \varphi'(x)h + \frac{\langle \varphi''(x)h, h \rangle}{2}$$
$$\varphi(x-h) \simeq \varphi(x) - \varphi'(x)h + \frac{\langle \varphi''(x)h, h \rangle}{2}$$

By making the average, we obtain the following formula:

$$\frac{\varphi(x+h) + \varphi(x-h)}{2} = \varphi(x) + \frac{\langle \varphi''(x)h, h \rangle}{2}$$

Thus, thinking a bit, we are led to the conclusion in the statement.

Good news, we can now talk about mechanical waves. Here is the result:

THEOREM 2.7. The wave equation in \mathbb{R}^N is

$$\ddot{\varphi} = v^2 \Delta \varphi$$

where v > 0 is the propagation speed.

PROOF. Before everything, the equation in the statement is of course what comes out of experiments. However, allowing us a bit of imagination, and trust in this imagination, we can mathematically "prove" this equation, by discretizing, as follows:

(1) Let us first consider the 1D case. In order to understand the propagation of waves, we will model \mathbb{R} as a network of balls, with springs between them, as follows:

 $\cdots \times \times \bullet \times \times \cdots$

Now let us send an impulse, and see how the balls will be moving. For this purpose, we zoom on one ball. The situation here is as follows, l being the spring length:

$$\cdots \bullet_{\varphi(x-l)} \times \times \bullet_{\varphi(x)} \times \times \bullet_{\varphi(x+l)} \cdots \cdots$$

We have two forces acting at x. First is the Newton motion force, mass times acceleration, which is as follows, with m being the mass of each ball:

$$F_n = m \cdot \ddot{\varphi}(x)$$
2A. WAVE EQUATION

And second is the Hooke force, displacement of the spring, times spring constant. Since we have two springs at x, this is as follows, k being the spring constant:

$$F_h = F_h^r - F_h^l$$

= $k(\varphi(x+l) - \varphi(x)) - k(\varphi(x) - \varphi(x-l))$
= $k(\varphi(x+l) - 2\varphi(x) + \varphi(x-l))$

We conclude that the equation of motion, in our model, is as follows:

$$m \cdot \ddot{\varphi}(x) = k(\varphi(x+l) - 2\varphi(x) + \varphi(x-l))$$

(2) Now let us take the limit of our model, as to reach to continuum. For this purpose we will assume that our system consists of B >> 0 balls, having a total mass M, and spanning a total distance L. Thus, our previous infinitesimal parameters are as follows, with K being the spring constant of the total system, which is of course lower than k:

$$m = \frac{M}{B}$$
 , $k = KB$, $l = \frac{L}{B}$

With these changes, our equation of motion found in (1) reads:

$$\ddot{\varphi}(x) = \frac{KB^2}{M}(\varphi(x+l) - 2\varphi(x) + \varphi(x-l))$$

Now observe that this equation can be written, more conveniently, as follows:

$$\ddot{\varphi}(x) = \frac{KL^2}{M} \cdot \frac{\varphi(x+l) - 2\varphi(x) + \varphi(x-l)}{l^2}$$

With $N \to \infty$, and therefore $l \to 0$, we obtain in this way:

$$\ddot{\varphi}(x) = \frac{KL^2}{M} \cdot \frac{d^2\varphi}{dx^2}(x)$$

We are therefore led to the wave equation in the statement, which is $\ddot{\varphi} = v^2 \varphi''$ in our present N = 1 dimensional case, the propagation speed being $v = \sqrt{K/M} \cdot L$.

(3) In 2 dimensions now, the same argument carries on. Indeed, we can use here a lattice model as follows, with all the edges standing for small springs:



2. WAVES, LIGHT

As before in one dimension, we send an impulse, and we zoom on one ball. The situation here is as follows, with l being the spring length:



We have two forces acting at (x, y). First is the Newton motion force, mass times acceleration, which is as follows, with m being the mass of each ball:

$$F_n = m \cdot \ddot{\varphi}(x, y)$$

And second is the Hooke force, displacement of the spring, times spring constant. Since we have four springs at (x, y), this is as follows, k being the spring constant:

$$F_h = F_h^r - F_h^l + F_h^u - F_h^d$$

= $k(\varphi(x+l,y) - \varphi(x,y)) - k(\varphi(x,y) - \varphi(x-l,y))$
+ $k(\varphi(x,y+l) - \varphi(x,y)) - k(\varphi(x,y) - \varphi(x,y-l))$
= $k(\varphi(x+l,y) - 2\varphi(x,y) + \varphi(x-l,y))$
+ $k(\varphi(x,y+l) - 2\varphi(x,y) + \varphi(x,y-l))$

We conclude that the equation of motion, in our model, is as follows:

$$\begin{aligned} m \cdot \ddot{\varphi}(x,y) &= k(\varphi(x+l,y) - 2\varphi(x,y) + \varphi(x-l,y)) \\ &+ k(\varphi(x,y+l) - 2\varphi(x,y) + \varphi(x,y-l)) \end{aligned}$$

(4) Now let us take the limit of our model, as to reach to continuum. For this purpose we will assume that our system consists of $B^2 >> 0$ balls, having a total mass M, and spanning a total area L^2 . Thus, our previous infinitesimal parameters are as follows, with K being the spring constant of the total system, taken to be equal to k:

$$m = \frac{M}{B^2}$$
 , $k = K$, $l = \frac{L}{B}$

With these changes, our equation of motion found in (3) reads:

$$\begin{split} \ddot{\varphi}(x,y) &= \frac{KB^2}{M}(\varphi(x+l,y) - 2\varphi(x,y) + \varphi(x-l,y)) \\ &+ \frac{KB^2}{M}(\varphi(x,y+l) - 2\varphi(x,y) + \varphi(x,y-l)) \end{split}$$

2B. LIGHT CREATION

Now observe that this equation can be written, more conveniently, as follows:

$$\begin{split} \ddot{\varphi}(x,y) &= \frac{KL^2}{M} \times \frac{\varphi(x+l,y) - 2\varphi(x,y) + \varphi(x-l,y)}{l^2} \\ &+ \frac{KL^2}{M} \times \frac{\varphi(x,y+l) - 2\varphi(x,y) + \varphi(x,y-l)}{l^2} \end{split}$$

With $N \to \infty$, and therefore $l \to 0$, we obtain in this way:

$$\ddot{\varphi}(x,y) = \frac{KL^2}{M} \left(\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2}\right)(x,y)$$

As a conclusion to this, we are led to the following wave equation in two dimensions, with $v = \sqrt{K/M} \cdot L$ being the propagation speed of our wave:

$$\ddot{\varphi}(x,y) = v^2 \left(\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2}\right)(x,y)$$

But we recognize at right the Laplace operator, and we are done. As before in 1D, there is of course some discussion to be made here, arguing that our spring model in (3) is indeed the correct one. But do not worry, experiments confirm our findings.

(5) In 3 dimensions now, which is the case of the main interest, corresponding to our real-life world, the same argument carries over, and the wave equation is as follows:

$$\ddot{\varphi}(x,y,z) = v^2 \left(\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} \right) (x,y,z)$$

(6) Finally, the same argument, namely a lattice model, carries on in arbitrary N dimensions, and the wave equation here is as follows:

$$\ddot{\varphi}(x_1,\ldots,x_N) = v^2 \sum_{i=1}^N \frac{d^2\varphi}{dx_i^2}(x_1,\ldots,x_N)$$

Thus, we are led to the conclusion in the statement.

Observe that there are some subtleties in the above, in relation with our conventions for the total spring constant K, which varies with the dimension N. Also, once we are in $N \ge 2$ dimensions, modifying the springs in our lattice model, as to allow a dissymmetry between horizontal and vertical, either at the level of spring lengths, or spring constants, will most likely lead to different results. We will be back to these issues, later.

2b. Light creation

Back now to electromagnetism, let us focus on the Maxwell equations, and try to make appear c, and waves. And here, surprise, things are quite simple, as follows:

2. WAVES, LIGHT

THEOREM 2.8. In regions of space where there is no charge or current present the Maxwell equations for electrodynamics read

$$\label{eq:phi} \begin{split} &< \nabla, E > = < \nabla, B > = 0 \\ &\nabla \times E = - \dot{B} \quad , \quad \nabla \times B = \dot{E}/c^2 \end{split}$$

and both the electric field E and magnetic field B are subject to the wave equation

$$\ddot{\varphi} = c^2 \Delta \varphi$$

with $\Delta = \sum_i d^2/dx_i^2$ being the Laplace operator, and c the speed of light.

PROOF. Under the circumstances in the statement, namely no charge or current present, the Maxwell equations simply read, taking into account $\mu_0 \varepsilon_0 = 1/c^2$:

$$\langle \nabla, E \rangle = \langle \nabla, B \rangle = 0$$

 $\nabla \times E = -\dot{B}$, $\nabla \times B = \dot{E}/c^2$

Now by applying the curl operator to the last two equations, we obtain:

$$\nabla \times (\nabla \times E) = -\nabla \times \dot{B} = -(\nabla \times B)' = -\dot{E}/c^2$$
$$\nabla \times (\nabla \times B) = \nabla \times \dot{E}/c^2 = (\nabla \times E)'/c^2 = -\ddot{B}/c^2$$

But the double curl operator is subject to the following formula:

$$\nabla \times (\nabla \times \varphi) = \nabla < \nabla, \varphi > -\Delta \varphi$$

Now by using the first two equations, we are led to the conclusion in the statement. \Box

So, what is light? Light is the wave predicted by Theorem 2.8, traveling in vacuum at the maximum possible speed, c, and with an important extra property being that it depends on a real positive parameter, that can be called, upon taste, frequency, wavelength, or color. And in what regards the creation of light, the mechanism here is as follows:

FACT 2.9. An accelerating or decelerating charge produces electromagnetic radiation, called light, whose frequency and wavelength can be explicitly computed.

This phenomenon can be observed is a variety of situations, such as the usual light bulbs, where electrons get decelerated by the filament, acting as a resistor, or in usual fire, which is a chemical reaction, with the electrons moving around, as they do in any chemical reaction, or in more complicated machinery like nuclear plants, particle accelerators, and so on, leading there to all sorts of eerie glows, of various colors.

Getting back now to Fact 2.9, in its general form, as stated above, this is something that can be deduced via some math, based on the Maxwell equations.

Many other things involving c can be said, regarding the Maxwell equations, with for instance the invariance of these equations under the Lorentz transformations of spacetime, and with these eventually coming, via Einstein's theory of relativity, from v < c. But all this is rather advanced material, for later. For now, the above will do.

2C. FREQUENCY, COLORS

2c. Frequency, colors

Moving ahead, let us go back to the wave equation found in Theorem 2.7, and try to understand its simplest solutions. In 1D, the situation is as follows:

THEOREM 2.10. The 1D wave equation, with speed v, namely

$$\ddot{\varphi} = v^2 \, \frac{d^2 \varphi}{dx^2}$$

has as basic solutions the following functions,

$$\varphi(x) = A\cos(kx - wt + \delta)$$

with A being called amplitude, $kx - wt + \delta$ being called the phase, k being the wave number, w being the angular frequency, and δ being the phase constant. We have

$$\lambda = \frac{2\pi}{k} \quad , \quad T = \frac{2\pi}{kv} \quad , \quad \nu = \frac{1}{T} \quad , \quad w = 2\pi\nu$$

relating the wavelength λ , period T, frequency ν , and angular frequency w. Moreover, any solution of the wave equation appears as a linear combination of such basic solutions.

PROOF. There are several things going on here, the idea being as follows:

(1) Our first claim is that the function φ in the statement satisfies indeed the wave equation, with speed v = w/k. For this purpose, observe that we have:

$$\ddot{\varphi} = -w^2 \varphi \quad , \quad \frac{d^2 \varphi}{dx^2} = -k^2 \varphi$$

Thus, the wave equation is indeed satisfied, with speed v = w/k:

$$\ddot{\varphi} = \left(\frac{w}{k}\right)^2 \frac{d^2\varphi}{dx^2} = v^2 \frac{d^2\varphi}{dx^2}$$

(2) Regarding now the other things in the statement, all this is basically terminology, which is very natural, when thinking how $\varphi(x) = A\cos(kx - wt + \delta)$ propagates.

(3) Finally, the last assertion is something standard, coming from Fourier analysis, that we will not really need, in what follows. \Box

As a first observation, the above result invites the use of complex numbers. Indeed, we can write the solutions that we found in a more convenient way, as follows:

$$\varphi(x) = Re\left[A e^{i(kx - wt + \delta)}\right]$$

And we can in fact do even better, by absorbing the quantity $e^{i\delta}$ into the amplitude A, which becomes now a complex number, and writing our formula as:

$$\varphi = Re(\widetilde{\varphi}) \quad , \quad \widetilde{\varphi} = \widetilde{A}e^{i(kx-wt)}$$

Moving ahead now towards electromagnetism and 3D, let us formulate:

2. WAVES, LIGHT

DEFINITION 2.11. A monochromatic plane wave is a solution of the 3D wave equation which moves in only 1 direction, making it in practice a solution of the 1D wave equation, and which is of the special from found in Theorem 2.10, with no frequencies mixed.

In other words, we are making here two assumptions on our wave. First is the 1dimensionality assumption, which gets us into the framework of Theorem 2.10. And second is the assumption, in connection with the Fourier decomposition result from the end of Theorem 2.10, that our solution is of "pure" type, meaning a wave having a welldefined wavelenght and frequency, instead of being a "packet" of such pure waves.

All this is still mathematics, and making now the connection with physics and electromagnetism, and more specifically with Theorem 2.10 and Fact 2.9, we have:

FACT 2.12. Physically speaking, a monochromatic plane wave is the electromagnetic radiation appearing as in Theorem 2.8 and Fact 2.9, via equations of type

$$E = Re(\widetilde{E})$$
 : $\widetilde{E} = \widetilde{E}_0 e^{i(\langle k, x \rangle - wt)}$

$$B = Re(\widetilde{B}) \quad : \quad \widetilde{B} = \widetilde{B}_0 e^{i(\langle k, x \rangle - wt)}$$

with the wave number being now a vector, $k \in \mathbb{R}^3$. Moreover, it is possible to add to this an extra parameter, accounting for the possible polarization of the wave.

To be more precise, what we are doing here is to import the conclusions of our mathematical discussion so far, from Theorem 2.10 and Definition 2.11, into the context of our original physics discussion, from Fact 2.9. And also to add an extra twist coming from physics, and more specifically from the notion of polarization.

In any case, we have now a decent intuition about what light is, and more on this later, and let us discuss now the examples. The idea is that we have various types of light, depending on frequency and wavelength. These are normally referred to as "electromagnetic waves", but for keeping things simple and luminous, we will keep using

2D. POLARIZATION

the familiar term "light". The classification, in a rough form, is as follows:

Frequency	Type	Wavelength
	_	
$10^{18} - 10^{20}$	γ rays	$10^{-12} - 10^{-10}$
$10^{16} - 10^{18}$	X - rays	$10^{-10} - 10^{-8}$
$10^{15} - 10^{16}$	UV	$10^{-8} - 10^{-7}$
	_	
$10^{14} - 10^{15}$	blue	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	yellow	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	red	$10^{-7} - 10^{-6}$
	_	
$10^{11} - 10^{14}$	IR	$10^{-6} - 10^{-3}$
$10^9 - 10^{11}$	microwave	$10^{-3} - 10^{-1}$
$1 - 10^9$	radio	$10^{-1} - 10^8$

Observe the tiny space occupied by the visible light, all colors there, and the many more missing, being squeezed under the $10^{14} - 10^{15}$ frequency banner. Here is a zoom on that part, with of course the remark that all this, colors, is something subjective:

Frequency $THz = 10^{12} Hz$	Color	Wavelength $nm = 10^{-9} m$
	_	
670 - 790	violet	380 - 450
620 - 670	blue	450 - 485
600 - 620	cyan	485 - 500
530 - 600	green	500 - 565
510 - 530	yellow	565 - 590
480 - 510	orange	590 - 625
400 - 480	red	625 - 750

Outside visible light we have, as you probably know it, UV on higher frequencies, and IR on lower frequencies. At the high frequency end we have X-rays, that you surely know about too, and γ rays, which are usually associated with various bad things, such as thunderstorms, solar flares, and small bugs with our nuclear energy technology.

As for the lower frequency end of the scale, first we have microwaves, but if you love physics and chemistry you should learn some cooking, that's first-class chemistry, that you can practice every day. And then we have all sorts of radio wavelenghts, including FM, followed by AM, and then by several more obscure low-frequency waves.

2d. Polarization

Polarization.

2. WAVES, LIGHT

2e. Exercises

Exercises:

EXERCISE 2.13.

EXERCISE 2.14.

EXERCISE 2.15.

Exercise 2.16.

EXERCISE 2.17.

EXERCISE 2.18.

Exercise 2.19.

EXERCISE 2.20.

Bonus exercise.

CHAPTER 3

Light and heat

3a. Resistors, Joule

Time now to discuss the relation between light and heat. For this purpose, we will go back to our usual guiding device, since the beginning of this book, namely the light bulb. Ignoring Led bulbs and other modern gadgets, we will need here a usual, old-style light bulb, and even better, a very ancient one, needing some time to heat, then produce light. So, question for us now, how does all this procedure, mixing light and heat, work?

In practice, a first question is that of understanding the current density J flowing through a given material. The answer here is given by Ohm's law, as follows:

FACT 3.1 (Ohm's law). The current density J is given by

 $J=\sigma E$

where σ is a constant, called conductivity of the material.

We are already a bit familiar with this, with our notion of ideal conductor corresponding to $\sigma = \infty$, and our notion of ideal insulator corresponding to $\sigma = 0$. In real life, however, we have of course $\sigma \in (0, \infty)$. Here are 3 + 3 + 3 basic examples, at 20° C and 1 atm, consisting of 3 conductors, 3 semiconductors and 3 insulators, and with σ being replaced by its inverse $\rho = 1/\sigma$, called resistivity, more employed in engineering:

Silver	:	1.59×10^{-8}
Iron	:	9.61×10^{-8}
Graphite	:	1.6×10^{-5}
	_	
Seawater	:	0.2
Diamond	:	2.7
Silicon	:	2500
	_	
Water	:	8300
Glass	:	$10^9 - 10^{14}$
Teflon	:	$10^{22} - 10^{24}$

Getting back now to Ohm's law, a more familiar version of it is as follows, expressing the total current flowing from one electrode to the other in terms of the potential difference

3. LIGHT AND HEAT

between them, or rather vice versa, and with $R \sim \rho$ being the resistance, which depends, besides on ρ , on the precise configuration of the resistor to be crossed:

V = IR

With this second formulation of the Ohm law in hand, we can now formulate as well, following Joule, a formula in regards with energy, as follows:

FACT 3.2 (Joule heating law). The work done by the electric force is

 $P = VI = I^2 R$

with this being understood as corresponding to heating the resistor.

As usual, we refer to our standard undergraduate books, such as Griffiths [44], for more on all this, and to an engineering book for even more.

3b. Planck, quanta

Consider a black body, that is, a body at thermal equilibrium, assumed to be at temperature T. This body radiates heat, and we are interested in computing the energy density of the radiation $\mathcal{E}(\nu, T)$, around a given frequency ν of this radiation.

Quite surprisingly, the intuitive and honest modeling of the problem, and the subsequent math, done honestly too, lead to a spectacularly wrong result, as follows:

THEOREM 3.3. We have the Rayleigh-Jeans formula for the energy density

$$\mathcal{E}(\nu,T) = \frac{8\pi bT}{c^3} \,\nu^2$$

where b is the Boltzmann constant, leading globally to the divergent integral

$$\mathcal{E} = \frac{8\pi bTV}{c^3} \int_0^\infty \nu^2 \, d\nu$$

over a volume V, with this divergence phenomenon being called UV catastrophe.

PROOF. This is arguably the most famous wrong result in the history of physics, so we will spend some time in trying to understand its proof. And with the comment that this will be no waste of time, because the fix, found later by Max Planck, uses exactly the same ideas and computations, but with an unexpected twist at the end.

(1) Our starting point are the equations for the electromagnetic radiation, that we will now regard as heat, as formulated before, namely:

 $E = Re(\widetilde{E}) \quad : \quad \widetilde{E} = e_n e^{i(\langle k_n, x \rangle - w_n t)}$ $B = Re(\widetilde{B}) \quad : \quad \widetilde{B} = b_n e^{i(\langle k_n, x \rangle - w_n t)}$

Here n is a certain parameter, that will appear later on, and that we can for the moment ignore. Now inserting this data into the Maxwell equations gives the following formulae, connecting the parameters, that we will use several times in what follows:

$$k_n \times b_n + \frac{w_n}{c} e_n = 0$$

$$k_n \times e_n - \frac{w_n}{c} b_n = 0$$

$$\langle k_n, e_n \rangle = \langle k_n, b_n \rangle = 0$$

(2) Let us compute the electromagnetic energy in a finite volume $V = L^3$. We will use here the well-known fact, coming from classical electrodynamics, that the energy density in radiation is $(||E||^2 + ||B||^2)/8\pi$. Thus, the energy we are looking for is given by:

$$\mathcal{E} = \frac{1}{8\pi} \int_{V} ||E||^2 + ||B||^2$$

(3) In order to compute this integral, let us better model our question. Due to obvious periodicity reasons, the wave number k and the angular frequency w must be of the following form, with $n \in \mathbb{Z}^3$ being a vector with integer components:

$$k_n = \frac{2\pi}{L} \cdot n \quad , \quad w_n = c||k_n|$$

Thus, the electric and magnetic fields in our enclosure $V = L^3$ appear as linear combinations as follows, for certain vectors $e_n, b_n \perp n$, related by the formulae in (1):

$$E = Re(\widetilde{E})$$
 : $\widetilde{E} = \sum_{n} e_n e^{i(\langle k_n, x \rangle - w_n t)}$

$$B = Re(\widetilde{B})$$
 : $\widetilde{B} = \sum_{n} b_n e^{i(\langle k_n, x \rangle - w_n t)}$

(4) According to the above formula of E, we have:

$$\begin{aligned} ||E||^2 &= ||Re(\widetilde{E})||^2 \\ &= \frac{1}{4} \left| \left| \sum_n e_n e^{i(\langle k_n, x \rangle - w_n t)} + \bar{e}_n e^{-i(\langle k_n, x \rangle - w_n t)} \right| \right|^2 \\ &= \frac{1}{4} \sum_{nm} \langle e_n, e_m \rangle e^{i(\langle k_n - k_m, x \rangle - (w_n - w_m) t)} \\ &+ \frac{1}{4} \sum_{nm} \langle e_n, \bar{e}_m \rangle e^{i(\langle k_n + k_m, x \rangle - (w_n + w_m) t)} \\ &+ \frac{1}{4} \sum_{nm} \langle \bar{e}_n, e_m \rangle e^{i(\langle k_n + k_m, x \rangle + (w_n + w_m) t)} \\ &+ \frac{1}{4} \sum_{nm} \langle \bar{e}_n, \bar{e}_m \rangle e^{i(\langle k_n - k_m, x \rangle + (w_n - w_m) t)} \end{aligned}$$

(5) Now by integrating, we obtain the following formula:

$$\frac{1}{V} \int_{V} ||E||^{2} = \frac{1}{4} \sum_{n} \langle e_{n}, e_{n} \rangle + \frac{1}{4} \sum_{n} \langle e_{n}, \bar{e}_{-n} \rangle e^{-2iw_{n}t} + \frac{1}{4} \sum_{n} \langle \bar{e}_{n}, e_{-n} \rangle e^{2iw_{n}t} + \frac{1}{4} \sum_{n} \langle \bar{e}_{n}, \bar{e}_{n} \rangle$$

(6) Similarly, according to the above formula of B, we have:

$$\begin{aligned} \frac{1}{V} \int_{V} ||B||^{2} &= \frac{1}{4} \sum_{n} < b_{n}, b_{n} > + \frac{1}{4} \sum_{n} < b_{n}, \bar{b}_{-n} > e^{-2iw_{n}t} \\ &+ \frac{1}{4} \sum_{n} < \bar{b}_{n}, b_{-n} > e^{2iw_{n}t} + \frac{1}{4} \sum_{n} < \bar{b}_{n}, \bar{b}_{n} > \end{aligned}$$

(7) Before summing the integrals that we found, let us use the formulae connecting the parameters k_n, e_n, b_n found in (1) above, namely:

$$k_n \times b_n + \frac{w_n}{c} e_n = 0$$
$$k_n \times e_n - \frac{w_n}{c} b_n = 0$$
$$< k_n, e_n \ge < k_n, b_n \ge 0$$

By using these formulae, we first obtain the following identity:

$$\langle b_n, b_n \rangle = \frac{c^2}{w_n^2} \langle k_n \times e_n, k_n \times e_n \rangle$$
$$= \frac{c^2 ||k_n||^2}{w_n^2} \langle e_n, e_n \rangle$$
$$= \langle e_n, e_n \rangle$$

Similarly, we have we well the following identity:

$$< b_n, \bar{b}_{-n} > = \frac{c^2}{w_n^2} < k_n \times e_n, k_{-n} \times \bar{e}_n >$$

 $= -\frac{c^2 ||k_n||^2}{w_n^2} < e_n, \bar{e}_{-n} >$
 $= - < e_n, \bar{e}_{-n} >$

Also similarly, we have as well the following identity:

$$< \bar{b}_n, b_{-n} > = \frac{c^2}{w_n^2} < k_n \times \bar{e}_n, k_{-n} \times e_n >$$

$$= -\frac{c^2 ||k_n||^2}{w_n^2} < \bar{e}_n, e_{-n} >$$

$$= - < \bar{e}_n, e_{-n} >$$

Finally, we have as well the following identity:

$$\langle \bar{b}_n, \bar{b}_n \rangle = \frac{c^2}{w_n^2} \langle k_n \times \bar{e}_n, k_n \times \bar{e}_n \rangle$$
$$= \frac{c^2 ||k_n||^2}{w_n^2} \langle \bar{e}_n, \bar{e}_n \rangle$$
$$= \langle \bar{e}_n, \bar{e}_n \rangle$$

(8) We conclude that when summing the integrals computed in (5) and (6), all the terms involving phases will cancel, and we obtain the following formula:

$$\frac{1}{V} \int_{V} ||E||^{2} + ||B||^{2} = \frac{1}{2} \sum_{n} \langle e_{n}, e_{n} \rangle + \frac{1}{2} \sum_{n} \langle \bar{e}_{n}, \bar{e}_{n} \rangle$$

Now by multiplying everything by $V/8\pi$, as explained in (2), we obtain:

$$\mathcal{E} = \frac{V}{16\pi} \sum_{n} \left(\langle e_n, e_n \rangle + \langle \bar{e}_n, \bar{e}_n \rangle \right)$$

3. LIGHT AND HEAT

(9) The point now is that, by computing this sum, we are led to the Rayleigh-Jeans formula in the statement for the corresponding radiation energy density, namely:

$$\mathcal{E}(\nu,T) = \frac{8\pi bT}{c^3} \,\nu^2$$

(10) And this is certainly wrong, because the total energy which is radiated by our black body, all over the frequency spectrum, follows to be:

$$\mathcal{E} = \frac{8\pi bTV}{c^3} \int_0^\infty \nu^2 \, d\nu = \infty$$

More precisely, the Rayleigh-Jeans formula works quite well all across the frequency spectrum, in particular fitting well with the known data, expect for the UV range, where things diverge. And with this phenomenon being called "UV catastrophe". \Box

Fortunately, the solution to the UV catastrophe, and to the black body problem in general, was found a few years later by Max Planck, his result being as follows:

THEOREM 3.4. The correct formula for the black body radiation, obtained by assuming that energy is quantized, is the Planck formula

$$\mathcal{E}(\nu,T) \, d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3 d\nu}{e^{h\nu/bt} - 1}$$

with h being a new constant, called Planck constant, given by

$$h = 6.626\ 070\ 15 \times 10^{-34}$$

as per the latest SI regulations. The Planck formula fits with all known data, fits as well with the Rayleigh-Jeans formula outside the UV range, and globally leads to

$$\mathcal{E} = \int_0^\infty \mathcal{E}(\nu, T) \, d\nu = a T^4$$

with the radiation energy constant on the right being given by:

$$a = \frac{16\pi^8 b^4}{15h^3c^3}$$

PROOF. This is something quite technical, obtained along the lines of the proof of Theorem 3.4, by counting in a new way, by assuming that energy is quantized. \Box

Regarding applications, a very interesting continuation of Planck's work concerns the black body radiation of the early universe, with the microwave part of it, via a Doppler shift, still permeating the space that we live in. And with this phenomenon, called "cosmic microwave background", being at the origin of all modern cosmology.

3C. ATOMIC THEORY

3c. Atomic theory

There is a long story with the modern atomic theory, with everything coming from hydrogen, and involving many discoveries of many people, around 1890-1900. First on our list is the following discovery, which actually came second, by Lyman in 1906:

FACT 3.5 (Lyman). The hydrogen atom has spectral lines given by the formula

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{n^2}\right)$$

where $R \simeq 1.097 \times 10^7$ and $n \ge 2$, which are as follows,

n	Name	Wavelength	Color
	_	—	
2	α	121.567	UV
3	β	102.572	UV
4	γ	97.254	UV
÷	:	:	÷
∞	limit	91.175	UV

called Lyman series of the hydrogen atom.

Observe that all the Lyman series lies in UV, which is invisible to the human eye. The first discovery, which was the big one, and the breakthrough, was by Balmer, the founding father of all this, back in 1885, in the visible range, as follows:

FACT 3.6 (Balmer). The hydrogen atom has spectral lines given by the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{n^2}\right)$$

where $R \simeq 1.097 \times 10^7$ and $n \ge 3$, which are as follows,

n	Name	Wavelength	Color
	_	—	
3	α	656.279	red
4	β	486.135	aqua
5	γ	434.047	blue
6	δ	410.173	violet
7	ε	397.007	UV
:	:	:	÷
∞	limit	346.600	UV

called Balmer series of the hydrogen atom.

So, this was Balmer's original result, which started everything, and with his original wavelength formula being in fact something equivalent to the above formula, but a bit more complicated, as follows, with $B \simeq 3.645 \times 10^{-7}$ being the Balmer constant:

$$\lambda = \frac{Bn^2}{n^2 - 4}$$

As a third main result now, this time in IR, due to Paschen in 1908, we have:

FACT 3.7 (Paschen). The hydrogen atom has spectral lines given by the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{9} - \frac{1}{n^2}\right)$$

where $R \simeq 1.097 \times 10^7$ and $n \ge 4$, which are as follows, \sim Name Wavelength Color

n	Name	Wavelength	Colo
	—	—	
4	α	1875	IR
5	β	1282	IR
6	γ	1094	IR
÷	:	:	÷
∞	limit	820.4	IR

called Paschen series of the hydrogen atom.

Observe the striking similarity between the above three results. In fact, we have here the following fundamental, grand result, due to Rydberg in 1888, based on the Balmer series, and with later contributions by Ritz in 1908, using the Lyman series as well:

CONCLUSION 3.8 (Rydberg, Ritz). The spectral lines of the hydrogen atom are given by the Rydberg formula, depending on integer parameters $n_1 < n_2$,

$$\frac{1}{\lambda_{n_1n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

with R being the Rydberg constant for hydrogen, which is as follows:

$$R \simeq 1.096\ 775\ 83 \times 10^7$$

These spectral lines combine according to the Ritz-Rydberg principle, as follows:

$$\frac{1}{\lambda_{n_1 n_2}} + \frac{1}{\lambda_{n_2 n_3}} = \frac{1}{\lambda_{n_1 n_3}}$$

Similar formulae hold for other atoms, with suitable fine-tunings of R.

Here the first part, the Rydberg formula, generalizes the results of Lyman, Balmer, Paschen, which appear at $n_1 = 1, 2, 3$, at least retrospectively. The Rydberg formula predicts further spectral lines, appearing at $n_1 = 4, 5, 6, \ldots$, and these were discovered

3C. ATOMIC THEORY

later, by Brackett in 1922, Pfund in 1924, Humphreys in 1953, and others aftwerwards, with all these extra lines being in far IR. The simplified complete table is as follows:

n_1	n_2	Series name	Wavelength $n_2 = \infty$	Color $n_2 = \infty$
		—	—	
1	$2-\infty$	Lyman	91.13 nm	UV
2	$3-\infty$	Balmer	$364.51~\mathrm{nm}$	UV
3	$4-\infty$	Paschen	820.14 nm	IR
		_	_	
4	$5-\infty$	Brackett	1458.03 nm	far IR
5	$6-\infty$	Pfund	2278.17 nm	far IR
6	$7-\infty$	Humphreys	$3280.56~\mathrm{nm}$	far IR
:	:	:	÷	:

Regarding the last assertion, concerning other elements, this is something conjectured and partly verified by Ritz, and fully verified and clarified later, via many experiments, the fine-tuning of R being basically $R \to RZ^2$, where Z is the atomic number.

But from a theoretical physics viewpoint, the main result remains the middle assertion, called Ritz-Rydberg combination principle. This is something at the same time extremely simple, and completely puzzling, the informal conclusion being as follows:

THOUGHT 3.9. The simplest observables of the hydrogen atom, combining via

$$\frac{1}{\lambda_{n_1n_2}} + \frac{1}{\lambda_{n_2n_3}} = \frac{1}{\lambda_{n_1n_3}}$$

look like quite weird quantities. Why wouldn't they just sum normally.

Getting now to quantum mechanics, and to our dreams about it, formulated before, well, good news, we have some serious data here. These spectral lines are basic and beautiful, obviously of quantized type, and in order to get started with our theory, we first need to solve the puzzle of the Ritz-Rydberg combination principle. But, how to do this? Fortunately, matrix theory comes to the rescue, as follows:

THOUGHT 3.10. The Ritz-Rydberg combination principle reminds the formula

$$e_{n_1 n_2} e_{n_2 n_3} = e_{n_1 n_3}$$

for the usual matrix units, which are the elementary matrices given by

 $e_{ij}: e_j \to e_i$

perhaps taken in infinite dimensions, as to allow infinite-ranging indices.

This looks certainly very interesting, and following now Heisenberg, we can start dreaming of something more precise, in relation with the above, as follows:

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THOUGHT 3.11. Observables in quantum mechanics should be some sort of infinite matrices, generalizing the Lyman, Balmer, Paschen lines of the hydrogen atom, and multiplying between them as the matrices do, as to produce further observables.

Now back to more concrete things, as a main problem that we would like to solve, we have the understanding the intimate structure of matter, at the atomic level.

There is of course a long story here, regarding the intimate structure of matter, going back centuries and even millennia ago, and our presentation here will be quite simplified. As a starting point, since we need a starting point for all this, let us agree on:

CLAIM 3.12. Ordinary matter is made of small particles called atoms, with each atom appearing as a mix of even smaller particles, namely protons +, neutrons 0 and electrons -, with the same number of protons + and electrons -.

As a first observation, this is something which does not look obvious at all, with probably lots of work, by many people, being involved, as to lead to this claim. And so it is. The story goes back to the discovery of charges and electricity, which were attributed to a small particle, the electron -. Now since matter is by default neutral, this naturally leads to the consideration to the proton +, having the same charge as the electron.

But, as a natural question, why should be these electrons - and protons + that small? And also, what about the neutron 0? These are not easy questions, and the fact that it is so came from several clever experiments, due to Thomson, Rutherford and others.

Let us first recall that, due to somewhat obvious reasons, careful experiments with tiny particles are practically impossible. This is because of the resolution of our scientific machinery, go put an electron on a balance, then measure its weight, that sort of thing certainly looks impossible. In addition, never know with the electrons, these tiny little beasts might be more clever than us, and escape our attempts to capture them.

However, and here comes the point, all sorts of brutal experiments, such as bombarding matter with other pieces of matter, accelerated to the extremes, or submitting it to huge electric and magnetic fields, do work. And it is such kind of experiments, , "peeling off" protons +, neutrons 0 and electrons – from matter, and observing them, that led to the conclusion that these small beasts +, 0, - exist indeed, in agreement with Claim 3.12.

In addition to this, of particular importance was as well, at that time around 1900, the radioactivity theory of Becquerel and Pierre and Marie Curie, involving this time such small beasts, or perhaps some related radiation, peeling off by themselves, in heavy elements such as uranium $_{92}$ U, polonium $_{84}$ Po and radium $_{88}$ Ra. And there was also Einstein's work on the photoelectric effect, light interacting with matter, suggesting that even light itself might have associated to it some kind of particle, called photon.

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All this goes of course beyond what Claim 3.12 says, with some further particles involved, and more on this later, but as a general idea, all this deluge of small particle findings, all coming around 1900-1910, further solidified Claim 3.12.

So, taking now Claim 3.12 for granted, how are then the atoms organized, as mixtures of protons +, neutrons 0 and electrons -? The answer here lies again in the abovementioned "brutal" experiments of Thomson, Rutherford and others, which not only proved Claim 3.12, but led to an improved version of it, as follows:

CLAIM 3.13. The atoms are formed by a core of protons + and neutrons 0, surrounded by a cloud of electrons -, gravitating around the core.

This is a considerable advance, because we are now into familiar territory, namely some kind of mechanics. And with this in mind, all the pieces of our puzzle start fitting together, and we are led to the following grand conclusion:

CLAIM 3.14 (Bohr and others). The atoms are formed by a core of protons and neutrons, surrounded by a cloud of electrons, basically obeying to a modified version of electromagnetism. And with a fine mechanism involved, as follows:

- (1) The electrons are free to move only on certain specified elliptic orbits, labelled $1, 2, 3, \ldots$, situated at certain specific heights.
- (2) The electrons can jump or fall between orbits $n_1 < n_2$, absorbing or emitting light and heat, that is, electromagnetic waves, as accelerating charges.
- (3) The energy of such a wave, coming from $n_1 \rightarrow n_2$ or $n_2 \rightarrow n_1$, is given, via the Planck viewpoint, by the Rydberg formula, applied with $n_1 < n_2$.
- (4) The simplest such jumps are those observed by Lyman, Balmer, Paschen. And multiple jumps explain the Ritz-Rydberg formula.

And isn't this beautiful. Moreover, some further claims, also by Bohr and others, are that the theory can be further extended and fine-tuned as to explain many other phenomena, such as the findings of Einstein on the photoelectric effect, of Becquerel and Pierre and Marie Curie on radioactivity, and many more.

And the story is not over here. Following now Heisenberg, the next claim is that the underlying math in all the above can lead to a beautiful axiomatization of quantum mechanics, as a "matrix mechanics", along the lines of Thought 3.11.

3d. Chemistry, fire

Chemistry, fire.

3. LIGHT AND HEAT

3e. Exercises

Exercises:

EXERCISE 3.15.

EXERCISE 3.16.

Exercise 3.17.

EXERCISE 3.18.

Exercise 3.19.

Exercise 3.20.

Exercise 3.21.

EXERCISE 3.22.

Bonus exercise.

CHAPTER 4

Solar light

4a. Nuclei, fusion

We have kept the best, the very familiar Solar light, and perhaps some other forms of light too, for the end of this introductory Part I. Beware, dangerous physics to come.

Let us start with a continuation of the atomic theory, discussed in chapter 3. We mostly focused there on hydrogen, but as already mentioned, basically the same ideas apply to the heavier atoms too. And the point is that these heavier atoms, or at least the heavier atoms, or chemical elements, presently known to us, having atomic number $Z = 1, \ldots, 118$, can be arranged in a table, called periodic table, as follows:

	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	$\frac{\mathrm{H}}{\mathrm{1}}$																		$\frac{\text{He}}{2}$
2	$\frac{\text{Li}}{3}$	$\frac{\mathrm{Be}}{4}$												$\frac{\mathrm{B}}{5}$	$\frac{C}{6}$	$\frac{\mathrm{N}}{\mathrm{7}}$	$\frac{O}{8}$	$\frac{\mathrm{F}}{\mathrm{9}}$	$\frac{\mathrm{Ne}}{10}$
3	$\frac{\text{Na}}{11}$	$\frac{\mathrm{Mg}}{12}$												$\frac{\mathrm{Al}}{13}$	$\frac{\mathrm{Si}}{14}$	$\frac{P}{15}$	$\frac{\mathrm{S}}{16}$	$\frac{\text{Cl}}{17}$	$\frac{\mathrm{Ar}}{18}$
4	$\frac{\mathrm{K}}{\mathrm{19}}$	$\frac{\mathrm{Ca}}{20}$		$\frac{\mathrm{Sc}}{21}$	$\frac{\mathrm{Ti}}{22}$	$\frac{\mathrm{V}}{23}$	$\frac{\mathrm{Cr}}{24}$	$\frac{\mathrm{Mn}}{25}$	$\frac{\text{Fe}}{26}$	$\frac{\text{Co}}{27}$	$\frac{\mathrm{Ni}}{28}$	$\frac{\mathrm{Cu}}{29}$	$\frac{\mathrm{Zn}}{\mathrm{30}}$	$\frac{\text{Ga}}{31}$	$\frac{\text{Ge}}{32}$	$\frac{\mathrm{As}}{33}$	$\frac{\text{Se}}{34}$	$\frac{\mathrm{Br}}{35}$	$\frac{\mathrm{Kr}}{36}$
5	$\frac{\mathrm{Rb}}{37}$	$\frac{\mathrm{Sr}}{38}$		$\frac{Y}{39}$	$\frac{\mathrm{Zr}}{40}$	$\frac{\mathrm{Nb}}{41}$	$\frac{\mathrm{Mo}}{42}$	$\frac{\mathrm{Tc}}{43}$	$\frac{\mathrm{Ru}}{44}$	$\frac{\mathrm{Rh}}{45}$	$\frac{\mathrm{Pd}}{46}$	$\frac{\mathrm{Ag}}{47}$	$\frac{\mathrm{Cd}}{48}$	$\frac{\text{In}}{49}$	$\frac{\mathrm{Sn}}{50}$	$\frac{\mathrm{Sb}}{51}$	$\frac{\text{Te}}{52}$	$\frac{\mathrm{I}}{53}$	$\frac{Xe}{54}$
6	$\frac{\mathrm{Cs}}{55}$	$\frac{\text{Ba}}{56}$	l	$\frac{\mathrm{Lu}}{71}$	$\frac{\mathrm{Hf}}{\mathrm{72}}$	$\frac{\mathrm{Ta}}{73}$	$\frac{W}{74}$	$\frac{\text{Re}}{75}$	$\frac{\mathrm{Os}}{76}$	$\frac{\mathrm{Ir}}{77}$	$\frac{\mathrm{Pt}}{78}$	$\frac{\mathrm{Au}}{\mathrm{79}}$	$\frac{\mathrm{Hg}}{\mathrm{80}}$	$\frac{\mathrm{Tl}}{81}$	$\frac{\mathrm{Pb}}{\mathrm{82}}$	$\frac{\mathrm{Bi}}{83}$	$\frac{\mathrm{Po}}{\mathrm{84}}$	$\frac{\mathrm{At}}{\mathrm{85}}$	$\frac{\mathrm{Rn}}{86}$
7	$\frac{\mathrm{Fr}}{87}$	$\frac{\mathrm{Ra}}{88}$	a	$\frac{\mathrm{Lr}}{103}$	$\frac{\mathrm{Rf}}{104}$	$\frac{\mathrm{Db}}{\mathrm{105}}$	$\frac{Sg}{106}$	$\frac{\mathrm{Bh}}{107}$	$\frac{\rm Hs}{108}$	$\frac{\mathrm{Mt}}{\mathrm{109}}$	$\frac{\mathrm{Ds}}{110}$	$\frac{\mathrm{Rg}}{111}$	$\frac{\mathrm{Cn}}{112}$	$\frac{\mathrm{Nh}}{113}$	$\frac{\mathrm{Fl}}{114}$	$\frac{\mathrm{Mc}}{115}$	$\frac{\mathrm{Lv}}{116}$	$\frac{\mathrm{Ts}}{117}$	<u>Og</u> 118
			l:	$\frac{\text{La}}{57}$	$\frac{\text{Ce}}{58}$	$\frac{\Pr}{59}$	$\frac{\mathrm{Nd}}{60}$	$\frac{\mathrm{Pm}}{61}$	$\frac{\mathrm{Sm}}{62}$	$\frac{\mathrm{Eu}}{63}$	$\frac{\mathrm{Gd}}{64}$	$\frac{\mathrm{Tb}}{65}$	$\frac{\mathrm{Dy}}{66}$	$\frac{\mathrm{Ho}}{67}$	$\frac{\mathrm{Er}}{68}$	$\frac{\mathrm{Tm}}{69}$	$\frac{\mathrm{Yb}}{70}$		
			a:	$\frac{Ac}{89}$	$\frac{\mathrm{Th}}{90}$	$\frac{\mathrm{Pa}}{91}$	$\frac{\mathrm{U}}{92}$	$\frac{\mathrm{Np}}{93}$	$\frac{\mathrm{Pu}}{94}$	$\frac{\mathrm{Am}}{95}$	$\frac{\mathrm{Cm}}{96}$	$\frac{\mathrm{Bk}}{97}$	$\frac{\mathrm{Cf}}{98}$	$\frac{\mathrm{Es}}{99}$	$\frac{\mathrm{Fm}}{100}$	$\frac{\mathrm{Md}}{\mathrm{101}}$	$\frac{\text{No}}{102}$		

Here the horizontal parameter $1, \ldots, 18$ is called the group, and the vertical parameter $1, \ldots, 7$ is called the period. The two rows on the bottom consist of lanthanum ${}_{57}$ La and its followers, called lanthanides, and of actinium ${}_{89}$ Ac and its followers, called actinides. These are to be inserted in the main table, where indicated, lanthanides between barium ${}_{56}$ Ba and lutetium ${}_{71}$ Lu, and actinides between radium ${}_{88}$ Ra and lawrencium ${}_{103}$ Lr.

Thus, the periodic table, when correctly drawn, but no one does that because of obvious typographical reasons, is in fact a 7×32 table. Note here that, according to our 7×18 convention, which is the standard one, lanthanides and actinides don't have a group number $1, \ldots, 18$. Their group is by definition "lanthanides" and "actinides".

4b. Solar light

Many things can be said about the chemical elements, and their various properties, and we have already got a bit into this, at the end of the previous chapter, when talking about fire. In what concerns us, in this chapter, in relation with Solar light, we will be mostly interested in the good old hydrogen $_1$ H, and in helium $_2$ He too.

The point indeed is that hydrogen $_1$ H, but then also helium $_2$ He, and heavier elements too, can be subject to nuclear fusion, and this is what makes our Sun work, and shine.

The numerics of the sunlight are, as you might know, quite frightening.

4c. Einstein, relativity

As a continuation of the above, let us discuss the relativity theory of Einstein. This will bring answers to many questions about light, that we still have left.

Based on experiments by Fizeau, then Michelson-Morley and others, to be described in a moment, and by some physics by Maxwell and Lorentz too, in relation with the above, that we will discuss later too, Einstein came upon the following principles:

FACT 4.1 (Einstein principles). The following happen:

- (1) Light travels in vacuum at a finite speed, $c < \infty$.
- (2) This speed c is the same for all inertial observers.
- (3) In non-vacuum, the light speed is lower, v < c.
- (4) Nothing can travel faster than light, $v \neq c$.

The point now is that, obviously, something is wrong here. Indeed, assuming for instance that we have a train, running in vacuum at speed v > 0, and someone on board

lights a flashlight * towards the locomotive, then an observer \circ on the ground will see the light travelling at speed c + v > c, which is a contradiction:



Summarizing, Fact 4.1 implies c + v = c, so contradicts classical mechanics, which therefore needs a fix. But the fix in 1D is straightforward, as follows:

THEOREM 4.2. If we sum the speeds according to the Einstein formula

$$u +_e v = \frac{u + v}{1 + uv}$$

in c = 1 units, then the Galileo formula still holds, approximately, for low speeds

 $u +_e v \simeq u + v$

and if we have u = 1 or v = 1, the resulting sum is $u +_e v = 1$.

PROOF. All this is self-explanatory, and clear from definitions, and with the Einstein formula of $u +_e v$ itself being just the obvious solution to our c + v = c puzzle. To be more precise, if we plug in u = 1 in the above summation formula, we obtain as result:

$$1 +_e v = \frac{1+v}{1+v} = 1$$

Thus, we are led to the conclusion in the statement.

Importantly, all this agrees with the following key experiment of Fizeau:

EXPERIMENT 4.3 (Fizeau, 1851). Assume that light moves through a liquid at speed u < c. Then, when this liquid moves through a tube at speed v > 0,



the observed speed of light is not the Galilean $u +_g v = u + v$, but rather

$$u +_f v = u + v \left(1 - \frac{1}{n^2} \right)$$

where n = c/u is the index of refraction of the liquid.

You must agree with me that this looks very good, especially in what regards the first part, with the observed speed by Fizeau being not the Galilean one, $u +_f v \neq u +_g v$, but then with the second part too, with Fizeau's sum $u +_f v$ looking quite similar to the Einstein sum $u +_e v$. So, let us do now the math, and compare what Fizeau and Einstein say. The result here, which is certainly a success, if you are a bit familiar with the difficulties of experimental physics, say via daily cooking at home, is as follows:

THEOREM 4.4. The Fizeau speed summation, which in c = 1 units is

$$u +_{f} v = u + v - u^{2}v \simeq (u + v)(1 - uv)$$

is compatible with the Einstein speed summation, which in c = 1 units is

$$u +_e v = \frac{u+v}{1+uv} \simeq (u+v)(1-uv)$$

with the approximations coming from u >> v, and from $1/(1+x) \simeq 1-x$.

PROOF. This is something rather self-explanatory, but let us work out the details. In c = 1 units the index of refraction of the liquid is n = 1/u, and we have:

$$u +_{f} v = u + v \left(1 - \frac{1}{(1/u)^{2}} \right)$$

= $u + v (1 - u^{2})$
= $u + v - u^{2}v$
 $\simeq u + v - u^{2}v - uv^{2}$
= $(u + v)(1 - uv)$

To be more precise here, we have used, for the approximation at the end:

 $v << u \implies uv^2 << u, v, u^2v$

As for the processing of the Einstein formula, this simply uses $1/(1+x) \simeq 1-x$. \Box

Getting back now to the formula from Theorem 4.2, that formula, while looking very simple, is in fact quite subtle, and must be handled with care. Indeed, we have:

THEOREM 4.5. The Einstein speed summation, written in c = 1 units as

$$u +_e v = \frac{u + v}{1 + uv}$$

has the following properties:

- (1) u, v < 1 implies $u +_e v < 1$.
- (2) $u +_e v = v +_e u$.
- (3) $(u +_e v) +_e w = u +_e (v +_e w).$
- (4) However, $\lambda u +_e \lambda v = \lambda (u +_e v)$ fails.

PROOF. All these assertions are elementary, as follows:

(1) This follows from the following formula, valid for any speeds u, v:

$$1 - u +_e v = 1 - \frac{u + v}{1 + uv} = \frac{(1 - u)(1 - v)}{1 + uv}$$

Indeed, we deduce from this that u, v < 1 implies $u +_e v < 1$, as claimed.

(2) This is clear, coming from the following observation:

$$u +_e v = \frac{u + v}{1 + uv} = \frac{v + u}{1 + vu} = v +_e u$$

(3) We have indeed the following computation, for the first sum:

$$(u +_e v) +_e w = \frac{\frac{u+v}{1+uv} + w}{1 + \frac{u+v}{1+uv} \cdot w} = \frac{u+v+w+uvw}{1+uv+uw+vw}$$

As for the second sum, this is as follows, given by the same formula:

$$u +_{e} (v +_{e} w) = \frac{u + \frac{v + w}{1 + vw}}{1 + u \cdot \frac{v + w}{1 + vw}} = \frac{u + v + w + uvw}{1 + uv + uw + vw}$$

(4) This is clear, with the remark however that it works at $\lambda = -1, 0, 1$, or when u = 0, v = 0, or u + v = 0. To be more precise, we have the following computation:

$$\begin{aligned} \lambda u +_e \lambda v &= \lambda (u +_e v) &\iff \frac{\lambda u + \lambda v}{1 + \lambda^2 u v} = \lambda \cdot \frac{u + v}{1 + u v} \\ &\iff \lambda (u + v) = 0, \text{ or } \lambda^2 u v = u v \\ &\iff \lambda = 0, \text{ or } u + v = 0, \text{ or } \lambda^2 = 1, \text{ or } u v = 0 \end{aligned}$$

Thus, we are led to the above conclusions.

All the above is very nice, but remember, takes place in 1D. So, time now to get seriously to work, and see what all this becomes in 3D. And here, we have:

THEOREM 4.6. When defining the Einstein speed summation in 3D as

$$u +_{e} v = \frac{1}{1 + \langle u, v \rangle} \left(u + v + \frac{u \times (u \times v)}{1 + \sqrt{1 - ||u||^2}} \right)$$

in c = 1 units, the following happen:

- (1) When $u \sim v$, we recover the previous 1D formula.
- (2) We have $||u||, ||v|| < 1 \implies ||u +_e v|| < 1.$
- (3) When ||u|| = 1, we have $u +_e v = u$.
- (4) When ||v|| = 1, we have $||u +_e v|| = 1$.
- (5) However, ||v|| = 1 does not imply $u +_e v = v$.
- (6) Also, the formula $u +_e v = v +_e u$ fails.

PROOF. As before with the 1D formula, this is something that can be guessed, as the simplest solution to our c + v = c puzzle, and will leave this guessing part as an instructive, long exercise. As for the various assertions, their proof goes as follows:

(1) This is because $u \sim v$ implies $u \times v = 0$, so the correction term disappears.

(2) In order to simplify notation, let us set $\delta = \sqrt{1 - ||u||^2}$, which is the inverse of the quantity $\gamma = 1/\sqrt{1 - ||u||^2}$. With this convention, we have:

$$u_{+e} v = \frac{1}{1+\langle u, v \rangle} \left(u + v + \frac{\langle u, v \rangle u - ||u||^2 v}{1+\delta} \right)$$
$$= \frac{(1+\delta+\langle u, v \rangle)u + (1+\delta-||u||^2)v}{(1+\langle u, v \rangle)(1+\delta)}$$

Taking now the squared norm and computing gives the following formula:

$$||u +_e v||^2 = \frac{(1+\delta)^2 ||u + v||^2 + (||u||^2 - 2(1+\delta))(||u||^2 ||v||^2 - \langle u, v \rangle^2)}{(1+\langle u, v \rangle)^2 (1+\delta)^2}$$

But this formula can be further processed by using $\delta = \sqrt{1 - ||u||^2}$, and by navigating through the various quantities which appear, we obtain, as a final product:

$$||u +_e v||^2 = \frac{||u + v||^2 - ||u||^2 ||v||^2 + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2}$$

But this type of formula is exactly what we need, for what we want to do. Indeed, by assuming ||u||, ||v|| < 1, we have the following estimate:

$$\begin{split} ||u+_e v||^2 < 1 &\iff ||u+v||^2 - ||u||^2 ||v||^2 + \langle u, v \rangle^2 < (1+\langle u, v \rangle)^2 \\ &\iff ||u+v||^2 - ||u||^2 ||v||^2 < 1 + 2 < u, v \rangle \\ &\iff ||u||^2 + ||v||^2 - ||u||^2 ||v||^2 < 1 \\ &\iff (1-||u||^2)(1-||v||^2) > 0 \end{split}$$

Thus, we are led to the conclusion in the statement.

(3) This follows indeed from $u \times (u \times v) = \langle u, v \rangle = \langle u, u \rangle v$.

(4) This comes from the squared norm formula established in the proof of (2) above, because when assuming ||v|| = 1, we obtain:

$$\begin{aligned} ||u +_e v||^2 &= \frac{||u + v||^2 - ||u||^2 + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2} \\ &= \frac{||u||^2 + 1 + 2 \langle u, v \rangle - ||u||^2 + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2} \\ &= \frac{1 + 2 \langle u, v \rangle + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2} \\ &= 1 \end{aligned}$$

(5) This is clear, from the obvious lack of symmetry of our formula.

(6) This is again clear, from the obvious lack of symmetry of our formula.

Very nice, but all the above was just some mathematics, coming by looking for the simplest possible solution to our c + v = c puzzle. And as good news now, coming from various experiments of Fizeau, then Michelson-Morley and others, we have:

FACT 4.7. The above speed addition formulae are correct, physically speaking.

Moving forward, time now to draw some concrete conclusions, from the above speed computations. Since speed v = d/t is distance over time, we must fine-tune distance d, or time t, or both. Let us first discuss, following as usual Einstein, what happens to time t. Here the result, which might seem quite surprising, at a first glance, is as follows:

THEOREM 4.8. Relativistic time is subject to Lorentz dilation

$$t \to \gamma t$$

where the number $\gamma \geq 1$, called Lorentz factor, is given by the formula

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

with v being the moving speed, at which time is measured.

PROOF. Assume indeed that we have a train, moving to the right with speed v, through vacuum. In order to compute the height h of the train, the passenger onboard switches on the ceiling light bulb, measures the time t that the light needs to hit the floor, by travelling at speed c, and concludes that the train height is h = ct:



On the other hand, an observer on the ground will see here something different, namely a right triangle, with on the vertical the height of the train h, on the horizontal the distance vT that the train has traveled, and on the hypotenuse the distance cT that light has travelled, with T being the duration of the event, according to his watch:



Now by Pythagoras applied to this triangle, we have:

$$h^2 + (vT)^2 = (cT)^2$$

Thus, the observer on the ground will reach to the following formula for h:

$$h = \sqrt{c^2 - v^2} \cdot T$$

But h must be the same for both observers, so we have the following formula:

$$\sqrt{c^2 - v^2} \cdot T = ct$$

It follows that the two times t and T are indeed not equal, and are related by:

$$T = \frac{ct}{\sqrt{c^2 - v^2}} = \frac{t}{\sqrt{1 - v^2/c^2}} = \gamma t$$

Thus, we are led to the formula in the statement.

Let us discuss now what happens to length. Intuitively, since speed is distance/time, and since time gets dilated, we can somehow expect distance to get dilated too. However, this is wrong, and after due thinking and computations, what we have is in fact:

THEOREM 4.9. Relativistic length is subject to Lorentz contraction

$$L \to L/\gamma$$

where the number $\gamma \geq 1$, called Lorentz factor, is given by the usual formula

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

with v being the moving speed, at which length is measured.

PROOF. As before in the proof of Theorem 4.8, meaning in the same train travelling at speed v, in vacuum, imagine now that the passenger wants to measure the length L of the car. For this purpose he switches on the light bulb, now at the rear of the car, and

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measures the time t needed for the light to reach the front of the car, and get reflected back by a mirror installed there, according to the following scheme:



He concludes that, as marked above, the length L of the car is given by:

$$L = \frac{ct}{2}$$

Now viewed from the ground, the duration of the event is $T = T_1 + T_2$, where $T_1 > T_2$ are respectively the time needed for the light to travel forward, among others for beating v, and the time for the light to travel back, helped this time by v. More precisely, if l denotes the length of the train car viewed from the ground, the formula of T is:

$$T = T_1 + T_2 = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2lc}{c^2 - v^2}$$

With this data, the formula $T = \gamma t$ of time dilation established before reads:

$$\frac{2lc}{c^2 - v^2} = \gamma t = \frac{2\gamma L}{c}$$

Thus, the two lengths L and l are indeed not equal, and related by:

$$l = \frac{\gamma L(c^2 - v^2)}{c^2} = \gamma L\left(1 - \frac{v^2}{c^2}\right) = \frac{\gamma L}{\gamma^2} = \frac{L}{\gamma}$$

Thus, we are led to the conclusion in the statement.

With this discussed, time to get to the real thing, see what happens to our usual \mathbb{R}^4 . The result here, which is something fundamental, based on the above, is as follows:

THEOREM 4.10. In the context of a relativistic object moving with speed v along the x axis, the frame change is given by the Lorentz transformation

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^{2})$$

with $\gamma = 1/\sqrt{1 - v^2/c^2}$ being as usual the Lorentz factor.

PROOF. We know that, with respect to the non-relativistic formulae, x is subject to the Lorentz dilation by γ , and we obtain as desired:

$$x' = \gamma(x - vt)$$

Regarding y, z, these are obviously unchanged, so done with these too. Finally, regarding time t, we can use here the reverse Lorentz transformation, given by:

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$

By using the formula of x' we can compute t', and we obtain the following formula:

$$t' = \frac{x - \gamma x'}{\gamma v}$$
$$= \frac{x - \gamma^2 (x - vt)}{\gamma v}$$
$$= \frac{\gamma^2 vt + (1 - \gamma^2) x}{\gamma v}$$

On the other hand, we have the following computation:

$$\gamma^2 = \frac{c^2}{c^2 - v^2} \implies \gamma^2(c^2 - v^2) = c^2 \implies (\gamma^2 - 1)c^2 = \gamma^2 v^2$$

Thus we can finish the computation of t' as follows:

$$t' = \frac{\gamma^2 v t + (1 - \gamma^2) x}{\gamma v}$$
$$= \frac{\gamma^2 v t - \gamma^2 v^2 x / c^2}{\gamma v}$$
$$= \gamma \left(t - \frac{v x}{c^2} \right)$$

We are therefore led to the conclusion in the statement.

Now back to electomagnetism, we have the following key result, due to Lorentz:

THEOREM 4.11. The Maxwell equations are invariant under Lorentz transformations

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^{2})$$

with $\gamma = 1/\sqrt{1-v^2/c^2}$ being as usual the Lorentz factor.

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PROOF. As a first comment, this result, due to Lorentz, working on electromagnetism, was established some time before Einstein's relativity theory, and provided strong evidence for that. As for the proof, consider an electromagnetic field (E, B). This is altered by a Lorentz transformation into a field (E', B'), the equations for E' being as follows:

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$$E'_{x} = E_{x}$$
$$E'_{y} = \gamma(E_{y} - vB_{z})$$
$$E'_{z} = \gamma(E_{z} + vB_{y})$$

As for the equations of B', these are quite similar, as follows:

$$B'_{x} = B_{x}$$
$$B'_{y} = \gamma \left(B_{y} + \frac{v}{c^{2}} E_{z} \right)$$
$$B'_{z} = \gamma \left(B_{z} - \frac{v}{c^{2}} E_{y} \right)$$

In order to do the math, consider the following matrices, with $\beta = v/c$ as usual:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix} , \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta\gamma \\ 0 & \beta\gamma & 0 \end{pmatrix}$$

In terms of these matrices, the formulae for the new field (E', B') read:

$$E' = DE + cMB$$
$$B' = DB - \frac{M}{c}E$$

But this is already not that bad, and starting from these formulae, it is possible to prove that (E', B') satisfies as well the Maxwell equations, as desired.

4d. Atomic bombs

Let us discuss now mass and energy, in the relativistic context. Things here are quite tricky, and as a first objective we would like to fix the momentum conservation equations for the plastic collisions, from classical mechanics, namely:

$$m = m_1 + m_2$$
$$mv = m_1v_1 + m_2v_2$$

This cannot really be done with bare hands, and by this meaning with mathematics only, but with some help from experiments, the conclusion is as follows:

FACT 4.12. When defining the relativistic mass of an object of rest mass m > 0, moving at speed v, by the formula

$$M = \gamma m \quad : \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

this relativistic mass M, and the corresponding relativistic momentum P = Mv, are both conserved during collisions.

In other words, the situation here is a bit similar to that of the Galileo addition vs Einstein addition for speeds. The collision equations given above are in fact low-speed approximations of the correct, relativistic equations, which are as follows:

$$M = M_1 + M_2$$
$$Mv = M_1v_1 + M_2v_2$$

With his done, it remains to discuss kinetic energy. You have certainly heard of the formula
$$E = mc^2$$
, which might actually well be on your T-shirt, now as you read this book, and in this case here is the explanation for it, in relation with the above:

THEOREM 4.13. The relativistic energy of an object of rest mass m > 0,

$$\mathcal{E} = Mc^2$$
 : $M = \gamma m$

which is conserved, as being a multiple of M, can be written as $\mathcal{E} = E + T$, with

$$E = mc^2$$

being its v = 0 component, called rest energy of m, and with

$$T = (1 - \gamma)mc^2 \simeq \frac{mv^2}{2}$$

being called relativistic kinetic energy of m.

PROOF. All this is a bit abstract, coming from Fact 4.12, as follows:

(1) Given an object of rest mass m > 0, consider its relativistic mass $M = \gamma m$, as appearing in Fact 4.12, and then consider the following quantity:

$$\mathcal{E} = Mc^2$$

We know from Fact 4.12 that the relativistic mass M is conserved, so $\mathcal{E} = Mc^2$ is conserved too. In view of this, is makes somehow sense to call \mathcal{E} energy. There is of course no clear reason for doing that, but let's just do it, and we'll understand later.

(2) Let us compute \mathcal{E} . This quantity is by definition given by:

$$\mathcal{E} = Mc^2 = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

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Since $1/\sqrt{1-x} \simeq 1 + x/2$ for x small, by calculus, we obtain, for v small:

$$\mathcal{E} \simeq mc^2 \left(1 + \frac{v^2}{2c^2} \right) = mc^2 + \frac{mv^2}{2}$$

And, good news here, we recognize at right the kinetic energy of m.

(3) But this leads to the conclusions in the statement. Indeed, we are certainly dealing with some sort of energies here, and so calling the above quantity \mathcal{E} relativistic energy is legitimate, and calling $E = mc^2$ rest energy is legitimate too. Finally, the difference between these two energies $T = \mathcal{E} - E$ follows to be given by:

$$T = (1 - \gamma)mc^2 \simeq \frac{mv^2}{2}$$

Thus, calling T relativistic kinetic energy is legitimate too, and we are done. \Box

Now let us study a bit more the main energy formula above, $E = mc^2$, and try to understand its numerics, and consequences. At the terrestrial level, we have:

THEOREM 4.14. An atomic bomb based on a glass of water releasing all its $E = mc^2$ energy is equivalent to the Giza Pyramid hitting you at 30 km/s.

PROOF. When converting $E = mc^2$ into kinetic energy of a body M, the formula is:

$$mc^2 = \frac{Mv^2}{2} \implies v = \sqrt{\frac{2m}{M}}c$$

In our case the glass of water m, glass included, is about 300 grams, and the Giza Pyramid M is about 6 million tons. Thus the impact speed is:

$$v = \sqrt{\frac{2 \times 3 \times 10^{-1}}{6 \times 10^9}} c = \frac{c}{10^5} = 30,000 \ m/s$$

As already mentioned before, relax, and do not worry about this. We will see however in a moment, that certain atomic bombs, based on other materials, can however be constructed. As for the glass of water, as long as it stays far away from a big source of energy, of stellar or atomic type, it will certainly not explode. \Box

Summarizing, we have some evidence here for the existence of atomic bombs, having mighty power, and producing a lot of light. But, how to construct such bombs?

In order to discuss this, let us go back to the periodic table, and look for radioactive elements. The story here is long, twisted, and particularly fascinating:

(1) The fact that uranium $_{92}$ U, which is not that uncommon in the earth, at certain places, is naturally radioactive has been known since ages, with various items such as pottery being decorated with uranium based paint, as to glow in the dark.

(2) Official science started to investigate this phenomenon quite late, at the very end of the 19th century, with the work of Henri Becquerel and Marie Curie on uranium salts. A bit later, many others joined, and there was particular excitement in regards with polonium $_{84}$ Po and radium $_{88}$ Ra. In fact, looking a bit retrospectively, there are 8 culprits of this type, relatively light chemical elements having no stable isotopes, namely technetium, promethium, bismuth, polonium, astatine, radon, francium and radium:

 $_{43}$ Tc $_{61}$ Pm $_{83}$ Bi $_{84}$ Po $_{85}$ At $_{86}$ Rn $_{87}$ Fr $_{88}$ Ra

(3) As for uranium itself, this is part of a series of 15 heavy elements, called actinides, coming right after radium in the periodic table, which are actinium, thorium, protactinium, uranium, neptunium, plutonium, americium, curium, berkelium, californium, einsteinium, fermium, mendelevium, nobelium and lawrencium:

$$_{89}Ac \quad _{90}Th \quad _{91}Pa \quad _{92}U \quad _{93}Np \quad _{94}Pu \quad _{95}Am \quad _{96}Cm \\ _{97}Bk \quad _{98}Cf \quad _{99}Es \quad _{100}Fm \quad _{101}Md \quad _{102}No \quad _{103}Lr$$

(4) All this, notably with polonium, radium and uranium, was happening at the beginning of the 20th century, roughly at the same when Rydberg, Planck, Einstein, later joined by Bohr, Thomson, Rutherford and many others were trying to make some sense of quantum mechanics, and develop a reasonable atomic theory. And this soon became job done, with Heisenberg and then Schrödinger, later joined by De Broglie, Pauli, Dirac and others, developing a mathematical theory of quantum mechanics, and proving the Bohr model for the hydrogen atom, and in fact for all the other atoms too, true.

(5) However, the mechanism of radioactivity proved to be much harder to understand than the basic atomic functioning itself, with the reason for this coming from the fact that, in contrast to the latter which deals with a modified version of electromagnetism, negative electrons spinning around a positive nucleus, the former involves the nucleus only, and some subtle forces keeping it together, namely the weak and strong force. Understanding these two new forces, and putting them in the context of quantum mechanics, required a lot of new physics, coming from numerous civilian and military experiments, around WW2 and afterwards, and a lot of theoretical efforts too, with the whole story being over, if ever, in the 1970s, with the Standard Model for particle physics.

(6) Despite all this, the open questions remain numerous, especially in what concerns the continuation of the periodic table, beyond actinides. Here there are presently known 15 more elements, called super-heavy, which are rutherfordium, dubnium, seaborgium, bohrium, hassium, meitnerium, darmstadtium, roentgenium, copernicium, nihonium, flerovium, moscovium, livermorium, tennessine and oganesson:

$_{104}\mathrm{Rf}$	$_{105}\mathrm{Db}$	$_{106}\mathrm{Sg}$	$_{107}\mathrm{Bh}$	$_{108}\mathrm{Hs}$	$_{109}\mathrm{Mt}$	$_{110}\mathrm{Ds}$	$_{111}\mathrm{Rg}$
	$_{112}Cn$	$_{113}\mathrm{Nh}$	$_{114}\mathrm{Fl}$	$_{115}\mathrm{Mc}$	$_{116}\mathrm{Lv}$	$_{117}\mathrm{Ts}$	$_{118}\mathrm{Og}$

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(7) To be more precise, a first challenging problem is that of understanding the chemistry of these elements, which are too short-lived for chemistry experiments, and with the known theoretical models being too complex. As for the other challenging problem, this is of course going beyond oganesson, either with experiments, or with theory.

In order to understand the mathematics of radioactivity, and nuclear fission, we need to know more about particle decay. Let us start with something basic, as follows:

THEOREM 4.15. In the context of decay, the quantity to look at is the decay rate λ , which is the probability per unit time that the particle will disintegrate. With this:

- (1) The number of particles remaining at time t > 0 is $N_t = e^{-\lambda t} N_0$.
- (2) The mean lifetime of a particle is $\tau = 1/\lambda$.
- (3) The half-life of the substance is $t_{1/2} = (\log 2)/\lambda$.

PROOF. As said above, this is basic probability, as follows:

(1) In mathematical terms, our definition of the decay rate reads:

$$\frac{dN}{dt} = -\lambda N$$

By integrating, we are led to the formula in the statement, namely:

$$N_t = e^{-\lambda t} N_0$$

(2) Let us first convert what we have into a probability law. We have:

$$\int_0^\infty N_t dt = \int_0^\infty N_0 e^{-\lambda t} dt = \frac{N_0}{\lambda}$$

Thus, the density of the probability decay function is given by:

$$f(t) = \frac{\lambda}{N_0} \cdot N_0 e^{-\lambda t} = \lambda e^{-\lambda t}$$

We can now compute the mean lifetime, by integrating by parts, as follows:

$$\tau = \langle t \rangle$$

$$= \int_{0}^{\infty} tf(t)dt$$

$$= \int_{0}^{\infty} \lambda t e^{-\lambda t} dt$$

$$= \int_{0}^{\infty} t(-e^{-\lambda t})' dt$$

$$= \int_{0}^{\infty} e^{-\lambda t} dt$$

$$= \frac{1}{\lambda}$$

(3) Finally, regarding the half-life, this is by definition the time $t_{1/2}$ required for the decaying quantity to fall to one-half of its initial value. Mathematically, this means:

$$N_t = 2^{-\frac{\iota}{t_{1/2}}} N_0$$

Now by comparing with $N_t = e^{-\lambda t} N_0$, this gives $t_{1/2} = (\log 2)/\lambda$, as stated.

Now back to elementary particles, as a main principle regarding particle decay, following Fermi and others, we have the following simple and useful fact:

PRINCIPLE 4.16 (Fermi Golden Rule). In the context of a particle physics decay, $*_0 \rightarrow *_1 + \ldots + *_n$, the decay rate is given by

$$\lambda = \int |M|^2 dp$$

with $M = M(p_0, \ldots, p_n)$ being the amplitude of the interaction, and with the integration being restricted to the part of the phase space allowed by basic physics.

Obviously, several things going on here, that will take us some time, to understand. To start with, the above Golden Rule looks quite reasonable, namely getting λ by integrating something on the phase space. It remains to understand two things, namely what the formula of the amplitude M is, and where does the integration exactly take place.

Leaving the formula of the amplitude M for later, let us try to answer the second question, regarding the allowed phase space. According to the Golden Rule, that is simply the phase space allowed by basic physics, and here that basic physics is:

ADDENDUM 4.17. In the above context, the basic physics is as follows:

- (1) The total energy and momentum must be conserved.
- (2) Each outgoing particle must keep its mass constant.
- (3) Each outgoing particle must have positive energy.

Summarizing, all common sense things that we have here. In mathematical terms now, it is better to add integrands corresponding to the above conditions (1,2,3), instead of exactly specifying the allowed state space. And with both (1) and (2) requiring Dirac masses δ , and with (3) requiring a Heaviside function $H = \chi_{(0,\infty)}$, we are led to:

PRINCIPLE 4.18 (Golden Rule 2). In the context of a particle physics decay, $*_0 \rightarrow *_1 + \ldots + *_n$, the decay rate is given by

$$\lambda = \int |M|^2 \delta\left(p_0 - \sum_{i=1}^n p_i\right) \prod_{i=1}^n \delta(p_i^2 - m_i^2 c^2) H(p_i^0) dp$$

with $M = M(p_0, \ldots, p_n)$ being the amplitude of the interaction.
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Which looks quite neat, but there is actually a subtlety here, in relation with the Dirac masses, which take as arguments squares of variables, instead of the variables themselves. In order to clarify this, let us make the following computation, with a > 0:

$$\begin{split} \int_{\mathbb{R}} f(x)\delta(x^2 - a^2)dx &= \int_{-\infty}^0 f(x)\delta(x^2 - a^2)dx + \int_0^\infty f(x)\delta(x^2 - a^2)dx \\ &= \int_{-\infty}^a f(y - a)\delta(y^2 - 2ay)dy + \int_{-a}^\infty f(y + a)\delta(y^2 + 2ay)dy \\ &\simeq \int_{-\infty}^a f(y - a)\delta(-2ay)dy + \int_{-a}^\infty f(y + a)\delta(2ay)dy \\ &= \int_{-\infty}^{2a^2} f\left(\frac{z}{2a} - a\right)\delta(-z)\frac{dz}{2a} + \int_{-2a^2}^\infty f\left(\frac{z}{2a} + a\right)\delta(z)\frac{dz}{2a} \\ &= \frac{f(-a)}{2a} + \frac{f(a)}{2a} \\ &= \int_R f(x)\frac{\delta(x - a) + \delta(x + a)}{2a}dx \end{split}$$

Sounds like physics, you would say, and in answer, yes physics that is, but in any case, we have in this way the definition for our quadratic Dirac masses, as follows:

$$\delta(x^2 - a^2) = \frac{\delta(x - a) + \delta(x + a)}{2a}$$

With this understood, and before getting into what the amplitude M is, let us make some normalizations. Here these normalizations are, and you will have to believe me here, all of them are made for good reasons, as we will discover in a moment:

(1) We have $\lambda \sim S$, with $S = 1/\prod_i (m_i!)$, where $m_i \in \mathbb{N}$ with $\sum m_i = n$ are the multiplicities of the output particles, and it is better to leave S outside the integral.

(2) Also, $\lambda \sim 1/(2hm_0)$, with *h* being as usual the reduced Planck constant, and m_0 being the initial mass, and it is better to leave $1/(2hm_0)$ outside the integral too.

(3) Each Dirac mass δ behaves better in computations when multiplied by a 2π factor. Also, each individual dp_i symbol behaves better when divided by a 2π factor.

Now by doing all these normalizations, which amounts in correspondingly rescaling the amplitude M, and with this being certainly not a big deal, because we haven't even talked yet about what this amplitude M is, so free to do this, we are led to:

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PRINCIPLE 4.19 (Golden Rule 3). In the context of a particle physics decay, $*_0 \rightarrow *_1 + \ldots + *_n$, the decay rate is given by

$$\lambda = \frac{S}{2hm_0} \int |\mathcal{M}|^2 (2\pi)^4 \delta \left(p_0 - \sum_{i=1}^n p_i \right) \prod_{i=1}^n 2\pi \delta(p_i^2 - m_i^2 c^2) H(p_i^0) \frac{dp_i}{(2\pi)^4}$$

with $\mathcal{M} = \mathcal{M}(p_0, \ldots, p_n)$ being the normalized amplitude of the interaction, and with $S = 1/\prod_i (m_i!)$, where $m_i \in \mathbb{N}$ with $\sum m_i = n$ are the multiplicities of the output.

And good news, this will be normally the final form of the Golden Rule for decays, that we will be using, in what follows. In practice, however, we will see in a moment that the integration with respect to time is easy to perform, and this will lead to yet another formulation of the Golden Rule, which is the most useful one, for applications.

Before that, however, some philosophical comments. The Golden Rule has become now something quite complicated, and there is still a discussion about \mathcal{M} , which will certainly bring its part of complicated mathematics. But remember that, in the end, everything comes from Principle 4.16, which is something quite simple. So, no fear.

This being said, even when looking at Principle 4.16, you might wonder, is that really correct, and where that really comes from. In answer, common sense as explained above, then lots of experiments too, confirming it, or rather confirming the formula of \mathcal{M} , that we haven't talked about yet, and finally quantum field theory, which is something advanced, that can actually prove this Golden Rule, starting from simple principles.

Back to our business now, we will take Principle 4.19 for granted, and further build on it, with examples, the formula of \mathcal{M} , and more. Before that, however, let us do what was suggested above, namely integrating with respect to time. This leads to:

THEOREM 4.20 (Golden Rule 4). In the context of a particle physics decay, $*_0 \rightarrow *_1 + \ldots + *_n$, the decay rate is given, in standard $\tilde{p} = (E/c, p)$ notation, by

$$\lambda = \frac{S}{2hm_0} \int |\mathcal{M}|^2 (2\pi)^4 \delta \left(\tilde{p}_0 - \sum_{i=1}^n \tilde{p}_i \right) \prod_{i=1}^n \frac{1}{2\sqrt{||p_i||^2 + m_i^2 c^2}} \cdot \frac{dp_i}{(2\pi)^3}$$

with $\mathcal{M} = \mathcal{M}(p_0, \ldots, p_n)$ being the normalized amplitude, $S = 1/\prod_i (m_i!)$ being the statistical factor, and with the convention $E_i/c = \sqrt{||p_i||^2 + m_i^2 c^2}$, both in \mathcal{M} and δ .

PROOF. We use the general formula from Principle 4.19, which is as follows:

$$\lambda = \frac{S}{2hm_0} \int |\mathcal{M}|^2 (2\pi)^4 \delta\left(\tilde{p}_0 - \sum_{i=1}^n \tilde{p}_i\right) \prod_{i=1}^n 2\pi \delta(\tilde{p}_i^2 - m_i^2 c^2) H\left(\frac{E_i}{c}\right) \frac{d\tilde{p}_i}{(2\pi)^4}$$

4D. ATOMIC BOMBS

In order to further process this formula, let us look at each of the n products on the right. According to our conventions for quadratic Dirac masses, explained after Principle 4.18, the Dirac mass appearing there is given by the following formula:

$$\begin{split} \delta(\tilde{p}_i^2 - m_i^2 c^2) &= \delta\left(\frac{E_i^2}{c^2} - ||p_i||^2 - m_i^2 c^2\right) \\ &= \delta\left(\left(\frac{E_i}{c}\right)^2 - \left(\sqrt{||p_i||^2 + m_i^2 c^2}\right)^2\right) \\ &= \frac{\delta\left(\frac{E_i}{c} - \sqrt{||p_i||^2 + m_i^2 c^2}\right) + \delta\left(\frac{E_i}{c} + \sqrt{||p_i||^2 + m_i^2 c^2}\right)}{2\sqrt{||p_i||^2 + m_i^2 c^2}} \end{split}$$

Thus we have two possibilities, and since the Heaviside term $H(E_i/c)$ equals 1 on the first one, and vanishes on the second one, we are led to the following formula:

$$\delta(\tilde{p}_i^2 - m_i^2 c^2) H\left(\frac{E_i}{c}\right) = \frac{\delta\left(\frac{E_i}{c} - \sqrt{||p_i||^2 + m_i^2 c^2}\right)}{2\sqrt{||p_i||^2 + m_i^2 c^2}}$$

But this leads to the conclusion in the statement.

As an illustration, for two-particle decays many things simplify, and we have:

THEOREM 4.21. For two-particle decays, $*_0 \rightarrow *_1 + *_2$, the Golden Rule reads

$$\lambda = \frac{S||p||}{8\pi h m_0^2 c} \, |\mathcal{M}|^2$$

with \mathcal{M} being the amplitude, ||p|| being the magnitude of either outgoing momentum,

$$||p|| = \frac{c}{2m_0}\sqrt{m_0^4 + m_1^4 + m_2^4 - 2m_0^2m_1^2 - 2m_0^2m_2^2 - 2m_1^2m_2^2}$$

and the statistical factor being S = 1 if $*_1 \neq *_2$, and S = 1/2 if $*_1 = *_2$.

PROOF. In the case of two-particle decays, the formula in Theorem 4.20 takes the following form, with the statistical factor S being the one in the statement:

$$\lambda = \frac{S}{2hm_0} \int |\mathcal{M}|^2 (2\pi)^4 \delta\left(\tilde{p}_0 - \tilde{p}_1 - \tilde{p}_2\right) \prod_{i=1}^2 \frac{1}{2\sqrt{||p_i||^2 + m_i^2 c^2}} \cdot \frac{dp_i}{(2\pi)^3}$$
$$= \frac{S}{32\pi^2 hm_0} \int |\mathcal{M}|^2 \frac{\delta\left(\tilde{p}_0 - \tilde{p}_1 - \tilde{p}_2\right)}{\sqrt{||p_1||^2 + m_1^2 c^2} \sqrt{||p_2||^2 + m_2^2 c^2}} \, dp_1 dp_2$$

Let us look now at the Dirac function. This decomposes over components, as follows:

$$\delta(\tilde{p}_0 - \tilde{p}_1 - \tilde{p}_2) = \delta\left(\frac{E_0}{c} - \frac{E_1}{c} - \frac{E_2}{c}\right)\delta(p_0 - p_1 - p_2)$$

4. SOLAR LIGHT

With the particle $*_0$ being supposed to be at rest, we have the following formulae:

$$\frac{E_0}{c} = m_0 c \quad , \quad p_0 = 0$$

On the other hand, recall from Theorem 4.20 that the machinery there leads to:

$$\frac{E_1}{c} = \sqrt{||p_1||^2 + m_1^2 c^2} \quad , \quad \frac{E_2}{c} = \sqrt{||p_2||^2 + m_2^2 c^2}$$

Thus, the above Dirac mass is in fact given by the following formula:

$$\delta\left(\tilde{p}_0 - \tilde{p}_1 - \tilde{p}_2\right) = \delta\left(m_0 c - \sqrt{||p_1||^2 + m_1^2 c^2} - \sqrt{||p_2||^2 + m_2^2 c^2}\right)\delta(p_1 + p_2)$$

Getting back now to the formula of the decay rate, that becomes:

$$\lambda = \frac{S}{32\pi^2 h m_0} \int |\mathcal{M}|^2 \frac{\delta \left(m_0 c - \sqrt{||p_1||^2 + m_1^2 c^2} - \sqrt{||p_2||^2 + m_2^2 c^2} \right)}{\sqrt{||p_1||^2 + m_1^2 c^2} \sqrt{||p_2||^2 + m_2^2 c^2}} \times \delta(p_1 + p_2) \, dp_1 dp_2$$

Since we must have $p_2 = -p_1$, this expression further simplifies to:

$$\lambda = \frac{S}{32\pi^2 h m_0} \int |\mathcal{M}|^2 \frac{\delta \left(m_0 c - \sqrt{||p_1||^2 + m_1^2 c^2} - \sqrt{||p_1||^2 + m_2^2 c^2} \right)}{\sqrt{||p_1||^2 + m_1^2 c^2} \sqrt{||p_1||^2 + m_2^2 c^2}} \, dp_1$$

In spherical coordinates, this expression takes the following form:

$$\lambda = \frac{S}{32\pi^2 h m_0} \int |\mathcal{M}|^2 \frac{\delta \left(m_0 c - \sqrt{r^2 + m_1^2 c^2} - \sqrt{r^2 + m_2^2 c^2} \right)}{\sqrt{r^2 + m_1^2 c^2} \sqrt{r^2 + m_2^2 c^2}} r^2 \sin s \, dr ds dt$$

The point now is that by physics, to be explained later, the amplitude must be of the form $\mathcal{M} = \mathcal{M}(r)$. Thus the angular integrals contribute with factors as follows:

$$\int_0^\pi \sin s \, ds = 2 \quad , \quad \int_0^{2t} dt = 2\pi$$

We conclude that in the end we are left with a real integral, over r, as follows:

$$\lambda = \frac{S}{8\pi h m_0} \int_0^\infty |\mathcal{M}|^2 \frac{\delta \left(m_0 c - \sqrt{r^2 + m_1^2 c^2} - \sqrt{r^2 + m_2^2 c^2} \right)}{\sqrt{r^2 + m_1^2 c^2} \sqrt{r^2 + m_2^2 c^2}} r^2 dr$$

In order to compute this integral, consider the following variable:

$$u = \sqrt{r^2 + m_1^2 c^2} + \sqrt{r^2 + m_2^2 c^2}$$

Now observe that by differentiating, we obtain the following formula:

$$\begin{aligned} \frac{du}{dr} &= \frac{2r}{2\sqrt{r^2 + m_1^2 c^2}} + \frac{2r}{2\sqrt{r^2 + m_2^2 c^2}} \\ &= r\left(\frac{1}{\sqrt{r^2 + m_1^2 c^2}} + \frac{1}{\sqrt{r^2 + m_2^2 c^2}}\right) \\ &= \frac{ur}{\sqrt{r^2 + m_1^2 c^2}\sqrt{r^2 + m_2^2 c^2}} \end{aligned}$$

Thus, in terms of this new variable u, we have the following formula:

$$\lambda = \frac{S}{8\pi h m_0} \int_{m_1 c + m_2 c}^{\infty} |\mathcal{M}|^2 \delta(m_0 c - u) \frac{r}{u} du$$
$$= \frac{Sr}{8\pi h m_0^2 c} |\mathcal{M}|^2$$

Here in the last formula r stands for the value of the variable r evaluated at the place where the Dirac mass takes the value 1, that we can compute as follows:

$$u = m_0 c \iff \sqrt{r^2 + m_1^2 c^2} + \sqrt{r^2 + m_2^2 c^2} = m_0 c$$

$$\iff r^2 + m_1^2 c^2 = r^2 + m_2^2 c^2 + m_0^2 c^2 - 2m_0 c \sqrt{r^2 + m_2^2 c^2}$$

$$\iff 2m_0 \sqrt{r^2 + m_2^2 c^2} = (m_0^2 - m_1^2 + m_2^2) c$$

$$\iff 4m_0^2 (r^2 + m_2^2 c^2) = (m_0^2 - m_1^2 + m_2^2)^2 c^2$$

$$\iff 4m_0^2 r^2 = ((m_0^2 - m_1^2 + m_2^2)^2 - 4m_0^2 m_2^2) c^2$$

$$\iff r = \frac{c}{2m_0} \sqrt{m_0^4 + m_1^4 + m_2^4 - 2m_0^2 m_1^2 - 2m_0^2 m_2^2 - 2m_1^2 m_2^2}$$

Thus, we are led to the conclusion in the statement.

With the above discussed, it remains to say what the amplitude \mathcal{M} of the interactions is. However, this is something quite tricky. In order to discuss this, it is convenient to enlarge attention to scattering, and to have as starting point the mechanics of the hydrogen atom, with the electron e moving around the proton p. Obviously, this is some

form of scattering, $e + p \rightarrow e + p$, so let us draw right away diagram for this situation:



You might say, end of the story. However, this is wrong. The whole point in advanced quantum mechanics lies in the corrections to the hydrogen atom, and this leads to:

IDEA 4.22. Even for simple situations, like the hydrogen atom $e + p \rightarrow e + p$, the interactions should come in a hierarchic way, with the basic order 0 diagram



being followed by order 1, order 2 and so on diagrams, and with the corresponding amplitude \mathcal{M} being computed accordingly, as a power series in a certain variable α .

In practice now, there is a considerable amount of work lying ahead, for making all this really work. Getting started now, inspired by the above, let us formulate:

DEFINITION 4.23. A Feynman diagram for a multiple scattering and decay event $a_1 + \ldots + a_m \rightarrow b_1 + \ldots + b_m$ is a diagram of type



with c_i , d_i being short-lived particles appearing in the event, and with the middle box being allowed to contain any such configuration of temporary particles too.

4D. ATOMIC BOMBS

Very nice all this, and getting now to work for good, so many things to be done. Let us start with the general recipe, and we will understand later what this really means:

PRINCIPLE 4.24. The amplitude \mathcal{M} coming from a given Feynman diagram F can be computed as follows:

- (1) Label each vertex with the corresponding four-momentum vector.
- (2) Put factors -ig at each vertex, g being the coupling constant.
- (3) Install Feynman propagators $\frac{1}{p_j^2 m_j^2 c^2}$ on each internal line.
- (4) Install rescaled Dirac masses $(2\pi)^4 \delta(\sum_i p_i)$ at each vertex.
- (5) Put integration factors $\frac{dp_j}{(2\pi)^4}$ on each internal line, and integrate.
- (6) Erase the global $(2\pi)^4 \delta(\sum_i p_i)$ factor appearing after integrating.
- (7) Multiply the answer by i. That is your amplitude \mathcal{M} .

Sounds exciting, doesn't it. Obviously, this will take us some time to understand. However, as some preliminary observations, what we are doing is quite simple, namely:

(1) The amplitude \mathcal{M} appears by integrating over the state space, with conservation of energy and momentum being taken into account, at each internal vertex.

(2) What we are integrating are, basically, modulo some rescalings and other mathematical manipulations, the Feynman propagators, on each internal line.

(3) And with these mathematical manipulations including, crucially, the one at the end, namely erasing the final energy and momentum conservation term.

Which looks quite reasonable, physically speaking. Importantly, the coupling constant g, appearing in the above, is something very concrete and numeric, given by:

RULE 4.25. In quantum electrodynamics, the coupling constant is

$$g = \sqrt{4\pi\alpha}$$

with $\alpha \simeq 1/137$ being the fine structure constant.

As an example now for the above, chosen as simple as possible, we have:

THEOREM 4.26. For a two-particle decay $*_0 \rightarrow *_1 + *_2$ the order 0 amplitude is $\mathcal{M} = g$, which gives via the Golden Rule an order 0 decay rate of

$$\lambda = \frac{S||p||\alpha}{2hm_0^2c}$$

with ||p|| being the magnitude of either outgoing momentum,

$$||p|| = \frac{c}{2m_0}\sqrt{m_0^4 + m_1^4 + m_2^4 - 2m_0^2m_1^2 - 2m_0^2m_2^2 - 2m_1^2m_2^2}$$

and the statistical factor being S = 1 if $*_1 \neq *_2$, and S = 1/2 if $*_1 = *_2$.

4. SOLAR LIGHT

PROOF. We know from Theorem 4.21 that for two-particle decays, $*_0 \rightarrow *_1 + *_2$, the Golden Rule takes the following form, with S and ||p|| being as in the statement:

$$\lambda = \frac{S||p||}{8\pi h m_0^2 c} |\mathcal{M}|^2$$

In order to compute the amplitude \mathcal{M} , we use Principle 4.24. At order 0 we only have one Feynman diagram, which is the obvious one, namely:



Now let us apply Principle 4.24. We have a -ig factor, no propagators, then a $(2\pi)^4 \delta(p_0 - p_1 - p_2)$ factor which appears and dissapears, and so we get, right away:

$$\mathcal{M} = i(-ig) = g$$

Thus $|\mathcal{M}|^2 = g^2 = 4\pi\alpha$, which gives the formula of λ in the statement.

4e. Exercises

Exercises:

EXERCISE 4.27.

EXERCISE 4.28.

EXERCISE 4.29.

EXERCISE 4.30.

EXERCISE 4.31.

EXERCISE 4.32.

EXERCISE 4.33.

EXERCISE 4.34.

Bonus exercise.

Part II

Optics, colors

One night in Bangkok makes a hard man humble Not much between despair and ecstasy One night in Bangkok and the tough guys tumble Can't be too careful with your company

Basic optics

5a. Snell law

Back now to our usual business, light and heat, and with the aim of understanding the colors, with all the above in hand, we can do some optics. Light usually comes in "bundles", with waves of several wavelenghts coming at the same time, from the same source, and the first challenge is that of separating these wavelenghts.

Let us start with the following fact, that we know well since chapter 2:

THEOREM 5.1. In regions of space where there is no charge or current present the Maxwell equations for electrodynamics read

$$<\nabla, E>=<\nabla, B>=0$$

$$\nabla\times E=-\dot{B} \quad , \quad \nabla\times B=\dot{E}/c^2$$

and both the electric field
$$E$$
 and magnetic field B are subject to the wave equation

$$\ddot{\varphi} = c^2 \Delta \varphi$$

with $\Delta = \sum_i d^2/dx_i^2$ being the Laplace operator, and c the speed of light.

PROOF. This is something that we know well, the idea being that in regions of space where there is no charge or current present the Maxwell equations take the above form, and then that, by applying the curl operator to the last two equations, we obtain:

$$\nabla \times (\nabla \times E) = -\nabla \times \dot{B} = -(\nabla \times B)' = -\ddot{E}/c^2$$
$$\nabla \times (\nabla \times B) = \nabla \times \dot{E}/c^2 = (\nabla \times E)'/c^2 = -\ddot{B}/c^2$$

But the double curl operator is subject to the following formula:

$$\nabla\times(\nabla\times\varphi)=\nabla<\nabla,\varphi>-\Delta\varphi$$

Now by using the first two equations, we are led to the conclusion in the statement. \Box

The point now, which is of key importance in relation with material science and optics, is that Theorem 5.1 can be generalized in the following way:

5. BASIC OPTICS

FACT 5.2. Inside a linear, homogeneous medium, where there is no free charge or current present, the Maxwell equations for electrodynamics read

$$\langle \nabla, E \rangle = \langle \nabla, B \rangle = 0$$

 $\nabla \times E = -\dot{B} , \quad \nabla \times B = \varepsilon \mu \dot{E}$

with E, B being as before the electric and the magnetic field, and with $\varepsilon > \varepsilon_0$ and $\mu > \mu_0$ being the electric permittivity and magnetic permeability of the medium.

Observe that this is precisely the first part of Theorem 5.1, with the vacuum constants ε_0, μ_0 being replaced by their versions ε, μ , concerning the medium in question. In what regards now the second part of Theorem 5.1, we have here the following result:

THEOREM 5.3. Inside a linear, homogeneous medium, where there is no free charge or free current present, both E and B are subject to the wave equation

$$\ddot{\varphi} = v^2 \Delta \varphi$$

with v being the speed of light inside the medium, given by

$$v = \frac{c}{n}$$
 : $n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$

with the quantity on the right n > 1 being called refraction index of the medium.

PROOF. This is something that we know well in vacuum, from the above, and the proof in general is identical, with the resulting speed being:

$$v = \frac{1}{\sqrt{\varepsilon\mu}}$$

But this formula can be written is a more familiar from, as above.

As a first observation here, while the above is something quite trivial, mathematically speaking, from the physical viewpoint we are here into complicated things. Materials can be transparent or opaque, with the distinction between them being something very subtle, and advanced, and Theorem 5.3 obviously deals with the transparent case.

In short, we are here inside advanced materials theory, that we cannot really understand, with our knowledge so far. In what follows we will be interested in transparent materials only, such as glass. Regarding the other materials, such as rock, let us just mention that light disapears inside them, converted into heat. Of course glass heats too when light crosses it, with this being related to v < c inside it. More on this later.

Next in line, and for interest for us, we have:

FACT 5.4. When traveling through a material, and hitting a new material, some of the light gets reflected, at the same angle, and some of it gets refracted, at a different angle, depending both on the old and the new material, and on the wavelength.

5B. MIRRORS, LENSES

Again, this is something deep, and very old as well, and there are many things that can be said here, ranging from various computations based on the Maxwell equations, to all sorts of considerations belonging to advanced materials theory.

As a basic formula here, we have the famous Snell law, which relates the incidence angle θ_1 to the refraction angle θ_2 , via the following simple formula:

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{n_1(\lambda)}{n_2(\lambda)}$$

Here $n_i(\lambda)$ are the refraction indices of the two materials, adjusted for the wavelength, and with this adjustment for wavelength being the whole point, which is something quite complicated. For an introduction to all this, we refer for instance to Griffiths [44].

As a simple consequence of the above, we have:

THEOREM 5.5. Light can be decomposed, by using a prism.

PROOF. This follows from Fact 5.4. Indeed, when hitting a piece of glass, provided that the hitting angle is not 90° , the light will decompose over the wavelengths present, with the corresponding refraction angles depending on these wavelengths. And we can capture these split components at the exit from the piece of glass, again deviated a bit, provided that the exit surface is not parallel to the entry surface. And the simplest device doing the job, that is, having two non-parallel faces, is a prism.

With this in hand, we can now talk about spectroscopy:

FACT 5.6. We can study events via spectroscopy, by capturing the light the event has produced, decomposing it with a prism, carefully recording its "spectral signature", consisting of the wavelenghts present, and their density, and then doing some reverse engineering, consisting in reconstructing the event out of its spectral signature.

This is the main principle of spectroscopy, and applications, of all kinds, abound. In practice, the mathematical tool needed for doing the "reverse engineering" mentioned above is the Fourier transform, which allows the decomposition of packets of waves, into monochromatic components. Finally, let us mention too that, needless to say, the event can be reconstructed only partially out of its spectral signature.

In fact, we have already met spectroscopy and its applications, when talking atoms and their spectral lines. In what follows, we will heavily use this method.

5b. Mirrors, lenses

Mirrors, lenses.

5. BASIC OPTICS

5c. Telescopes

Telescopes.

5d. Microscopes

Microscopes.

5e. Exercises

Exercises:

EXERCISE 5.7.

EXERCISE 5.8.

EXERCISE 5.9. EXERCISE 5.10.

Exercise 5.11.

Exercise 5.12.

EXERCISE 5.13.

Exercise 5.14.

Bonus exercise.

Into colors

6a. Into colors

Into colors.

6b.

6c.

6d.

6e. Exercises

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

Exercise 6.4.

Exercise 6.5.

EXERCISE 6.6.

EXERCISE 6.7.

EXERCISE 6.8.

Spectroscopy

7a. Spectroscopy

Spectroscopy.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

- EXERCISE 7.4.
- Exercise 7.5.

Exercise 7.6.

EXERCISE 7.7.

EXERCISE 7.8.

Optics, revised

8a. Optics, revised

Optics, revised.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

Exercise 8.4.

Exercise 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Part III

Math of colors

In between What I find is pleasing and I'm feeling fine Love is so confusing There's no peace of mind

Space of colors

9a. Space of colors

Space of colors.

9b.

9c.

9d.

9e. Exercises

Exercises:

- EXERCISE 9.1.
- EXERCISE 9.2.

EXERCISE 9.3.

- Exercise 9.4.
- EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Some mathematics

10a. Some mathematics

Some mathematics.

10b.

10c.

10d.

10e. Exercises

Exercises:

- Exercise 10.1.
- EXERCISE 10.2.
- EXERCISE 10.3.
- EXERCISE 10.4.
- EXERCISE 10.5.
- EXERCISE 10.6.
- EXERCISE 10.7.

EXERCISE 10.8.

Practical models

11a. Practical models

Practical models.

11b.

11c.

11d.

11e. Exercises

Exercises:

- Exercise 11.1.
- EXERCISE 11.2.
- EXERCISE 11.3.
- EXERCISE 11.4.
- EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Engineering matters

12a. Engineering matters

Engineering matters.

12b.

12c.

12d.

12e. Exercises

Exercises:

- Exercise 12.1.
- EXERCISE 12.2.
- EXERCISE 12.3.
- EXERCISE 12.4.
- EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Part IV

Smell and taste

Don't give up Cause somewhere There's a place Where we belong

Sound, hearing

13a. Sound, hearing

Sound, hearing.

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

EXERCISE 13.2.

Exercise 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Chemistry, smells

14a. Chemistry, smells

Chemistry, smells.

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

Exercise 14.7.

EXERCISE 14.8.
CHAPTER 15

Cooking and taste

15a. Cooking and taste

Cooking and taste.

15b.

15c.

15d.

15e. Exercises

Exercises:

- Exercise 15.1.
- EXERCISE 15.2.
- EXERCISE 15.3.
- Exercise 15.4.
- Exercise 15.5.

Exercise 15.6.

EXERCISE 15.7.

EXERCISE 15.8.

Bonus exercise.

CHAPTER 16

Mechanics, touch

16a. Mechanics, touch

Mechanics, touch.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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