THE REPRESENTATION THEORY OF FREE ORTHOGONAL QUANTUM GROUPS

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Abstract. We find, for each $n \geq 2$, the class of $n \times n$ compact quantum groups whose representation theory is similar to that of $SU(2)$: this is the class of "free analogues of $O(n)$" constructed by Van Daele and Wang.

Let us first recall the definition of the compact matrix quantum groups [6], [8]: these are the pairs (G, u) formed by a unital C^*-algebra G and a matrix $u \in M_n(G)$ such that:

(1) The coefficients of u generate a dense $*$-subalgebra $G_s \subset G$.
(2) There exists a C^*-morphism $\delta : G \to G \otimes G$ such that $(Id \otimes \delta)u = u_{12}u_{13}$.
(3) The matrices u and $\bar{u}$ are both invertible.

We call representation of (G, u) any invertible matrix $r \in M_k(G_s)$ such that $(Id \otimes \delta) = r_{12}r_{13}$. In [6] Woronowicz shows that any compact matrix quantum group has a Haar measure, and develops a Peter-Weyl type theory for its representations. We will freely use the notations and results from [6].

Wang [5], then Van Daele and Wang [3] have recently constructed compact matrix quantum groups having universality properties similar to those of $U(n)$ and $O(n)$:

The unitary case. Let (G, u) be a compact matrix quantum group. Then any representation of G is equivalent to a unitary representation. We can therefore suppose (up to similarity) that u and $F\bar{u}F^{-1}$ are unitaries, for a certain scalar matrix F.

We can define, for any $n \in \mathbb{N}$ and any $F \in GL(n)$, the universal C^*-algebra $A_u(F)$ generated by variables $(u_{ij})_{1 \leq i,j \leq n}$, with the relations making unitaries the matrices u and $F\bar{u}F^{-1}$. It is easy to see that $(A_u(F), u)$ is a compact matrix quantum group.

The orthogonal case. Let (G, u), with u being unitary, and assume that we have $u \sim \bar{u}$ as representations, so that there exists a scalar matrix F such that $u = F\bar{u}F^{-1}$. We have then $\bar{u} = \bar{F}u\bar{F}^{-1}$, and so $u = (F\bar{F})u(F\bar{F})^{-1}$. Thus if u is irreducible, then $F\bar{F} = c \in \mathbb{R}$ (because $F\bar{F} = c \in \mathbb{C} \implies \bar{F}F = \bar{c} \implies c \in \mathbb{R}$).

We can define, for any $n \in \mathbb{N}$ and any $F \in GL(n)$ satisfying $F\bar{F} = c \in \mathbb{R}$, the universal C^*-algebra $A_o(F)$ generated by variables $(u_{ij})_{1 \leq i,j \leq n}$, with the relations $u = F\bar{u}F^{-1} =$ unitary. It is easy to see that $(A_o(F), u)$ is a compact matrix quantum group.

Remark. We have $A_o\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = C(SU(2))$. In fact, it is easy to see that $S_\mu U(2) = A_o\left(\begin{smallmatrix} 0 & 1 \\ -\mu^{-1} & 0 \end{smallmatrix}\right)$, and that any $A_o(F)$ with $F \in GL(2)$ is similar to a certain $S_\mu U(2)$ (see [3]).

The above link between $SU(2)$ and the algebras $A_o(F)$ can be extended at $n \geq 3$:

Theorem 1. *Let $n \in \mathbb{N}$, and let $F \in GL(n)$ such that $F\bar{F} = c \in \mathbb{R}$. Then the irreducible representations of $A_o(F)$ are self-adjoint, and can be indexed by $\mathbb{N}$, with $r_0 = 1$, $r_1 = u$ and*

$$r_k r_s = r_{|k-s|} + r_{|k-s|+2} + \ldots + r_{k+s-2} + r_{k+s}$$

(i.e. the same formulae as for the representations of $SU(2)$).

In order to prove this result, we begin with some considerations regarding the concrete complete monoidal W^*-category (see [7]) $X(F)$ of representations of $A_o(F)$. In terms of monoidal categories, the relation $u \sim \bar{u}$ tells us that:

– $X(F)$ is the completion of the subcategory $Y(F)$ having as objects $1, u, u^2, u^3, \ldots$

– $Y(F)$ contains a certain morphism, intertwining 1 and u^2.

These two conditions allow one to fully reconstruct $Y(F)$ (see also the corresponding construction for $S_\mu U(N)$ from [7]):

Proposition 1. *Let $H = \mathbb{C}^n$, with standard basis $\{e_i\}$. For $r, s \in \mathbb{N}$ we define the sets $Mor(r,s) \subset B(H^{\otimes r}, H^{\otimes s})$ of linear combinations of (composable) products of maps of type $Id_{H^{\otimes k}}$ or $Id_{H^{\otimes k}} \otimes E \otimes Id_{H^{\otimes p}}$ or $Id_{H^{\otimes k}} \otimes E^* \otimes Id_{H^{\otimes p}}$, where $E \in Mor(0,2)$ is the linear map $1 \to \sum F_{ji} e_i \otimes e_j$. Then, each $Mor(r,s)$ equals $Mor(u^r, u^s) \subset B(H^{\otimes r}, H^{\otimes s})$.*

Proof. $Z(F) = \{\mathbb{N}, +, \{H^{\otimes r}\}_{r\in\mathbb{N}}, \{Mor(r,s)\}_{r,s\in\mathbb{N}}\}$ is clearly a concrete monoidal W^*-category, generated by 1. If we denote by $k : H \to H$ the antilinear involution given by $\lambda e_i \to \bar{\lambda} e_i$ and $j = k\bar{F}$, then $t_j, t_{j^{-1}} \in Mor(0,2)$, and so $1 = \bar{1}$ inside $Z(F)$ (see [7], page 39). By the duality theorem (Theorem 1.3 in [7]) the universal $Z(F)$-admissible pair is a compact matrix quantum group defined by the same universal property as $A_o(F)$, and so is $A_o(F)$ itself. Thus $Y(F) = Z(F)$, and so $Mor(u^r, u^s) = Mor(r,s)$, for any $r, s \in \mathbb{N}$. □

In order to compute the above monoidal W^*-category, we will need:

Lemma 1. *$(E^* \otimes Id_H)(Id_H \otimes E) = cId_H$.*

Proof. This is clear from the definition of E, and from $F\bar{F} = c$. □

We study now the spaces $Mor(k,k)$. For $s = 1, \ldots, k-1$ let us define:

$$f_s = ||E(1)||^{-2} Id_{H^{\otimes s-1}} \otimes EE^* \otimes Id_{H^{\otimes k-s-1}}$$

An elementary computation based on Lemma 1 above shows that:

(1) $f_s^2 = f_s^* = f_s$, $\forall 1 \leq s \leq k-1$.

(2) $f_s f_t = f_t f_s$, $\forall 1 \leq s,t \leq k-1$ with $|s-t| \geq 2$.

(3) $\beta f_s f_t f_s = f_s$, $\forall 1 \leq s,t \leq k-1$ with $|s-t| = 1$, with $\beta = c^{-2}||E(1)||^4$.

We recall that the Temperley-Lieb algebra $A_{\beta,k}$ is defined with generators $1, f_1, \ldots, f_{k-1}$ and the above relations (see [2]). In order to make the link with $A_{\beta,k}$, we will need:

Proposition 2. *The elements $1, f_1, \ldots, f_{k-1}$ generate $Mor(k,k)$, as a $\mathbb{C}$-algebra.*

Proof. Let $I(p) = Id_{H^{\otimes p}}$ and $V(p,q) = I(p) \otimes E \otimes I(q)$. By using Lemma 1, we can see that any morphism of $Y(F)$ appears as a linear combination of maps of type $I(.)$ or of type $V(.,.) \circ \ldots \circ V(.,.) \circ V^*(.,.) \circ \ldots \circ V^*(.,.)$. In particular, the elements of $Mor(k,k)$ are linear combinations of $I(k)$ and of maps of the following form:

$$(*) \qquad V(p_m,q_m) \circ \ldots \circ V(p_1,q_1) \circ V^*(r_1,s_1) \circ \ldots \circ V^*(r_m,s_m)$$

Let U be the set of morphisms of $Y(F)$ which are linear combinations of maps of the form $I(m)$ or of the form $U(m,q,p) = I(m) \otimes E \otimes I(q) \otimes E^* \otimes I(p)$ or of the form $U(m,q,p)^*$. We show by recurrence that any map of the form $(*)$ belongs to U.

Indeed, let us suppose that this is true at $m \in \mathbb{N}$, and pick an arbitrary map of the form $A = V(p_{m+1},q_{m+1}) \circ \ldots \circ V(p_1,q_1) \circ V^*(r_1,s_1) \circ \ldots \circ V^*(r_{m+1},s_{m+1})$. By recurrence, we have $A = V(p_{m+1},q_{m+1}) \circ T \circ V^*(r_{m+1},s_{m+1})$, for a certain $T \in U$.

It is clear that any product of the form $V(.,.) \circ U(.,.,.)^\sigma$, with $\sigma \in \{1,*\}$, can be written in the form $U(.,.,.)^\sigma \circ V(.,.)$. By using a recurrence, we conclude that any product of the form $V(.,.) \circ T$ with $T \in U$ can be written as a sum $\sum T_i \circ V(a_i,b_i)$ with $T_i \in U$. Thus A is of the form $\sum T_i \circ V(.,.) \circ V^*(.,.)$, with $T_i \in U$. But, each product of the type $V(.,.) \circ V^*(.,.)$ being in U, we conclude that we have $A \in U$.

Summarizing, we have proved that $Mor(k,k)$ is contained in U, and so is the set of linear combinations of products of maps of type $I(k)$ or $U(m,p,q)$ or $U(m,q,p)^*$, with $m+q+p = k-2$. Now since $f_1,\ldots,f_{k-1}$ are self-adjoint, it remains to prove that $U(m,p,q)$ with $m+q+p = k-2$ belongs to the algebra generated by $f_1,\ldots,f_{k-1}$. But this is clear from the formula $c^q||E(1)||^{-2q-2}U(m,p,q) = f_{m+1}f_{m+2}\ldots f_{m+q}f_{m+q+1}$, which can be shown by recurrence on q, by using Lemma 1. □

As a consequence of the above result, we obtain:

Corollary 1. *$Mor(k,k)$ is a quotient of $A_{\beta,k}$.*

Remark. By counting the reduced words in $A_{\beta,k}$ we have $\dim(A_{\beta,k}) \leq C_k := \frac{1}{k+1}\binom{2k}{k}$, with these latter numbers being the Catalan numbers (see [2], Aside 4.1.4). Thus, we have $\dim(Mor(u^k,u^k)) = \dim(Mor(k,k)) \leq \dim(A_{\beta,k}) \leq C_k$, for any k.

In the particular case of $A_o\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = C(SU(2))$ we have equality everywhere. This is well-known, and can be proved as well as follows. Let v be the fundamental representation of $SU(2)$, with character denoted f, and let h be the Haar integration over $SU(2)$. The, the last formula from the first appendix in [6] gives $h((f/2)^{2k}) = \frac{2}{\pi}\int_{-1}^1 x^{2k}\sqrt{1-x^2}dx$. In addition, $h((f/2)^{2k+1}) = 0$, and so $f/2 \in (C(SU(2)),h)$ is a semicircular variable in the sense of Voiculescu [4]. Thus $dim(Mor(v^k,v^k)) = 4^k h((f/2)^{2k}) = 4^k\gamma_{0,1}(X^{2k})$, where $\gamma_{0,1}$ is the semicircle law, whose moments can be computed by using 3.3 and 3.4 in [4] and the residue formula, as follows: $\gamma_{0,1}(X^{2k}) = \frac{1}{2k+1} \cdot \frac{1}{2\pi i}\int_T (z^{-1}+z/4)^{2k+1} = 4^{-k}C_k$.

Now back to our quantum group setting, we obtain from this:

Corollary 2. *Let u,v be the fundamental representations of $A_o(F), SU(2)$, respectively. Then* $\dim(Mor(u^k,u^k)) \leq \dim(Mor(v^k,v^k))$.

With these results in hand, we can now prove Theorem 1:

Proof. (of Theorem 1) Let $\{\chi_k\}_{k\in\mathbb{N}}$ be the characters of the irreducible representations of $SU(2)$. The linear space $A \subset C(SU(2))$ spanned by these characters is then a $\mathbb{C}$-algebra, which is isomorphic to $\mathbb{C}[X]$, via $X \to \chi_1$. By recurrence on k, we can find integers $a(k,s) \in \mathbb{N}$ such that $a(k,k) = 1$ and $\chi_1^k = \sum_{s=0}^k a(k,s)\chi_s$.

Since A is a polynomial algebra on χ_1, we can define a morphism $\Psi : A \to A_o(F)$ by $\chi_1 \to f_1$, where f_1 is the character of the fundamental representation of $A_o(F)$. The elements $f_k = \Psi(\chi_k) \in A_o(F)$ verify then $f_k f_s = f_{|k-s|} + f_{|k-s|+2} + \ldots + f_{k+s}$.

We show now by recurrence on k that each f_k is the character of an irreducible representation r_k of $A_o(F)$, non-equivalent to $r_0, \ldots, r_{k-1}$. At $k = 0, 1$ this is clear. Assume now that the result holds at $k-1$. We have $f_{k-2}f_1 = f_{k-3} + f_{k-1}$, and so $r_{k-2}r_1 = r_{k-3} + r_{k-1}$, which gives $r_{k-1} \subset r_{k-2}r_1$. Now since r_{k-2} is by recurrence irreducible, by Frobenius reciprocity we have $r_{k-2} \subset r_{k-1}r_1$, so there exists a representation r_k such that $r_{k-1}r_1 = r_{k-2} + r_k$. Since $f_{k-1}f_1 = f_{k-2} + f_k$, the character of r_k is f_k.

Now since Ψ is a morphism, we have $f_1^k = \sum_{s=0}^k a(k,s) f_s$ and so $\dim(Mor(u^k, u^k)) \geq \sum_{s=0}^k a(k,s)^2$, with equality when r_k is irreducible, and non-equivalent to $r_1, \ldots, r_{k-1}$. But $\sum_{s=0}^k a(k,s)^2 = \dim(Mor(v^k, v^k))$, and by Corollary 2, we have equality.

Finally, since any irreducible representation of $A_o(F)$ must appear in some tensor power of u, and we have a formula for decomposing each u^k into sums of representations r_s, we conclude that these representations r_s are all the irreducible representations of $A_o(F)$. □

Remarks. (1) By recurrence on k, we have $\dim(r_k) = (x^{k+1} - y^{k+1})/(x-y)$, where x, y are the solutions of $X^2 - nX + 1 = 0$. At $n = 2$ we have $\dim(r_k) = k+1$.

(2) The proof of Theorem 1 shows that the commutant of u^k is precisely $A_{\beta,k}$. This can be used in order to prove some simplicity results, in the spirit of [1].

Finally, Theorem 1 has the following converse, which follows from the definition of $A_o(F)$, and from Theorem 1 itself:

Theorem 2. *If the irreducible representations of a compact quantum group G are self-adjoint, and can be indexed by $\mathbb{N}$, with $r_0 = 1$, $r_1 = u$ and $r_k r_s = r_{|k-s|} + r_{|k-s|+2} + \ldots + r_{k+s-2} + r_{k+s}$, then G_{red} must be similar to a certain $A_o(F)_{red}$.*

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