Diagrams and algebras

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

2010 *Mathematics Subject Classification.* 81R50

Key words and phrases. Diagrams, Algebra

Abstract. This is an introduction to the various algebras formed by diagrams, and vice versa, and their applications to questions from quantum physics. We first discuss the basic questions of topology, followed by a discussion of the simplest diagrammatic objects appearing, namely braids and partitions, and with a look into the associated algebras too. Then we go on a detailed study of knot invariants, following Alexander, Jones, Witten and others. We then discuss the relation between diagrams and representation theory, following Schur-Weyl, Brauer and others, and with a look into quantum groups and planar algebras too. Finally, we provide an introduction to the Feynman diagrams and their applications, and the various combinatorial algebras associated to them.

Preface

We certainly live in 3 dimensions, but our understanding of this world is rather 2-dimensional. Information gets to us, be that via sight, or hearing and so on, as some sort of 2-dimensional picture, and it is hard to say something reliable about the distance between us and the objects and phenomena that we see, hear, smell and so on. For instance a pungent odor of rotten eggs usually means that the rotten eggs are close, but is that really correct, it might well happen that the rotten eggs are somewhere out in space, as a chemical weapon, brought by the little green men from Mars, attacking us.

In mathematics and physics, which are normally about 3 dimensions, we have of course a lot of troubles in fighting with this 2-dimensionality of our thought. Who was not dreamed, for instance, to be able to manipulate $N \times N \times N$ matrices, the way we do it for the usual $N \times N$ matrices, with lots of useful theorems about them. Well, this is unfortunately not really possible, for us humans, so stuck with usual linear algebra.

Nevermind. With this lesson learned, we can do our best in mathematics and physics by using 2-dimensional methods, which in practice means, using "diagrams". And there are countless such useful diagrams, invented by mankind since ages, for all sorts of practical purposes. In addition, in the present modern times, a bit of abstract algebra can help too, the idea being that the formal linear combinations of such diagrams, which usually form an algebra, can be more powerful tools than the diagrams themselves.

This book is an introduction to the various algebras formed by diagrams, and vice versa, and their applications to questions from quantum physics. We have tried to keep things as simple and elementary as possible, usually relying only on some basic knowledge of undergraduate mathematics, with the organization, in 4 parts, being as follows:

(1) We first discuss the basic questions of topology, concerned with bodies and their shapes, followed by a discussion of the simplest diagrammatic objects appearing, namely braids and partitions, and with a look into the associated algebras too.

(2) Then we go on a detailed study of knot invariants, following Alexander, Jones, Witten and others. Our approach here will be closely following the 1980s work of Jones, and its ramifications, with a regular look backwards, and forward.

(3) We then discuss the relation between diagrams and representation theory, first for the finite and compact groups, following Schur-Weyl, Brauer and others, and then with a look into quantum groups, random matrices and planar algebras too.

(4) Finally, we will get into quantum physics, with the aim of applying the techniques that we learned. We will provide here an introduction to the Feynman diagrams and their applications, and the various combinatorial algebras associated to them.

Many thanks to my colleagues, collaborators, and to various books and internet too, there is so much to be learned about diagrams, from so many places, and with this being a never-ending story. Thanks as well to my cats, for some help with the physics.

Cergy, March 2025
Teo Banica

Contents

Part I

Topology, diagrams

All the way to Jackson
I don't think I'll miss you much
All the way to Jackson
I don't think I'll miss you much

CHAPTER 1

Topology

1a. Topological spaces

Welcome to topology. Before getting started with our mathematics, we need spaces. We will use here a very general definition, as folllows:

Definition 1.1. *A topological space is a set X, given with a collection of subsets $E \subset X$ called open sets, satisfying what we can expect from the open sets, namely:*

(1) *X itself is open, and so is $\emptyset$.*
(2) *Union of open sets is open.*
(3) *Finite intersection of open sets is open.*

We will see examples in a moment, but before anything, it looks like we forgot the closed sets, which are often more important than the open sets, when doing analysis, in our axiomatization. Good point, and in answer, we have the following result:

Proposition 1.2. *Given a topological space X, call a subset $F \subset X$ closed when its complement $F^c \subset X$ is open. Then, the closed sets have the following properties:*

(1) *X itself is closed, and so is $\emptyset$.*
(2) *Finite union of closed sets is closed.*
(3) *Intersection of closed sets is closed.*

Proof. This is something trivial, which follows by talking the complement of the various axioms from Definition 1.1. As an important remark, however, observe the symmetry between the present statement and Definition 1.1, which shows that, whenever needed, these statements can be interchanged. That is, if needed, we can define a topological space X via its closed sets $F \subset X$, which must satisfy the present conditions (1,2,3), and then we can talk about the open sets $E \subset X$ too, as being the complements of the closed sets, and these open sets satisfy then the conditions (1,2,3) in Definition 1.1. □

In practice now, for geometry and analysis, we will mostly need the case where X is a metric space, with the remark however that the abstract topological spaces as above are something quite interesting, in relation with all sorts of geometry.

So, getting right away to the metric spaces, we have here the following result, making the link with Definition 1.1 and Proposition 1.2, and providing us with examples:

THEOREM 1.3. *Given a metric space X, call a subset $E \subset X$ open if any $x \in E$ has a ball around it belonging to U, and call a subset $F \subset X$ closed if its complement $F^c \subset X$ is open. Then, the following happen, showing that X is a topological space:*

1. *If E_i are open, then $\cup_i E_i$ is open.*
2. *If F_i are closed, then $\cap_i F_i$ is closed.*
3. *If $E_1, \ldots, E_n$ are open, then $\cap_i E_i$ is open.*
4. *If $F_1, \ldots, F_n$ are closed, then $\cup_i F_i$ is closed.*

Moreover, both (3) and (4) can fail for infinite intersections and unions.

PROOF. We have several things to be proved, the idea being as follows:

(1) This is clear from definitions, because any point $x \in \cup_i E_i$ must satisfy $x \in E_i$ for some i, and so has a ball around it belonging to E_i, and so to $\cup_i E_i$.

(2) This follows from (1), by using the following well-known set theory formula:

$$\left(\bigcup_i E_i\right)^c = \bigcap_i E_i^c$$

(3) Given an arbitrary point $x \in \cap_i E_i$, we have $x \in E_i$ for any i, and so we have a ball $B_x(r_i) \subset E_i$ for any i. Now with this in hand, let us set:

$$B = B_x(r_1) \cap \ldots \cap B_x(r_n)$$

As a first observation, this is a ball around x, $B = B_x(r)$, of radius given by:

$$r = \min(r_1, \ldots, r_n)$$

But this ball belongs to all the E_i, and so belongs to their intersection $\cap_i E_i$. We conclude that the intersection $\cap_i E_i$ is open, as desired.

(4) This follows from (3), by using the following well-known set theory formula:

$$\left(\bigcap_i E_i\right)^c = \bigcup_i E_i^c$$

(5) Finally, in what regards the counterexamples at the end, we will leave their construction, which is something very elementary, as an instructive exercise. □

Getting back now to the general topological spaces, from Definition 1.1 and Proposition 1.2, we can do analysis on them, inspired a bit by Theorem 1.3, and we have:

PROPOSITION 1.4. *For a set $E \in X$, the following are equivalent:*

1. *E is closed in our sense, meaning that E^c is open.*
2. *We have $x_n \to x, x_n \in E \implies x \in E$.*

PROOF. For metric spaces, we can prove this by double implication, as follows:

(1) $\implies$ (2) Assume by contradiction $x_n \to x, x_n \in E$ with $x \notin E$. Since we have $x \in E^c$, which is open, we can pick a ball $B_x(r) \subset E^c$. But this contradicts our convergence assumption $x_n \to x$, so we are done with this implication.

(2) $\implies$ (1) Assume by contradiction that E is not closed in our sense, meaning that E^c is not open. Thus, we can find $x \in E^c$ such that there is no ball $B_x(r) \subset E^c$. But with $r = 1/n$ this provides us with a point $x_n \in B_x(1/n) \cap E$, and since we have $x_n \to x$, this contradicts our assumption (2). Thus, we are done with this implication too. □

Still in relation with open and closed sets, we have as well:

DEFINITION 1.5. *Let X be a metric space, and $E \subset X$ be a subset.*

1. *The interior $E^\circ \subset E$ is the set of points $x \in E$ which admit around them open balls $B_x(r) \subset E$.*
2. *The closure $E \subset \bar{E}$ is the set of points $x \in X$ which appear as limits of sequences $x_n \to x$, with $x \in E$.*

These notions are quite interesting, because they make sense for any set E. That is, when E is open, that is open and end of the story, and when E is closed, that is closed and end of the story. In general, however, a set $E \subset X$ is not open or closed, and what we can best do to it, in order to study with our tools, is to "squeeze" it, as follows:

$$E^\circ \subset E \subset \bar{E}$$

In practice now, in order to use the above notions, we need to know a number of things, including that fact that E open implies $E^\circ = E$, the fact that E closed implies $\bar{E} = E$, and many more such results. But all this can be done, and the useful statement here, summarizing all that we need to know about interiors and closures, is as follows:

PROPOSITION 1.6. *Let X be a metric space, and $E \subset X$ be a subset.*

1. *The interior $E^\circ \subset E$ is the biggest open set contained in E.*
2. *The closure $E \subset \bar{E}$ is the smallest closed set containing E.*

PROOF. We have several things to be proved, the idea being as follows:

(1) Let us first prove that the interior E° is open. For this purpose, pick $x \in E^\circ$. We know that we have a ball $B_x(r) \subset E$, and since this ball is open, it follows that we have $B_x(r) \subset E^\circ$. Thus, the interior E° is open, as claimed.

(2) Let us prove now that the closure $\bar{E}$ is closed. For this purpose, we will prove that the complement $\bar{E}^c$ is open. So, pick $x \in \bar{E}^c$. Then x cannot appear as a limit of a sequence $x_n \to x$ with $x_n \in E$, so we have a ball $B_x(r) \subset \bar{E}^c$, as desired.

(3) Finally, the maximality and minimality assertions regarding E° and $\bar{E}$ are both routine too, coming from definitions, and we will leave them as exercises. □

As an application of the theory developed above, and more specifically of the notion of closure from Definition 1.5, we can talk as well about density, as follows:

Definition 1.7. *We say that a subset $E \subset X$ is dense when:*

$$\bar{E} = X$$

That is, any point of X must appear as a limit of points of E.

Obviously, this is something which is in tune with what we know so far from this book, and with the intuitive notion of density. As a basic example, we have $\bar{\mathbb{Q}} = \mathbb{R}$.

Moving ahead now, again in analogy with what we know about $X = \mathbb{R}, \mathbb{C}$, we can talk about compact sets, and about connected sets. Let us start with:

Definition 1.8. *A set $K \subset X$ is called compact if any cover with open sets*

$$K \subset \bigcup_i E_i$$

has a finite subcover, $K \subset (E_{i_1} \cup \ldots \cup E_{i_n})$.

This might seem overly abstract, but our claim is that this is the correct definition, and that there is no way of doing otherwise. The point indeed is that we have:

Proposition 1.9. *Given an infinite set X with the discrete distance on it, namely $d(p,q) = 1 - \delta_{pq}$, which can be modeled as the basis of a suitable Hilbert space,*

$$X = \{e_x\}_{x \in X} \subset l^2(X)$$

this set is closed and bounded, but not compact.

Proof. Here the first part, regarding the modelling of X, that we will actually not really need, is something that we already know. Regarding now the second part:

(1) X being the total space, it is by definition closed. As a remark here, that we will need later, since the points of X are obviously open, any subset $E \subset X$ is open, and by taking complements, any set $E \subset X$ is closed as well.

(2) X is also bounded, because all distances are smaller than 1.

(3) However, our set X is not compact, because its points being open, as noted above, $X = \cup_{x \in X}\{x\}$ is an open cover, having no finite subcover. □

Let us develop now the theory of compact sets. We first have the following result:

Proposition 1.10. *The following hold:*

1. *Compact implies closed.*
2. *Closed inside compact is compact.*
3. *Compact intersected with closed is compact.*

PROOF. These assertions are all clear from definitions, as follows:

(1) Assume that $K \subset X$ is compact, and let us prove that K is closed. For this purpose, we will prove that K^c is open. So, pick $p \in K^c$. For any $q \in K$ we set $r = d(p,q)/3$, and we consider the following balls, separating p and q:

$$U_q = B_p(r) \quad , \quad V_q = B_q(r)$$

We have then $K \subset \cup_{q \in K} V_q$, so we can pick a finite subcover, as follows:

$$K \subset (V_{q_1} \cup \ldots \cup V_{q_n})$$

With this done, consider the following intersection:

$$U = U_{q_1} \cap \ldots \cap U_{q_n}$$

This intersection is then a ball around p, and since this ball avoids $V_{q_1}, \ldots, V_{q_n}$, it avoids the whole K. Thus, we have proved that K^c is open at p, as desired.

(2) Assume that $F \subset K$ is closed, with $K \subset X$ being compact. For proving our result, we can assume, by replacing X with K, that we have $X = K$. In order to prove now that F is compact, consider an open cover of it, as follows:

$$F \subset \bigcup_i E_i$$

By adding the set F^c, which is open, to this cover, we obtain a cover of K. Now since K is compact, we can extract from this a finite subcover Ω, and there are two cases:

– If $F^c \in \Omega$, by removing F^c from Ω we obtain a finite cover of F, as desired.

– If $F^c \notin \Omega$, we are done too, because in this case Ω is a finite cover of F.

(3) This follows from (1) and (2), because if $K \subset X$ is compact, and $F \subset X$ is closed, then $K \cap F \subset K$ is closed inside a compact, so it is compact. □

As a second batch of results, which are useful as well, we have:

PROPOSITION 1.11. *The following hold:*

1. *If $K_i \subset X$ are compact, satisfying $K_{i_1} \cap \ldots \cap K_{i_n} \neq \emptyset$, then $\cap_i K_i \neq \emptyset$.*
2. *If $K_1 \supset K_2 \supset K_3 \supset \ldots$ are non-empty compacts, then $\cap_i K_i \neq \emptyset$.*
3. *If K is compact, and $E \subset K$ is infinite, then E has a limit point in K.*
4. *If K is compact, any sequence $\{x_n\} \subset K$ has a limit point in K.*
5. *If K is compact, any $\{x_n\} \subset K$ has a subsequence which converges in K.*

PROOF. Again, these are elementary results, which can be proved as follows:

(1) Assume by contradiction $\cap_i K_i = \emptyset$, and let us pick $K_1 \in \{K_i\}$. Since any $x \in K_1$ is not in $\cap_i K_i$, there is an index i such that $x \in K_i^c$, and we conclude that we have:

$$K_1 \subset \bigcup_{i \neq 1} K_i^c$$

But this can be regarded as being an open cover of K_1, that we know to be compact, so we can extract from it a finite subcover, as follows:

$$K_1 \subset \left(K_{i_1}^c \cup \ldots \cup K_{i_n}^c\right)$$

Now observe that this latter subcover tells us that we have:

$$K_1 \cap K_{i_1} \cap \ldots \cap K_{i_n} = \emptyset$$

But this contradicts our intersection assumption in the statement, and we are done.

(2) This is a particular case of (1), proved above.

(3) We prove this by contradiction. So, assume that E has no limit point in K. This means that any $p \in K$ can be isolated from the rest of E by a certain open ball $V_p = B_p(r)$, and in both the cases that can appear, $p \in E$ or $p \notin E$, we have:

$$|V_p \cap E| = 0, 1$$

Now observe that these sets V_p form an open cover of K, and so of E. But due to $|V_p \cap E| = 0, 1$ and to $|E| = \infty$, this open cover of E has no finite subcover. Thus the same cover, regarded now as cover of K, has no finite subcover either, contradiction.

(4) This follows from (3) that we just proved, with $E = \{x_n\}$.

(5) This is a reformulation of (4), that we just proved. □

Getting now to some more exciting theory, here is a key result about compactness, which is less trivial, and that we will need on a regular basis, in what follows:

THEOREM 1.12. *For a subset $K \subset \mathbb{R}^N$, the following are equivalent:*

(1) *K is closed and bounded.*
(2) *K is compact.*
(3) *Any infinite subset $E \subset K$ has a limiting point in K.*

PROOF. This is something quite tricky, the idea being as follows:

(1) $\implies$ (2) As a first task, in order to establish this implication, let us prove that any product of closed intervals, as follows, is indeed compact:

$$J = \prod_{i=1}^{N} [a_i, b_i] \subset \mathbb{R}^N$$

We can assume by linearity that we are dealing with the unit cube:

$$C_1 = \prod_{i=1}^{N} [0, 1] \subset \mathbb{R}^N$$

In order to prove that C_1 is compact, we proceed by contradiction. So, assume that we have an open cover as follows, having no finite subcover:

$$C_1\subset\bigcup_iE_i$$

Now let us cut C_1 into 2^N small cubes, in the obvious way, over the N coordinate axes. Then at least one of these small cubes, which are all covered by $\cup_iE_i$ too, has no finite subcover. So, let us call $C_2\subset C_1$ one of these small cubes, having no finite subcover:

$$C_2\subset\bigcup_iE_i$$

We can then cut C_2 into 2^N small cubes, and by the same reasoning, we obtain a smaller cube $C_3\subset C_2$ having no finite subcover. And so on by recurrence, and we end up with a decreasing sequence of cubes, as follows, having no finite subcover:

$$C_1\supset C_2\supset C_3\supset\ldots$$

Now since these decreasing cubes have edge size $1,1/2,1/4,\ldots$, their intersection must be a point. So, let us call p this point, defined by the following formula:

$$\{p\}=\bigcap_kC_k$$

But this point p must be covered by $\cup_iE_i$, so we can find an index i such that:

$$p\in E_i$$

Now observe that E_i must contain a whole ball around p, and so starting from a certain $K\in\mathbb N$, all the cubes C_k will be contained in this ball, and so in E_i:

$$C_k\subset E_i\quad,\quad\forall k\geq K$$

But this is a contradiction, because C_K, and in fact the smaller cubes C_k with $k>K$ as well, were assumed to have no finite subcover. Thus, we have proved our claim.

(1) $\implies$ (2), continuation. But with this claim in hand, the result is now clear. Indeed, assume that $K\subset\mathbb R^N$ is closed and bounded. Then, since K is bounded, we can view it as a subset as a suitable big cube, of the following form:

$$K\subset\prod_{i=1}^N[-M,M]\subset\mathbb R^N$$

But, what we have here is a closed subset inside a compact set, that follows to be compact, as desired.

(2) $\implies$ (3) This is something that we already know, not needing $K\subset\mathbb R^N$.

(3) $\implies$ (1) We have to prove that K as in the statement is both closed and bounded, and we will do both these things by contradiction, as follows:

– Assume first that K is not closed. But this means that we can find a point $x \notin K$ which is a limiting point of K. Now let us pick $x_n \in K$, with $x_n \to x$, and consider the set $E = \{x_n\}$. According to our assumption, E must have a limiting point in K. But this limiting point can only be x, which is not in K, contradiction.

– Assume now that K is not bounded. But this means that we can find points $x_n \in K$ satisfying $||x_n|| \to \infty$, and if we consider the set $E = \{x_n\}$, then again this set must have a limiting point in K, which is impossible, so we have our contradiction, as desired. □

So long for compactness. As a last piece of general topology, in our metric space framework, we can talk as well about connectedness, as follows:

Definition 1.13. *We can talk about connected sets $E \subset X$, as follows:*

1. *We say that E is connected if it cannot be separated as $E = E_1 \cup E_2$, with the components E_1, E_2 satisfying $E_1 \cap \bar{E}_2 = \bar{E}_1 \cap E_2 = \emptyset$.*
2. *We say that E is path connected if any two points $p, q \in E$ can be joined by a path, meaning a continuous $f : [0.1] \to X$, with $f(0) = p$, $f(1) = q$.*

All this looks a bit technical, and indeed it is. To start with, (1) is something quite natural, but the separation condition there $E_1 \cap \bar{E}_2 = \bar{E}_1 \cap E_2 = \emptyset$ can be weakened into $E_1 \cap E_2 = \emptyset$, or strengthened into $\bar{E}_1 \cap \bar{E}_2 = \emptyset$, depeding on purposes, and with our (1) as formulated being the good compromise, for most purposes. As for (2), this condition is obviously something stronger, and we have in fact the following implications:

$$\text{convex} \implies \text{path connected} \implies \text{connected}$$

The problem, however, is that connected does not imply path connected, and there are as well various counterexamples in relation with the various versions of (1) that can be formulated, as explained above. In any case, once these questions clarified, the idea is that any set E can be written as a disjoint union of connected components, as follows:

$$E = \bigsqcup_i E_i$$

Getting back now to more concrete things, that is, calculus, we have:

Theorem 1.14. *Assuming that $f : X \to Y$ is continuous, the following happen:*

1. *If O is open, then $f^{-1}(O)$ is open.*
2. *If C is closed, then $f^{-1}(C)$ is closed.*
3. *If K is compact, then $f(K)$ is compact.*
4. *If E is connected, then $f(E)$ is connected.*

Proof. This is something fundamental, which can be proved as follows:

(1) This is clear from the definition of continuity, written with ε, δ. In fact, the converse holds too, in the sense that if $f^{-1}(\text{open}) = \text{open}$, then f must be continuous.

(2) This follows from (1), by taking complements. And again, the converse holds too, in the sense that if $f^{-1}(\text{closed}) = \text{closed}$, then f must be continuous.

(3) Given an open cover $f(K) \subset \cup_i E_i$, we have by using (1) an open cover $K \subset \cup_i f^{-1}(E_i)$, and so by compactness of K, a finite subcover $K \subset f^{-1}(E_{i_1}) \cup \ldots \cup f^{-1}(E_{i_n})$, and so finally a finite subcover $f(K) \subset E_{i_1} \cup \ldots \cup E_{i_n}$, as desired.

(4) This can be proved via the same trick as for (3). Indeed, any separation of $f(E)$ into two parts can be returned via f^{-1} into a separation of E into two parts, contradiction. □

As a comment here, Theorem 1.14 generalizes, and in a clever way, many things that we know from one-variable calculus. Of particular interest is (3), which shows in particular that any continuous function on a compact space $f : X \to \mathbb{R}$ attains its minimum and its maximum, and then (4), which can be regarded as being a general mean value theorem. As for (1) and (2), these are useful in everyday life, and we will see examples of this.

1b. Homotopy groups

Time now to start investigating the shape of our topological spaces. Let us start with something that we know from the above, namely:

Definition 1.15. *A topological space X is called connected when any two points $x, y \in X$ can be connected by a path. That is, given any two points $x, y \in X$, we can find a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.*

The problem is now, given a connected space X, how to count its "holes". And this is quite subtle problem, because as examples of such spaces we have:

(1) The sphere, the donut, the double-holed donut, the triple-holed donut, and so on. These spaces are quite simple, and intuition suggests to declare that the number of holes of the N-holed donut is, and you guessed right, N.

(2) However, we have as well as example the empty sphere, I mean just the crust of the sphere, and while this obviously falls into the class of "one-holed spaces", this is not the same thing as a donut, its hole being of different nature.

(3) As another example, consider again the sphere, but this time with two tunnels drilled into it, in the shape of a cross. Whether that missing cross should account for 1 hole, or for 2 holes, or for something in between, I will leave it up to you.

Summarizing, things are quite tricky, suggesting that the "number of holes" of a topological space X is not an actual number, but rather something more complicated.

Now with this in mind, let us formulate the following definition:

Definition 1.16. *The homotopy group $\pi_1(X)$ of a connected space X is the group of loops based at a given point $* \in X$, with the following conventions,*

(1) *Two such loops are identified when one can pass continuously from one loop to the other, via a family of loops indexed by $t \in [0,1]$,*
(2) *The composition of two such loops is the obvious one, namely is the loop obtaining by following the first loop, then the second loop,*
(3) *The unit loop is the null loop at $*$, which stays there, and the inverse of a given loop is the loop itself, followed backwards,*

with the remark that the group $\pi_1(X)$ defined in this way does not depend on the choice of the given point $* \in X$, where the loops are based.

This definition is obviously something non-trivial, based on some preliminary thinking on the subject, the technical details being as follows:

– The fact that the set $\pi_1(X)$ defined as above is indeed a group is obvious, with all the group axioms being clear from definitions.

– Obvious as well is the fact that, since X is assumed to be connected, this group does not depend on the choice of the given point $* \in X$, where the loops are based.

As basic examples now, for spaces having "no holes", such as $\mathbb{R}$ itself, or $\mathbb{R}^N$, and so on, we have $\pi_1 = \{1\}$. In fact, having no holes can only mean, by definition, $\pi_1 = \{1\}$:

Definition 1.17. *A space is called simply connected when:*

$$\pi_1 = \{1\}$$

That is, any loop inside our space must be contractible.

So, this will be our starting definition, for the considerations in this section. As further illustrations for Definition 1.16, here are now a few basic computations:

Theorem 1.18. *We have the following computations of homotopy groups:*

(1) *For the circle, we have $\pi_1 = \mathbb{Z}$.*
(2) *For the torus, we have $\pi_1 = \mathbb{Z} \times \mathbb{Z}$.*
(3) *For the disk minus 2 points, we have $\pi_1 = F_2$.*
(4) *In fact, for the disk minus N points, we have $\pi_1 = F_N$.*

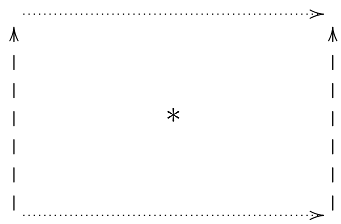
Proof. These results are all standard, as follows:

(1) The first assertion is clear, because a loop on the circle must wind $n \in \mathbb{Z}$ times around the center, and this parameter $n \in \mathbb{Z}$ uniquely determines the loop, up to the identification in Definition 1.16. Thus, the homotopy group of the circle is the group of such parameters $n \in \mathbb{Z}$, which is of course the group $\mathbb{Z}$ itself.

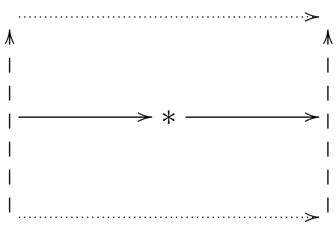
(2) In what regards now the second assertion, the torus being a product of two circles, we are led to the conclusion that its homotopy group must be some kind of product of $\mathbb{Z}$ with itself. But pictures show that the two standard generators of $\mathbb{Z}$, and so the two copies of $\mathbb{Z}$ themselves, commute, $gh = hg$, so we obtain the product of $\mathbb{Z}$ with itself, subject to commutation, which is the usual product $\mathbb{Z} \times \mathbb{Z}$:

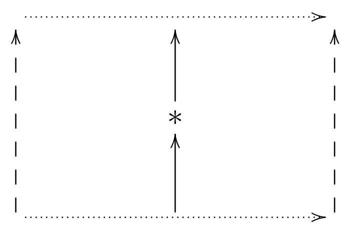
$$\left\langle g,h \middle| gh=hg \right\rangle = \mathbb{Z} \times \mathbb{Z}$$

It is actually instructive here to work out explicitly the proof of the commutation relation. We can use the usual drawing convention for the torus, namely:

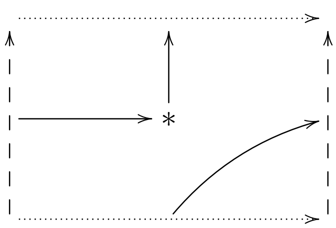


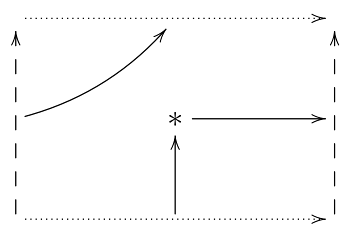
The standard generators g, h of the homotopy group are then as follows:



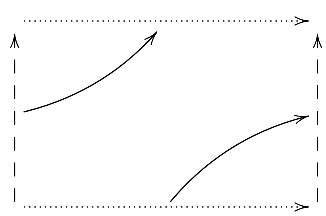


Regarding now the two compositions gh, hg, these are as follows:





But these two pictures coincide, up to homotopy, with the following picture:



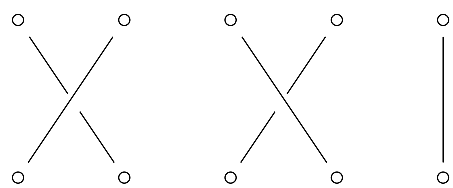
Thus we have indeed $gh = hg$, as desired, which gives the formula in (2).

(3) Regarding now the disk minus 2 points, the result here is quite clear, because the homotopy group is generated by the 2 loops around the 2 missing points, and these 2 loops are obviously free, algebrically speaking. Thus, we obtain a free product of the group $\mathbb{Z}$ with itself, which is by definition the free group on 2 generators F_2.

(4) This is again clear, because the homotopy group is generated here by the N loops around the N missing points, which are free, algebrically speaking. Thus, we obtain a N-fold free product of $\mathbb{Z}$ with itself, which is the free group on N generators F_N. □

As another interesting example, which is a bit more complicated, we have:

THEOREM 1.19. *The braid group B_k, which is the group of diagrams of type*



with composition by vertical concatenation, in the homotopy group of

$$X = (\mathbb{C}^k - \Delta)/S_k$$

with $\Delta \subset \mathbb{C}^k$ standing for the points z satisfying $z_i = z_j$ for some $i \neq j$.

PROOF. This is something quite self-explanatory, and many other things can be said here. We will be back to this later, when discussing knot invariants. □

There are many other things that can be said about homotopy groups, notably about their behavior with respect to all sorts of product and gluing operations for the topological spaces, in the spirit of those that we met in Theorem 1.18 and Theorem 1.19.

1c. Surfaces, genus

We can talk about the genus of a surface, $g \in \mathbb{N}$, as being its number of holes:

FACT 1.20. *We can talk about the genus of a surface*

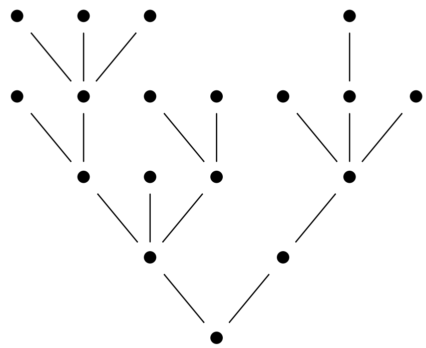
$$g \in \mathbb{N}$$

as being its number of holes.

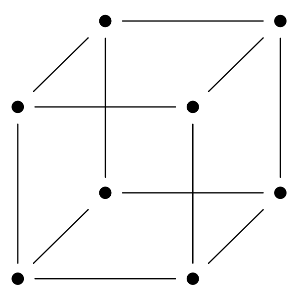
In order to be fully rigorous here, there are many possible approaches, ranging from elementary to advanced, depending on how much geometric you want to be. The best answer, which is however a bit complicated, involves complex analysis, and the notion of Riemann surface. Indeed, it is for such surfaces that the genus is best understood.

1d. Graph theory

All the above leads us into many things, mostly from discrete mathematics, via triangulations, and in particular, into graphs. Some graphs can be drawn without crossings in the plane, and we call them planar. For instance the fact that trees are planar is obvious, and as an illustration, here is some sort of "random" tree, which is clearly planar:

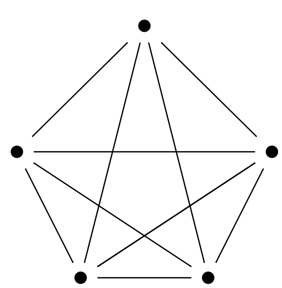


Of course, there are many other interesting examples of planar graphs, as for instance the cube graph, and up to you to tell me why this graph is planar:



However, not all graphs are planar. In order to find basic examples of non-planar graphs, we can look at simplices, and we are led to the following result:

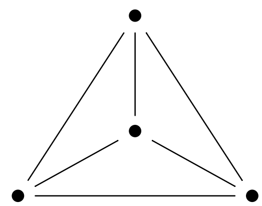
PROPOSITION 1.21. *When looking at simplices, the segment K_2, the triangle K_3 and the tetrahedron K_4 are planar. However, the next simplex K_5, namely*



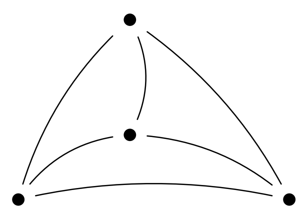
is not planar. Nor are the higher simplices, K_N with $N \geq 6$, planar.

Proof. This is something quite elementary and intuitive, as follows:

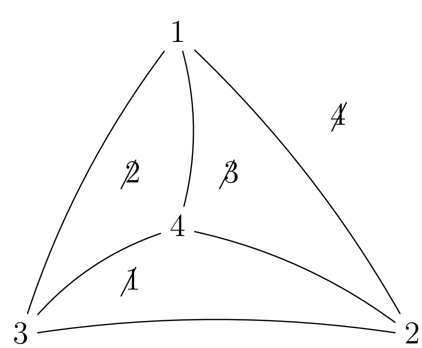
(1) The graphs K_2, K_3, K_4 are indeed planar, with this being clear for K_2, K_3, and with the planarity of K_4 being shown by the following picture for it:



(2) Regarding now the non-planarity of K_5, let us try to manufacture an intuitive proof for this. In order to draw K_5 in a planar way, we first have to draw its subgraph K_4 in a planar way, and it is pretty much clear that this can only be done as a variation of the above picture, from (1), with curved edges this time, as follows:



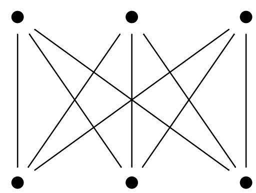
But with this in hand, it is clear that there is no room in the plane for our 5th vertex, as to avoid crossings. Indeed, we have 4 possible regions in the plane for this 5th vertex, and each of them is forbidden by the edge towards a certain vertex, as follows:



(3) Finally, the fact that the graphs K_N with $N \geq 6$ are not planar either follows from the fact that their subgraphs K_5 are not planar, that we know from (2). □

In order to find some further examples of non-planar graphs, we can look as well at the bipartite simplices, and we are led to the following result:

PROPOSITION 1.22. *When looking at bipartite simplices, the square $K_{2,2}$ is planar, and so are all the graphs $K_{2,N}$. However, the next such graph, namely $K_{3,3}$,*

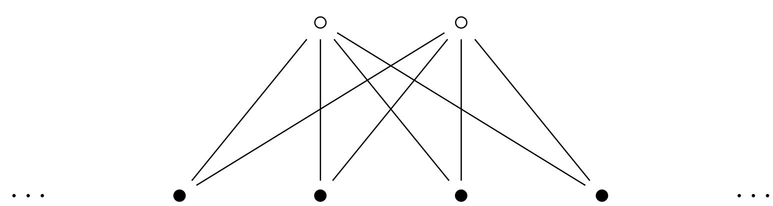


called "utility graph" is not planar. Nor are planar the graphs $K_{M,N}$, for any $M, N \geq 3$.

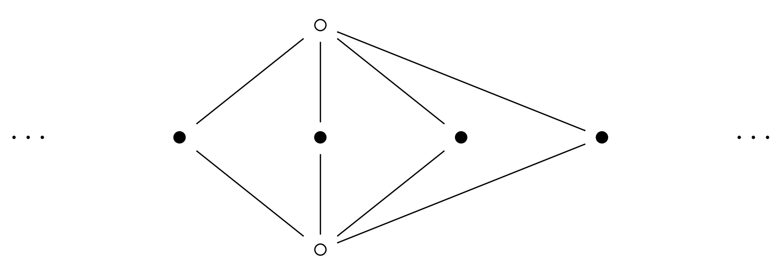
PROOF. Again, this is something elementary and intuitive, as follows:

(1) In what regards the first bipartite simplex, which is $K_{2,2}$, this is indeed the square, which is of course a planar graph, as shown by the following equality:

(2) Regarding now the bipartite simplex $K_{2,N}$ with $N \geq 2$ arbitrary, this graph looks at follows, with N vertices in the lower row:

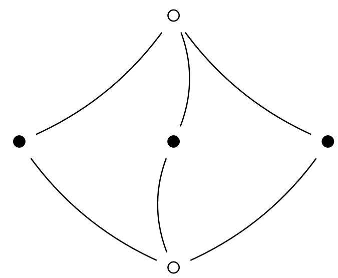


But this graph is planar too, because we can draw it in the following way:

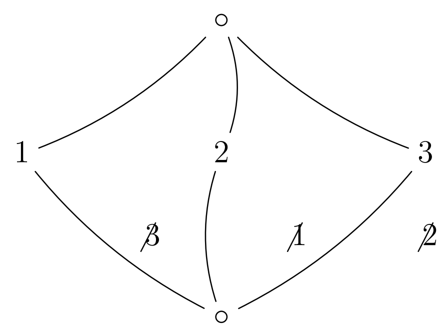


(3) Regarding now $K_{3,3}$, as before with the simplex K_5, the result here is quite clear by thinking a bit, and drawing pictures. To be more precise, reasoning by contradiction,

we first have to draw its subgraph $K_{2,3}$ in a planar way, and this is done as follows:



(4) But now, as before with K_5, it is clear that there is no room in the plane for our 6th vertex, as to avoid crossings. Indeed, we have 3 regions in the plane for this 6th vertex, and each of them is forbidden by the edge towards a certain vertex, as follows:



Thus, theorem proved for the utility graph $K_{3,3}$, via the same method as for K_5.

(5) Still talking $K_{3,3}$, let us mention that this is called indeed "utility graph", as said above, due to a certain story with it. The story involves 3 companies, selling gas, water and electricity to 3 customers, and looking for a way to arrange their underground tubes and wires as not to cross. Thus, they are looking to implement their "utillity graph", which is $K_{3,3}$, in a planar way, and unfortunately, this is not possible.

(6) And as further comments on $K_{3,3}$, quite remarkably, in recent years the stocks of the above-mentioned 3 companies have skyrocketed, apparently due to very good business done by their Saturn ring branches, which were able to considerably cut from their costs. But are we here for talking about economy, or about mathematics.

(7) Finally, the bipartite simplex $K_{M,N}$ with $M,N\geq 3$ is not planar either, because it contains $K_{3,3}$. Thus, we are led to the conclusions in the statement. □

As a first main result now about the planar graphs, we have:

THEOREM 1.23. *The fact that a graph X is non-planar can be checked as follows:*

1. *Kuratowski criterion: X contains a subdivision of K_5 or $K_{3,3}$.*
2. *Wagner criterion: X has a minor of type K_5 or $K_{3,3}$.*

PROOF. This is obviously something quite powerful, when thinking at the potential applications, and non-trivial to prove as well, the idea being as follows:

(1) Regarding the Kuratowski criterion, the convention is that "subdivision" means graph obtained by inserting vertices into edges, e.g. replacing $\bullet - \bullet$ with $\bullet - \bullet - \bullet$.

(2) Regarding the Wagner criterion, the convention there is that "minor" means graph obtained by contracting certain edges into vertices.

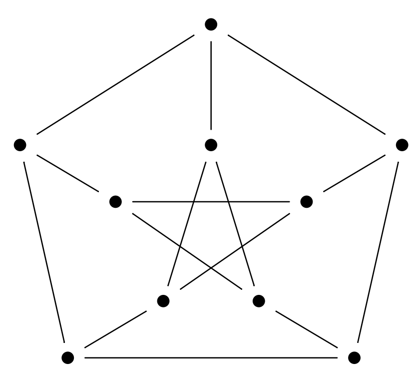
(3) Regarding now the proofs, the Kuratowski and Wagner criteria are more or less equivalent, and their proof is via standard, although long, recurrence methods.

(4) In short, non-trivial, but rather routine results that we have here, and we will leave finding and studying their complete proofs as an instructive exercise.

(5) Finally, let us mention that, often in practice, Wagner works a bit better than Kuratowski. More on this in a moment, when discussing examples. □

Regarding now the applications of the Kuratowski and Wagner criteria, things are quite tricky here, because most of the graphs that we met so far in this book are trees and other planar graphs, for which these criteria are not needed. We have as well the graphs K_N and $K_{M,N}$, to which these criteria apply trivially. Thus, for illustrations, we have to go to more complicated graphs, and as a standard example here, we have:

PROPOSITION 1.24. *The Petersen graph P, namely*



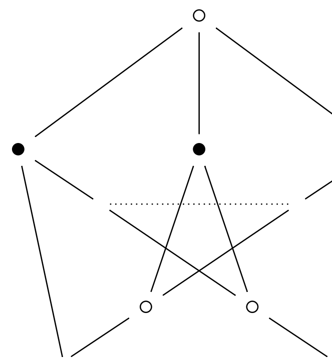
is not planar, the reasons for this being as follows:

(1) *Kuratowski: P contains no subdivision of K_5, but contains a subdivision of $K_{3,3}$.*
(2) *Wagner: P has both K_5 and $K_{3,3}$ as minors.*

PROOF. We have four things to be proved, all instructive, the idea being as follows:

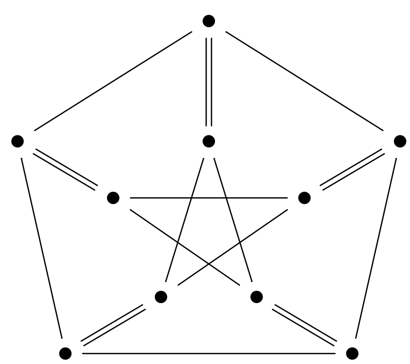
(1) To start with, P contains no subdivision of K_5, because P has valence 3, while K_5 has valence $4 > 3$. Thus, game over with the Kuratowski criterion using K_5.

(2) On the other hand, regarding $K_{3,3}$, this has valence 3, exactly as P, so there is a chance for the Kuratowski criterion using $K_{3,3}$ to apply to P. And this is indeed the case, showing that P is not planar, with the subdivision of $K_{3,3}$ being obtained as follows:



To be more precise, ignoring the dotted edges, what we have here is indeed a subdivision of $K_{3,3}$, obtained from $K_{3,3}$ by inserting 4 vertices into 4 certain edges.

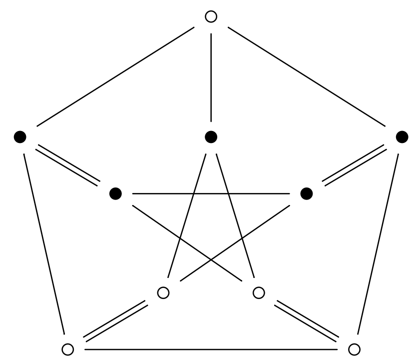
(3) Regarding now Wagner, in contrast with Kuratowski, and better than it, this applies to P by using K_5, with the K_5 minor of P being obtained as follows:



To be more precise, the convention here is that we identify the vertices joined by = edges, and this procedure obviously producing the graph K_5, we have K_5 as minor.

(4) Finally, still regarding Wagner, and adding to the power of this criterion, this applies to P as well by using $K_{3,3}$, with the $K_{3,3}$ minor of P being obtained as follows,

again with the above conventions, namely identifying the vertices joined by = edges:



Thus, we are led to the various conclusions in the statement. □

As a second main result now about the planar graphs, we have:

THEOREM 1.25. *For a connected planar graph we have the Euler formula*

$$v - e + f = 2$$

with v, e, f being the number of vertices, edges and faces.

PROOF. Given a connected planar graph, drawn in a planar way, without crossings, we can certainly talk about v and e, as for any graph, and also about f, as being the number of faces that our graph has, in our picture, with these including by definition the outer face too, the one going to ∞. As an example here, for a triangle we have $v = e = 3$ and $f = 2$, and we conclude that the Euler formula holds indeed, as:

$$3 - 3 + 2 = 2$$

More generally now, the Euler formula holds for any N-gon graph, as:

$$N - N + 2 = 2$$

But this shows that the Euler formula holds at $f = 2$, and by a standard recurrence on f, we conclude that this formula is valid at any $f \in \mathbb{N}$, as desired. □

As a third main result now about the planar graphs, we have:

THEOREM 1.26. *Any planar graph has the following properties:*

(1) *It is vertex 4-colorable.*
(2) *It is a 4-partite graph.*

PROOF. This is something quite difficult, definitely beyond our reach, in this book, but do not hesitate to look it up, and learn more about it. □

As you can see, the theory of planar graphs can vary a lot, with Theorem 1.25 being something trivial, Theorem 1.23 being something quite tricky, and Theorem 1.26 being something of extreme difficulty. Quite fascinating all this, hope you agree with me.

Switching topics now, let us get into the following question:

QUESTION 1.27. *What are the graphs which are not planar, but can be however drawn on a torus? Also, what about graphs which can be drawn on higher surfaces, having $g \geq 2$ holes, instead of the $g = 0$ holes of the sphere, and the $g = 1$ hole of the torus?*

As a first result on this subject, generalizing Theorem 1.25, we have:

THEOREM 1.28. *For a connected graph of genus $g \in \mathbb{N}$ we have the Euler formula*

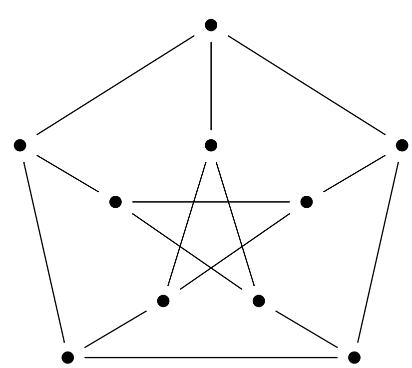
$$v - e + f = 2 - 2g$$

with v, e, f being the number of vertices, edges and faces.

PROOF. This comes as a continuation of Theorem 1.25, dealing with the case $g = 0$, and assuming that you have read in detail the proof there, to put it in this way, you will certainly have no troubles now in understanding the present extension, to genus $g \in \mathbb{N}$. □

But all this might seem a bit abstract. In practice, passed the planar graphs, $g = 0$, that we understand quite well, the next problem comes in understanding the toral graphs, $g = 1$, with a main example here being the Petersen graph, which is as follows:

THEOREM 1.29. *The Petersen graph, namely*

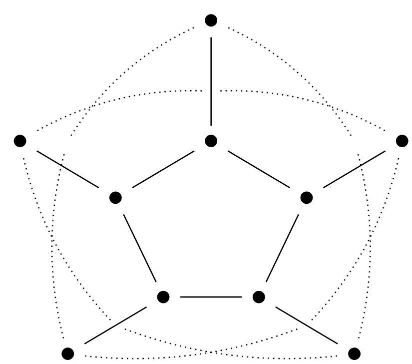


is toral, and the Euler formula for it reads $10 - 15 + 5 = 0$.

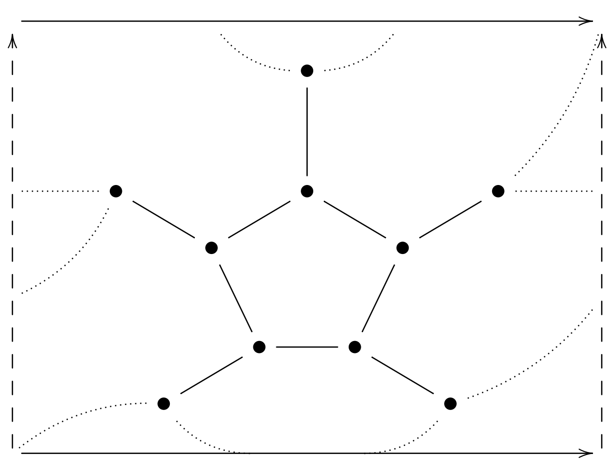
PROOF. There are several things going on here, the idea being as follows:

(1) The fact that this graph is indeed not planar can be best seen by using the Wagner criterion from Theorem 1.23, with both the graphs K_5 and $K_{3,3}$ being minors of it, and we have already talked about this, with full details, in Proposition 1.24.

(2) Regarding now the toral graph assertion, this requires some skill. By inverting the two pentagons, in the obvious way, the Petersen graph becomes as follows:

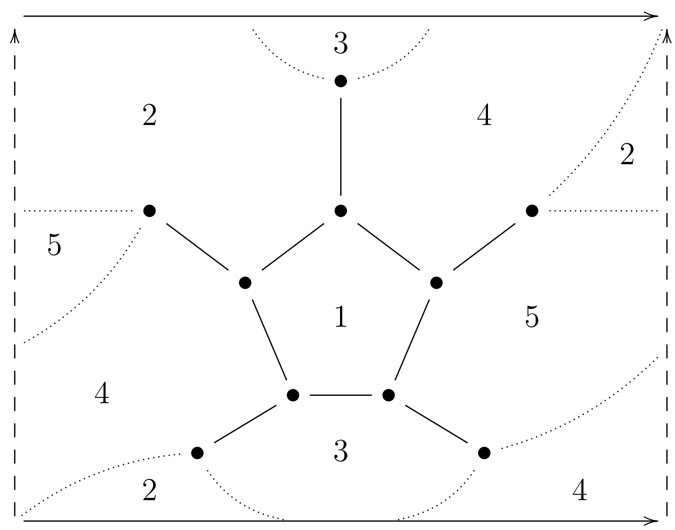


But now, we can keep the two pentagons and the solid edges, and send flying the various dotted edges, on suitable directions on a torus, as follows:



Observe the usage of the lower left vertex, which is identified with the upper right vertex, and in fact with the other two vertices of the rectangle as well, according to our gluing conventions for the torus. In any case, job done, and torality proved.

(3) In order to finish, we still have to count the number of faces, in order to check the Euler formula. But there are 5 faces, as shown by the following picture:



Thus the Euler formula holds indeed in our embedding, as $10 - 15 + 5 = 0$. □

Many other things can be said about the Petersen graph, and about other toral graphs, of similar type, or of general type. We will be back to this.

1e. Exercises

Exercises:

Exercise 1.30.

Exercise 1.31.

Exercise 1.32.

Exercise 1.33.

Exercise 1.34.

Exercise 1.35.

Exercise 1.36.

Exercise 1.37.

Bonus exercise.

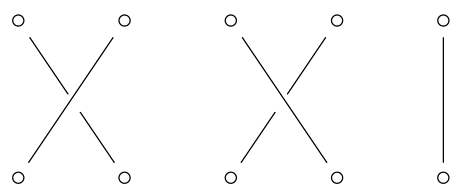
CHAPTER 2

Braids

2a. Braids

At a more advanced level now, we will need the following key observation, making the connection with group theory, and algebra in general, due to Alexander:

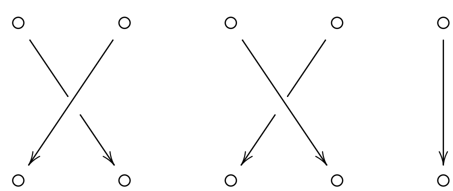
THEOREM 2.1. *Any knot or link can be thought of as being the closure of a braid,*



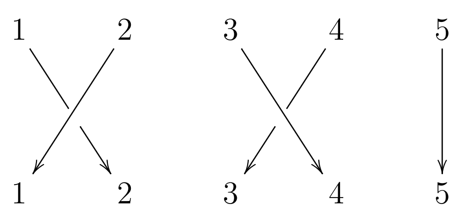
with the braids forming a group B_k, called braid group.

PROOF. Again, this is something quite self-explanatory, as follows:

(1) Consider indeed the braids with k strings, with the convention that things go from up to down. For instance the braid in the statement should be thought of as being:

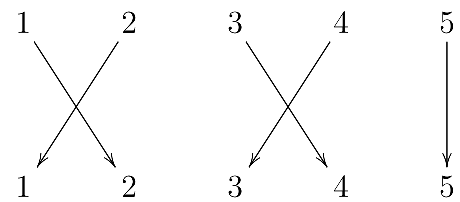


But, with this convention, braids become some sort of permutations of $\{1,\ldots,k\}$, which are decorated at the level of crossings, with for instance the above braid corresponding to the following permutation of $\{1,2,3,4,5\}$, with due decorations:



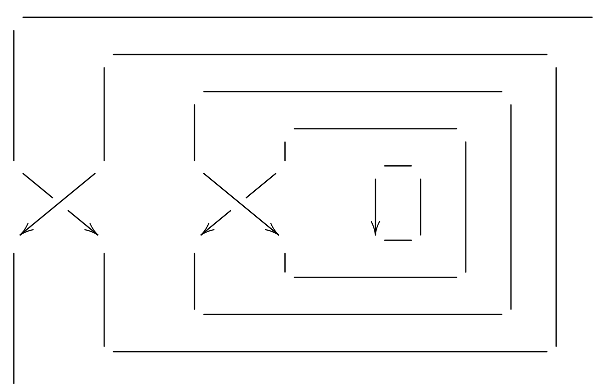
In any case, we can see in this picture that B_k is indeed a group, with composition law similar to that of the permutations in S_k, that is, going from up to down.

(2) Moreover, we can also see in this picture that we have a surjective group morphism $B_k \to S_k$, obtained by forgetting the decorations, at the level of crossings. For instance the braid pictured above is mapped in this way to the following permutation in S_5:



It is possible to do some more algebra here, in relation with the morphism $B_k \to S_k$, but we will not need this in what follows. We will keep in mind, from the above, the fact that "braids are not exactly permutations, but they compose like permutations".

(3) Regarding now the closure operation in the statement, this consists by definition in adding semicircles at right, which makes our braid into a certain oriented link. As an illustration, the closure of the braid pictured above is the following link:



(4) This was for the precise statement of the theorem, and in what regards now the proof, this can be done by some sort of cut and paste procedure, or recurrence if you prefer. As before with such things, we will leave this as an easy exercise for you. □

Many interesting things can be said about the braid group B_k, as for instance:

THEOREM 2.2. *The braid group B_k has the following properties:*

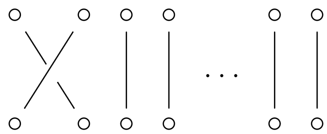
(1) *It is generated by variables $g_1, \ldots, g_{k-1}$, with the following relations:*

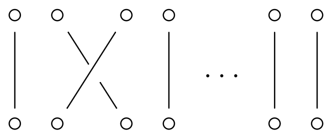
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad , \quad g_i g_j = g_j g_i \text{ for } |i-j| \geq 2$$

(2) *It is the homotopy group of $X = (\mathbb{C}^k - \Delta)/S_k$, with $\Delta \subset \mathbb{C}^k$ standing for the points z satisfying $z_i = z_j$ for some $i \neq j$.*

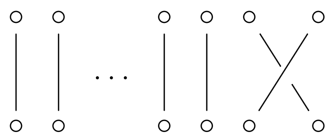
PROOF. These are things that we will not really need here, as follows:

(1) In order to prove this assertion, due to Artin, consider the following braids:

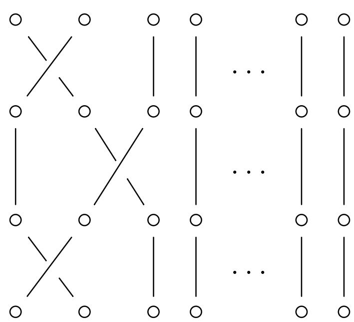
$g_1 =$ 

$g_2 =$ 

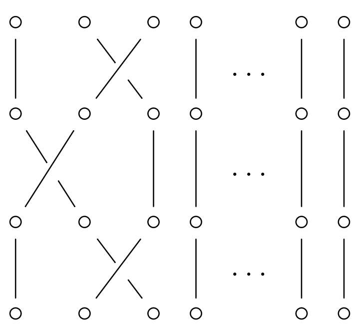
$\vdots$

$g_{k-1} =$ 

We have then $g_i g_j = g_j g_i$, for $|i-j| \geq 2$. As for the relation $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, by translation it is enough to check this at $i = 1$. And here, we first have:

$g_1 g_2 g_1 =$ 

On the other hand, we have as well the following computation:

$g_2 g_1 g_2 =$ 

Now since the above two pictures are identical, up to isotopy, we have $g_1 g_2 g_1 = g_2 g_1 g_2$, as desired. Thus, the braid group B_k is indeed generated by elements $g_1, \ldots, g_{k-1}$ with the relations in the statement, and in what regards now the proof of universality, this can only be something quite routine, and we will leave this as an instructive exercise.

(2) This is something quite self-explanatory, based on the general homotopy group material from chapter 1, and we will leave this as an easy exercise for you.

(3) Finally, before leaving the subject, let us mention that the Artin relations in (1) are something very useful, in order to construct explicit matrix representations of B_k. For instance, it can be shown that the braid group B_k is linear, and well, we will leave this as usual as an exercise for you, meaning either solve it, or look it up. □

2b.

2c.

2d.

2e. Exercises

Exercises:

Exercise 2.3.

Exercise 2.4.

Exercise 2.5.

Exercise 2.6.

Exercise 2.7.

Exercise 2.8.

Exercise 2.9.

Exercise 2.10.

Bonus exercise.

CHAPTER 3

Partitions

3a. Partitions

We have seen in the previous chapter that the study of the braids suggests introducing a number of related algebras, and notably, of the Temperley-Lieb algebra.

In order to introduce the Temperley-Lieb algebra, many options are on the table. Group theory, topology, statistical mechanics, as in the original paper of Temperley and Lieb, operator algebras, quantum mechanics, probability theory, random matrices, and many more, all these can be useful in order to introduce you to this algebra.

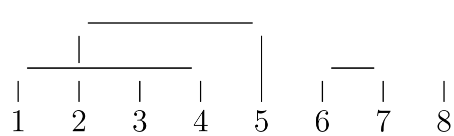
However, for "creation" purposes in mathematics, nothing beats set theory, and the mighty empty set $\emptyset$. So, here is the story that we intend to tell, to start with:

FACT 3.1. *The legend has it that $\emptyset$ produced $\mathbb{N}$ and mathematics, by recursion. In fact, $\emptyset$ produced the Temperley-Lieb algebra too, by recursion and partition.*

Very nice, so eventually, we have a plan. We will talk here about $\emptyset$ and its various creations, including sets, partitions, and later about the Temperley-Lieb algebra too.

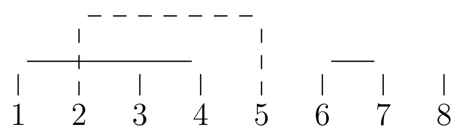
Getting started for good now, as a first definition for this chapter, we have:

DEFINITION 3.2. *We denote by $P(k)$ the set of partitions of $\{1,\ldots,k\}$, with these partitions $\pi\in P(k)$ being most conveniently being drawn as diagrams,*



with the strings joining the numbers belonging to the same block of π. That is, the above diagram represents the partition $\{1,\ldots,8\}=\{1,3,4\}\cup\{2,5\}\cup\{6,7\}\cup\{8\}$.

Observe that there is a bit of care to be taken with this convention, in respect to the crossings. We can either proceed as above, with the $\{2,5\}$ block being respresented "under" the block $\{1,3,4\}$, or use different types of strings, as for instance:



Both these conventions are good in practice, and we will be mostly using here the first one, that from Definition 3.3.

Now, let us study these partitions. And here, surprise, instead of pulling a theorem, as you would expect, and believe me I would have liked as well to have a quick theorem, to start my book, we must formulate something quite modest, as follows:

PROPOSITION 3.3. *The Bell numbers $B_k = |P(k)|$ satisfy the recurrence relation*

$$B_{k+1} = \sum_s \binom{k}{s} B_{k-s}$$

with initial data $B_0 = 1$, $B_1 = 1$, and are numerically as follows:

$$1,\ 1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ 21147,\ 115975,\ 678570,\ \ldots$$

However, there is no mathematical formula for B_k.

PROOF. There are several things going on here, the idea being as follows:

(1) Experiments first, before anything, let us compute a few Bell numbers. Obviously $B_1 = 1$, and then we have $B_2 = 2$, the partitions being as follows:

$$|\,| \quad , \quad \sqcap$$

Next, we have $B_3 = 5$, the partitions at $k = 3$ being as follows:

$$|\,|\,| \quad , \quad \sqcap\,| \quad , \quad \text{╓╖} \quad , \quad |\,\sqcap \quad , \quad \sqcap\!\sqcap$$

At $k = 4$ now, things become more complex, and it is better to trick. We can count the partitions up to permutations of the corresponding diagrams, and with this convention made, here are the relevant partitions and their multiplicities, leading to $B_4 = 15$:

$$|\,|\,|\,| \times 1 \quad , \quad \sqcap\,|\,| \times 6 \quad , \quad \sqcap\,\sqcap \times 3 \quad , \quad \sqcap\!\sqcap\,| \times 4 \quad , \quad \sqcap\!\sqcap\!\sqcap \times 1$$

The same method works at $k = 5$, with the block distributions and multiplicities, which are simpler to draw than partitions, being as follows, leading to $B_5 = 52$:

$$11111 \to 1\ ,\ 2111 \to 10\ ,\ 221 \to 15\ ,\ 311 \to 10\ ,\ 32 \to 10\ ,\ 41 \to 5\ ,\ 5 \to 1$$

As for the case $k = 6$, where $B_6 = 203$, we will leave this as an instructive exercise.

(2) Let us try now to find a recurrence for these Bell numbers. Since a partition of $\{1,\ldots,k+1\}$ appears by choosing s partners for 1, among the k numbers available, and then partitioning the $k-s$ elements left, we have the following formula:

$$B_{k+1}=\sum_s\binom{k}{s}B_{k-s}$$

Observe that this formula forces us to talk about $B_0=1$, as done in the statement.

(3) As for the last assertion, regarding the non-computability of the Bell numbers, take this as a physics fact. Mankind has tried to find a formula for these numbers, had not found anything, and we are reporting here this finding, which is of course rock-solid. □

All the above does not look very good. We seem to be going on some sort of wrong way with our partitions, most likely into one of the numerous fringe branches of mathematics.

However, as a ray of light, we have the following theorem, connecting the partitions and Bell numbers to the central objects in discrete probability, the Poisson laws:

THEOREM 3.4. *The moments of the Poisson law are the Bell numbers:*

$$p_1=\frac{1}{e}\sum_{k\in\mathbb{N}}\frac{\delta_k}{k!}\quad:\quad M_k(p_1)=|P(k)|$$

More generally, the moments of the Poisson law of parameter $t>0$ are as follows,

$$p_t=e^{-t}\sum_{k\in\mathbb{N}}\frac{t^k}{k!}\,\delta_k\quad:\quad M_k(p_t)=\sum_{\pi\in P(k)}t^{|\pi|}$$

where $|.|$ is the number of blocks.

PROOF. The moments of p_1 are given by the following formula:

$$M_k=\frac{1}{e}\sum_r\frac{r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned}M_{k+1}&=\frac{1}{e}\sum_r\frac{r^k}{r!}\left(1+\frac{1}{r}\right)^k\\&=\frac{1}{e}\sum_r\frac{r^k}{r!}\sum_s\binom{k}{s}r^{-s}\\&=\sum_s\binom{k}{s}M_{k-s}\end{aligned}$$

But the Bell numbers $B_k = |P(k)|$ satisfy the same recurrence, so we have $M_k = B_k$, as claimed. Next, the moments of p_t with $t > 0$ are given by:

$$N_k = e^{-t}\sum_r \frac{t^r r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned}
N_{k+1} &= e^{-t}\sum_r \frac{t^{r+1}r^k}{r!}\left(1+\frac{1}{r}\right)^k\\
&= e^{-t}\sum_r \frac{t^{r+1}r^k}{r!}\sum_s \binom{k}{s}r^{-s}\\
&= t\sum_s \binom{k}{s}N_{k-s}
\end{aligned}$$

But the numbers $S_k = \sum_{\pi\in P(k)} t^{|\pi|}$ are easily seen to satisfy the same recurrence, with the same initial values, namely t and $t + t^2$, so we have $N_k = S_k$, as claimed. □

Summarizing, we have some partition mathematics going on, for sure, but with the Poisson laws being something quite deep, we are not exactly into the simple and conceptual things we were wishing for. Let us record our conclusions as follows:

CONCLUSION 3.5. *The set partitions $\pi \in P(k)$ are something quite complicated, and better not mess with them, unless doing advanced probability.*

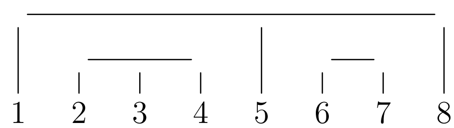
Shall we give up? Certainly not. When looking for a bug in our theory, after some thinking, that bug is in fact in Definition 3.3 and in the comments afterwards, regarding the annoyance caused by the crossings, when drawing our partitions. So, forgetting all the mathematics that we know, back to primary school, and we would prefer our partitions to be noncrossing, as for us to be able to draw them quicker. Obviously.

This might sound of course overly futile, but you know what, sometimes kids are right, and adults are wrong. So, let us record this thought, as follows:

THOUGHT 3.6. *The partitions $\pi \in P(k)$ look like non-topological objects, but when it comes to drawing them, they are topological, with the crossings causing the mess.*

And in what follows, we will trust this thought. Which teaches us something very simple, namely that in order to reach to simpler objects, we must remove the crossings. So, let us update Definition 3.3, in wishing for a better theory, as follows:

Definition 3.7. *We denote by $NC(k)$ the set of noncrossing partitions of $\{1,\ldots,k\}$, that is, of the partitions $\pi\in P(k)$ which can be drawn as noncrossing diagrams,*



with the strings joining as usual the numbers belonging to the same block of π. The above diagram represents the partition $\{1,\ldots,8\}=\{1,5,8\}\cup\{2,3,4\}\cup\{6,7\}$.

And surprise here, with this definition in hand, everything illuminates. To start with, the numbers $C_k=|NC(k)|$, called Catalan numbers, are computable, and very interesting. There are many things to be said here, and as a first result on the subject, we have:

Theorem 3.8. *The Catalan numbers $C_k=|NC(k)|$ satisfy the recurrence relation*

$$C_{k+1}=\sum_{a+b=k}C_aC_b$$

with initial data $C_0=1$, $C_1=1$, and are numerically as follows:

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ \ldots$$

Moreover, these numbers are given by the formula

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

coming from the fact that $f(z)=\sum_kC_kz^k$ satisfies $zf^2-f+1=0$.

Proof. As before with the Bell numbers, there is no hurry in proving this, and we will take our time, with experiments first, comments, and then proofs:

(1) To start with, let us compute a few Catalan numbers. At $k=1,2,3$ all the partitions are obviously noncrossing, so we have $C_k=B_k$ here, that is:

$$C_1=1\quad,\quad C_2=2\quad,\quad C_3=5$$

At $k=4$ now, we have exactly 1 crossing partition, namely $\sqcap\!\!\!\sqcap$, and we obtain:

$$C_4=B_4-1=14$$

At $k=5$, we can recycle the count for the Bell numbers, from the proof of Proposition 1.4. Taking into account the crossings, this goes as follows, yielding $C_5=42$:

$$11111\to1\ ,\ 2111\to10\ ,\ 221\to\not{15}\,10\ ,\ 311\to10\ ,\ 32\to\not{10}\,5\ ,\ 41\to5\ ,\ 5\to1$$

As for the case $k=6$, where $C_6=132$, we will leave this as an instructive exercise.

(2) Before getting into abstract mathematics, let us record a numeric comparison between the Bell and the Catalan numbers. The table here is as follows:

k	1	2	3	4	5	6	7	8	9	10
B_k	1	2	5	15	52	203	877	4140	21147	115975
C_k	1	2	5	14	42	132	429	1430	4862	16796
$B_k - C_k$	0	0	0	1	10	71	448	2710	16285	99179

This table is quite interesting, definitely showing that we are dealing with different beasts here, the point being that, with $k \to \infty$, most of the partitions appear crossing.

(3) Getting now to general theory, let us try to find a recurrence for the Catalan numbers. In order to construct a noncrossing partition of $\{1, \ldots, k+1\}$, we must choose a number of partners for 1, and by looking at the partner which appears the most at right, we are led to the following recurrence formula for the Catalan numbers:

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

Observe that this formula forces us to talk about $C_0 = 1$, as done in the statement.

(4) In order to solve our recurrence, consider the generating series of the Catalan numbers, $f(z) = \sum_{k \geq 0} C_k z^k$. In terms of this generating series, our recurrence gives:

$$\begin{aligned} zf^2 &= \sum_{a,b \geq 0} C_a C_b z^{a+b+1} \\ &= \sum_{k \geq 1} \sum_{a+b=k-1} C_a C_b z^k \\ &= \sum_{k \geq 1} C_k z^k \\ &= f - 1 \end{aligned}$$

(5) By solving the equation $zf^2 - f + 1 = 0$ found above, and choosing the solution which is bounded at $z = 0$, we obtain the following formula for our series:

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

(6) In order to compute now this function, we use the generalized binomial formula, which is as follows, with $p \in \mathbb{R}$ being an arbitrary exponent, and with $|t| < 1$:

$$(1+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} t^k$$

To be more precise, this formula, which generalizes the usual binomial formula, holds indeed due to the Taylor formula, with the binomial coefficients being given by:

$$\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}$$

(7) For the exponent $p = 1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{1/2}{k} &= \frac{1/2(-1/2)(-3/2)\ldots(3/2-k)}{k!}\\ &= (-1)^{k-1}\frac{1\cdot 3\cdot 5\ldots(2k-3)}{2^k k!}\\ &= (-1)^{k-1}\frac{(2k-2)!}{2^{k-1}(k-1)!2^k k!}\\ &= \frac{(-1)^{k-1}}{2^{2k-1}}\cdot\frac{1}{k}\binom{2k-2}{k-1}\\ &= -2\left(\frac{-1}{4}\right)^k\cdot\frac{1}{k}\binom{2k-2}{k-1}\end{aligned}$$

(8) Thus the generalized binomial formula at exponent $p = 1/2$ reads:

$$\sqrt{1+t} = 1 - 2\sum_{k=1}^\infty \frac{1}{k}\binom{2k-2}{k-1}\left(\frac{-t}{4}\right)^k$$

But with $t = -4z$ we obtain from this the following formula:

$$\sqrt{1-4z} = 1 - 2\sum_{k=1}^\infty \frac{1}{k}\binom{2k-2}{k-1} z^k$$

(9) Now back to our series f, we obtain the following formula for it:

$$\begin{aligned}f(z) &= \frac{1-\sqrt{1-4z}}{2z}\\ &= \sum_{k=1}^\infty \frac{1}{k}\binom{2k-2}{k-1}z^{k-1}\\ &= \sum_{k=0}^\infty \frac{1}{k+1}\binom{2k}{k}z^k\end{aligned}$$

(10) Thus the Catalan numbers are given by the formula the statement, namely:

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

So done, we have now proof for everything claimed in the statement. □

The above was quite exciting, but the occurrence of heavy calculus at the end can be interpreted as good or bad news, depending on your mathematical knowledge and philosophy, and mood. Personally I tend to take such things as good news, whenever I see calculus showing up in abstract algebra questions, I say to myself "calculus, saved".

But this is of course something subjective, assuming that calculus is indeed the foundation of mathematics, which, while most likely true, remains something debatable.

So, here is as well a bijective proof for the formula of C_k, that I sort of love too, while considering however that this is no match for our previous $\sqrt{1-4z}$ beauties:

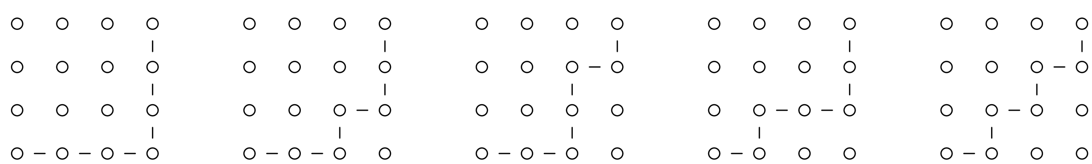
THEOREM 3.9. *The Catalan numbers are given by the formula*

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

with this being seen also by counting the length $2k$ Dyck paths in the plane.

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, the length $2k$ Dyck paths in the plane are by definition the paths from $(0,0)$ to (k,k), marching North-East over the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, by staying inside the square $[0,k] \times [0,k]$, and staying as well under the diagonal of this square. As an example, here are the 5 possible Dyck paths at $k=3$:



(2) In practice, counting a bit, and we will leave this as an exercise, shows that the number of such paths is as follows, exactly as the Catalan numbers:

$$1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ \ldots$$

Now forgetting all the math that we know, from Theorem 3.8, let us denote by C_k' the number of such length $2k$ Dyck paths. We have to do two things, namely prove that these numbers C_k' equal indeed the Catalan numbers $C_k = |NC(k)|$, and then, do some sort of direct counting, as to reach to the formula for $C_k = C_k'$ in the statement.

(3) In what concerns the first question, this is easy settled. Indeed, when looking at the point where our Dyck path last intersects the diagonal, we are led to the following recurrence relation for the number of such paths, exactly as for the Catalan numbers:

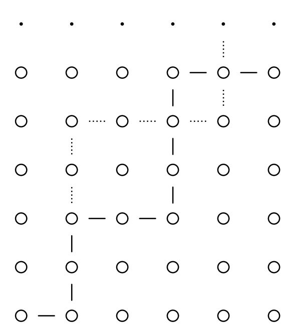
$$C_{k+1}' = \sum_{a+b=k} C_a' C_b'$$

Moreover, the initial data being $C_1'=1$, $C_2'=2$, we conclude that we have:

$$C_k'=C_k$$

(4) Let us count now the Dyck paths in the plane. For this purpose, we use a trick. If we ignore the assumption that our path must stay under the diagonal of the square, we have $\binom{2k}{k}$ such paths. And among these, we have the "good" ones, those that we want to count, and then the "bad" ones, those that we want to ignore.

(5) So, let us count the bad paths, those crossing the diagonal of the square, and reaching the higher diagonal next to it, the one joining $(0,1)$ and $(k,k+1)$. In order to count these, the trick is to "flip" their bad part over that higher diagonal, as follows:



(6) Now observe that, as it is obvious on the above picture, due to the flipping, the flipped bad path will no longer end in (k,k), but rather in $(k-1,k+1)$. Moreover, more is true, in the sense that, by thinking a bit, we see that the flipped bad paths are precisely those ending in $(k-1,k+1)$. Thus, we can count these flipped bad paths, and so the bad paths, and so the good paths too, and so good news, we are done.

(7) To finish now, by putting everything together, we have:

$$\begin{aligned} C_k' &= \binom{2k}{k}-\binom{2k}{k-1}\\ &= \binom{2k}{k}-\frac{k}{k+1}\binom{2k}{k}\\ &= \frac{1}{k+1}\binom{2k}{k} \end{aligned}$$

Thus, we are led to the formula in the statement. □

Still with me, I hope, after all this delicate counting work. And with the comment that all this was in fact the tip of the iceberg, with the Catalan numbers being involved in many other things, which are all interesting, and good to know. More on this later in this chapter, and then later in this book, on several occasions.

To conclude now this opening section, good work done for the day, with some interesting theorems, and as a summary of our findings so far, let us record:

Conclusion 3.10. *The following happen, in relation with partitions:*

(1) *The noncrossing partitions $\pi \in NC(k)$ are simpler objects than the arbitrary partitions $\pi \in P(k)$, potentially leading to interesting mathematics.*
(2) *And with this coming from the fact that, as we can see when practically drawing partitions, there is less topology involved in $NC(k)$ than in $P(k)$.*

To be more precise, all this is of course quite subjective, with (1) coming by comparing Proposition 3.4, which is ugly, with Theorem 3.8, and with Theorem 3.9 too, which are both beautiful, and with (2), which certainly contradicts our instant intuition, but go draw some partitions first, coming from Conclusion 3.5 and Thought 3.6.

Of course, I can hear you screaming, what is the point with all these subjective comments. In answer, despite the formal simplicity of both $P(k)$ and $NC(k)$, we are in fact into uncharted territory, not far from quantum mechanics. And love, hate, and subjectiveness in general can only help, a bit in the same way as in quantum mechanics.

But probably the best here, in connection with formal mathematics vs subjectivity, is to quote Hermann Weyl, one of the best mathematicans and physicists ever:

Weyl 3.11. *Among the correct and the beautiful, I always chose the beautiful.*

Finally, also in relation with this, a big, modern conjecture in physics is that at very small scales, somewhere between quarks and the Planck scale, with both ends not excluded, free geometry, coming from $NC(k)$, rules, and produces via thermodynamic limits the higher theories, including our usual, continuous geometry, coming from $P(k)$.

In short, Conclusion 3.10 is something quite deep, and if looking for a good prize in mathematics or physics, simply work some more on that. But more on all this later.

Looking at what we did so far with the Catalan numbers, and looking for more occurrences of these numbers, we are led to some sort of combinatorial wonderland. Many things can be said here, and for the purposes of our present book, let us record:

Theorem 3.12. *The Catalan numbers C_k count:*

(1) *The noncrossing partitions of* $1,\ldots,k$.
(2) *The noncrossing pairings of* $1,\ldots,2k$.
(3) *The length $2k$ loops on $\mathbb{N}$, based at* 0.
(4) *The length $2k$ Dyck paths in the plane.*

Proof. All this is standard combinatorics, the idea being as follows:

(1) This is something that we know well from the above, standing for us, with the approach that we used here, as a definition for the Catalan numbers C_k.

(2) This is something quite surprising, having no crossing counterpart, in the sense that the pairings $P_2(2k)$ of the set $\{1,\ldots,2k\}$ are by no means related to the partitions $P(k)$ of the set $\{1,\ldots,k\}$, hope we agree on this. However, by some kind of magic, when restricting the attention to the noncrossing partitions, all this works. In order to understand this phenomenon, let us begin with some examples. Let us set:

$$C_k' = |NC_2(2k)|$$

We have then $C_1' = 1$, $C_2' = 2$, coming from the following noncrossing pairings:

$$\cap \quad , \quad \cap\cap \quad , \quad ⋒$$

At $k = 3$ now, we have $C_3' = 5$, the noncrossing pairings being as follows:

$\cap\cap\cap$, $\cap⋒$, $⋒\cap$, ,  , 

And then $C_4' = 14$, $C_5' = 42$ so on, we obtain the Catalan numbers. In order now to prove this, we have two choices. First, we can try to establish a bijection as follows:

$$NC(k) \simeq NC_2(2k)$$

However, we will leave this for later, because this bijection will be in fact so important for us, that it is worth a separate treatment, with a dedicated theorem, coming with full details, comments, examples, pictures and so on. In the meantime, we can establish as well $C_k' = C_k$ by recurrence, as follows. In order to construct a noncrossing pairing of $\{1,\ldots,2k+2\}$ we must choose a partner x for the first number, 1, and then pair in a noncrossing way the $2k$ elements left, by avoiding the string $1-x$. Thus, we have:

$$C_k' = \sum_{a+b=k} C_a' C_b'$$

Since the initial data is $C_1' = 1$, $C_2' = 2$, we conclude that we have, as claimed:

$$C_k' = C_k$$

(3) This is something interesting too, which will end up in clarifying our probability work, started with Theorem 3.5. To begin with, some examples. If we denote by C_k'' the number of $2k$ loops on $\mathbb{N}$, based at 0, we first have $C_1'' = 1$, the only loop here being:

$$0 - 1 - 0$$

Then we have $C_2'' = 2$, due to two possible loops, namely:

$$0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0$$

Then we have $C''_3 = 5$, the possible loops here being as follows:

$$0 - 1 - 0 - 1 - 0 - 1 - 0$$

$$0 - 1 - 0 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 1 - 0 - 1 - 0$$

$$0 - 1 - 2 - 1 - 2 - 1 - 0$$

$$0 - 1 - 2 - 3 - 2 - 1 - 0$$

And then $C''_4 = 14$, $C''_5 = 42$ so on, we obtain the Catalan numbers. In order now to formally prove this, we can either establish a bijection with the partitions in (1), or with the pairings in (2), or pull out a formal proof, by showing that our numbers satisfy:

$$C''_k = \sum_{a+b=k} C''_a C''_b$$

But all three proofs work, and we will leave them as an instructive exercise.

(4) In what regards the Dyck paths, we already know from Theorem 3.10 that these are counted by the Catalan numbers, so done. However, if looking for some good exercises in combinatorics, prove that these Dyck paths are in bijection with the partitions in (1), and also with the pairings in (2), and also with the paths on $\mathbb{N}$ in (3). Enjoy. □

Getting back now to our philosophical considerations, regarding the creation of sets and mathematics, starting with $\emptyset$, what we have in Theorem 3.12 is quite exciting, suggesting a rivalry between noncrossing partitions and pairings. So, let us formulate:

QUESTION 3.13. *What is the correct object among:*

(1) *The set $NC(k)$ of noncrossing partitions of $\{1, \ldots, k\}$.*
(2) *The set $NC_2(2k)$ of noncrossing pairings of $\{1, \ldots, 2k\}$.*

Here the term "correct" should be taken in the sense of Weyl 3.11, meaning potentially more conceptual, potentially more useful, and in a word, since we cannot rely on mathematics that we don't have yet, simply meaning more beautiful.

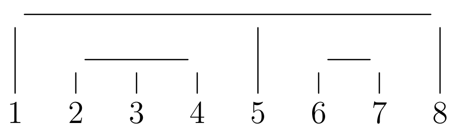
In order to deal with this question, let us first understand the bijection between our sets, which was something left open in the proof of Theorem 3.12. We have here:

THEOREM 3.14. *We have a bijection as follows,*

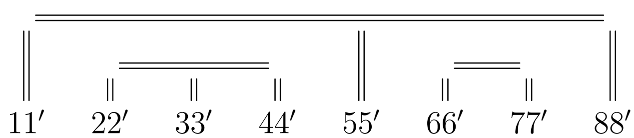
$$NC(k) \simeq NC_2(2k)$$

obtained by fattening the partitions, and by shrinking the pairings.

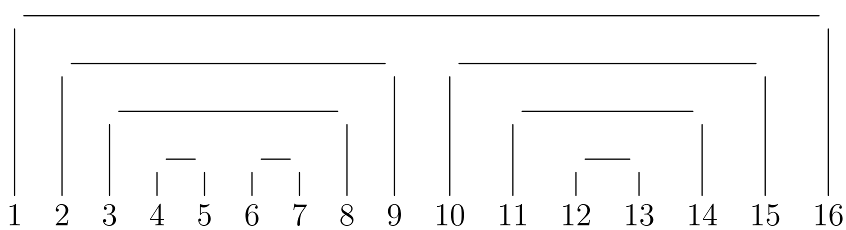
Proof. This is something self-explanatory, and in order to see how this works, let us discuss an example. Consider a noncrossing partition, say the one in Definition 3.7:



Now let us "fatten" this partition, by doubling everything, as follows:



We can see emerging here a noncrossing pairing, and by relabeling the points $1,\ldots,16$, and properly redrawing the picture, what we have is indeed a noncrossing pairing:



As for the reverse operation, that is obviously obtained by "shrinking" our pairing, by collapsing pairs of consecutive neighbors, that is, by identifying $1=2$, then $3=4$, then $5=6$, and so on. Thus, we are led to the conclusion in the statement. $\square$

With this done, let us get back to Question 3.13, which remains to be answered.

Not an easy choice, but remembering from Conclusion 3.5 and Thought 3.6 that we hate crossings, which after all appear when drawing any partition $\pi\in NC(k)-NC_{12}(k)$, with 12 standing here for "singletons and pairings", we have a naive answer, as follows:

Answer 3.15. *Pairings are better than partitions, because they are easier to draw, therefore suggesting that they contain less complex information.*

However, all this remains subjective, and since switching from partitions to pairings can amount in an earthquake, hitting all the mathematics that we did so far in this book, let us doublecheck our answer, by some alternative means.

And here, thinking a bit, the best is to go to the usual, crossing partitions. And good news, we have here the following result, which is something quite conceptual:

THEOREM 3.16. *The number of pairings of* $\{1,\ldots,k\}$ *is zero when* k *is odd, and is*

$$|P_2(k)|=k!!$$

when k *is even, with* $k!!=(k-1)(k-3)\ldots$ *Also, the moments of the normal law are*

$$g_1=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\quad:\quad M_k(g_1)=|P_2(k)|$$

and more generally, the moments of the normal law of parameter $t>0$ *are*

$$g_t=\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx\quad:\quad M_k(g_t)=\sum_{\pi\in P_2(k)}t^{|\pi|}$$

with $|.|$ *standing as usual for the number of blocks.*

PROOF. There are several things going on here, the idea being as follows:

(1) First, in what regards the count, assuming that k is even, in order to construct a pairing of $\{1,\ldots,k\}$ we must choose a partner for 1, and use a pairing of the $k-2$ elements left. Thus, we are led by recurrence to the formula in the statement, namely:

$$|P_2(k)|=(k-1)(k-3)(k-5)\ldots$$

(2) Regarding the moments of the standard normal law g_1, the odd ones vanish because the density is even, and the even ones can be computed as follows:

$$\begin{aligned}
M_k &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}x^ke^{-x^2/2}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(x^{k-1})\left(-e^{-x^2/2}\right)'dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(k-1)x^{k-2}e^{-x^2/2}dx\\
&= (k-1)\times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}x^{k-2}e^{-x^2/2}dx\\
&= (k-1)M_{k-2}
\end{aligned}$$

Thus by recurrence, we are led to the formula in the statement.

(3) Finally, regarding the moments of the normal law g_t with $t>0$, we can get them either from (2) via a change of variable, or by redoing the computation, which gives:

$$M_k=t(k-1)M_{k-2}$$

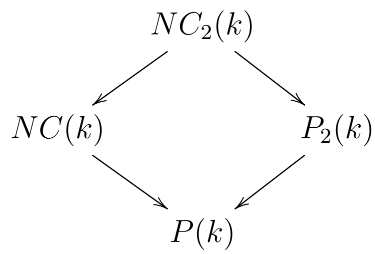
Thus, we are led to the following formula for these moments:

$$M_k=t^{k/2}|P_2(k)|$$

But this can be reformulated more fancily as in the statement, as to make the link with what we have in Theorem 3.5, the point being that the number of blocks of any $\pi \in P_2(k)$ is of course $|\pi| = k/2$. Thus, we are led to the conclusions in the statement. $\square$

All the above is quite exciting, and time now to face the truth. We got it all wrong with our partitions, be them crossing or noncrossing, the good objects are obviously the pairings, and more specifically, the noncrossing pairings. Let us record this, as follows:

Conclusion 3.17. *The correct hierarchy of the various sets of partitions is*



with $NC_2(k)$ being the king, for a multitude of reasons, explained above.

To be more precise here, the "multitude of reasons" evoked above include the primary school drawing of our partitions, the mathematical count of these partitions, and also the probabilistic aspects of these partitions, with in each case $NC_2(k)$, sometimes helped by its close subordinates, namely $NC(k)$ and $P_2(k)$, clearly beating $P(k)$.

What to do now, in view of all this? As always when it comes to discovering new things, blowing up previous mathematics that you did, with sweat and tears, relax and enjoy. There are actually 3 things to be done, in relation with all this, namely:

(1) Rewrite what we know about Catalan numbers, with $NC_2(k)$ coming first.

(2) Explore further algebraic properties of $NC_2(k)$, by playing with pairings.

(3) Have done as well the probabillistic aspects of $NC_2(k)$, and of $NC(k)$ too.

In what follows we will leave (1) as a thought exercise, with this being just a matter of meditating a bit at what we did in this chapter, and how this reorganizes with $NC_2(k)$ coming first. Regarding (2), we will certainly jump on this, and develop this next. As for (3), no hurry here, and we will leave this for the end of this chapter. With the good news, coming in advance, that we will reach in this way to the central laws in random matrix theory, namely those of Wigner and Marchenko-Pastur, and with this providing us with some solid evidence that we are on our way in doing some good physics, with all this.

3b.

3c.

3d.

3e. Exercises

Exercises:

EXERCISE 3.18.

EXERCISE 3.19.

EXERCISE 3.20.

EXERCISE 3.21.

EXERCISE 3.22.

EXERCISE 3.23.

EXERCISE 3.24.

EXERCISE 3.25.

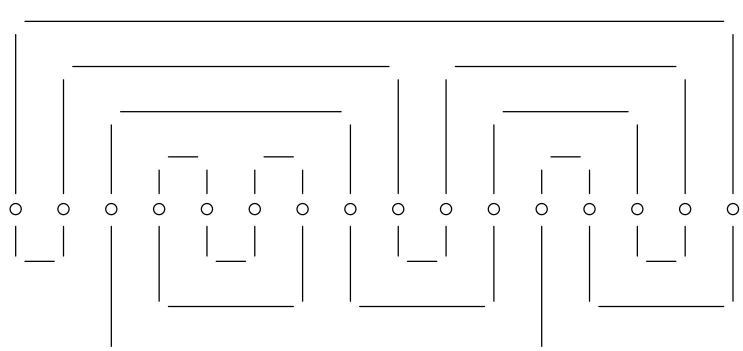
Bonus exercise.

CHAPTER 4

Algebras

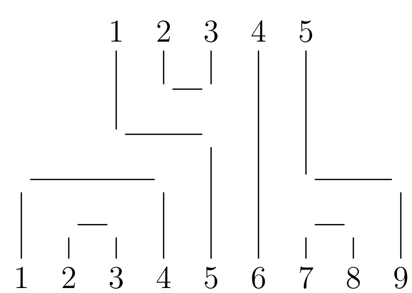
4a. Algebras

In order to further advance, the idea is to use the obvious algebraic operation on the pairings in $NC_2(k)$, obtained by superposing such pairings. This leads to some interesting diagrams, known as "meanders", and here is an illustrating example:



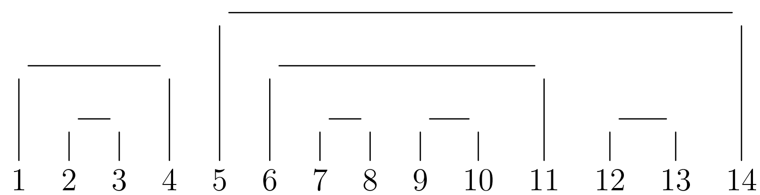
However, we can in fact do better than this. Remember category theory, telling us that for conceptual mathematics, we need objects, and arrows between them? We can do this in our context, by formulating first the following definition:

DEFINITION 4.1. *We denote by $NC_2(k,l)$ the set of noncrossing pairings between an upper row of k points, and a lower row of l points, with for instance*



being an element of $NC_2(5,9)$. With the remark that at $k=0$ we obtain the former $NC_2(l)$, and that at $l=0$ we obtain the former $NC_2(k)$, written upside down.

Observe that we have $NC_2(k,l) = \emptyset$ when $k+l$ is odd. As another key remark, the above definition brings in fact nothing new, combinatorially speaking, because we can always rotate the upper legs, say via $\curvearrowright$, as to reach to a diagram in $NC_2(k+l)$. As an illustration, the rotated version of the pairing in Definition 4.1 looks as follows:



Thus, no need for new counting results of anything, we are ready to go with more algebra. Now with the above definition in hand, we can formulate:

Definition 4.2. *The Temperley-Lieb category TL_N° has the positive integers $\mathbb{N}$ as objects, with the space of arrows $k \to l$ being the formal span*

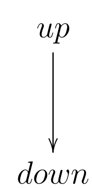
$$TL_N^\circ(k,l) = span(NC_2(k,l))$$

and with the composition of arrows appearing by composing the pairings, in the obvious way, with the rule $\bigcirc = N$, for the closed loops that might appear.

This definition is something quite subtle, hiding several non-trivial things, and is worth a detailed discussion, our comments about it being as follows:

(1) First of all, our scalars in this book will be complex numbers, $\lambda \in \mathbb{C}$, and the "formal span" in the above must be understood in this sense, namely abstract complex vector space spanned by the elements of $NC_2(k,l)$. Of course it is possible to use an arbitrary field, at least at this stage of things, but remember that we are interested in quantum mechanics, and related mathematics, where the field of scalars is $\mathbb{C}$.

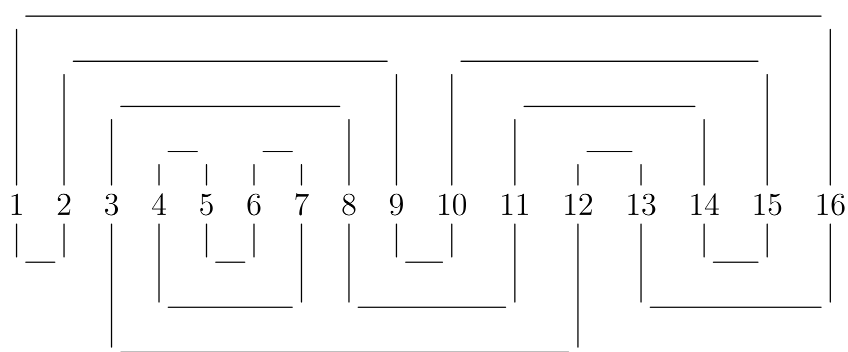
(2) Regarding the composition of arrows, this is by obvious vertical concatenation, with the convention, for here and for the rest of this book, that things go "from up to down". And with this convention coming from pure laziness, why pushing things from left to right, when we can have gravity work for us, pulling them from up to down:



(3) Less poetically, this "from up to down" convention is also useful for purely mathematical purposes, because the left-right direction will be reserved for the intervention of sums Σ and scalars $\lambda \in \mathbb{C}$, while the up-down direction will be reserved for "action". But

of course, you might argue that this is a bit poetical, too. To which I will answer, give up with your cool and poetry, and your math will soon become some total garbage.

(4) More seriously now, let us discuss what happens with the closed circles, when concatenating. As an example here, let us consider the meander pictured before:



According to our conventions, this meander appears as the product $\pi\sigma \in NC_2(0,0)$ between the upper pairing $\sigma \in NC_2(0,16)$ and the lower pairing $\pi \in NC_2(16,0)$. But, what is the value of this product? We have two loops appearing, namely:

$$1 - 2 - 9 - 10 - 15 - 14 - 11 - 8 - 3 - 12 - 13 - 16$$

$$4 - 5 - 6 - 7$$

Thus, according to Definition 4.2, the value of this meander is N^2, with one N for each of the above loops, and with these two values of N multiplying each other.

(5) The same discussion applies to an arbitrary composition $\pi\sigma \in NC_2(k,m)$ between an upper pairing $\sigma \in NC_2(k,l)$ and a lower pairing $\pi \in NC_2(l,m)$, with a certain number of loops appearing in this way, each contributing with a multiplicative factor N.

(6) Finally, in Definition 4.2 the value of the circle $N = \bigcirc$ can be pretty much anything, but due to some positivity reasons to become clear later, we will assume in what follows $N \in [1,\infty)$. Also, we will call this parameter N the "index", with the precise reasons for calling this index to become clear later, too, as this books develops.

With all this discussed, what is next? More category theory I guess, and matter of having a theorem formulated too, instead of definitions only, let us formulate:

THEOREM 4.3. *The Temperley-Lieb category TL_N° is a tensor $*$-category, with:*
1. *Composition of arrows: by vertical concatenation.*
2. *Tensoring of arrows: by horizontal concatenation.*
3. *Star operation: by turning the arrows upside-down.*

Proof. This is more of a definition, disguised as a theorem. To be more precise, we already know about (1), from Definition 4.2, and we can talk as well about (2) and (3), constructed as above, with (2) using of course multiplicativity with respect to the scalars, and with (3) using antimultiplicativity with respect to the scalars:

$$\left(\sum_i\lambda_i\pi_i\right)\otimes\left(\sum_j\mu_j\sigma_j\right)=\sum_{ij}\lambda_i\mu_j\pi_i\otimes\sigma_j$$

$$\left(\sum_i\lambda_i\pi_i\right)^*=\sum_i\bar{\lambda}_i\pi_i^*$$

And the point now is that our three operations are compatible with each other via all sorts of compatibility formulae, which are all clear from definitions, with the conclusion being that what we have a tensor $*$-category, as stated. We will leave the details here, basically amounting in figuring out what a tensor $*$-category exactly is, as an exercise. □

In order to further understand the category TL_N°, let us focus on its diagonal part, formed by the End spaces of various objects. With the convention that these End spaces embed into each other by adding bars at right, this is a graded algebra, as follows:

$$TL_N=\bigcup_{k\geq0}TL_N^\circ(k,k)$$

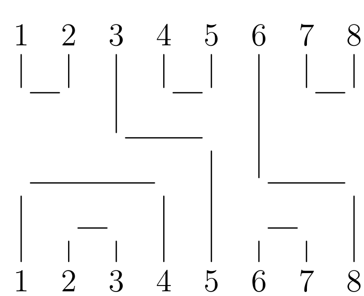
Moreover, for further fine-tuning our study, let us actually focus on the individual components of this graded algebra. These components will play a key role in what follows, and they are worth a dedicated definition, and new notation and name, as follows:

Definition 4.4. *The Temperley-Lieb algebra $TL_N(k)$ is the formal span*

$$TL_N(k)=span(NC_2(k,k))$$

with multiplication coming by concatenating, with the rule $\bigcirc=N$.

In other words, $TL_N(k)$ appears as the formal span of the noncrossing pairings between an upper row of k points, and a lower row of k points, with multiplication coming by concatenating, with $\bigcirc=N$. As an example, here is a basis element of $TL_N(8)$:



Getting back now to what we know about TL_N°, from Theorem 4.3, the tensor product operation makes sense in the context of the diagonal algebra TL_N, but does not apply to its individual components $TL_N(k)$. However, the involution is useful, and we have:

THEOREM 4.5. *The Temperley-Lieb algebra $TL_N(k)$ is a $*$-algebra, with involution coming by turning the diagrams upside-down.*

PROOF. This is something trivial, which follows from Theorem 4.3, and can be verified as well directly, and we will leave this as an instructive exercise. □

There are many things that we can do, as a continuation of the above. First, we can further study the Temperley-Lieb algebra $TL_N(k)$, for instance with a multimatrix decomposition for it, and also with a study of its natural trace $tr : TL_N(k) \to \mathbb{C}$, obtained by "closing" the diagrams in the obvious way. We will leave all this for later.

Also, we can do many algebraic and topological things with $TL_N(k)$, such as working out a number of selected Brauer theorems, for groups or quantum groups, or constructing some selected knot invariants. Again, we will leave all this for later in this book.

For the end of this chapter, however, let us do something analytic, that was left open in the above. We would like to solve the following question:

QUESTION 4.6. *We know that $P_2(k)$ corresponds to the normal laws g_t, and that $P(k)$ corresponds to the Poisson laws p_t. What about $NC_2(k)$ and $NC(k)$?*

Observe that this is related indeed to the Temperley-Lieb algebra TL_N, because we can define the Poincaré series of this graded algebra as follows:

$$\begin{aligned} f(z) &= \sum_{k\geq 0}\dim(TL_N(k))z^k \\ &= \sum_{k\geq 0}|NC_2(k,k)|z^k \\ &= \sum_{k\geq 0}|NC_2(2k)|z^k \\ &= \sum_{k\geq 0}|NC(k)|z^k \end{aligned}$$

Thus, we can reformulate Question 4.6 in a more fancy way, as follows:

QUESTION 4.7. *What are the measures π_1,γ_1 having the Poincaré series*

$$f(z) = \sum_{k\geq 0}\dim(TL_N(k))z^k$$

and its version $g(z) = f(z^2)$ as Stieltjes transforms? What about π_t,γ_t, with $t > 0$?

Here we are assuming a bit of familiarity with advanced algebra and probability, but clarifying this fancy blurb being not a pressing issue, we can always do this later in this book, no worries for that, let us get back now to work, and do some computations. We have the following result, in the spirit of the results from chapter 3:

THEOREM 4.8. *The moments of the Wigner semicircle law are*

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx \quad : \quad M_k(\gamma_1) = |NC_2(k)|$$

and the moments of the Marchenko-Pastur law are

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx \quad : \quad M_k(\pi_1) = |NC(k)|$$

and in addition, we have suitable $t > 0$ analogues of both these results.

PROOF. This follows as usual, via calculus, the idea being as follows:

(1) Regarding the two moment formulae in the statement, these both follow by doing some standard calculus, which shows that the moments in question satisfy the needed recurrence formulae, and we will leave the proofs here as an instructive exercise.

(2) Alternatively, and answering a question that you surely have in mind, you can also come upon the measures in the statement via the Stieltjes inversion formula, which states that the density of a real probability measure μ can be recaptured from its sequence of moments $\{M_k\}_{k\geq 0}$ by setting $G(\xi) = \xi^{-1} + M_1\xi^{-2} + M_2\xi^{-3} + \ldots$, and then:

$$d\mu(x) = \lim_{t\searrow 0} -\frac{1}{\pi}\, Im\,(G(x+it)) \cdot dx$$

(3) So, exercise for you to work out all this, Stieltjes inversion at $t = 1$, as to reach to γ_1, π_1, and then at general $t > 0$ too, with the desired moment formula for γ_t, π_t being the usual one, namely $M_k = \sum_{\pi\in D(k)} t^{|\pi|}$, with $D = NC_2, NC$ respectively. □

Let us discuss now the positivity of the trace, first for the Temperley-Lieb algebra, following Di Francesco, and then in more general situations, following Jones.

Let us begin with some standard combinatorics, as follows:

DEFINITION 4.9. *Let $P(k)$ be the set of partitions of $\{1,\ldots,k\}$, and $\pi,\sigma \in P(k)$.*

1. *We write $\pi \leq \sigma$ if each block of π is contained in a block of σ.*
2. *We let $\pi \vee \sigma \in P(k)$ be the partition obtained by superposing π,σ.*

Also, we denote by $|.|$ the number of blocks of the partitions $\pi \in P(k)$.

As an illustration here, at $k = 2$ we have $P(2) = \{||, \sqcap\}$, and we have:

$$|| \leq \sqcap$$

Also, at $k = 3$ we have $P(3) = \{|||, \sqcap|, \sqcap\hspace{-0.45em}|, |\sqcap, \sqcap\sqcap\}$, and the order relation is as follows:

$$||| \ \leq \ \sqcap| \ , \ \sqcap\hspace{-0.45em}| \ , \ |\sqcap \ \leq \ \sqcap\sqcap$$

In relation with our linear independence questions, the idea will be that of using:

Proposition 4.10. *The Gram matrix of the vectors ξ_π is given by the formula*

$$< \xi_\pi, \xi_\sigma >= N^{|\pi \vee \sigma|}$$

where $\vee$ is the superposition operation, and $|.|$ is the number of blocks.

Proof. According to the formula of the vectors ξ_π, we have:

$$\begin{aligned} < \xi_\pi, \xi_\sigma > &= \sum_{i_1 \ldots i_k} \delta_\pi(i_1, \ldots, i_k) \delta_\sigma(i_1, \ldots, i_k) \\ &= \sum_{i_1 \ldots i_k} \delta_{\pi \vee \sigma}(i_1, \ldots, i_k) = N^{|\pi \vee \sigma|} \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In order to study the Gram matrix $G_k(\pi, \sigma) = N^{|\pi \vee \sigma|}$, and more specifically to compute its determinant, we will use several standard facts about the partitions. We have:

Definition 4.11. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ -\sum_{\pi \leq \tau < \sigma} \mu(\pi, \tau) & \text{if } \pi < \sigma \\ 0 & \text{if } \pi \not\leq \sigma \end{cases}$$

with the construction being performed by recurrence.

As an illustration here, for $P(2) = \{||, \sqcap\}$, we have by definition:

$$\mu(||, ||) = \mu(\sqcap, \sqcap) = 1$$

Also, $|| < \sqcap$, with no intermediate partition in between, so we obtain:

$$\mu(||, \sqcap) = -\mu(||, ||) = -1$$

Finally, we have $\sqcap \not\leq ||$, and so we have as well the following formula:

$$\mu(\sqcap, ||) = 0$$

Thus, as a conclusion, we have computed the Möbius matrix $M_2(\pi, \sigma) = \mu(\pi, \sigma)$ of the lattice $P(2) = \{||, \sqcap\}$, the formula being as follows:

$$M_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which states that the following happens:

$$f(\sigma)=\sum_{\pi\leq\sigma}g(\pi)\qquad\Longrightarrow\qquad g(\sigma)=\sum_{\pi\leq\sigma}\mu(\pi,\sigma)f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

THEOREM 4.12. *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1 & \text{if } \pi\leq\sigma\\ 0 & \text{if } \pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.

PROOF. This is well-known, coming for instance from the fact that A_k is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 4.11. □

As an illustration, for $P(2)$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following key result:

PROPOSITION 4.13. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G_k=A_kL_k$, where

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1) & \text{if } \sigma\leq\pi\\ 0 & \text{otherwise}\end{cases}$$

and where A_k is the adjacency matrix of $P(k)$.

PROOF. We have the following computation, based on Proposition 4.10:

$$\begin{aligned}G_k(\pi,\sigma)&=N^{|\pi\vee\sigma|}\\&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According now to the definition of A_k, L_k, this formula reads:

$$\begin{aligned} G_k(\pi,\sigma) &= \sum_{\tau\geq\pi} L_k(\tau,\sigma) \\ &= \sum_\tau A_k(\pi,\tau)L_k(\tau,\sigma) \\ &= (A_kL_k)(\pi,\sigma) \end{aligned}$$

Thus, we are led to the formula in the statement. □

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix} N^2 & N \\ N & N \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^2-N & 0 \\ N & N \end{pmatrix}$$

We are led in this way to the following formula, due to Lindstöm:

THEOREM 4.14. *The determinant of the Gram matrix G_k is given by*

$$\det(G_k) = \prod_{\pi\in P(k)} \frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N<k$ we obtain 0.

PROOF. If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, A_k is upper triangular, and L_k is lower triangular. Thus, we have:

$$\begin{aligned} \det(G_k) &= \det(A_k)\det(L_k) \\ &= \det(L_k) \\ &= \prod_\pi L_k(\pi,\pi) \\ &= \prod_\pi N(N-1)\ldots(N-|\pi|+1) \end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us discuss now the case of the orthogonal group O_N. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. We have then:

THEOREM 4.15. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN}) = \prod_{|\lambda|=k/2} f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda)=\prod_{(i,j)\in\lambda}(N+2j-i-1)$.

PROOF. This follows from some technical results of Zinn-Justin. Indeed, it is known from there that the Gram matrix is diagonalizable, as follows:

$$G_{kN} = \sum_{|\lambda|=k/2} f_N(\lambda) P_{2\lambda}$$

To be more precise, here $1 = \sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are by definition those in the statement. Now since we have $Tr(P_{2\lambda}) = f^{2\lambda}$, this gives the result. □

For the free orthogonal and symmetric groups, coming from noncrossing pairings and partitions, the results, by Di Francesco [22], are substantially more complicated.

In order to discuss this, we will need the following standard fact, in relation with the fattening and shrinking of noncrossing partitions, that we met in chapter 3:

PROPOSITION 4.16. *The Gram matrices of the sets of partitions*

$$NC_2(2k) \simeq NC(k)$$

are related by the following formula,

$$G_{2k,n}(\pi, \sigma) = n^k (\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi', \sigma')$$

where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}.

PROOF. In the context of the bijection $NC_2(2k) \simeq NC(k)$, we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we are led to the formula in the statement. □

Now back to our explicit computation questions, let us begin our study with some examples. We first have the following result, which is elementary:

PROPOSITION 4.17. *The first Gram matrices and determinants for O_N^+ are*

$$\det \begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix} = N^2(N^2 - 1)$$

$$\det \begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix} = N^5(N^2 - 1)^4(N^2 - 2)$$

with the matrices being written by using the lexicographic order on $NC_2(2k)$.

PROOF. The formula at $k=2$, where $NC_2(4)=\{\sqcap\sqcap,\bigcap\!\!\!\!\!\cap\}$, is clear from definitions. At $k=3$ however, things are tricky. The partitions here are as follows:

$$NC(3)=\{|||,\sqcap|,\sqcap\!\!\!\!|\;,|\sqcap,\sqcap\!\sqcap\}$$

The Gram matrix and its determinant are, according to Theorem 4.14:

$$\det\begin{pmatrix}N^3&N^2&N^2&N^2&N\\N^2&N^2&N&N&N\\N^2&N&N^2&N&N\\N^2&N&N&N^2&N\\N&N&N&N&N\end{pmatrix}=N^5(N-1)^4(N-2)$$

By shrinking the partitions into pairings, we obtain, for $NC_2(6)$:

$$\begin{aligned}\det(G_{6N})&=\frac{1}{N^2\sqrt{N}}\times N^{10}(N^2-1)^4(N^2-2)\times\frac{1}{N^2\sqrt{N}}\\&=N^5(N^2-1)^4(N^2-2)\end{aligned}$$

Thus, we have obtained the formula in the statement. □

In general, following Di Francesco [22], we have the following result:

THEOREM 4.18. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb{Z}$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb{Z}$.

PROOF. This is something quite technical, obtained by using a decomposition as follows of the Gram matrix G_{kN}, with the matrix T_{kN} being lower triangular:

$$G_{kN}=T_{kN}T_{kN}^t$$

Thus, a bit as in the proof of the Lindstöm formula, we obtain the result, but the problem lies however in the construction of T_{kN}, which is non-trivial. See [22]. □

Regarding S_N^+, we have here the following formula, also from Di Francesco [22]:

THEOREM 4.19. *The determinant of the Gram matrix for S_N^+ is given by*

$$\det(G_{kN})=(\sqrt{N})^{a_k}\prod_{r=1}^k P_r(\sqrt{N})^{d_{kr}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb{Z}$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb{Z}$, and where $a_k=\sum_{\pi\in\mathcal{P}(k)}(2|\pi|-k)$.

PROOF. This follows indeed from Theorem 4.18, by fattening the pairings. □

4b.

4c.

4d.

4e. Exercises

Exercises:

EXERCISE 4.20.

EXERCISE 4.21.

EXERCISE 4.22.

EXERCISE 4.23.

EXERCISE 4.24.

EXERCISE 4.25.

EXERCISE 4.26.

EXERCISE 4.27.

Bonus exercise.

Part II

Knot invariants

The stars shine so bright
But they're fading after dawn
There is magic
In Kingston Town

CHAPTER 5

Knots and links

5a. Knots and links

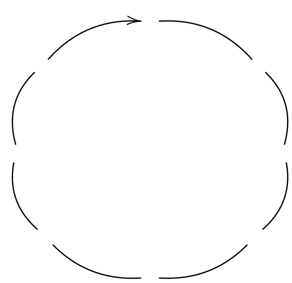
Leaving the graphs and related topological spaces aside, let us focus now on the simplest objects of topology, which are the knots. Knots are something very familiar, from the real life, and mathematically, it is most convenient to define them as follows:

DEFINITION 5.1. *A knot is a smooth closed curve in* $\mathbb{R}^3$*,*

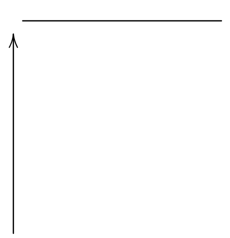
$$\gamma : \mathbb{T} \to \mathbb{R}^3$$

regarded modulo smooth transformations of $\mathbb{R}^3$*.*

Observe that our knots are by definition oriented. The reverse knot $z \to \gamma(z^{-1})$ can be isomorphic or not to the original knot $z \to \gamma(z)$, and we will discuss this in a moment. At the level of examples, we first have the unknot, represented as follows:



For typographical reasons, it is most convenient to represent our knots by squarish diagrams, with these being far easier to type in Latex, the computer program used for writing math books, with the unknot for instance being represented as follows:

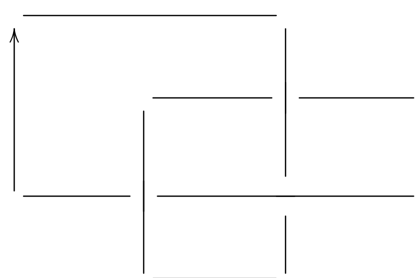


The unknot is already a quite interesting mathematical object, suggesting a lot of exciting mathematical questions, for the most quite difficult, as follows:

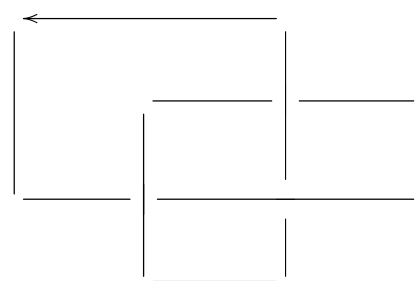
QUESTIONS 5.2. *In relation with the unknot:*

(1) *Given a closed curve in $\mathbb{R}^3$, say given via its algebraic equations, can we decide if it is tied or not?*
(2) *Perhaps simpler, given the 2D picture of a knot, can we decide if the knot is tied or not?*
(3) *Experience with cables and ropes shows that a random closed curve is usually tied. But, can we really prove this?*

Obviously, difficult questions, and as you can see, knot theory is not an easy thing. But do not worry, we will manage to find our way through this jungle, and even come up with some mathematics for it. Going ahead now with examples, as the simplest possible true knot, meaning tied knot, we have the trefoil knot, which looks as follows:



We also have the opposite trefoil knot, obtained by reversing the orientation, whose picture is identical to that of the trefoil knot, save for the orientation of the arrow:



As before with the unknot, while the trefoil knot might look quite trivial, when it comes to formal mathematics regarding it, we are quickly led into delicate questions. Let us formulate a few intuitive observations about it, as follows:

FACT 5.3. *In relation with the trefoil knot:*

(1) *This knot is indeed tied, that is, not isomorphic to the unknot.*
(2) *The trefoil knot and its opposite knot are not isomorphic.*

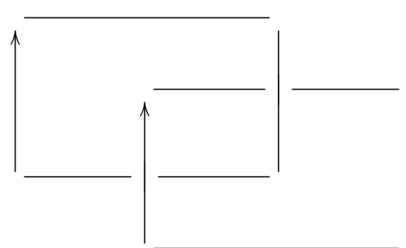
To be more precise, here (1) is something which definitely holds, as we know it from real life, but if looking for a formal proof for this, based on Definition 5.1, we will certainly run into troubles. As for (2), here again we are looking for troubles, because when playing with two trefoil knots, made from rope, with opposite arrows marked on them, we certainly see that our two beasts are not identical, but go find a formal proof for that.

In short, as before with the unknot, modesty. For the moment, let us keep exploring the subject, by recording as Questions and Facts things that we see and feel, but cannot prove yet, mathematically, based on Definition 5.1 alone, due to a lack of tools.

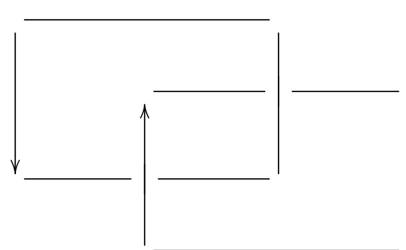
Getting back now to Definition 5.1, as stated, it is convenient to allow, in relation with certain mathematical questions, links in our discussion:

Definition 5.4. *A link is a collection of disjoint knots in* $\mathbb{R}^3$, *taken as usual oriented, and regarded as usual up to isotopy.*

As before with the knots, which can be truly knotted or not, there is a discussion here with respect to the links, which can be truly linked or not, and with orientation involved too. Drawing some pictures here, with some basic examples, is very instructive, the idea being that two or several basic unknots can be linked in many possible ways. For instance, as simplest non-trivial link, made of two unknots, which are indeed linked, we have:



By reversing the orientation of one unknot, we have as well the following link:



This was for the story of two linked unknots, which is easily seen to stop here, with the above two links, but when trying to link N unknots, with $N = 3, 4, 5, \ldots$, many things can happen. Which leads us into the following philosophical question:

Question 5.5. *Mathematically speaking, which are simpler to enumerate,*

(1) *Usual knots, that is, links with one component,*
(2) *Or links with several components, all being unknots,*

and this, in order to have some business started, for the links?

And with this being probably enough, as preliminary experimental work, time now to draw some conclusions. Obviously, what we have so far, namely Questions 5.2, Fact 5.3 and Question 5.5, is extremely interesting, at the core of everything that can be called "geometry". And by further thinking a bit, at how knots and links can be tied, in so many fascinating ways, we are led to the following philosophical conclusion:

Conclusion 5.6. *Knots and links are to geometry and topology what prime numbers are to number theory.*

Very nice all this, we are now certainly motivated for studying the knots and links, and time for some mathematics. But the question is, how to get started?

In view of the above, this is not an easy question. Fortunately, graph theory comes to the rescue, via to the following simple fact, which will be our starting point:

Fact 5.7. *The plane projection of a knot or link is something similar to an oriented graph with* 4*-valent vertices, except for the fact that we have some extra data at the vertices, telling us, about the* 2 *strings crossing there, which goes up and which goes down.*

Based on this, let us try to construct some knot invariants. A natural idea is that of defining the invariant on the 2D picture of the knot, that is, on a plane projection of the knot, and then proving that the invariant is indeed independent on the chosen plane.

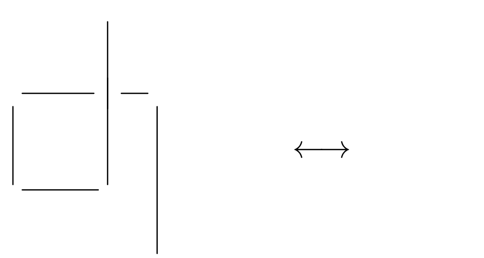
This method rests on the following technical result, which is well-known:

Theorem 5.8. *Two pictures correspond to plane projections of the same knot or link precisely when they differ by a sequence of Reidemeister moves, namely:*

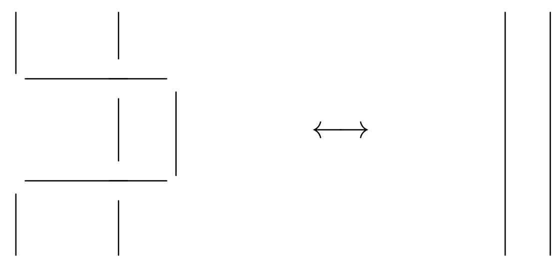
(1) *Moves of type I, given by* $\propto\ \leftrightarrow\ |$.
(2) *Moves of type II, given by* $≬\ \leftrightarrow\ )($.
(3) *Moves of type III, given by* $\triangle\ \leftrightarrow\ \nabla$.

Proof. This is something very standard, as follows:

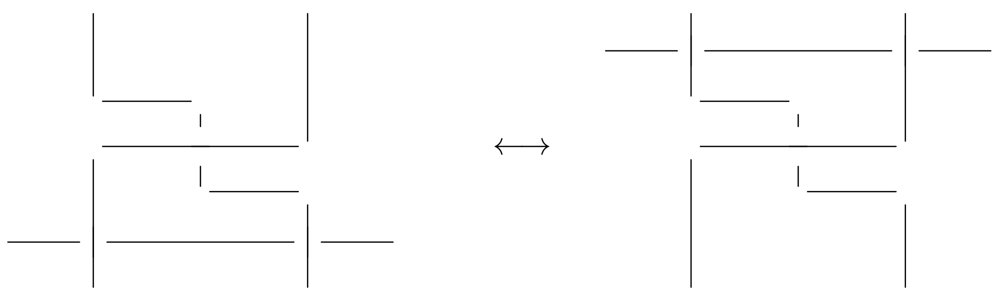
(1) To start with, the Reidemeister moves of type I are by definition as follows:



(2) Regarding the Reidemeister moves of type II, these are by definition as follows:



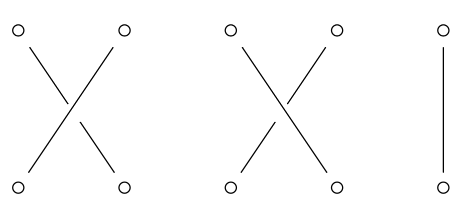
(3) As for the Reidemeister moves of type III, these are by definition as follows:



(4) This was for the precise statement of the theorem, and in what regards now the proof, this is somewhat clear from definitions, and in practice, this can be done by some sort of cut and paste procedure, or recurrence if you prefer, easy exercise for you. □

At a more advanced level now, we will need the following key observation, making the connection with group theory, and algebra in general, due to Alexander:

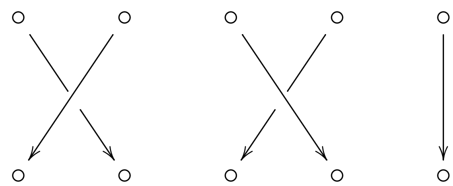
THEOREM 5.9. *Any knot or link can be thought of as being the closure of a braid,*



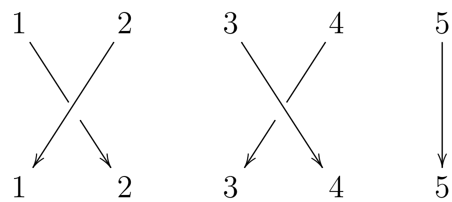
with the braids forming a group B_k, called braid group.

PROOF. Again, this is something quite self-explanatory, as follows:

(1) Consider indeed the braids with k strings, with the convention that things go from up to down. For instance the braid in the statement should be thought of as being:

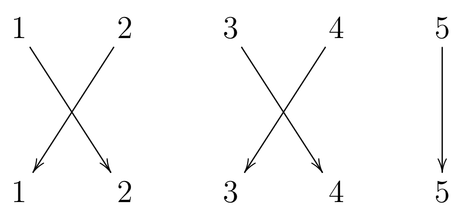


But, with this convention, braids become some sort of permutations of $\{1,\ldots,k\}$, which are decorated at the level of crossings, with for instance the above braid corresponding to the following permutation of $\{1,2,3,4,5\}$, with due decorations:



In any case, we can see in this picture that B_k is indeed a group, with composition law similar to that of the permutations in S_k, that is, going from up to down.

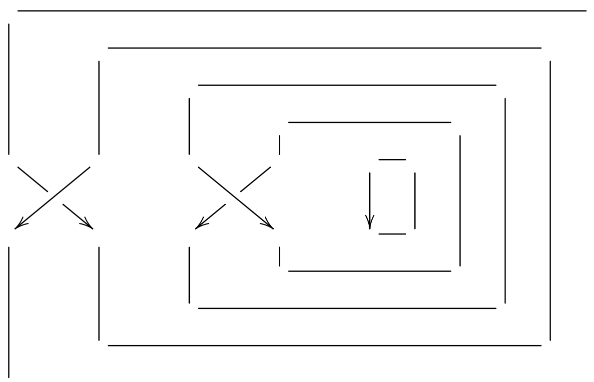
(2) Moreover, we can also see in this picture that we have a surjective group morphism $B_k \to S_k$, obtained by forgetting the decorations, at the level of crossings. For instance the braid pictured above is mapped in this way to the following permutation in S_5:



It is possible to do some more algebra here, in relation with the morphism $B_k \to S_k$, but we will not need this in what follows. We will keep in mind, from the above, the fact that "braids are not exactly permutations, but they compose like permutations".

(3) Regarding now the closure operation in the statement, this consists by definition in adding semicircles at right, which makes our braid into a certain oriented link. As an

illustration, the closure of the braid pictured above is the following link:



(4) This was for the precise statement of the theorem, and in what regards now the proof, this can be done by some sort of cut and paste procedure, or recurrence if you prefer. As before with Theorem 5.8, we will leave this as an easy exercise for you. □

Many interesting things can be said about the braid group B_k, as for instance:

Theorem 5.10. *The braid group B_k has the following properties:*

(1) *It is generated by variables $g_1, \ldots, g_{k-1}$, with the following relations:*

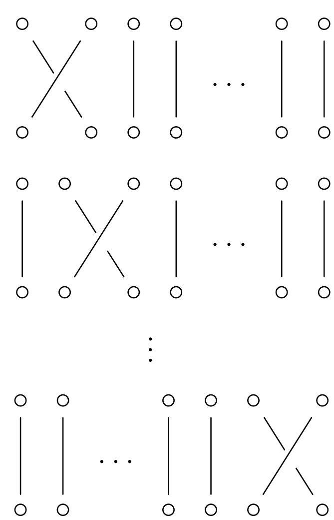
$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad , \quad g_i g_j = g_j g_i \text{ for } |i-j| \geq 2$$

(2) *It is the homotopy group of $X = (\mathbb{C}^k - \Delta)/S_k$, with $\Delta \subset \mathbb{C}^k$ standing for the points z satisfying $z_i = z_j$ for some $i \neq j$.*

Proof. These are things that we will not really need here, as follows:

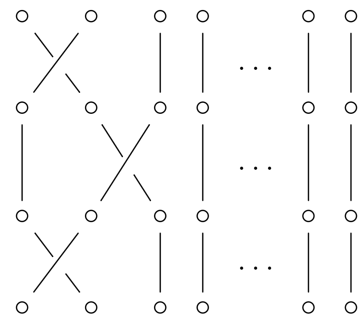
(1) In order to prove this assertion, due to Artin, consider the following braids:

$g_1 =$, $g_2 =$, …, $g_{k-1} =$



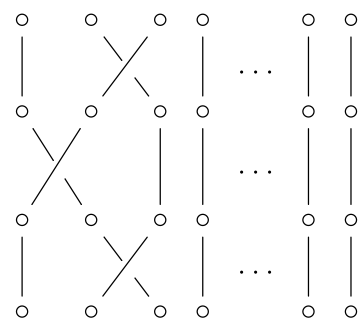
We have then $g_ig_j = g_jg_i$, for $|i-j|\geq 2$. As for the relation $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$, by translation it is enough to check this at $i=1$. And here, we first have:

$g_1g_2g_1 =$



On the other hand, we have as well the following computation:

$g_2g_1g_2 =$



Now since the above two pictures are identical, up to isotopy, we have $g_1g_2g_1 = g_2g_1g_2$, as desired. Thus, the braid group B_k is indeed generated by elements $g_1,\ldots,g_{k-1}$ with the relations in the statement, and in what regards now the proof of universality, this can only be something quite routine, and we will leave this as an instructive exercise.

(2) This is something quite self-explanatory, based on the general homotopy group material from chapter 1, and we will leave this as an easy exercise for you.

(3) Finally, before leaving the subject, let us mention that the Artin relations in (1) are something very useful, in order to construct explicit matrix representations of B_k. For instance, it can be shown that the braid group B_k is linear, and well, we will leave this as usual as an exercise for you, meaning either solve it, or look it up. □

Getting back now to knots and links, a quick comparison between our main results so far, namely Theorem 5.8 due to Reidemeister, and then Theorem 5.9 due to Alexander, suggests the following question, whose answer will certainly advance us:

QUESTION 5.11. *What is the analogue of the Reidemeister theorem, in the context of braids? That is, when do two braids produce, via closing, the same link?*

And this is, and we insist, a very good question, because assuming that we have an answer to it, no need afterwards to bother with plane projections, decorated graphs, Reidemeister moves, and amateurish topology in general, it will be all about groups and algebra. Which groups and algebra questions, you guessed right, we will eat them raw.

In answer now, we have the following theorem, due to Markov:

THEOREM 5.12. *Two elements of the full braid group, obtained as the increasing union of the various braid groups, with embeddings given by $\beta \to \beta\,|$,*

$$B_\infty = \bigsqcup_{k=1}^{\infty} B_k$$

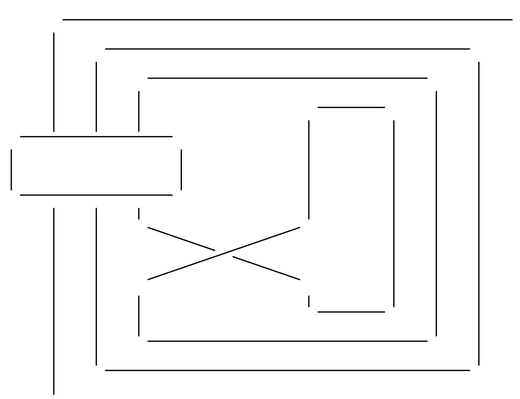
produce the same link, via closing, when one can pass from one to another via:

(1) *Conjugation: $\beta \to \alpha\beta\alpha^{-1}$.*
(2) *Markov move: $\beta \to g_k^{\pm 1}\beta$.*

PROOF. This is a version of the Reidemeister theorem, the idea being as follows:

(1) To start with, it is clear that conjugating a braid, $\beta \to \alpha\beta\alpha^{-1}$, will produce the same link after closing, because we can pull the α, α^{-1} to the right, in the obvious way, and there on the right, these α, α^{-1} will annihilate, according to $\alpha\alpha^{-1} = 1$.

(2) Regarding now the Markov move from the statement, with $\beta \in B_k \subset B_{k+1}$ and with $g_1, \ldots, g_k \in B_{k+1}$ being the standard Artin generators, from Theorem 5.10 and its proof, this is the tricky move, which is worth a proof. Taking $k = 3$ for an illustration, and representing $\beta \in B_3$ by a box, the link obtained by closing $g_4\beta$ is as follows, which is obviously the same link as the one obtained by closing β, and the same goes for $g_4^{-1}\beta$:



(3) Thus, the links produced by braids are indeed invariant under the two moves in the statement. As for the proof of the converse, this comes from the Reidemeister theorem, applied in the context of the Alexander theorem, or perhaps simpler, by reasoning directly, a bit as in the proof of the Reidemeister theorem. We will leave this as an exercise. □

5b.

5c.

5d.

5e. Exercises

Exercises:

Exercise 5.13.

Exercise 5.14.

Exercise 5.15.

Exercise 5.16.

Exercise 5.17.

Exercise 5.18.

Exercise 5.19.

Exercise 5.20.

Bonus exercise.

CHAPTER 6

Jones polynomial

6a. Jones polynomial

As explained before, the Markov theorem is exactly what we need, in order to reformulate everything in terms of groups and algebra. To be more precise, looking now more in detail at what the Markov theorem exactly says, we are led to the following strategy:

STRATEGY 6.1. *In order to construct numeric invariants for knots and links:*

(1) *We must map B_∞ somewhere, and then apply the trace.*
(2) *And if the trace is preserved by Markov moves, it's a win.*

You get the point, if we are do (1) then, by using the trace property $tr(ab) = tr(ba)$ of the trace, we will have $tr(\beta) = tr(\alpha\beta\alpha^{-1})$, in agreement with what the Markov theorem first requires. And if we do (2) too, whatever that condition exactly means, and more on this in a moment, we will have as well $tr(\beta) = tr(g_k^{\pm 1}\beta)$, in agreement with what the Markov theorem fully requires, so we will have our invariant for knots and links.

This sound very good, but before getting into details, let us be a bit megalomaniac, and add two more ambitious points to our war plan, as follows:

ADDENDUM 6.2. *Our victory will be total, with a highly reliable invariant, if:*

(1) *The representation and trace are faithful as possible.*
(2) *And they depend, if possible, on several parameters.*

Here (1) and (2) are obviously related, because the more parameters we have in (2), the more chances for our constructions in (1) to be faithful will be. In short, what we are wishing here for is an invariant which distinguishes well between various knots and links, and this can only come via a mixture of faithfulness, and parameters involved.

So long for the plan, and in practice now, getting back to what Strategy 6.1 says, we are faced right away with a problem, coming from the fact that B_∞ is not that easy to represent. You might actually already know this, if you have struggled a bit with the exercise that I left for you, at the end of the previous chapter. So, we are led to:

QUESTION 6.3. *How to represent the braid group B_∞?*

So, this was the question that Reidemeister, Alexander, Markov, Artin and the others were fighting with, a long time ago, in the first half of the 20th century. Quite surprisingly, the answer to it came very late, in the 80s, from Jones [57], with inspiration from operator algebras, and more specifically, from his previous paper [56] about subfactors.

Retrospectively looking at all this, what really matters in Jones' answer to Question 6.3 is the algebra constructed by Temperley and Lieb, in the context of questions from statistical mechanics. But then, by looking even more retrospectively at all this, we can even say that the answer to Question 6.3 comes from nothing at all, meaning basic category theory. So, this will be our approach in what follows, with our answer being:

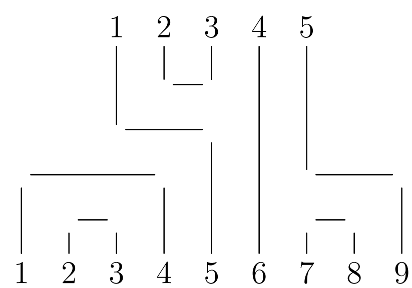
Answer 6.4. *Thinking well, B_∞ is self-represented, without help from the outside.*

So, ready for some category theory? We first need objects, and our set of objects will be the good old $\mathbb{N}$. As for the arrows, somehow in relation with topology and braids, we will choose something very simple too, with our definition being as follows:

Definition 6.5. *The Temperley-Lieb category TL_N has the positive integers $\mathbb{N}$ as objects, with the space of arrows $k \to l$ being the formal span*

$$TL_N(k,l) = span(NC_2(k,l))$$

of noncrossing pairings between an upper row of k points, and a lower row of l points



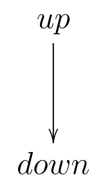
and with the composition of arrows appearing by composing the pairings, in the obvious way, with the rule $\bigcirc = N$, for the closed loops that might appear.

This definition is something quite subtle, hiding several non-trivial things, and is worth a detailed discussion, our comments about it being as follows:

(1) First of all, our scalars in this chapter will be complex numbers, $\lambda \in \mathbb{C}$, and the "formal span" in the above must be understood in this sense, namely abstract complex vector space spanned by the elements of $NC_2(k,l)$. Of course it is possible to use an arbitrary field, at least at this stage of things, but remember that we are interested in quantum mechanics, and related mathematics, where the field of scalars is $\mathbb{C}$.

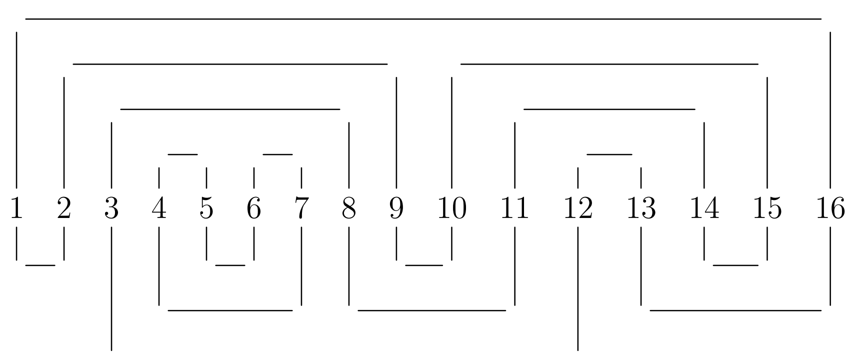
(2) Regarding the composition of arrows, this is by vertical concatenation, with our usual convention that things go "from up to down". And with this coming from care for

our planet, and for entropy at the galactic level, I mean why pushing things from left to right, when we can have gravity work for us, pulling them from up to down:



(3) Less poetically, this "from up to down" convention is also useful for purely mathematical purposes, because the left-right direction will be reserved for the intervention of sums Σ and scalars $\lambda \in \mathbb{C}$, while the up-down direction will be reserved for "action".

(4) Let us discuss now what happens with the closed circles, when concatenating. As an example, let us consider a full capping of noncrossing pairings, also called meander:



According to our conventions, this meander appears as the product $\pi\sigma \in NC_2(0,0)$ between the upper pairing $\sigma \in NC_2(0,16)$ and the lower pairing $\pi \in NC_2(16,0)$. But, what is the value of this product? We have two loops appearing, namely:

$$1 - 2 - 9 - 10 - 15 - 14 - 11 - 8 - 3 - 12 - 13 - 16$$

$$4 - 5 - 6 - 7$$

Thus, according to Definition 6.5, the value of this meander is N^2, with one N for each of the above loops, and with these two values of N multiplying each other.

(5) The same discussion applies to an arbitrary composition $\pi\sigma \in NC_2(k,m)$ between an upper pairing $\sigma \in NC_2(k,l)$ and a lower pairing $\pi \in NC_2(l,m)$, with a certain number of loops appearing in this way, each contributing with a multiplicative factor N.

(6) Finally, in Definition 6.5 the value of the circle $N = \bigcirc$ can be pretty much anything, but due to some positivity reasons to become clear later, we will assume in what follows $N \in [1,\infty)$. Also, we will call this parameter N the "index", with the precise reasons for calling this index to become clear later, too, as this book develops.

With all this discussed, what is next? More category theory I guess, and matter of having a theorem formulated too, instead of definitions only, let us formulate:

THEOREM 6.6. *The Temperley-Lieb category TL_N is a tensor $*$-category, with:*

(1) *Composition of arrows: by vertical concatenation.*
(2) *Tensoring of arrows: by horizontal concatenation.*
(3) *Star operation: by turning the arrows upside-down.*

PROOF. This is more of a definition, disguised as a theorem. To be more precise, we already know about (1), from Definition 6.5, and we can talk as well about (2) and (3), constructed as above, with (2) using of course multiplicativity with respect to the scalars, and with (3) using antimultiplicativity with respect to the scalars:

$$\left(\sum_i \lambda_i\pi_i\right)\otimes\left(\sum_j \mu_j\sigma_j\right)=\sum_{ij}\lambda_i\mu_j\pi_i\otimes\sigma_j$$

$$\left(\sum_i \lambda_i\pi_i\right)^*=\sum_i \bar{\lambda}_i\pi_i^*$$

And the point now is that our three operations are compatible with each other via all sorts of compatibility formulae, which are all clear from definitions, with the conclusion being that what we have a tensor $*$-category, as stated. We will leave the details here, basically amounting in figuring out what a tensor $*$-category exactly is, as an exercise. □

In order to further understand the category TL_N, let us focus on its diagonal part, formed by the End spaces of various objects. With the convention that these End spaces embed into each other by adding bars at right, this is a graded algebra, as follows:

$$\Delta TL_N=\bigcup_{k\geq 0}TL_N(k,k)$$

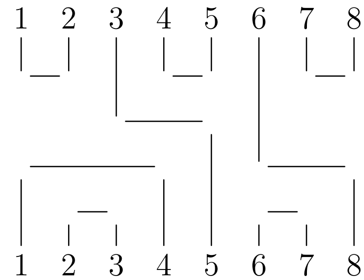
Moreover, for further fine-tuning our study, let us actually focus on the individual components of this graded algebra. These components will play a key role in what follows, and they are worth a dedicated definition, and new notation and name, as follows:

DEFINITION 6.7. *The Temperley-Lieb algebra $TL_N(k)$ is the formal span*

$$TL_N(k)=span(NC_2(k,k))$$

with multiplication coming by concatenating, with the rule $\bigcirc=N$.

In other words, $TL_N(k)$ appears as the formal span of the noncrossing pairings between an upper row of k points, and a lower row of k points, with multiplication coming by concatenating, with $\bigcirc = N$. As an example, here is a basis element of $TL_N(8)$:



Getting back now to what we know about TL_N, from Theorem 6.6, the tensor product operation makes sense in the context of the diagonal algebra ΔTL_N, but does not apply to its individual components $TL_N(k)$. However, the involution is useful, and we have:

PROPOSITION 6.8. *The Temperley-Lieb algebra $TL_N(k)$ is a $*$-algebra, with involution coming by turning the diagrams upside-down.*

PROOF. This is something trivial, which follows from Theorem 6.6, and can be verified as well directly, and we will leave this as an instructive exercise. □

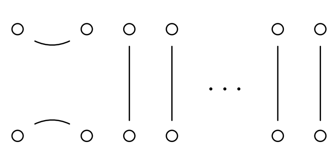
Getting back now to knots and links, we first have to make the connection between braids and Temperley-Lieb diagrams. But this can be done as follows:

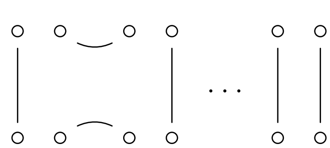
THEOREM 6.9. *The following happen:*

(1) *We have a braid group representation $B_k \to TL_N(k)$, mapping standard generators to standard generators.*
(2) *We have a trace $tr : TL_N(k) \to \mathbb{C}$, obtained by closing the diagrams, which is positive, and has a suitable Markov invariance property.*

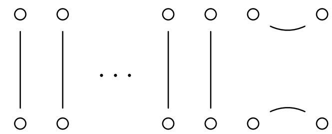
PROOF. Again, this is something quite intuitive, with the generators in (1) being by definition the standard ones, on both sides, and with the closing operation in (2) being similar to the one for braids, from chapter 5. To be more precise:

(1) The idea here is to map the Artin generators of the braid group to suitable modifications of the following Temperley-Lieb diagrams, called Jones projections:

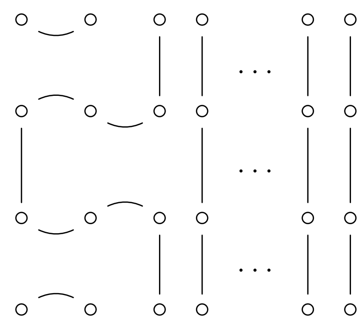
$e_1 =$ 

$e_2 =$ 

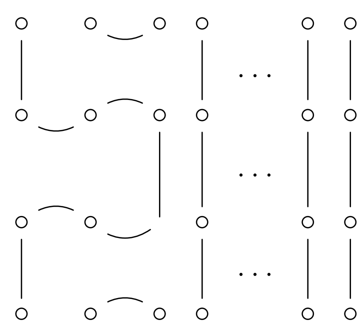
$$\vdots$$

$e_{k-1} =$ 

As a first observation, these diagrams satisfy $e_i^2 = Ne_i$, with $N = \bigcirc$ being as usual the value of the circle, so it is rather the rescaled versions $f_i = e_i/N$ which are projections, but we will not bother with this, and use our terminology above. Next, our Jones projections certainly satisfy the Artin relations $e_ie_j = e_je_i$, for $|i-j| \geq 2$. Our claim now is that is that we have as well the formula $e_ie_{i\pm 1}e_i = e_i$. Indeed, by translation it is enough to check $e_ie_{i+1}e_i = e_i$ at $i = 1$, and this follows from the following computation:

$e_1e_2e_1 =$  $= e_1$

As for the verification of the relation $e_2e_1e_2 = e_2$, this is similar, as follows:

$e_2e_1e_2 =$  $= e_2$

Now with the relations $e_ie_{i\pm 1}e_i = e_i$ in hand, let us try to reach to the Artin relations $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$. For this purpose, let us set $g_i = te_i - 1$. We have then:

$$\begin{aligned}
g_ig_{i+1}g_i &= (te_i - 1)(te_{i+1} - 1)(te_i - 1) \\
&= t^3e_i - t^2(Ne_i + e_ie_{i+1} + e_{i+1}e_i) + t(2e_i + e_{i+1}) - 1 \\
&= t(t^2 - Nt + 2)e_i + te_{i+1} - t^2(e_ie_{i+1} + e_{i+1}e_i)
\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned} g_{i+1}g_ig_{i+1} &= (te_{i+1}-1)(te_i-1)(te_{i+1}-1) \\ &= t^3e_{i+1}-t^2(Ne_{i+1}+e_ie_{i+1}+e_{i+1}e_i)+t(2e_{i+1}+e_i)-1 \\ &= t(t^2-Nt+2)e_{i+1}+te_i-t^2(e_ie_{i+1}+e_{i+1}e_i) \end{aligned}$$

Thus with $t^2-Nt+1=0$ we have a representation $B_k\to TL_N(k)$, as desired.

(2) This is something more subtle, especially in what regards the positivity properties of the trace $tr:TL_N(k)\to\mathbb{C}$, which requires a bit more mathematics. So, no hurry with this, and we will discuss all this, and applications, in the remainder of this chapter. □

Now back to the knots and links, we have all the needed ingredients. Indeed, we can now put everything together, and we obtain, following Jones:

THEOREM 6.10. *We can define the Jones polynomial of an oriented knot or link as being the image of the corresponding braid producing it via the map*

$$tr:B_k\to TL_N(k)\to\mathbb{C}$$

with the following change of variables:

$$N=q^{1/2}+q^{-1/2}$$

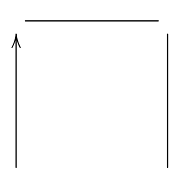
We obtain a Laurent polynomial in $q^{1/2}$, which is an invariant, up to planar isotopy.

PROOF. There is a long story here, the idea being as follows:

(1) To start with, the result follows indeed by combining the above ingredients, the idea being that the various algebraic properties of $tr:TL_N(k)\to\mathbb{C}$ are exactly what is needed for the above composition, up to a normalization, to be invariant under the Reidemeister moves of type I, II, III, and so to produce indeed a knot invariant.

(2) More specifically, the result follows from Theorem 6.6, combined with what we have in Theorem 6.9, which is now fully proved, with the positivity part coming from chapter 4, and with the change of variables $N=q^{1/2}+q^{-1/2}$ in the statement coming from the equation $t^2-Nt+1=0$ that we found in the proof of Theorem 6.9.

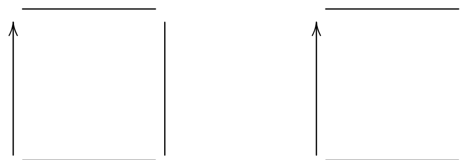
(3) As an illustration for how this works, consider first the unknot:



For this knot, or rather unknot, the corresponding Jones polynomial is:

$$V=1$$

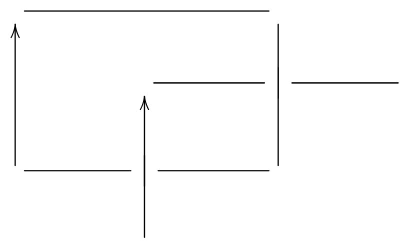
(4) Next, let us look at the link formed by two unlinked unknots:



For this link, or rather unlink, the corresponding Jones polynomial is:

$$V = -q^{-1/2} - q^{1/2}$$

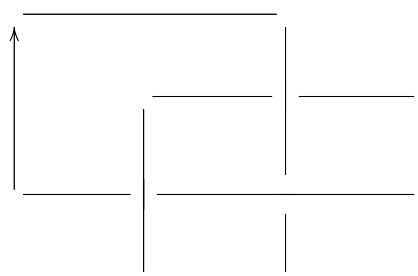
(5) Next, let us look at the link formed by two linked unknots, namely:



For this link, the corresponding Jones polynomial is given by:

$$V = q^{1/2} + q^{5/2}$$

(6) Finally, let us look at the trefoil knot, which is as follows:



For this knot, the corresponding Jones polynomial is as follows:

$$V = q + q^3 - q^4$$

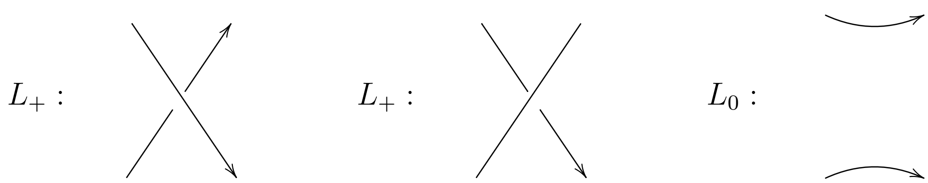
Observe that, as previously for the unknot, this is a Laurent polynomial in q. This is part of a more general phenomenon, the point being that for knots, or more generally for links having an odd number of components, we get a Laurent polynomial in q.

(7) In practice now, far more things can be said, about this. For instance the change of variables $N = q^{1/2} + q^{-1/2}$ in the statement is something well-known in planar algebras, and with all this being related to operator algebras and subfactor theory. More on this later in this book, when discussing subfactors and planar algebras.

(8) From a purely topological perspective, however, nothing beats the skein relation interpretation of the Jones polynomial $V_L(q)$, which is as follows, with L_+, L_-, L_0 being knots, or rather links, differing at exactly 1 crossing, in the 3 possible ways:

$$q^{-1}V_{L_+} - qV_{L_-} = (q^{1/2} + q^{-1/2})V_{L_0}$$

To be more precise, here are the conventions for L_+, L_-, L_0, that you need to know, in order to play with the above formula, and compute Jones polynomials at wish:



As for the proof of the above formula, this comes from our definition of the Jones polynomial, because thinking well, "unclosing" links as to get braids, and then closing Temperley-Lieb diagrams as to get scalars, as required by the construction of $V_L(q)$, seemingly is some sort of identity operation, but the whole point comes from the fact that the Artin braids $g_1, \ldots, g_{k-1}$ and the Jones projections $e_1, \ldots, e_{k-1}$ differ precisely by a crossing being replaced by a non-crossing. Exercise for you, to figure out all this.

(9) In short, up to you to learn all this, in detail, and its generalizations too, with link polynomials defined more generally via relations of the following type:

$$xP_{L_+} + yP_{L_-} + zP_{L_0} = 0$$

Equivalently, we can define these more general invariants by using various versions of the Temperley-Lieb algebra. As usual, check here the papers of Jones [**56**], [**57**], [**58**].

(10) With the comment here that, among all these invariants, Jones polynomial included, the first came, historically, the Alexander polynomial. However, from a modern point of view, the Alexander polynomial is something more complicated than the Jones polynomial, which remains the central invariant of knots and links.

(11) As another comment, with all this pure mathematics digested, physics strikes back, via a very interesting relation with statistical mechanics, happening in 2D as well, the idea being that "interactions happen at crossings", and it is these interactions that produce the knot invariant, as a kind of partition function. See Jones [**59**], [**60**].

(12) Quite remarkably, the above invariants can be directly understood in 3D as well, in a purely geometric way, with elegance, and no need for 2D projection. But this is a more complicated story, involving ideas from quantum field theory. See Witten [**98**]. □

6b.

6c.

6d.

6e. Exercises

Exercises:

Exercise 6.11.

Exercise 6.12.

Exercise 6.13.

Exercise 6.14.

Exercise 6.15.

Exercise 6.16.

Exercise 6.17.

Exercise 6.18.

Bonus exercise.

CHAPTER 7

Further invariants

7a. Further invariants

7b.

7c.

7d.

7e. Exercises

Exercises:

EXERCISE 7.1.

EXERCISE 7.2.

EXERCISE 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

EXERCISE 7.7.

EXERCISE 7.8.

Bonus exercise.

CHAPTER 8

Mechanical aspects

8a. Mechanical aspects

8b.

8c.

8d.

8e. Exercises

Exercises:

Exercise 8.1.

Exercise 8.2.

Exercise 8.3.

Exercise 8.4.

Exercise 8.5.

Exercise 8.6.

Exercise 8.7.

Exercise 8.8.

Bonus exercise.

Part III

Diagram algebras

Bahama, bahama mama
Got the biggest house in town bahama mama
Bahama, bahama mama
But her trouble's getting down bahama mama

CHAPTER 9

Group theory

9a. Group theory

Generally speaking, no matter on what we want to do with our group, we must compute the spaces $Fix(v^{\otimes k})$. It is technically convenient to slightly enlarge the class of spaces to be computed, by talking about Tannakian categories, as follows:

Definition 9.1. *The Tannakian category associated to a closed subgroup $G \subset_v U_N$ is the collection $C_G = (C_G(k,l))$ of vector spaces*

$$C_G(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

where the representations $v^{\otimes k}$ with $k = \circ \bullet \bullet \circ \ldots$ colored integer, defined by

$$v^{\otimes \emptyset} = 1 \quad , \quad v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v}$$

and multiplicativity, $v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$, are the Peter-Weyl representations.

Let us make a summary of what we have so far, regarding these spaces $C_G(k,l)$. In order to formulate our result, let us start with the following definition:

Definition 9.2. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C = (C(k,l))$ of linear spaces*

$$C(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

1. *$S, T \in C$ implies $S \otimes T \in C$.*
2. *If $S, T \in C$ are composable, then $ST \in C$.*
3. *$T \in C$ implies $T^* \in C$.*
4. *$C(k,k)$ contains the identity operator.*
5. *$C(\emptyset, k)$ with $k = \circ\bullet, \bullet\circ$ contain the operator $R : 1 \to \sum_i e_i \otimes e_i$.*
6. *$C(kl, lk)$ with $k, l = \circ, \bullet$ contain the flip operator $\Sigma : a \otimes b \to b \otimes a$.*

Here the tensor power Hilbert spaces $H^{\otimes k}$, with $k = \circ \bullet \bullet \circ \ldots$ being a colored integer, are defined by the following formulae, and multiplicativity:

$$H^{\otimes \emptyset} = \mathbb{C} \quad , \quad H^{\otimes \circ} = H \quad , \quad H^{\otimes \bullet} = \bar{H} \simeq H$$

With these conventions, we have the following result, summarizing our knowledge on the subject, coming from the results established in the above:

THEOREM 9.3. *For a closed subgroup $G \subset_v U_N$, the associated Tannakian category*

$$C_G(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

is a tensor category over the Hilbert space $H = \mathbb{C}^N$.

PROOF. We know that the fundamental representation v acts on the Hilbert space $H = \mathbb{C}^N$, and that its conjugate $\bar{v}$ acts on the Hilbert space $\bar{H} = \mathbb{C}^N$. Now by multiplicativity we conclude that any Peter-Weyl representation $v^{\otimes k}$ acts on the Hilbert space $H^{\otimes k}$, and so that we have embeddings as in Definition 9.2, as follows:

$$C_G(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

Regarding now the fact that the axioms (1-6) in Definition 9.2 are indeed satisfied, this is something that we basically already know. To be more precise, (1-4) are clear, and (5) follows from the fact that each element $g \in G$ is a unitary, which gives:

$$R \in Hom(1, g \otimes \bar{g}) \quad , \quad R \in Hom(1, \bar{g} \otimes g)$$

As for (6), this is something trivial, coming from the fact that the matrix coefficients $g \to g_{ij}$ and their complex conjugates $g \to \bar{g}_{ij}$ commute with each other. □

Our purpose now will be that of showing that any closed subgroup $G \subset U_N$ is uniquely determined by its Tannakian category $C_G = (C_G(k,l))$. This result, known as Tannakian duality, is something quite deep, and extremely useful. Indeed, the idea is that what we would have here is a "linearization" of G, allowing us to do combinatorics, and to ultimately reach to concrete and powerful results, regarding G itself. We first have:

THEOREM 9.4. *Given a tensor category $C = (C(k,l))$ over a finite dimensional Hilbert space $H \simeq \mathbb{C}^N$, the following construction,*

$$G_C = \left\{ g \in U_N \middle| Tg^{\otimes k} = g^{\otimes l} T \ , \ \forall k,l, \forall T \in C(k,l) \right\}$$

produces a closed subgroup $G_C \subset U_N$.

PROOF. This is something elementary, with the fact that the closed subset $G_C \subset U_N$ constructed in the statement is indeed stable under the multiplication, unit and inversion operation for the unitary matrices $g \in U_N$ being clear from definitions. □

We can now formulate the Tannakian duality result, as follows:

THEOREM 9.5. *The above Tannakian constructions*

$$G \to C_G \quad , \quad C \to G_C$$

are bijective, and inverse to each other.

PROOF. This is something quite technical, obtained by doing some abstract algebra, and for details here, we refer to the Tannakian duality literature. The whole subject is actually, in modern times, for the most part of quantum algebra, and you can consult here various quantum group papers and books, for details on the above. □

In order to reach now to more concrete things, following Brauer, we have:

Definition 9.6. *Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l. A collection of subsets*

$$D=\bigsqcup_{k,l}D(k,l)$$

with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*
6. *The sets $P(k,\bar{k})$ with $|k|=2$ contain the crossing partition $\slash\!\!\!\backslash$.*

There are many examples of such categories, as for instance the category of all pairings P_2, or of all matching pairings $\mathcal{P}_2$. We will be back to examples in a moment.

Let us formulate as well the following definition:

Definition 9.7. *Given a partition $\pi\in P(k,l)$ and an integer $N\in\mathbb{N}$, we can construct a linear map between tensor powers of $\mathbb{C}^N$,*

$$T_\pi:(\mathbb{C}^N)^{\otimes k}\to(\mathbb{C}^N)^{\otimes l}$$

by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb{C}^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the coefficients on the right being Kronecker type symbols,

$$\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}\in\{0,1\}$$

whose values depend on whether the indices fit or not.

To be more precise, we put the indices of i,j on the legs of π, in the obvious way. In case all the blocks of π contain equal indices of i,j, we set $\delta_\pi\binom{i}{j}=1$. Otherwise, we set $\delta_\pi\binom{i}{j}=0$. The relation with the Tannakian categories comes from:

Proposition 9.8. *The assignement $\pi\to T_\pi$ is categorical, in the sense that*

$$T_\pi\otimes T_\nu=T_{[\pi\nu]}\quad,\quad T_\pi T_\nu=N^{c(\pi,\nu)}T_{[^\nu_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\nu)$ are certain integers, coming from the erased components in the middle.

Proof. This is something elementary, the computations being as follows:

(1) The concatenation axiom can be checked as follows:

$$\begin{aligned}
& (T_\pi \otimes T_\nu)(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r}) \\
= & \sum_{j_1\ldots j_q} \sum_{l_1\ldots l_s} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} \delta_\nu \begin{pmatrix} k_1 & \ldots & k_r \\ l_1 & \ldots & l_s \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s} \\
= & \sum_{j_1\ldots j_q} \sum_{l_1\ldots l_s} \delta_{[\pi\nu]} \begin{pmatrix} i_1 & \ldots & i_p & k_1 & \ldots & k_r \\ j_1 & \ldots & j_q & l_1 & \ldots & l_s \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_q} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s} \\
= & T_{[\pi\nu]}(e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e_{k_1} \otimes \ldots \otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom can be checked as follows:

$$\begin{aligned}
& T_\pi T_\nu(e_{i_1} \otimes \ldots \otimes e_{i_p}) \\
= & \sum_{j_1\ldots j_q} \delta_\nu \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} \sum_{k_1\ldots k_r} \delta_\pi \begin{pmatrix} j_1 & \ldots & j_q \\ k_1 & \ldots & k_r \end{pmatrix} e_{k_1} \otimes \ldots \otimes e_{k_r} \\
= & \sum_{k_1\ldots k_r} N^{c(\pi,\nu)} \delta_{[^\nu_\pi]} \begin{pmatrix} i_1 & \ldots & i_p \\ k_1 & \ldots & k_r \end{pmatrix} e_{k_1} \otimes \ldots \otimes e_{k_r} \\
= & N^{c(\pi,\nu)} T_{[^\nu_\pi]}(e_{i_1} \otimes \ldots \otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom can be checked as follows:

$$\begin{aligned}
& T_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q}) \\
= & \sum_{i_1\ldots i_p} < T_\pi^*(e_{j_1} \otimes \ldots \otimes e_{j_q}), e_{i_1} \otimes \ldots \otimes e_{i_p} > e_{i_1} \otimes \ldots \otimes e_{i_p} \\
= & \sum_{i_1\ldots i_p} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_p \\ j_1 & \ldots & j_q \end{pmatrix} e_{i_1} \otimes \ldots \otimes e_{i_p} \\
= & T_{\pi^*}(e_{j_1} \otimes \ldots \otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

In relation now with the groups, we have the following result:

Theorem 9.9. *Each category of partitions $D = (D(k,l))$ produces a family of compact groups $G = (G_N)$, with $G_N \subset_v U_N$, via the formula*

$$Hom(v^{\otimes k}, v^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

and the Tannakian duality correspondence.

Proof. Given an integer $N \in \mathbb{N}$, consider the correspondence $\pi \to T_\pi$ constructed in Definition 9.7, and then the collection of linear spaces in the statement, namely:

$$C(k,l) = span\left(T_\pi\Big|\pi \in D(k,l)\right)$$

According to Proposition 9.8, and to our axioms for the categories of partitions, from Definition 9.6, this collection of spaces $C = (C(k,l))$ satisfies the axioms for the Tannakian categories, from Definition 9.2. Thus the Tannakian duality result, Theorem 9.5, applies, and provides us with a closed subgroup $G_N \subset_v U_N$ such that:

$$C(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

Thus, we are led to the conclusion in the statement. □

We can now formulate a key definition, as follows:

Definition 9.10. *A closed subgroup $G \subset_v U_N$ is called easy when we have*

$$Hom(v^{\otimes k}, v^{\otimes l}) = span\left(T_\pi\Big|\pi \in D(k,l)\right)$$

for any colored integers k, l, for a certain category of partitions $D \subset P$.

The notion of easiness goes back to the results of Brauer regarding the orthogonal group O_N, and the unitary group U_N, which can be formulated as follows:

Theorem 9.11. *We have the following results:*

(1) *U_N is easy, coming from the category of matching pairings $\mathcal{P}_2$.*
(2) *O_N is easy too, coming from the category of all pairings P_2.*

Proof. This is something very standard, the idea being as follows:

(1) The group U_N being defined via the relations $v^* = v^{-1}$, $v^t = \bar{v}^{-1}$, the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =< \overset{\cap}{\circ\bullet}\ ,\ \overset{\cap}{\bullet\circ} >= \mathcal{P}_2$$

(2) The group $O_N \subset U_N$ being defined by imposing the relations $v_{ij} = \bar{v}_{ij}$, the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D =< \mathcal{P}_2, \substack{\circ\\ |\\ \bullet}, \substack{\bullet\\ |\\ \circ} >= P_2$$

Thus, we are led to the conclusion in the statement. □

Beyond this, a first natural question is that of computing the easy group associated to the category P itself, and we have here the following Brauer type theorem:

Theorem 9.12. *The symmetric group S_N, regarded as group of unitary matrices,*

$$S_N \subset O_N \subset U_N$$

via the permutation matrices, is easy, coming from the category of all partitions P.

PROOF. Consider the easy group $G\subset O_N$ coming from the category of all partitions P. Since P is generated by the one-block partition $Y\in P(2,1)$, we have:

$$C(G)=C(O_N)\Big/\Big\langle T_Y\in Hom(v^{\otimes 2},v)\Big\rangle$$

The linear map associated to Y is given by the following formula:

$$T_Y(e_i\otimes e_j)=\delta_{ij}e_i$$

Thus, the relation defining the above group $G\subset O_N$ reformulates as follows:

$$T_Y\in Hom(v^{\otimes 2},v)\iff v_{ij}v_{ik}=\delta_{jk}v_{ij},\forall i,j,k$$

In other words, the elements v_{ij} must be projections, and these projections must be pairwise orthogonal on the rows of $v=(v_{ij})$. We conclude that $G\subset O_N$ is the subgroup of matrices $g\in O_N$ having the property $g_{ij}\in\{0,1\}$. Thus we have $G=S_N$, as claimed. $\square$

Many other things can be said, along these lines.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.13.

EXERCISE 9.14.

EXERCISE 9.15.

EXERCISE 9.16.

EXERCISE 9.17.

EXERCISE 9.18.

EXERCISE 9.19.

EXERCISE 9.20.

Bonus exercise.

CHAPTER 10

Quantum groups

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

EXERCISE 10.8.

Bonus exercise.

CHAPTER 11

Random matrices

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.

Exercise 11.4.

Exercise 11.5.

Exercise 11.6.

Exercise 11.7.

Exercise 11.8.

Bonus exercise.

CHAPTER 12

Planar algebras

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Analytic aspects

My girl, my girl, don't lie to me
Tell me where did you sleep last night
In the pines, in the pines where the sun don't ever shine
I would shiver the whole night through

CHAPTER 13

Quantum physics

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Bonus exercise.

CHAPTER 14

Feynman diagrams

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.

CHAPTER 15

Diagram algebras

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

Open questions

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

Bibliography

[1] E. Abe, Hopf algebras, Cambridge Univ. Press (1980).

[2] V.I. Arnold, Mathematical methods of classical mechanics, Springer (1974).

[3] V.I. Arnold, Lectures on partial differential equations, Springer (1997).

[4] V.I. Arnold, Catastrophe theory, Springer (1974).

[5] M.F. Atiyah, K-theory, CRC Press (1964).

[6] M.F. Atiyah, The geometry and physics of knots, Cambridge Univ. Press (1990).

[7] M.F. Atiyah and I.G. MacDonald, Introduction to commutative algebra, Addison-Wesley (1969).

[8] T. Banica, Linear algebra and group theory (2024).

[9] T. Banica, Basic quantum algebra (2025).

[10] T. Banica, Introduction to modern physics (2025).

[11] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[12] N. Berline, E. Getzler and M. Vergne, Heat kernels and Dirac operators, Springer (2004).

[13] B. Blackadar, K-theory for operator algebras, Cambridge Univ. Press (1986).

[14] A.H. Chamseddine and A. Connes, The spectral action principle, *Comm. Math. Phys.* **186** (1997), 731–750.

[15] V. Chari and A. Pressley, A guide to quantum groups, Cambridge Univ. Press (1994).

[16] B. Collins and P. Śniady, Integration with respect to the Haar measure on the unitary, orthogonal and symplectic group, *Comm. Math. Phys.* **264** (2006), 773–795.

[17] A. Connes, Noncommutative geometry, Academic Press (1994).

[18] A. Connes and M. Marcolli, Noncommutative geometry, quantum fields and motives, AMS (2008).

[19] W.N. Cottingham and D.A. Greenwood, An introduction to the standard model of particle physics, Cambridge Univ. Press (2012).

[20] P. Deligne, Catégories tannakiennes, in "The Grothendieck Festschrift" (2007), 111–195.

[21] P.A.M. Dirac, Principles of quantum mechanics, Oxford Univ. Press (1930).

[22] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.

[23] M.P. do Carmo, Differential geometry of curves and surfaces, Dover (1976).

[24] M.P. do Carmo, Riemannian geometry, Birkhäuser (1992).

[25] S. Doplicher and J.E. Roberts, A new duality theory for compact groups, *Invent. Math.* **98** (1989), 157–218.

[26] S.K. Donaldson, Riemann surfaces, Oxford Univ. Press (2004).

[27] V.G. Drinfeld, Quantum groups, Proc. ICM Berkeley (1986), 798–820.

[28] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).

[29] A. Einstein, Relativity: the special and the general theory, Dover (1916).

[30] P. Etingof, S. Gelaki, D. Nikshych and V. Ostrik, Tensor categories, AMS (2016).

[31] L.C. Evans, Partial differential equations, AMS (1998).

[32] L. Faddeev, Instructive history of the quantum inverse scattering method, *Acta Appl. Math.* **39** (1995), 69–84.

[33] L. Faddeev, N. Reshetikhin and L. Takhtadzhyan, Quantization of Lie groups and Lie algebras, *Leningrad Math. J.* **1** (1990), 193–225.

[34] E. Fermi, Thermodynamics, Dover (1937).

[35] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics, Caltech (1963).

[36] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals, Dover (1965).

[37] P. Flajolet and R. Sedgewick, Analytic combinatorics, Cambridge Univ. Press (2009).

[38] A.P. French, Special relativity, Taylor and Francis (1968).

[39] W. Fulton, Algebraic topology, Springer (1995).

[40] W. Fulton and J. Harris, Representation theory, Springer (1991).

[41] C. Godsil and G. Royle, Algebraic graph theory, Springer (2001).

[42] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, Cambridge Univ. Press (2012).

[43] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).

[44] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).

[45] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).

[46] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley (1994).

[47] J. Harris, Algebraic geometry, Springer (1992).

[48] R. Hartshorne, Algebraic geometry, Springer (1977).

[49] A. Hatcher, Algebraic topology, Cambridge Univ. Press (2002).

[50] H. Hofer and E. Zehnder, Symplectic invariants and Hamiltonian dynamics, Birkhäuser (1994).

[51] L. Hörmander, The analysis of linear partial differential operators, Springer (1983).

[52] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge Univ. Press (1985).

[53] K. Huang, Introduction to statistical physics, CRC Press (2001).

[54] J.E. Humphreys, Introduction to Lie algebras and representation theory, Springer (1972).

[55] J.E. Humphreys, Linear algebraic groups, Springer (1975).

[56] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[57] V.F.R. Jones, A polynomial invariant for knots via von Neumann algebras, *Bull. Amer. Math. Soc.* **12** (1985), 103–111.

[58] V.F.R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math.* **126** (1987), 335–388.

[59] V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math.* **137** (1989), 311–334.

[60] V.F.R. Jones, Planar algebras I (1999).

[61] C. Kassel, Quantum groups, Springer (1995).

[62] T. Kibble and F.H. Berkshire, Classical mechanics, Imperial College Press (1966).

[63] A. Kirillov Jr., On an inner product in modular tensor categories, *J. Amer. Math. Soc.* **9** (1996), 1135–1169.

[64] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).

[65] L.D. Landau and E.M. Lifshitz, Course of theoretical physics, Pergamon Press (1960).

[66] G. Landi, An introduction to noncommutative spaces and their geometry, Springer (1997).

[67] S. Lang, Algebra, Addison-Wesley (1993).

[68] P. Lax, Linear algebra and its applications, Wiley (2007).

[69] P. Lax, Functional analysis, Wiley (2002).

[70] F. Lusztig, Introduction to quantum groups, Birkhäuser (1993).

[71] S. Majid, Foundations of quantum group theory, Cambridge Univ. Press (1995).

[72] Y.I. Manin, Quantum groups and noncommutative geometry, Springer (2018).

[73] D. McDuff and D. Salamon, Introduction to symplectic topology, Oxford Univ. Press (2017).

[74] M.L. Mehta, Random matrices, Elsevier (1967).

[75] S. Montgomery, Hopf algebras and their actions on rings, AMS (1993).

[76] J. Nash, The imbedding problem for Riemannian manifolds, *Ann. of Math.* **63** (1956), 20–63.

[77] P. Petersen, Linear algebra, Springer (2012).

[78] P. Petersen, Riemannian geometry, Springer (2006).

[79] D.E. Radford, Hopf algebras, World Scientific (2011).

[80] W. Rudin, Principles of mathematical analysis, McGraw-Hill (1964).

[81] W. Rudin, Real and complex analysis, McGraw-Hill (1966).

[82] W. Schlag, A course in complex analysis and Riemann surfaces, AMS (2014).

[83] D.V. Schroeder, An introduction to thermal physics, Oxford Univ. Press (1999).

[84] J.P. Serre, Linear representations of finite groups, Springer (1977).

[85] I.R. Shafarevich, Basic algebraic geometry, Springer (1974).

[86] G.C. Shephard and J.A. Todd, Finite unitary reflection groups, *Canad. J. Math.* **6** (1954), 274–304.

[87] M.E. Sweedler, Hopf algebras, W.A. Benjamin (1969).

[88] C.H. Taubes, Differential geometry, Oxford Univ. Press (2011).

[89] J.R. Taylor, Classical mechanics, Univ. Science Books (2003).

[90] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, AMS (1992).

[91] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).

[92] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[93] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).

[94] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).

[95] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).

[96] H. Weyl, The classical groups: their invariants and representations, Princeton Univ. Press (1939).

[97] H. Weyl, Space, time, matter, Princeton Univ. Press (1918).

[98] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–399.

[99] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[100] B. Zwiebach, A first course in string theory, Cambridge Univ. Press (2004).

Index