Free differential equations

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Abstract. This is an introduction to free PDE and their potential applications, to questions from theoretical physics. The space $\mathbb{R}^N$ has no free analogue, but the unit sphere $S^{N-1}_\mathbb{R}\subset\mathbb{R}^N$ does have a free analogue, denoted $S^{N-1}_{\mathbb{R},+}$, which appears via free coordinates $x_1,\ldots,x_N$ which are self-adjoint, $x_i=x_i^*$, and satisfy $\sum_ix_i^2=1$. More generally, we can talk about various submanifolds $X\subset S^{N-1}_{\mathbb{R},+}$, which under suitable assumptions have a Laplace operator Δ. We first discuss the Laplace equation $\Delta f=0$ in this setting, and then various free analogues of well-known PDE from physics. The mathematics here is quite interesting, suggesting among others the existence of a "free electrodynamics" theory, conjecturally related to questions in QCD.

Preface

As you surely know, the main question in theoretical physics is that of improving the Standard Model, which dates back to the 70s. Although there have been many interesting discoveries recently, in the "quantum" direction, often accompanied by new engineering feats, the truth remains that our basic knowledge of quantum theory goes back to that old model from the 70s. And as long as we remain unable to improve that model, our flagship quantum technologies, such as nuclear power, will basically remain stuck.

You probably know too that theoretical physicists are not alone in struggling with this question, because large branches of mathematics try to solve this problem too. Indeed, this is certainly true for most people doing PDE or probability, who often get involved, openly, into such questions. As for pure mathematics, that is not as pure as it might seem, because its main architects from the 70s and 80s, such as Atiyah, Connes, Jones and others, were having precisely these Standard Model questions in mind.

Needless to say, there is hurry with this. Not necessarily because of new quantum technologies to be discovered, we are a bit overwhelmed with new technology in recent years, aren't we, but also for the survival of our academic system, as we know it, mathematics and theoretical physics being basically stuck for 50 years being not a good thing.

The aim of the present book is to present one of the many speculations that can be made, in connection with such questions. Importantly, while not yet really connected to physics, these speculations are quite fresh, going back to the 10s and early 20s, and so are a sort of a "start-up" operation, whose potential remains to be determined.

The starting point is the start of quantum mechanics, as we know it from Heisenberg and others. As you zoom down, to the level of protons, electrons and neutrons, things become noncommutative. And this leads to the natural idea that, maybe, if we zoom further down, things might perhaps drastically simplify, and become free.

At the first glance, this might sound like a worthless, wild speculation. However, there is in fact increasing evidence for this. To start with, linguistically at least, it is known since 1973 that quarks are subject to "asymptotic freedom", and whether that famous freedom is the same as mathematical freeness, remains to be determined.

More concretely now, Connes and collaborators have done a lot of work on the Standard Model in their noncommutative geometry formulation, and one of the features of their formalism is that it allows the construction of a "free gauge group" of the Standard Model. Via some standard twisting results, acting on the QED part is S_4^+, and acting on the QCD part is S_9^+. This is quite interesting, suggesting that QED and QCD, suitably twisted, might be some sort of Yang-Mills theories based on S_4^+, S_9^+, respectively.

Another approach, with the theory here going back to work of Yang-Baxter, Faddeev and the Leningrad School, then Drinfeld-Jimbo, and especially Jones and others, is via statistical mechanics and lattice models. Again, this leads to quantum groups, which are traditionally deformed with the help of a parameter $q \in \mathbb{C}$, but which can be as well undeformed, and rather free, depending on which precise model you are looking at.

Yet another approach, and facet of the problem, which is the one that we will describe in this book, is via some sort of "reverse engineering". Indeed, let us temporarily forget about physics. Mathematically then, a free sphere $S^{N-1}_{\mathbb{R},+}$ is not hard to construct, and afterwards you can simply go ahead with mathematics, developed without thinking much: free manifolds, free Laplace operator, free harmonic functions, free PDE. In short, free everything, and the question which appears at the end, mainly coming from free PDE, is whether that new mathematics corresponds to some sort of "free physics", and then, importantly, whether that free physics is true physics, at very small scales, or not.

This sounds quite reasonable, or at least fun, and we will have here a look, at all this. The conjecture at the end will be that there should be a kind of "free electrodynamics" theory, very related to QCD. However, a bit as before with the above-mentioned other approaches, this remains just a facet of the problem. Further advancing, and then putting all the pieces of the puzzle together, remains of course an open problem.

The book mainly contains mathematics developed in the 10s, and early 20s, and some folklore too. Many thanks go to my colleagues, young or old, having contributed to the theory discussed here, or to some wrong, but inspiring rival theories. Thanks as well to my sister Valeria, who's a mathematician like me, but doing hardline PDE, and I will certainly find a way to talk about her exciting work with Luis, in this book. Finally, many thanks to my PDE colleagues in Cergy, and cats at home, nothing better as work environment, than being surrounded by various apex predators.

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Part I

Free manifolds

I'm coming home
I'm coming back down tonight
Cause I've been hypnotized
By the lights

CHAPTER 1

Free spheres

1a. Free tori

Welcome to freeness. We will be interested in this book in developing free geometry and analysis, with the hope that all this might be related to physics, at very small scales, quarks or below. Before anything, all this is well-known to be complicated business, and technically, it is an open problem. So, we will use a trick, developing first as much free geometry and analysis as we can, hard work done in the dark, a bit like miners working in a mine, and only afterwards, towards the end of the book, we will go to the surface, and look at all this under the light of true physics, see if we have some diamonds or not. With diamond meaning free PDE having an interesting physical meaning.

In short, expect a lot of mathematics, at least to start with, correct as mathematics usually comes, but not necessarily very logical, also as mathematics usually comes.

Helping with writing, however, will be my cat assistant, who knows some physics. Usually cats won't tell, at that level of wisdom you admire this world as it was created, with bigger animals eating smaller animals, evolution and so on. However, I have my own tricks, and although I'm very slow, and with a lame diet by his standards, cat ranks me somewhere higher than dogs and bears, and is sometimes willing to help.

And good news, cat is here, so let's ask him how to get started:

Cat 1.1. *Normally for high speed physics and freeness, you need to be fast and free yourself. But yes, do some math, and start with what you know.*

Thanks cat, I was kind of expecting this, but the advice at the end is really helpful. I was twisting my mind with looking for a free analogue of $\mathbb{R}^N$, for developing afterwards free geometry and analysis inside, sort of a nice program, as any mathematician would do. But, as cat says, let's better relax, and start with what we know.

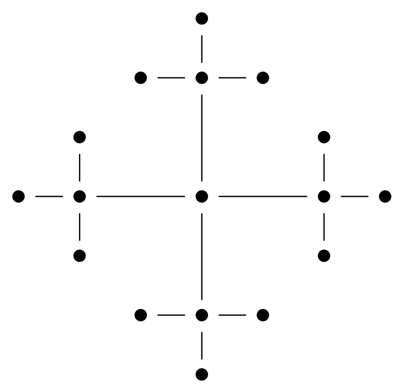
So, what's free? The simplest free object in mathematics is the free group F_N:

Definition 1.2. *The free group F_N is the infinite group*

$$F_N = \left\langle g_1, \ldots, g_N \middle| \emptyset \right\rangle$$

generated by N variables $g_1, \ldots, g_N$, with no relations between them.

This might look a bit abstract, but no worries, F_N has some interesting mathematics, coming right away, if you have some knowledge in discrete groups, and know how to look for interesting questions. For instance if you want to draw the Cayley graph of F_N, whose vertices are the elements of F_N, with edges $h - k$ drawn when $h = g_i^{\pm 1}k$ for some i, you will end up with an interesting picture, which at $N = 2$ looks like this:



And this type of graph certainly has interesting mathematics. One good question for instance is that of computing the number of length $2k$ loops based at the root. Another question, which is in fact equivalent, via moments, is that of computing the Kesten measure of F_N, which is that of the following variable in the group algebra of F_N:

$$\chi = g_1 + \ldots + g_N$$

All this looks very good, we most likely have here our first object of free geometry, the above graph, regarded as some sort of "manifold", and mathematically speaking, this manifold is as good and interesting as manifolds can get. However, before going ahead with loops and Kesten, let's ask the cat, who's still around. Not that I need help with math, but sometimes a piece of recognition from a fellow physics colleague, for a bright idea like this, can bring pleasure. To my surprise, however, cat answers:

Cat 1.3. *You got it wrong with your math, that graph is not continuous, even by alien standards. It's the dual of F_N which is a free manifold.*

Thanks cat, and interesting remark, indeed. In fact, I was too quick in developing free geometry, and forgot to think at classical geometry first. Here, if there is an interesting formula in relation with free groups and manifolds, this is the following formula, with $\mathbb{T}_N = \mathbb{T}^N$ being the usual torus, and with $\mathbb{Z}^N$ being the free abelian group:

$$\mathbb{T}_N = \widehat{\mathbb{Z}^N}$$

Thus, getting back now to our free group F_N, which is the free analogue of $\mathbb{Z}^N$, it is its dual $\widehat{F_N}$ which is a free manifold, and more specifically the free analogue of $\mathbb{T}^N$. Which is a nice finding, so let us formulate our conclusions as follows:

Definition 1.4. *The free torus* $\mathbb{T}_N^+$ *is the dual of the free group* F_N*,*

$$\mathbb{T}_N^+ = \widehat{F_N}$$

in analogy with the fact that the usual torus $\mathbb{T}_N = \mathbb{T}^N$ *appears as*

$$\mathbb{T}_N = \widehat{\mathbb{Z}^N}$$

with on the right the group $\mathbb{Z}^N$ *being the free abelian group.*

It is of course possible to formulate things more precisely, and we will be back to this in a moment, but before that, isn't this a bit too abstract? But the point here is that no, at the level of questions to be solved, these remain the same, as for instance the computation of the Kesten measure, which is now a "function" on the free torus:

$$\chi \in C(\mathbb{T}_N^+)$$

In fact, this function is the main character of $\mathbb{T}_N^+$, regarded as a compact quantum group, and so our Kesten problem suddenly becomes something very conceptual, namely the computation of the law of the main character of $\mathbb{T}_N^+$. Which is very nice.

Before getting into details regarding all this, recall that $\mathbb{R}^N$ is as interesting as $\mathbb{C}^N$. So, let us formulate as well the real version of Definition 1.4, as follows:

Definition 1.5. *The free real torus, or free cube,* T_N^+ *is the dual*

$$T_N^+ = \widehat{L_N}$$

of the group $L_N = F_N / < g_i^2 = 1 >$*, in analogy with the fact that the usual cube is*

$$T_N = \widehat{\mathbb{Z}_2^N}$$

with on the right the group $\mathbb{Z}_2^N$ *being the free real abelian group.*

Here the "real" at the end stands for the fact that the generators must satisfy the real reflection condition $g^2 = 1$. As for the fact that "real torus = cube", as stated, this needs some thinking, and in the hope that, after such thinking, you will agree with me that there is indeed a standard torus inside $\mathbb{R}^N$, and that is the unit cube.

As before with the free complex torus $\mathbb{T}_N^+$, there is some mathematics to be done with the free real torus T_N^+, for instance in relation with the law of $\chi = g_1 + \ldots + g_N$.

Summarizing, all this sounds good, we have a beginning of free geometry, both real and complex, worth developing, by knowing at least what the torus of each theory is. In practice now, at the level of details, in order to talk about $\mathbb{T}_N^+ = \widehat{F_N}$ and $T_N^+ = \widehat{L_N}$ we need an extension of the usual Pontrjagin duality theory for the abelian groups, and this is best done via operator algebras, and the related notion of compact quantum group.

So, in order to fully understand all this, let us start with operator algebras.

You have probably already heard about infinite matrices, operators and operator algebras, from Heisenberg, Schrödinger, Dirac and others. As a starting point for all this, we need a complex Hilbert space H, with the main example in mind being the space $H = L^2(\mathbb{R}^3)$ of the wave functions of the electron. So, let us formulate:

DEFINITION 1.6. *A Hilbert space is a complex vector space H, given with a scalar product $< x, y >$, satisfying the following conditions:*

(1) *$< x, y >$ is linear in x, and antilinear in y.*
(2) *$\overline{< x, y >} =< y, x >$, for any x, y.*
(3) *$< x, x >> 0$, for any $x \neq 0$.*
(4) *H is complete with respect to the norm $||x|| = \sqrt{< x, x >}$.*

This looks nice and correct, with the remark that (4) assumes that you know about Cauchy-Schwarz, but thinking well, I'm using here mathematicians' convention for scalar products, linear at left, and aren't we supposed to do as Dirac and other physicists do, with the scalar products linear at right. And making a decision here does not seem to be an easy question, shall we trade the usefulness of Dirac's bras and kets $< x|$ and $|y >$ for mathematical simplicity, I mean what's simple and linear must come first.

I'm afraid I will have to disturb again the cat. And cat says:

CAT 1.7. *Bras and kets are made to interact, and love each other, and that vertical bar is a bad idea, preventing the physics to happen.*

Interesting remark, so if I understand well $< x|y >$ being a bad idea, and I fully agree with this because that vertical bar | slows down computations anyway, we are left with $< x, y >$, and free to choose the linearity as we like. So, Definition 1.6 is correct.

Moving ahead, we need to talk about operators. Again, you might have heard of these from Heisenberg, Schrödinger, Dirac and others, and with the theory being quite complicated to read and digest, because these operators, while fortunately self-adjoint, are unfortunately unbounded. However, cat who's still around, declares:

CAT 1.8. *Self-adjoint and unbounded operators are nice, but not fast enough. For fast physics, you need non-self-adjoint, bounded operators.*

Thanks cat, this sounds good, and again agrees with my mathematical intuition, the bounded operators are the simplest, and who cares about self-adjointness, and I would be even happier not to get into that, I prefer these bounded operators to be arbitrary.

So, bounded operators. These are in fact quite tricky to study, even when taken arbitrary, and after some work, we can formulate, as a first theorem for our book:

THEOREM 1.9. *The linear operators $T : H \to H$ which are bounded, meaning that*

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, having the following properties:

(1) *$B(H)$ is complete with respect to $||.||$, so we have a Banach algebra.*
(2) *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

In addition, the norm and involution are related by the formula $||TT^*|| = ||T||^2$.

PROOF. The fact that we have an algebra is clear, and the completness comes from the fact that, assuming that $\{T_n\} \subset B(H)$ is Cauchy, then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Regarding $T \to T^*$, this comes from the fact that $\varphi(x) =< Tx, y >$ being a linear form $\varphi : H \to \mathbb{C}$, we must have $\varphi(x) =< x, T^*y >$, for a certain vector $T^*y \in H$. Thus we have a well-defined involution $T \to T^*$, which stays inside $B(H)$, because:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y > \\ &= ||T^*|| \end{aligned}$$

Regarding now the last assertion, observe first that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, so we are done. $\square$

Observe that when H comes with an orthonormal basis $\{e_i\}_{i\in I}$, the linear map $T \to M$ given by $M_{ij} =< Te_j, e_i >$ produces an embedding as follows:

$$B(H) \subset M_I(\mathbb{C})$$

Moreover, in this picture the operation $T \to T^*$ takes a very simple form, namely:

$$(M^*)_{ij} = \overline{M}_{ji}$$

However, with examples like Schrödinger's wave function space $H = L^2(\mathbb{R}^3)$ in mind, it is better in general not to use bases, and accept Theorem 1.9 as stated.

Moving ahead, the conditions found in Theorem 1.9 suggest formulating:

Definition 1.10. *A C^*-algebra is a complex algebra A, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra.*
2. *An involution $a \to a^*$, satisfying $||aa^*|| = ||a||^2$.*

As basic examples, we have $B(H)$ itself, as well as any norm closed $*$-subalgebra $A \subset B(H)$. It is possible to prove that any C^*-algebra appears in this way, but we will not need in what follows this deep result, called GNS theorem after Gelfand, Naimark, Segal. So, let us simply agree that, by definition, the C^*-algebras A are some sort of "generalized operator algebras", and their elements $a \in A$ can be thought of as being some kind of "generalized operators", on some Hilbert space which is not present.

In practice, this vague idea is all that we need. Indeed, by taking some inspiration from linear algebra, we can emulate spectral theory in our setting, as follows:

Proposition 1.11. *Given $a \in A$, define its spectrum as being the set*

$$\sigma(a) = \left\{\lambda \in \mathbb{C}\Big|a - \lambda \notin A^{-1}\right\}$$

and its spectral radius $\rho(a)$ as the radius of the smallest centered disk containing $\sigma(a)$.

1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

Proof. The first claim is that for any polynomial $f \in \mathbb{C}[X]$, and more generally for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$, we have:

$$\sigma(f(a)) = f(\sigma(a))$$

This indeed something well-known for the usual matrices, and in general, the proof is similar. Regarding now the assertions in the statement, these all follow from this:

(1) This comes from the following formula, valid when $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

(2) Assuming $a^* = a^{-1}$, if we denote by D the unit disk, we have, by using (1):

$$||a|| = 1 \implies \sigma(a) \subset D$$

$$||a^{-1}|| = 1 \implies \sigma(a^{-1}) \subset D$$

On the other hand, by using the rational function $f(z) = z^{-1}$, we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Now by putting everything together we obtain, as desired:

$$\sigma(a)\subset D\cap D^{-1}=\mathbb T$$

(3) This follows from (2), by using the rational function $f(z)=(z+it)/(z-it)$. Indeed, for $t>>0$ we have the following computation:

$$\left(\frac{a+it}{a-it}\right)^*=\frac{a-it}{a+it}=\left(\frac{a+it}{a-it}\right)^{-1}$$

Thus the element $f(a)$ is a unitary, and by using (2) its spectrum is contained in $\mathbb T$. We conclude from this that we have:

$$f(\sigma(a))=\sigma(f(a))\subset\mathbb T$$

But this shows that we have $\sigma(a)\subset f^{-1}(\mathbb T)=\mathbb R$, as desired.

(4) We already know that we have $\rho(a)\leq||a||$, for any $a\in A$. For the reverse inequality, when a is normal, we fix a number $\rho>\rho(a)$. We have then:

$$\begin{aligned}\int_{|z|=\rho}\frac{z^n}{z-a}\,dz&=\int_{|z|=\rho}\sum_{k=0}^\infty z^{n-k-1}a^k\,dz\\&=\sum_{k=0}^\infty\left(\int_{|z|=\rho}z^{n-k-1}dz\right)a^k\\&=a^{n-1}\end{aligned}$$

By applying the norm and taking n-th roots we obtain from this formula:

$$\rho\geq\lim_{n\to\infty}||a^n||^{1/n}$$

When $a=a^*$ we have $||a^n||=||a||^n$ for any exponent of type $n=2^k$, by using the C^*-algebra condition $||aa^*||=||a||^2$, and by taking n-th roots we get, as desired:

$$\rho(a)\geq||a||$$

In the general normal case now, $aa^*=a^*a$, we have $a^n(a^n)^*=(aa^*)^n$, and by using this, along with the result for self-adjoints, applied to aa^*, we obtain:

$$\begin{aligned}\rho(a)&\geq\lim_{n\to\infty}||a^n||^{1/n}=\sqrt{\lim_{n\to\infty}||a^n(a^n)^*||^{1/n}}\\&=\sqrt{\lim_{n\to\infty}||(aa^*)^n||^{1/n}}=\sqrt{\rho(aa^*)}\\&=\sqrt{||a||^2}=||a||\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Generally speaking, Proposition 1.11 is all that you need for doing further operator algebras, only military grade weapons there. As a main application, we have:

THEOREM 1.12 (Gelfand). *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a commutative C^*-algebra, with structure as follows:*

(1) *The norm is the usual sup norm, $||f|| = \sup_{x\in X}|f(x)|$.*
(2) *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

Conversely, any commutative C^*-algebra is of the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.

PROOF. Given a commutative C^*-algebra A, we can define indeed X to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras:

$$ev : A \to C(X)$$

We first prove that ev is involutive. We use the following formula:

$$a = \frac{a+a^*}{2} - i \cdot \frac{i(a-a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$. So, claim proved. Also, since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The Gelfand theorem suggests formulating the following definition:

DEFINITION 1.13. *Given a C^*-algebra A, not necessarily commutative, we write*

$$A = C(X)$$

and call the abstract object X a "compact quantum space".

This might look quite revolutionary, but in practice, this definition changes nothing to what we have been doing so far, namely studying the C^*-algebras. So, we will keep studying the C^*-algebras, but by using the above fancy quantum space terminology. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

Taking a break now from all this, mathematics endlessly building and self-replicating, once started, like some sort of monster, shall we perhaps think a bit at the physical meaning of all this. I am particularly concerned by the fact that our quantum spaces are compact, if there is one good space for math and physics, that is $\mathbb{R}^N$, which is obviously not compact, so shouldn't be our quantum spaces not compact either.

This does not look like an obvious question, so time to ask the cat. And cat says:

Cat 1.14. *The strong force is confined, expect mathematical freeness to be confined too. As strange as this might sound, linguistically speaking.*

Thanks cat, but this sounds a bit too deep, to the point that I cannot tell if it's a joke or not. In any case, I take it as an encouragement, so we'll go for confinement and compactness, as a continuation of the above, and may the strong force be with us.

So, getting back now to our operator algebra machinery, what's next? Actually, now that we have our definition for the quantum spaces, good time to get back towards Definitions 1.4 and 1.5. In order to understand what that free tori are, we will need:

Theorem 1.15. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^* = g^{-1}$.*

1. *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
2. *When Γ is abelian, we have an isomorphism $C^*(\Gamma) \simeq C(G)$, where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.*

Proof. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma] \subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h) = gh$ that $g \to \pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g \in \Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A = C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A = C(X)$, with:

$$X = Spec(A)$$

But the spectrum $X = Spec(A)$, consisting of the characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result. □

The above result suggests the following definition:

Definition 1.16. *Given a discrete group Γ, the compact quantum space G given by*

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Good news, this definition is exactly what we need, in order to understand the meaning of Definitions 1.4 and 1.5. To be more precise, we have the following result:

PROPOSITION 1.17. *The basic tori are all group duals, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} \quad = \quad \begin{array}{ccc} \widehat{L_N} & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N \end{array}$$

where $F_N = \mathbb{Z}^{*N}$ is the free group on N generators, and $L_N = \mathbb{Z}_2^{*N}$ is its real version.

PROOF. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 1.15 (2), this gives the result. □

Moving ahead, now that we have our formalism, we can start developing free geometry. As a first objective, we would like to better understand the relation between the classical and free tori. In order to discuss this, let us introduce the following notion:

DEFINITION 1.18. *Given a compact quantum space X, its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I = < [a,b] >$$

In this situation, we also say that X appears as a “liberation” of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space.

In relation now with our tori, we have the following result:

THEOREM 1.19. *We have inclusions between the various tori, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and the free tori on top appear as liberations of the tori on the bottom.

PROOF. This is indeed clear from definitions, because commutativity of a group algebra means precisely that the group in question is abelian. □

As a conclusion now to all this, we have a beginning of free geometry, both real and complex, by knowing at least what the torus of each theory is. And with our construction being definitely the good one, for the simple reason that the main problems in the analysis of the free groups correspond in this way the main questions in our free geometry.

1b. Free spheres

In order to extend now the free geometries that we have, real and complex, let us begin with the spheres. Following [11], we have the following notions:

Definition 1.20. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_i x_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^*=\sum_i x_i^*x_i=1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

Here the fact that these algebras are indeed well-defined comes from the following estimate, which shows that the biggest C^*-norms on these $*$-algebras are bounded:

$$||x_i||^2=||x_ix_i^*||\leq\left|\left|\sum_i x_ix_i^*\right|\right|=1$$

As a first result now, regarding the above free spheres, we have:

Theorem 1.21. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{matrix}S^{N-1}_{\mathbb{R},+}&\rightarrow&S^{N-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb{R}}&\rightarrow&S^{N-1}_{\mathbb{C}}\end{matrix}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{matrix}C(S^{N-1}_{\mathbb{R},+})&\leftarrow&C(S^{N-1}_{\mathbb{C},+})\\ \downarrow&&\downarrow\\ C(S^{N-1}_{\mathbb{R}})&\leftarrow&C(S^{N-1}_{\mathbb{C}})\end{matrix}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_\mathbb{R})=C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_i x_i^2=1\right)$$

$$C(S^{N-1}_\mathbb{C})=C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^*=\sum_i x_i^*x_i=1\right)$$

It is enough to establish the second isomorphism. So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_\mathbb{C}$ satisfy the defining relations for A, we have a quotient map of as follows:

$$A\to C(S^{N-1}_\mathbb{C})$$

Conversely, let us write $A=C(S)$, by using the Gelfand theorem. The variables $x_1,\ldots,x_N$ become in this way true coordinates, providing us with an embedding $S\subset\mathbb{C}^N$. Also, the quadratic relations become $\sum_i|x_i|^2=1$, so we have $S\subset S^{N-1}_\mathbb{C}$. Thus, we have a quotient map $C(S^{N-1}_\mathbb{C})\to A$, as desired, and this gives all the results. $\square$

By using the free spheres constructed above, we can now formulate:

Definition 1.22. *A real algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i\in\mathbb{C}<x_1,\ldots,x_N>$$

We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinates $x_1,\ldots,x_N$.

As a basic example here, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_\mathbb{C},S^{N-1}_\mathbb{R}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem:

Theorem 1.23. *If $X\subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class}=\left\{x\in S^{N-1}_\mathbb{C}\Big|f_i(x_1,\ldots,x_N)=0\right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class})\to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_{\mathbb{C}}$. Now since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

DEFINITION 1.24. *We agree to identify two real algebraic submanifolds $X, Y \subset S^{N-1}_{\mathbb{C},+}$ when we have a $*$-algebra isomorphism between $*$-algebras of coordinates*

$$f : \mathcal{C}(Y) \to \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

We will see later the reasons for making this convention, coming from amenability. Now back to the tori, as constructed before, we can see that these are examples of algebraic manifolds, in the sense of Definition 1.22. In fact, we have the following result:

THEOREM 1.25. *The four main quantum spheres produce the main quantum tori*

$$\begin{matrix} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

via the formula $T = S \cap \mathbb{T}_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb{C},+}$.

PROOF. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb{T}_N^+ = S^{N-1}_{\mathbb{C},+} \cap \mathbb{T}_N^+$. But this is something trivial, because we have $\mathbb{T}_N^+ \subset S^{N-1}_{\mathbb{C},+}$.

(2) Free real case. Here the formula in the statement reads $T_N^+ = S^{N-1}_{\mathbb{R},+} \cap \mathbb{T}_N^+$. But this is clear as well, the real version of $\mathbb{T}_N^+$ being T_N^+.

(3) Classical complex case. Here the formula in the statement reads $\mathbb{T}_N = S^{N-1}_{\mathbb{C}} \cap \mathbb{T}_N^+$. But this is clear as well, the classical version of $\mathbb{T}_N^+$ being $\mathbb{T}_N$.

(4) Classical real case. Here the formula in the statement reads $T_N = S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). □

1c. Free rotations

In order to better understand the structure of $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}$, we need to talk about free rotations. Following Woronowicz [99], [100], let us start with:

Definition 1.26. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras $\Delta:A\to A\otimes A$, $\varepsilon:A\to\mathbb{C}$, $S:A\to A^{opp}$.

The morphisms Δ,ε,S are called comultiplication, counit and antipode. We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. We have the following result, which justifies the terminology and axioms:

Proposition 1.27. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi)=(g,h)\to\varphi(gh)\quad,\quad\varepsilon(\varphi)=\varphi(1)\quad,\quad S(\varphi)=g\to\varphi(g^{-1})$$

(2) *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g)=g\otimes g\quad,\quad\varepsilon(g)=1\quad,\quad S(g)=g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. This is something very standard, the idea being as follows:

(1) Given $G\subset U_N$, we can set $A=C(G)$, which is a Woronowicz algebra, together with the matrix $u=(u_{ij})$ formed by coordinates of G, given by:

$$g=\begin{pmatrix}u_{11}(g)&\dots&u_{1N}(g)\\ \vdots&&\vdots\\ u_{N1}(g)&\dots&u_{NN}(g)\end{pmatrix}$$

Conversely, if (A,u) is a commutative Woronowicz algebra, by using the Gelfand theorem we can write $A=C(X)$, with X being a certain compact space. The coordinates u_{ij} give then an embedding $X\subset M_N(\mathbb{C})$, and since the matrix $u=(u_{ij})$ is unitary we actually obtain an embedding $X\subset U_N$, and finally by using the maps Δ,ε,S we conclude that our compact subspace $X\subset U_N$ is in fact a compact Lie group, as desired.

(2) Consider a finitely generated group $F_N\to\Gamma$. We can set $A=C^*(\Gamma)$, which is by definition the completion of the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by

$g^* = g^{-1}$, for any $g \in \Gamma$, with respect to the biggest C^*-norm, and we obtain a Woronowicz algebra, together with the diagonal matrix formed by the generators of Γ:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Conversely, if (A, u) is a cocommutative Woronowicz algebra, the Peter-Weyl theory of Woronowicz, to be explained below, shows that the irreducible corepresentations of A are all 1-dimensional, and form a group Γ, and so we have $A = C^*(\Gamma)$, as desired. □

In general now, the structural maps Δ, ε, S have the following properties:

PROPOSITION 1.28. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$\begin{aligned} (\Delta \otimes id)\Delta &= (id \otimes \Delta)\Delta \\ (\varepsilon \otimes id)\Delta &= (id \otimes \varepsilon)\Delta = id \end{aligned}$$

(2) *S satisfies the antipode axiom, on the $*$-subalgebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

PROOF. The two comultiplication axioms follow from:

$$\begin{aligned} (\Delta \otimes id)\Delta(u_{ij}) &= (id \otimes \Delta)\Delta(u_{ij}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj} \\ (\varepsilon \otimes id)\Delta(u_{ij}) &= (id \otimes \varepsilon)\Delta(u_{ij}) = u_{ij} \end{aligned}$$

As for the antipode formulae, the verification here is similar. □

Summarizing, the Woronowicz algebras appear to have nice properties. In view of Proposition 1.27 and Proposition 1.28, we can formulate the following definition:

DEFINITION 1.29. *Given a Woronowicz algebra A, we formally write*

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

In relation with this, there are actually some analytic subtleties, coming from amenability, so our objects must be divided by a certain equivalence relation, for everything to work fine. To be more precise, we agree to write $(A, u) = (B, v)$ when there is a $*$-algebra isomorphism as follows, mapping standard coordinates to standard coordinates:

$$< u_{ij} > \simeq < v_{ij} > \quad , \quad u_{ij} \to v_{ij}$$

Moving ahead now, let us call now corepresentation of A any unitary matrix $v\in M_n(A)$ satisfying the same conditions as those satisfied by u, namely:

$$\Delta(v_{ij})=\sum_k v_{ik}\otimes v_{kj}\quad,\quad\varepsilon(v_{ij})=\delta_{ij}\quad,\quad S(v_{ij})=v_{ji}^*$$

These corepresentations can be thought of as corresponding representations of the underlying compact quantum group G. Following Woronowicz [99], we have:

Theorem 1.30. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb C)\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb C^n|v\xi=\xi\}$.

Proof. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, by linearity we can assume that a is the coefficient of corepresentation, $a=(\tau\otimes id)v$. But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the space $Fix(v)=\{\xi\in\mathbb C^n|v\xi=\xi\}$. The point now is that when $\varphi\in A^*$ is faithful, by using a positivity trick, one can prove that we have $P_\varphi=P$. Thus our linear form $\int_\varphi$ is independent of φ, and is given on coefficients $a=(\tau\otimes id)v$ by:

$$\left(id\otimes\int_\varphi\right)v=P$$

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. ☐

Consider the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

We have then the following result, also due to Woronowicz [99]:

THEOREM 1.31. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. $\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$*, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [99], the idea being as follows:

(1) Given $v \in M_n(A)$, its intertwiner algebra $End(v) = \{T \in M_n(\mathbb{C}) | Tv = vT\}$ is a finite dimensional C^*-algebra, and so decomposes as $End(v) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$. But this gives a decomposition of type $v = v_1 + \ldots + v_r$, as desired.

(2) Consider indeed the Peter-Weyl corepresentations, $u^{\otimes k}$ with k colored integer, defined by $u^{\otimes \emptyset} = 1$, $u^{\otimes \circ} = u$, $u^{\otimes \bullet} = \bar{u}$ and multiplicativity. The coefficients of these corepresentations span the dense algebra $\mathcal{A}$, and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is technically a $*$-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id \otimes \int_G)v$ is the orthogonal projection P_v onto the space $Fix(v)$, for any corepresentation v.

(4) Let us define indeed the character of $v \in M_n(A)$ to be the matrix trace, $\chi_v = Tr(v)$. Since this character is a coefficient of v, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id \otimes \int_G)v = P_v$. □

We can now talk about free rotations. Following Wang [89], we have:

PROPOSITION 1.32. *The following universal algebras are Woronowicz algebras,*

$$C(O_N^+) = C^* \left((u_{ij})_{i,j=1,\ldots,N} \Big| u = \bar{u}, u^t = u^{-1} \right)$$

$$C(U_N^+) = C^* \left((u_{ij})_{i,j=1,\ldots,N} \Big| u^* = u^{-1}, u^t = \bar{u}^{-1} \right)$$

so the underlying spaces O_N^+, U_N^+ are compact quantum groups.

PROOF. This follows from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Thus, we can indeed define morphisms Δ, ε, S as in Definition 1.26, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$, and this gives the result. □

In order to discuss now the relation with the spheres, we will need:

PROPOSITION 1.33. *Given an algebraic manifold $X\subset S^{N-1}_{\mathbb C,+}$, the category of the closed subgroups $G\subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

defines a morphism of C^*-algebras $\Phi:C(X)\to C(X)\otimes C(G)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

PROOF. Assume indeed that our manifold $X\subset S^{N-1}_{\mathbb C,+}$ comes as follows:

$$C(X)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_\alpha(x_1,\ldots,x_N)=0\Big\rangle$$

In order to prove the result, consider the following variables:

$$X_i=\sum_j x_j\otimes u_{ji}\in C(X)\otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G=G^+(X)$ appears as:

$$C(G)=C(U_N^+)\Big/\Big\langle f_\alpha(X_1,\ldots,X_N)=0\Big\rangle$$

In order to prove this, pick one of the defining polynomials, and write it as follows:

$$f_\alpha(x_1,\ldots,x_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\cdot x_{i_1^r}\ldots x_{i_{s_r}^r}$$

With $X_i=\sum_j x_j\otimes u_{ji}$ as above, we have the following formula:

$$f_\alpha(X_1,\ldots,X_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\sum_{j_1^r\ldots j_{s_r}^r}x_{j_1^r}\ldots x_{j_{s_r}^r}\otimes u_{j_1^ri_1^r}\ldots u_{j_{s_r}^ri_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f_\alpha(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a closed subspace $G\subset U_N^+$, with a universal map, as follows:

$$\Phi:C(X)\to C(X)\otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider as well the following elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f_\alpha(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f_\alpha(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f_\alpha(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. □

We can now formulate a result about spheres and rotations, as follows:

THEOREM 1.34. *The quantum isometry groups of the basic spheres are*

$$\begin{matrix}S^{N-1}_{\mathbb R,+}&\longrightarrow&S^{N-1}_{\mathbb C,+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb R}&\longrightarrow&S^{N-1}_{\mathbb C}\end{matrix}\qquad\to\qquad\begin{matrix}O_N^+&\longrightarrow&U_N^+\\ \uparrow&&\uparrow\\ O_N&\longrightarrow&U_N\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

PROOF. We have 4 results to be proved, the idea being as follows:

$\underline{S^{N-1}_{\mathbb C,+}}$. Let us first construct an action $U_N^+\curvearrowright S^{N-1}_{\mathbb C,+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb C,+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

By using the biunitarity of u, we have the following computation:

$$\sum_i X_iX_i^*=\sum_{ijk}x_jx_k^*\otimes u_{ji}u_{ki}^*=\sum_j x_jx_j^*\otimes 1=1\otimes 1$$

Once again by using the biunitarity of u, we have as well:

$$\sum_i X_i^*X_i=\sum_{ijk}x_j^*x_k\otimes u_{ji}^*u_{ki}=\sum_j x_j^*x_j\otimes 1=1\otimes 1$$

Thus we have an action $U_N^+\curvearrowright S^{N-1}_{\mathbb C,+}$, which gives $G^+(S^{N-1}_{\mathbb C,+})=U_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb R,+}}$. Let us first construct an action $O_N^+\curvearrowright S^{N-1}_{\mathbb R,+}$. We already know that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb C,+}$, so we just have to check that these variables are self-adjoint. But this is clear from $u=\bar u$, as follows:

$$X_i^*=\sum_j x_j^*\otimes u_{ji}^*=\sum_j x_j\otimes u_{ji}=X_i$$

Conversely, assume that we have an action $G\curvearrowright S^{N-1}_{\mathbb R,+}$, with $G\subset U_N^+$. The variables $X_i=\sum_j x_j\otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u=\bar u$. Thus our quantum group must satisfy $G\subset O_N^+$, as desired.

$\underline{S^{N-1}_\mathbb{C}}$. The fact that we have an action $U_N\curvearrowright S^{N-1}_\mathbb{C}$ is clear. Conversely, assume that we have an action $G\curvearrowright S^{N-1}_\mathbb{C}$, with $G\subset U_N^+$. We must prove that this implies $G\subset U_N$, and we will use a standard trick of Bhowmick-Goswami [11]. We have:

$$\Phi(x_i)=\sum_jx_j\otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_ix_k)=\sum_{jl}x_jx_l\otimes u_{ji}u_{lk}$$

$$\Phi(x_kx_i)=\sum_{jl}x_lx_j\otimes u_{lk}u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_ix_k)=\sum_{j<l}x_jx_l\otimes(u_{ji}u_{lk}+u_{li}u_{jk})+\sum_jx_j^2\otimes u_{ji}u_{jk}$$

$$\Phi(x_ix_k)=\sum_{j<l}x_jx_l\otimes(u_{lk}u_{ji}+u_{jk}u_{li})+\sum_jx_j^2\otimes u_{jk}u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji}u_{lk}+u_{li}u_{jk}=u_{lk}u_{ji}+u_{jk}u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^*u_{ij}^*+u_{kj}^*u_{il}^*=u_{ij}^*u_{kl}^*+u_{il}^*u_{kj}^*$$

$$u_{ij}u_{kl}+u_{il}u_{kj}=u_{kl}u_{ij}+u_{kj}u_{il}$$

$$u_{ji}u_{lk}+u_{jk}u_{li}=u_{lk}u_{ji}+u_{li}u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li}u_{jk}=u_{jk}u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_ix_k^*)=\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*$$

$$\Phi(x_k^*x_i)=\sum_{jl}x_l^*x_j\otimes u_{lk}^*u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*=\sum_{jl}x_jx_l^*\otimes u_{lk}^*u_{ji}$$

Thus we have $u_{ji}u_{lk}^* = u_{lk}^*u_{ji}$, and so $G \subset U_N$, as claimed.

$\underline{S^{N-1}_{\mathbb{R}}}$. The fact that we have an action $O_N \curvearrowright S^{N-1}_{\mathbb{R}}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned} G \curvearrowright S^{N-1}_{\mathbb{R}} &\implies G \curvearrowright S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C}} \\ &\implies G \subset O_N^+, U_N \\ &\implies G \subset O_N^+ \cap U_N = O_N \end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_{\mathbb{R}}) = O_N$, as desired. □

1d. Fine structure

Let us discuss now the correspondence $U \to S$. In the classical case the situation is very simple, because the sphere $S = S^{N-1}$ appears by rotating the point $x = (1, 0, \ldots, 0)$ by the isometries in $U = U_N$. Moreover, the stabilizer of this action is the subgroup $U_{N-1} \subset U_N$ acting on the last $N-1$ coordinates, and so the sphere $S = S^{N-1}$ appears from the corresponding rotation group $U = U_N$ as an homogeneous space, as follows:

$$S^{N-1} = U_N/U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U \to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \to u_{1i}$$

In general now, the straightforward homogeneous space interpretation of S as above fails. However, we can have some theory going by using the functional analytic viewpoint, with an embedding $x_i \to u_{1i}$ as above. Let us start with the following result:

THEOREM 1.35. *For the basic spheres, we have a diagram as follows,*

$$\begin{array}{ccc} C(S) & \xrightarrow{\ \Phi\ } & C(S) \otimes C(U) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha \otimes id} \\ C(U) & \xrightarrow{\ \Delta\ } & C(U) \otimes C(U) \end{array}$$

where on top $\Phi(x_i) = \sum_j x_j \otimes u_{ji}$, *and on the left* $\alpha(x_i) = u_{1i}$.

PROOF. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows, on these coordinates:

$$\begin{array}{ccc}
x_i & \longrightarrow & \sum_j x_j \otimes u_{ji} \\
\downarrow & & \downarrow \\
u_{1i} & \longrightarrow & \sum_j u_{1j} \otimes u_{ji}
\end{array}$$

Thus by linearity and multiplicativity, the whole the diagram commutes, and this leads to the conclusion in the statement. □

The point now is that, by further building on the above result, we obtain the desired correspondence $U \to S$, and some useful integration results as well. All this is explained in [7], and we will be back to this in chapter 3 below, in a more general setting.

At the level of the fine structure of the free spheres $S^{N-1}_{\mathbb{R},+}, S^{N-1}_{\mathbb{C},+}$ now, we have some obvious formal eigenspaces for the Laplace operator, and a Weingarten integration formula as well, both coming from the representation theory of O_N^+, U_N^+. Moreover, it is possible to get beyond this, with a full construction of a Laplace operator, meaning eigenvalues too, and with all this again coming from the representation theory of O_N^+, U_N^+.

Regarding other possible invariants, orientability does not work, the Dirac operator does not exist, smoothness does not work either, and in what regards K-theory, with our free objects we are a bit too far away from the traditional "reasonable" range of K-theory, usually requiring amenability, or at least some form of K-amenability.

However, after some thinking, maybe including some physical thoughts too, in connection with what is smoothness and is that wished or not, in the present situation, all this is normal. So, no worries, and as we will soon discover, we will get away with the tools that we have, namely Laplace operator and the Weingarten formula, which are not that bad, technically speaking, for all the problems that we will choose to solve.

1e. Exercises

Exercises.

CHAPTER 2

Free rotations

2a. Diagrams, easiness

We have so far a beginning of free geometry, in the real case with a triple of basic objects $(S^{N-1}_{\mathbb{R},+},O_N^+,T_N^+)$, and in the complex case with objects $(S^{N-1}_{\mathbb{C},+},U_N^+,\mathbb{T}_N^+)$. This is not bad, and our purpose in what follows will be that of expanding these two collections of objects, from 3 items each, to 10, 100, 1000, or as many as we can, and the more the merrier, in the name of pure mathematics, where new objects are always welcome.

This being said, what to start with? Leaving aside the tori, which are just duals of discrete groups, and as old as modern mathematics, we face a choice between spheres S, and rotation groups U. As a first observation, these two types of objects are closely related, because in the classical case, given a sphere S, we can recover U as being its isometry group, and conversely, given a group U, we can recover S just by rotating a point. And, as seen in chapter 1, the situation is quite similar in the free case.

This being said, spheres S are not the same thing as rotation groups U, and we will have to make a choice. Normally spheres S look a bit more important, but on the other hand physics, or even mathematics, tell us that no matter what we want to do, of advanced type, about either S or U, we will always end up in struggling with U.

So, we will go for U, and our goal in this chapter will be that of better understanding O_N^+,U_N^+, and also look for more free quantum groups, as many as we can find. And regarding spheres S and other such manifolds, we will leave this for later. Sounds good, doesn't it? Before getting into this, however, let us check with physics and cat:

Cat 2.1. *Gauge invariance gives you everything. But don't forget to do some manifolds too, all our kittens learn that, and it's good learning.*

Thanks cat, this is a pleasure to hear, and in tune with my mathematical intuition. Getting started now, we would like to have a better understanding of the liberation operations that we have, $O_N\to O_N^+$ and $U_N\to U_N^+$, and also have more examples of liberation operations of the same type, $G_N\to G_N^+$. And then, once we will have enough theory and examples, look for classification results for the free quantum groups $\{G_N^+\}$.

Let us start with the construction of more examples, which is certainly a very exciting business, and leave the abstractions for later. Following Wang [89], we first have:

Proposition 2.2. *Consider the symmetric group S_N, viewed as permutation group of the N coordinate axes of $\mathbb{R}^N$. The coordinate functions on $S_N \subset O_N$ are given by*

$$u_{ij} = \chi\left(\sigma \in G\middle|\sigma(j) = i\right)$$

and the matrix $u = (u_{ij})$ that these functions form is magic, in the sense that its entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column.

Proof. The action of S_N on the standard basis $e_1, \ldots, e_N \in \mathbb{R}^N$ being given by $\sigma : e_j \to e_{\sigma(j)}$, this gives the formula of u_{ij} in the statement. As for the fact that the matrix $u = (u_{ij})$ that these functions form is magic, this is clear. □

With a bit more effort, we obtain the following nice characterization of S_N:

Proposition 2.3. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\middle|u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

Proof. The universal algebra A in the statement being commutative, by the Gelfand theorem it must be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates form a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 entry on each row and each column, and so $X = S_N$. Thus we have proved the first assertion, and the second assertion is clear as well. □

Still following Wang [89], we can now liberate S_N, as follows:

Theorem 2.4. *The following universal C^*-algebra, with magic meaning as usual formed by projections ($p^2 = p^* = p$), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\middle|u = \text{magic}\right)$$

is a Woronowicz algebra, with comultiplication, counit and antipode given by:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

Thus the space S_N^+ is a compact quantum group, called quantum permutation group.

PROOF. As a first observation, the universal C^*-algebra in the statement is indeed well-defined, because the conditions $p^2 = p^* = p$ satisfied by the coordinates give:

$$||u_{ij}|| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps Δ, ε, S given by the formulae in the statement. Consider the following matrices:

$$u^\Delta_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u^\varepsilon_{ij} = \delta_{ij} \quad , \quad u^S_{ij} = u_{ji}$$

Our claim is that, since u is magic, so are these three matrices. Indeed, regarding u^Δ, its entries are idempotents, as shown by the following computation:

$$(u^\Delta_{ij})^2 = \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj} = \sum_{kl} \delta_{kl}u_{ik} \otimes \delta_{kl}u_{kj} = u^\Delta_{ij}$$

These elements are self-adjoint as well, as shown by the following computation:

$$(u^\Delta_{ij})^* = \sum_k u^*_{ik} \otimes u^*_{kj} = \sum_k u_{ik} \otimes u_{kj} = u^\Delta_{ij}$$

The row and column sums for the matrix u^Δ can be computed as follows:

$$\sum_j u^\Delta_{ij} = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1$$

$$\sum_i u^\Delta_{ij} = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1$$

Thus, u^Δ is magic. Regarding now u^ε, u^S, these matrices are magic too, and this for obvious reasons. Thus, all our three matrices $u^\Delta, u^\varepsilon, u^S$ are magic, so we can define Δ, ε, S by the formulae in the statement, by using the universality property of $C(S_N^+)$. □

Our first task now is to make sure that Theorem 2.4 produces indeed a new quantum group, which does not collapse to S_N. Still following Wang [89], we have:

THEOREM 2.5. *We have an embedding $S_N \subset S_N^+$, given at the algebra level by:*

$$u_{ij} \to \chi\left(\sigma \in S_N \Big| \sigma(j) = i\right)$$

This is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. The fact that we have indeed an embedding as above follows from Proposition 2.3. Observe that in fact more is true, because our results above give:

$$C(S_N) = C(S_N^+) \Big/ \Big\langle ab = ba \Big\rangle$$

Thus, the inclusion $S_N \subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N = 2$. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+ = S_2$.

Case $N = 3$. By using the same argument as in the $N = 2$ case, and the symmetries of the problem, it is enough to check that u_{11}, u_{22} commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain that we have as well $u_{22}u_{11} = u_{11}u_{22}u_{11}$. Thus, we obtain $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N = 4$. Consider the following matrix, with p, q being projections:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and we can choose $p, q \in B(H)$ as for the algebra $< p, q >$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N \geq 5$. Here we can use the standard embedding $S_4^+ \subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u \to \begin{pmatrix} u & 0 \\ 0 & 1_{N-4} \end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N \geq 5$ has these two properties as well. □

With the above results in hand, we can introduce as well quantum reflections:

Theorem 2.6. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(H_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N} \middle| u_{ij} = u_{ij}^*,\ (u_{ij}^2) = \text{magic}\right) \\ C(K_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N} \middle| [u_{ij}, u_{ij}^*] = 0,\ (u_{ij}u_{ij}^*) = \text{magic}\right) \end{aligned}$$

which appear as liberations of the reflection groups $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$.

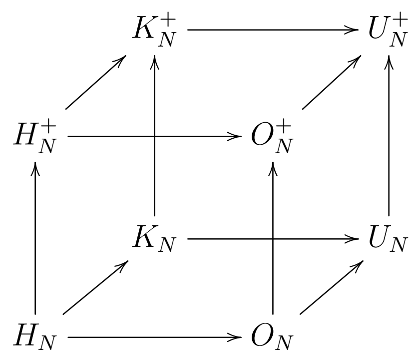
PROOF. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, and with the second assertion being trivial. Let us also mention that, in analogy with $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$, we have decomposition results as follows:

$$H_N^+ = \mathbb{Z}_2 \wr_* S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_* S_N^+$$

To be more precise, here $\wr_*$ is a free wreath product, and these formulae can be established a bit as in the classical case. For more on all this, we refer to [7]. □

All the above is very nice, and as a conclusion to all this, let us record the following result, which collects and refines the various liberation statements formulated above:

THEOREM 2.7. *The quantum unitary and reflection groups are as follows,*



and in this diagram, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

PROOF. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For some further details on all this, we refer to [7]. □

As a comment here, observe that the symmetric group S_N and its free analogue S_N^+, while certainly being very interesting objects, had not made the cut for appearing in the above almighty cube, called "standard cube" in quantum algebra. However, this is something quite natural, because S_N and S_N^+ are objects on their own, neither real or complex, and for practical purposes, like ours with our cube, these quantum groups must be replaced with H_N, H_N^+ in the real case, and with K_N, K_N^+ in the free case.

Actually I'm not quite sure about this, time to ask the cat. Who says:

CAT 2.8. *Do not worry, the high speed world is projective anyway, and it is better to use reflections instead of permutations.*

Thanks cat, not that I really understand what you say, but it fits with my purposes and cube, which looks really cool. But I will keep this in mind, and discuss later the relation between affine and projective geometry, in the free setting, that is promised.

With this done, let us get now into the second question that we were having, namely the conceptual understanding of the various liberation operations $G_N \to G_N^+$. In order to discuss this, we will need Tannakian duality, and Brauer type theorems. Let us start with Tannakian duality. This is a rather abstract statement, as follows:

THEOREM 2.9. *The following operations are inverse to each other:*

(1) *The construction $G \to C$, which associates to a closed subgroup $G \subset_u U_N^+$ the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
(2) *The construction $C \to G$, associating to a tensor category C the closed subgroup $G \subset_u U_N^+$ coming from the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. We have indeed a construction $G \to C_G$, whose output is a subcategory of the tensor C^*-category of finite dimensional Hilbert spaces, as follows:

$$(C_G)_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

We have as well a construction $C \to G_C$, obtained by setting:

$$C(G_C) = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C_{kl} \right\rangle$$

Regarding now the bijection claim, some elementary algebra shows that $C = C_{G_C}$ implies $G = G_{C_G}$, and that $C \subset C_{G_C}$ is automatic. Thus we are left with proving:

$$C_{G_C} \subset C$$

But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. □

The above result is something quite abstract, yet powerful. We will see applications of it in a moment, in the form of Brauer theorems for S_N, O_N, U_N and S_N^+, O_N^+, U_N^+, and other quantum groups. In order to formulate these Brauer theorems, let us start with:

DEFINITION 2.10. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A collection of sets*

$$D = \bigsqcup_{k,l} D(k,l)$$

with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:

(1) *Stability under the horizontal concatenation, $(\pi, \sigma) \to [\pi\sigma]$.*
(2) *Stability under the vertical concatenation, $(\pi, \sigma) \to [^\sigma_\pi]$.*
(3) *Stability under the upside-down turning, $\pi \to \pi^*$.*
(4) *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
(5) *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

As a basic example, we have the category of all partitions P itself. Other basic examples are the category of pairings P_2, and the categories NC, NC_2 of noncrossing partitions, and pairings. We have as well the category $\mathcal{P}_2$ of pairings which are "matching", in the sense that they connect $\circ-\circ$, $\bullet-\bullet$ on the vertical, and $\circ-\bullet$ on the horizontal, and its subcategory $\mathcal{NC}_2\subset\mathcal{P}_2$ consisting of the noncrossing matching pairings.

There are many other examples, and we will be back to this, gradually, in what follows. Regarding now the relation with the Tannakian categories, this comes from:

Proposition 2.11. *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. The concatenation axiom follows from the following computation:

$$\begin{aligned}&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\ =&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\ =&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\ =&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\end{aligned}$$

As for the composition and involution axioms, their proof is similar. □

In relation now with quantum groups, we have the following result:

Theorem 2.12. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and so a closed subgroup $G_N\subset_u U_N^+$.

Proof. Let call C_{kl} the spaces on the right. By using the axioms in Definition 2.10, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 2.11, we see that $C = (C_{kl})$ is a Tannakian category. Thus Theorem 2.9 applies, and gives the result. $\square$

We can now formulate a key definition, as follows:

Definition 2.13. *A compact quantum group G_N is called easy when we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for any colored integers k, l, for a certain category of partitions $D \subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool. In relation now with the orthogonal, unitary and symmetric quantum groups, here is the result:

Theorem 2.14. *The basic quantum permutation and rotation groups,*

$$\begin{array}{ccccc} S_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N & \longrightarrow & U_N \end{array}$$

are all easy, the corresponding categories of partitions being as follows,

$$\begin{array}{ccccc} NC & \longleftarrow & NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P & \longleftarrow & P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

with 2 standing for pairings, NC for noncrossing, and calligraphic for matching.

Proof. This is something quite fundamental, the proof being as follows:

(1) The quantum group U_N^+ is defined via the following relations:

$$u^* = u^{-1} \quad , \quad u^t = \bar{u}^{-1}$$

But, by doing some elementary computations, these relations tell us precisely that the following two operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi = \overset{\cap}{\circ\bullet} \ , \ \overset{\cap}{\bullet\circ}$$

Thus, the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = < \overset{\cap}{\circ\bullet} \ , \ \overset{\cap}{\bullet\circ} > = \mathcal{N}C_2$$

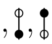
(2) The subgroup $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

Thus, the following operators must be in the associated Tannakian category C:

$T_\pi \quad : \quad \pi =$ 

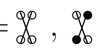
We conclude that the Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$D = < \mathcal{NC}_2,$  $> = NC_2$

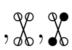
(3) The subgroup $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0 \quad , \quad [u_{ij}, \bar{u}_{kl}] = 0$$

Thus, the following operators must be in the associated Tannakian category C:

$T_\pi \quad : \quad \pi =$ 

Thus the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$D = < \mathcal{NC}_2,$  $> = \mathcal{P}_2$

(4) In order to deal now with O_N, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us that O_N is indeed easy, coming from:

$$D = < NC_2, \mathcal{P}_2 > = P_2$$

(5) We know that the subgroup $S_N^+ \subset O_N^+$ appears as follows:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle u = \text{magic} \Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y \in P(2,1)$$

Given a corepresentation u, we have the following formulae:

$$(T_Y u^{\otimes 2})_{i,jk} = \sum_{lm} (T_Y)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

$$(u T_Y)_{i,jk} = \sum_l u_{il} (T_Y)_{l,jk} = \delta_{jk} u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle T_Y \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D =< Y >= NC$$

(6) Finally, in order to deal with S_N, we can use the following formula:

$$S_N = S_N^+ \cap O_N$$

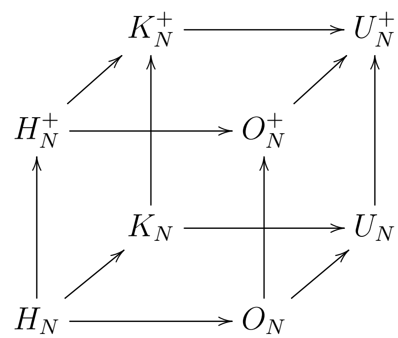
At the categorical level, this tells us that S_N is indeed easy, coming from:

$$D =< NC, P_2 >= P$$

Thus, we are led to the conclusions in the statement. □

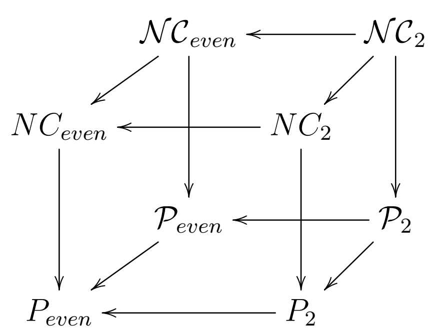
Moving ahead, we can upgrade what we have into a cube result, as follows:

THEOREM 2.15. *The basic quantum unitary and reflection groups,*



are all easy, and the corresponding categories of partitions form an intersection diagram.

PROOF. The precise claim is that the categories are as follows, with P_{even} being the category of partitions having even blocks, and with $\mathcal{P}_{even}(k,l) \subset P_{even}(k,l)$ consisting of the partitions satisfying $\#\circ = \#\bullet$ in each block, when flattening the partition:



But this is something that we already know for the right face, from Theorem 2.14, and in what regards the left face, the proof here is similar, by using the results for S_N, S_N^+ from that same Theorem 2.14. As for the last assertion, this is something trivial. □

The above results are something quite deep, and we will see in what follows countless applications of them. As a first such application, rather philosophical, we have:

THEOREM 2.16. *The constructions $G_N \to G_N^+$ with $G = O, U, S, H, K$ are easy quantum group liberations, in the sense that they come from the construction*

$$D \to D \cap NC$$

at the level of the associated categories of partitions.

PROOF. This is clear indeed from Theorem 2.14 and Theorem 2.15, and from the following trivial equalities, connecting the categories found there:

$$NC_2 = P_2 \cap NC \quad , \quad \mathcal{NC}_2 = \mathcal{P}_2 \cap NC$$

$$NC = P \cap NC$$

$$NC_{even} = P_{even} \cap NC \quad , \quad \mathcal{NC}_{even} = \mathcal{P}_{even} \cap NC$$

Thus, we are led to the conclusion in the statement. □

The above result is quite nice, because the various constructions $G_N \to G_N^+$ that we made so far, although natural, were something quite ad-hoc. Now all this is no longer ad-hoc, and the next time that we will have to liberate a subgroup $G_N \subset U_N$, we know what the recipe is, namely check if G_N is easy, and if so, simply define $G_N^+ \subset U_N^+$ as being the easy quantum group coming from the category $D = D_G \cap NC$.

2b. Uniformity, characters

In general, the study of the free quantum groups, in the "easy" sense explained above, is something quite complex. In order to cut a bit from complexity, we will use:

PROPOSITION 2.17. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D \subset P$, the following conditions are equivalent:*

1. *$G_{N-1} = G_N \cap U_{N-1}^+$, via the embedding $U_{N-1}^+ \subset U_N^+$ given by $u \to diag(u, 1)$.*
2. *$G_{N-1} = G_N \cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+ \subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

PROOF. We use the general easiness theory, as explained above:

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N \subset G_N$, which makes everything S_N-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given a subgroup $K \subset U_{N-1}^+$, with fundamental corepresentation u, consider the $N \times N$ matrix $v = diag(u, 1)$. Our claim is that for any $\pi \in P(k)$ we have:

$$\xi_\pi \in Fix(v^{\otimes k}) \iff \xi_{\pi'} \in Fix(v^{\otimes k'}), \forall \pi' \in P(k'), \pi' \subset \pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}
\xi_\pi\in Fix(v^{\otimes k}) &\iff (v^{\otimes k}\xi_\pi)_{i_1\ldots i_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\
&\iff \sum_j(v^{\otimes k})_{i_1\ldots i_k,j_1\ldots j_k}(\xi_\pi)_{j_1\ldots j_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\
&\iff \sum_j\delta_\pi(j_1,\ldots,j_k)v_{i_1j_1}\ldots v_{i_kj_k}=\delta_\pi(i_1,\ldots,i_k),\forall i
\end{aligned}$$

Now let us recall that our corepresentation has the special form $v=diag(u,1)$. We conclude from this that for any index $a\in\{1,\ldots,k\}$, we must have:

$$i_a=N\implies j_a=N$$

With this observation in hand, if we denote by i',j' the multi-indices obtained from i,j obtained by erasing all the above $i_a=j_a=N$ values, and by $k'\leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'}\delta_\pi(j_1,\ldots,j_k)(v^{\otimes k'})_{i'j'}=\delta_\pi(i_1,\ldots,i_k),\forall i$$

Here the index j is by definition obtained from j' by filling with N values. In order to finish now, we have two cases, depending on i, as follows:

<u>Case 1</u>. Assume that the index set $\{a|i_a=N\}$ corresponds to a certain subpartition $\pi'\subset\pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'}\delta_\pi(j'_1,\ldots,j'_{k'})(v^{\otimes k'})_{i'j'}=\delta_\pi(i'_1,\ldots,i'_{k'})$$

<u>Case 2</u>. Assume now the opposite, namely that the set $\{a|i_a=N\}$ does not correspond to a subpartition $\pi'\subset\pi$. In this case the indices mix, and our formula reads:

$$0=0$$

Thus, we are led to $\xi_{\pi'}\in Fix(v^{\otimes k'})$, for any subpartition $\pi'\subset\pi$, as claimed. Thus our claim is proved, and with this in hand, the result follows from Tannakian duality. □

Based on the above result, let us formulate the following definition:

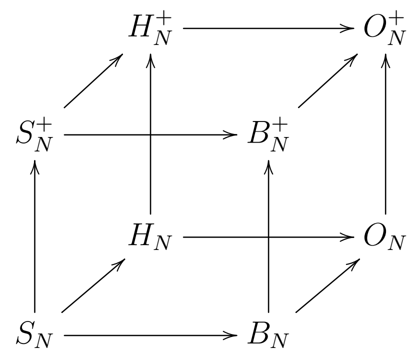
DEFINITION 2.18. *An easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, is called uniform when we have, for any $N\in\mathbb N$:*

$$G_{N-1}=G_N\cap U_{N-1}^+$$

Equivalently, D must be stable under the operation which consists in removing blocks.

For classification purposes the uniformity axiom is something very natural and useful, substantially cutting from complexity, and we have the following result:

Theorem 2.19. *The classical and free uniform orthogonal easy quantum groups are*



with B_N, B_N^+ being the classical and quantum bistochastic groups.

Proof. There are several things to be proved, the idea being as follows:

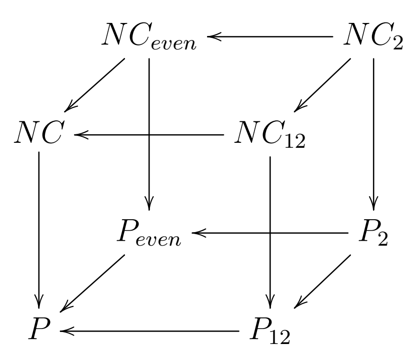
(1) We first recall that the bistochastic group $B_N \subset O_N$ consists of the orthogonal matrices whose entries sum up to 1 on each row, or equivalently, sum up to 1 on each column. Thus, if we denote by $\xi \in \mathbb{C}^N$ the all-one vector, we have:

$$B_N = \left\{ U \in O_N \Big| U\xi = \xi \right\}$$

Based on this, we can construct a free analogue of B_N as follows, and the fact that we obtain indeed a quantum group follows exactly as for O_N^+, U_N^+:

$$C(B_N^+) = C(O_N^+) \Big/ \left\langle u\xi = \xi \right\rangle$$

(2) Since the relation $u\xi = \xi$ reads $T_| \in Fix(u)$, with $| \in P(0,1)$ being the singleton partition, we conclude that B_N, B_N^+ are easy, coming from the categories P_{12}, NC_{12} of singletons and pairings, and noncrossing singletons and pairings. Thus, all the quantum groups in the statement are easy, the corresponding categories of partitions being:



(3) Regarding now the classification, consider an easy quantum group $S_N \subset G_N \subset O_N$. This must come from a category $P_2 \subset D \subset P$, and if we assume $G = (G_N)$ to be uniform,

then D is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in D. Our claim is that the admissible sets are as follows:

– $L = \{2\}$, producing O_N.

– $L = \{1, 2\}$, producing B_N.

– $L = \{2, 4, 6, \ldots\}$, producing H_N.

– $L = \{1, 2, 3, \ldots\}$, producing S_N.

(4) Indeed, in one sense, this follows from our easiness results for O_N, B_N, H_N, S_N. In the other sense now, assume that $L \subset \mathbb{N}$ is such that the set P_L consisting of partitions whose sizes of the blocks belong to L is a category of partitions. We know from the axioms of the categories of partitions that the semicircle $\cap$ must be in the category, so we have $2 \in L$. We claim that the following conditions must be satisfied as well:

$$k, l \in L,\ k > l \implies k - l \in L$$

$$k \in L,\ k \geq 2 \implies 2k - 2 \in L$$

(5) Indeed, we will prove that both conditions follow from the axioms of the categories of partitions. Let us denote by $b_k \in P(0, k)$ the one-block partition:

$$b_k = \left\{ \begin{matrix} \sqcap\!\sqcap & \ldots & \sqcap \\ 1\ \ 2 & \ldots & k \end{matrix} \right\}$$

For $k > l$, we can write b_{k-l} in the following way:

$$b_{k-l} = \left\{ \begin{matrix} \sqcap\!\sqcap & \ldots & \ldots & \ldots & \ldots & \sqcap \\ 1\ \ 2 & \ldots & l & l+1 & \ldots & k \\ \sqcup\!\sqcup & \ldots & \sqcup & | & \ldots & | \\ & & & 1 & \ldots & k-l \end{matrix} \right\}$$

In other words, we have the following formula:

$$b_{k-l} = (b_l^* \otimes |^{\otimes k-l}) b_k$$

Since all the terms of this composition are in P_L, we have $b_{k-l} \in P_L$, and this proves our first claim. As for the second claim, this can be proved in a similar way, by capping two adjacent k-blocks with a 2-block, in the middle.

(6) With these conditions in hand, we can conclude in the following way:

<u>Case 1</u>. Assume $1 \in L$. By using the first condition with $l = 1$ we get:

$$k \in L \implies k - 1 \in L$$

This condition shows that we must have $L=\{1,2,\ldots,m\}$, for a certain number $m\in\{1,2,\ldots,\infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} m\in L &\implies 2m-2\in L\\ &\implies 2m-2\leq m\\ &\implies m\in\{1,2,\infty\}\end{aligned}$$

The case $m=1$ being excluded by the condition $2\in L$, we reach to one of the two sets producing the groups S_N,B_N.

<u>Case 2</u>. Assume $1\notin L$. By using the first condition with $l=2$ we get:

$$k\in L\implies k-2\in L$$

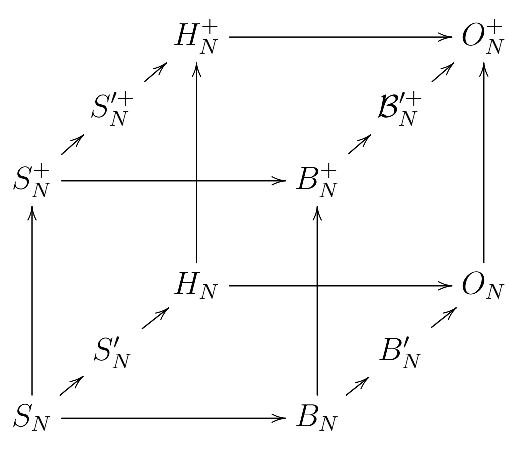
This condition shows that we must have $L=\{2,4,\ldots,2p\}$, for a certain number $p\in\{1,2,\ldots,\infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} 2p\in L &\implies 4p-2\in L\\ &\implies 4p-2\leq 2p\\ &\implies p\in\{1,\infty\}\end{aligned}$$

Thus L must be one of the two sets producing O_N,H_N, and we are done. In the free case, $S_N^+\subset G_N\subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time O_N^+,B_N^+,H_N^+,S_N^+. □

When removing the uniformity axiom things become more complicated, as follows:

Theorem 2.20. *The classical and free orthogonal easy quantum groups are*



with $S_N'=S_N\times\mathbb{Z}_2$, $B_N'=B_N\times\mathbb{Z}_2$, and with $S_N'^+,\mathcal{B}_N'^+$ being their liberations, where $\mathcal{B}_N'^+$ stands for the two possible such liberations, $B_N'^+\subset B_N''^+$.

Proof. The idea here is that of jointly classifying the "classical" categories of partitions $P_2\subset D\subset P$, and the "free" ones $NC_2\subset D\subset NC$:

(1) At the classical level this leads, via a study which is quite similar to that from the proof of Theorem 2.19, to 2 more groups, namely S_N', B_N'.

(2) At the free level we obtain 3 more quantum groups, $S_N'^+, B_N'^+, B_N''^+$, with the inclusion $B_N'^+ \subset B_N''^+$, which is something a bit surprising, being best thought of as coming from an inclusion $B_N' \subset B_N''$, which happens to be an isomorphism. □

It is possible to obtain similar results in the general unitary case, first with a quite simple statement, regarding the uniform case, and then with something more complicated, regarding the non-uniform case. We refer here to the paper of Tarrago-Weber [80].

Importantly, the uniformity assumption has some interesting analytic consequences, making the link with the Bercovici-Pata bijection [19]. In order to discuss this, we first need to know how to integrate on the easy quantum groups, and we have here:

Theorem 2.21. *Assuming that a closed subgroup $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1} \ldots u_{i_kj_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

where $\delta \in \{0,1\}$ are the usual Kronecker type symbols, and where the Weingarten matrix $W_{kN} = G_{kN}^{-1}$ is the inverse of the Gram matrix $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$.

Proof. We know from the general theory in chapter 1 that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

In order to prove the result, consider the following linear map:

$$E(x) = \sum_{\pi \in D(k)} < x, \xi_\pi > \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse on $Fix(u^{\otimes k})$ of the restriction of E. But this restriction is the linear map given by G_{kN}, and so W is the linear map given by W_{kN}, and this gives the result. □

In relation now with characters, we have the following moment formula:

Proposition 2.22. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11} + \ldots + u_{ss})^k = Tr(W_{kN}G_{ks})$$

where G_{kN} and $W_{kN} = G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

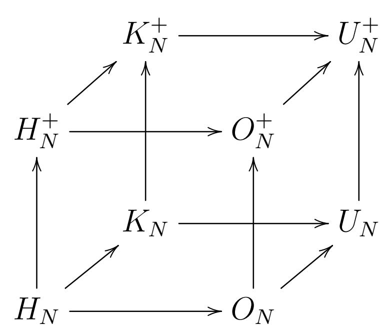
Proof. We have indeed the following computation:

$$\begin{aligned}\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int u_{i_1i_1}\ldots u_{i_ki_k}\\ &= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\pi(i)\delta_\sigma(i)\\ &= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\ &= Tr(W_{kN}G_{ks})\end{aligned}$$

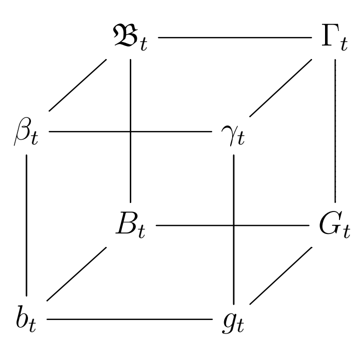
Thus, we have obtained the formula in the statement. □

With the above general theory in hand, we can now formulate our character results for the main examples of uniform easy quantum groups, as follows:

Theorem 2.23. *For the main quantum rotation and reflection groups,*

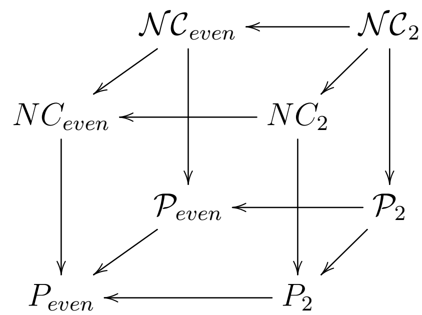


the corresponding truncated characters follow with $N\to\infty$ the laws



which are the main limiting laws in classical and free probability.

PROOF. We know from Theorem 2.15 that the above quantum groups are all easy, coming from the following categories of partitions:



Now by using Proposition 2.22, we obtain the following formula:

$$\lim_{N\to\infty}\int_{G_N}\chi_t^k=\sum_{\pi\in D(k)}t^{|\pi|}$$

But this gives the laws in the statement, via some standard calculus. □

2c. Temperley-Lieb

All the above is sweet, and there are many other things that can be said, along the same lines, about the liberation operations $G_N\to G_N^+$, using easiness and partitions. This being said, we are rather interested in free quantum groups, so we do not need partitions with crossings, and this leads us to a quite puzzling question, as follows:

QUESTION 2.24. *Among the many objects which are in bijection with the noncrossing partitions, which are the most adapted to the study of the free quantum groups?*

To be more precise here, in order to give you a taste on what this question is about, you have surely heard for instance about the Catalan numbers:

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

These Catalan numbers count the partitions in $NC(k)$, but they count as well a zillion other interesting things, just ask and any expert in combinatorics will probably get you stuck for 1 hour in the coffee room, in explaining you all this, and our problem is, among these zillion things, what are the best for the study of free quantum groups.

This does not look obvious, and so time to ask the cat. And cat says:

CAT 2.25. *You're getting old, double the strings as to have Temperley-Lieb diagrams, as in the heyday of free quantum group theory.*

Thanks cat, and yes indeed, age does not help much with knowledge and memory, in fact Question 2.24 is something that I already thought about, some 30 years ago, when developing the basic theory of free quantum groups. Following Temperley-Lieb, who by the way were first-class physicists, and then Jones, who was a first-class physicist too, and many others, including myself when younger, not to forget cat of course, we will of course go for this, doubling strings and using Temperley-Lieb diagrams.

Let us start with the following result, which is well-known:

Proposition 2.26. *We have a bijection $NC(k) \simeq NC_2(2k)$, as follows:*

1. *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the above two operations are indeed inverse to each other is clear, by drawing pictures, and computing the corresponding compositions. □

With the above result in hand, we can axiomatize the free quantum groups, in terms of Temperley-Lieb diagrams NC_2, and say many interesting things about them, based on the work of Jones and others on subfactor theory and planar algebras [64].

We can compute representations and their fusion rules, Cayley graphs, growth exponents, laws of characters and more, by using diagrams, and more specifically Temperley-Lieb diagrams NC_2, which are quite often the most adapted, to our questions.

As a basic example for what can be done here, regarding O_N^+, we have:

Theorem 2.27. *The irreducible representations of O_N^+ with $N \geq 2$ can be labelled by positive integers, r_k with $k \in \mathbb{N}$, the fusion rules for these representations are*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

and the dimensions are $\dim r_k = (q^{k+1} - q^{-k-1})/(q - q^{-1})$, *with* $q + q^{-1} = N$.

Proof. The idea is to skilfully recycle the well-known proof for SU_2. Our claim is that we can construct, by recurrence on $k \in \mathbb{N}$, a sequence $r_0, r_1, r_2, \ldots$ of irreducible, self-adjoint and distinct representations of O_N^+, satisfying:

$$r_0 = 1 \quad , \quad r_1 = u \quad , \quad r_{k-1} \otimes r_1 = r_{k-2} + r_k$$

In order to do so, we can use the formula $r_{k-2} \otimes r_1 = r_{k-3} + r_{k-1}$ and Frobenius duality, and we conclude there exists a certain representation r_k such that:

$$r_{k-1} \otimes r_1 = r_{k-2} + r_k$$

As a first observation, r_k is self-adjoint, because its character is a certain polynomial with integer coefficients in χ, which is self-adjoint. In order to prove now that r_k is irreducible, and non-equivalent to $r_0,\ldots,r_{k-1}$, let us split as before $u^{\otimes k}$, as follows:

$$u^{\otimes k}=c_kr_k+c_{k-2}r_{k-2}+c_{k-4}r_{k-4}+\ldots$$

The point now is that we have the following equalities and inequalities:

$$C_k=\sum_i c_i^2\leq\dim(End(u^{\otimes k}))\leq|NC_2(k,k)|=C_k$$

Indeed, the equality at left is clear as before, then comes a standard inequality, then an inequality coming from easiness, then a standard equality. Thus, we have equality, so r_k is irreducible, and non-equivalent to $r_{k-2},r_{k-4},\ldots$ Moreover, r_k is not equivalent to $r_{k-1},r_{k-3},\ldots$ either, by using the same argument as for SU_2, and the end of the proof is exactly as for SU_2. As for dimensions, by recurrence we obtain, with $q+q^{-1}=N$:

$$\dim r_k=q^k+q^{k-2}+\ldots+q^{-k+2}+q^{-k}$$

But this gives the dimension formula in the statement, and we are done. □

It is possible to use similar methods for the other main examples of free quantum groups, and do many other things, in relation with the Temperley-Lieb algebra.

2d. Meander determinants

We discuss now, following Di Francesco [40] and others, the computation of the Gram determinants for the free quantum groups, which is a very interesting question, related to many things. But let us start with S_N and other classical groups. We will need:

Definition 2.28. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi,\sigma)=\begin{cases}1&\text{if }\pi=\sigma\\-\sum_{\pi\leq\tau<\sigma}\mu(\pi,\tau)&\text{if }\pi<\sigma\\0&\text{if }\pi\not\leq\sigma\end{cases}$$

with the construction being performed by recurrence.

As an illustration here, for $P(2)=\{||,\sqcap\}$, we have by definition:

$$\mu(||,||)=\mu(\sqcap,\sqcap)=1$$

Also, $||<\sqcap$, with no intermediate partition in between, so we obtain:

$$\mu(||,\sqcap)=-\mu(||,||)=-1$$

Finally, we have $\sqcap\not\leq||$, and so we have as well the following formula:

$$\mu(\sqcap,||)=0$$

We will need the Möbius inversion formula, which can be formulated as follows:

THEOREM 2.29. *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1 & \text{if } \pi\leq\sigma\\ 0 & \text{if } \pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.

PROOF. This is well-known, coming from the fact that A_k is upper triangular. Indeed, when inverting, we are led into the recurrence for μ, from Definition 2.28. □

As an illustration, for $P(2)$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

PROPOSITION 2.30. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G_k=A_kL_k$, where

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1) & \text{if } \sigma\leq\pi\\ 0 & \text{otherwise}\end{cases}$$

and where A_k is the adjacency matrix of $P(k)$.

PROOF. We have indeed the following computation:

$$\begin{aligned}G_k(\pi,\sigma)&=N^{|\pi\vee\sigma|}\\&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According now to the definition of A_k,L_k, this formula reads:

$$\begin{aligned}G_k(\pi,\sigma)&=\sum_{\tau\geq\pi}L_k(\tau,\sigma)\\&=\sum_{\tau}A_k(\pi,\tau)L_k(\tau,\sigma)\\&=(A_kL_k)(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix}N^2&N\\N&N\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\N&N\end{pmatrix}$$

We are led in this way to the following formula, due to Lindstöm:

THEOREM 2.31. *The determinant of the Gram matrix G_k is given by*

$$\det(G_k)=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N<k$ we obtain 0*.*

PROOF. If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, A_k is upper triangular, and L_k is lower triangular. Thus, we have:

$$\begin{aligned}\det(G_k)&=\det(A_k)\det(L_k)\\&=\det(L_k)\\&=\prod_\pi L_k(\pi,\pi)\\&=\prod_\pi N(N-1)\ldots(N-|\pi|+1)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us discuss as well the case of the orthogonal group O_N. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. We have then:

THEOREM 2.32. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN})=\prod_{|\lambda|=k/2}f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda)=\prod_{(i,j)\in\lambda}(N+2j-i-1)$.

PROOF. For the group O_N the Gram matrix is diagonalizable, as follows:

$$G_{kN}=\sum_{|\lambda|=k/2}f_N(\lambda)P_{2\lambda}$$

Here $1=\sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda})=f^{2\lambda}$, this gives the formula in the statement. □

In order to deal now with O_N^+,S_N^+, we will need the following fact:

Proposition 2.33. *The Gram matrices of $NC_2(2k) \simeq NC(k)$ are related by*

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1}G_{k,n^2}\Delta_{kn}^{-1})(\pi',\sigma')$$

where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}.

Proof. In the context of the bijection from Proposition 2.26, we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we are led to the formula in the statement. □

Now back to O_N^+, S_N^+, let us begin with some examples. We first have:

Proposition 2.34. *The first Gram matrices and determinants for O_N^+ are*

$$\det\begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix} = N^2(N^2-1)$$

$$\det\begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix} = N^5(N^2-1)^4(N^2-2)$$

with the matrices being written by using the lexicographic order on $NC_2(2k)$.

Proof. The formula at $k = 2$, where $NC_2(4) = \{\sqcap\sqcap, \Cap\}$, is clear from definitions. At $k = 3$ however, things are tricky. The partitions here are as follows:

$$NC(3) = \{|||, \sqcap|, \sqcap\!\!\!\mid\ , |\sqcap, \sqcap\!\!\sqcap\}$$

The Gram matrix and its determinant are, according to Theorem 2.31:

$$\det\begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^2 & N & N & N \\ N^2 & N & N^2 & N & N \\ N^2 & N & N & N^2 & N \\ N & N & N & N & N \end{pmatrix} = N^5(N-1)^4(N-2)$$

By using Proposition 2.33, the Gram determinant of $NC_2(6)$ is given by:

$$\begin{aligned} \det(G_{6N}) &= \frac{1}{N^2\sqrt{N}} \times N^{10}(N^2-1)^4(N^2-2) \times \frac{1}{N^2\sqrt{N}} \\ &= N^5(N^2-1)^4(N^2-2) \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In general, such tricks won't work, because $NC(k)$ is strictly smaller than $P(k)$ at $k\geq 4$. However, following Di Francesco [40], we have the following result:

THEOREM 2.35. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb Z$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb Z$.

PROOF. This is something quite technical, obtained by using a decomposition as follows of the Gram matrix G_{kN}, with the matrix T_{kN} being lower triangular:

$$G_{kN}=T_{kN}T_{kN}^t$$

Thus, a bit as in the proof of the Lindstöm formula, we obtain the result, but the problem lies however in the construction of T_{kN}, which is non-trivial. See [40]. □

With this in hand, we have as well a similar formula for S_N^+, obtained from Theorem 2.35 via Proposition 2.33. For the other free quantum groups, the computations can be done as well. For more on all this, we refer to [40] and related papers.

2e. Exercises

Exercises.

CHAPTER 3

Free manifolds

3a. Quotient spaces

Let us begin with some generalities regarding the quotient spaces, and more general homogeneous spaces. Regarding the quotients, we have the following construction:

PROPOSITION 3.1. *Given a quantum subgroup $H\subset G$, with associated quotient map $\rho:C(G)\to C(H)$, if we define the quotient space $X=G/H$ by setting*

$$C(X)=\left\{f\in C(G)\Big|(\rho\otimes id)\Delta f=1\otimes f\right\}$$

then we have a coaction map as follows,

$$\Phi:C(X)\to C(X)\otimes C(G)$$

obtained as the restriction of the comultiplication of $C(G)$. In the classical case, we obtain in this way the usual quotient space $X=G/H$.

PROOF. Observe that the linear subspace $C(X)\subset C(G)$ defined in the statement is indeed a subalgebra, because it is defined via a relation of type $\varphi(f)=\psi(f)$, with both φ,ψ being morphisms of algebras. Observe also that in the classical case we obtain the algebra of continuous functions on the quotient space $X=G/H$, because:

$$\begin{aligned}(\rho\otimes id)\Delta f=1\otimes f&\iff(\rho\otimes id)\Delta f(h,g)=(1\otimes f)(h,g),\forall h\in H,\forall g\in G\\&\iff f(hg)=f(g),\forall h\in H,\forall g\in G\\&\iff f(hg)=f(kg),\forall h,k\in H,\forall g\in G\end{aligned}$$

Regarding now the construction of Φ, observe that for $f\in C(X)$ we have:

$$\begin{aligned}(\rho\otimes id\otimes id)(\Delta\otimes id)\Delta f&=(\rho\otimes id\otimes id)(id\otimes\Delta)\Delta f\\&=(id\otimes\Delta)(\rho\otimes id)\Delta f\\&=(id\otimes\Delta)(1\otimes f)\\&=1\otimes\Delta f\end{aligned}$$

Thus the condition $f\in C(X)$ implies $\Delta f\in C(X)\otimes C(G)$, and this gives the existence of Φ. Finally, the other assertions are all clear. □

As an illustration, following [13], in the group dual case we have:

Proposition 3.2. *Assume that $G=\widehat{\Gamma}$ is a discrete group dual.*

(1) *The quantum subgroups of G are $H=\widehat{\Lambda}$, with $\Gamma\to\Lambda$ being a quotient group.*
(2) *For such a quantum subgroup $\widehat{\Lambda}\subset\widehat{\Gamma}$, we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, where:*

$$\Theta=\ker(\Gamma\to\Lambda)$$

Proof. This is well-known, the idea being as follows:

(1) In one sense, this is clear. Conversely, since the algebra $C(G)=C^*(\Gamma)$ is cocommutative, so are all its quotients, and this gives the result.

(2) Consider a quotient map $r:\Gamma\to\Lambda$, and denote by $\rho:C^*(\Gamma)\to C^*(\Lambda)$ its extension. Consider a group algebra element, written as follows:

$$f=\sum_{g\in\Gamma}\lambda_g\cdot g\in C^*(\Gamma)$$

We have then the following computation:

$$\begin{aligned}
f\in C(\widehat{\Gamma}/\widehat{\Lambda})
&\iff (\rho\otimes id)\Delta(f)=1\otimes f\\
&\iff \sum_{g\in\Gamma}\lambda_g\cdot r(g)\otimes g=\sum_{g\in\Gamma}\lambda_g\cdot 1\otimes g\\
&\iff \lambda_g\cdot r(g)=\lambda_g\cdot 1,\forall g\in\Gamma\\
&\iff supp(f)\subset\ker(r)
\end{aligned}$$

But this means that we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, with $\Theta=\ker(\Gamma\to\Lambda)$, as claimed. □

Given two compact quantum spaces X,Y, we say that X is a quotient space of Y when we have an embedding of C^*-algebras $\alpha:C(X)\subset C(Y)$. We have:

Definition 3.3. *We call a quotient space $G\to X$ homogeneous when*

$$\Delta(C(X))\subset C(X)\otimes C(G)$$

where $\Delta:C(G)\to C(G)\otimes C(G)$ is the comultiplication map.

In other words, an homogeneous quotient space $G\to X$ is a quantum space coming from a subalgebra $C(X)\subset C(G)$, which is stable under the comultiplication. The relation with the quotient spaces from Proposition 3.1 is as follows:

Theorem 3.4. *The following results hold:*

(1) *The quotient spaces $X=G/H$ are homogeneous.*
(2) *In the classical case, any homogeneous space is of type G/H.*
(3) *In general, there are homogeneous spaces which are not of type G/H.*

Proof. Once again these results are well-known, the proof being as follows:

(1) This is clear from Proposition 3.1.

(2) Consider a quotient map $p : G \to X$. The invariance condition in the statement tells us that we must have an action $G \curvearrowright X$, given by:

$$g(p(g')) = p(gg')$$

Thus, we have the following implication:

$$p(g') = p(g'') \implies p(gg') = p(gg''),\ \forall g \in G$$

Now observe that the following subset $H \subset G$ is a subgroup:

$$H = \left\{ g \in G \Big| p(g) = p(1) \right\}$$

Indeed, $g, h \in H$ implies that we have:

$$p(gh) = p(g) = p(1)$$

Thus we have $gh \in H$, and the other axioms are satisfied as well. Our claim now is that we have an identification $X = G/H$, obtained as follows:

$$p(g) \to Hg$$

Indeed, the map $p(g) \to Hg$ is well-defined and bijective, because $p(g) = p(g')$ is equivalent to $p(g^{-1}g') = p(1)$, and so to $Hg = Hg'$, as desired.

(3) Given a discrete group Γ and an arbitrary subgroup $\Theta \subset \Gamma$, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is homogeneous. Now by using Proposition 3.2, we can see that if $\Theta \subset \Gamma$ is not normal, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is not of the form G/H. $\square$

With the above formalism in hand, let us try now to understand the general properties of the homogeneous spaces $G \to X$, in the sense of Theorem 3.4. We have:

Proposition 3.5. *Assume that a quotient space $G \to X$ is homogeneous.*

(1) *We have a coaction map as follows, obtained as restriction of Δ:*

$$\Phi : C(X) \to C(X) \otimes C(G)$$

(2) *We have the following implication:*

$$\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$$

(3) *We have as well the following formula:*

$$\left(id \otimes \int_G \right) \Phi f = \int_G f$$

(4) *The restriction of $\int_G$ is the unique unital form satisfying:*

$$(\tau \otimes id)\Phi = \tau(.)1$$

PROOF. These results are all elementary, the proof being as follows:

(1) This is clear from definitions, because Δ itself is a coaction.

(2) Assume that $f\in C(G)$ satisfies $\Delta(f)=f\otimes 1$. By applying the counit we obtain:

$$(\varepsilon\otimes id)\Delta f=(\varepsilon\otimes id)(f\otimes 1)$$

We conclude from this that we have $f=\varepsilon(f)1$, as desired.

(3) The formula in the statement, $(id\otimes\int_G)\Phi f=\int_G f$, follows indeed from the left invariance property of the Haar functional of $C(G)$, namely:

$$\left(id\otimes\int_G\right)\Delta f=\int_G f$$

(4) We use here the right invariance of the Haar functional of $C(G)$, namely:

$$\left(\int_G\otimes id\right)\Delta f=\int_G f$$

Indeed, we obtain from this that $tr=(\int_G)_{|C(X)}$ is G-invariant, in the sense that:

$$(tr\otimes id)\Phi f=tr(f)1$$

Conversely, assuming that $\tau:C(X)\to\mathbb C$ satisfies $(\tau\otimes id)\Phi f=\tau(f)1$, we have:

$$\begin{aligned}\left(\tau\otimes\int_G\right)\Phi(f)&=\int_G(\tau\otimes id)\Phi(f)\\&=\int_G(\tau(f)1)\\&=\tau(f)\end{aligned}$$

On the other hand, we can compute the same quantity as follows:

$$\begin{aligned}\left(\tau\otimes\int_G\right)\Phi(f)&=\tau\left(id\otimes\int_G\right)\Phi(f)\\&=\tau(tr(f)1)\\&=tr(f)\end{aligned}$$

Thus we have $\tau(f)=tr(f)$ for any $f\in C(X)$, and this finishes the proof. □

Summarizing, we have a notion of noncommutative homogeneous space, which perfectly covers the classical case. In general, however, the group dual case shows that our formalism is more general than that of the quotient spaces G/H.

Let us discuss now an extra issue, of analytic nature. The point indeed is that for one of the most basic examples of actions, namely $O_N^+\curvearrowright S^{N-1}_{\mathbb R,+}$, the associated morphism $\alpha:C(X)\to C(G)$ is not injective. The same is true for other basic actions, in the free setting. In order to include such examples, we must relax our axioms:

Definition 3.6. *An extended homogeneous space over a compact quantum group G consists of a morphism of C^*-algebras, and a coaction map, as follows,*

$$\alpha : C(X) \to C(G)$$

$$\Phi : C(X) \to C(X) \otimes C(G)$$

such that the following diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id} \\ C(G) & \xrightarrow{\Delta} & C(G) \otimes C(G) \end{array}$$

and such that the following diagram commutes as well

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X) \otimes C(G) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle id\otimes\int} \\ C(G) & \xrightarrow{\int(.)1} & C(X) \end{array}$$

where $\int$ is the Haar integration over G. We write then $G \to X$.

As a first observation, when the morphism α is injective we obtain an homogeneous space in the previous sense. The examples with α not injective, which motivate the above formalism, include the standard action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, and the standard action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$. Here are a few general remarks on the above axioms:

Proposition 3.7. *Assume that we have morphisms of C^*-algebras*

$$\alpha : C(X) \to C(G)$$

$$\Phi : C(X) \to C(X) \otimes C(G)$$

satisfying the coassociativity condition $(\alpha \otimes id)\Phi = \Delta\alpha$.

1. *If α is injective on a dense $*$-subalgebra $A \subset C(X)$, and $\Phi(A) \subset A \otimes C(G)$, then Φ is automatically a coaction map, and is unique.*
2. *The ergodicity type condition $(id \otimes \int)\Phi = \int \alpha(.)1$ is equivalent to the existence of a linear form $\lambda : C(X) \to \mathbb{C}$ such that $(id \otimes \int)\Phi = \lambda(.)1$.*

Proof. This is something elementary, the idea being as follows:

(1) Assuming that we have a dense $*$-subalgebra $A \subset C(X)$ as in the statement, satisying $\Phi(A) \subset A \otimes C(G)$, the restriction $\Phi_{|A}$ is given by:

$$\Phi_{|A} = (\alpha_{|A} \otimes id)^{-1}\Delta\alpha_{|A}$$

This restriction and is therefore coassociative, and unique. By continuity, the morphism Φ itself follows to be coassociative and unique, as desired.

(2) Assuming $(id\otimes\int)\Phi=\lambda(.)1$, we have:

$$\left(\alpha\otimes\int\right)\Phi=\lambda(.)1$$

On the other hand, we have as well the following formula:

$$\left(\alpha\otimes\int\right)\Phi=\left(id\otimes\int\right)\Delta\alpha=\int\alpha(.)1$$

Thus we obtain $\lambda=\int\alpha$, as claimed. □

Given an extended homogeneous space $G\to X$ in our sense, with associated map $\alpha:C(X)\to C(G)$, we can consider the image of this latter map:

$$\alpha:C(X)\to C(Y)\subset C(G)$$

Equivalently, at the level of the associated noncommutative spaces, we can factorize the corresponding quotient map $G\to Y\subset X$. With these conventions, we have:

Proposition 3.8. *Consider an extended homogeneous space $G\to X$.*

1. *$\Phi(f)=f\otimes 1\implies f\in\mathbb{C}1$.*
2. *$tr=\int\alpha$ is the unique unital G-invariant form on $C(X)$.*
3. *The image space obtained by factorizing, $G\to Y$, is homogeneous.*

Proof. We have several assertions to be proved, the idea being as follows:

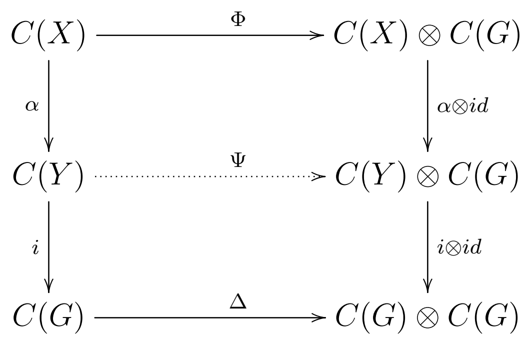
(1) This follows indeed from $(id\otimes\int)\Phi(f)=\int\alpha(f)1$, which gives $f=\int\alpha(f)1$.

(2) The fact that $tr=\int\alpha$ is indeed G-invariant can be checked as follows:

$$\begin{aligned}(tr\otimes id)\Phi f&=(\textstyle\int\alpha\otimes id)\Phi f\\&=(\textstyle\int\otimes id)\Delta\alpha f\\&=\textstyle\int\alpha(f)1\\&=tr(f)1\end{aligned}$$

As for the uniqueness assertion, this follows as before.

(3) The condition $(\alpha\otimes id)\Phi = \Delta\alpha$, together with the fact that i is injective, allows us to factorize Δ into a morphism Ψ, as follows:

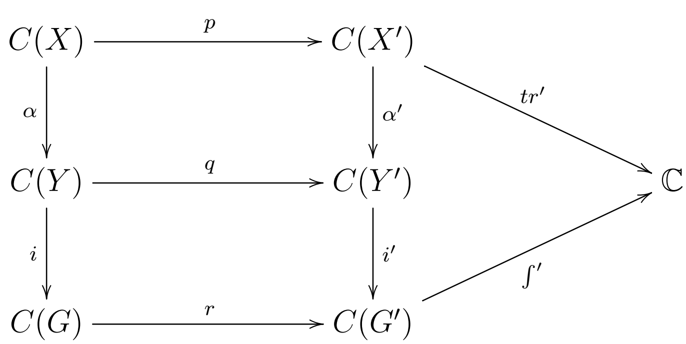


Thus the image space $G\to Y$ is indeed homogeneous, and we are done. □

Finally, we have the following result:

THEOREM 3.9. *Let $G\to X$ be an extended homogeneous space, and construct quotients $X\to X'$, $G\to G'$ by performing the GNS construction with respect to $\int\alpha,\int$. Then α factorizes into an inclusion $\alpha': C(X')\to C(G')$, and we have an homogeneous space.*

PROOF. We factorize $G\to Y\subset X$ as above. By performing the GNS construction with respect to $\int i\alpha,\int i,\int$, we obtain a diagram as follows:



Indeed, with $tr=\int\alpha$, the GNS quotient maps p,q,r are defined respectively by:

$$\begin{aligned}\ker p &= \left\{f\in C(X)\Big| tr(f^*f)=0\right\}\\ \ker q &= \left\{f\in C(Y)\Big| \textstyle\int(f^*f)=0\right\}\\ \ker r &= \left\{f\in C(G)\Big| \textstyle\int(f^*f)=0\right\}\end{aligned}$$

Next, we can define factorizations i',α' as above. Observe that i' is injective, and that α' is surjective. Our claim now is that α' is injective as well. Indeed:

$$\begin{aligned}\alpha'p(f)=0 &\implies q\alpha(f)=0\\ &\implies \int\alpha(f^*f)=0\\ &\implies tr(f^*f)=0\\ &\implies p(f)=0\end{aligned}$$

We conclude that we have $X'=Y'$, and this gives the result. □

Summarizing, the basic homogeneous space theory from the classical case extends to the quantum group setting, with a few twists, both of algebraic and analytic nature.

3b. Partial isometries

Let us discuss now some basic examples of homogeneous spaces, in our sense above. We begin with a study in the classical case. Our starting point will be:

DEFINITION 3.10. *Associated to any integers $L\leq M,N$ are the spaces*

$$O_{MN}^L=\left\{T:E\to F\text{ isometry}\Big|E\subset\mathbb R^N,F\subset\mathbb R^M,\dim_{\mathbb R}E=L\right\}$$

$$U_{MN}^L=\left\{T:E\to F\text{ isometry}\Big|E\subset\mathbb C^N,F\subset\mathbb C^M,\dim_{\mathbb C}E=L\right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L=M=N$ we obtain the groups O_N,U_N:

$$O_{NN}^N=O_N\quad,\quad U_{NN}^N=U_N$$

Another interesting specialization is $L=M=1$. Here the elements of O_{1N}^1 are the isometries $T:E\to\mathbb R$, with $E\subset\mathbb R^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1)\in\mathbb R^N$, which must belong to $S^{N-1}_{\mathbb R}$. Thus, we have $O_{1N}^1=S^{N-1}_{\mathbb R}$. Similarly, in the complex case we have $U_{1N}^1=S^{N-1}_{\mathbb C}$, and so our results here are:

$$O_{1N}^1=S^{N-1}_{\mathbb R}\quad,\quad U_{1N}^1=S^{N-1}_{\mathbb C}$$

Yet another interesting specialization is $L=N=1$. Here the elements of O_{1N}^1 are the isometries $T:\mathbb R\to F$, with $F\subset\mathbb R^M$ one-dimensional. But such an isometry is uniquely determined by $T(1)\in\mathbb R^M$, which must belong to $S^{M-1}_{\mathbb R}$. Thus, we have $O_{M1}^1=S^{M-1}_{\mathbb R}$. Similarly, in the complex case we have $U_{M1}^1=S^{M-1}_{\mathbb C}$, and so our results here are:

$$O_{M1}^1=S^{M-1}_{\mathbb R}\quad,\quad U_{M1}^1=S^{M-1}_{\mathbb C}$$

In general, the most convenient is to view the elements of O_{MN}^L,U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 3.11. *We have identifications of compact spaces*

$$O_{MN}^L \simeq \left\{ U \in M_{M\times N}(\mathbb{R}) \Big| UU^t = \text{projection of trace } L \right\}$$

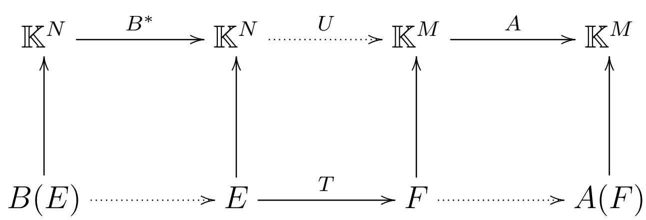
$$U_{MN}^L \simeq \left\{ U \in M_{M\times N}(\mathbb{C}) \Big| UU^* = \text{projection of trace } L \right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T : E \to F$ with their corresponding extensions $U : \mathbb{R}^N \to \mathbb{R}^M$, $U : \mathbb{C}^N \to \mathbb{C}^M$, obtained by setting $U_{E^\perp} = 0$. Then, we can identify these latter maps U with the corresponding rectangular matrices. $\square$

As an illustration, at $L = M = N$ we recover in this way the usual matrix description of O_N, U_N. Also, at $L = M = 1$ we obtain the usual description of $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$, as row spaces over the corresponding groups O_N, U_N. Finally, at $L = N = 1$ we obtain the usual description of $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$, as column spaces over the corresponding groups O_N, U_N.

Now back to the general case, observe that the isometries $T : E \to F$, or rather their extensions $U : \mathbb{K}^N \to \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:



With the identifications in Proposition 3.11 made, the precise statement here is:

Proposition 3.12. *We have action maps as follows, which are both transitive,*

$$O_M \times O_N \curvearrowright O_{MN}^L \quad , \quad (A,B)U = AUB^t$$

$$U_M \times U_N \curvearrowright U_{MN}^L \quad , \quad (A,B)U = AUB^*$$

whose stabilizers are the groups $O_L \times O_{M-L} \times O_{N-L}$ and $U_L \times U_{M-L} \times U_{N-L}$.

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since $(A,B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix} x & * \\ 0 & a \end{pmatrix} \quad , \quad B = \begin{pmatrix} x & 0 \\ * & b \end{pmatrix}$$

Now since A,B are both unitaries, these matrices follow to be block-diagonal, and so:

$$G=\left\{(A,B)\Big|A=\begin{pmatrix}x&0\\0&a\end{pmatrix},B=\begin{pmatrix}x&0\\0&b\end{pmatrix}\right\}$$

The stabilizer of U is then parametrized by triples (x,a,b) belonging respectively to:

$$O_L\times O_{M-L}\times O_{N-L}$$

$$U_L\times U_{M-L}\times U_{N-L}$$

Thus, we are led to the conclusion in the statement. □

Finally, let us work out the quotient space description of O_{MN}^L,U_{MN}^L. We have here:

Theorem 3.13. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned}O_{MN}^L&=&(O_M\times O_N)/(O_L\times O_{M-L}\times O_{N-L})\\U_{MN}^L&=&(U_M\times U_N)/(U_L\times U_{M-L}\times U_{N-L})\end{aligned}$$

with the quotient maps being given by $(A,B)\to AUB^*$, *where* $U=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

Proof. This is just a reformulation of Proposition 3.12, by taking into account the fact that the fixed point used in the proof there was $U=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. □

Once again, the basic examples here come from the cases $L=M=N$ and $L=M=1$. At $L=M=N$ the quotient spaces at right are respectively:

$$O_N\quad,\quad U_N$$

At $L=M=1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1}\quad,\quad U_N/U_{N-1}$$

In fact, in the general orthogonal $L=M$ case we obtain the following spaces:

$$O_{MN}^M=(O_M\times O_N)/(O_M\times O_{N-M})=O_N/O_{N-M}$$

Also, in the general unitary $L=M$ case we obtain the following spaces:

$$U_{MN}^M=(U_M\times U_N)/(U_M\times U_{N-M})=U_N/U_{N-M}$$

Similarly, the examples coming from the cases $L=M=N$ and $L=N=1$ are particular cases of the general $L=N$ case, where we obtain the following spaces:

$$O_{MN}^N=(O_M\times O_N)/(O_M\times O_{M-N})=O_N/O_{M-N}$$

In the unitary case, we obtain the following spaces:

$$U_{MN}^N=(U_M\times U_N)/(U_M\times U_{M-N})=U_N/U_{M-N}$$

Summarizing, we have basic homogeneous spaces, unifying the spheres with the rotation groups. The point now is that we can liberate the spaces O_{MN}^L,U_{MN}^L, as follows:

Definition 3.14. *Associated to any integers $L \leq M, N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|u=\bar{u}, uu^t = \text{projection of trace } L\right)\\ C(U_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|uu^*, \bar{u}u^t = \text{projections of trace } L\right)\end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij} u_{ij}u_{ij}^* = \sum_{ij} u_{ij}^* u_{ij} = L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+}\\ \uparrow & & \uparrow\\ O_{MN}^{L} & \longrightarrow & U_{MN}^{L}\end{array}$$

Observe that at $L = M = 1$ we obtain the following diagram:

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+}\\ \uparrow & & \uparrow\\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}}\end{array}$$

Also, at $L = N = 1$ we obtain the following diagram:

$$\begin{array}{ccc} S^{M-1}_{\mathbb{R},+} & \longrightarrow & S^{M-1}_{\mathbb{C},+}\\ \uparrow & & \uparrow\\ S^{M-1}_{\mathbb{R}} & \longrightarrow & S^{M-1}_{\mathbb{C}}\end{array}$$

In addition to this, we have as well the following result:

Proposition 3.15. *At $L=M=N$ we obtain the diagram*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

consisting of the groups O_N, U_N, and their liberations.

Proof. We recall that the various quantum groups in the statement are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$C(O_N^\times) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=u^tu=1\right)$$

$$C(U_N^\times) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1,\bar{u}u^t=u^t\bar{u}=1\right)$$

On the other hand, according to Proposition 3.11 and to Definition 3.14, we have the following presentation results:

$$C(O_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace } N\right)$$

$$C(U_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace } N\right)$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. We use as well the following formulae:

$$Tr(uu^*)=Tr(u^t\bar{u})$$

$$Tr(\bar{u}u^t)=Tr(u^*u)$$

We therefore obtain the following formulae:

$$C(O_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t,u^tu=\text{projections of trace } N\right)$$

$$C(U_{NN}^{N\times}) = C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,u^*u,\bar{u}u^t,u^t\bar{u}=\text{projections of trace } N\right)$$

Now observe that, in tensor product notation, the conditions at right are all of the form $(tr\otimes id)p=1$. Thus, p must be follows, for the above conditions:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

We therefore obtain that, for any faithful state φ, we have:

$$(tr\otimes\varphi)(1-p)=0$$

It follows from this that the following projections must be all equal to the identity:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

But this leads to the conclusion in the statement. □

Regarding now the homogeneous space structure of $O_{MN}^{L\times}, U_{MN}^{L\times}$, the situation here is a bit more complicated in the free case than in the classical case, due to a number of algebraic and analytic issues. We first have the following result:

PROPOSITION 3.16. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times \times U_N^\times \curvearrowright U_{MN}^{L\times}$, given by $u_{ij} \to \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$.*
(2) *We have a map $U_M^\times \times U_N^\times \to U_{MN}^{L\times}$, given by $u_{ij} \to \sum_{r\leq L} a_{ri} \otimes b_{rj}^*$.*

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

PROOF. In the classical case, consider the following action and quotient maps:

$$U_M \times U_N \curvearrowright U_{MN}^L$$

$$U_M \times U_N \to U_{MN}^L$$

The transposes of these two maps are as follows, where $J = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi &\to ((U,A,B) \to \varphi(AUB^*))\\ \varphi &\to ((A,B) \to \varphi(AJB^*))\end{aligned}$$

But with $\varphi = u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u = u$, let us set:

$$U_{ij} = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

We have then the following computation:

$$\begin{aligned}(UU^*U)_{ij} &= \sum_{pq}\sum_{klmnst} u_{kl}u_{mn}^*u_{st} \otimes a_{ki}a_{mq}^*a_{sq} \otimes b_{lp}^*b_{np}b_{tj}^*\\ &= \sum_{klmt} u_{kl}u_{ml}^*u_{mt} \otimes a_{ki} \otimes b_{tj}^*\\ &= \sum_{kt} u_{kt} \otimes a_{ki} \otimes b_{tj}^*\\ &= U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}\sum_{ij} U_{ij}U_{ij}^* &= \sum_{ij}\sum_{klst} u_{kl}u_{st}^* \otimes a_{ki}a_{si}^* \otimes b_{lj}^*b_{tj}\\ &= \sum_{kl} u_{kl}u_{kl}^* \otimes 1 \otimes 1\\ &= L\end{aligned}$$

(2) Assuming $uu^*u=u$, let us set:

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We have then the following computation:

$$\begin{aligned}(VV^*V)_{ij}&=&\sum_{pq}\sum_{x,y,z\leq L}a_{xi}a_{yq}^*a_{zq}\otimes b_{xp}^*b_{yp}b_{zj}^*\\&=&\sum_{x\leq L}a_{xi}\otimes b_{xj}^*\\&=&V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}V_{ij}V_{ij}^*&=&\sum_{ij}\sum_{r,s\leq L}a_{ri}a_{si}^*\otimes b_{rj}^*b_{sj}\\&=&\sum_{l\leq L}1\\&=&L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. $\square$

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X=G/H$, the associated action and quotient maps are given by:

$$\begin{cases}a:X\times G\to X & : \quad (Hg,h)\to Hgh\\ p:G\to X & : \quad g\to Hg\end{cases}$$

Thus we have $a(p(g),h)=p(gh)$. In our context, a similar result holds:

THEOREM 3.17. *With $G=G_M\times G_N$ and $X=G_{MN}^L$, where $G_N=O_N^\times,U_N^\times$, we have*

$$\begin{array}{ccc}G\times G & \xrightarrow{\ m\ } & G\\ {\scriptstyle p\times id}\downarrow & & \downarrow{\scriptstyle p}\\ X\times G & \xrightarrow{\ a\ } & X\end{array}$$

where a,p are the action map and the map constructed in Proposition 3.16.

PROOF. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ, α are morphisms of algebras induced by a, p:

$$\begin{array}{ccc} C(X) & \xrightarrow{\Phi} & C(X \times G) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha \otimes id} \\ C(G) & \xrightarrow{\Delta} & C(G \times G) \end{array}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned} (\alpha \otimes id)\Phi(u_{ij}) &= (\alpha \otimes id)\sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^* \\ &= \sum_{kl}\sum_{r \leq L} a_{rk} \otimes b_{rl}^* \otimes a_{ki} \otimes b_{lj}^* \end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned} \Delta\pi(u_{ij}) &= F_{23}(\Delta \otimes \Delta)\sum_{r \leq L} a_{ri} \otimes b_{rj}^* \\ &= F_{23}\left(\sum_{r \leq L}\sum_{kl} a_{rk} \otimes a_{ki} \otimes b_{rl}^* \otimes b_{lj}^*\right) \end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. □

3c. Affine spaces

We discuss now an abstract extension of the constructions of manifolds that we have so far. The idea will be that of looking at certain classes of algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$, which are homogeneous spaces, of a certain special type. We have:

DEFINITION 3.18. *An affine homogeneous space over a closed subgroup $G \subset U_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$, such that there exists an index set $I \subset \{1, \ldots, N\}$ such that*

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}}\sum_{j \in I} u_{ji} \quad , \quad \Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

define morphisms of C^*-algebras, satisfying the following condition,

$$\left(id \otimes \int_G\right)\Phi = \int_G \alpha(.)1$$

called ergodicity condition for the action.

To start with, as a basic example, $O_N^+\to S^{N-1}_{\mathbb R,+}$ is indeed affine in our sense, with $I=\{1\}$. The same goes for $U_N^+\to S^{N-1}_{\mathbb C,+}$, which is affine as well, also with $I=\{1\}$.

Observe that the $1/\sqrt{|I|}$ constant appearing above is the correct one, because:

$$\begin{aligned}\sum_i\left(\sum_{j\in I}u_{ji}\right)\left(\sum_{k\in I}u_{ki}\right)^* &= \sum_i\sum_{j,k\in I}u_{ji}u_{ki}^*\\ &= \sum_{j,k\in I}(uu^*)_{jk}\\ &= |I|\end{aligned}$$

As a first general result about such spaces, we have:

PROPOSITION 3.19. *Consider an affine homogeneous space X, as above.*
(1) *The coaction condition $(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$ is satisfied.*
(2) *We have as well the formula $(\alpha\otimes id)\Phi=\Delta\alpha$.*

PROOF. The coaction condition is clear. For the second formula, we first have:

$$\begin{aligned}(\alpha\otimes id)\Phi(x_i) &= \sum_k\alpha(x_k)\otimes u_{ki}\\ &= \frac{1}{\sqrt{|I|}}\sum_k\sum_{j\in I}u_{jk}\otimes u_{ki}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\Delta\alpha(x_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\Delta(u_{ji})\\ &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\end{aligned}$$

Thus, by linearity, multiplicativity and continuity, we obtain the result. □

Summarizing, the terminology in Definition 3.18 is justified, in the sense that what we have there are indeed certain homogeneous spaces, of very special, "affine" type. As a second result regarding such spaces, which closes the discussion in the case where α is injective, which is something that happens in many cases, we have:

THEOREM 3.20. *When α is injective we must have $X=X_{G,I}^{min}$, where:*

$$C(X_{G,I}^{min})=\left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)$$

Moreover, $X_{G,I}^{min}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

PROOF. The first assertion is clear from definitions. Regarding now the second assertion, consider the variables in the statement:

$$X_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\in C(G)$$

In order to prove that we have $X_{G,I}^{min}\subset S^{N-1}_{\mathbb{C},+}$, observe first that we have:

$$\begin{aligned}\sum_iX_iX_i^* &= \frac{1}{|I|}\sum_i\sum_{j,k\in I}u_{ji}u_{ki}^*\\ &= \frac{1}{|I|}\sum_{j,k\in I}(uu^*)_{jk}\\ &= 1\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\sum_iX_i^*X_i &= \frac{1}{|I|}\sum_i\sum_{j,k\in I}u_{ji}^*u_{ki}\\ &= \frac{1}{|I|}\sum_{j,k\in I}(\bar{u}u^t)_{jk}\\ &= 1\end{aligned}$$

Thus $X_{G,I}^{min}\subset S^{N-1}_{\mathbb{C},+}$. Finally, observe that we have:

$$\begin{aligned}\Delta(X_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\\ &= \sum_kX_k\otimes u_{ki}\end{aligned}$$

Thus we have indeed a coaction map, given by $\Phi=\Delta$. As for the ergodicity condition, namely $(id\otimes\int_G)\Delta=\int_G(.)1$, this holds as well, by definition of the integration functional $\int_G$. Thus, our axioms for affine homogeneous spaces are indeed satisfied. □

Our purpose now will be to show that the affine homogeneous spaces appear as follows, a bit in the same way as the discrete group algebras:

$$X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$$

We make the standard convention that all the tensor exponents k are "colored integers", that is, $k=e_1\ldots e_k$ with $e_i\in\{\circ,\bullet\}$, with $\circ$ corresponding to the usual variables, and with $\bullet$ corresponding to their adjoints. With this convention, we have:

Proposition 3.21. *The ergodicity condition, namely*

$$\left(id\otimes\int_G\right)\Phi=\int_G\alpha(.)1$$

is equivalent to the condition

$$(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\quad,\quad\forall k,\forall i_1,\ldots,i_k$$

where P is the matrix formed by the Peter-Weyl integrals of exponent k,

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

and where $(x^{\otimes k})_{i_1\ldots i_k}=x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}$.

Proof. We have the following computation:

$$\begin{aligned}\left(id\otimes\int_G\right)\Phi(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k})&=\sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\&=\sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}(x^{\otimes k})_{j_1\ldots j_k}\\&=(Px^{\otimes k})_{i_1\ldots i_k}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\int_G\alpha(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k})&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a consequence, we have the following result:

Theorem 3.22. *We must have $X\subset X_{G,I}^{max}$, as subsets of $S^{N-1}_{\mathbb{C},+}$, where:*

$$C(X_{G,I}^{max})=C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle$$

Moreover, $X_{G,I}^{max}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

PROOF. Let us first prove that we have an action $G \curvearrowright X_{G,I}^{max}$. We must show here that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $X_{G,I}^{max}$. We have:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{l_1\ldots l_k} P_{i_1\ldots i_k,l_1\ldots l_k}(X^{\otimes k})_{l_1\ldots l_k}\\ &= \sum_{l_1\ldots l_k} P_{i_1\ldots i_k,l_1\ldots l_k}\sum_{j_1\ldots j_k} x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes u_{j_1l_1}^{e_1}\ldots u_{j_kl_k}^{e_k}\\ &= \sum_{j_1\ldots j_k} x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes (u^{\otimes k}P^t)_{j_1\ldots j_k,i_1\ldots i_k}\end{aligned}$$

Since by Peter-Weyl the transpose of $P_{i_1\ldots i_k,j_1\ldots j_k} = \int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$ is the orthogonal projection onto $Fix(u^{\otimes k})$, we have $u^{\otimes k}P^t = P^t$. We therefore obtain:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{j_1\ldots j_k} P_{i_1\ldots i_k,j_1\ldots j_k} x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I} P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

Thus we have an action $G \curvearrowright X_{G,I}^{max}$, and since this action is ergodic by Proposition 7.21, we have an affine homogeneous space, as claimed. $\square$

We can now merge the results that we have, and we obtain:

THEOREM 3.23. *Given a closed quantum subgroup $G \subset U_N^+$, and a set $I \subset \{1,\ldots,N\}$, if we consider the following C^*-subalgebra and the following quotient C^*-algebra,*

$$\begin{aligned}C(X_{G,I}^{min}) &= \left\langle \frac{1}{\sqrt{|I|}}\sum_{j\in I} u_{ji}\Big| i = 1,\ldots,N\right\rangle \subset C(G)\\ C(X_{G,I}^{max}) &= C(S_{\mathbb{C},+}^{N-1})\Big/\left\langle (Px^{\otimes k})_{i_1\ldots i_k} = \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I} P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle\end{aligned}$$

then we have maps as follows,

$$G \to X_{G,I}^{min} \subset X_{G,I}^{max} \subset S_{\mathbb{C},+}^{N-1}$$

the space $G \to X_{G,I}^{max}$ is affine homogeneous, and any affine homogeneous space $G \to X$ appears as an intermediate space $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$.

PROOF. This follows indeed from the various results that we have, namely Theorem 3.20 and Theorem 3.22, regarding the minimal and maximal constructions. $\square$

At the level of the general theory, we will need one more general result, namely an extension of the Weingarten integration formula [28], [93], as follows:

Theorem 3.24. *Assuming that $G\to X$ is an affine homogeneous space, with index set $I\subset\{1,\ldots,N\}$, the Haar integration functional $\int_X=\int_G\alpha$ is given by*

$$\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_{\pi,\sigma\in D}K_I(\pi)\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)$$

where $\{\xi_\pi|\pi\in D\}$ is a basis of $Fix(u^{\otimes k})$, $W_{kN}=G_{kN}^{-1}$ with

$$G_{kN}(\pi,\sigma)=<\xi_\pi,\xi_\sigma>$$

is the associated Weingarten matrix, and:

$$K_I(\pi)=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}(\xi_\pi)_{j_1\ldots j_k}$$

Proof. By using the Weingarten formula for the quantum group G, in its abstract form, coming from Peter-Weyl theory, as discussed in chapter 2, we have:

$$\begin{aligned}\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

But this gives the formula in the statement, and we are done. □

Let us go back now to the "minimal vs maximal" discussion. Here is a natural example of an intermediate space $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$, which will be of interest for us:

Theorem 3.25. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following quotient algebra*

$$C(X_{G,I}^{med})=C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle\sum_{j_1\ldots j_k}\xi_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\xi_{j_1\ldots j_k}\Big|\forall k,\forall\xi\in Fix(u^{\otimes k})\right\rangle$$

we obtain in this way an affine homogeneous space $G\to X_{G,I}$.

Proof. We know from Theorem 3.22 that $X_{G,I}^{max}\subset S^{N-1}_{\mathbb{C},+}$ is constructed by imposing to the standard coordinates the conditions $Px^{\otimes k}=P^I$, where:

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

$$P^I_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}$$

According to the Weingarten integration formula for G, we have:

$$(Px^{\otimes k})_{i_1\dots i_k} = \sum_{j_1\dots j_k}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\dots j_k}\overline{(\xi_\sigma)}_{i_1\dots i_k}W_{kN}(\pi,\sigma)x_{j_1}^{e_1}\dots x_{j_k}^{e_k}$$

$$P^I_{i_1\dots i_k} = \frac{1}{\sqrt{|I|^k}}\sum_{j_1\dots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\dots j_k}\overline{(\xi_\sigma)}_{i_1\dots i_k}W_{kN}(\pi,\sigma)$$

Thus $X_{G,I}^{med}\subset X_{G,I}^{max}$, and the other assertions are standard as well. □

We can now put everything together, as follows:

THEOREM 3.26. *Given a closed subgroup $G\subset U_N^+$, and a subset $I\subset\{1,\dots,N\}$, the affine homogeneous spaces over G, with index set I, have the following properties:*

1. *These are exactly the intermediate subspaces $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$ on which G acts affinely, with the action being ergodic.*
2. *For the minimal and maximal spaces $X_{G,I}^{min}$ and $X_{G,I}^{max}$, as well as for the intermediate space $X_{G,I}^{med}$ constructed above, these conditions are satisfied.*
3. *By performing the GNS construction with respect to the Haar integration functional $\int_X=\int_G\alpha$ we obtain the minimal space $X_{G,I}^{min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them $X_{G,I}$.

PROOF. This follows indeed by combining the various results and observations formulated above. As before, for full details on all these facts, we refer to [7]. □

3d. Axiomatization

We would first like to see what happens for the classical groups, and for the group duals. In the classical case, the result is as follows:

THEOREM 3.27. *In the classical case, $G\subset U_N$, there is only one affine homogeneous space, for each index set $I=\{1,\dots,N\}$, namely the quotient space*

$$X=G/(G\cap C_N^I)$$

where $C_N^I\subset U_N$ is the group of unitaries fixing the following vector,

$$\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$$

which generalizes the complex bistochastic group, $C_N\subset U_N$.

PROOF. Consider an affine homogeneous space $G\to X$. It is elementary to see, using our axioms, that X is indeed classical. We will first prove that we have $X=X_{G,I}^{min}$, and then we will prove that $X_{G,I}^{min}$ equals the quotient space in the statement.

(1) We use the well-known fact that the functional $E = (id \otimes \int_G)\Phi$ is the projection onto the fixed point algebra of the action, given by:

$$C(X)^\Phi = \left\{ f \in C(X) \Big| \Phi(f) = f \otimes 1 \right\}$$

Thus our ergodicity condition, namely $E = \int_G \alpha(.)1$, shows that we must have:

$$C(X)^\Phi = \mathbb{C}1$$

But in the classical case the condition $\Phi(f) = f \otimes 1$ reformulates as:

$$f(gx) = f(x) \quad , \quad \forall g \in G, x \in X$$

Thus, we recover in this way the usual ergodicity condition, stating that whenever a function $f \in C(X)$ is constant on the orbits of the action, it must be constant. Now observe that for an affine action, the orbits are closed. Thus an affine action which is ergodic must be transitive, and we deduce from this that we have:

$$X = X_{G,I}^{min}$$

(2) We know that the inclusion $C(X) \subset C(G)$ comes via:

$$x_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji}$$

Thus, the quotient map $p : G \to X \subset S_{\mathbb{C}}^{N-1}$ is given by the following formula:

$$p(g) = \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} g_{ji} \right)_i$$

In particular, the image of the unit matrix $1 \in G$ is the following vector:

$$p(1) = \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} \delta_{ji} \right)_i = \left(\frac{1}{\sqrt{|I|}} \delta_{i \in I} \right)_i = \xi_I$$

But this gives the quotient space result in the statement.

(3) Finally, regarding the last assertion, stating that our group $C_N^I \subset U_N$ generalizes the complex bishochastic group $C_N \subset U_N$, this is more of a comment, coming from definitions. Indeed, C_N consists by definition of the unitary matrices $g \in U_N$ which are bistochastic, meaning having the same sums on rows and columns. But this bistochasticity condition is equivalent to the following condition, with ξ being the all-1 vector:

$$g\xi = \xi$$

Thus, our group $C_N^I \subset U_N$ generalizes indeed the group $C_N \subset U_N$, as claimed. □

Let us discuss now the group dual case. Here we have the following result:

THEOREM 3.28. *In the group dual case, $G=\widehat{\Gamma}$ with $\Gamma=<g_1,\ldots,g_N>$, we have*

$$X=\widehat{\Gamma}_I\quad:\quad\Gamma_I=<g_i|i\in I>\subset\Gamma$$

for any affine homogeneous space X, when identifying full and reduced group algebras.

PROOF. Assume indeed that we have an affine homogeneous space $G\to X$. In terms of the rescaled coordinates $h_i=\sqrt{|I|}x_i$, our axioms for α,Φ read:

$$\alpha(h_i)=\delta_{i\in I}g_i\quad,\quad\Phi(h_i)=h_i\otimes g_i$$

As for the ergodicity condition, this translates as follows:

$$\begin{aligned}
&\left(id\otimes\int_G\right)\Phi(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p})=\int_G\alpha(h_{i_1}^{e_p}\ldots h_{i_p}^{e_p})\\
\iff\quad&\left(id\otimes\int_G\right)(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}\otimes g_{i_1}^{e_1}\ldots g_{i_p}^{e_p})=\int_G\delta_{i_1\in I}\ldots\delta_{i_p\in I}g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}\\
\iff\quad&\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}\delta_{i_1\in I}\ldots\delta_{i_p\in I}\\
\iff\quad&\left[g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\implies h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{i_1\in I}\ldots\delta_{i_p\in I}\right]
\end{aligned}$$

Now observe that from $g_ig_i^*=g_i^*g_i=1$ we obtain in this way:

$$h_ih_i^*=h_i^*h_i=\delta_{i\in I}$$

Thus the elements h_i vanish for $i\notin I$, and are unitaries for $i\in I$. We conclude that we have $X=\widehat{\Lambda}$, where $\Lambda=<h_i|i\in I>$ is the group generated by these unitaries. In order to finish now the proof, our claim is that for indices $i_x\in I$ we have:

$$g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\iff h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=1$$

Indeed, $\implies$ comes from the ergodicity condition, as processed above, and $\Longleftarrow$ comes from the existence of the morphism α, which is given by $\alpha(h_i)=g_i$, for $i\in I$. □

Let us go back now to the general case, and discuss a number of further axiomatization issues, based on the examples that we have. We will need the following result:

PROPOSITION 3.29. *The closed subspace $C_N^{I+}\subset U_N^+$ defined via*

$$C(C_N^{I+})=C(U_N^+)\Big/\langle u\xi_I=\xi_I\rangle$$

where $\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$, is a compact quantum group.

PROOF. We must check Woronowicz's axioms, and the proof goes as follows:

(1) Let us set $U_{ij}=\sum_k u_{ik}\otimes u_{kj}$. We have then:

$$(U\xi_I)_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik}\otimes u_{kj}=\sum_k u_{ik}\otimes(u\xi_I)_k$$

Since the vector ξ_I is by definition fixed by u, we obtain:

$$(U\xi_I)_i = \frac{1}{\sqrt{|I|}}\sum_{k\in I} u_{ik}\otimes 1 = (\xi_I)_i\otimes 1$$

Thus we can define indeed a comultiplication map, by $\Delta(u_{ij}) = U_{ij}$.

(2) In order to construct the counit map, $\varepsilon(u_{ij}) = \delta_{ij}$, we must prove that the identity matrix $1 = (\delta_{ij})_{ij}$ satisfies $1\xi_I = \xi_I$. But this is clear.

(3) In order to construct the antipode, $S(u_{ij}) = u_{ji}^*$, we must prove that the adjoint matrix $u^* = (u_{ji}^*)_{ij}$ satisfies $u^*\xi_I = \xi_I$. But this is clear from $u\xi_I = \xi_I$. □

Based on the computations that we have so far, we can formulate:

THEOREM 3.30. *Given a closed quantum subgroup $G \subset U_N^+$ and a set $I \subset \{1,\ldots,N\}$, we have a quotient map and an inclusion map as follows:*

$$G/(G\cap C_N^{I+}) \to X_{G,I}^{min} \subset X_{G,I}^{max}$$

These maps are both isomorphisms in the classical case. In general, they are both proper.

PROOF. Consider the quantum group $H = G\cap C_N^{I+}$, which is by definition such that at the level of the corresponding algebras, we have:

$$C(H) = C(G)\Big/\left<u\xi_I = \xi_I\right>$$

In order to construct a quotient map $G/H \to X_{G,I}^{min}$, we must check that the defining relations for $C(G/H)$ hold for the standard generators $x_i \in C(X_{G,I}^{min})$. But if we denote by $\rho : C(G) \to C(H)$ the quotient map, then we have, as desired:

$$\begin{aligned}(id\otimes\rho)\Delta x_i &= (id\otimes\rho)\left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik}\otimes u_{kj}\right)\\ &= \sum_k u_{ik}\otimes(\xi_I)_k\\ &= x_i\otimes 1\end{aligned}$$

In the classical case, Theorem 3.28 shows that both the maps in the statement are isomorphisms. For the group duals, however, these maps are not isomorphisms, in general. This follows indeed from Theorem 3.29, and from the general theory in [**13**]. □

3e. Exercises

Exercises.

CHAPTER 4

Free space

4a. Projective space

We discuss in this chapter several things that can be done, going beyond the sphere setting. First we will discuss free projective geometry, which is by definition compact, and so can be developed in full generality, without norm restrictions. Then, at the end of the chapter, we will go back to the affine setting, with some further results.

As a first topic that we would like to discuss, which historically speaking, was at the beginning of everything, we have the following remarkable isomorphism:

$$PO_N^+ = PU_N^+$$

In order to get started, let us first discuss the classical case, and more specifically the precise relation between the orthogonal group O_N, and the unitary group U_N. Contrary to the passage $\mathbb{R}^N \to \mathbb{C}^N$, or to the passage $S^{N-1}_\mathbb{R} \to S^{N-1}_\mathbb{C}$, which are both elementary, the passage $O_N \to U_N$ cannot be understood directly. In order to understand this passage we must pass through the corresponding Lie algebras, a follows:

THEOREM 4.1. *The passage $O_N \to U_N$ appears via Lie algebra complexification,*

$$O_N \to \mathfrak{o}_N \to \mathfrak{u}_n \to U_N$$

with the Lie algebra $\mathfrak{u}_N$ being a complexification of the Lie algebra $\mathfrak{o}_N$.

PROOF. This is something rather philosophical, and advanced as well, that we will not really need here, the idea being as follows:

(1) The unitary and orthogonal groups U_N, O_N are both Lie groups, in the sense that they are smooth manifolds. The corresponding Lie algebras $\mathfrak{u}_N, \mathfrak{o}_N$, which are by definition the respective tangent spaces at 1, can be computed by differentiating the equations defining U_N, O_N, with the conclusion being as follows:

$$\mathfrak{u}_N = \left\{ A \in M_N(\mathbb{C}) \Big| A^* = -A \right\}$$

$$\mathfrak{o}_N = \left\{ B \in M_N(\mathbb{R}) \Big| B^t = -B \right\}$$

(2) This was for the correspondences $U_N \to \mathfrak{u}_N$ and $O_N \to \mathfrak{o}_N$. In the other sense, the correspondences $\mathfrak{u}_N \to U_N$ and $\mathfrak{o}_N \to O_N$ appear by exponentiation, the result here

stating that, around 1, the unitary matrices can be written as $U=e^A$, with $A\in\mathfrak{u}_N$, and the orthogonal matrices can be written as $U=e^B$, with $B\in\mathfrak{o}_N$.

(3) In view of all this, in order to understand the passage $O_N\to U_N$ it is enough to understand the passage $\mathfrak{o}_N\to\mathfrak{u}_N$. But, in view of the above formulae for $\mathfrak{o}_N,\mathfrak{u}_N$, this is basically an elementary linear algebra problem. Indeed, let us pick an arbitrary matrix $A\in M_N(\mathbb{C})$, and write it as follows, with $B,C\in M_N(\mathbb{R})$:

$$A=B+iC$$

In terms of B,C, the equation $A^*=-A$ defining the Lie algebra $\mathfrak{u}_N$ reads:

$$B^t=-B\quad,\quad C^t=C$$

(4) As a first observation, we must have $B\in\mathfrak{o}_N$. Regarding now C, let us decompose this matrix as follows, with D being its diagonal, and C' being the reminder:

$$C=D+C'$$

The matrix C' being symmetric with 0 on the diagonal, by swithcing all the signs below the main diagonal we obtain a certain matrix $C'_-\in\mathfrak{o}_N$. Thus, we have decomposed $A\in\mathfrak{u}_N$ as follows, with $B,C'\in\mathfrak{o}_N$, and with $D\in M_N(\mathbb{R})$ being diagonal:

$$A=B+iD+iC'_-$$

(5) As a conclusion now, we have shown that we have a direct sum decomposition of real linear spaces as follows, with $\Delta\subset M_N(\mathbb{R})$ being the diagonal matrices:

$$\mathfrak{u}_N\simeq\mathfrak{o}_N\oplus\Delta\oplus\mathfrak{o}_N$$

Thus, we can stop our study here, and say that we have reached the conclusion in the statement, namely that $\mathfrak{u}_N$ appears as a "complexification" of $\mathfrak{o}_N$. □

As before with many other things, that we will not really need in what follows, this was just an introduction to the subject. More can be found in any Lie group book. In the free case now, the situation is much simpler, and we have:

THEOREM 4.2. *The passage $O_N^+\to U_N^+$ appears via free complexification,*

$$U_N^+=\widetilde{O_N^+}$$

where the free complexification of a pair (G,u) is the pair $(\widetilde{G},\widetilde{u})$ with

$$C(\widetilde{G})=<zu_{ij}>\subset C(\mathbb{T})*C(G)\quad,\quad \widetilde{u}=zu$$

where $z\in C(\mathbb{T})$ is the standard generator, given by $x\to x$ for any $x\in\mathbb{T}$.

PROOF. We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of U_N^+:

$$O_N^+ \subset \widetilde{O_N^+} \subset U_N^+$$

We must prove that the embedding on the right is an isomorphism, and there are several ways of doing this, all instructive, as follows:

(1) If we denote by v, u the fundamental corepresentations of O_N^+, U_N^+, we have:

$$Fix(v^{\otimes k}) = span\left(\xi_\pi\Big|\pi \in NC_2(k)\right)$$

$$Fix(u^{\otimes k}) = span\left(\xi_\pi\Big|\pi \in \mathcal{NC}_2(k)\right)$$

Moreover, the above vectors ξ_π are known to be linearly independent at $N \geq 2$, and so the above results provide us with bases, and we obtain:

$$\dim(Fix(v^{\otimes k})) = |NC_2(k)| \quad , \quad \dim(Fix(u^{\otimes k})) = |\mathcal{NC}_2(k)|$$

Now since integrating the character of a corepresentation amounts in counting the fixed points, the above two formulae can be rewritten as follows:

$$\int_{O_N^+} \chi_v^k = |NC_2(k)| \quad , \quad \int_{U_N^+} \chi_u^k = |\mathcal{NC}_2(k)|$$

But this shows, via standard free probability theory, that χ_v must follow the Winger semicircle law γ_1, and that χ_u must follow the Voiculescu circular law Γ_1:

$$\chi_v \sim \gamma_1 \quad , \quad \chi_u \sim \Gamma_1$$

On the other hand, by [87], when freely multiplying a semicircular variable by a Haar unitary we obtain a circular variable. Thus, the main character of $\widetilde{O_N^+}$ is circular:

$$\chi_{zv} \sim \Gamma_1$$

Now by forgetting about circular variables and free probability, the conclusion is that the inclusion $\widetilde{O_N^+} \subset U_N^+$ preserves the law of the main character:

$$law(\chi_{zv}) = law(u)$$

Thus by Peter-Weyl we obtain that the inclusion $\widetilde{O_N^+} \subset U_N^+$ must be an isomorphism, modulo the usual equivalence relation for quantum groups.

(2) A version of the above proof, not using any prior free probability knowledge, makes use of the easiness property of O_N^+, U_N^+ only, namely:

$$Hom(v^{\otimes k}, v^{\otimes l}) = span\left(\xi_\pi\Big|\pi \in NC_2(k,l)\right)$$

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\xi_\pi\Big|\pi \in \mathcal{NC}_2(k,l)\right)$$

Indeed, let us look at the following inclusions of quantum groups:

$$O_N^+ \subset \widetilde{O_N^+} \subset U_N^+$$

At the level of the associated Hom spaces we obtain reverse inclusions, as follows:

$$Hom(v^{\otimes k}, v^{\otimes l}) \supset Hom((zv)^{\otimes k}, (zv)^{\otimes l}) \supset Hom(u^{\otimes k}, u^{\otimes l})$$

The spaces on the left and on the right are known from easiness, the result being that these spaces are as follows:

$$span\left(T_\pi\Big|\pi \in NC_2(k,l)\right) \supset span\left(T_\pi\Big|\pi \in \mathcal{NC}_2(k,l)\right)$$

Regarding the spaces in the middle, these are obtained from those on the left by "coloring", so we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O_N^+} \subset U_N^+$ is an isomorphism, modulo the usual equivalence relation. □

As an interesting consequence of the above result, we have:

Theorem 4.3. *We have an identification as follows,*

$$PO_N^+ = PU_N^+$$

modulo the usual equivalence relation for compact quantum groups.

Proof. As before, we have several proofs for this result, as follows:

(1) This follows from Theorem 4.2, because we have:

$$PU_N^+ = P\widetilde{O_N^+} = PO_N^+$$

(2) We can deduce this as well directly. With notations as before, we have:

$$Hom\left((v \otimes v)^k, (v \otimes v)^l\right) = span\left(T_\pi\Big|\pi \in NC_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

$$Hom\left((u \otimes \bar{u})^k, (u \otimes \bar{u})^l\right) = span\left(T_\pi\Big|\pi \in \mathcal{NC}_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusion $PO_N^+ \subset PU_N^+$ preserves the corresponding Tannakian categories, and so must be an isomorphism. □

As a conclusion, the passage $O_N^+ \to U_N^+$ is something much simpler than the passage $O_N \to U_N$, with this ultimately coming from the fact that the combinatorics of O_N^+, U_N^+ is something much simpler than the combinatorics of O_N, U_N. In addition, all this leads as well to the interesting conclusion that the free projective geometry does not fall into real and complex, but is rather unique and "scalarless". We will be back to this.

Let us discuss now the projective spaces. Our starting point is the following functional analytic description of the real and complex projective spaces $P^{N-1}_\mathbb{R}, P^{N-1}_\mathbb{C}$:

Proposition 4.4. *We have presentation results as follows,*

$$C(P^{N-1}_\mathbb{R}) = C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=\bar{p}=p^t=p^2,Tr(p)=1\right)$$
$$C(P^{N-1}_\mathbb{C}) = C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

for the algebras of continuous functions on the real and complex projective spaces.

Proof. We use the fact that the projective spaces $P^{N-1}_\mathbb{R}, P^{N-1}_\mathbb{C}$ can be respectively identified with the spaces of rank one projections in $M_N(\mathbb{R}), M_N(\mathbb{C})$. With this picture in mind, it is clear that we have arrows $\leftarrow$. In order to construct now arrows $\to$, consider the universal algebras on the right, A_R, A_C. These algebras being both commutative, by the Gelfand theorem we can write, with X_R, X_C being certain compact spaces:

$$A_R = C(X_R) \quad , \quad A_C = C(X_C)$$

Now by using the coordinate functions p_{ij}, we conclude that X_R, X_C are certain spaces of rank one projections in $M_N(\mathbb{R}), M_N(\mathbb{C})$. In other words, we have embeddings:

$$X_R \subset P^{N-1}_\mathbb{R} \quad , \quad X_C \subset P^{N-1}_\mathbb{C}$$

By transposing we obtain arrows $\to$, as desired. □

The above result suggests the following definition:

Definition 4.5. *Associated to any $N \in \mathbb{N}$ is the following universal algebra,*

$$C(P^{N-1}_+) = C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

whose abstract spectrum is called "free projective space".

Observe that, according to our presentation results for the real and complex projective spaces $P^{N-1}_\mathbb{R}$ and $P^{N-1}_\mathbb{C}$, we have embeddings of compact quantum spaces, as follows:

$$P^{N-1}_\mathbb{R} \subset P^{N-1}_\mathbb{C} \subset P^{N-1}_+$$

Our first goal will be that of explaining why, in analogy with the uniqueness of the quantum group $PO_N^+ = PU_N^+$, the free projective space P^{N-1}_+ is unique, and scalarless.

Let us first discuss the relation with the spheres. Given a closed subset $X \subset S^{N-1}_{\mathbb{R},+}$, its projective version is by definition the quotient space $X \to PX$ determined by the fact that $C(PX) \subset C(X)$ is the subalgebra generated by the following variables:

$$p_{ij} = x_i x_j$$

In order to discuss the relation with the spheres, it is convenient to neglect the material regarding the complex and hybrid cases, the projective versions of such spheres bringing nothing new. Thus, we are left with the 3 real spheres, and we have:

Theorem 4.6. *The projective versions of the* 3 *real spheres are as follows,*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+}\\
\downarrow & & \downarrow & & \downarrow\\
P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{+}
\end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

Proof. The assertion at left is true by definition. For the assertion at right, we have to prove that the variables $p_{ij} = z_i z_j$ over the free sphere $S^{N-1}_{\mathbb{R},+}$ satisfy the defining relations for $C(P^{N-1}_+)$, from Definition 4.5, namely:

$$p = p^* = p^2 \quad , \quad Tr(p) = 1$$

We first have the following computation:

$$(p^*)_{ij} = p^*_{ji} = (z_j z_i)^* = z_i z_j = p_{ij}$$

We have as well the following computation:

$$(p^2)_{ij} = \sum_k p_{ik} p_{kj} = \sum_k z_i z_k^2 z_j = z_i z_j = p_{ij}$$

Finally, we have as well the following computation:

$$Tr(p) = \sum_k p_{kk} = \sum_k z_k^2 = 1$$

Regarding now $PS^{N-1}_{\mathbb{R},*} = P^{N-1}_{\mathbb{C}}$, the inclusion "$\subset$" follows from $abcd = cbad = cbda$. In the other sense now, the point is that we have a matrix model, as follows:

$$\pi : C(S^{N-1}_{\mathbb{R},*}) \to M_2(C(S^{N-1}_{\mathbb{C}})) \quad , \quad x_i \to \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

But this gives the missing inclusion "$\supset$", and we are done. See [11]. □

In addition to the above result, let us mention that, as already discussed above, passing to the complex case brings nothing new. This is because the projective version of the free complex sphere is equal to the free projective space constructed above:

$$PS^{N-1}_{\mathbb{C},+} = P^{N-1}_+$$

And the same goes for the "hybrid" spheres. For details on all this, we refer to [7].

Following [12], we can axiomatize our various projective spaces, as follows:

DEFINITION 4.7. *A monomial projective space is a closed subset $P\subset P^{N-1}_+$ obtained via relations of type*

$$p_{i_1i_2}\ldots p_{i_{k-1}i_k}=p_{i_{\sigma(1)}i_{\sigma(2)}}\ldots p_{i_{\sigma(k-1)}i_{\sigma(k)}},\ \forall(i_1,\ldots,i_k)\in\{1,\ldots,N\}^k$$

with σ ranging over a certain subset of the infinite symmetric group

$$S_\infty=\bigcup_{k\in2\mathbb N}S_k$$

which is stable under the operation $\sigma\to|\sigma|$.

Here the stability under the operation $\sigma\to|\sigma|$ means that if the above relation associated to σ holds, then the following relation, associated to $|\sigma|$, must hold as well:

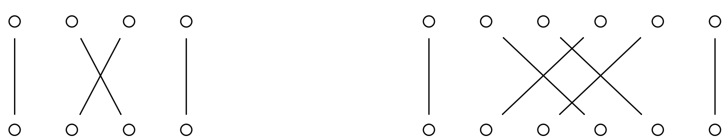
$$p_{i_0i_1}\ldots p_{i_ki_{k+1}}=p_{i_0i_{\sigma(1)}}p_{i_{\sigma(2)}i_{\sigma(3)}}\ldots p_{i_{\sigma(k-2)}i_{\sigma(k-1)}}p_{i_{\sigma(k)}i_{k+1}}$$

As an illustration, the basic projective spaces are all monomial:

PROPOSITION 4.8. *The 3 projective spaces are all monomial, with the permutations*

producing respectively the spaces $P^{N-1}_{\mathbb R},P^{N-1}_{\mathbb C}$, and with no relation needed for P^{N-1}_+.

PROOF. We must divide the algebra $C(P^{N-1}_+)$ by the relations associated to the diagrams in the statement, as well as those associated to their shifted versions, given by:



(1) The basic crossing, and its shifted version, produce the following relations:

$$p_{ab}=p_{ba}$$

$$p_{ab}p_{cd}=p_{ac}p_{bd}$$

Now by using these relations several times, we obtain the following formula:

$$p_{ab}p_{cd}=p_{ac}p_{bd}=p_{ca}p_{db}=p_{cd}p_{ab}$$

Thus, the space produced by the basic crossing is classical, $P\subset P^{N-1}_{\mathbb C}$. By using one more time the relations $p_{ab}=p_{ba}$ we conclude that we have $P=P^{N-1}_{\mathbb R}$, as claimed.

(2) The fattened crossing, and its shifted version, produce the following relations:

$$p_{ab}p_{cd}=p_{cd}p_{ab}$$

$$p_{ab}p_{cd}p_{ef}=p_{ad}p_{eb}p_{cf}$$

The first relations tell us that the projective space must be classical, $P\subset P^{N-1}_\mathbb{C}$. Now observe that with $p_{ij}=z_i\bar{z}_j$, the second relations read:

$$z_a\bar{z}_bz_c\bar{z}_dz_e\bar{z}_f=z_a\bar{z}_dz_e\bar{z}_bz_c\bar{z}_f$$

Since these relations are automatic, we have $P=P^{N-1}_\mathbb{C}$, and we are done. □

Following [12], we can now formulate our classification result, as follows:

THEOREM 4.9. *The basic projective spaces, namely*

$$P^{N-1}_\mathbb{R}\subset P^{N-1}_\mathbb{C}\subset P^{N-1}_+$$

are the only monomial ones.

PROOF. We follow the proof from the affine case. Let $\mathcal{R}_\sigma$ be the collection of relations associated to a permutation $\sigma\in S_k$ with $k\in 2\mathbb{N}$, as in Definition 4.7. We fix a monomial projective space $P\subset P^{N-1}_+$, and we associate to it subsets $G_k\subset S_k$, as follows:

$$G_k=\begin{cases}\{\sigma\in S_k|\mathcal{R}_\sigma\text{ hold over }P\}&(k\text{ even})\\ \{\sigma\in S_k|\mathcal{R}_{|\sigma}\text{ hold over }P\}&(k\text{ odd})\end{cases}$$

As in the affine case, we obtain in this way a filtered group $G=(G_k)$, which is stable under removing outer strings, and under removing neighboring strings. Thus the computations from the affine case apply, and show that we have only 3 possible situations, corresponding to the 3 projective spaces in Proposition 4.8. See [12]. □

Let us discuss now similar results for the projective quantum groups. Given a closed subgroup $G\subset O_N^+$, its projective version $G\to PG$ is by definition given by the fact that $C(PG)\subset C(G)$ is the subalgebra generated by the following variables:

$$w_{ij,ab}=u_{ia}u_{jb}$$

In the classical case we recover in this way the usual projective version:

$$PG=G/(G\cap\mathbb{Z}_2^N)$$

We have the following key result:

THEOREM 4.10. *The quantum group O_N^* is the unique intermediate easy quantum group $O_N\subset G\subset O_N^+$. Moreover, in the non-easy case, the following happen:*

(1) *The group inclusion $\mathbb{T}O_N\subset U_N$ is maximal.*
(2) *The group inclusion $PO_N\subset PU_N$ is maximal.*
(3) *The quantum group inclusion $O_N\subset O_N^*$ is maximal.*

PROOF. The first assertion comes by classifying the categories of pairings, and then:

(1) This can be obtained by using standard Lie group methods.

(2) This follows from (1), by taking projective versions.

(3) This follows from (2), via standard algebraic lifting results. □

Our claim now is that, under suitable assumptions, PU_N is the only intermediate object $PO_N\subset G\subset PO_N^+$. In order to formulate a precise statement here, we will need:

DEFINITION 4.11. *A projective category of pairings is a collection of subsets*

$$NC_2(2k,2l)\subset E(k,l)\subset P_2(2k,2l)$$

stable under the usual categorical operations, and satisfying $\sigma\in E\implies|\sigma|\in E$.

As basic examples for this notion, we have the following projective categories of pairings, where P_2^* is the category of matching pairings:

$$NC_2\subset P_2^*\subset P_2$$

This follows indeed from definitions. Now with the above notion in hand, we can formulate the following projective analogue of the notion of easiness:

DEFINITION 4.12. *An intermediate compact quantum group*

$$PO_N\subset H\subset PO_N^+$$

is called projectively easy when its Tannakian category

$$span(NC_2(2k,2l))\subset Hom(v^{\otimes k},v^{\otimes l})\subset span(P_2(2k,2l))$$

comes via via the following formula, using the standard $\pi\to T_\pi$ *construction,*

$$Hom(v^{\otimes k},v^{\otimes l})=span(E(k,l))$$

for a certain projective category of pairings $E=(E(k,l))$.

Thus, we have a projective notion of easiness. Observe that, given an easy quantum group $O_N\subset G\subset O_N^+$, its projective version $PO_N\subset PG\subset PO_N^+$ is projectively easy in our sense. In particular the basic projective quantum groups $PO_N\subset PU_N\subset PO_N^+$ are all projectively easy in our sense, coming from the categories $NC_2\subset P_2^*\subset P_2$.

We have in fact the following general result, from [12]:

THEOREM 4.13. *We have a bijective correspondence between the affine and projective categories of partitions, given by the operation*

$$G\to PG$$

at the level of the corresponding affine and projective easy quantum groups.

PROOF. The construction of correspondence $D\to E$ is clear, simply by setting:

$$E(k,l)=D(2k,2l)$$

Indeed, due to the axioms in Definition 4.11, the conditions in Definition 4.12 are satisfied. Conversely, given $E=(E(k,l))$ as in Definition 4.12, we can set:

$$D(k,l)=\begin{cases}E(k,l)&(k,l\text{ even})\\\{\sigma:|\sigma\in E(k+1,l+1)\}&(k,l\text{ odd})\end{cases}$$

Our claim is that $D=(D(k,l))$ is a category of partitions. Indeed:

(1) The composition action is clear. Indeed, when looking at the numbers of legs involved, in the even case this is clear, and in the odd case, this follows from:

$$\begin{aligned}|\sigma,|\sigma'\in E &\implies |^\sigma_\tau\in E\\ &\implies {}^\sigma_\tau\in D\end{aligned}$$

(2) For the tensor product axiom, we have 4 cases to be investigated, depending on the parity of the number of legs of σ,τ, as follows:

– The even/even case is clear.

– The odd/even case follows from the following computation:

$$\begin{aligned}|\sigma,\tau\in E &\implies |\sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

– Regarding now the even/odd case, this can be solved as follows:

$$\begin{aligned}\sigma,|\tau\in E &\implies |\sigma|,|\tau\in E\\ &\implies |\sigma||\tau\in E\\ &\implies |\sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

– As for the remaining odd/odd case, here the computation is as follows:

$$\begin{aligned}|\sigma,|\tau\in E &\implies ||\sigma|,|\tau\in E\\ &\implies ||\sigma||\tau\in E\\ &\implies \sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

(3) Finally, the conjugation axiom is clear from definitions. It is also clear that both compositions $D\to E\to D$ and $E\to D\to E$ are the identities, as claimed. As for the quantum group assertion, this is clear as well from definitions. □

Now back to uniqueness issues, we have here the following result, also from [12]:

THEOREM 4.14. *We have the following results:*

(1) *O_N^* is the only intermediate easy quantum group $O_N\subset G\subset O_N^+$.*
(2) *PU_N is the only intermediate projectively easy quantum group $PO_N\subset G\subset PO_N^+$.*

PROOF. The idea here is as follows:

(1) The assertion regarding $O_N\subset O_N^*\subset O_N^+$ is from [14], and this is something that we already know, explained in chapter 2.

(2) The assertion regarding $PO_N\subset PU_N\subset PO_N^+$ follows from the classification result in (1), and from the duality in Theorem 4.13. □

Summarizing, we have analogues of the various affine classification results, with the remark that everything becomes simpler in the projective setting.

Our next goal will be that of finding projective versions of the quantum isometry group results that we have in the affine setting. We use the following action formalism, which is quite similar to the affine action formalism introduced in chapter 1:

DEFINITION 4.15. *Consider a closed subgroup of the free orthogonal group, $G\subset O_N^+$, and a closed subset of the free real sphere, $X\subset S^{N-1}_{\mathbb{R},+}$.*

(1) *We write $G\curvearrowright X$ when we have a morphism of C^*-algebras, as follows:*

$$\Phi:C(X)\to C(X)\otimes C(G)$$

$$\Phi(z_i)=\sum_a z_a\otimes u_{ai}$$

(2) *We write $PG\curvearrowright PX$ when we have a morphism of C^*-algebras, as follows:*

$$\Phi:C(PX)\to C(PX)\otimes C(PG)$$

$$\Phi(z_iz_j)=\sum_a z_az_b\otimes u_{ai}u_{bj}$$

Observe that the above morphisms Φ, if they exist, are automatically coaction maps. Observe also that an affine action $G\curvearrowright X$ produces a projective action $PG\curvearrowright PX$. Let us also mention that given an algebraic subset $X\subset S^{N-1}_{\mathbb{R},+}$, it is routine to prove that there exist indeed universal quantum groups $G\subset O_N^+$ acting as (1), and as in (2). We have the following result, from [11] and related papers, with respect to the above notions:

THEOREM 4.16. *The quantum isometry groups of basic spheres and projective spaces,*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+}\\
\downarrow & & \downarrow & & \downarrow\\
P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{+}
\end{array}$$

are the following affine and projective quantum groups,

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+\\
\downarrow & & \downarrow & & \downarrow\\
PO_N & \longrightarrow & PU_N & \longrightarrow & PO_N^+
\end{array}$$

with respect to the affine and projective action notions introduced above.

PROOF. The fact that the 3 quantum groups on top act affinely on the corresponding 3 spheres is known since [11], and is elementary, explained before. By restriction, the 3 quantum groups on the bottom follow to act on the corresponding 3 projective spaces. We must prove now that all these actions are universal. At right there is nothing to prove, so we are left with studying the actions on $S^{N-1}_\mathbb{R},S^{N-1}_{\mathbb{R},*}$ and on $P^{N-1}_\mathbb{R},P^{N-1}_\mathbb{C}$.

$\underline{P^{N-1}_\mathbb{R}}$. Consider the following projective coordinates:

$$p_{ij}=z_iz_j\quad,\quad w_{ij,ab}=u_{ai}u_{bj}$$

In terms of these projective coordinates, the coaction map is given by:

$$\Phi(p_{ij})=\sum_{ab}p_{ab}\otimes w_{ij,ab}$$

Thus, we have the following formulae:

$$\begin{aligned}\Phi(p_{ij})&=\sum_{a<b}p_{ab}\otimes(w_{ij,ab}+w_{ij,ba})+\sum_a p_{aa}\otimes w_{ij,aa}\\ \Phi(p_{ji})&=\sum_{a<b}p_{ab}\otimes(w_{ji,ab}+w_{ji,ba})+\sum_a p_{aa}\otimes w_{ji,aa}\end{aligned}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{ab}=z_az_b$ for $a\leq b$, we conclude that we must have:

$$w_{ij,ab}+w_{ij,ba}=w_{ji,ab}+w_{ji,ba}$$

Let us apply now the antipode to this formula. For this purpose, observe that:

$$\begin{aligned}S(w_{ij,ab})&=S(u_{ai}u_{bj})\\ &=S(u_{bj})S(u_{ai})\\ &=u_{jb}u_{ia}\\ &=w_{ba,ji}\end{aligned}$$

Thus by applying the antipode we obtain:

$$w_{ba,ji}+w_{ab,ji}=w_{ba,ij}+w_{ab,ij}$$

By relabelling, we obtain the following formula:

$$w_{ji,ba}+w_{ij,ba}=w_{ji,ab}+w_{ij,ab}$$

Now by comparing with the original relation, we obtain:

$$w_{ij,ab}=w_{ji,ba}$$

But, with $w_{ij,ab}=u_{ai}u_{bj}$, this formula reads:

$$u_{ai}u_{bj}=u_{bj}u_{ai}$$

Thus $G\subset O_N$, and it follows that we have $PG\subset PO_N$, as claimed.

$\underline{P^{N-1}_\mathbb{C}}$. Consider a coaction map, written as follows, with $p_{ab}=z_a\bar{z}_b$:

$$\Phi(p_{ij})=\sum_{ab}p_{ab}\otimes u_{ai}u_{bj}$$

The idea here will be that of using the following formula:

$$p_{ab}p_{cd}=p_{ad}p_{cb}$$

We have the following formulae:

$$\Phi(p_{ij}p_{kl}) = \sum_{abcd}p_{ab}p_{cd}\otimes u_{ai}u_{bj}u_{ck}u_{dl}$$

$$\Phi(p_{il}p_{kj}) = \sum_{abcd}p_{ad}p_{cb}\otimes u_{ai}u_{dl}u_{ck}u_{bj}$$

The terms at left being equal, and the last terms at right being equal too, we deduce that, with $[a,b,c]=abc-cba$, we must have the following formula:

$$\sum_{abcd}u_{ai}[u_{bj},u_{ck},u_{dl}]\otimes p_{ab}p_{cd}=0$$

Now since the quantities $p_{ab}p_{cd}=z_a\bar{z}_bz_c\bar{z}_d$ at right depend only on the numbers $|\{a,c\}|,|\{b,d\}|\in\{1,2\}$, and this dependence produces the only possible linear relations between the variables $p_{ab}p_{cd}$, we are led to $2\times 2=4$ equations, as follows:

(1) $u_{ai}[u_{bj},u_{ak},u_{bl}]=0$, $\forall a,b$.

(2) $u_{ai}[u_{bj},u_{ak},u_{dl}]+u_{ai}[u_{dj},u_{ak},u_{bl}]=0$, $\forall a$, $\forall b\neq d$.

(3) $u_{ai}[u_{bj},u_{ck},u_{bl}]+u_{ci}[u_{bj},u_{ak},u_{bl}]=0$, $\forall a\neq c$, $\forall b$.

(4) $u_{ai}[u_{bj},u_{ck},u_{dl}]+u_{ai}[u_{dj},u_{ck},u_{bl}]+u_{ci}[u_{bj},u_{ak},u_{dl}]+u_{ci}[u_{dj},u_{ak},u_{bl}]=0$, $\forall a\neq c,b\neq d$.

We will need in fact only the first two formulae. Since (1) corresponds to (2) at $b=d$, we conclude that (1,2) are equivalent to (2), with no restriction on the indices. By multiplying now this formula to the left by u_{ai}, and then summing over i, we obtain:

$$[u_{bj},u_{ak},u_{dl}]+[u_{dj},u_{ak},u_{bl}]=0$$

We use now the antipode/relabel trick from [11]. By applying the antipode we obtain:

$$[u_{ld},u_{ka},u_{jb}]+[u_{lb},u_{ka},u_{jd}]=0$$

By relabelling we obtain the following formula:

$$[u_{dl},u_{ak},u_{bj}]+[u_{dj},u_{ak},u_{bl}]=0$$

Now by comparing with the original relation, we obtain:

$$[u_{bj},u_{ak},u_{dl}]=[u_{dj},u_{ak},u_{bl}]=0$$

Thus $G\subset O_N^*$, and it follows that we have $PG\subset PU_N$, as desired. $\square$

The above results can be probably improved. As an example, let us say that a closed subgroup $G\subset U_N^+$ acts projectively on PX when we have a coaction map as follows:

$$\Phi(z_iz_j)=\sum_{ab}z_az_b\otimes u_{ai}u_{bj}^*$$

The above proof can be adapted, by putting $*$ signs where needed, and Theorem 4.16 still holds, in this setting. However, establishing general universality results, involving arbitrary subgroups $H\subset PO_N^+$, looks like a quite non-trivial question.

4b. Grassmannians

In order to develop free projective geometry, a first piece of work is that of developing a theory of free Grassmannians, free flag manifolds, and free Stiefel manifolds, based on the affine theory of the spaces of quantum partial isometries, from chapter 3. To be more precise, the definition of the free Grassmannians is straightforward, as follows, and the definition of the free flag manifolds and free Stiefel manifolds is very similar:

$$C(Gr_{LN}^+)=C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=L\right)$$

Most of the arguments from the affine case carry over in the projective setting, and with solid and useful affine results to rely upon being available from chapter 3.

We would like to end this discussion with something refreshing, namely a preliminary study of the free analogue of $P^2_\mathbb{R}$. We recall that the projective space $P^{N-1}_\mathbb{R}$ is the space of lines in $\mathbb{R}^N$ passing through the origin, the basic examples being as follows:

(1) At $N=2$ each such a line, in $\mathbb{R}^2$ passing through the origin, corresponds to 2 opposite points on the unit circle $\mathbb{T}\subset\mathbb{R}^2$. Thus, $P^1_\mathbb{R}$ corresponds to the upper semicircle of $\mathbb{T}$, with the endpoints identified, and so we obtain a circle, $P^1_\mathbb{R}=\mathbb{T}$.

(2) At $N=3$ the situation is similar, with $P^2_\mathbb{R}$ corresponding to the upper hemisphere of the sphere $S^2_\mathbb{R}\subset\mathbb{R}^3$, with the points on the equator identified via $x=-x$. Topologically speaking, we can deform if we want the upper hemisphere into a square, with the equator becoming the boundary of this square, and in this picture, the $x=-x$ identification corresponds to the "identify opposite edges, with opposite orientations" folding method for the square, leading to a space $P^2_\mathbb{R}$ which is obviously not embeddable into $\mathbb{R}^3$.

In what follows we will be interested in the free analogue P^2_+ of this projective space $P^2_\mathbb{R}$. Our main motivation comes from the fact that, according to the work of Bhowmick-D'Andrea-Dabrowski [20], later on continued with Das [21], the quantum isometry group $PO_3^+=PU_3^+$ of the free projective space P^2_+ acts on the quark part of the Standard Model spectral triple, in Chamseddine-Connes formulation [26], [27].

We recall that the free projective space is defined by the following formula:

$$C(P^{N-1}_+)=C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

Let us first discuss, as a warm-up, the 2D case. Here the above matrix of projective coordinates is as follows, with $a=a^*$, $b=b^*$, $a+b=1$:

$$p=\begin{pmatrix}a&c\\ c^*&b\end{pmatrix}$$

We have the following computation:

$$p^2=\begin{pmatrix}a&c\\ c^*&b\end{pmatrix}\begin{pmatrix}a&c\\ c^*&b\end{pmatrix}=\begin{pmatrix}a^2+cc^*&ac+cb\\ c^*a+bc^*&c^*c+b^2\end{pmatrix}$$

Thus, the equations to be satisfied are as follows:

$$a^2+cc^*=a$$

$$b^2+c^*c=b$$

$$ac+cb=c$$

$$c^*a+bc^*=c^*$$

The 4th equation is the conjugate of the 3rd equation, so we remove it. By using $a+b=1$, the remaining equations can be written as:

$$cc^*=c^*c=ab$$

$$ac+ca=0$$

We have several explicit models for this, using the spheres $S^1_{\mathbb R,+}$ and $S^1_{\mathbb C,+}$, as well as the first row spaces of O_2^+ and U_2^+, which ultimately lead us to SU_2 and $\bar{S}U_2$. These models are known to be all equivalent under Haar, and the question is whether they are identical. Thus, we must do computations as above in all models, and compare. These are all interesting questions, whose precise answers are not known, so far.

In the 3D case now, that of projective space P^2_+, that we are mainly interested in here, the matrix of coordinates is as follows, with r,s,t self-adjoint, $r+s+t=1$:

$$p=\begin{pmatrix}r&a&b\\ a^*&s&c\\ b^*&c^*&t\end{pmatrix}$$

The square of this matrix is given by:

$$p^2=\begin{pmatrix}r&a&b\\ a^*&s&c\\ b^*&c^*&t\end{pmatrix}\begin{pmatrix}r&a&b\\ a^*&s&c\\ b^*&c^*&t\end{pmatrix}$$

We obtain the following formula:

$$p^2=\begin{pmatrix}r^2+aa^*+bb^*&ra+as+bc^*&rb+ac+bt\\ a^*r+sa^*+cb^*&a^*a+s^2+cc^*&a^*b+sc+ct\\ b^*r+c^*a^*+tb^*&b^*a+c^*s+tc^*&b^*b+c^*c+t^2\end{pmatrix}$$

On the diagonal, the equations for $p^2=p$ are as follows:

$$aa^*+bb^*=r-r^2$$

$$a^*a+cc^*=s-s^2$$

$$b^*b+c^*c=t-t^2$$

On the off-diagonal upper part, the equations for $p^2=p$ are as follows:

$$ra+as+bc^*=a$$

$$rb+ac+bt=b$$

$$a^*b+sc+ct=c$$

On the off-diagonal lower part, the equations for $p^2=p$ are those above, conjugated. Thus, we have 6 equations. The first problem is that of using $r+s+t=1$, in order to make these equations look better. Again, many interesting questions here.

4c. Lifting questions

The are many interesting lifting questions, between affine and projective geometry, with all sorts of half-liberations involved when lifting, and also within affine geometry itself, in connection with the free analogue of the stereographic projection.

So, what is $\mathbb{R}^N_+$? There are several approaches to this problem, and in each case we are looking for a triple (A,Δ,h) consisting of an operator algebra A, typically a non-unital C^*-algebra, then a comultiplication Δ, understood to come accompanied by maps ε,S too, and then a Haar integration functional h. As a starting point, we have:

1. Products. Using $\mathbb{R}^N=(\mathbb{R})^N$. At the algebra level we have $C_0(\mathbb{R}^N)=C_0(\mathbb{R})^{\otimes N}$, and this suggests setting $C_0(\mathbb{R}^N_+)=C_0(\mathbb{R})^{*N}$. Thus we have a well-defined algebra A, and we have a comultiplication Δ too. The problem is with the Haar integration h. Our belief is that this problem can be solved by using suitable $N\times N$ matrix models, with our algebra A appearing on the diagonal. This looks quite tricky.

2. Polar coordinates. Using $[0,\infty)\times S^{N-1}_\mathbb{R}\to\mathbb{R}^N$. At the algebra level we have $C_0(\mathbb{R}^N)\subset C_0[0,\infty)\otimes C(S^{N-1}_\mathbb{R})$, and the very first question is that of understanding what the subalgebra $C_0(\mathbb{R}^N)$ exactly is. Since the quotient map $[0,\infty)\times S^{N-1}_\mathbb{R}\to\mathbb{R}^N$, given by $(r,x)\to rx$, has the property $0x=0y$ for any x,y, this suggests that $C_0(\mathbb{R}^N)\subset C_0[0,\infty)\otimes C(S^{N-1}_\mathbb{R})$ consists of functions such that $f(0,x)$ does not depend on x. It is not very clear what this means, algebrically. Once this difficulty solved, we can probably

go ahead and construct something similar in the free case, $C_0(\mathbb{R}^N_+)\subset C_0[0,\infty)*C(S^{N-1}_{\mathbb{R},+})$, then look for a comultiplication Δ, and a Haar functional h.

2b. An alternative approach here would be by using $\mathbb{R}^N-\{0\}=(0,\infty)\times S^{N-1}_\mathbb{R}$. Here we have at the algebra level $C_0(\mathbb{R}^N-\{0\})=C_0(0,\infty)\otimes C(S^{N-1}_\mathbb{R})$, so at least we have a clearly defined algebra, that we can generalize right away to the free settting, in the form of something of type $C_0(\mathbb{R}^N_+-\{0\})=C_0(0,\infty)*C(S^{N-1}_{\mathbb{R},+})$. However, we cannot really investigate the Δ problem in this setting, so we run once again into a difficulty, namely constructing the correct lifts $C_0(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N_+)$. This being said, the question of investigating the Haar functional h seems to make sense, even in this "$-\{0\}$" setting, meaning without solving the lifting problem. This is actually quite unclear.

2c. Yet another alternative approach would be by using $P^{N-1}_\mathbb{R}$ instead of $S^{N-1}_\mathbb{R}$. The first question here is that of understanding the precise relation between the spaces $\mathbb{R}\times P^{N-1}_\mathbb{R}$ and $\mathbb{R}^N$, which is probably something well-known, but looks quite geometric and tricky. Assuming this geometric problem solved, we can probably have $C_0(\mathbb{R}^N)$ constructed afterwards in terms of $C_0(\mathbb{R})\otimes C(P^{N-1}_\mathbb{R})$, and then at the free level, we can have $C_0(\mathbb{R}^N_+)$ constructed in terms of $C_0(\mathbb{R})*C(P^{N-1}_{\mathbb{R},+})$, and then look for Δ, and for h.

2d. In fact, in modern terms, we are looking for a "free suspension of the free sphere".

3. Compactification. Using $\mathbb{R}^N=S^N_\mathbb{R}-\{\infty\}$. To be more precise, we want to use the fact that $S^N_\mathbb{R}$ appears as the 1-point compactification of $\mathbb{R}^N$, with the isomorphism being the standard stereographic projection map. This might look like a weird idea, because it is not group-theoretical at all, the main feature of the stereographic projection being the fact that it is conformal, preserving angles, and so useful in geometry, but not in group theory. This being said, this is an idea to be explored too, especially since the formula for h should be not that complicated, and here are some preliminary computations:

Let us start with some abstract considerations. The 1-point compactification of $\mathbb{R}^N$ is indeed the sphere $S^N_\mathbb{R}$, and for precise formulae and everything, to be given later, the best is to say that the 1-point compactification of $\mathbb{R}^N=\mathbb{R}^N\times\{0\}\subset\mathbb{R}^{N+1}$ is the unit sphere $S^N_\mathbb{R}\subset\mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty=(1,0,\ldots,0)$. Also, we make the convention that the coordinates on $\mathbb{R}^{N+1}$ are denoted $x_0,\ldots,x_N$.

In functional analysis terms, we have a diagram as follows, with all horizontal maps being inclusions, with the bar on $C_0(\mathbb{R}^N)$ standing for unitization, and with the 0 subscript to $C(S^N_\mathbb{R})$ standing for taking the ideal generated by the first coordinate x_0:

$$\begin{array}{ccccc} C_0(\mathbb{R}^N) & \longrightarrow & \bar{C}_0(\mathbb{R}^N) & \longrightarrow & C_b(\mathbb{R}^N) \\ \| & & \| & & \\ C(S^N_\mathbb{R})_0 & \longrightarrow & C(S^N_\mathbb{R}) & & \end{array}$$

In view of our motivations, this is not bad, because in the free case we can normally talk as well about the ideal $C(S^N_{\mathbb{R},+})_0 \subset C(S^N_{\mathbb{R},+})$ generated by the first coordinate x_0. The problem is whether we can declare this ideal to be $C_0(\mathbb{R}^N_+)$, with a Δ and h.

In order to comment on this, let us do some computations, in the classical case. We first need the precise formulae of the isomorphism $\mathbb{R}^N \simeq S^N_\mathbb{R} - \{\infty\}$, obtained in practice by identifying $\mathbb{R}^N = \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ with the unit sphere $S^N_\mathbb{R} \subset \mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty = (1, 0, \ldots, 0)$, via the stereographic projection. That is, we need the precise formulae of two inverse maps, as follows:

$$\Phi : \mathbb{R}^N \to S^N_\mathbb{R} - \{\infty\}$$

$$\Psi : S^N_\mathbb{R} - \{\infty\} \to \mathbb{R}^N$$

In one sense we must have $\Phi(v) = t(0, v) + (1-t)(1, 0)$, with $t \in (0, 1)$ being such that $||\Phi(v)|| = 1$. The equation here is $(1-t)^2 + t^2||v||^2 = 1$, which simplifies to $t^2(1+||v||^2) = 2t$, with solution $t = \frac{2}{1+||v||^2}$, and so the formula of Φ is as follows:

$$\Phi(v) = (1, 0) + \frac{2}{1 + ||v||^2}(-1, v)$$

In the other sense we must have $(0, \Psi(c, x)) = \alpha(c, x) + (1 - \alpha)(1, 0)$ for a certain $\alpha \in \mathbb{R}$, and from $\alpha c + 1 - \alpha = 0$ we get $\alpha = \frac{1}{1-c}$, so the formula of Ψ is as follows:

$$\Psi(c, x) = \frac{x}{1 - c}$$

Here, as before, and in what follows too, we use $\mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R}^N$, with the coordinate of $\mathbb{R}$ denoted x_0, and with the coordinates of $\mathbb{R}^N$ denoted $x_1, \ldots, x_N$.

Let us discuss now the Δ problematics. We can transport the group structure of $\mathbb{R}^N$ to a group structure on $S^N_\mathbb{R} - \{\infty\}$, as follows:

$$p \cdot q = \Phi(\Psi(p) + \Psi(q))$$

In view of the above formulae of Φ,Ψ, the multiplication on $S^N_\mathbb{R}-\{\infty\}$ that we obtain is given by the following formula:

$$\begin{aligned}(c,x)\cdot(d,y)&=\Phi(\Psi(c,x)+\Psi(d,y))\\&=\Phi\left(\frac{x}{1-c}+\frac{y}{1-d}\right)\\&=(1,0)+\frac{2}{1+t}\left(-1,\frac{x}{1-c}+\frac{y}{1-d}\right)\end{aligned}$$

Here the parameter t is given by the following formula:

$$t=\left|\left|\frac{x}{1-c}+\frac{y}{1-d}\right|\right|^2$$

Now by transposing, we obtain a comultiplication map as follows, with $C(S^N_\mathbb{R})_0\subset C(S^N_\mathbb{R})$ being the ideal generated by the first coordinate x_0:

$$\Delta:C(S^N_\mathbb{R})_0\to C(S^N_\mathbb{R})_0\otimes C(S^N_\mathbb{R})_0$$

$$f\to\Big[(c,x),(d,y)\to f((c,x)\cdot(d,y))\Big]$$

The problem is that of slowly working out the details of this map Δ, on various products of coordinates and so on, and see if we can get a decent formula for Δ out of this, and then if this formula has a free generalization or not.

Let us discuss now the Haar problematics, which is the point where we wanted to get, where things might get simpler. As before with Δ, we can transport the Haar integration over $\mathbb{R}^N$ into an integration over $S^N_\mathbb{R}-\{\infty\}$, according to the following formula:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{\mathbb{R}^N}f(\Phi(v))dv$$

In practice, according to the above formula of Φ, the precise formula is:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{\mathbb{R}^N}f\left((1,0)+\frac{2}{1+||v||^2}(-1,v)\right)dv$$

Passed the details of this formula, which might look quite complicated, the transport of the Haar integration over $\mathbb{R}^N$ into an integration over $S^N_\mathbb{R}-\{\infty\}$ looks like something quite simple. Indeed, the measure on $S^N_\mathbb{R}-\{\infty\}$ should not be that far from the usual Haar measure of $S^N_\mathbb{R}$, with just a density added on the x_0 direction, and this because both measures, the transported one on $S^N_\mathbb{R}-\{\infty\}$, and the Haar one on $S^N_\mathbb{R}$, are invariant under the action of O_N, acting on the coordinates $x_1,\ldots,x_N$.

In short, we should have a formula as follows, with on the right the integration being the usual Haar one on $S^N_\mathbb{R}$, and with $\varphi:[-1,1]\to(0,\infty)$ being a certain density:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{S^N_\mathbb{R}}f(x)\varphi(x_0)dx$$

Assuming all this understood, and φ explicitely computed, the extension to the free case would be probably quite routine, our conjecture being that the integration on $\mathbb{R}^N_+$, in a "free stereographic picture", should be just a modification of the usual Weingarten formula for $S^N_{\mathbb{R},+}$, via a horizontal density $\psi:[-1,1]\to(0,\infty)$, appearing as the free version of $\varphi:[-1,1]\to(0,\infty)$, in the sense of the Bercovici-Pata bijection.

4d. Sums of squares

Another way of "escaping" from spheres, in the affine setting, is via various sums of squares, chosen to be more complicated than those defining the spheres. In order to discuss this, let us first study the compact hypersurfaces $X\subset\mathbb{R}^N_+$. These hypersurfaces fit into the C^*-algebra formalism, their definition being as follows:

Definition 4.17. *A real compact hypersurface in N variables, denoted $X_f\subset\mathbb{R}^N_+$, is the abstract spectrum of a universal C^*-algebra of the following type,*

$$C(X_f)=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,f(x_1,\ldots,x_N)=0\right)$$

with the noncommutative polynomial $f\in\mathbb{R}<x_1,\ldots,x_N>$ being such the maximal C^*-norm on the complex $*$-algebra $\mathbb{C}<x_1,\ldots,x_N>/(f)$ is bounded.

As a first result here, coming from the Gelfand theorem, we have:

Theorem 4.18. *In order for X_f to exist, the real algebraic manifold*

$$X_f^\times=\left\{x\in\mathbb{R}^N\Big|f(x_1,\ldots,x_N)=0\right\}$$

must be compact. In addition, in this case we have $||x_i||_\times\leq||x_i||$, for any i.

Proof. Assuming that X_f exists, the Gelfand theorem shows that the algebra of continuous functions on the manifold $X_f^\times$ in the statement appears as:

$$C(X_f^\times)=C(X_f)\Big/\Big\langle[x_i,x_j]=0\Big\rangle$$

Thus we have an embedding of compact quantum spaces $X_f^\times\subset X_f$, and the norm estimate is clear as well, because such embeddings increase the norms. □

Let us first discuss the quadratic case. We have here:

PROPOSITION 4.19. *Given a quadratic polynomial $f \in \mathbb{R} < x_1, \ldots, x_N >$, written as*

$$f = \sum_{ij} A_{ij} x_i x_j + \sum_i B_i x_i + C$$

the following conditions are equivalent:

1. X_f *exists.*
2. $X_f^\times$ *is compact.*
3. *The symmetric matrix* $Q = \frac{A+A^t}{2}$ *is positive or negative.*

PROOF. The implication (1) $\implies$ (2) being known from Theorem 4.18, and the implication (2) $\iff$ (3) being well-known, we are left with proving (3) $\implies$ (1). As a first remark here, by applying the adjoint, our manifold X_f is defined by:

$$\begin{cases} \sum_{ij} A_{ij} x_i x_j + \sum_i B_i x_i + C = 0 \\ \sum_{ij} A_{ij} x_j x_i + \sum_i B_i x_i + C = 0 \end{cases}$$

In terms of $P = \frac{A-A^t}{2}$ and $Q = \frac{A+A^t}{2}$, these equations can be written as:

$$\begin{cases} \sum_{ij} P_{ij} x_i x_j = 0 \\ \sum_{ij} Q_{ij} x_i x_j + \sum_i B_i x_i + C = 0 \end{cases}$$

Let us first examine the second equation. When regarding x as a column vector, and B as a row vector, this equation becomes an equality of 1×1 matrices, as follows:

$$x^t Q x + Bx + C = 0$$

Now let us assume that Q is positive or negative. Up to a sign change, we can assume $Q > 0$. We can write $Q = UDU^t$, with $D = diag(d_i)$ and $d_i > 0$, and with $U \in O_N$. In terms of the vector $y = U^t x$, and with $E = BU$, our equation becomes:

$$y^t D y + Ey + C = 0$$

By reverting back to sums and indices, this equation reads:

$$\sum_i d_i y_i^2 + \sum_i e_i y_i + C = 0$$

Now by making squares, this equation takes the following form:

$$\sum_i d_i \left(y_i + \frac{e_i}{2d_i} \right)^2 = c$$

By positivity, we deduce that we have the following estimate:

$$\left\| y_i + \frac{e_i}{2d_i} \right\|^2 \leq \frac{|c|}{d_i}$$

Thus our hypersurface X_f is well-defined, and we are done. □

We have in fact the following result:

Theorem 4.20. *Up to linear changes of coordinates, the free compact quadrics in* $\mathbb{R}^N_+$ *are the empty set, the point, the standard free sphere* $S^{N-1}_{\mathbb{R},+}$*, defined by*

$$\sum_i x_i^2 = 1$$

and some intermediate spheres $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$*, which can be explicitly characterized. Moreover, for all these free quadrics, we have* $||x_i|| = ||x_i||_\times$*, for any* i*.*

Proof. We use the computations from the proof of Proposition 4.19. The first equation there, making appear the matrix $P = \frac{A-A^t}{2}$, is as follows:

$$\sum_{ij} P_{ij} x_i x_j = 0$$

As for the second equation, up to a linear change of the coordinates, this reads:

$$\sum_i z_i^2 = c$$

At $c < 0$ we obtain the empty set. At $c = 0$ we must have $z = 0$, and depending on whether the first equation is satisfied or not, we obtain either a point, or the empty set. At $c > 0$ now, we can assume by rescaling $c = 1$, and our second equation reads:

$$X_f \subset S^{N-1}_{\mathbb{R},+}$$

As a conclusion, the solutions here are certain subspaces $S \subset S^{N-1}_{\mathbb{R},+}$ which appear via equations of type $\sum_{ij} P_{ij} x_i x_j = 0$, with $P \in M_N(\mathbb{R})$ being antisymmetric, and with $x_1, \ldots, x_N$ appearing via $z_1, \ldots, z_N$ via a linear change of variables. Now observe that when redoing the above computation with $X_f^\times$ at the place of X_f, we obtain $X_f = S^{N-1}_{\mathbb{R}}$, and this, because the equations $\sum_{ij} P_{ij} x_i x_j = 0$ are trivial for commuting variables. We conclude that our subspaces $S \subset S^{N-1}_{\mathbb{R},+}$ must satisfy:

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$$

Thus, we are left with investigating which such subspaces can indeed be solutions. Observe that both the extreme cases can appear as solutions, as shown by:

$$\begin{aligned} X_{2x^2+y^2+\frac{3}{2}xy+\frac{1}{2}yx} &= S^1_{\mathbb{R}} \\ X_{2x^2+y^2+xy+yx} &= S^1_{\mathbb{R},+} \end{aligned}$$

Finally, the last assertion is clear for the empty set and for the point, and for the remaining hypersurfaces, this follows from $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$. □

Here is now yet another version of Proposition 4.19:

Proposition 4.21. *Given M real linear functions $L_1,\ldots,L_M$ in N noncommuting variables $x_1,\ldots,x_N$, the following are equivalent:*

(1) *$\sum_k L_k(x_1,\ldots,x_N)^2=1$ defines a compact hypersurface in $\mathbb R^N$.*
(2) *$\sum_k L_k(x_1,\ldots,x_N)^2=1$ defines a compact quantum hypersurface.*
(3) *The matrix formed by the coefficients of $L_1,\ldots,L_M$ has rank N.*

Proof. The equivalence (1) $\iff$ (2) follows from the equivalence (1) $\iff$ (2) in Proposition 4.19, because the surfaces under investigation are quadrics. As for the equivalence (2) $\iff$ (3), this is well-known. More precisely, our equation read:

$$\begin{aligned}1&=\sum_k L_k(x_1,\ldots,x_N)^2\\&=\sum_k\sum_i L_{ki}x_i\sum_j L_{kj}x_j\\&=\sum_{ij}(L^tL)_{ij}x_ix_j\end{aligned}$$

Thus, in the context of Proposition 4.19, the underlying square matrix $A\in M_N(\mathbb R)$ is given by $A=L^tL$. It follows that we have $Q=A=L^tL$, and so the condition $Q>0$ is equivalent to L^tL being invertible, and so to L to have rank N, as claimed. □

In order to construct more examples, we will need the following basic fact:

Proposition 4.22. *In a C^*-algebra we have*

$$||x||^p\leq 1\implies ||x||\leq 1$$

for any self-adjoint element x.

Proof. With $n\in\mathbb N$ being such that $2^n\geq p$, we have:

$$||x||^{2^n}=||x^2||^{2^{n-1}}=\ldots=||x^{2^n}||\leq||x^p||\cdot||x^{2^n-p}||\leq 1\cdot||x||^{2^n-p}$$

Thus, we obtain $||x||^p\leq 1$, and so $||x||\leq 1$, as desired. □

As an application, we have the following construction:

Proposition 4.23. *Given integers $p_i\in\mathbb N$, the equation*

$$\sum_i x_i^{2p_i}=1$$

defines a noncommutative hypersurface.

Proof. This follows indeed from Proposition 4.22, by positivity. □

More generally, we have the following result, covering our various examples, so far:

PROPOSITION 4.24. *Given M real linear functions $L_1,\ldots,L_M$ in N noncommuting variables $x_1,\ldots,x_N$, and exponents $p_1,\ldots,p_M\in\mathbb N$, the equation*

$$\sum_k L_k(x_1,\ldots,x_N)^{2p_i}=1$$

defines a quantum hypersurface, provided that the $M\times N$ matrix formed by the coefficients of $L_1,\ldots,L_M$ has rank N.

PROOF. By positivity, imposing the above equation leads to:

$$||L_k(x_1,\ldots,x_N)||\leq 1\quad,\quad\forall k$$

We are therefore left with the problem of uniformly bounding the norms $||x_i||$, and normally we can proceed here exactly as in the classical case. □

More generally now, we have the following result:

THEOREM 4.25. *General construction of hypersurfaces, via equations of type*

$$\sum_k L_kL_k^*=1$$

with $L_k\in\mathbb R<x_1,\ldots,x_N>$, improving the construction from Proposition 4.24.

PROOF. This does not look obvious at all. As usual, there are some norm estimates to be worked out too, in relation with the basic inequality $||x_i||_\times\leq||x_i||$. □

Going beyond the above looks like a non-trivial question.

4e. Exercises

Exercises.

Part II

Free harmonics

Melody, tempo, harmony
I'll do it for me, to be new myself
Melody, tempo, harmony
Guitar base for me, lead and back vocals

CHAPTER 5

Laplace operator

5a.

5b.

5c.

5d.

5e. Exercises

CHAPTER 6

Harmonic functions

6a.

6b.

6c.

6d.

6e. Exercises

CHAPTER 7

Smooth structure

7a.

7b.

7c.

7d.

7e. Exercises

CHAPTER 8

Quotient spaces

8a.

8b.

8c.

8d.

8e. Exercises

Part III

Free equations

Open to everything happy and sad
Seeing the good, même si tout va si mal
Voir le soleil quand la nuit nous accable
Oh, pour un jour croire aux dieux, croire aux fables

CHAPTER 9

9a.

9b.

9c.

9d.

9e. Exercises

CHAPTER 10

10a.

10b.

10c.

10d.

10e. Exercises

CHAPTER 11

11a.

11b.

11c.

11d.

11e. Exercises

CHAPTER 12

12a.

12b.

12c.

12d.

12e. Exercises

Part IV

Free physics

She never drinks the water and
Makes you order French champagne
Once you've had a taste of her
You'll never be the same

CHAPTER 13

13a.

13b.

13c.

13d.

13e. Exercises

CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

CHAPTER 15

15a.

15b.

15c.

15d.

15e. Exercises

CHAPTER 16

16a.

16b.

16c.

16d.

16e. Exercises

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