Introduction to economics

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Abstract. This is an introduction to probability and game theory, social science, and economy and finance, by assuming basic calculus known. We first discuss the probability basics, motivated by simple questions, involving coins, dice and cards. Then we discuss the social science basics, with a look at life, human beings, history of mankind, and money. With this done, we get into economy and its mathematical modelling, first at large, by using various basic models, and then into its more modern aspects, involving present-day finance, and the mathematical methods for dealing with it.

Preface

You have certainly heard about probability and finance, and in fact already have some solid experience with it, coming from the real life. A game of poker played with friends, for instance, can teach you many things, how probabilities can be big or small, how they must be sometimes computed, or at least evaluated, in order to win, and also, importantly, how so many human factors must be taken into account, into your approach.

Perhaps you had the occasion of playing at the casino too. Pretty much the same fun here, with on the upside the casino housing and food, usually a bargain, and on the downside the play itself, where, as you probably know, on average, the casino wins.

You might perhaps have some experience with the stock market, or with crypto, too. Similar principles and fun as the casino, although this time, no food offered.

In short, good experience, and good experimental data, and the problem is, can we make a scientific theory, out of this? Normally yes, that theory being called "probability and finance", and with the present book being here to introduce you, to all this.

Some disclaimers, first. This book will not teach you how to win money, but rather what money is, and with this being something quite subtle, requiring good knowledge in both probability theory, and in social science, that we will both explain. So, this will be our plan, talking about probability, talking about social science, and then doing some mathematics for economy and finance, first in the old-style way, and then in more modern ways. As for winning money itself, this is something that we will not discuss here, but pay a good dinner to that old high-school friend who is now doing politics, and guy will certainly tell you all you need to know, when and what to buy, taxes and so on.

The book is organized in four parts, as follows:

I - Basic probability. This is an all-around introduction to probability, of somewhat hybrid type, lying between what we call at the university "probability and statistics", basic course mainly destined to biologists, geologists, economists and philosophers, and "measure theory", more advanced class, coming after calculus, destined to mathematicians, physicists, chemists and engineers. Both approaches are useful and interesting, and we will be doing here something in between, simply based on basic calculus.

II - Life, economy. This is something independent from Part I, all-around introduction to social science. We will first discuss life, biology and Darwin, matter of knowing who we are, and what we're doing on this planet. Then we will go into the history of mankind, with particular attention to the organization of society, economy and money, starting from old texts like the Bible and the Quran, up to more modern times. We will mostly rely here on Karl Marx, although Freud will make an appearance too, at the end.

III - Economy models. This will be our first attempt of fusion between Part I and Part II, that is, understanding economy via mathematics and probability. Our discussion here will be mostly at the abstract level, first by talking about mathematical games, equations for these games, stability and chaos, and other such notions that are needed, and then by applying this technology to very simple models of economy, with the main aim of looking for the outcome, that is, stability or chaos, or if you prefer, peace or war.

IV - Modern finance. This will be a continuation of Part III, and our second attempt of fusion between Part I and Part II, that is, doing economy via probability, but this time in the context of modern finance. The discussion here will be quite challenging, in view of the many financial opportunities offered by the modern world, such as the stock market, and the complexity of the various modern financial products. We will discuss all this, and at the end, we will have a look at the global economy, and sovereign debts.

In the hope that you will find this useful, and as a second disclaimer, I should probably mention that I am myself not an economist or a social scientist, but rather a mathematician, and to be more precise, mostly calculus professor in my teaching, advanced quantum physicist in my research, and amateur scientist at large, in the evenings. That is, Part I and Part III, which are mostly mathematical, are definitely my business, and are written in a professional way, I believe, while Parts II and IV, which are mostly social science, are more amateurish, "way a quantum physicist sees all this" being the idea there.

Many thanks go to my math and physics collaborators, who quite often are like me, amateur scientists at large in the evenings, for countless dinners and discussions, sometimes featuring all sorts of crazy ideas and theories, some of which will be reported here. Thanks as well to the internet, it is such a pleasure to have direct, instant access to all sorts of interesting conspiracy theories. Finally, many thanks to my cats, no idea about their economy knowledge, but when it comes to negotiating, they always win.

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Part I

Basic games

We were both young when I first saw you
I close my eyes and the flashback starts
I'm standing there
On a balcony in summer air

CHAPTER 1

Coins and dice

1a. Flipping coins

Welcome to probability. Probability theory can come in many flavors, and in our present academic system, worldwide, we basically have classes labeled "probability and statistics", which are introductory, for scientists at large, and then classes labeled "measure theory", which more advanced, assuming calculus known, and usually destined to hard scientists, such as mathematicians, physicists, chemists and engineers.

From the outside this looks like a reasonable system, take first a "probability and statistics" class, matter of learning the basics, and then, if needing more, take later a "measure theory" class, in order to fine-tune your knowledge, and learn further things. In practice, however, there is a bug with this, because "probability and statistics" is usually taught by using certain concepts, notations, theorems and so on, and then "measure theory" is taught by using completely different concepts, notations, theorems and so on. In a word, totally confusing all this, certainly us academics to be put at blame, and as cherry on the cake, all this happens, as many other things nowadays, worldwide.

The present Part I of this book is an introduction to "probability and statistics", that is, the probability basics that we will need later on, when doing economy. However, we have written it mostly by using concepts and notations from "measure theory", which are more advanced, and believe me, better. This way, you will certainly have no troubles in understanding what both camps are saying. Also, technically speaking, certain aspects of economy, to be investigated later on, can be linked to advanced math and physics disciplines such as random matrices, or thermodynamics, whose understanding rather requires the measure theory approach. So, we will be prepared for this too.

Getting started now, many things can be learned by flipping coins, and recording your findings. Let us start with something very basic, as follows:

FACT 1.1. *The probability of winning when flipping a coin is* $1/2$.

Obvious you would say, but there are some subtleties here, even in this simplest possible probability question. The first thing is that I said "winning", like everyone says when it comes to flipping coins, but winning against whom?

So, this is a first subtlety. Flipping a coin is best regarded as being a game, with you choosing between heads and tails, let us say heads, then flipping the coin, and winning if heads. But now, that we talked about a game, you need a partner for your game. That is, you are not playing a game alone, but with someone else, who wins when it's tails.

Which brings us into a second question, winning what? Many options here, like winning apples, or oranges, or luxury cars, assuming that both you and your partner have a considerable stock of those. Or why not, for making the game even more exciting, the right to slap your partner, or why not pulling a knife, and killing your partner.

So, what to choose? The answer here is money, that is what money is made for, for simplifying such things, transactions between humans. In the hope that we agree on this, and now with this discussion made, let us record our findings, as follows:

Conclusion 1.2. *Flipping a coin is best regarded as being a game, between you and a partner, the rules being:*

(1) *Every time it is heads, you win* \$1 *from your partner.*
(2) *Every time it is tails, your partner wins* \$1 *from you.*

With this conclusion recorded, we can see now more clearly what is behind coin flipping. Obviously, all sorts of interesting things that we can explore, and we will do that, and with the main question, which is surely on everyone's mind, being:

Question 1.3. *Who wins?*

So, let us study now this question. What we know so far about flipping coins are Fact 1.1 and Conclusion 1.2, and with these being independent things, because the number 1/2, which was the main content of Fact 1.2, does not appear in Conclusion 1.2. So, it is now a matter of understanding how the game axiomatized in Conclusion 1.2 evolves over the time, taking into account the 1/2 mathematics from Fact 1.1.

Here are a few preliminary observations, about this, and we will call this "Proposition", as mathematicians do for their statements, coming with full mathematical proof:

Proposition 1.4. *When flipping a coin k times, the following happen,*

(1) *The probability of you winning* $\$k$ *is* $1/2^k$.
(2) *The probability of you winning* $\$k-1$ *is* 0.
(3) *The probability of you winning* $\$k-2$ *is* $k/2^k$.
(4) *The probability of you winning* $\$k-3$ *is again* 0.
(5) *The probability of you winning* $\$k-4$ *is* $k(k-1)/2^{k+1}$.

and so on, with the probability increasing, up to the tie situation, and then decreasing.

Proof. This follows indeed from some simple mathematics, as follows:

(1) You winning \$$k$ means you winning every time, over k attempts, so your probability here is $P = (1/2) \times \ldots \times (1/2)$, with k terms in the product, which reads $P = 1/2^k$.

(2) The point here is that you cannot win \$$k-1$, exactly. Indeed, you must lose at least once, and so you profit will be $\leq (k-1) - 1 = k-2$.

(3) Here we have a similar computation as in (1). For winning \$$k-2$ you need to lose exactly once, and since there are k possibilities of losing exactly once, $P = k/2^k$.

(4) Here the situation is similar to that in (2). Indeed, for winning exactly \$$k-3$ you would certainly need to lose twice, so you profit will be $\leq (k-2) - 2 = k-4$.

(5) With the same reasoning as in (3), here you need to lose exactly twice, and since there are $k(k-1)/2$ possibilities of losing exactly twice, $P = k(k-1)/2^{k+1}$.

(6) Finally, regarding the last assertion, which is a bit informal, we will leave the clarification here, both statement and proof, to you, as an instructive exercise. □

Obviously, some interesting mathematics is going on here, that needs to be better understood, before tackling Question 1.3. As a first requirement, we must review the theory of binomial numbers, which obviously appear in the above. Let us start with:

Theorem 1.5. *The number of possibilities of choosing k objects among n objects is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

called binomial number, where $n! = 1 \cdot 2 \cdot 3 \ldots (n-2)(n-1)n$, called "factorial n".

Proof. Imagine a set consisting of n objects. We have n possibilities for choosing our 1st object, then $n-1$ possibilities for choosing our 2nd object, out of the $n-1$ objects left, and so on up to $n-k+1$ possibilities for choosing our k-th object, out of the $n-k+1$ objects left. Since the possibilities multiply, the total number of choices is:

$$\begin{aligned} N &= n(n-1)\ldots(n-k+1) \\ &= n(n-1)\ldots(n-k+1) \cdot \frac{(n-k)(n-k-1)\ldots 2 \cdot 1}{(n-k)(n-k-1)\ldots 2 \cdot 1} \\ &= \frac{n(n-1)\ldots 2 \cdot 1}{(n-k)(n-k-1)\ldots 2 \cdot 1} \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

However, thinking a bit, the number N that we computed is in fact the number of possibilities of choosing k ordered objects among n objects. Thus, we must divide everything by the number M of orderings of the k objects that we chose:

$$\binom{n}{k}=\frac{N}{M}$$

In order to compute now the missing number M, imagine a set consisting of k objects. We have then k choices for the object to be designated #1, then $k-1$ choices for the object to be designated #2, and so on up to 1 choice for the object to be designated $\#k$. We conclude that we have $M=k(k-1)\ldots 2\cdot 1=k!$, and so that we have:

$$\binom{n}{k}=\frac{n!/(n-k)!}{k!}=\frac{n!}{k!(n-k)!}$$

Thus, we are led to the conclusion in the statement. □

The above was quite tricky, and as a key complement to it, we must add:

CONVENTION 1.6. *By definition, we have the formula*

$$0!=1$$

and this, in order to have $\binom{n}{n}=1$*, as we should.*

To be more precise here, we certainly have $\binom{n}{n}=1$, because there is exactly 1 choice for n objects among n objects. Thus, we must declare that we have $0!=1$, as for the following computation, based on the formula in Theorem 1.5, to work indeed:

$$\binom{n}{n}=\frac{n!}{n!0!}=\frac{n!}{n!\times 1}=1$$

Going ahead now with more mathematics, we have the following key result:

THEOREM 1.7. *We have the binomial formula*

$$(a+b)^n=\sum_{k=0}^n\binom{n}{k}a^kb^{n-k}$$

valid for any two numbers $a,b\in\mathbb{R}$.

PROOF. We have to compute the following quantity, with n terms in the product:

$$(a+b)^n=(a+b)(a+b)\ldots(a+b)$$

When expanding, we obtain a certain sum of products of a,b variables, with each such product being a quantity of type a^kb^{n-k}. Thus, we have a formula as follows:

$$(a+b)^n=\sum_{k=0}^n C_ka^kb^{n-k}$$

In order to finish, it remains to compute the coefficients C_k. But, according to our product formula, C_k is the number of choices for the k needed a variables among the n available a variables. Thus, according to Theorem 1.5, we have:

$$C_k = \binom{n}{k}$$

We are therefore led to the formula in the statement. □

Theorem 1.7 is something quite interesting, so let us doublecheck it with some numerics. At small values of n we obtain the following formulae, which are all correct:

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

$$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$

$$\vdots$$

Now observe that in these formulae, say for memorization purposes, the powers of the a,b variables are something very simple, that can be recovered right away. What matters are the coefficients, which are the binomial coefficients $\binom{n}{k}$, which form a triangle. So, it is enough to memorize this triangle, and this can be done by using:

THEOREM 1.8. *The Pascal triangle, formed by the binomial coefficients* $\binom{n}{k}$,

$$1$$

$$1\ ,\ 1$$

$$1\ ,\ 2\ ,\ 1$$

$$1\ ,\ 3\ ,\ 3\ ,\ 1$$

$$1\ ,\ 4\ ,\ 6\ ,\ 4\ ,\ 1$$

$$1\ ,\ 5\ ,\ 10\ ,\ 10\ ,\ 5\ ,\ 1$$

$$1\ ,\ 6\ ,\ 15\ ,\ 20\ ,\ 15\ ,\ 6\ ,\ 1$$

$$\vdots$$

has the property that each entry is the sum of the two entries above it.

PROOF. In practice, the theorem states that the following formula must hold:

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

There are many ways of proving this formula, all instructive, as follows:

(1) Brute-force computation. We have indeed, as desired:

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k}&=\frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\&=\frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right)\\&=\frac{(n-1)!}{(k-1)!(n-k-1)!}\cdot\frac{n}{k(n-k)}\\&=\binom{n}{k}\end{aligned}$$

(2) Algebraic proof. We have the following formula, to start with:

$$(a+b)^n=(a+b)^{n-1}(a+b)$$

By using now the binomial formula, this formula becomes:

$$\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}=\left[\sum_{r=0}^{n-1}\binom{n-1}{r}a^rb^{n-1-r}\right](a+b)$$

Now let us perform the multiplication on the right. We obtain a certain sum of terms of type a^kb^{n-k}, and to be more precise, each such a^kb^{n-k} term can either come from the $\binom{n-1}{k-1}$ terms $a^{k-1}b^{n-k}$ multiplied by a, or from the $\binom{n-1}{k}$ terms a^kb^{n-1-k} multiplied by b. Thus, the coefficient of a^kb^{n-k} on the right is $\binom{n-1}{k-1}+\binom{n-1}{k}$, as desired.

(3) Combinatorics. Let us count k objects among n objects, with one of the n objects having a hat on top. Obviously, the hat has nothing to do with the count, and we obtain $\binom{n}{k}$. On the other hand, we can say that there are two possibilities. Either the object with hat is counted, and we have $\binom{n-1}{k-1}$ possibilities here, or the object with hat is not counted, and we have $\binom{n-1}{k}$ possibilities here. Thus $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$, as desired. □

There are many more things that can be said about binomial coefficients, with all sorts of interesting formulae, but the idea is always the same, namely that in order to find such formulae you have a choice between algebra and combinatorics, and that when it comes to proofs, the brute-force computation method is useful too. In practice, the best is to master all 3 techniques. Among others, because you will have in this way 3 different methods, for making sure that your formulae are correct indeed.

Getting back now to probability, the above mathematics is all we need. We are led to the following Theorem, generalizing our findings from Proposition 1.4:

THEOREM 1.9. *When flipping a coin k times what you can win are quantities of type $\$k-2s$, with $s=0,1,\ldots,k$, with the probability for this to happen being:*

$$P(k-2s)=\frac{1}{2^k}\binom{k}{s}$$

Geometrically, your winning curve starts with probability $1/2^k$ of winning $-\$k$, then increases up to the tie situation, and then decreases, up to probability $1/2^k$ of winning $\$k$.

PROOF. All this is quite clear, by fine-tuning our various observations from Proposition 1.4 and its proof, the whole point here being that, in order for you to win $k-s$ times and lose s times, over your k attempts, the number of possibilities is:

$$\binom{k}{s}=\frac{k!}{s!(k-s)!}$$

Thus, by dividing now by 2^k, which is the total number of possibilities, for the whole game, we are led to the probability in the statement, namely:

$$P(k-2s)=\frac{1}{2^k}\binom{k}{s}$$

Shall we doublecheck this? Sure yes, doublecheking is the first thing to be done, when you come across a theorem, in your mathematics. As a first check, the sum of probabilities that we found should be 1, which is intuitive, right, and 1 that is, as shown by:

$$\begin{aligned}\sum_{s=0}^k P(k-2s) &= \sum_{s=0}^k\frac{1}{2^k}\binom{k}{s}\\ &= \frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}\\ &= \frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}1^s1^{k-s}\\ &= \frac{1}{2^k}(1+1)^k\\ &= \frac{1}{2^k}\times 2^k\\ &= 1\end{aligned}$$

But shall we really trust this. Imagine for instance that you play your game for \$1000 instead of \$1 as basic gain, your life is obviously at stake, so all this is worth a second doublecheck, before being used in practice. So, as second doublecheck, let us verify that, on average, what you win is exactly \$0, which is something very intuitive, the game itself obviously not favoring you, nor your partner. But this can be checked as follows:

$$\begin{aligned}
\sum_{s=0}^k P(k-2s)\times(k-2s) &= \frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}(k-2s)\\
&= \frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}(k-s)-\frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}s\\
&= \frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}(k-s)-\frac{1}{2^k}\sum_{t=0}^k\binom{k}{k-t}(k-t)\\
&= \frac{1}{2^k}\sum_{s=0}^k\binom{k}{s}(k-s)-\frac{1}{2^k}\sum_{t=0}^k\binom{k}{t}(k-t)\\
&= 0
\end{aligned}$$

Here we have used a change of indices, namely $s = k - t$, along with the following formula, which is clear from the definition of binomial coefficients:

$$\binom{k}{t}=\binom{k}{k-t}$$

Summarizing, we have a good theorem here, proved, doublechecked and triplechecked, as per the highest scientific standards, ready to be used in practice. □

With Theorem 1.9 in hand, we are somehow done with math, and time now to turn to Question 1.3. Let us first examine a more concrete question, namely:

QUESTION 1.10. *What and how do you win, depending on your strategy?*

However, this appears to be a bit of a bad question, at least in the context of our very simple flipping game, because you have not so many options for developing a strategy. To be more precise, the only thing that you can do, as strategy, is that of pulling off the game, once you won enough money. And even this is something debatable, because you pulling off at the moment of your choice assumes that the rules are biased, favoring you. So, well, let us do this, and reformulate our strategy question as follows:

QUESTION 1.11. *Assuming that the rules are biased, favoring you, by allowing you to pull off at any moment of your choice, what is your best strategy?*

And with this, we are now straight into popular mathematics, because everyone in a casino, or buying lottery tickets, or doing stocks or crypto in front of a computer, thinks about such things, and at a highest possible seriousness level. You won't mess up things with your own money, hardly won via hard daily labor, won't you.

In relation with this, the legend goes that what you have to do is play and play, until you reached a sum of money that you fixed as objective in advance, say \$100. Then you pull off, with the money in your pocket. Simple like that.

So, let us see how this works. To start with, this can only work, I mean just play and play, as indicated above, and you will certainly end up with \$100 in your pocket, no question about it. However, this might take some precious time t, and the mathematics, based on our formula in Theorem 1.9, shows that this time t is as follows:

Time spent playing	Probability to win
$t = 100$	$1/2^{100} = 0.000$
$t = 102$	$102/2^{102} = 0.000$
$t = 104$	$\binom{104}{2}/2^{104} = 0.000$
$t = 106$	$\binom{106}{3}/2^{106} = 0.000$
$t = 108$	$\binom{108}{4}/2^{108} = 0.000$
$t = 110$	$\binom{110}{5}/2^{110} = 0.000$
$\vdots$	$\vdots$

Which does not loook very good, hope you agree with me. Obviously, we are here into some sort of very abstract math, not corresponding to anything in the real life. So, in order to reach to something more reasonable, good moment to remember that:

FACT 1.12. *Time is money.*

In view of this, let us downgrade our ambitions, and only wish to win a modest \$10. Here we reach to a more reasonable winning scheme, as follows:

Time spent playing	Probability to win
$t = 10$	$1/2^{10} = 0.001$
$t = 12$	$12/2^{12} = 0.003$
$t = 14$	$\binom{14}{2}/2^{14} = 0.006$
$t = 16$	$\binom{16}{3}/2^{16} = 0.009$
$t = 18$	$\binom{18}{4}/2^{18} = 0.012$
$t = 20$	$\binom{20}{5}/2^{20} = 0.015$
$\vdots$	$\vdots$

However, this is still not interesting, financially speaking, so in order to reach to something more viable, let us further downgrade our ambitions, and only wish to win a very modest \$5. And here, we reach to something more attractive, as follows:

Time spent playing	Probability to win
$t = 5$	$1/2^5 = 0.031$
$t = 7$	$7/2^7 = 0.055$
$t = 9$	$\binom{9}{2}/2^9 = 0.070$
$t = 11$	$\binom{11}{3}/2^{11} = 0.081$
$t = 13$	$\binom{13}{4}/2^{13} = 0.087$
$t = 15$	$\binom{15}{5}/2^{15} = 0.092$
$\vdots$	$\vdots$

But this still does not look very good, so going now for the true way of reason, let us simply wish to win a tiny \$3. And here, the situation becomes as follows:

Time spent playing	Probability to win
$t = 3$	$1/2^3 = 0.125$
$t = 5$	$5/2^5 = 0.156$
$t = 7$	$\binom{7}{2}/2^7 = 0.164$
$t = 9$	$\binom{9}{3}/2^9 = 0.164$
$t = 11$	$\binom{11}{4}/2^{11} = 0.163$
$t = 13$	$\binom{13}{5}/2^{13} = 0.157$
$\vdots$	$\vdots$

Which is sort of reasonable, but not really, observe for instance that the probabilities on the right start decreasing, and before putting this scheme into practice, we must probably do some more math, make sure that these probabilities won't start to decrease very sharply, which might complicate our business, and so on.

Moving ahead now, we talked in the above about "time is money", which is something that must be taken into account, but thinking well, what really matters in all this is the maximum amount of money that you can afford to lose. Which is something quite subtle, not included in our modelling above. So, let us further reformulate our strategy question, by making it more realistic, in touch with what happens in the real life, as follows:

QUESTION 1.13. *What is your best strategy, assuming that the game is asymmetric:*

(1) *With the rules being biased, favoring you, by allowing you to pull off from the game, at any moment of your choice.*
(2) *With the capital being unequal, favoring your partner, who has N money that he can afford to lose, compared to your $n < N$ money.*
(3) *And perhaps with a fee for playing the game too, again favoring your partner, to be paid by you, and this because N, n are normally secret.*

And good news, this is the good, final question, which perfectly makes sense, and is fully realistic. There is some math to be done here, but we will rather defer the discussion to Part III below, when we will systematically investigate such games. However, we can solve a simple case now, namely that when your partner has endless money:

$$N = \infty$$

A player having this feature is called "the bank", and with this convention made, the answer to our various questions, and notably to Question 1.3 that we started with, is:

ANSWER 1.14. *The bank wins.*

To be more precise here, as already mentioned, we can certainly do some math here, and we will certainly do this later. But, for our purposes now, in this opening chapter, which are mostly introductory and philosophical, the simplest is to argue that, in your situation, when you have $\$n$ and you lose $\$n$, say with $n = 1,000,000$ for having a precise figure, you are dead, say with this coming from fentanyl overdose, after reaching the street, after your bankruptcy. So, your strategy of pulling off once you won a precise sum of money, say \$100, is certainly flawed, because you can meet death on the way:

Time spent playing	Probability to win	Other outcomes
$t = 100$	small	losing
$t = 102$	small	losing
$t = 104$	small	losing
$\vdots$	$\vdots$	$\vdots$
$t = 1,000,000$	attractive	death
$t = 1,000,002$	attractive	death
$t = 1,000,004$	attractive	death
$\vdots$	$\vdots$	$\vdots$

As for the variations of this strategy, these can be certainly investigated too, but it is quite clear that all this will not lead to anything good, because originally you were there happy, looking for a strategy for winning the game, but all of the sudden, the rule (2) from Question 1.13 puts you in a very defensive situation, more caring about your life, than of winning the game. So, it is pretty much clear that we are led to Answer 1.14.

We will come back with details on all this, in Part III below, as promised.

As a conclusion now to all this, and leaving aside the precise coin game that we were playing, "be the bank" is the winning strategy. But what exactly is the bank, how to be the bank, and so on. Stay tuned, we have the whole book for discussing this.

1b. Rolling dice

At a more advanced level, we can roll dice. The difference with the coins comes from the fact that the $1/2-1/2$ basic probabilities at coins, which quite often can lead to coincidences and confusions, get now replaced by a fully readable $1/6-5/6$.

To be more precise, let us first convene for the following rules for the game:

RULES 1.15. *Rolling the die is played with the following rules:*

(1) *Every time it is* $1,2,3,4,5$*, your partner wins* \$1 *from you.*
(2) *And every time it is* 6*, you win* \$5 *from your partner.*

Of course, you might say that this is not very standard, but hey, we are just doing some math here, and we will complicate the rules later on, no worries for that. Now with these rules agreed on, we have the following analogue of Theorem 1.9:

THEOREM 1.16. *When rolling a die k times what you can win are quantities of type* $\$6w-k$*, with* $w=0,1,\ldots,k$*, with the probability for this to happen being:*

$$P(6w-k)=\frac{5^{k-w}}{6^k}\binom{k}{w}$$

Geometrically, your winning curve starts with probability $(5/6)^k$ *of losing* $\$k$*, then increases, up to some point, and then decreases, up to probability* $1/6^k$ *of winning* $\$5k$*.*

PROOF. There are several things going on here, the idea being as follows:

(1) When rolling the die k times, what will happen is that you will win w times and lose l times, with $k=w+l$. And in this situation, your profit will be, as stated:

$$\begin{aligned}\$ &= 5w-l\\ &= 5w-(k-w)\\ &= 6w-k\end{aligned}$$

(2) As for the probability for this to happen, this is the total number of possibilities for you to win w times, which is $5^{k-w}\binom{k}{w}$, because this amounts in choosing the w times when you will win, among k, then multiplying by 5^{k-w} possibilities, at places where your partner wins, and finally dividing by the total number of possibilities, which is 6^k:

$$P(6w-k)=\frac{5^{k-w}}{6^k}\binom{k}{w}$$

(3) As usual when doing complicated math, let us doublecheck all this, matter of being sure that we did not mess up our counting. First, the sum of all probabilities involved

must be 1, and 1 that sum is, as shown by the following computation:

$$\begin{aligned}
\sum_{w=0}^{k} P(6w-k) &= \sum_{w=0}^{k} \frac{5^{k-w}}{6^k}\binom{k}{w}5^{k-w} \\
&= \frac{1}{6^k}\sum_{w=0}^{k}\binom{k}{w} \\
&= \frac{1}{6^k}\sum_{w=0}^{k}\binom{k}{w}1^w 5^{k-w} \\
&= \frac{1}{6^k}(1+5)^k \\
&= \frac{1}{6^k}\times 6^k \\
&= 1
\end{aligned}$$

(4) Let us triplecheck this as well. Obviously, Rules 1.15 do not favor you, nor your partner, so on average, you should win 0. And this is the case indeed, because:

$$\begin{aligned}
\sum_{w=0}^{k} P(6w-k)\times(6w-k) &= \frac{1}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k}{w}(6w-k) \\
&= \frac{1}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k}{w}5w - \frac{1}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k}{w}(k-w) \\
&= \frac{5}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k}{w}w - \frac{1}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k}{w}(k-w) \\
&= \frac{5k}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k-1}{w-1} - \frac{k}{6^k}\sum_{w=0}^{k} 5^{k-w}\binom{k-1}{w} \\
&= \frac{5k}{6^k}(1+5)^{k-1} - \frac{5k}{6^k}(1+5)^{k-1} \\
&= 0
\end{aligned}$$

(5) This last computation was hot, wasn't it, but triplechecks are mandatory. In any case theorem proved, and the final conclusions in the statement are clear too. □

Quite interestingly, Theorem 1.16 is best seen, both at the level of the statement, and of the proof, from the viewpoint of your partner. Let us record this, as follows:

Theorem 1.17. *When rolling a die k times what you can win are quantities of type $\$5k-6l$, with $l=0,1,\ldots,k$, with the probability for this to happen being:*

$$P(5k-6l)=\frac{5^l}{6^k}\binom{k}{l}$$

Geometrically, your winning curve starts with probability $(5/6)^k$ of losing $\$k$, then increases, up to some point, and then decreases, up to probability $1/6^k$ of winning $\$5k$.

Proof. As before, when rolling the die k times, you will win w times and lose l times, with $k=w+l$. And in this situation, your profit will be, as stated:

$$\begin{aligned}\$ &= 5w-l\\ &= 5(k-l)-l\\ &= 5k-6l\end{aligned}$$

As for the rest, we already know all this from Theorem 1.16, but the point is that the proof of Theorem 1.16 becomes slightly simpler when using l instead of w. □

Now with Theorem 1.17 in hand, it is quite clear that the basic $1/6-5/6$ probabilities at dice can be repaced with something of type $p-(1-p)$, with $p\in[0,1]$ being arbitrary. We are led in this way to the following notions, which are quite general:

Definition 1.18. *Given $p\in[0,1]$, the Bernoulli law of parameter p is given by:*

$$P(\text{win})=p\quad,\quad P(\text{lose})=1-p$$

More generally, the k-th binomial law of parameter p, with $k\in\mathbb{N}$, is given by

$$P(s)=p^s(1-p)^{k-s}\binom{k}{s}$$

with the Bernoulli law appearing at $k=1$, with $s=1,0$ here standing for win and lose.

To be more precise, what we call here "law" is something intuitive, based on what we did before with coins and dice, basically standing for "outcome of a game". As a first observation, the Bernoulli law generalizes indeed what we did before with coins and dice, which come respectively from the following choices of the parameter $p\in[0,1]$:

$$p_{coin}=1/2\quad,\quad p_{die}=1/6$$

Observe also that the last assertion holds indeed, because at $k=1$ the binomial law is as follows, coinciding indeed with the Bernoulli law of parameter p:

$$P(1)=p\quad,\quad P(0)=1-p$$

Finally, regarding the binomial law, observe that is indeed a "law", or what we can expect from a game, because the various probabilities sum up to 1, as they should:

$$\begin{aligned}\sum_{s=0}^{k} P(s) &= \sum_{s=0}^{k} p^s(1-p)^{k-s}\binom{k}{s} \\ &= (p+(1-p))^k \\ &= 1\end{aligned}$$

Let us try now to better understand the relation between the Bernoulli and binomial laws. Indeed, we know from both coins and dice that the Bernoulli laws produce the binomial laws, simply by iterating the game, from 1 throw to $k \in \mathbb{N}$ throws.

The reasons behind this obviously come from the "independence" of our coin or dice throwings, when iterating. Let us record this finding, as follows:

CONCLUSION 1.19. *The Bernoulli laws produce the binomial laws, by iterating the game, via the independence of the throws.*

Of course, this finding is something quite intuitive, and temporary, and it still remains to work out the precise mathematics of independence, producing the explicit formula of the binomial laws, out of the explicit formula of the Bernoulli laws. We will discuss this later, but coming a bit in advance, here is the answer to this question:

(1) The idea is to encapsulate the data from Definition 1.18 into so-called "probability measures" associated to the Bernoulli and binomial laws. For the Bernoulli law, the corresponding measure is as follows, with the δ symbols standing for Dirac masses:

$$\mu_{ber} = (1-p)\delta_0 + p\delta_1$$

As for the binomial law, here the measure is as follows, constructed in a similar way, you get the point I hope, again with the δ symbols standing for Dirac masses:

$$\mu_{bin} = \sum_{s=0}^{k} p^s(1-p)^{k-s}\binom{k}{s}\delta_s$$

(2) Getting now to independence, and to our finding from Conclusion 1.19, the mathematics there is that of the following formula, with $*$ standing for the convolution operation for real measures, which on Dirac masses is simply given by $\delta_x * \delta_y = \delta_{x+y}$:

$$\mu_{bin} = \underbrace{\mu_{ber} * \ldots * \mu_{ber}}_{k\ terms}$$

(3) To be more precise, this latter formula does hold indeed, as a straightforward application of the binomial formula, the formal proof being as follows:

$$\begin{aligned}\mu_{ber}^{*k} &= \big((1-p)\delta_0+p\delta_1\big)^{*k}\\ &= \sum_{s=0}^k p^s(1-p)^{k-s}\binom{k}{s}\delta_0^{*(k-s)}*\delta_1^{*s}\\ &= \sum_{s=0}^k p^s(1-p)^{k-s}\binom{k}{s}\delta_s\\ &= \mu_{bin}\end{aligned}$$

All this is very nice, and is perhaps worth a reformulation of Conclusion 1.19. We reach in this way to a quite drastic statement, as follows:

Conclusion 1.20. *Most of what we did with coins and dice reduces to the formula*

$$\mu_{ber}^{*k}=\mu_{bin}$$

relating the Bernoulli and binomial laws, via the convolution operation $*$.

And isn't this magic. We have proof here for the abstract power of mathematics. Or perhaps of physics, because the Dirac masses, involved in all this, come from Dirac.

Moving ahead now, in relation with all this, it is very useful, in practice, to be able to estimate the binomials and factorials. And here, we have the following formula:

Theorem 1.21. *We have the Stirling formula*

$$N!\simeq\left(\frac{N}{e}\right)^N\sqrt{2\pi N}$$

valid in the $N\to\infty$ *limit.*

Proof. This is something quite tricky, based on heavier calculus methods that will get into later, in chapter 2, but assuming some knowledge here, the idea is as follows:

(1) Let us first see what we can get with Riemann sums. We have:

$$\begin{aligned}\log(N!) &= \sum_{k=1}^N\log k\\ &\approx \int_1^N\log x\,dx\\ &= N\log N-N+1\end{aligned}$$

By exponentiating, this gives the following estimate, which is not bad:

$$N! \approx \left(\frac{N}{e}\right)^N \cdot e$$

(2) We can improve our estimate by replacing the rectangles from the Riemann sum approach to the integrals by trapezoids. In practice, this gives the following estimate:

$$\begin{aligned}
\log(N!) &= \sum_{k=1}^N \log k \\
&\approx \int_1^N \log x\,dx + \frac{\log 1 + \log N}{2} \\
&= N\log N - N + 1 + \frac{\log N}{2}
\end{aligned}$$

By exponentiating, this gives the following estimate, which gets us closer:

$$N! \approx \left(\frac{N}{e}\right)^N \cdot e \cdot \sqrt{N}$$

(3) In order to conclude, we must take some kind of mathematical magnifier, and carefully estimate the error made in (2). Fortunately, this mathematical magnifier exists, called Euler-Maclaurin formula, and after some tough computations, we get to:

$$N! \simeq \left(\frac{N}{e}\right)^N \sqrt{2\pi N}$$

(4) However, all this remains a bit complicated, so we would like to present now an alternative approach to (3), which also misses some details, but better does the job, explaining where the $\sqrt{2\pi}$ factor comes from. First, by partial integration we have:

$$N! = \int_0^\infty x^N e^{-x} dx$$

Since the integrand is sharply peaked at $x = N$, as you can see by computing the derivative of $\log(x^N e^{-x})$, this suggests writing $x = N + y$, and we obtain:

$$\begin{aligned}
\log(x^N e^{-x}) &= N\log x - x \\
&= N\log(N+y) - (N+y) \\
&= N\log N + N\log\left(1 + \frac{y}{N}\right) - (N+y) \\
&\simeq N\log N + N\left(\frac{y}{N} - \frac{y^2}{2N^2}\right) - (N+y) \\
&= N\log N - N - \frac{y^2}{2N}
\end{aligned}$$

By exponentiating, we obtain from this the following estimate:

$$x^N e^{-x} \simeq \left(\frac{N}{e}\right)^N e^{-y^2/2N}$$

Now by integrating, and using the Gauss formula, we obtain from this:

$$\begin{aligned}
N! &= \int_0^\infty x^N e^{-x} dx \\
&\simeq \int_{-N}^N \left(\frac{N}{e}\right)^N e^{-y^2/2N}\, dy \\
&\simeq \left(\frac{N}{e}\right)^N \int_{\mathbb{R}} e^{-y^2/2N}\, dy \\
&= \left(\frac{N}{e}\right)^N \sqrt{2N} \int_{\mathbb{R}} e^{-z^2}\, dz \\
&= \left(\frac{N}{e}\right)^N \sqrt{2\pi N}
\end{aligned}$$

Thus, we have proved the Stirling formula, as formulated in the statement. □

With the above formula in hand, we have many useful applications, such as:

THEOREM 1.22. *We have the following estimate for the binomial coefficients,*

$$\binom{N}{K} \simeq \left(\frac{1}{t^t(1-t)^{1-t}}\right)^N \frac{1}{\sqrt{2\pi t(1-t)N}}$$

in the $K \simeq tN \to \infty$ limit, with $t \in (0,1]$. In particular we have

$$\binom{2N}{N} \simeq \frac{4^N}{\sqrt{\pi N}}$$

in the $N \to \infty$ limit, for the central binomial coefficients.

PROOF. All this is very standard, by using the Stirling formula etablished above, for the various factorials which appear, the idea being as follows:

(1) This follows from the definition of the binomial coefficients, namely:

$$\begin{aligned}\binom{N}{K} &= \frac{N!}{K!(N-K)!}\\ &\simeq \left(\frac{N}{e}\right)^N\sqrt{2\pi N}\left(\frac{e}{K}\right)^K\frac{1}{\sqrt{2\pi K}}\left(\frac{e}{N-K}\right)^{N-K}\frac{1}{\sqrt{2\pi(N-K)}}\\ &= \frac{N^N}{K^K(N-K)^{N-K}}\sqrt{\frac{N}{2\pi K(N-K)}}\\ &\simeq \frac{N^N}{(tN)^{tN}((1-t)N)^{(1-t)N}}\sqrt{\frac{N}{2\pi tN(1-t)N}}\\ &= \left(\frac{1}{t^t(1-t)^{1-t}}\right)^N\frac{1}{\sqrt{2\pi t(1-t)N}}\end{aligned}$$

Thus, we are led to the conclusion in the statement.

(2) This estimate follows from a similar computation, as follows:

$$\begin{aligned}\binom{2N}{N} &= \frac{(2N)!}{N!N!}\\ &\simeq \left(\frac{2N}{e}\right)^{2N}\sqrt{4\pi N}\left(\frac{e}{N}\right)^{2N}\frac{1}{2\pi N}\\ &= \frac{4^N}{\sqrt{\pi N}}\end{aligned}$$

Alternatively, we can take $t = 1/2$ in (1), then rescale. Indeed, we have:

$$\begin{aligned}\binom{N}{[N/2]} &\simeq \left(\frac{1}{(\frac{1}{2})^{1/2}(\frac{1}{2})^{1/2}}\right)^N\frac{1}{\sqrt{2\pi\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot N}}\\ &= 2^N\sqrt{\frac{2}{\pi N}}\end{aligned}$$

Thus with the change $N \to 2N$ we obtain the formula in the statement. □

Finally, and getting back to games, and what these can teach us, we can play too with a revolver, which has 6 shots, a bit as a die. And with me coming from Eastern Europe, we will play of course Russian roulette. And with the results processed via Stirling. We will certainly have an exercise about this, at the end of this chapter.

1c. Playing cards

We have learned so far many things, by playing with coins and dice. At an even more advanced level, which is that of playing cards, we have, to start with:

THEOREM 1.23. *The probabilities at poker are as follows:*

(1) *One pair:* 0.533.
(2) *Two pairs:* 0.120.
(3) *Three of a kind:* 0.053.
(4) *Full house:* 0.006.
(5) *Straight:* 0.005.
(6) *Four of a kind:* 0.001.
(7) *Flush:* 0.000.
(8) *Straight flush:* 0.000.

PROOF. Let us consider indeed our deck of 32 cards, $7, 8, 9, 10, J, Q, K, A$. The total number of possibilities for a poker hand is:

$$\binom{32}{5} = \frac{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28}{2 \cdot 3 \cdot 4 \cdot 5} = 32 \cdot 31 \cdot 29 \cdot 7$$

(1) For having a pair, the number of possibilities is:

$$N = \binom{8}{1}\binom{4}{2} \times \binom{7}{3}\binom{4}{1}^3 = 8 \cdot 6 \cdot 35 \cdot 64$$

Thus, the probability of having a pair is:

$$P = \frac{8 \cdot 6 \cdot 35 \cdot 64}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{6 \cdot 5 \cdot 16}{31 \cdot 29} = \frac{480}{899} = 0.533$$

(2) For having two pairs, the number of possibilities is:

$$N = \binom{8}{2}\binom{4}{2}^2 \times \binom{24}{1} = 28 \cdot 36 \cdot 24$$

Thus, the probability of having two pairs is:

$$P = \frac{28 \cdot 36 \cdot 24}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{36 \cdot 3}{31 \cdot 29} = \frac{108}{899} = 0.120$$

(3) For having three of a kind, the number of possibilities is:

$$N = \binom{8}{1}\binom{4}{3} \times \binom{7}{2}\binom{4}{1}^2 = 8 \cdot 4 \cdot 21 \cdot 16$$

Thus, the probability of having three of a kind is:

$$P = \frac{8 \cdot 4 \cdot 21 \cdot 16}{32 \cdot 31 \cdot 29 \cdot 7} = \frac{3 \cdot 16}{31 \cdot 29} = \frac{48}{899} = 0.053$$

(4) For having full house, the number of possibilities is:

$$N=\binom{8}{1}\binom{4}{3}\times\binom{7}{1}\binom{4}{2}=8\cdot 4\cdot 7\cdot 6$$

Thus, the probability of having full house is:

$$P=\frac{8\cdot 4\cdot 7\cdot 6}{32\cdot 31\cdot 29\cdot 7}=\frac{6}{31\cdot 29}=\frac{6}{899}=0.006$$

(5) For having a straight, the number of possibilities is:

$$N=4\left[\binom{4}{1}^4-4\right]=16\cdot 63$$

Thus, the probability of having a straight is:

$$P=\frac{16\cdot 63}{32\cdot 31\cdot 29\cdot 7}=\frac{9}{2\cdot 31\cdot 29}=\frac{9}{1798}=0.005$$

(6) For having four of a kind, the number of possibilities is:

$$N=\binom{8}{1}\binom{4}{4}\times\binom{7}{1}\binom{4}{1}=8\cdot 7\cdot 4$$

Thus, the probability of having four of a kind is:

$$P=\frac{8\cdot 7\cdot 4}{32\cdot 31\cdot 29\cdot 7}=\frac{1}{31\cdot 29}=\frac{1}{899}=0.001$$

(7) For having a flush, the number of possibilities is:

$$N=4\left[\binom{8}{4}-4\right]=4\cdot 66$$

Thus, the probability of having a flush is:

$$P=\frac{4\cdot 66}{32\cdot 31\cdot 29\cdot 7}=\frac{33}{4\cdot 31\cdot 29\cdot 7}=\frac{9}{25172}=0.000$$

(8) For having a straight flush, the number of possibilities is:

$$N=4\cdot 4$$

Thus, the probability of having a straight flush is:

$$P=\frac{4\cdot 4}{32\cdot 31\cdot 29\cdot 7}=\frac{1}{2\cdot 31\cdot 29\cdot 7}=\frac{1}{12586}=0.000$$

Thus, we have obtained the numbers in the statement. □

Many other things can be said, about playing cards, with friends or at the casino, and even more interesting things can be said about cheating, shuffling decks of cards, and so on, with a lot of interesting mathematics involved, sometimes coming from magicians.

1d. Conclusions

Good news, with our knowledge, coming from all the games that we played, we can now axiomatize discrete probability. We can do this in two possible ways, as follows:

(1) Pedestrian way. We can talk about events, and their probability to happen, in the obvious way, and with the number of events being finite, $N < \infty$, or even countable, $N = \infty$. All this is quite self-explanatory, and we will not further insist on this.

(2) Emperor way. Thinking well at what we did in this chapter, especially in relation with the Bernoulli and binomial laws, in the end, it is all about a discrete probability measure, and what can be done with it, with this mesure being something as follows:

$$\mu = \sum_i \lambda_i \delta_{x_i}$$

To be more precise, here the data consists of real numbers $x_i \in \mathbb{R}$, which can be countably many, and of parameters $\lambda_i \geq 0$, summing up to one, $\sum_i \lambda_i = 1$. And with all this being, again, rather self-explanatory, after all we did in this chapter.

In what follows we will opt, and no surprise here, for the Emperor way. However, technically speaking, it is best to wait a bit before doing that, because we still have to talk about continuous probability. So, we will defer the axiomatization for the end of chapter 2, after some calculus discussion, in relation with continuous probability.

1e. Exercises

Exercises:

Exercise 1.24.

Exercise 1.25.

Exercise 1.26.

Exercise 1.27.

Exercise 1.28.

Exercise 1.29.

Exercise 1.30.

Exercise 1.31.

Bonus exercise.

CHAPTER 2

Laws, independence

2a. Calculus, revised

We have seen so far that discrete games and probability are basically a matter of correctly modelling your problem, and then going ahead with the mathematics, which is basically binomials and factorials. Dealing with these binomials and factorials can be of course a quite difficult question, depending on the complexity of your original problem, sometimes requiring a deep know-how, acquired over the time, tricky computer simulations, and so on. However, from a conceptual perspective, things remain quite simple, with basically no need for further foundations, and deep math theorems.

As a joke here, in view of this, a pure mathematics colleague of mine, and there are quite a few of them in the academia, would probably say "discrete probability is trivial". Beware of making such jokes, in front of your probability professor, including myself, and in case you are indeed a pure mathematician, with such dirty thoughts, have a quick look at chapter 4 below, where we will go back to discrete probability, dealing that time with advanced aspects. If you find all that stuff trivial, on the spot, I will eat my hat.

With this discussed, let us turn now to continuous probability. This is where the "true" probability theory happens, for most of the sciences, including physics, chemistry, biology and so on, because all the readings that you can obtain, from your instruments measuring temperature, pressure an so on, obviously vary in a continuous way. And the same is true, in a certain sense, for economy, because if you consider global problems, involving $N >> 0$ people, the data that you will get will be basically continuous.

So, what is continuous probability? Not clear at all, and certainly something far more complicated than discrete probability, because we are faced right away with:

PARADOX 2.1. *When picking a random number $x \in [0,1]$, the probability for that number to be a given $y \in [0,1]$, that we chose in advance, is obviously:*

$$P(x=y)=0$$

On the other hand, if everyone in this universe plays this game, with each having their own y chosen in advance, so that $\{y\}=[0,1]$, then some will win, $P(x=y)>0$.

And here, it depends on you, and your mathematical skills. Sure I know that some of you can refute this paradox, but some other probably cannot, and even those who can refute this paradox, do not tease me, because if I think a bit, I can surely come up with something more diabolic, along the same lines, that you will not be able to refute.

In short, we are all in need of a crash course in continuous probability, for never ever be bothered by Paradox 2.1, and its countless versions, most likely invented by the Devil. With "all of us" actually including myself, because as quantum physicist, believe me, I face such things on a daily basis, coming from these damn elementary particles that we have there, playing games with us, and sure yes, sometimes I get fooled too.

So, continuous probability. This will take us some time to develop, but coming a bit in advance, here is the eventual answer to Paradox 2.1, that we will reach to:

ANSWER 2.2. *The probability for a randomly picked $x \in [0,1]$ to be equal to a given $y \in [0,1]$, that we chose in advance, is given by the formula*

$$P(x=y)=\int_y^y 1\,dx$$

which, after computation of the integral, yields the answer

$$P(x=y)=0$$

and this is all that can be seriously said, on this matter. Period.

Which sounds very good, hope you agree with me. That is, having a mathematical theory enabling us to formulate such an authoritative answer is certainly worth the effort, and this is what we will do in what follows, develop this theory, no matter what it takes. In fact, this theory will be our main mathematical weapon, afterwards.

A quick look at the math in Answer 2.2 clearly reveals what we have to do, namely develop the theory of integrals, and then connect it with probability questions. Moreover, and coming too a bit in advance, the hard part is actually developing the theory of integrals, with all the rest being quite straightforward. Let us record this as follows:

CONCLUSION 2.3. *Continuous probability is the same thing as integration*

$$\int f(x)dx = ?$$

in one or several variables. If we are able to master this, we win.

In practice now, all this leads us into calculus. You certainly know well about derivatives, but what about integrals? We will review now the theory of integrals, by assuming the theory of derivatives known. In case you need first to revise the derivatives, or if you take now the decision to learn full calculus, in its entirety, many good books are available, as for instance those of Lax-Terrell [67], [68], which in addition are finance-friendly. For a physics-friendly version of the story, you can check my own book [6]. And if you are brave enough for math-friendly books, the old style, go with Rudin [82], [83].

So, integrals. There are several possible viewpoints on the integral, which are all useful, and good to know. To start with, we have something very simple, as follows:

DEFINITION 2.4. *The integral of a continuous function $f:[a,b]\to\mathbb{R}$, denoted*

$$\int_a^b f(x)dx$$

is the area below the graph of f, signed $+$ where $f\geq 0$, and signed $-$ where $f\leq 0$.

Let us do now some computations. In order to compute areas, and so integrals of functions, we can use our geometric knowledge. Here are some basic results:

PROPOSITION 2.5. *We have the following results:*

(1) *When f is linear, we have the following formula:*

$$\int_a^b f(x)dx=(b-a)\cdot\frac{f(a)+f(b)}{2}$$

(2) *In fact, when f is piecewise linear on $[a=a_1,a_2,\ldots,a_n=b]$, we have:*

$$\int_a^b f(x)dx=\sum_{i=1}^{n-1}(a_{i+1}-a_i)\cdot\frac{f(a_i)+f(a_{i+1})}{2}$$

(3) *We have as well the formula $\int_{-1}^1\sqrt{1-x^2}\,dx=\pi/2$.*

PROOF. These results all follow from basic geometry, as follows:

(1) Assuming $f\geq 0$, we must compute the area of a trapezoid having sides $f(a),f(b)$, and height $b-a$. But this is the same as the area of a rectangle having side $(f(a)+f(b))/2$ and height $b-a$, and we obtain $(b-a)(f(a)+f(b))/2$, as claimed.

(2) This is clear indeed from the formula found in (1), by additivity.

(3) The integral in the statement is by definition the area of the upper unit half-disc. But since the area of the whole unit disc is π, this half-disc area is $\pi/2$. □

As an interesting observation, (2) in the above result makes it quite clear that f does not necessarily need to be continuous, in order to talk about its integral. Indeed, assuming that f is piecewise linear on $[a=a_1,a_2,\ldots,a_n=b]$, but not necessarily continuous, we can still talk about its integral, in the obvious way, and we have the following formula:

$$\int_a^b f(x)dx=\sum_{i=1}^{n-1}(a_{i+1}-a_i)\cdot\frac{f(a_i+)+f(a_{i+1}-)}{2}$$

Based on this observation, let us upgrade our formalism, as follows:

Definition 2.6. *We say that a function $f:[a,b]\to\mathbb{R}$ is integrable when the area below its graph is computable. In this case we denote by*

$$\int_a^b f(x)dx$$

this area, signed $+$ where $f\geq 0$, and signed $-$ where $f\leq 0$.

As basic examples of integrable functions, we have the continuous ones. We will soon see that this is indeed true, coming with mathematical proof. As further examples, we have the functions which are piecewise linear, or more generally piecewise continuous. We will also see, later, as another class of examples, that the piecewise monotone functions are integrable. But more on this later, let us not bother for the moment with all this.

Back to work now, here are some general results regarding the integrals:

Proposition 2.7. *We have the following formulae,*

$$\int_a^b f(x)+g(x)dx=\int_a^b f(x)dx+\int_a^b g(x)dx$$

$$\int_a^b \lambda f(x)=\lambda\int_a^b f(x)$$

valid for any functions f,g and any scalar $\lambda\in\mathbb{R}$.

Proof. Both these formulae are indeed clear from definitions. □

Moving ahead now, passed the above results, which are of purely algebraic and geometric nature, and perhaps a few more of the same type, which are all quite trivial and that we we will not get into here, we must do some analysis, in order to compute integrals. This is something quite tricky, and we have here the following result:

Theorem 2.8. *We have the Riemann integration formula,*

$$\int_a^b f(x)dx=(b-a)\times\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f\left(a+\frac{b-a}{N}\cdot k\right)$$

which can serve as a definition for the integral.

Proof. This is standard, by drawing rectangles. We have indeed the following formula, which can stand as a definition for the signed area below the graph of f:

$$\int_a^b f(x)dx=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N\frac{b-a}{N}\cdot f\left(a+\frac{b-a}{N}\cdot k\right)$$

Thus, we are led to the formula in the statement. □

Observe that the above formula suggests that $\int_a^b f(x)dx$ is the length of the interval $[a,b]$, namely $b-a$, times the average of f on the interval $[a,b]$. Thinking a bit, this is indeed something true, with no need for Riemann sums, coming directly from Definition 2.4, because area means side times average height. Thus, we can formulate:

THEOREM 2.9. *The integral of a function $f:[a,b]\to\mathbb R$ is given by*

$$\int_a^b f(x)dx=(b-a)\times A(f)$$

where $A(f)$ is the average of f over the interval $[a,b]$.

PROOF. As explained above, this is clear from Definition 2.4, via some geometric thinking. Alternatively, this is something which certainly comes from Theorem 2.8. □

The point of view in Theorem 2.9 is something quite useful, and as an illustration for this, let us review the results that we already have, by using this interpretation. First, we have the formula for linear functions from Proposition 2.5, namely:

$$\int_a^b f(x)dx=(b-a)\cdot\frac{f(a)+f(b)}{2}$$

But this formula is totally obvious with our new viewpoint, from Theorem 2.9. The same goes for the results in Proposition 2.7, which become even more obvious with the viewpoint from Theorem 2.9. However, not everything trivializes in this way, and the result which is left, namely the formula $\int_{-1}^1\sqrt{1-x^2}\,dx=\pi/2$ from Proposition 2.5 (3), not only does not trivialize, but becomes quite opaque with our new philosophy.

Going ahead with more interpretations of the integral, we have:

THEOREM 2.10. *We have the Monte Carlo integration formula,*

$$\int_a^b f(x)dx=(b-a)\times\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f(x_i)$$

with $x_1,\ldots,x_N\in[a,b]$ being random.

PROOF. We recall from Theorem 2.9 that the idea is to use a formula as follows, with the points $x_1,\ldots,x_N\in[a,b]$ being uniformly distributed:

$$\int_a^b f(x)dx=(b-a)\times\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f(x_i)$$

But this works as well when the points $x_1,\ldots,x_N\in[a,b]$ are randomly distributed, for somewhat obvious reasons, and this gives the result. □

Observe that the Monte Carlo integration works better than Riemann integration, for instance when trying to improve the estimate, via $N \to N+1$. Indeed, in the context of Riemann integration, assume that we managed to find an estimate as follows, which in practice requires computing N values of our function f, and making their average:

$$\int_a^b f(x)dx \simeq \frac{b-a}{N}\sum_{k=1}^N f\left(a+\frac{b-a}{N}\cdot k\right)$$

In order to improve this estimate, any extra computed value of our function $f(y)$ will be unuseful. For improving our formula, what we need are N extra values of our function, $f(y_1),\ldots,f(y_N)$, with the points $y_1,\ldots,y_N$ being precisely the midpoints of the previous division of $[a,b]$, so that we can write an improvement of our formula, as follows:

$$\int_a^b f(x)dx \simeq \frac{b-a}{2N}\sum_{k=1}^{2N} f\left(a+\frac{b-a}{2N}\cdot k\right)$$

With Monte Carlo, things are far more flexible. Assume indeed that we managed to find an estimate as follows, which again requires computing N values of our function:

$$\int_a^b f(x)dx \simeq \frac{b-a}{N}\sum_{k=1}^N f(x_i)$$

Now if we want to improve this, any extra computed value of our function $f(y)$ will be helpful, because we can set $x_{n+1}=y$, and improve our estimate as follows:

$$\int_a^b f(x)dx \simeq \frac{b-a}{N+1}\sum_{k=1}^{N+1} f(x_i)$$

And isn't this potentially useful, and powerful, when thinking at practically computing integrals, either by hand, or by using a computer. Let us record this finding as follows:

CONCLUSION 2.11. *Monte Carlo integration works better than Riemann integration, when it comes to computing as usual, by estimating, and refining the estimate, and this because Monte Carlo can benefit, right away, from every single new input.*

As another interesting feature of Monte Carlo integration, this works much better than Riemann integration, for functions having various symmetries, because Riemann integration can get "fooled" by these symmetries, while Monte Carlo remains strong.

As an example for this phenomeon, chosen to be quite drastic, let us attempt to integrate, via both Riemann and Monte Carlo, the following function $f:[0,\pi]\to\mathbb{R}$:

$$f(x)=\Big|\sin(120x)\Big|$$

The first few Riemann sums for this function are then as follows:

$$I_2(f) = \frac{\pi}{2}(|\sin 0| + |\sin 60\pi|) = 0$$

$$I_3(f) = \frac{\pi}{3}(|\sin 0| + |\sin 40\pi| + |\sin 80\pi|) = 0$$

$$I_4(f) = \frac{\pi}{4}(|\sin 0| + |\sin 30\pi| + |\sin 60\pi| + |\sin 90\pi|) = 0$$

$$I_5(f) = \frac{\pi}{5}(|\sin 0| + |\sin 24\pi| + |\sin 48\pi| + |\sin 72\pi| + |\sin 96\pi|) = 0$$

$$I_6(f) = \frac{\pi}{6}(|\sin 0| + |\sin 20\pi| + |\sin 40\pi| + |\sin 60\pi| + |\sin 80\pi| + |\sin 100\pi|) = 0$$

$$\vdots$$

Based on this evidence, we will conclude, obviously, that we have:

$$\int_0^\pi f(x)dx = 0$$

With Monte Carlo, however, such things cannot happen. Indeed, since there are finitely many points $x \in [0,\pi]$ having the property $\sin(120x) = 0$, a random point $x \in [0,\pi]$ will have the property $|\sin(120x)| > 0$, so Monte Carlo will give, at any $N \in \mathbb{N}$:

$$\int_0^\pi f(x)dx \simeq \frac{b-a}{N}\sum_{k=1}^N f(x_i) > 0$$

Again, this is something interesting, when practically computing integrals, either by hand, or by using a computer. So, let us record, as a complement to Conclusion 2.11:

Conclusion 2.12. *Monte Carlo integration is smarter than Riemann integration, because the symmetries of the function can fool Riemann, but not Monte Carlo.*

Our purpose now will be to understand which functions $f : \mathbb{R} \to \mathbb{R}$ are integrable, and how to compute their integrals. For this purpose, the Riemann formula in Theorem 2.8 will be our favorite tool. We first have the following technical result:

Theorem 2.13. *The following functions are integrable:*

(1) *The piecewise continuous functions.*
(2) *The piecewise monotone functions.*

Proof. This is indeed something quite standard, coming from the definition of the integral as a limit of Riemann sums, via some routine analysis:

(1) It is enough to prove the first assertion for a function $f : [a,b] \to \mathbb{R}$ which is continuous, and our claim here is that this follows from the uniform continuity of f. To be more precise, given $\varepsilon > 0$, let us choose $\delta > 0$ such that the following happens:

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

In order to prove the result, let us pick two arbitrary divisions of the interval $[a,b]$, not necessarily uniform, denoted as follows:

$$I=[a=a_1<a_2<\ldots<a_n=b]$$

$$I'=[a=a_1'<a_2'<\ldots<a_m'=b]$$

Our claim, which will prove the result, is that if these divisions are sharp enough, of resolution $<\delta/2$, then the associated Riemann sums $\Sigma_I(f),\Sigma_{I'}(f)$ are close within ε:

$$a_{i+1}-a_i<\frac{\delta}{2}\ ,\ a_{i+1}'-a_i'<\delta_2\implies\Big|\Sigma_I(f)-\Sigma_{I'}(f)\Big|<\varepsilon$$

(2) In order to prove this claim, let us denote by l the length of the intervals on the real line. Our assumption is that the lengths of the divisions I,I' satisfy:

$$l\big([a_i,a_{i+1}]\big)<\frac{\delta}{2}\quad,\quad l\big([a_i',a_{i+1}']\big)<\frac{\delta}{2}$$

Now let us intersect the intervals of our divisions I,I', and set:

$$l_{ij}=l\big([a_i,a_{i+1}]\cap[a_j',a_{j+1}']\big)$$

The difference of Riemann sums that we are interested in is then given by:

$$\begin{aligned}\Big|\Sigma_I(f)-\Sigma_{I'}(f)\Big| &= \left|\sum_{ij}l_{ij}f(a_i)-\sum_{ij}l_{ij}f(a_j')\right|\\ &= \left|\sum_{ij}l_{ij}(f(a_i)-f(a_j'))\right|\end{aligned}$$

(3) Now let us estimate $f(a_i)-f(a_j')$. Since in the case $l_{ij}=0$ we do not need this estimate, we can assume $l_{ij}>0$. Now by remembering what the definition of the numbers l_{ij} was, we conclude that we have at least one point $x\in\mathbb{R}$ satisfying:

$$x\in[a_i,a_{i+1}]\cap[a_j',a_{j+1}']$$

But then, by using this point x and our assumption on I,I' involving δ, we get:

$$\begin{aligned}|a_i-a_j'| &\leq |a_i-x|+|x-a_j'|\\ &\leq \frac{\delta}{2}+\frac{\delta}{2}\\ &= \delta\end{aligned}$$

Thus, according to our definition of δ from (1), in relation to ε, we get:

$$|f(a_i)-f(a_j')|<\varepsilon$$

(4) But this is what we need, in order to finish. Indeed, with the estimate that we found, we can finish the computation started in (2), as follows:

$$\begin{aligned}\left|\Sigma_I(f)-\Sigma_{I'}(f)\right| &= \left|\sum_{ij} l_{ij}(f(a_i)-f(a'_j))\right|\\ &\leq \varepsilon\sum_{ij} l_{ij}\\ &= \varepsilon(b-a)\end{aligned}$$

Thus our two Riemann sums are close enough, provided that they are both chosen to be fine enough, and this finishes the proof of the first assertion.

(5) Regarding now the second assertion, this is something more technical, that we will not really need in what follows. We will leave the proof here, which uses similar ideas to those in the proof of (1) above, namely subdivisions and estimates, as an exercise. □

Going ahead with more theory, let us establish now some abstract properties of the integration operation. We already know from Proposition 2.7 that the integrals behave well with respect to sums and multiplication by scalars, the formulae being as follows:

$$\int_a^b f(x)+g(x)dx=\int_a^b f(x)dx+\int_a^b g(x)dx$$

$$\int_a^b \lambda f(x)=\lambda\int_a^b f(x)$$

Along the same lines, but at a more advanced level, we have the following result, which is equally useful, in practice, for the concrete computation of integrals:

THEOREM 2.14. *The integrals behave well with respect to taking limits,*

$$\int_a^b\left(\lim_{n\to\infty} f_n(x)\right)dx=\lim_{n\to\infty}\int_a^b f_n(x)dx$$

and with respect to taking infinite sums as well,

$$\int_a^b\left(\sum_{n=0}^\infty f_n(x)\right)dx=\sum_{n=0}^\infty\int_a^b f_n(x)dx$$

with both these formulae being valid, under mild assumptions.

PROOF. This is something very standard. To be more precise, (1) follows by using the known facts about sequences, via Riemann sums, and then (2) follows as a particular case of (1). We will leave the clarification of all this as an instructive exercise. □

Finally, still at the general level, let us record as well the following result:

THEOREM 2.15. *Given a continuous function $f:[a,b]\to\mathbb R$, we have*

$$\exists c\in[a,b]\quad,\quad\int_a^bf(x)dx=(b-a)f(c)$$

with this being called mean value property.

PROOF. Our claim is that this follows from the following trivial estimate:

$$\min(f)\leq f\leq\max(f)$$

Indeed, by integrating this over $[a,b]$, we obtain the following estimate:

$$(b-a)\min(f)\leq\int_a^bf(x)dx\leq(b-a)\max(f)$$

Now observe that this latter estimate can be written as follows:

$$\min(f)\leq\frac{\int_a^bf(x)dx}{b-a}\leq\max(f)$$

Since f must takes all values on $[\min(f),\max(f)]$, we get a $c\in[a,b]$ such that:

$$\frac{\int_a^bf(x)dx}{b-a}=f(c)$$

Thus, we are led to the conclusion in the statement. □

2b. Computing integrals

At the level of examples, which is what matters the most, let us first look at the simplest functions that we know, namely the power functions $f(x)=x^p$. Here we have:

THEOREM 2.16. *We have the integration formula*

$$\int_a^bx^pdx=\frac{b^{p+1}-a^{p+1}}{p+1}$$

valid at $p=0,1,2,3$.

PROOF. This is something quite tricky, the idea being as follows:

(1) By linearity we can assume that our interval $[a,b]$ is of the form $[0,c]$, and the formula that we want to establish is as follows:

$$\int_0^cx^pdx=\frac{c^{p+1}}{p+1}$$

(2) We can further assume $c=1$, and by expressing the left term as a Riemann sum, we are in need of the following estimate, in the $N\to\infty$ limit:

$$1^p+2^p+\ldots+N^p\simeq\frac{N^{p+1}}{p+1}$$

(3) So, let us try to prove this. At $p=0$, obviously nothing to do, because we have the following formula, which is exact, and which proves our estimate:

$$1^0+2^0+\ldots+N^0=N$$

(4) At $p=1$ now, we are confronted with a well-known question, namely the computation of $1+2+\ldots+N$. But this is simplest done by arguing that the average of the numbers $1,2,\ldots,N$ being the number in the middle, we have:

$$\frac{1+2+\ldots+N}{N}=\frac{N+1}{2}$$

Thus, we obtain the following formula, which again solves our question:

$$1+2+\ldots+N=\frac{N(N+1)}{2}\simeq\frac{N^2}{2}$$

(5) At $p=2$ now, go compute $1^2+2^2+\ldots+N^2$. This is not obvious at all, so as a preliminary here, let us go back to the case $p=1$, and try to find a new proof there, which might have some chances to extend at $p=2$. The trick is to use 2D geometry. Indeed, consider the following picture, with stacks going from 1 to N:

$$\begin{matrix}
&&&&\square\\
&&&&\vdots\\
&&\square&\ldots&\square\\
&\square&\square&\ldots&\square\\
\square&\square&\square&\ldots&\square
\end{matrix}$$

Now if we take two copies of this, and put them one on the top of the other, with a twist, in the obvious way, we obtain a rectangle having size $N\times(N+1)$. Thus:

$$2(1+2+\ldots+N)=N(N+1)$$

But this gives the same formula as before, solving our question, namely:

$$1+2+\ldots+N=\frac{N(N+1)}{2}\simeq\frac{N^2}{2}$$

(6) Armed with this new method, let us attack now the case $p=2$. Here we obviously need to do some 3D geometry, namely taking the picture P formed by a succession of solid squares, having sizes 1×1, 2×2, 3×3, and so on up to $N\times N$. Some quick thinking suggests that stacking 3 copies of P, with some obvious twists, will lead us to a parallelepiped. But this is not exactly true, and some further thinking shows that what we have to do is to add 3 more copies of P, leading to the following formula:

$$1^2+2^2+\ldots+N^2=\frac{N(N+1)(2N+1)}{6}$$

Or at least, that's how the legend goes. In practice, the above formula holds indeed, and you can check it for instance by recurrence, and this solves our problem:

$$1^2 + 2^2 + \ldots + N^2 \simeq \frac{2N^3}{6} = \frac{N^3}{3}$$

(7) At $p = 3$ now, the legend goes that by deeply thinking in 4D we are led to the following formula, a bit as in the cases $p = 1, 2$, explained above:

$$1^3 + 2^3 + \ldots + N^3 = \left(\frac{N(N+1)}{2}\right)^2$$

Alternatively, assuming that the gods of combinatorics are with us, we can see right away the following formula, which coupled with (4) gives the result:

$$1^3 + 2^3 + \ldots + N^3 = (1 + 2 + \ldots + N)^2$$

In any case, in practice, the above formula holds indeed, and you can check it for instance by recurrence, and this solves our problem:

$$1^3 + 2^3 + \ldots + N^3 \simeq \frac{N^4}{4}$$

(8) Thus, we proved our theorem. Of course, I can hear you screaming what about $p = 4$ and higher. But the thing is that, by a strange twist of fate, there is no exact formula for $1^p + 2^p + \ldots + N^p$, at $p = 4$ and higher. Thus, game over. □

Let us look now at some other functions. And here, as good news, we have:

Theorem 2.17. *We have the following integration formula,*

$$\int_a^b e^x dx = e^b - e^a$$

valid for any two real numbers $a < b$.

Proof. This follows indeed from the Riemann integration formula, because:

$$\begin{aligned}
\int_a^b e^x dx &= \lim_{N\to\infty} \frac{e^a + e^{a+(b-a)/N} + e^{a+2(b-a)/N} + \ldots + e^{a+(N-1)(b-a)/N}}{N} \\
&= \lim_{N\to\infty} \frac{e^a}{N} \cdot \left(1 + e^{(b-a)/N} + e^{2(b-a)/N} + \ldots + e^{(N-1)(b-a)/N}\right) \\
&= \lim_{N\to\infty} \frac{e^a}{N} \cdot \frac{e^{b-a} - 1}{e^{(b-a)/N} - 1} \\
&= (e^b - e^a) \lim_{N\to\infty} \frac{1}{N\left(e^{(b-a)/N} - 1\right)} \\
&= e^b - e^a
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The problem is now, what to do with what we have, namely Theorem 2.16 and Theorem 2.17. But these suggest connecting integrals and derivatives, according to:

$$\left(\frac{x^{p+1}}{p+1}\right)' = x^p \quad , \quad (e^x)' = e^x$$

So, eureka, we have our idea. Moving ahead now, following this idea, we first have the following result, called fundamental theorem of calculus:

THEOREM 2.18. *Given a continuous function $f : [a,b] \to \mathbb{R}$, if we set*

$$F(x) = \int_a^x f(s)ds$$

then $F' = f$. That is, the derivative of the integral is the function itself.

PROOF. This follows from the Riemann integration picture, and more specifically, from the mean value property from Theorem 2.15. Indeed, we have:

$$\frac{F(x+t) - F(x)}{t} = \frac{1}{t}\int_x^{x+t} f(x)dx$$

On the other hand, our function f being continuous, by using the mean value property from Theorem 2.12, we can find a number $c \in [x, x+t]$ such that:

$$\frac{1}{t}\int_x^{x+t} f(x)dx = f(x)$$

Thus, putting our formulae together, we conclude that we have:

$$\frac{F(x+t) - F(x)}{t} = f(c)$$

Now with $t \to 0$, no matter how the number $c \in [x, x+t]$ varies, one thing that we can be sure about is that we have $c \to x$. Thus, by continuity of f, we obtain:

$$\lim_{t\to 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

But this means exactly that we have $F' = f$, and we are done. ☐

We have as well the following result, which is something equivalent, and a hair more beautiful, also called fundamental theorem of calculus:

THEOREM 2.19. *Given a function $F : \mathbb{R} \to \mathbb{R}$, we have*

$$\int_a^b F'(x)dx = F(b) - F(a)$$

for any interval $[a,b]$.

Proof. As already mentioned, this is something which follows from Theorem 2.18, and is in fact equivalent to it. Indeed, consider the following function:

$$G(s)=\int_a^s F'(x)dx$$

By using Theorem 2.15 we have $G'=F'$, and so our functions F,G differ by a constant. But with $s=a$ we have $G(a)=0$, and so the constant is $F(a)$, and we get:

$$F(s)=G(s)+F(a)$$

Now with $s=b$ this gives $F(b)=G(b)+F(a)$, which reads:

$$F(b)=\int_a^b F'(x)dx+F(a)$$

Thus, we are led to the conclusion in the statement. □

There are many other equivalent formulations of the fundamental theorem of calculus, and countless applications as well. As an illustration for all this, we have:

Theorem 2.20. *We have the following integration formulae,*

$$\int_a^b x^p dx=\frac{b^{p+1}-a^{p+1}}{p+1}\quad,\quad\int_a^b\frac{1}{x}\,dx=\log\left(\frac{b}{a}\right)$$

$$\int_a^b\sin x\,dx=\cos a-\cos b\quad,\quad\int_a^b\cos x\,dx=\sin b-\sin a$$

$$\int_a^b e^x dx=e^b-e^a\quad,\quad\int_a^b\log x\,dx=b\log b-a\log a-b+a$$

all obtained, in case you ever forget them, via the fundamental theorem of calculus.

Proof. We already know two of these formulae, namely the one for powers from Theorem 2.16, and the one for exponentials Theorem 2.17, but the best is to do everything, using the fundamental theorem of calculus. The computations go as follows:

(1) With $F(x)=x^{p+1}$ we have $F'(x)=px^p$, and we get, as desired:

$$\int_a^b px^p\,dx=b^{p+1}-a^{p+1}$$

(2) Observe first that the formula (1) does not work at $p=-1$. However, here we can use $F(x)=\log x$, having as derivative $F'(x)=1/x$, which gives, as desired:

$$\int_a^b\frac{1}{x}\,dx=\log b-\log a=\log\left(\frac{b}{a}\right)$$

(3) With $F(x) = \cos x$ we have $F'(x) = -\sin x$, and we get, as desired:

$$\int_a^b -\sin x\,dx = \cos b - \cos a$$

(4) With $F(x) = \sin x$ we have $F'(x) = \cos x$, and we get, as desired:

$$\int_a^b \cos x\,dx = \sin b - \sin a$$

(5) With $F(x) = e^x$ we have $F'(x) = e^x$, and we get, as desired:

$$\int_a^b e^x\,dx = e^b - e^a$$

(6) This is something more tricky. We are looking for a function satisfying:

$$F'(x) = \log x$$

This does not look doable, but fortunately the answer to such things can be found on the internet. But, what if the internet connection is down? So, let us think a bit, and try to solve our problem. Speaking logarithm and derivatives, what we know is:

$$(\log x)' = \frac{1}{x}$$

But then, in order to make appear log on the right, the idea is quite clear, namely multiplying on the left by x. We obtain in this way the following formula:

$$(x\log x)' = 1 \cdot \log x + x \cdot \frac{1}{x} = \log x + 1$$

We are almost there, all we have to do now is to substract x from the left, as to get:

$$(x\log x - x)' = \log x$$

But this this formula in hand, we can go back to our problem, and we get:

$$\begin{aligned}\int_a^b \log x\,dx &= (b\log b - b) - (a\log a - a)\\ &= b\log b - a\log a - b + a\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Getting back now to theory, inspired by the above, let us formulate:

DEFINITION 2.21. *Given a function f, we call primitive of f any function F satisfying:*

$$F' = f$$

We denote such primitives by $\int f$, and also call them indefinite integrals.

Observe that the primitives are unique up to an additive constant, in the sense that if F is a primitive, then so is $F+c$, for any $c \in \mathbb{R}$, and conversely, if F, G are two primitives, then we must have $G = F + c$, for some $c \in \mathbb{R}$, with this coming from a basic result about derivatives, saying that the derivative vanishes when the function is constant.

As for the convention at the end, $F = \int f$, this comes from the fundamental theorem of calculus, which can be written as follows, by using this convention:

$$\int_a^b f(x)dx = \left(\int f\right)(b) - \left(\int f\right)(a)$$

By the way, observe that there is no contradiction here, coming from the indeterminacy of $\int f$. Indeed, when adding a constant $c \in \mathbb{R}$ to the chosen primitive $\int f$, when conputing the above difference the c quantities will cancel, and we will obtain the same result.

We can now reformulate Theorem 2.20 in a more digest form, as follows:

THEOREM 2.22. *We have the following formulae for primitives,*

$$\int x^p = \frac{x^{p+1}}{p+1} \quad , \quad \int \frac{1}{x} = \log x$$

$$\int \sin x = -\cos x \quad , \quad \int \cos x = \sin x$$

$$\int e^x = e^x \quad , \quad \int \log x = x \log x - x$$

allowing us to compute the corresponding definite integrals too.

PROOF. Here the various formulae in the statement follow from Theorem 2.20, and the last assertion comes from the integration formula given after Definition 2.21. □

Getting back now to theory, we have the following key result:

THEOREM 2.23. *We have the formula*

$$\int f'g + \int fg' = fg$$

called integration by parts.

PROOF. This follows by integrating the Leibnitz formula, namely:

$$(fg)' = f'g + fg'$$

Indeed, with our convention for primitives, this gives the formula in the statement. □

It is then possible to pass to usual integrals, and we obtain a formula here as well, as follows, also called integration by parts, with the convention $[\varphi]_a^b = \varphi(b) - \varphi(a)$:

$$\int_a^b f'g + \int_a^b fg' = \Big[fg\Big]_a^b$$

In practice, the most interesting case is that when fg vanishes on the boundary $\{a, b\}$ of our interval, leading to the following formula:

$$\int_a^b f'g = -\int_a^b fg'$$

Examples of this usually come with $[a, b] = [-\infty, \infty]$, and more on this soon. Now still at the theoretical level, we have as well the following result:

THEOREM 2.24. *We have the change of variable formula*

$$\int_a^b f(x)dx = \int_c^d f(\varphi(t))\varphi'(t)dt$$

where $c = \varphi^{-1}(a)$ and $d = \varphi^{-1}(b)$.

PROOF. This follows with $f = F'$, from the following differentiation rule, that you know well from basic calculus, and whose proof is something elementary:

$$(F\varphi)'(t) = F'(\varphi(t))\varphi'(t)$$

Indeed, by integrating between c and d, we obtain the result. □

As a main application now of our theory, in relation with advanced calculus, and more specifically with the Taylor formula, that you surely know, we have:

THEOREM 2.25. *Given a function $f : \mathbb{R} \to \mathbb{R}$, we have the formula*

$$f(x+t) = \sum_{k=0}^n \frac{f^{(k)}(x)}{k!}\, t^k + \int_x^{x+t} \frac{f^{(n+1)}(s)}{n!}(x+t-s)^n\, ds$$

called Taylor formula with integral formula for the remainder.

PROOF. This is something which looks a bit complicated, so we will first do some verifications, and then we will go for the proof in general:

(1) At $n = 0$ the formula in the statement is as follows, and certainly holds, due to the fundamental theorem of calculus, which gives $\int_x^{x+t} f'(s)ds = f(x+t) - f(x)$:

$$f(x+t) = f(x) + \int_x^{x+t} f'(s)ds$$

(2) At $n = 1$, the formula in the statement becomes more complicated, as follows:

$$f(x+t) = f(x) + f'(x)t + \int_x^{x+t} f''(s)(x+t-s)ds$$

As a first observation, this formula holds indeed for the linear functions, where we have $f(x+t)=f(x)+f'(x)t$, and $f''=0$. So, let us try $f(x)=x^2$. Here we have:

$$f(x+t)-f(x)-f'(x)t=(x+t)^2-x^2-2xt=t^2$$

On the other hand, the integral remainder is given by the same formula, namely:

$$\begin{aligned}\int_x^{x+t}f''(s)(x+t-s)ds&=2\int_x^{x+t}(x+t-s)ds\\&=2t(x+t)-2\int_x^{x+t}sds\\&=2t(x+t)-((x+t)^2-x^2)\\&=2tx+2t^2-2tx-t^2\\&=t^2\end{aligned}$$

(3) Still at $n=1$, let us try now to prove the formula in the statement, in general. Since what we have to prove is an equality, this cannot be that hard, and the first thought goes towards differentiating. But this method works indeed, and we obtain the result.

(4) In general, the proof is similar, by differentiating, the computations being similar to those at $n=1$. Thus, we are led to the formula in the statement. □

Very good all this, in a word, integration in one variable, and its applications, understood. However, we are not done yet, because we still have to discuss integration in several variables. Things here are quite tricky, the general idea being as follows:

Principle 2.26. *Functions of several variables can be integrated too, by performing multiple integrals with respect to all the variables,*

$$\int_{\mathbb R^N}f(x)dx=\int_{\mathbb R}\ldots\int_{\mathbb R}f(x)\,dx_1\ldots dx_N$$

and in this setting, most formulae that we know still work, with a notable exception being the change of variable formula, which in several variables reads

$$\int_E f(x)dx=\int_{\varphi^{-1}(E)}f(\varphi(t))|J_\varphi(t)|dt$$

with the J_φ quantity on the right, called Jacobian, being given by:

$$J_\varphi(t)=\det\left[\left(\frac{d\varphi_i}{dx_j}(x)\right)_{ij}\right]$$

However, there are technical assumptions to be satisfied, a bit everywhere, in several variables, and all this is to be taken with extreme care.

Well, as you can see, some non-trivial things going on here. In practice, in order to deal with all this, multivariable integration, your choices are as follows:

(1) The good choice: learn all this the old way, slowly, with full details, over 2-3 months, say from Rudin [82]. This is what I did myself, as a student, and I survived, and so did many of my colleagues, from the X Generation, and they survived too.

(2) The standard choice: learn all this quickly, over 2-3 weeks, from a modern calculus book, like Lax-Terrell [68], or mine [6]. These will teach you what works, quickly, and at least will warn you, about certain things where you have to be very careful.

(3) The lazy choice: well, you learned all you need to know, from Principle 2.26, but please, doublecheck and triplecheck the validity of your results, at the end, for instance if you plan to use your formulae for building planes or bridges, please do it for me.

As an illustration now for our multivariable integration technology, we have:

THEOREM 2.27. *The volume of the unit sphere is:*

$$V = \frac{4\pi}{3}$$

More generally, the volume of the sphere of radius R is $V = 4\pi R^3/3$.

PROOF. The equation of the unit sphere is as follows:

$$x^2 + y^2 + z^2 = 1$$

Thus, the range of the first coordinate x is as follows:

$$x \in [-1, 1]$$

Now when this first coordinate x is fixed, the other coordinates y, z vary on a circle, given by the equation $y^2 + z^2 = 1 - x^2$, and so having radius as follows:

$$r(x) = \sqrt{1 - x^2}$$

Thus, the vertical slice of our sphere at x has area as follows:

$$a(x) = \pi r(x)^2 = \pi(1 - x^2)$$

We conclude from this discussion that the volume of the sphere is given by:

$$\begin{aligned}
V &= \int_{-1}^1 a(x)\,dx \\
&= \pi\int_{-1}^1 1-x^2\,dx \\
&= \pi\int_{-1}^1 \left(x-\frac{x^3}{3}\right)'dx \\
&= \pi\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right] \\
&= \pi\left(\frac{2}{3}+\frac{2}{3}\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

Finally, the last assertion is clear too, by multiplying everything by R, which amounts in multiplying the final result of our volume computation by R^3. ☐

As a comment here, according to our general guidelines, the formula in Theorem 2.27, based solely on Principle 2.26, must be doublechecked and triplechecked, before being ready to use. For a doublecheck, try getting that via spherical coordinates, that you surely know about. As for the triplecheck, measure at home the volume of a ball, by plunging it into a container of water, and measuring the displacement of the water.

2c. Laws and densities

As an application of the integration theory developed above, let us develop now continuous probability. You surely know a bit, from the real life, what continuous probability is. But in practice, when trying to axiomatize this, in mathematical terms, things can be quite tricky. So, here comes our point, the definition saving us is as follows:

Definition 2.28. *A probability density is a function $f:\mathbb{R}\to\mathbb{R}$ satisfying:*

(1) $f\geq 0$.
(2) $\int f(x)dx=1$.

Observe the obvious relation with discrete probability theory, where the probability for something to happen is always positive, $P\geq 0$, and where the overall probability for something to happen, with this meaning for one of the possible events to happen, is of course $\Sigma P=1$, and this because life goes on, and something must happen, right.

In short, what we are proposing with Definition 2.28 is some sort of continuous analogue of basic probability theory, coming from coins, dice and cards, that we learned in chapter 1. Moving now ahead, let us formulate, as a continuation of Definition 2.28:

DEFINITION 2.29. *Given a probability density function $f:\mathbb{R}\to\mathbb{R}$, we set*

$$P(X\in[a,b])=\int_a^b f(x)dx$$

and call this probability for our variable to belong to $[a,b]$.

With this, we are now one step closer to what we know from coins, dice, cards and so on. Indeed, we have now a random variable X, that we can try to study. The first questions regard the mean and variance, which are constructed as follows:

DEFINITION 2.30. *Given a variable X, its mean is the following quantity:*

$$M=\int_{\mathbb{R}} xf(x)\,dx$$

More generally, we can say that the k-th moment of X is the following quantity:

$$M_k=\int_{\mathbb{R}} x^k f(x)\,dx$$

With this in hand, the variance of X is the quantity $V=M_2-M_1^2$.

As an application of all this, let us have a look into what happens when our density is the constant function $f=1$, on the interval $[0,1]$. We are led in this way to:

THEOREM 2.31. *When the density of X is $f=1$, on the interval $[0,1]$:*

(1) *The mean is $M=1/2$.*
(2) *The variance is $V=1/12$.*
(3) *Also, we have $P(X=y)=0$, obviously, for any $y\in[0,1]$.*

PROOF. All this looks quite easy, and you may wonder why we called this Theorem instead of Proposition, but wait for it. The proof goes as follows:

(1) The computation of the mean, based on Definition 2.30, is as follows:

$$M=\int_0^1 x\,dx=\frac{1}{2}$$

(2) The computation of the variance, also based on Definition 2.30, is as follows:

$$V=\int_0^1 x^2\,dx-\left(\int_0^1 x\,dx\right)^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

(3) We have kept the best for the end. According to Definition 2.29, we have:

$$P(X=y)=\int_y^y 1\,dx=0$$

But, this solves Paradox 2.1, as previously announced in Answer 2.2. Great. □

Moving forward now, as a last theoretical job, let us try to unify the discrete probability from chapter 1 with the continuous probability here. We can do this as follows:

PRINCIPLE 2.32. *Discrete probability can be unified with continuous probability by allowing Dirac masses δ_x into our density functions, with the integration rule being*

$$\int_{\mathbb{R}} g(x)\,d\delta_x = g(x)$$

and with this being a physicists' trick. Alternatively, mathematically, we can develop measure theory, basically leading to the same conclusion.

Obviously, this is something quite tricky, the idea at the end being that, mathematically, we can indeed develop measure theory, and you have here for instance Rudin [82], [83] for details, with the conclusion being that the "probability measures" on $\mathbb{R}$, in some suitable, very intuitive sense, appear as usual densities, as in Definition 2.29, complemented by a linear combination of Dirac masses, as in Principle 2.32:

$$d\mu(x) = \lambda f(x)dx + \sum_i \nu_i \delta_{x_i}$$

To be more precise, here all the parameters are positive, $\lambda, \nu_i \geq 0$, and sum up to 1, as for our measure μ to be indeed positive, and of total mass 1, as it should.

With this formalism in hand, time now to update our various notions above. With the idea in mind of doing things a bit abstractly, we can do this as follows:

DEFINITION 2.33. *Let X be a probability space, that is, a space with a probability measure, and with the corresponding integration denoted $\mathbb{E}$, and called expectation.*

1. *The random variables are the real functions $f \in L^\infty(X)$.*
2. *The moments of such a variable are the numbers $M_k(f) = \mathbb{E}(f^k)$.*
3. *The law of such a variable is the measure given by $M_k(f) = \int_{\mathbb{R}} x^k d\mu_f(x)$.*

Here the fact that μ_f exists indeed is well-known. Indeed, by linearity, we would like to have a probability measure making hold the following formula, for any $P \in \mathbb{R}[X]$:

$$\mathbb{E}(P(f)) = \int_{\mathbb{R}} P(x)d\mu_f(x)$$

By using a standard continuity argument, it is enough to have this formula for the characteristic functions χ_I of the measurable sets of real numbers $I \subset \mathbb{R}$:

$$\mathbb{E}(\chi_I(f)) = \int_{\mathbb{R}} \chi_I(x)d\mu_f(x)$$

But this latter formula, which reads $\mathbb{P}(f \in I) = \mu_f(I)$, can serve as a definition for μ_f, and we are done. Alternatively, assuming some familiarity with measure theory, μ_f is the push-forward of the probability measure on X, via the function $f : X \to \mathbb{R}$.

2d. Independence, Fourier

Next in line, we need to talk about independence. We can do this as follows:

DEFINITION 2.34. *Two variables $f,g\in L^\infty(X)$ are called independent when*

$$\mathbb E(f^kg^l)=\mathbb E(f^k)\,\mathbb E(g^l)$$

happens, for any $k,l\in\mathbb N$.

Again, this definition hides some non-trivial things. Indeed, by linearity, we would like to have a formula as follows, valid for any polynomials $P,Q\in\mathbb R[X]$:

$$\mathbb E[P(f)Q(g)]=\mathbb E[P(f)]\,\mathbb E[Q(g)]$$

By using a continuity argument, it is enough to have this formula for characteristic functions χ_I,χ_J of the measurable sets of real numbers $I,J\subset\mathbb R$:

$$\mathbb E[\chi_I(f)\chi_J(g)]=\mathbb E[\chi_I(f)]\,\mathbb E[\chi_J(g)]$$

Thus, we are led to the usual definition of independence, namely:

$$\mathbb P(f\in I,g\in J)=\mathbb P(f\in I)\,\mathbb P(g\in J)$$

All this might seem a bit abstract, but in practice, the idea is of course that f,g must be independent, in an intuitive, real-life sense. As a first result now, we have:

THEOREM 2.35. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$\mu_{f+g}=\mu_f*\mu_g$$

where $*$ is the convolution of real probability measures.

PROOF. We have the following computation, using the independence of f,g:

$$\begin{aligned}M_k(f+g)&=\mathbb E((f+g)^k)\\&=\sum_r\binom{k}{r}\mathbb E(f^rg^{k-r})\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

On the other hand, by using the Fubini theorem, we have as well:

$$\begin{aligned}\int_{\mathbb R}x^kd(\mu_f*\mu_g)(x)&=\int_{\mathbb R\times\mathbb R}(x+y)^kd\mu_f(x)d\mu_g(y)\\&=\sum_r\binom{k}{r}\int_{\mathbb R}x^rd\mu_f(x)\int_{\mathbb R}y^{k-r}d\mu_g(y)\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

Thus μ_{f+g} and $\mu_f * \mu_g$ have the same moments, so they coincide, as desired. □

Here is now a second result on independence, which is something more advanced:

THEOREM 2.36. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$F_{f+g}=F_fF_g$$

where $F_f(x)=\mathbb E(e^{ixf})$ is the Fourier transform.

PROOF. We have the following computation, using Theorem 2.35 and Fubini:

$$\begin{aligned}F_{f+g}(x)&=\int_{\mathbb R}e^{ixz}d\mu_{f+g}(z)\\&=\int_{\mathbb R}e^{ixz}d(\mu_f*\mu_g)(z)\\&=\int_{\mathbb R\times\mathbb R}e^{ix(z+t)}d\mu_f(z)d\mu_g(t)\\&=\int_{\mathbb R}e^{ixz}d\mu_f(z)\int_{\mathbb R}e^{ixt}d\mu_g(t)\\&=F_f(x)F_g(x)\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

This was for the foundations of probability theory, quickly explained. For further reading, a classical book is Feller [33]. A nice, more modern book is Durrett [25].

2e. Exercises

Exercises:

EXERCISE 2.37.

EXERCISE 2.38.

EXERCISE 2.39.

EXERCISE 2.40.

EXERCISE 2.41.

EXERCISE 2.42.

EXERCISE 2.43.

EXERCISE 2.44.

Bonus exercise.

CHAPTER 3

Normal laws

3a. Normal laws

The main result in classical probability is the Central Limit Theorem (CLT), that we will explain now. Let us start with a very natural question, that you will certain face in any science that you will be doing, be that mathematics, physics, chemistry, engineering in all its flavors, computer science, biology, economy, and so on:

QUESTION 3.1. *What is the normal law, coming from "normal" measurements?*

That is, we want a formula for what comes out of various real-life measurements, such as measuring the temperature of the room, over some time, or the pressure of a tyre, or why not, recording what comes out by grading a calculus exam. And here, intuition suggests that we should get some kind of bell-shaped curve, with most students doing average, and then with this average dropping on both sides, towards good and bad.

Now let us think a bit, on how these students actually produce the bell-shaped curve. Since students' contributions to the various exercises, and so to this curve, are rather independent, barring of course any cheating, we are led to the following conclusion:

CONCLUSION 3.2. *The normal law is the bell-shaped curve coming out of a "central limiting" procedure, consisting in adding i.i.d. variables.*

Summarizing, we are in need of a "central limiting theorem", telling us what the normal law is. However, doing this with bare hands is a bit complicated, so we will do instead some reverse engineering. And with the comment that, as calculus teachers, we are of course entitled to cheat a bit. Following Gauss, we first have:

THEOREM 3.3. *We have the following formula,*

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

called Gauss integral formula.

PROOF. Let I be the integral in the statement. By using polar coordinates, namely $x = r\cos t$, $y = r\sin t$, with the corresponding Jacobian being r, we have:

$$\begin{aligned} I^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2}dxdy \\ &= \int_0^{2\pi}\int_0^\infty e^{-r^2}r\,drdt \\ &= 2\pi\int_0^\infty\left(-\frac{e^{-r^2}}{2}\right)'dr \\ &= \pi \end{aligned}$$

Thus, we are led to the formula in the statement. □

As a comment here, the above proof, going to 2D in order to prove a 1D result, might seem quite puzzling. But this is the simplest proof of the Gauss formula, believe me, the point being that you cannot compute the primitive of e^{-x^2}, that is how life is.

Along the same lines, but at a more advanced level, the point is that we have Gauss laws and integral formulae in all dimensions $N \in \mathbb{N}$, and, by a strange twist of fate, the "simplest" such dimension, at least in relation with our Gauss type questions, is $N = 2$, as shown by the above simple computation, instead of the $N = 1$ that we would normally expect. But more on this later, after learning about the Gaussian laws.

As yet another comment, there is a similarity here with the question, discussed in chapter 2, of computing areas and volumes. Indeed, there is a dimensionality parameter $N \in \mathbb{N}$ involved here, and contrary to general mathematics, where $N = 1$ is the simplest dimension, where the theory of integration is first developed, by computing areas, and then extended via Fubini to higher dimensions $N = 2, 3, 4, \ldots$, in the real life $N = 2$, corresponding to computing volumes, in in fact the simplest dimension. Indeed, just plunge your volume into a body of water, then measure the water displacement in order to have your volume, and that is it. Far simpler than computing areas.

Back to work, we can now introduce candidates for the normal distributions:

DEFINITION 3.4. *The normal law of parameter* 1 *is the following measure:*

$$g_1 = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

More generally, the normal law of parameter $t > 0$ is the following measure:

$$g_t = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx$$

These are also called Gaussian distributions, with "g" standing for Gauss.

These laws are usually denoted $\mathcal{N}(0,1)$ and $\mathcal{N}(0,t)$, with the parameters being the mean and variance, but we will not need such complications here, all our theory using centered laws. Observe also that our laws have indeed mass 1, as they should, due to:

$$\int_{\mathbb{R}} e^{-x^2/2t}dx = \int_{\mathbb{R}} e^{-y^2}\sqrt{2t}\,dy = \sqrt{2\pi t}$$

Speaking variance of the normal laws, as a first result, we have:

PROPOSITION 3.5. *We have the variance formula*

$$V(g_t) = t$$

valid for any $t > 0$.

PROOF. The first moment is 0, because our normal law g_t is centered. As for the second moment, this can be computed as follows:

$$\begin{aligned}
M_2 &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^2 e^{-x^2/2t}dx \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} (tx)\left(-e^{-x^2/2t}\right)' dx \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} te^{-x^2/2t}dx \\
&= t
\end{aligned}$$

We conclude from this that the variance is $V = M_2 = t$. □

Here is another result about the normal laws, which is widely useful in practice:

THEOREM 3.6. *We have the following formula, valid for any* $t > 0$*:*

$$F_{g_t}(x) = e^{-tx^2/2}$$

In particular, the normal laws satisfy $g_s * g_t = g_{s+t}$*, for any* $s,t > 0$.

PROOF. The Fourier transform formula can be established as follows:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-z^2/2t+ixz}dz \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(z/\sqrt{2t}-\sqrt{t/2}\,iz)^2-tx^2/2}dz \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-y^2-tx^2/2}\sqrt{2t}\,dy \\
&= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\int_{\mathbb{R}} e^{-y^2}dy \\
&= e^{-tx^2/2}
\end{aligned}$$

As for $g_s * g_t = g_{s+t}$, this follows from the fact that $\log F_{g_t}$ is linear in t. □

3b. Central limits

We are now ready to state and prove the CLT, as follows:

THEOREM 3.7 (CLT). *Given real variables $f_1,f_2,f_3,\ldots\in L^\infty(X)$ which are i.i.d., centered, and with common variance $t>0$, we have*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i\sim g_t$$

with $n\to\infty$, in moments.

PROOF. In terms of moments, the Fourier transform is given by:

$$\begin{aligned}F_f(x) &= \mathbb{E}\left(\sum_{r=0}^\infty\frac{(ixf)^r}{r!}\right)\\ &= \sum_{r=0}^\infty\frac{(ix)^r\mathbb{E}(f^r)}{r!}\\ &= \sum_{r=0}^\infty\frac{i^rM_r(f)}{r!}\,x^r\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}F(x) &= \left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n\\ &= \left[1-\frac{tx^2}{2n}+O(n^{-2})\right]^n\\ &\simeq e^{-tx^2/2}\end{aligned}$$

But this function being the Fourier transform of g_t, we obtain the result. □

Let us discuss now some further properties of the normal law. We first have:

PROPOSITION 3.8. *The even moments of the normal law are the numbers*

$$M_k(g_t)=t^{k/2}\times k!!$$

where $k!!=(k-1)(k-3)(k-5)\ldots$, and the odd moments vanish.

PROOF. We have the following computation, valid for any integer $k\in\mathbb{N}$:

$$\begin{aligned}
M_k &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}y^ke^{-y^2/2t}dy\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}(ty^{k-1})\left(-e^{-y^2/2t}\right)'dy\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}t(k-1)y^{k-2}e^{-y^2/2t}dy\\
&= t(k-1)\times\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}y^{k-2}e^{-y^2/2t}dy\\
&= t(k-1)M_{k-2}
\end{aligned}$$

Now recall from the proof of Proposition 3.5 that we have $M_0=1$, $M_1=0$. Thus by recurrence, we are led to the formula in the statement. □

We have the following alternative formulation of the above result:

PROPOSITION 3.9. *The moments of the normal law are the numbers*

$$M_k(g_t)=t^{k/2}|P_2(k)|$$

where $P_2(k)$ is the set of pairings of $\{1,\ldots,k\}$.

PROOF. Let us count the pairings of $\{1,\ldots,k\}$. In order to have such a pairing, we must pair 1 with one of the numbers $2,\ldots,k$, and then use a pairing of the remaining $k-2$ numbers. Thus, we have the following recurrence formula:

$$|P_2(k)|=(k-1)|P_2(k-2)|$$

As for the initial data, this is $P_1=0$, $P_2=1$. Thus, we are led to the result. □

We are not done yet, and here is one more improvement of the above:

THEOREM 3.10. *The moments of the normal law are the numbers*

$$M_k(g_t)=\sum_{\pi\in P_2(k)}t^{|\pi|}$$

where $P_2(k)$ is the set of pairings of $\{1,\ldots,k\}$, and $|.|$ is the number of blocks.

PROOF. This follows indeed from Proposition 3.9, because the number of blocks of a pairing of $\{1,\ldots,k\}$ is trivially $k/2$, independently of the pairing. □

We will see later on that many other interesting probability distributions are subject to similar formulae regarding their moments, involving partitions.

Before that, however, let us go into a digression on how measures can be reconstructed, out of their moments. This is something quite interesting, that we will need on a regular basis, in what follows, so definitely good knowledge. In addition, in relation with the normal laws, we will reach in this way to an alternative way of answering our questions from the beginning of this chapter, without cheating, as we did in the above.

Our question is called "moment problem", and as a first and main result about it, we have the following theorem, called Stieltjes inversion formula:

THEOREM 3.11. *The density of a real probability measure μ can be recaptured from the sequence of moments $\{M_k\}_{k\geq 0}$ via the Stieltjes inversion formula*

$$d\mu(x) = \lim_{t\searrow 0} -\frac{1}{\pi}\, Im\,(G(x+it)) \cdot dx$$

where the function on the right, given in terms of moments by

$$G(\xi) = \xi^{-1} + M_1\xi^{-2} + M_2\xi^{-3} + \ldots$$

is the Cauchy transform of the measure μ.

PROOF. The Cauchy transform of our measure μ is given by:

$$\begin{aligned} G(\xi) &= \xi^{-1}\sum_{k=0}^{\infty} M_k\xi^{-k} \\ &= \int_{\mathbb{R}} \frac{\xi^{-1}}{1-\xi^{-1}y}\, d\mu(y) \\ &= \int_{\mathbb{R}} \frac{1}{\xi - y}\, d\mu(y) \end{aligned}$$

Now with $\xi = x + it$, we obtain the following formula:

$$\begin{aligned} Im(G(x+it)) &= \int_{\mathbb{R}} Im\left(\frac{1}{x-y+it}\right) d\mu(y) \\ &= \int_{\mathbb{R}} \frac{1}{2i}\left(\frac{1}{x-y+it} - \frac{1}{x-y-it}\right) d\mu(y) \\ &= -\int_{\mathbb{R}} \frac{t}{(x-y)^2+t^2}\, d\mu(y) \end{aligned}$$

By integrating over $[a,b]$ we obtain, with the change of variables $x=y+tz$:

$$\begin{aligned}\int_a^b Im(G(x+it))dx &= -\int_{\mathbb{R}}\int_a^b \frac{t}{(x-y)^2+t^2}\,dx\,d\mu(y)\\ &= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{t}{(tz)^2+t^2}\,t\,dz\,d\mu(y)\\ &= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{1}{1+z^2}\,dz\,d\mu(y)\\ &= -\int_{\mathbb{R}}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)d\mu(y)\end{aligned}$$

Now observe that with $t\searrow 0$ we have:

$$\lim_{t\searrow 0}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)=\begin{cases}\frac{\pi}{2}-\frac{\pi}{2}=0 & (y<a)\\ \frac{\pi}{2}-0=\frac{\pi}{2} & (y=a)\\ \frac{\pi}{2}-(-\frac{\pi}{2})=\pi & (a<y<b)\\ 0-(-\frac{\pi}{2})=\frac{\pi}{2} & (y=b)\\ -\frac{\pi}{2}-(-\frac{\pi}{2})=0 & (y>b)\end{cases}$$

We therefore obtain the following formula:

$$\lim_{t\searrow 0}\int_a^b Im(G(x+it))dx=-\pi\left(\mu(a,b)+\frac{\mu(a)+\mu(b)}{2}\right)$$

Thus, we are led to the conclusion in the statement. □

Before getting further, let us mention that the above result does not fully solve the moment problem, because we still have the question of understanding when a sequence of numbers $M_1,M_2,M_3,\ldots$ can be the moments of a measure μ. We have here:

THEOREM 3.12. *A sequence of numbers $M_0,M_1,M_2,M_3,\ldots\in\mathbb{R}$, with $M_0=1$, is the series of moments of a real probability measure μ precisely when:*

$$\begin{vmatrix}M_0\end{vmatrix}\geq 0\quad,\quad\begin{vmatrix}M_0&M_1\\ M_1&M_2\end{vmatrix}\geq 0\quad,\quad\begin{vmatrix}M_0&M_1&M_2\\ M_1&M_2&M_3\\ M_2&M_3&M_4\end{vmatrix}\geq 0\quad,\quad\ldots$$

That is, the associated Hankel determinants must be all positive.

PROOF. This is something a bit more advanced, the idea being as follows:

(1) As a first observation, the positivity conditions in the statement tell us that the following associated linear forms must be positive:

$$\sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}\geq 0$$

(2) But this is something very classical, in one sense the result being elementary, coming from the following computation, which shows that we have positivity indeed:

$$\begin{aligned}\int_{\mathbb{R}}\left|\sum_{i=1}^n c_ix^i\right|^2 d\mu(x) &= \int_{\mathbb{R}}\sum_{i,j=1}^n c_i\bar{c}_jx^{i+j}d\mu(x)\\ &= \sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}\end{aligned}$$

(3) As for the other sense, here the result comes once again from the above formula, this time via some standard functional analysis. □

Getting back now to Stieltjes inversion, we have, as a first application:

THEOREM 3.13. *The measure having as moments the numbers*

$$M_k = t^{k/2} \times k!!$$

is the normal law of parameter t, having density as follows:

$$g_t = \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx$$

Moreover, based on this, we can rewrite probability, with the CLT coming first.

PROOF. This is certainly something that we know, but my point comes from what is said at the end, rewriting probability theory, with the CLT coming first. So, let us forget all the probability theory that we know, or almost, and have as starting point the Stieltjes inversion formula, which is something purely mathematical, coming from complex analysis. With this, we can develop the theory of normal variables, as follows:

(1) First, let us look more in detail at the proof of the CLT, or rather at the beginning of that proof, making no reference to the normal variables. After some thinking and computations, that we will leave here as an instructive exercise, we conclude that the "normal laws" that we are looking for, coming via performing a central limit, should be precisely the laws having as sequence of moments the numbers $M_k = t^{k/2} \times k!!$.

(2) But by Stieltjes inversion, with $M_k = t^{k/2} \times k!!$ as data, we come upon the measure g_t in the statement, and with the computation here being an instructive calculus exercise, again left to you reader. Thus we "rediscovered" these normal laws, and with the remark that in this way, we short-circuit the Gauss integral formula too, because the laws coming via Stieltjes inversion are by definition probability measures, of mass 1. □

All this is quite interesting, philosophically speaking. In fact, thinking at the CLT and at the normal variables, in all possible ways, interchanging definitions and theorems, playing with the dimension $N \in \mathbb{N}$ where your variables live, which was $N = 1$ in the above, save for some $N = 2$ in the proof of the Gauss formula, why not throwing in some physics too, say in relation with the heat diffusion equation, and the heat kernel, and so on, is an very useful pastime, teaching you many things. We will be back to this.

As another basic application of the Stieltjes formula, let us solve the moment problem for the Catalan numbers C_k, and for the central binomial coefficients D_k. We first have:

THEOREM 3.14. *The real measure having as even moments the Catalan numbers, $C_k = \frac{1}{k+1}\binom{2k}{k}$, and having all odd moments* 0 *is the measure*

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

called Wigner semicircle law on $[-2,2]$.

PROOF. In order to apply the inversion formula, our starting point will be a standard calculus formula for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^\infty C_k z^k = \frac{1-\sqrt{1-4z}}{2z}$$

By using this formula with $z=\xi^{-2}$, we obtain the following formula:

$$\begin{aligned}G(\xi) &= \xi^{-1}\sum_{k=0}^\infty C_k\xi^{-2k}\\ &= \xi^{-1}\cdot\frac{1-\sqrt{1-4\xi^{-2}}}{2\xi^{-2}}\\ &= \frac{\xi}{2}\left(1-\sqrt{1-4\xi^{-2}}\right)\\ &= \frac{\xi}{2}-\frac{1}{2}\sqrt{\xi^2-4}\end{aligned}$$

Now let us apply Theorem 3.11. The study here goes as follows:

(1) According to the general philosophy of the Stieltjes formula, the first term, namely $\xi/2$, which is "trivial", will not contribute to the density.

(2) As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{\xi^2-4}$ will be replaced by the "dual" square root $\sqrt{4-x^2}\,dx$, and that we have to multiply everything by $-1/\pi$.

(3) As a conclusion, by Stieltjes inversion we obtain the following density:

$$d\mu(x) = -\frac{1}{\pi}\cdot -\frac{1}{2}\sqrt{4-x^2}\,dx = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

Thus, we have obtained the mesure in the statement, and we are done. □

We have the following version of the above result:

THEOREM 3.15. *The real measure having as sequence of moments the Catalan numbers, $C_k = \frac{1}{k+1}\binom{2k}{k}$, is the measure*

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

called Marchenko-Pastur law on $[0,4]$.

PROOF. As before, we use the standard formula for the generating series of the Catalan numbers. With $z = \xi^{-1}$ in that formula, we obtain the following formula:

$$\begin{aligned} G(\xi) &= \xi^{-1}\sum_{k=0}^\infty C_k\xi^{-k} \\ &= \xi^{-1}\cdot\frac{1-\sqrt{1-4\xi^{-1}}}{2\xi^{-1}} \\ &= \frac{1}{2}\left(1-\sqrt{1-4\xi^{-1}}\right) \\ &= \frac{1}{2}-\frac{1}{2}\sqrt{1-4\xi^{-1}} \end{aligned}$$

With this in hand, let us apply now the Stieltjes inversion formula, from Theorem 3.11. We obtain, a bit as before in Theorem 3.14, the following density:

$$d\mu(x) = -\frac{1}{\pi}\cdot -\frac{1}{2}\sqrt{4x^{-1}-1}\,dx = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

Thus, we are led to the conclusion in the statement. □

Regarding now the central binomial coefficients, we have here:

THEOREM 3.16. *The real probability measure having as moments the central binomial coefficients, $D_k = \binom{2k}{k}$, is the measure*

$$\alpha_1 = \frac{1}{\pi\sqrt{x(4-x)}}\,dx$$

called arcsine law on $[0,4]$.

PROOF. We have the following computation, using some standard formulae:

$$\begin{aligned}
G(\xi) &= \xi^{-1}\sum_{k=0}^\infty D_k\xi^{-k}\\
&= \frac{1}{\xi}\sum_{k=0}^\infty D_k\left(-\frac{t}{4}\right)^k\\
&= \frac{1}{\xi}\cdot\frac{1}{\sqrt{1-4/\xi}}\\
&= \frac{1}{\sqrt{\xi(\xi-4)}}
\end{aligned}$$

But this gives the density in the statement, via Theorem 3.11. □

Finally, we have the following version of the above result:

THEOREM 3.17. *The real probability measure having as moments the middle binomial coefficients, $E_k=\binom{k}{[k/2]}$, is the following law on $[-2,2]$,*

$$\sigma_1=\frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\,dx$$

called "square root" of the arcsine law on $[0,4]$.

PROOF. In terms of the central binomial coefficients D_k, we have:

$$E_{2k}=\binom{2k}{k}=\frac{(2k)!}{k!k!}=D_k$$

$$E_{2k-1}=\binom{2k-1}{k}=\frac{(2k-1)!}{k!(k-1)!}=\frac{D_k}{2}$$

Standard calculus based on the Taylor formula for $(1+t)^{-1/2}$ gives:

$$\frac{1}{2x}\left(\sqrt{\frac{1+2x}{1-2x}}-1\right)=\sum_{k=0}^\infty E_kx^k$$

With $x=\xi^{-1}$ we obtain the following formula for the Cauchy transform:

$$\begin{aligned} G(\xi) &= \xi^{-1}\sum_{k=0}^\infty E_k\xi^{-k} \\ &= \frac{1}{\xi}\left(\sqrt{\frac{1+2/\xi}{1-2/\xi}}-1\right) \\ &= \frac{1}{\xi}\left(\sqrt{\frac{\xi+2}{\xi-2}}-1\right) \end{aligned}$$

By Stieltjes inversion we obtain the density in the statement. □

You might probably wonder, what is the point with all these latter measures. These are in fact the main laws in Random Matrix Theory (RMT), that we will discuss later.

3c. Complex variables

We have seen so far a number of interesting results regarding the normal laws, and their geometric interpretation. As a last topic of this chapter, let us discuss now the complex analogues of all this. To start with, we have the following definition:

Definition 3.18. *The complex Gaussian law of parameter $t>0$ is*

$$G_t = law\left(\frac{1}{\sqrt{2}}(a+ib)\right)$$

where a,b are independent, each following the law g_t.

As in the real case, these measures form convolution semigroups:

Theorem 3.19. *The complex Gaussian laws have the property*

$$G_s * G_t = G_{s+t}$$

for any $s,t>0$, and so they form a convolution semigroup.

Proof. This follows indeed from the real result, namely $g_s*g_t=g_{s+t}$, established in Theorem 3.6, simply by taking real and imaginary parts. □

We have as well the following complex analogue of the CLT:

Theorem 3.20 (CCLT). *Given complex variables $f_1,f_2,f_3,\ldots\in L^\infty(X)$ which are i.i.d., centered, and with common variance $t>0$, we have*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i \sim G_t$$

with $n\to\infty$, in moments.

Proof. This follows indeed from the real CLT, established in Theorem 3.7, simply by taking the real and imaginary parts of all variables involved. □

Regarding now the moments, the situation is more complicated than in the real case, because in order to have good results, we have to deal with both the complex variables, and their conjugates. Let us formulate the following definition:

Definition 3.21. *The moments a complex variable $f \in L^\infty(X)$ are the numbers*

$$M_k = \mathbb{E}(f^k)$$

depending on colored integers $k = \circ\bullet\bullet\circ\ldots$, with the conventions

$$f^\emptyset = 1 \quad , \quad f^\circ = f \quad , \quad f^\bullet = \bar{f}$$

and multiplicativity, in order to define the colored powers f^k.

Observe that, since $f, \bar{f}$ commute, we can permute terms, and restrict the attention to exponents of type $k = \ldots\circ\circ\circ\bullet\bullet\bullet\bullet\ldots$, if we want to. However, our result about the complex Gaussian laws, and other complex laws, later on, will actually look better without doing is, and so we will use Definition 3.21 as stated. We first have:

Theorem 3.22. *The moments of the complex normal law are given by*

$$M_k(G_t) = \begin{cases} t^p p! & (k \text{ uniform, of length } 2p) \\ 0 & (k \text{ not uniform}) \end{cases}$$

where $k = \circ\bullet\bullet\circ\ldots$ is called uniform when it contains the same number of $\circ$ and $\bullet$.

Proof. We must compute the moments, with respect to colored integer exponents $k = \circ\bullet\bullet\circ\ldots$, of the variable from Definition 3.18, namely:

$$f = \frac{1}{\sqrt{2}}(a + ib)$$

We can assume that we are in the case $t = 1$, and the proof here goes as follows:

(1) As a first observation, in the case where our exponent $k = \circ\bullet\bullet\circ\ldots$ is not uniform, a standard rotation argument shows that the corresponding moment of f vanishes. To be more precise, the variable $f' = wf$ is complex Gaussian too, for any complex number $w \in \mathbb{T}$, and from $M_k(f) = M_k(f')$ we obtain $M_k(f) = 0$, in this case.

(2) In the uniform case now, where the exponent $k=\circ\bullet\bullet\circ\ldots$ consists of p copies of $\circ$ and p copies of $\bullet$, the corresponding moment can be computed as follows:

$$\begin{aligned}
M_k &= \int (f\bar{f})^p \\
&= \frac{1}{2^p}\int (a^2+b^2)^p \\
&= \frac{1}{2^p}\sum_r \binom{p}{r}\int a^{2r}\int b^{2p-2r} \\
&= \frac{1}{2^p}\sum_r \binom{p}{r}(2r)!!(2p-2r)!! \\
&= \frac{1}{2^p}\sum_r \frac{p!}{r!(p-r)!}\cdot\frac{(2r)!}{2^r r!}\cdot\frac{(2p-2r)!}{2^{p-r}(p-r)!} \\
&= \frac{p!}{4^p}\sum_r \binom{2r}{r}\binom{2p-2r}{p-r}
\end{aligned}$$

(3) In order to finish now the computation, let us recall that we have the following formula, coming from the generalized binomial formula, or from the Taylor formula:

$$\frac{1}{\sqrt{1+t}}=\sum_{q=0}^\infty \binom{2q}{q}\left(\frac{-t}{4}\right)^q$$

By taking the square of this series, we obtain the following formula:

$$\begin{aligned}
\frac{1}{1+t} &= \sum_{qr}\binom{2q}{q}\binom{2r}{r}\left(\frac{-t}{4}\right)^{q+r} \\
&= \sum_p \left(\frac{-t}{4}\right)^p \sum_r \binom{2r}{r}\binom{2p-2r}{p-r}
\end{aligned}$$

Now by looking at the coefficient of t^p on both sides, we conclude that the sum on the right equals 4^p. Thus, we can finish the moment computation in (2), as follows:

$$M_k=\frac{p!}{4^p}\times 4^p=p!$$

We are therefore led to the conclusion in the statement. □

As before with the real Gaussian laws, a better-looking statement is in terms of partitions. Given a colored integer $k=\circ\bullet\bullet\circ\ldots$, we say that a pairing $\pi\in P_2(k)$ is matching when it pairs $\circ-\bullet$ symbols. With this convention, we have the following result:

THEOREM 3.23. *The moments of the complex normal law are the numbers*

$$M_k(G_t)=\sum_{\pi\in\mathcal{P}_2(k)}t^{|\pi|}$$

where $\mathcal{P}_2(k)$ are the matching pairings of $\{1,\ldots,k\}$, and $|.|$ is the number of blocks.

PROOF. This is a reformulation of Theorem 3.22. Indeed, we can assume that we are in the case $t=1$, and here we know from Theorem 3.22 that the moments are:

$$M_k=\begin{cases}(|k|/2)! & (k\text{ uniform})\\ 0 & (k\text{ not uniform})\end{cases}$$

On the other hand, the numbers $|\mathcal{P}_2(k)|$ are given by exactly the same formula. Indeed, in order to have a matching pairing of k, our exponent $k=\circ\bullet\bullet\circ\ldots$ must be uniform, consisting of p copies of $\circ$ and p copies of $\bullet$, with $p=|k|/2$. But then the matching pairings of k correspond to the permutations of the $\bullet$ symbols, as to be matched with $\circ$ symbols, and so we have $p!$ such pairings. Thus, we have the same formula as for the moments of f, and we are led to the conclusion in the statement. □

In practice, we also need to know how to compute joint moments of independent normal variables. We have here the following result, to be used later on:

THEOREM 3.24 (Wick formula). *Given independent variables f_i, each following the complex normal law G_t, with $t>0$ being a fixed parameter, we have the formula*

$$\mathbb{E}\left(f_{i_1}^{k_1}\ldots f_{i_s}^{k_s}\right)=t^{s/2}\#\left\{\pi\in\mathcal{P}_2(k)\Big|\pi\leq\ker i\right\}$$

where $k=k_1\ldots k_s$ and $i=i_1\ldots i_s$, for the joint moments of these variables, where $\pi\leq\ker i$ means that the indices of i must fit into the blocks of π, in the obvious way.

PROOF. This is something well-known, which can be proved as follows:

(1) Let us first discuss the case where we have a single variable f, which amounts in taking $f_i=f$ for any i in the formula in the statement. What we have to compute here are the moments of f, with respect to colored integer exponents $k=\circ\bullet\bullet\circ\ldots$, and the formula in the statement tells us that these moments must be:

$$\mathbb{E}(f^k)=t^{|k|/2}|\mathcal{P}_2(k)|$$

But this is the formula in Theorem 3.23, so we are done with this case.

(2) In general now, when expanding the product $f_{i_1}^{k_1}\ldots f_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts in counting the partitions in the statement, with the condition $\pi\leq\ker i$ there standing for the fact that we are doing the various type (1) computations independently, and then making the product. □

The above statement is one of the possible formulations of the Wick formula, and there are in fact many other formulations, which are all useful. Here is one alternative such formulation, which is quite popular, and that we will often use in what follows:

THEOREM 3.25 (Wick formula 2). *Given independent variables f_i, each following the complex normal law G_t, with $t > 0$ being a fixed parameter, we have the formula*

$$\mathbb{E}\left(f_{i_1}\dots f_{i_k}f_{j_1}^*\dots f_{j_k}^*\right)=t^k\#\left\{\pi\in S_k\Big|i_{\pi(r)}=j_r,\forall r\right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from the usual Wick formula, from Theorem 3.24. With some changes in the indices and notations, the formula there reads:

$$\mathbb{E}\left(f_{I_1}^{K_1}\dots f_{I_s}^{K_s}\right)=t^{s/2}\#\left\{\sigma\in\mathcal{P}_2(K)\Big|\sigma\leq\ker I\right\}$$

Now observe that we have $\mathcal{P}_2(K)=\emptyset$, unless the colored integer $K=K_1\dots K_s$ is uniform, in the sense that it contains the same number of $\circ$ and $\bullet$ symbols. Up to permutations, the non-trivial case, where the moment is non-vanishing, is the case where the colored integer $K=K_1\dots K_s$ is of the following special form:

$$K=\underbrace{\circ\circ\dots\circ}_k\ \underbrace{\bullet\bullet\dots\bullet}_k$$

So, let us focus on this case, which is the non-trivial one. Here we have $s=2k$, and we can write the multi-index $I=I_1\dots I_s$ in the following way:

$$I=i_1\dots i_k\ j_1\dots j_k$$

With these changes made, the above usual Wick formula reads:

$$\mathbb{E}\left(f_{i_1}\dots f_{i_k}f_{j_1}^*\dots f_{j_k}^*\right)=t^k\#\left\{\sigma\in\mathcal{P}_2(K)\Big|\sigma\leq\ker(ij)\right\}$$

The point now is that the matching pairings $\sigma\in\mathcal{P}_2(K)$, with $K=\circ\dots\circ\bullet\dots\bullet$, of length $2k$, as above, correspond to the permutations $\pi\in S_k$, in the obvious way. With this identification made, the above modified usual Wick formula becomes:

$$\mathbb{E}\left(f_{i_1}\dots f_{i_k}f_{j_1}^*\dots f_{j_k}^*\right)=t^k\#\left\{\pi\in S_k\Big|i_{\pi(r)}=j_r,\forall r\right\}$$

Thus, we have reached to the formula in the statement, and we are done. □

Finally, here is one more formulation of the Wick formula, which is useful as well:

THEOREM 3.26 (Wick formula 3). *Given independent variables f_i, each following the complex normal law G_t, with $t > 0$ being a fixed parameter, we have the formula*

$$\mathbb{E}\left(f_{i_1}f_{j_1}^*\dots f_{i_k}f_{j_k}^*\right)=t^k\#\left\{\pi\in S_k\Big|i_{\pi(r)}=j_r,\forall r\right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from our second Wick formula, from Theorem 3.25, simply by permuting the terms, as to have an alternating sequence of plain and conjugate variables. Alternatively, we can start with Theorem 3.24, and then perform the same manipulations as in the proof of Theorem 3.25, but with the exponent being this time as follows:

$$K = \underbrace{\circ\bullet\circ\bullet\ldots\ldots\circ\bullet}_{2k}$$

Thus, we are led to the conclusion in the statement. □

3d. Random matrices

Finally, a word about random matrices, that we will need later. Let us start with:

DEFINITION 3.27. *A complex Gaussian matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

which has i.i.d. centered complex normal entries.

Here we use the notion of complex normal variable, introduced and studied before. To be more precise, let us recall that the complex Gaussian law of parameter $t > 0$ is by definition the following law, with a, b being independent, each following the law g_t:

$$G_t = law\left(\frac{1}{\sqrt{2}}(a + ib)\right)$$

With this notion in hand, the assumption in the above definition is that all the matrix entries Z_{ij} are independent, and follow this law G_t, for a fixed value of the parameter $t > 0$. We will see that the above matrices have an interesting, and "central" combinatorics, among all kinds of random matrices, with the study of the other random matrices being usually obtained as a modification of the study of the Gaussian matrices.

Here is now our first result, regarding the Gaussian matrices:

THEOREM 3.28. *Given a sequence of Gaussian random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, for some fixed $t > 0$, we have

$$M_k\left(\frac{Z_N}{\sqrt{N}}\right) \simeq t^{|k|/2}|\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ\bullet\bullet\circ\ldots$, in the $N \to \infty$ limit.

PROOF. This is something standard, which can be done as follows:

(1) We fix $N\in\mathbb N$, and we let $Z=Z_N$. Let us first compute the trace of Z^k. With $k=k_1\ldots k_s$, and with the convention $(ij)^\circ=ij,(ij)^\bullet=ji$, we have:

$$\begin{aligned}Tr(Z^k)&=Tr(Z^{k_1}\ldots Z^{k_s})\\&=\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z^{k_1})_{i_1i_2}(Z^{k_2})_{i_2i_3}\ldots(Z^{k_s})_{i_si_1}\\&=\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}\end{aligned}$$

(2) Next, we rescale our variable Z by a $\sqrt N$ factor, as in the statement, and we also replace the usual trace by its normalized version, $tr=Tr/N$. Our formula becomes:

$$tr\left(\left(\frac{Z}{\sqrt N}\right)^k\right)=\frac{1}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}$$

Thus, the moment that we are interested in is given by:

$$M_k\left(\frac{Z}{\sqrt N}\right)=\frac{1}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N\int_X(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}$$

(3) Let us apply now the Wick formula, from Theorem 3.24. We conclude that the moment that we are interested in is given by the following formula:

$$\begin{aligned}&M_k\left(\frac{Z}{\sqrt N}\right)\\=&\ \frac{t^{s/2}}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N\#\left\{\pi\in\mathcal P_2(k)\Big|\pi\leq\ker\left((i_1i_2)^{k_1},(i_2i_3)^{k_2},\ldots,(i_si_1)^{k_s}\right)\right\}\\=&\ t^{s/2}\sum_{\pi\in\mathcal P_2(k)}\frac{1}{N^{s/2+1}}\#\left\{i\in\{1,\ldots,N\}^s\Big|\pi\leq\ker\left((i_1i_2)^{k_1},(i_2i_3)^{k_2},\ldots,(i_si_1)^{k_s}\right)\right\}\end{aligned}$$

(4) Our claim now is that in the $N\to\infty$ limit the combinatorics of the above sum simplifies, with only the noncrossing partitions contributing to the sum, and with each of

them contributing precisely with a 1 factor, so that we will have, as desired:

$$\begin{aligned}
M_k\left(\frac{Z}{\sqrt{N}}\right) &= t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\Big(\delta_{\pi\in NC_2(k)}+O(N^{-1})\Big)\\
&\simeq t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\delta_{\pi\in NC_2(k)}\\
&= t^{s/2}|\mathcal{NC}_2(k)|
\end{aligned}$$

(5) In order to prove this, the first observation is that when k is not uniform, in the sense that it contains a different number of $\circ,\bullet$ symbols, we have $\mathcal{P}_2(k)=\emptyset$, and so:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=t^{s/2}|\mathcal{NC}_2(k)|=0$$

(6) Thus, we are left with the case where k is uniform. Let us examine first the case where k consists of an alternating sequence of $\circ$ and $\bullet$ symbols, as follows:

$$k=\underbrace{\circ\bullet\circ\bullet\ldots\ldots\circ\bullet}_{2p}$$

In this case it is convenient to relabel our multi-index $i=(i_1,\ldots,i_s)$, with $s=2p$, in the form $(j_1,l_1,j_2,l_2,\ldots,j_p,l_p)$. With this done, our moment formula becomes:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=t^p\sum_{\pi\in\mathcal{P}_2(k)}\frac{1}{N^{p+1}}\#\left\{j,l\in\{1,\ldots,N\}^p\Big|\pi\le\ker\left(j_1l_1,j_2l_1,j_2l_2,\ldots,j_1l_p\right)\right\}$$

Now observe that, with k being as above, we have an identification $\mathcal{P}_2(k)\simeq S_p$, obtained in the obvious way. With this done too, our moment formula becomes:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=t^p\sum_{\pi\in S_p}\frac{1}{N^{p+1}}\#\left\{j,l\in\{1,\ldots,N\}^p\Big|j_r=j_{\pi(r)+1},l_r=l_{\pi(r)},\forall r\right\}$$

(7) We are now ready to do our asymptotic study, and prove the claim in (4). Let indeed $\gamma\in S_p$ be the full cycle, which is by definition the following permutation:

$$\gamma=(1\,2\,\ldots\,p)$$

In terms of γ, the conditions $j_r=j_{\pi(r)+1}$ and $l_r=l_{\pi(r)}$ found above read:

$$\gamma\pi\le\ker j\quad,\quad\pi\le\ker l$$

Counting the number of free parameters in our moment formula, we obtain:

$$\begin{aligned}
M_k\left(\frac{Z}{\sqrt{N}}\right) &= \frac{t^p}{N^{p+1}}\sum_{\pi\in S_p}N^{|\pi|+|\gamma\pi|}\\
&= t^p\sum_{\pi\in S_p}N^{|\pi|+|\gamma\pi|-p-1}
\end{aligned}$$

(8) The point now is that the last exponent is well-known to be ≤ 0, with equality precisely when the permutation $\pi \in S_p$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \to \infty$ limit, as desired:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) \simeq t^p |\mathcal{NC}_2(k)|$$

This finishes the proof in the case of the exponents k which are alternating, and the case where k is an arbitrary uniform exponent is similar, by permuting everything. □

In practice now, the most important random matrices are in fact the real versions of the Gaussian matrices, called Wigner matrices, constructed as follows:

Definition 3.29. *A Wigner matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

which has i.i.d. centered complex normal entries, up to the constraint $Z = Z^*$.

This definition is something a bit compacted, and to be more precise here, a Wigner matrix is by definition a random matrix as follows, with the diagonal entries being real normal variables, $a_i \sim g_t$, for some $t > 0$, the upper diagonal entries being complex normal variables, $b_{ij} \sim G_t$, the lower diagonal entries being the conjugates of the upper diagonal entries, as indicated, and with all the variables a_i, b_{ij} being independent:

$$Z = \begin{pmatrix} a_1 & b_{12} & \ldots & \ldots & b_{1N} \\ \bar{b}_{12} & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{N-1} & b_{N-1,N} \\ \bar{b}_{1N} & \ldots & \ldots & \bar{b}_{N-1,N} & a_N \end{pmatrix}$$

This might look a bit complicated, but for many concrete applications, the Wigner matrices are in fact the central objects in random matrix theory, and in particular, they are often more important than the Gaussian ones. In fact, these are the random matrices which were first considered and investigated, a long time ago, by Wigner himself.

As before with the complex Gaussian matrices, we would like to compute the law of the Wigner matrices, in the $N \to \infty$ limit. But for this purpose, no need to use the Wick formula and do heavy combinatorics again, because we can use the following simple fact, making the connection with our previous computations for the Gaussian matrices:

Proposition 3.30. *Given a Gaussian matrix Z, with independent entries following the centered complex normal law G_t, with $t > 0$, if we write*

$$Z = \frac{1}{\sqrt{2}}(X + iY)$$

with X, Y being self-adjoint, then both X, Y are Wigner matrices, of parameter t.

Proof. This is something elementary, which can be done in two steps, as follows:

(1) As a first observation, the result holds at $N=1$. Indeed, here our Gaussian matrix Z is just a random variable, subject to the condition $Z\sim G_t$. But recall that the law G_t is by definition as follows, with X,Y being independent, each following the law g_t:

$$G_t=law\left(\frac{1}{\sqrt{2}}(X+iY)\right)$$

Thus in this case, $N=1$, the variables X,Y that we obtain in the statement, as rescaled real and imaginary parts of Z, are subject to the condition $X,Y\sim g_t$, and so are Wigner matrices of size $N=1$ and parameter $t>0$, as in Definition 3.29.

(2) In the general case now, $N\in\mathbb N$, the proof is similar, by using the basic behavior of the real and complex normal variables with respect to sums. □

By using now our previous computations for the Gaussian matrices, we obtain:

Theorem 3.31. *Given a sequence of Wigner random matrices*

$$Z_N\in M_N(L^\infty(X))$$

having independent G_t variables as entries, up to $Z_N=Z_N^*$, we have, with $N\to\infty$,

$$\frac{Z_N}{\sqrt{N}}\sim\gamma_t$$

the limiting measure being the Wigner semicircle law $\gamma_t=\frac{1}{2\pi t}\sqrt{4t-x^2}dx$.

Proof. We know from Theorem 3.28 that for a sequence of complex Gaussian matrices $Y_N\in M_N(L^\infty(X))$ we have the following formula, valid for any colored integer $K=\circ\bullet\bullet\circ\ldots$, with $\mathcal{NC}_2$ standing for noncrossing matching pairings:

$$M_K\left(\frac{Y_N}{\sqrt{N}}\right)\simeq t^{|K|/2}|\mathcal{NC}_2(K)|$$

By doing some combinatorics, we deduce from this that we have the following formula for the moments of the matrices $Re(Y_N)$, with respect to usual exponents, $k\in\mathbb N$:

$$\begin{aligned}M_k\left(\frac{Re(Y_N)}{\sqrt{N}}\right)&=2^{-k}\cdot M_k\left(\frac{Y_N}{\sqrt{N}}+\frac{Y_N^*}{\sqrt{N}}\right)\\&=2^{-k}\sum_{|K|=k}M_K\left(\frac{Y_N}{\sqrt{N}}\right)\\&\simeq 2^{-k}\sum_{|K|=k}t^{k/2}|\mathcal{NC}_2(K)|\\&=2^{-k}\cdot t^{k/2}\cdot 2^{k/2}|\mathcal{NC}_2(k)|\\&=2^{-k/2}\cdot t^{k/2}|NC_2(k)|\end{aligned}$$

Now since the matrices $Z_N = \sqrt{2}Re(Y_N)$ are of Wigner type, this gives the result, via the fact that the moments of γ_t are the Catalan numbers, from Theorem 3.14. □

Along the same lines, let us discuss now the complex Wishart matrices, which are the positive analogues of the Gaussian and Wigner matrices, constructed as follows:

DEFINITION 3.32. *A complex Wishart matrix is a random matrix of type*

$$W = YY^* \in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix, with entries following the law G_t.

Due to the formula $W = YY^*$, the Wishart matrices are positive, in the abstract positivity sense of linear algebra. The result regarding them is as follows:

THEOREM 3.33. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times N$ complex Gaussian of parameter 1, we have

$$\frac{W_N}{N} \sim \frac{1}{2\pi}\sqrt{4x^{-1} - 1}\,dx$$

with $N \to \infty$, with the limiting measure being the Marchenko-Pastur law π_1.

PROOF. There are several possible proofs for this result, as follows:

(1) A first method is by using the result that we have from the above, for the complex Gaussian matrices Y_N. Indeed, we know from there that we have the following formula, valid for any colored integer $K = \circ \bullet \bullet \circ \ldots$, in the $N \to \infty$ limit:

$$M_K\left(\frac{Y_N}{\sqrt{N}}\right) \simeq t^{|K|/2}|\mathcal{NC}_2(K)|$$

With $K = \circ \bullet \circ \bullet \ldots$, alternating word of length $2k$, with $k \in \mathbb{N}$, this gives:

$$M_k\left(\frac{Y_N Y_N^*}{N}\right) \simeq t^k|\mathcal{NC}_2(K)|$$

Thus, in terms of the Wishart matrix $W_N = Y_N Y_N^*$ we have, for any $k \in \mathbb{N}$:

$$M_k\left(\frac{W_N}{N}\right) \simeq t^k|\mathcal{NC}_2(K)|$$

The point now is that, by doing some combinatorics, we have:

$$|\mathcal{NC}_2(K)| = |NC_2(2k)| = C_k$$

Thus, we are led to the formula in the statement.

(2) A second method, that we will explain now as well, is by proving the result directly, starting from definitions. The matrix entries of our matrix $W = W_N$ are given by:

$$W_{ij} = \sum_{r=1}^N Y_{ir}\bar{Y}_{jr}$$

Thus, the normalized traces of powers of W are given by the following formula:

$$\begin{aligned} tr(W^k) &= \frac{1}{N}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N W_{i_1i_2}W_{i_2i_3}\ldots W_{i_ki_1}\\ &= \frac{1}{N}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N Y_{i_1r_1}\bar{Y}_{i_2r_1}Y_{i_2r_2}\bar{Y}_{i_3r_2}\ldots Y_{i_kr_k}\bar{Y}_{i_1r_k}\end{aligned}$$

By rescaling now W by a $1/N$ factor, as in the statement, we obtain:

$$tr\left(\left(\frac{W}{N}\right)^k\right) = \frac{1}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N Y_{i_1r_1}\bar{Y}_{i_2r_1}Y_{i_2r_2}\bar{Y}_{i_3r_2}\ldots Y_{i_kr_k}\bar{Y}_{i_1r_k}$$

By using now the Wick rule, we obtain the following formula for the moments, with $K = \circ\bullet\circ\bullet\ldots$, alternating word of length $2k$, and with $I = (i_1r_1, i_2r_1, \ldots, i_kr_k, i_1r_k)$:

$$\begin{aligned} M_k\left(\frac{W}{N}\right) &= \frac{t^k}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N \#\left\{\pi\in\mathcal{P}_2(K)\Big|\pi\leq\ker I\right\}\\ &= \frac{t^k}{N^{k+1}}\sum_{\pi\in\mathcal{P}_2(K)}\#\left\{i,r\in\{1,\ldots,N\}^k\Big|\pi\leq\ker I\right\}\end{aligned}$$

In order to compute this quantity, we use the standard bijection $\mathcal{P}_2(K)\simeq S_k$. By identifying the pairings $\pi\in\mathcal{P}_2(K)$ with their counterparts $\pi\in S_k$, we obtain:

$$M_k\left(\frac{W}{N}\right) = \frac{t^k}{N^{k+1}}\sum_{\pi\in S_k}\#\left\{i,r\in\{1,\ldots,N\}^k\Big|i_s = i_{\pi(s)+1}, r_s = r_{\pi(s)}, \forall s\right\}$$

Now let $\gamma\in S_k$ be the full cycle, $\gamma = (1\,2\,\ldots\,k)$. The general factor in the product computed above is then 1 precisely when following two conditions are satisfied:

$$\gamma\pi\leq\ker i\quad,\quad\pi\leq\ker r$$

Counting the number of free parameters in our moment formula, we obtain:

$$M_k\left(\frac{W}{N}\right) = t^k\sum_{\pi\in S_k}N^{|\pi|+|\gamma\pi|-k-1}$$

The point now is that the last exponent is well-known to be ≤ 0, with equality precisely when the permutation $\pi \in S_k$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \to \infty$ limit:

$$M_k\left(\frac{W}{N}\right) \simeq t^k C_k$$

Thus, we are led to the conclusion in the statement. □

Many other things can be said about the random matrices, notably with a number of more specialized results, yielding the arcsine laws too. We will be back to this.

So long for normal variables. All the above was of course quite quick. For a more detailed introduction, you can check Feller [33], or Durrett [25].

3e. Exercises

Exercises:

Exercise 3.34.

Exercise 3.35.

Exercise 3.36.

Exercise 3.37.

Exercise 3.38.

Exercise 3.39.

Exercise 3.40.

Exercise 3.41.

Bonus exercise.

CHAPTER 4

Poisson laws

4a. Poisson laws

We have seen so far that the centered normal laws g_t and their complex analogues G_t, which appear from the Central Limit Theorem (CLT), have interesting combinatorial properties, and appear in several group-theoretical and geometric contexts.

We discuss here the discrete counterpart of these results. The mathematics will involve the Poisson laws p_t, which appear via the Poisson Limit Theorem (PLT), and their generalized versions p_ν, called compound Poisson laws, which appear via the Compound Poisson Limit Theorem (CPLT). Let us start with the following definition:

Definition 4.1. *The Poisson law of parameter 1 is the following measure,*

$$p_1 = \frac{1}{e} \sum_{k \in \mathbb{N}} \frac{\delta_k}{k!}$$

and the Poisson law of parameter $t > 0$ is the following measure,

$$p_t = e^{-t} \sum_{k \in \mathbb{N}} \frac{t^k}{k!} \, \delta_k$$

with the letter "p" standing for Poisson.

We are using here, as before, some simplified notations for these laws, which are in tune with the notations g_t, G_t that we used for the centered Gaussian laws. Observe that our laws have indeed mass 1, as they should, due to the following key formula:

$$e^t = \sum_{k \in \mathbb{N}} \frac{t^k}{k!}$$

We will see in the moment why these measures appear a bit everywhere, in discrete contexts, the reasons for this coming from the Poisson Limit Theorem (PLT). Let us first develop some general theory. We first have the following result:

Theorem 4.2. *We have the following formula, for any $s, t > 0$,*

$$p_s * p_t = p_{s+t}$$

so the Poisson laws form a convolution semigroup.

Proof. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula, we obtain:

$$\begin{aligned} p_s * p_t &= e^{-s}\sum_k \frac{s^k}{k!}\,\delta_k * e^{-t}\sum_l \frac{t^l}{l!}\,\delta_l \\ &= e^{-s-t}\sum_n \delta_n \sum_{k+l=n}\frac{s^k t^l}{k!l!} \\ &= e^{-s-t}\sum_n \frac{\delta_n}{n!}\sum_{k+l=n}\frac{n!}{k!l!}s^k t^l \\ &= e^{-s-t}\sum_n \frac{(s+t)^n}{n!}\,\delta_n \\ &= p_{s+t} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Next in line, we have the following result, which is fundamental as well:

Theorem 4.3. *The Poisson laws appear as formal exponentials*

$$p_t = \sum_k \frac{t^k(\delta_1 - \delta_0)^{*k}}{k!}$$

with respect to the convolution of measures $*$.

Proof. By using the binomial formula, the measure on the right is:

$$\begin{aligned} \mu &= \sum_k \frac{t^k}{k!}\sum_{r+s=k}(-1)^s\frac{k!}{r!s!}\delta_r \\ &= \sum_k t^k \sum_{r+s=k}(-1)^s\frac{\delta_r}{r!s!} \\ &= \sum_r \frac{t^r\delta_r}{r!}\sum_s \frac{(-1)^s}{s!} \\ &= \frac{1}{e}\sum_r \frac{t^r\delta_r}{r!} \\ &= p_t \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the Fourier transform computation, this is as follows:

Theorem 4.4. *The Fourier transform of* p_t *is given by*

$$F_{p_t}(y) = \exp\left((e^{iy}-1)t\right)$$

for any $t > 0$.

PROOF. We have indeed the following computation:

$$\begin{aligned}
F_{p_t}(y) &= e^{-t}\sum_k\frac{t^k}{k!}F_{\delta_k}(y)\\
&= e^{-t}\sum_k\frac{t^k}{k!}\,e^{iky}\\
&= e^{-t}\sum_k\frac{(e^{iy}t)^k}{k!}\\
&= \exp(-t)\exp(e^{iy}t)\\
&= \exp\left((e^{iy}-1)t\right)
\end{aligned}$$

Thus, we obtain the formula in the statement. □

Observe that the above formula gives an alternative proof for Theorem 4.2, by using the fact that the logarithm of the Fourier transform linearizes convolution.

4b. Limits, moments

We can now establish the Poisson Limit Theorem, as follows:

THEOREM 4.5 (PLT). *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\,\delta_1\right)^{*n}\to p_t$$

for any $t>0$.

PROOF. Let us denote by ν_n the measure under the convolution sign, namely:

$$\nu_n=\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\,\delta_1$$

We have the following computation, for the Fourier transform of the limit:

$$\begin{aligned}
F_{\delta_r}(y)=e^{iry}\quad &\implies\quad F_{\nu_n}(y)=\left(1-\frac{t}{n}\right)+\frac{t}{n}\,e^{iy}\\
&\implies\quad F_{\nu_n^{*n}}(y)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}\,e^{iy}\right)^n\\
&\implies\quad F_{\nu_n^{*n}}(y)=\left(1+\frac{(e^{iy}-1)t}{n}\right)^n\\
&\implies\quad F(y)=\exp\left((e^{iy}-1)t\right)
\end{aligned}$$

Thus, we obtain indeed the Fourier transform of p_t, as desired. □

Speaking limits which produce the Poisson laws, we have as well the following interesting result, making the connection with combinatorics and group theory:

THEOREM 4.6. *The probability for a random $\sigma \in S_N$ to have no fixed points is*

$$P \simeq \frac{1}{e}$$

in the $N \to \infty$ limit, where $e = 2.718\ldots$ is the usual constant from analysis. More generally, the main character of S_N, which counts these permutations, given by

$$\chi = \sum_i \sigma_{ii}$$

via the standard embedding $S_N \subset O_N$, follows the Poisson law p_1, in the $N \to \infty$ limit. Even more generally, the truncated characters of S_N, given by

$$\chi = \sum_{i=1}^{[tN]} \sigma_{ii}$$

with $t > 0$, follow the Poisson laws p_t, in the $N \to \infty$ limit.

PROOF. Obviously, many things going on here. The idea is as follows:

(1) In order to prove the first assertion, which is the key, and probably the most puzzling one, we will use the inclusion-exclusion principle. Let us set:

$$S_N^k = \left\{\sigma \in S_N \Big| \sigma(k) = k\right\}$$

The set of permutations having no fixed points, called derangements, is then:

$$X_N = \left(\bigcup_k S_N^k\right)^c$$

Now the inclusion-exclusion principle tells us that we have:

$$\begin{aligned}
|X_N| &= \left|\left(\bigcup_k S_N^k\right)^c\right| \\
&= |S_N| - \sum_k |S_N^k| + \sum_{k<l} |S_N^k \cap S_N^l| - \ldots + (-1)^N \sum_{k_1<\ldots<k_N} |S_N^{k_1} \cup \ldots \cup S_N^{k_N}| \\
&= N! - N(N-1)! + \binom{N}{2}(N-2)! - \ldots + (-1)^N \binom{N}{N}(N-N)! \\
&= \sum_{r=0}^{N} (-1)^r \binom{N}{r}(N-r)!
\end{aligned}$$

Thus, the probability that we are interested in, for a random permutation $\sigma \in S_N$ to have no fixed points, is given by the following formula:

$$P = \frac{|X_N|}{N!} = \sum_{r=0}^{N} \frac{(-1)^r}{r!}$$

Since on the right we have the expansion of $1/e$, this gives the result.

(2) Let us construct now the main character of S_N, as in the statement. The permutation matrices being given by $\sigma_{ij} = \delta_{i\sigma(j)}$, we have the following formula:

$$\chi(\sigma) = \sum_i \delta_{\sigma(i)i} = \#\left\{i \in \{1, \ldots, N\}\Big|\sigma(i) = i\right\}$$

In order to establish now the asymptotic result in the statement, regarding these characters, we must prove the following formula, for any $r \in \mathbb{N}$, in the $N \to \infty$ limit:

$$P(\chi = r) \simeq \frac{1}{r!e}$$

We already know, from (1), that this formula holds at $r = 0$. In the general case now, we have to count the permutations $\sigma \in S_N$ having exactly r points. Now since having such a permutation amounts in choosing r points among $1, \ldots, N$, and then permuting the $N - r$ points left, without fixed points allowed, we have:

$$\begin{aligned}
\#\left\{\sigma \in S_N\Big|\chi(\sigma) = r\right\} &= \binom{N}{r}\#\left\{\sigma \in S_{N-r}\Big|\chi(\sigma) = 0\right\}\\
&= \frac{N!}{r!(N-r)!}\#\left\{\sigma \in S_{N-r}\Big|\chi(\sigma) = 0\right\}\\
&= N! \times \frac{1}{r!} \times \frac{\#\left\{\sigma \in S_{N-r}\Big|\chi(\sigma) = 0\right\}}{(N-r)!}
\end{aligned}$$

By dividing everything by $N!$, we obtain from this the following formula:

$$\frac{\#\left\{\sigma \in S_N\Big|\chi(\sigma) = r\right\}}{N!} = \frac{1}{r!} \times \frac{\#\left\{\sigma \in S_{N-r}\Big|\chi(\sigma) = 0\right\}}{(N-r)!}$$

Now by using the computation at $r = 0$, that we already have, from (1), it follows that with $N \to \infty$ we have the following estimate:

$$P(\chi = r) \simeq \frac{1}{r!} \cdot P(\chi = 0) \simeq \frac{1}{r!} \cdot \frac{1}{e}$$

Thus, we obtain as limiting measure the Poisson law of parameter 1, as stated.

(3) Finally, let us construct the truncated characters of S_N, as in the statement. As before in the case $t=1$, we have the following computation, coming from definitions:

$$\chi_t(\sigma)=\sum_{i=1}^{[tN]}\delta_{\sigma(i)i}=\#\left\{i\in\{1,\ldots,[tN]\}\Big|\sigma(i)=i\right\}$$

Also before in the proofs of (1) and (2), we obtain by inclusion-exclusion that:

$$\begin{aligned}P(\chi_t=0) &= \frac{1}{N!}\sum_{r=0}^{[tN]}(-1)^r\sum_{k_1<\ldots<k_r<[tN]}|S_N^{k_1}\cap\ldots\cap S_N^{k_r}|\\ &= \frac{1}{N!}\sum_{r=0}^{[tN]}(-1)^r\binom{[tN]}{r}(N-r)!\\ &= \sum_{r=0}^{[tN]}\frac{(-1)^r}{r!}\cdot\frac{[tN]!(N-r)!}{N!([tN]-r)!}\end{aligned}$$

Now with $N\to\infty$, we obtain from this the following estimate:

$$P(\chi_t=0)\simeq\sum_{r=0}^{[tN]}\frac{(-1)^r}{r!}\cdot t^r\simeq e^{-t}$$

More generally, by counting the permutations $\sigma\in S_N$ having exactly r fixed points among $1,\ldots,[tN]$, as in the proof of (2), we obtain:

$$P(\chi_t=r)\simeq\frac{t^r}{r!e^t}$$

Thus, we obtain in the limit a Poisson law of parameter t, as stated. □

Quite exciting, all the above. In fact, similar results can be obtained for O_N,U_N, involving this time the normal laws g_t,G_t, and for many other groups as well, be them continuous, or finite. For an introduction to this, you can check my book [7].

Regarding now the moments of the Poisson laws, the result, which is something fascinating too, and in a sense more complicated than for the normal laws, is as follows:

THEOREM 4.7. *The moments of p_1 are the Bell numbers,*

$$M_k(p_1)=|P(k)|$$

where $P(k)$ is the set of partitions of $\{1,\ldots,k\}$. More generally, we have

$$M_k(p_t)=\sum_{\pi\in P(k)}t^{|\pi|}$$

for any $t>0$, where $|.|$ is the number of blocks.

PROOF. We know that the moments of p_1 are given by the following formula:

$$M_k=\frac{1}{e}\sum_r\frac{r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned}M_{k+1}&=\frac{1}{e}\sum_r\frac{r^k}{r!}\left(1+\frac{1}{r}\right)^k\\&=\frac{1}{e}\sum_r\frac{r^k}{r!}\sum_s\binom{k}{s}r^{-s}\\&=\sum_s\binom{k}{s}M_{k-s}\end{aligned}$$

But the Bell numbers $B_k=|P(k)|$ satisfy the same recurrence, so we have $M_k=B_k$, as claimed. Next, we know that the moments of p_t with $t>0$ are given by:

$$N_k=e^{-t}\sum_r\frac{t^rr^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned}N_{k+1}&=e^{-t}\sum_r\frac{t^{r+1}r^k}{r!}\left(1+\frac{1}{r}\right)^k\\&=e^{-t}\sum_r\frac{t^{r+1}r^k}{r!}\sum_s\binom{k}{s}r^{-s}\\&=t\sum_s\binom{k}{s}N_{k-s}\end{aligned}$$

But the numbers $S_k=\sum_{\pi\in P(k)}t^{|\pi|}$ are easily seen to satisfy the same recurrence, with the same initial values, namely t and $t+t^2$, so we have $N_k=S_k$, as claimed. $\square$

4c. Compound Poisson

In order to further complicate our discrete probability matters, let us start with:

PROPOSITION 4.8. *Consider the hyperoctahedral group $H_N\subset O_N$, consisting of the various symmetries of the hypercube in $\mathbb{R}^N$.*

1. *H_N is the symmetry group of the N coordinate axes of $\mathbb{R}^N$.*
2. *H_N consists of the permutation-like matrices over $\{-1,0,1\}$.*
3. *We have the cardinality formula $|H_N|=2^NN!$.*
4. *We have a crossed product decomposition $H_N=S_N\rtimes\mathbb{Z}_2^N$.*
5. *We have a wreath product decomposition $H_N=\mathbb{Z}_2\wr S_N$.*

Proof. Consider indeed the standard cube in $\mathbb{R}^N$, which is by definition centered at 0, and having as vertices the points having coordinates ± 1.

(1) With the above picture of the cube in hand, it is clear that the symmetries of the cube coincide with the symmetries of the N coordinate axes of $\mathbb{R}^N$.

(2) Each of the permutations $\sigma \in S_N$ of the N coordinate axes of $\mathbb{R}^N$ can be further "decorated" by a sign vector $\varepsilon \in \{\pm 1\}^N$, consisting of the possible ± 1 flips which can be applied to each coordinate axis, at the arrival. In matrix terms, this gives the result.

(3) By using the above interpretation of H_N, we have the following formula:

$$|H_N| = |S_N| \cdot |\mathbb{Z}_2^N| = N! \cdot 2^N$$

(4) We know from (3) that at the level of cardinalities we have $|H_N| = |S_N \times \mathbb{Z}_2^N|$, and with a bit more work, we obtain that we have $H_N = S_N \rtimes \mathbb{Z}_2^N$, as claimed.

(5) This is simply a reformulation of (4), in terms of wreath products. $\square$

By doing now character computations, in the spirit of Theorem 4.6, we are led to:

Theorem 4.9. *For the hyperoctahedral group $H_N \subset O_N$, the law of the truncated character $\chi = g_{11} + \ldots + g_{ss}$ with $s = [tN]$ is, in the $N \to \infty$ limit, the measure*

$$b_t = e^{-t} \sum_{r=-\infty}^{\infty} \delta_r \sum_{p=0}^{\infty} \frac{(t/2)^{|r|+2p}}{(|r|+p)!p!}$$

called Bessel law of parameter $t > 0$.

Proof. We regard H_N as being the symmetry group of the graph $I_N = \{I^1, \ldots, I^N\}$ formed by n segments. The diagonal coefficients are then given by:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ 1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

Let us denote by F_g, R_g the number of segments among $\{I^1, \ldots, I^s\}$ which are fixed, respectively returned by an element $g \in H_N$. With this notation, we have:

$$u_{11} + \ldots + u_{ss} = F_g - R_g$$

We denote by P_N probabilities computed over H_N. The density of the law of the variable $u_{11} + \ldots + u_{ss}$ at a point $r \geq 0$ is then given by the following formula:

$$D(r) = P_N(F_g - R_g = r) = \sum_{p=0}^{\infty} P_N(F_g = r + p, R_g = p)$$

Assume first that we are in the case $t = 1$. We have the following computation:

$$\begin{aligned}\lim_{N\to\infty} D(r) &= \lim_{N\to\infty}\sum_{p=0}^\infty (1/2)^{r+2p}\binom{r+2p}{r+p} P_N(F_g+R_g=r+2p)\\ &= \sum_{p=0}^\infty (1/2)^{r+2p}\binom{r+2p}{r+p}\frac{1}{e(r+2p)!}\\ &= \frac{1}{e}\sum_{p=0}^\infty \frac{(1/2)^{r+2p}}{(r+p)!p!}\end{aligned}$$

The general case $0 < t \leq 1$ follows by performing some modifications in the above computation. Indeed, the asymptotic density can be computed as follows:

$$\begin{aligned}\lim_{N\to\infty} D(r) &= \lim_{N\to\infty}\sum_{p=0}^\infty (1/2)^{r+2p}\binom{r+2p}{r+p} P_N(F_g+R_g=r+2p)\\ &= \sum_{p=0}^\infty (1/2)^{r+2p}\binom{r+2p}{r+p}\frac{t^{r+2p}}{e^t(r+2p)!}\\ &= e^{-t}\sum_{p=0}^\infty \frac{(t/2)^{r+2p}}{(r+p)!p!}\end{aligned}$$

Together with $D(-r) = D(r)$, this gives the formula in the statement. □

The above result is quite interesting, because the densities that we found there are the following functions, called Bessel functions of the first kind:

$$f_r(t) = \sum_{p=0}^\infty \frac{t^{|r|+2p}}{(|r|+p)!p!}$$

Let us study now the Bessel laws. We first have the following result:

Theorem 4.10. *The Bessel laws b_t have the property*

$$b_s * b_t = b_{s+t}$$

so they form a truncated one-parameter semigroup with respect to convolution.

Proof. With $f_r(t)$ being the Bessel functions of the first kind, we have:

$$b_t = e^{-t}\sum_{r=-\infty}^\infty \delta_r\, f_r(t/2)$$

The Fourier transform of this measure b_t is given by:

$$F_{b_t}(y) = e^{-t}\sum_{r=-\infty}^{\infty} e^{iry}\, f_r(t/2)$$

We compute now the derivative with respect to the variable t:

$$F_{b_t}(y)' = -F_{b_t}(y) + \frac{e^{-t}}{2}\sum_{r=-\infty}^{\infty} e^{iry}\, f_r'(t/2)$$

On the other hand, the derivative of f_r with $r \geq 1$ is given by:

$$\begin{aligned}
f_r'(t) &= \sum_{p=0}^{\infty}\frac{(r+2p)t^{r+2p-1}}{(r+p)!p!}\\
&= \sum_{p=0}^{\infty}\frac{(r+p)t^{r+2p-1}}{(r+p)!p!} + \sum_{p=0}^{\infty}\frac{p\,t^{r+2p-1}}{(r+p)!p!}\\
&= \sum_{p=0}^{\infty}\frac{t^{r+2p-1}}{(r+p-1)!p!} + \sum_{p=1}^{\infty}\frac{t^{r+2p-1}}{(r+p)!(p-1)!}\\
&= \sum_{p=0}^{\infty}\frac{t^{(r-1)+2p}}{((r-1)+p)!p!} + \sum_{p=1}^{\infty}\frac{t^{(r+1)+2(p-1)}}{((r+1)+(p-1))!(p-1)!}\\
&= f_{r-1}(t) + f_{r+1}(t)
\end{aligned}$$

This computation works in fact for any r, and we obtain in this way:

$$\begin{aligned}
F_{b_t}(y)' &= -F_{b_t}(y) + \frac{e^{-t}}{2}\sum_{r=-\infty}^{\infty} e^{iry}(f_{r-1}(t/2) + f_{r+1}(t/2))\\
&= -F_{b_t}(y) + \frac{e^{-t}}{2}\sum_{r=-\infty}^{\infty} e^{i(r+1)y} f_r(t/2) + e^{i(r-1)y} f_r(t/2)\\
&= -F_{b_t}(y) + \frac{e^{iy}+e^{-iy}}{2}\,F_{b_t}(y)\\
&= \left(\frac{e^{iy}+e^{-iy}}{2} - 1\right) F_{b_t}(y)
\end{aligned}$$

By integrating, we obtain from this the following formula:

$$F_{b_t}(y) = \exp\left(\left(\frac{e^{iy}+e^{-iy}}{2} - 1\right)t\right)$$

Thus the log of the Fourier transform is linear in t, and we get the assertion. □

In order to further discuss all this, and extend the above results, we will need a number of standard probabilistic preliminaries. We have the following notion, extending the Poisson limit theory developed in the beginning of the present chapter:

DEFINITION 4.11. *Associated to any compactly supported positive measure ν on $\mathbb{C}$, not necessarily of mass 1, is the probability measure*

$$p_\nu = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{1}{n}\nu\right)^{*n}$$

where $t = mass(\nu)$, called compound Poisson law.

In what follows we will be mainly interested in the case where the measure ν is discrete, as is for instance the case for $\nu = t\delta_1$ with $t > 0$, which produces the Poisson laws. The following standard result allows one to detect compound Poisson laws:

PROPOSITION 4.12. *For $\nu = \sum_{i=1}^s t_i\delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{C}$, we have*

$$F_{p_\nu}(y) = \exp\left(\sum_{i=1}^s t_i(e^{iyz_i}-1)\right)$$

where F denotes the Fourier transform.

PROOF. Let η_n be the measure in Definition 4.11, under the convolution sign:

$$\eta_n = \left(1-\frac{t}{n}\right)\delta_0+\frac{1}{n}\nu$$

We have then the following computation:

$$F_{\eta_n}(y) = \left(1-\frac{t}{n}\right)+\frac{1}{n}\sum_{i=1}^s t_ie^{iyz_i} \implies F_{\eta_n^{*n}}(y) = \left(\left(1-\frac{t}{n}\right)+\frac{1}{n}\sum_{i=1}^s t_ie^{iyz_i}\right)^n$$
$$\implies F_{p_\nu}(y) = \exp\left(\sum_{i=1}^s t_i(e^{iyz_i}-1)\right)$$

Thus, we have obtained the formula in the statement. □

We have as well the following result, providing an alternative to Definition 4.11, and which will be our formulation here of the Compound Poisson Limit Theorem:

THEOREM 4.13 (CPLT). *For $\nu = \sum_{i=1}^s t_i\delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{C}$, we have*

$$p_\nu = \text{law}\left(\sum_{i=1}^s z_i\alpha_i\right)$$

where the variables α_i are Poisson (t_i), independent.

PROOF. Let α be the sum of Poisson variables in the statement, namely:

$$\alpha=\sum_{i=1}^s z_i\alpha_i$$

By using some standard Fourier transform formulae, we have:

$$\begin{aligned}F_{\alpha_i}(y)=\exp(t_i(e^{iy}-1)) &\implies F_{z_i\alpha_i}(y)=\exp(t_i(e^{iyz_i}-1))\\ &\implies F_\alpha(y)=\exp\left(\sum_{i=1}^s t_i(e^{iyz_i}-1)\right)\end{aligned}$$

Thus we have indeed the same formula as in Proposition 4.12, as desired. □

Summarizing, we have now a full generalization of the PLT. Getting back now to the Poisson and Bessel laws, with the above formalism in hand, we have:

THEOREM 4.14. *The Poisson and Bessel laws are compound Poisson laws,*

$$p_t=p_{t\delta_1}\quad,\quad b_t=p_{t\varepsilon}$$

where δ_1 is the Dirac mass at 1, and ε is the centered Bernoulli law, $\varepsilon=(\delta_1+\delta_{-1})/2$.

PROOF. We have two assertions here, the idea being as follows:

(1) The first assertion, regarding the Poisson law p_t, is clear from Definition 4.11, which for $\nu=t\delta_1$ takes the following form:

$$p_\nu=\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\,\delta_1\right)^{*n}$$

Indeed, according to the PLT, the limit on the right produces the Poisson law p_t, as desired. Alternatively, the result follows as well from Proposition 4.12, which gives:

$$F_{p_\nu}(y)=\exp\left(t(e^{iy}-1)\right)$$

But the simplest way of proving the result is by invoking Theorem 4.13, which tells us that for $\nu=t\delta_1$ we have $p_\nu=law(\alpha)$, with α being Poisson (t).

(2) Regarding the second assertion, concerning b_t, the most convenient here is to use the formula of the Fourier transform found in the proof of Theorem 4.10, namely:

$$F_{b_t}(y)=\exp\left(t\left(\frac{e^{iy}+e^{-iy}}{2}-1\right)\right)$$

On the other hand, the formula in Proposition 4.12 gives, for $\nu=t\varepsilon$:

$$F_{p_\nu}(y)=\exp\left(\frac{t}{2}(e^{iy}-1)+\frac{t}{2}(e^{-iy}-1)\right)$$

Thus, with $\nu=t\varepsilon$ we have $p_\nu=b_t$, as claimed. □

Moving ahead, what we have so far suggests formulating the following definition:

Definition 4.15. *The Bessel law of level $s\in\mathbb N\cup\{\infty\}$ and parameter $t>0$ is*

$$b_t^s=p_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s-th roots of unity. The measures

$$b_t=b_t^2\quad,\quad B_t=b_t^\infty$$

are called real Bessel law, and complex Bessel law.

Here the terminology comes, as already mentioned, from the fact that at $s=2$, the density of the measure $b_t=b_t^2$ is a Bessel function of the first kind, as follows:

$$f_r(t)=\sum_{p=0}^\infty\frac{t^{|r|+2p}}{(|r|+p)!p!}$$

In practice now, let us study the measures b_t^s in our standard way, meaning density, moments, Fourier, semigroup property, and limiting theorems. In what regards the limiting theorems, we know that the measures b_t^s appear by definition via the CPLT, so done with that, we know one thing. As a consequence of this, however, let us record:

Proposition 4.16. *The Bessel laws are given by*

$$b_t^s=\text{law}\left(\sum_{k=1}^s w^ka_k\right)$$

where $a_1,\ldots,a_s$ are Poisson (t) independent, and $w=e^{2\pi i/s}$.

Proof. At $s=1,2$ this is something that we already know, coming from Theorem 4.14 and its proof. In general, this follows from Theorem 4.13. □

For some further study of b_t^s, consider the level s exponential function, given by:

$$\exp_s z=\sum_{k=0}^\infty\frac{z^{sk}}{(sk)!}$$

We have then the following formula, in terms of the root of unity $w=e^{2\pi i/s}$:

$$\exp_s z=\frac{1}{s}\sum_{k=1}^s\exp(w^kz)$$

Observe that $\exp_1=\exp$ and $\exp_2=\cosh$. We have the following result:

Theorem 4.17. *The Fourier transform of b_t^s is given by*

$$\log F_t^s(z)=t\left(\exp_s z-1\right)$$

where $\exp_s z$ is as above. In particular we have the formula

$$b_t^s*b_{t'}^s=b_{t+t'}^s$$

so the measures b_t^s form a one-parameter convolution semigroup.

PROOF. Consider, as in Proposition 4.16, the variable $a=\sum_{k=1}^s w^ka_k$. We have then the following Fourier transform computation:

$$\log F_a(z)=\sum_{k=1}^s\log F_{a_k}(w^kz)=\sum_{k=1}^s\frac{t}{s}\left(\exp(w^kz)-1\right)$$

But this gives the following formula:

$$\log F_a(z)=t\left(\left(\frac{1}{s}\sum_{k=1}^s\exp(w^kz)\right)-1\right)=t\left(\exp_s z-1\right)$$

Now since b_t^s is the law of a, this gives the formula in the statement. As for the last assertion, this comes from the fact that the log of the Fourier transform is linear in t. □

We can compute now the density of b_t^s, as follows:

THEOREM 4.18. *We have the formula*

$$b_t^s=e^{-t}\sum_{c_1=0}^\infty\ldots\sum_{c_s=0}^\infty\frac{1}{c_1!\ldots c_s!}\left(\frac{t}{s}\right)^{c_1+\ldots+c_s}\delta\left(\sum_{k=1}^s w^kc_k\right)$$

where $w=e^{2\pi i/s}$, and the δ symbol is a Dirac mass.

PROOF. The Fourier transform of the measure on the right is given by:

$$\begin{aligned}F(z)&=e^{-t}\sum_{c_1=0}^\infty\ldots\sum_{c_s=0}^\infty\frac{1}{c_1!\ldots c_s!}\left(\frac{t}{s}\right)^{c_1+\ldots+c_s}F\delta\left(\sum_{k=1}^s w^kc_k\right)(z)\\&=e^{-t}\sum_{c_1=0}^\infty\ldots\sum_{c_s=0}^\infty\frac{1}{c_1!\ldots c_s!}\left(\frac{t}{s}\right)^{c_1+\ldots+c_s}\exp\left(\sum_{k=1}^s w^kc_kz\right)\\&=e^{-t}\sum_{r=0}^\infty\left(\frac{t}{s}\right)^r\sum_{\Sigma c_i=r}\frac{\exp\left(\sum_{k=1}^s w^kc_kz\right)}{c_1!\ldots c_s!}\end{aligned}$$

We multiply now by e^t, and we compute the derivative with respect to t:

$$\begin{aligned}(e^tF(z))'&=\sum_{r=1}^\infty\frac{r}{s}\left(\frac{t}{s}\right)^{r-1}\sum_{\Sigma c_i=r}\frac{\exp\left(\sum_{k=1}^s w^kc_kz\right)}{c_1!\ldots c_s!}\\&=\frac{1}{s}\sum_{r=1}^\infty\left(\frac{t}{s}\right)^{r-1}\sum_{\Sigma c_i=r}\left(\sum_{l=1}^s c_l\right)\frac{\exp\left(\sum_{k=1}^s w^kc_kz\right)}{c_1!\ldots c_s!}\\&=\frac{1}{s}\sum_{r=1}^\infty\left(\frac{t}{s}\right)^{r-1}\sum_{\Sigma c_i=r}\sum_{l=1}^s\frac{\exp\left(\sum_{k=1}^s w^kc_kz\right)}{c_1!\ldots c_{l-1}!(c_l-1)!c_{l+1}!\ldots c_s!}\end{aligned}$$

By using the variable $u = r - 1$, we obtain in this way:

$$\begin{aligned}(e^tF(z))' &= \frac{1}{s}\sum_{u=0}^\infty\left(\frac{t}{s}\right)^u\sum_{\Sigma d_i=u}\sum_{l=1}^s\frac{\exp\left(w^lz+\sum_{k=1}^s w^kd_kz\right)}{d_1!\ldots d_s!}\\ &= \left(\frac{1}{s}\sum_{l=1}^s\exp(w^lz)\right)\left(\sum_{u=0}^\infty\left(\frac{t}{s}\right)^u\sum_{\Sigma d_i=u}\frac{\exp\left(\sum_{k=1}^s w^kd_kz\right)}{d_1!\ldots d_s!}\right)\\ &= (\exp_s z)(e^tF(z))\end{aligned}$$

On the other hand, $\Phi(t) = \exp(t\exp_s z)$ satisfies the same equation, namely:

$$\Phi'(t) = (\exp_s z)\Phi(t)$$

Thus, we have the $e^tF(z) = \Phi(t)$, which gives the following formula:

$$\begin{aligned}\log F &= \log(e^{-t}\exp(t\exp_s z))\\ &= \log(\exp(t(\exp_s z - 1)))\\ &= t(\exp_s z - 1)\end{aligned}$$

Thus, we obtain the formulae in the statement. □

Many other interesting things can be said about the Bessel laws, and their various analogues, and for an introduction to all this, you can check my book [7].

4d. Cumulants

We will often need to do some further combinatorics, in relation with cumulants, following Rota. We have here the following well-known definition:

DEFINITION 4.19. *Associated to any real probability measure $\mu = \mu_f$ is the following modification of the logarithm of the Fourier transform $F_\mu(\xi) = \mathbb{E}(e^{i\xi f})$,*

$$K_\mu(\xi) = \log\mathbb{E}(e^{\xi f})$$

called cumulant-generating function. The Taylor coefficients $k_n(\mu)$ of this series, given by

$$K_\mu(\xi) = \sum_{n=1}^\infty k_n(\mu)\frac{\xi^n}{n!}$$

are called cumulants of the measure μ. We also use the notations k_f, K_f for these cumulants and their generating series, where f is a variable following the law μ.

In other words, the cumulants are more or less the coefficients of the logarithm of the Fourier transform $\log F_\mu$, up to some normalizations. To be more precise, we have $K_\mu(\xi) = \log F_\mu(-i\xi)$, so the formula relating $\log F_\mu$ to the cumulants $k_n(\mu)$ is:

$$\log F_\mu(-i\xi) = \sum_{n=1}^\infty k_n(\mu)\frac{\xi^n}{n!}$$

Equivalently, the formula relating $\log F_\mu$ to the cumulants $k_n(\mu)$ is:

$$\log F_\mu(\xi) = \sum_{n=1}^\infty k_n(\mu)\,\frac{(i\xi)^n}{n!}$$

We will see in a moment the reasons for the above normalizations, namely change of variables $\xi \to -i\xi$, and Taylor coefficients instead of plain coefficients, the idea being that for simple laws like g_t, p_t, we will obtain in this way very simple quantities.

As a first observation, the sequence of cumulants $k_1, k_2, k_3, \ldots$ appears as a modification of the sequence of moments $M_1, M_2, M_3, \ldots$, the numerics being as follows:

Proposition 4.20. *The sequence of cumulants $k_1, k_2, k_3, \ldots$ appears as a modification of the sequence of moments $M_1, M_2, M_3, \ldots$, and uniquely determines μ. We have*

$$\begin{aligned}
k_1 &= M_1\\
k_2 &= -M_1^2 + M_2\\
k_3 &= 2M_1^3 - 3M_1M_2 + M_3\\
k_4 &= -6M_1^4 + 12M_1^2M_2 - 3M_2^2 - 4M_1M_3 + M_4\\
&\vdots
\end{aligned}$$

in one sense, and in the other sense we have

$$\begin{aligned}
M_1 &= k_1\\
M_2 &= k_1^2 + k_2\\
M_3 &= k_1^3 + 3k_1k_2 + k_3\\
M_4 &= k_1^4 + 6k_1^2k_2 + 3k_2^2 + 4k_1k_3 + k_4\\
&\vdots
\end{aligned}$$

with in both cases the correspondence being polynomial, with integer coefficients.

Proof. Here all the theoretical assertions regarding moments and cumulants are clear from definitions, and the numerics are clear from definitions too. To be more precise, we know from Definition 4.19 that the cumulants are defined by the following formula:

$$\log \mathbb{E}(e^{\xi f}) = \sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}$$

By exponentiating, we obtain from this the following formula:

$$\mathbb{E}(e^{\xi f}) = \exp\left(\sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}\right)$$

Now by looking at the terms of order $1, 2, 3, 4$, this gives the above formulae. □

The interest in cumulants comes from the fact that $\log F_\mu$, and so the cumulants $k_n(\mu)$ too, linearize the convolution. To be more precise, we have the following result:

THEOREM 4.21. *The cumulants have the following properties:*
(1) $k_n(cf)=c^nk_n(f)$.
(2) $k_1(f+d)=k_1(f)+d$, *and* $k_n(f+d)=k_n(f)$ *for* $n>1$.
(3) $k_n(f+g)=k_n(f)+k_n(g)$, *if* f,g *are independent.*

PROOF. Here (1) and (2) are both clear from definitions, because we have the following computation, valid for any $c,d\in\mathbb R$, which gives the results:

$$\begin{aligned}K_{cf+d}(\xi) &= \log\mathbb E(e^{\xi(cf+d)})\\ &= \log[e^{\xi d}\cdot\mathbb E(e^{\xi cf})]\\ &= \xi d+K_f(c\xi)\end{aligned}$$

As for (3), this follows from the fact that the Fourier transform $F_f(\xi)=\mathbb E(e^{i\xi f})$ satisfies the following formula, whenever f,g are independent random variables:

$$F_{f+g}(\xi)=F_f(\xi)F_g(\xi)$$

Indeed, by applying the logarithm, we obtain the following formula:

$$\log F_{f+g}(\xi)=\log F_f(\xi)+\log F_g(\xi)$$

With the change of variables $\xi\to -i\xi$, we obtain the following formula:

$$K_{f+g}(\xi)=K_f(\xi)+K_g(\xi)$$

Thus, at the level of coefficients, we obtain $k_n(f+g)=k_n(f)+k_n(g)$, as claimed. □

At the level of the main examples now, we have the following result:

PROPOSITION 4.22. *The sequence of cumulants* $k_1,k_2,k_3,\ldots$ *is as follows:*
(1) *For* $\mu=\delta_c$ *the cumulants are* $c,0,0,\ldots$
(2) *For* $\mu=g_t$ *the cumulants are* $0,t,0,0,\ldots$
(3) *For* $\mu=p_t$ *the cumulants are* $t,t,t,\ldots$
(4) *For* $\mu=b_t$ *the cumulants are* $0,t,0,t,\ldots$

PROOF. We have 4 computations to be done, the idea being as follows:

(1) For $\mu=\delta_c$ we have the following computation:

$$\begin{aligned}K_\mu(\xi) &= \log\mathbb E(e^{c\xi})\\ &= \log(e^{c\xi})\\ &= c\xi\end{aligned}$$

But the plain coefficients of this series are the numbers $c,0,0,\ldots$, and so the Taylor coefficients of this series are these same numbers $c,0,0,\ldots$, as claimed.

(2) For $\mu=g_t$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log\exp\left[-t(-i\xi)^2/2\right] \\ &= t\xi^2/2 \end{aligned}$$

But the plain coefficients of this series are the numbers $0,t/2,0,0,\ldots$, and so the Taylor coefficients of this series are the numbers $0,t,0,0,\ldots$, as claimed.

(3) For $\mu=p_t$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log\exp\left[(e^{i(-i\xi)}-1)t\right] \\ &= (e^\xi-1)t \end{aligned}$$

But the plain coefficients of this series are the numbers $t/n!$, and so the Taylor coefficients of this series are the numbers $t,t,t,\ldots$, as claimed.

(4) For $\mu=b_t$ we have the following computation:

$$\begin{aligned} K_\mu(\xi) &= \log F_\mu(-i\xi) \\ &= \log\exp\left[\left(\frac{e^\xi+e^{-\xi}}{2}-1\right)t\right] \\ &= \left(\frac{e^\xi+e^{-\xi}}{2}-1\right)t \end{aligned}$$

But the plain coefficients of this series are the numbers $(1+(-1)^n)t/n!$, so the Taylor coefficients of this series are the numbers $0,t,0,t,\ldots$, as claimed. $\square$

At a more theoretical level, we have the following result, generalizing (3,4) above:

THEOREM 4.23. *For a compound Poisson law p_ν we have*

$$k_n(p_\nu)=M_n(\nu)$$

valid for any integer $n\geq 1$.

PROOF. We can assume, by using a continuity argument, that our measure ν is discrete, as follows, with $t_i>0$ and $z_i\in\mathbb{R}$, and with the sum being finite:

$$\nu=\sum_i t_i\delta_{z_i}$$

By using now the Fourier transform formula for p_ν, we obtain:

$$\begin{aligned}
K_{p_\nu}(\xi) &= \log F_{p_\nu}(-i\xi)\\
&= \log\exp\left[\sum_i t_i(e^{\xi z_i}-1)\right]\\
&= \sum_i t_i\sum_{n\geq 1}\frac{(\xi z_i)^n}{n!}\\
&= \sum_{n\geq 1}\frac{\xi^n}{n!}\sum_i t_i z_i^n\\
&= \sum_{n\geq 1}\frac{\xi^n}{n!}\,M_n(\nu)
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Getting back to theory now, the sequence of cumulants $k_1,k_2,k_3,\ldots$ appears as a modification of the sequence of moments $M_1,M_2,M_3,\ldots$, and understanding the relation between moments and cumulants will be our next task. Let us start with:

DEFINITION 4.24. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi,\nu)=\begin{cases}1 & \text{if } \pi=\nu\\ -\sum_{\pi\leq\tau<\nu}\mu(\pi,\tau) & \text{if } \pi<\nu\\ 0 & \text{if } \pi\not\leq\nu\end{cases}$$

with the construction being performed by recurrence.

As a first example, the Möbius function, or more conveniently matrix $M_{\pi\nu}=\mu(\pi,\nu)$, of the lattice $P(2)=\{||,\sqcap\}$ is very easy to compute, and looks as follows:

$$M=\begin{pmatrix}1&-1\\0&1\end{pmatrix}$$

At $k=3$ now, we have the following formula for the Möbius matrix $M_{\pi\nu}=\mu(\pi,\nu)$, once again written with the indices picked increasing in $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!\!|\;,|\sqcap,\sqcap\!\sqcap\}$:

$$M=\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}$$

In general, the Möbius matrix of $P(k)$ looks a bit like the above matrices at $k=2,3$, namely being upper triangular, with 1 on the diagonal, and so on.

The main interest in the Möbius function comes from the Möbius inversion formula, which in linear algebra terms can be formulated as follows:

THEOREM 4.25. *We have the following implication,*

$$f(\pi)=\sum_{\nu\leq\pi}g(\nu)\quad\Longrightarrow\quad g(\pi)=\sum_{\nu\leq\pi}\mu(\nu,\pi)f(\nu)$$

valid for any two functions $f,g:P(n)\to\mathbb{C}$.

PROOF. The above formula is in fact a linear algebra result, so let us start with some linear algebra. Consider the adjacency matrix of P, given by the following formula:

$$A_{\pi\nu}=\begin{cases}1&\text{if }\pi\leq\nu\\0&\text{if }\pi\not\leq\nu\end{cases}$$

Our claim is that the inverse of this matrix is the Möbius matrix of P, given by:

$$M_{\pi\nu}=\mu(\pi,\nu)$$

Indeed, the above matrix A is upper triangular, and when trying to invert it, we are led to the recurrence in Definition 4.24, so to the Möbius matrix M. Thus we have:

$$M=A^{-1}$$

Now by applying this equality of matrices to vectors, regarded as complex functions on $P(n)$, we are led to the inversion formula in the statement. □

As a first illustration, for $P(2)$ the formula $M=A^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

At $k=3$ now, the formula $M=A^{-1}$ for $P(3)$ reads:

$$\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}^{-1}$$

With these ingredients in hand, let us go back to probability. We first have:

DEFINITION 4.26. *We define quantities $M_\pi(f),k_\pi(f)$, depending on partitions*

$$\pi\in P(k)$$

by starting with $M_n(f),k_n(f)$, and using multiplicativity over the blocks.

To be more precise, the convention here is that for the one-block partition $1_n \in P(n)$, the corresponding moment and cumulant are the usual ones, namely:

$$M_{1_n}(f) = M_n(f) \quad , \quad k_{1_n}(f) = k_n(f)$$

Then, for an arbitrary partition $\pi \in P(k)$, we decompose this partition into blocks, having sizes $b_1, \ldots, b_s$, and we set, by multiplicativity over blocks:

$$M_\pi(f) = M_{b_1}(f) \ldots M_{b_s}(f) \quad , \quad k_\pi(f) = k_{b_1}(f) \ldots k_{b_s}(f)$$

With this convention, following Rota and others, we can now formulate a key result, fully clarifying the relation between moments and cumulants, as follows:

Theorem 4.27. *We have the moment-cumulant formulae*

$$M_n(f) = \sum_{\nu \in P(n)} k_\nu(f) \quad , \quad k_n(f) = \sum_{\nu \in P(n)} \mu(\nu, 1_n) M_\nu(f)$$

or, equivalently, we have the moment-cumulant formulae

$$M_\pi(f) = \sum_{\nu \leq \pi} k_\nu(f) \quad , \quad k_\pi(f) = \sum_{\nu \leq \pi} \mu(\nu, \pi) M_\nu(f)$$

where μ is the Möbius function of $P(n)$.

Proof. There are several things going on here, the idea being as follows:

(1) First, it is clear from our conventions, from Definition 4.26, that the first set of formulae is equivalent to the second set of formulae, by multiplicativity over blocks.

(2) The other observation is that, due to the Möbius inversion formula, from Theorem 4.25, in the second set of formulae, the two formulae there are in fact equivalent.

(3) Summarizing, the 4 formulae in the statement are all equivalent. In what follows we will focus on the first 2 formulae, which are the most useful, in practice.

(4) Let us first work out some examples. At $n = 1, 2, 3$ the moment formula gives the following equalities, which are in tune with the findings from Proposition 4.20:

$$M_1 = k_{|} = k_1$$

$$M_2 = k_{||} + k_{\sqcap} = k_1^2 + k_2$$

$$M_3 = k_{|||} + k_{\sqcap|} + k_{\sqcap\!\!\!|\,} + k_{|\sqcap} + k_{\sqcap\!\sqcap} = k_1^3 + 3k_1k_2 + k_3$$

At $n = 4$ now, which is a case which is of particular interest for certain considerations to follow, the computation is as follows, again in tune with Proposition 4.20:

$$\begin{aligned} M_4 &= k_{||||} + (\underbrace{k_{\sqcap||} + \ldots}_{6\ terms}) + (\underbrace{k_{\sqcap\sqcap} + \ldots}_{3\ terms}) + (\underbrace{k_{\sqcap\!\sqcap|} + \ldots}_{4\ terms}) + k_{\sqcap\!\sqcap\!\sqcap} \\ &= k_1^4 + 6k_1^2k_2 + 3k_2^2 + 4k_1k_3 + k_4 \end{aligned}$$

As for the cumulant formula, at $n=1,2,3$ this gives the following formulae for the cumulants, again in tune with the findings from Proposition 4.20:

$$k_1 = M_{|} = M_1$$

$$k_2 = (-1)M_{||} + M_{\sqcap} = -M_1^2 + M_2$$

$$k_3 = 2M_{|||} + (-1)M_{\sqcap|} + (-1)M_{\overset{\sqcap}{|}} + (-1)M_{|\sqcap} + M_{\sqcap\!\sqcap} = 2M_1^3 - 3M_1M_2 + M_3$$

Finally, at $n=4$, after computing the Möbius function of $P(4)$, we obtain the following formula for the fourth cumulant, again in tune with Proposition 4.20:

$$\begin{aligned} k_4 &= (-6)M_{||||} + 2(\underbrace{M_{\sqcap||} + \ldots}_{6\ terms}) + (-1)(\underbrace{M_{\sqcap\sqcap} + \ldots}_{3\ terms}) + (-1)(\underbrace{M_{\sqcap\!\sqcap|} + \ldots}_{4\ terms}) + M_{\sqcap\!\sqcap\!\sqcap} \\ &= -6M_1^4 + 12M_1^2M_2 - 3M_2^2 - 4M_1M_3 + M_4 \end{aligned}$$

(5) After all these preliminaries, time now to get to work, and prove the result. As mentioned above, our formulae are all equivalent, and it is enough to prove just one of them. We will prove in what follows the first formula, namely:

$$M_n(f) = \sum_{\nu \in P(n)} k_\nu(f)$$

(6) In order to do this, we use the very definition of the cumulants, namely:

$$\log \mathbb{E}(e^{\xi f}) = \sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}$$

By exponentiating, we obtain from this the following formula:

$$\mathbb{E}(e^{\xi f}) = \exp\left(\sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}\right)$$

(7) Let us first compute the function on the left. This is easily done, as follows:

$$\begin{aligned} \mathbb{E}(e^{\xi f}) &= \mathbb{E}\left(\sum_{n=0}^\infty \frac{(\xi f)^n}{n!}\right) \\ &= \sum_{n=0}^\infty M_n(f)\,\frac{\xi^n}{n!} \end{aligned}$$

(8) Regarding now the function on the right, this is given by:

$$\begin{aligned}\exp\left(\sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}\right) &= \sum_{p=0}^\infty \frac{\left(\sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}\right)^p}{p!}\\ &= \sum_{p=0}^\infty \frac{1}{p!}\sum_{s_1=1}^\infty k_{s_1}(f)\,\frac{\xi^{s_1}}{s_1!}\ldots\ldots\sum_{s_p=1}^\infty k_{s_p}(f)\,\frac{\xi^{s_p}}{s_p!}\\ &= \sum_{p=0}^\infty \frac{1}{p!}\sum_{s_1=1}^\infty\ldots\sum_{s_p=1}^\infty k_{s_1}(f)\ldots k_{s_p}(f)\,\frac{\xi^{s_1+\ldots+s_p}}{s_1!\ldots s_p!}\end{aligned}$$

(9) The point now is that all this leads us into partitions. Indeed, we are summing over indices $s_1,\ldots,s_p\in\mathbb N$, which can be thought of as corresponding to a partition of $n=s_1+\ldots+s_p$. So, let us rewrite our sum, as a sum over partitions. For this purpose, recall that the number of partitions $\nu\in P(n)$ having blocks of sizes $s_1,\ldots,s_p$ is:

$$\binom{n}{s_1,\ldots,s_p}=\frac{n!}{p_1!\ldots p_s!}$$

Also, when resumming over partitions, there will be a $p!$ factor as well, coming from the permutations of $s_1,\ldots,s_p$. Thus, our sum can be rewritten as follows:

$$\begin{aligned}\exp\left(\sum_{s=1}^\infty k_s(f)\,\frac{\xi^s}{s!}\right) &= \sum_{n=0}^\infty\sum_{p=0}^\infty\frac{1}{p!}\sum_{s_1+\ldots+s_p=n} k_{s_1}(f)\ldots k_{s_p}(f)\,\frac{\xi^n}{s_1!\ldots s_p!}\\ &= \sum_{n=0}^\infty\frac{\xi^n}{n!}\sum_{p=0}^\infty\frac{1}{p!}\sum_{s_1+\ldots+s_p=n}\binom{n}{s_1,\ldots,s_p}k_{s_1}(f)\ldots k_{s_p}(f)\\ &= \sum_{n=0}^\infty\frac{\xi^n}{n!}\sum_{\nu\in P(n)}k_\nu(f)\end{aligned}$$

(10) We are now in position to conclude. According to (6,7,9), we have:

$$\sum_{n=0}^\infty M_n(f)\,\frac{\xi^n}{n!}=\sum_{n=0}^\infty\frac{\xi^n}{n!}\sum_{\nu\in P(n)}k_\nu(f)$$

Thus, we have the following formula, valid for any $n\in\mathbb N$:

$$M_n(f)=\sum_{\nu\in P(n)}k_\nu(f)$$

We are therefore led to the conclusions in the statement. □

4e. Exercises

Exercises:

Exercise 4.28.

Exercise 4.29.

Exercise 4.30.

Exercise 4.31.

Exercise 4.32.

Exercise 4.33.

Exercise 4.34.

Exercise 4.35.

Bonus exercise.

Part II

Life, economy

Aqui se queda la clara
La entranable transparencia
De tu querida presencia
Comandante Che Guevara

CHAPTER 5

Life, Darwin

5a. Basic physics

Welcome to social science. The present Part II of this book will be an introduction to social science at large, vaguely focusing on organization of society and economic apects, independent from Part I. Afterwards, with our accumulated mathematical and social science knowledge, we will go in Part III and Part IV into the study of economy.

So, social science. The first thing to be known here is that humans are not machines, and you cannot apply to them the same methods as in hard sciences like mathematics, physics, chemistry or engineering, that is, all sorts of experiments, then quick, cold modelling based on these experiments, and finally calculus for solving your models.

What does work, although some exceptions to this rule exist, is the study of History. That is, History, or parts of it which are revelant to your social science work, is the "experiment", coming of course with a huge amount of data, and for free. And then, after studying History, you can come as usual with conclusions, models, and mathematics.

In what concerns us, with our economy motivations, we will be interested in pretty much everything. That is, pretty much everything that happened in the history of mankind, and even before, teaches us interesting things about what is life, who we are, life models, organization of society, money and economy. So, coming as first news, with respect to what has been said above about social sciences in general, economy turns to be a tricky business, requiring as preliminary a critical look at the whole History.

Generally speaking, the credit for such ideas, that understanding economy requires as preliminary a critical look at the whole History, goes to Karl Marx. In addition, the main opus of Marx, which is his 3-volume Capital [72], [73], [74], based on this idea, is quite a masterpiece, which remains hard to beat, even nowadays. To be more precise, the masterpiece in all this are the first two volumes [72], [73], which look in a first-class, objective scientific way, with economy motivations in mind, at the history of mankind. This study was brilliant when it came out, and as already said, is hard to beat, even nowadays. As for the third volume [74], this rather draws conclusions, and no surprise here, that conclusions are rather what we call today "marxism".

In short, what we were planning do to here, in Part II of the present book, is more or less what Marx exactly did in the first two volumes of Capital [**72**], [**73**], and so, no surprise, we will mostly rely on him, for our presentation. However, there will be a few differences, the plan for the present Part II of our book being as follows:

(1) In the present chapter 5 we will have a discussion about life, starting from the very beginnings, namely physics, chemistry and then biology, going from the first organisms, up to homo sapiens. The guiding figure here will be Charles Darwin, and with this having some important consequences afterwards, for instance in relation with the work of Marx, which certainly does not take enough into account the findings of Darwin.

(2) In chapters 6 and 7 we will discuss the history of mankind, from the Stone Age up to the modern times, with economy motivations in mind, basically by following Marx. To be more precise, chapter 6 will deal with the Stone, Bronze and Iron ages, followed by the Bible and the Quran, and then chapter 7 will deal with history afterwards, following Marx, and with a look into the modern implementations of marxism at the end.

(3) In chapter 8 we will discuss some other aspects of human behavior overlooked by Marx, coming as a complement to the Darwinism discussed in chapter 5, coming this time from the notion of "individualism", at large. There is a long story here, notably involving Kant, then Nietzsche, and our towering figure here will be Sigmund Freud. We will also discuss post-Freud developements, such as the work of Eliade, and Foucault.

In the hope that you will find this exciting. As a last remark about philosophy and organization, before getting to work, in case you are already familiar with all this, and wondering what thought and economics school I belong to, what my book advocates for, and so on, all very natural questions between connaisseurs, you have certainly figured out from the above what my plan is, for the present Part II of this book, namely sandwich organization, with leftwing views 6,7 surrounded by rightwing views 5,8. By the way, as a disclaimer here, I am not a social scientist, but just a modest quantum physicist, and this will be I guess the average way quantum physics sees social science, at large.

Getting to work now, as already said above, as a first goal, we would like to understand what life is, and more specifically, we have the following question to be solved:

Question 5.1. *What is life? How does life organize? Is there some sort of money and economy present, perhaps in some hidden form?*

In order to answer this question, we need a lot of science. On the menu, physics of all types, culminating with quantum mechanics, and the atomic theory, then chemistry and molecules, then organic molecules and life, and then a lot of biology, culminating with Darwin's findings, which will provide us with an answer to Question 5.1.

Getting started now, we need a crash course in general physics, going up to the atomic theory. This is something quite ambitious, our plan being as follows:

1. Classical mechanics.

2. Electrostatics.

3. Electrodynamics.

4. Relativity theory.

5. Quantum mechanics.

6. Atomic theory.

Generally speaking, all this can be learned from many places, with the classic being the books of Feynman [35], [36], [37], [38]. Alternatively, you can learn this from the equally lovely books of Griffiths [46], [47], [48], [49], or of Huang [52], [53], [54], [55], or at least these are, along with Feynman, my personal favorites. In case you already know some physics, and want to learn more, go with Weinberg [96], [97], [98], [99].

As another comment on physics books, all the above authors, Feynman, Griffiths, Huang and Weinberg, were mostly interested themselves in quantum mechanics, and certain less fashionable aspects of modern physics, such as classical mechanics, suffer a bit from this. So, for classical mechanics better go with Kibble [59] or Taylor [90] for learning the basics, and with Goldstein [44] or Landau-Lifshitz [63] for more advanced theory. And if you happen to be a mathematician, not afraid of difficult mathematics, go with Arnold [1], [2], [3], [4], who was the one on this planet best knowing classical mechanics, and who in addition knew how to write lovely mathematics books.

As yet another comment about learning physics, observe that thermodynamics and statistical mechanics, which are certainly some of the most exciting disciplines in physics, and in fact, in the context of modern physics, lie on par with quantum mechanics at spot #1, among physics trends on the internet, are missing from our above list (1-6). This is because we will not need these right now, for understanding the atoms and molecules. However, thermodynamics will come into play later in this book, when discussing economy, and some reading here in advance would be useful. Besides Feynman, Griffiths, Huang and Weinberg, you have here Fermi [34], and also Blundell and Blundell [12], Kadanoff [57], Pathria and Beale [79], Schroeder [85] and Steane [87], all good books.

Finally, if you are a social scientist not willing to read any technical books, sure yes, but check the popular physics books of Weinberg, he wrote quite a few of them, which are all very good, coming from someone who really knew well physics. Also, still speaking popular books, and as an advice for everyone now, be them mathematicians, physicists, other scientists or social scientists, get a copy of the book of Kumar [61], that is about quantum mechanics, very readable, and is my own favorite popular physics book.

Back to our business, looks like we forgot our to-do list (1-6), with all this discussion. Typical social science phenomenon, I guess. That list was as follows:

1. Classical mechanics.
2. Electrostatics.
3. Electrodynamics.
4. Relativity theory.
5. Quantum mechanics.
6. Atomic theory.

So, getting started for good now, at the beginnings of physics was classical mechanics, whose main findings can be summarized as follows:

Fact 5.2 (Classical mechanics). *The force of attraction between two bodies of masses m_1, m_2, having distance $d > 0$ between them, is given by*

$$||F|| = G \cdot \frac{m_1 m_2}{d^2}$$

where $G = 6.674 \times 10^{-11}$ is a constant. This force alters the trajectory of one body with respect to the other according to the following formula, a being the acceleration:

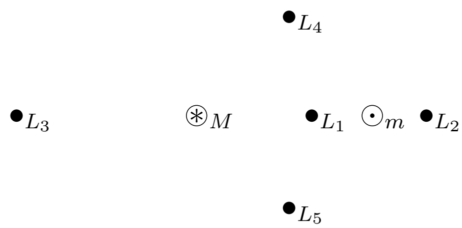
$$F = ma$$

This trajectory is a curve of degree 2, called conic, which can be an ellipsis, parabola, or hyperbola. However, for 3 or more objects, all this can lead to order, or chaos.

Here you surely know all this, with perhaps some gaps in the mathematics at the end, regarding conics and their classification. As an advice, learn this too from somewhere, this is all beautiful and useful mathematics, going back to the ancient Greeks.

Also, in what regards the last sentence, that is a very short summary of what happens for the N-body problem with $N \geq 3$, and as a piece of advice here, have a look at Earth scientific satellites and Lagrange points, in relation with $N = 3$, and also go on internet for more about $N = 3$, including weird solutions, all this is very interesting.

As a piece of advertisement here, the interesting problem at $N = 3$ is how to position a specialized scientific satellite, deep in space, and away from the dust and radiation of the usual orbits around the Earth, as to stay there, under the joint influence of the gravity of the Sun M and of the Earth m. And there are 5 possible solutions here, called Lagrange points L1-L5, whose positions with respect to M, m are as follows:



Moreover, and here comes another interesting point, L4, L5 are stable, in the sense that a satellite installed there will really stay there, regardless of the various tiny little things that might happen, like an asteroid passing by, while L1, L2, L3 are unstable, in the sense that a satellite installed there will need constant tiny adjustments, in order to really stay there. So, which one would you choose for installing your satellite?

You would probably say L4, L5, but this is precisely the wrong answer, because due to their stability, these points attract a lot of asteroids and space garbage, and our satellite will certainly not perform well there, in that crowd. So, with L4, L5 ruled out, and with L3 ruled out too, being too far, the correct choices are L1, L2. But here, you still need to learn a lot more mechanics, for understanding how to do this, in practice.

In what regards now electrostatics, this is again something very fundamental, that you know well, the basics here being summarized as follows:

Fact 5.3 (Electrostatics). *Ordinary matter is made of electrons $-$, protons $+$ and neutrons 0, with the number of $+,-$ being roughly equal. We set*

$$q = \#\{+\} - \#\{-\}$$

and if $q \neq 0$, we call this a charge. Any pair of charges $q_1, q_2 \in \mathbb{R}$ is then subject to a force as follows, which is attractive if $q_1q_2 < 0$ and repulsive if $q_1q_2 > 0$,

$$||F|| = K \cdot \frac{|q_1q_2|}{d^2}$$

where $K = 8.988 \times 10^9$. However, unlike in classical mechanics, $q_1 < 0$ will not spin around $q_2 > 0$ on an ellipsis, due to magnetism, relativity, and quantum mechanics.

Here you are certainly familiar with the Coulomb law formula in the statement, which is very similar to the Newton law formula from Fact 5.2. This normally suggests that when the force is attractive, $q_1q_2 < 0$, the negative charge, say an electron $-$, will spin around the positive charge, say a proton $+$, on an ellipsis. But this is well-known to be wrong, with the solution of this 2-body problem, which corresponds to the hydrogen atom, being far more complicated, due to the numerous reasons mentioned in the statement.

Regarding now electrodynamics, this comes as a continuation of electrostatics, with the aim of fixing some of the obvious bugs there, the basics being as follows:

FACT 5.4 (Electrodynamics). *Moving charges produce magnetic fields, and the dynamics of the electric fields E and magnetic fields B is governed by the formulae*

$$< \nabla, E >= \frac{\rho}{\varepsilon_0}$$

$$< \nabla, B >= 0$$

$$\nabla \times E = -\dot{B}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \dot{E}$$

called Maxwell equations. Also, accelerating or decelerating charges produce electromagnetic radiation, of various wavelengths, called light, of various colors.

Obviously, we are now into serious science here, with the Maxwell equations being something quite complicated, and the pride of 19th century physics, and still the nightmare of everyone using them. To start with, electrodynamics is the science of moving electrical charges. And the problem is that, unlike in classical mechanics, where the Newton law is good for both the static and the dynamic setting, the Coulomb law, which is actually very similar to the Newton law, does the job when the charges are static, but no longer describes well the situation when the charges are moving.

The problem comes from the fact that moving charges produce magnetism, and with this being visible when putting together two electric wires, which will attract or repel, depending on orientation. Thus, in contrast with classical mechanics, where static or dynamic problems are described by a unique field, the gravitational one, in electrodynamics we have two fields, namely the electric field E, and the magnetic field B.

Fortunately, there is a full set of equations relating the electric field E and the magnetic field B, those found by Maxwell and others, given above. Regarding the math, $< , >$ and $\times$ are the usual scalar and vector products on $\mathbb{R}^3$, the dots denote derivatives with respect to time, and ∇ is the gradient operator, or space derivative, given by:

$$\nabla = \begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{pmatrix}$$

As for the physics, the first formula is the Gauss law, ρ being the charge, and ε_0 being a constant, and with this Gauss law more or less replacing the Coulomb law from electrostatics. The second formula is something basic, and anonymous. The third formula is the Faraday law. As for the fourth formula, this is the Ampère law, as modified by Maxwell, with J being the volume current density, and μ_0 being a constant.

Importantly, in addition to what is said in Fact 5.4, it is also known that the constants there μ_0, ε_0, which are electrodynamic quantities, are subject to the following magic formula, due to Biot-Savart, with $c = 299\,792\,458$ m/s being the speed of light:

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

In what regards now the last sentence, this is something fundamental too, putting an end to centuries or even millenia of discussions, regarding the nature of light. Speaking light, here is the table coming from Fact 5.4, which is a must-know:

Frequency	Type	Wavelength
	–	
$10^{18} - 10^{20}$	γ rays	$10^{-12} - 10^{-10}$
$10^{16} - 10^{18}$	X – rays	$10^{-10} - 10^{-8}$
$10^{15} - 10^{16}$	UV	$10^{-8} - 10^{-7}$
	–	
$10^{14} - 10^{15}$	blue	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	yellow	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	red	$10^{-7} - 10^{-6}$
	–	
$10^{11} - 10^{14}$	IR	$10^{-6} - 10^{-3}$
$10^{9} - 10^{11}$	microwave	$10^{-3} - 10^{-1}$
$1 - 10^{9}$	radio	$10^{-1} - 10^{8}$

Observe the tiny space occupied by the visible light, all colors there, and the many more missing, being squeezed under the $10^{14} - 10^{15}$ frequency banner. Here is a zoom on that part, with of course the remark that all this, colors, is something subjective:

Frequency THz $= 10^{12}$ Hz	Color	Wavelength nm $= 10^{-9}$ m
	–	
670 – 790	violet	380 – 450
620 – 670	blue	450 – 485
600 – 620	cyan	485 – 500
530 – 600	green	500 – 565
510 – 530	yellow	565 – 590
480 – 510	orange	590 – 625
400 – 480	red	625 – 750

Hang on, we are not done yet with the Maxwell equations, and their consequences. Yet another feature of these equations is that these can be regarded as well as a precursor of Einstein's relativity theory, which can be summarized as follows:

FACT 5.5 (Relativity theory). *The speeds are bounded, $v < c$, by the speed of light in vaccuum, which is the same for all intertial observers, given by:*

$$c = 299\,792\,458 \text{ m/s}$$

In view of this, classical mechanics must be fixed, and the correct formula for the addition of speeds, guaranteeing $v < c$ for the sum, is Einstein's formula

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}/c^2}$$

which at small speeds reduces to the usual Galileo formula $v_{AC} = v_{AB} + v_{BC}$. Moreover, the improved theory is invariant under the space-time Lorentz transformation

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - vx/c^2)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, exactly as the Maxwell equations. Gravity can be added, too.

Obviously, many deep things going on here, and many other things can be said, for instance $E = mc^2$ comes from this too. This being said, the idea of Einstein is very simple, based only on $v < c$. Indeed, by rescaling things as to have $c = 1$, we are looking for a speed addition formula $(u, v) \to u +_e v$ satisfying the following condition:

$$u, v \leq 1 \implies u +_e v \leq 1$$

But here, thinking at the math, not many choices, with the obvious choice being:

$$u +_e v = \frac{u + v}{1 + uv}$$

And the miracle is that this formula, which is the one in the statement after rescaling by c, is indeed the correct one. With everything coming afterwards, namely Lorentz transformation, and gravity added, being more or less straightforward mathematics.

Finally, no discussion of relativity would be complete without a proof of $E = mc^2$. The idea here is that the relativistic energy of an object of rest mass $m > 0$ is as follows, making it clear that at speed $v = 0$, the energy should be $E = mc^2$:

$$\begin{aligned}
\mathcal{E} &= \frac{mc^2}{\sqrt{1 - v^2/c^2}} \\
&= mc^2\left(1 + \frac{v^2}{2c^2} + \ldots\right) \\
&= mc^2 + \frac{mv^2}{2} + \ldots
\end{aligned}$$

Now still speaking deep things, and going back to the Maxwell equations from Fact 5.4, although almighty, and compatible with relativity too, via the mathematics of the Lorentz transformation, these still do not solve the 2-body problem in electrodynamics, which is the functioning problem for the hydrogen atom. The problem comes from quantum mechanics, whose basic philosophy can be summarized as follows:

FACT 5.6 (Quantum mechanics). *Small particles like electrons and protons do not have clear positions and speeds. This is how things are, at that scale, and it is all about the probability of finding the particle here or there, and with this or that speed.*

This might seem overly vague, but sometimes a totally new and weird thought, of course in the hands of someone having the technical know-how, is enough to make science advance. Besides the above fact, which is something mathematical and theoretical, of key importance was the discovery, by Balmer, Rydberg and others, of the mechanism of the spectral lines of hydrogen H. These lines, depending on integer parameters $n_1 < n_2$, are given by the Rydberg formula, which is as follows, with $R = 1.096\ 775\ 83 \times 10^7$:

$$\frac{1}{\lambda_{n_1n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Interestingly, and perhaps reminding a bit speed addition in relativity, these spectral lines combine according to the Ritz-Rydberg principle, which is as follows:

$$\frac{1}{\lambda_{n_1n_2}} + \frac{1}{\lambda_{n_2n_3}} = \frac{1}{\lambda_{n_1n_3}}$$

In practice, all these lines came from the efforts of several people, namely Balmer in 1885, in the visible range, then Lyman in 1906 in UV, Paschen in 1908 in IR, and later Brackett in 1922, Pfund in 1924, Humphreys in 1953, and others aftwerwards, with all the extra lines being in far IR. The simplified complete table is as follows:

n_1	n_2	Series name	Wavelength $n_2 = \infty$	Color $n_2 = \infty$
		–	–	
1	$2 - \infty$	Lyman	91.13 nm	UV
2	$3 - \infty$	Balmer	364.51 nm	UV
3	$4 - \infty$	Paschen	820.14 nm	IR
		–	–	
4	$5 - \infty$	Brackett	1458.03 nm	far IR
5	$6 - \infty$	Pfund	2278.17 nm	far IR
6	$7 - \infty$	Humphreys	3280.56 nm	far IR
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now back to the Ritz-Rydberg principle, which is the main theoretical result in all this, this reminds the following multiplication formula for the usual matrix units $e_{ij} : e_j \to e_i$, perhaps taken in infinite dimensions, as to allow infinite-ranging indices:

$$e_{n_1n_2}e_{n_2n_3} = e_{n_1n_3}$$

But this is very interesting, suggesting that the observables of the hydrogen atom should be some sort of infinite matrices, making the link with Fact 5.6.

Obviously, what we have here is a first-class scientific puzzle. Based on all this, and on some earlier predictions of Bohr, who was the initiator of the whole program, Heisenberg and Schrödinger, and then De Broglie, Dirac, Pauli and others were able to solve this puzzle, and develop a quantum mechanics theory starting from Fact 5.6, with the main applications, to the functioning of hydrogen and of other atoms, being as follows:

FACT 5.7 (Atomic theory). *The atoms are formed by a core of protons and neutrons, surrounded by a cloud of electrons, basically obeying to a modified version of electromagnetism. And with a fine mechanism involved, as follows:*

(1) *The electrons are free to move only on certain specified elliptic orbits, labelled* $1, 2, 3, \ldots,$ *situated at certain specific heights.*
(2) *The electrons can jump or fall between orbits* $n_1 < n_2$*, absorbing or emitting light and heat, that is, electromagnetic waves, as accelerating charges.*
(3) *The energy of such a wave, coming from* $n_1 \to n_2$ *or* $n_2 \to n_1$*, is given, via the Planck viewpoint, by the Rydberg formula, applied with* $n_1 < n_2$*.*
(4) *The simplest such jumps are those observed by Lyman, Balmer, Paschen. And multiple jumps explain the Ritz-Rydberg formula.*

Still with me, I hope? We are certainly now into complicated physics, and even seem to be somewhere towards the end of science, as understandable by humans. But, thinking well, we are in fact only at the beginning, because Fact 5.7 is not that useful as such, for the simple reason that atoms usually don't come alone, but rather tend to attach to each other, and form molecules. So, with physics understood, welcome to chemistry.

5b. Molecules, cells

Getting into chemistry now, we first need a better understanding of the atoms, as described by Fact 5.7. The basics of chemistry can be summarized as follows:

FACT 5.8 (Basic chemistry). *Atoms can be labeled according to their atomic number, which is the number of protons in their nucleus, in practice*

$$Z = 1, \ldots, 118$$

and tend to attach to each other, and form molecules, with the electron distribution on the orbitals being responsible for this mechanism.

All this is very interesting, and truly corresponding to what happens in the real life, meaning at our scale, our usual temperature, our usual pressure, and so on. More precisely now, there are two assertions here. First is a continuation of Fact 5.7, namely more atomic physics, which leads to the conclusion that the known atoms, also called chemical elements, basically depend only on their atomic number $Z = 1, \ldots, 118$. These chemical elements can be arranged in a table, called periodic table, as follows:

	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	H 1																		He 2
2	Li 3	Be 4												B 5	C 6	N 7	O 8	F 9	Ne 10
3	Na 11	Mg 12												Al 13	Si 14	P 15	S 16	Cl 17	Ar 18
4	K 19	Ca 20		Sc 21	Ti 22	V 23	Cr 24	Mn 25	Fe 26	Co 27	Ni 28	Cu 29	Zn 30	Ga 31	Ge 32	As 33	Se 34	Br 35	Kr 36
5	Rb 37	Sr 38		Y 39	Zr 40	Nb 41	Mo 42	Tc 43	Ru 44	Rh 45	Pd 46	Ag 47	Cd 48	In 49	Sn 50	Sb 51	Te 52	I 53	Xe 54
6	Cs 55	Ba 56	l	Lu 71	Hf 72	Ta 73	W 74	Re 75	Os 76	Ir 77	Pt 78	Au 79	Hg 80	Tl 81	Pb 82	Bi 83	Po 84	At 85	Rn 86
7	Fr 87	Ra 88	a	Lr 103	Rf 104	Db 105	Sg 106	Bh 107	Hs 108	Mt 109	Ds 110	Rg 111	Cn 112	Nh 113	Fl 114	Mc 115	Lv 116	Ts 117	Og 118
			l:	La 57	Ce 58	Pr 59	Nd 60	Pm 61	Sm 62	Eu 63	Gd 64	Tb 65	Dy 66	Ho 67	Er 68	Tm 69	Yb 70		
			a:	Ac 89	Th 90	Pa 91	U 92	Np 93	Pu 94	Am 95	Cm 96	Bk 97	Cf 98	Es 99	Fm 100	Md 101	No 102		

Here the horizontal parameter $1, \ldots, 18$ is called the group, and the vertical parameter $1, \ldots, 7$ is called the period. The two rows on the bottom consist of lanthanum $_{57}$La and its followers, called lanthanides, and of actinium $_{89}$Ac and its followers, called actinides. These are to be inserted in the main table, where indicated, lanthanides between barium $_{56}$Ba and lutetium $_{71}$Lu, and actinides between radium $_{88}$Ra and lawrencium $_{103}$Lr.

Thus, the periodic table, when correctly drawn, but no one does that because of obvious typographical reasons, is in fact a 7×32 table. Note here that, according to our 7×18 convention, which is the standard one, lanthanides and actinides don't have a group number $1, \ldots, 18$. Their group is by definition "lanthanides" and "actinides".

In order to go now towards chemistry, as a first requirement, you cannot call yourself a chemist if not knowing all the elements up to krypton ${}_{36}Kr$, which are absolutely needed for everything, at least a little bit. The names of these elements are as follows:

(1) Hydrogen ${}_{1}H$, helium ${}_{2}He$.

(2) Lithium ${}_{3}Li$, berylium ${}_{4}Be$, boron ${}_{5}B$, carbon ${}_{6}C$, nitrogen ${}_{7}N$, oxygen ${}_{8}O$, fluorine ${}_{9}F$, neon ${}_{10}Ne$.

(3) Sodium ${}_{11}Na$, magnesium ${}_{12}Mg$, aluminium ${}_{13}Al$, silicon ${}_{14}Si$, phosphorus ${}_{15}P$, sulfur ${}_{16}S$, chlorine ${}_{17}Cl$, argon ${}_{18}Ar$.

(4) Potassium ${}_{19}K$, calcium ${}_{20}Ca$, scandium ${}_{21}Sc$, titanium ${}_{22}Ti$, vanadium ${}_{23}V$, and chromium ${}_{24}Cr$, manganese ${}_{25}Mn$, iron ${}_{26}Fe$, cobalt ${}_{27}Co$.

(5) Nickel ${}_{28}Ni$, copper ${}_{29}Cu$, zinc ${}_{30}Zn$, gallium ${}_{31}Ga$, germanium ${}_{32}Ge$, arsenic ${}_{33}As$, selenium ${}_{34}Se$, bromine ${}_{35}Br$, krypton ${}_{36}Kr$.

Observe that all names fit with the abbreviations, expect for sodium ${}_{11}Na$, coming from the Latin natrium, potassium ${}_{19}K$, coming from the Latin kalium, iron ${}_{26}Fe$ coming from the Latin ferrum, and also copper ${}_{29}Cu$, coming from the Latin cuprum.

In what regards the elements heavier than krypton ${}_{36}Kr$, it was heartbreaking to sort them out, I just love them all, but as a useful complement to the above list, we can at least list some remarkable elements, for various reasons, among them. These include:

(6) Noble gases: xenon ${}_{54}Xe$, radon ${}_{86}Rn$.

(7) Noble metals: silver ${}_{47}Ag$, iridium ${}_{77}Ir$, platinum ${}_{78}Pt$, gold ${}_{47}Au$.

(8) Heavy metals: mercury ${}_{80}Hg$, lead ${}_{82}Pb$.

(9) Radioactive: polonium ${}_{84}Po$, radium ${}_{88}Ra$, uranium ${}_{92}U$, plutonium ${}_{94}Pu$.

(10) Miscellaneous: rubidium ${}_{37}Rb$, strontium ${}_{38}Sr$, molybdenum ${}_{42}Mo$, technetium ${}_{43}Tc$, cadmium ${}_{48}Cd$, tin ${}_{50}Sn$, iodine ${}_{53}I$, caesium ${}_{55}Cs$, tungsten ${}_{74}Tu$, bismuth ${}_{83}Bi$, francium ${}_{87}Fr$, americium ${}_{95}Am$.

Here the abbreviations not fitting with English names come from the Latin or sometimes Greek argentum ${}_{47}Ag$, aurum ${}_{47}Au$, hydrargyrum ${}_{80}Hg$, plumbum ${}_{82}Pb$ and stannum ${}_{50}Sn$. The noble gases in (1) normally include oganesson ${}_{118}Og$ as well. The noble metals in (2) are something subjective. There are of course plenty of other heavy metals (3), or radioactive elements (4). As for the list in (5), this is something subjective, basically a mixture of well-known metals used in engineering, and some well-known bad guys in the context of nuclear fallout. Technetium ${}_{43}$Tc is a bizarre element, human-made.

Regarding now the second assertion in Fact 5.8, regarding the formation of molecules, this again comes from Fact 5.7, but via a more complicated mechanism. The idea here is that given two or several atoms, which can have the same atomic number Z or not, what happens is that, depending on their respective Z, these atoms might choose to share some electrons, with this coming somehow from less energy needed for functioning, in this new configuration. And so, we are led to clusters of atoms, called molecules.

As an example here, or rather counterexample, let us look at the group 18 elements, helium ${}_{2}$He, neon ${}_{10}$Ne, argon ${}_{18}$Ar, krypton ${}_{36}$Kr, xenon ${}_{54}$Xe and radon ${}_{86}$Rn. These are called noble gases, and are allergic to chemistry, because the group 18 elements are precisely those with all possible electron positions fully occupied, up to a certain $n \in \mathbb{N}$, which makes them very unfriendly to any chemistry proposition from the outside.

So long for the chemical elements, and the periodic table. Unfortunately business is business, and we will have to stop here, and go towards organic chemistry. We have:

Fact 5.9 (Organic chemistry). *Advanced molecules, called organic, are typically long and contain lots of carbon* ${}_{6}$C *and hydrogen* ${}_{1}$H*, which tend to team together.*

Here the formation mechanism is something quite complicated, relying on the remarkable properties of the carbon ${}_{6}$C and hydrogen ${}_{1}$H elements, which tend indeed to team together, and form all sorts of amazing molecules, which are typically very long.

Getting now to biology, the basics here can be summarized as follows:

Fact 5.10 (Basic biology). *Organic molecules tend to team together, as to form cells, which themselves tend to team together too, as to form advanced forms of life.*

Finally, as a last piece of theory, we need to talk about mutations, from a chemical viewpoint, with these being responsible for the evolution of the various forms of life.

5c. Charles Darwin

Charles Darwin.

5d. Homo sapiens

Homo sapiens.

5e. Exercises

Exercises:

Exercise 5.11.

Exercise 5.12.

Exercise 5.13.

Exercise 5.14.

Exercise 5.15.

Exercise 5.16.

Exercise 5.17.

Exercise 5.18.

Bonus exercise.

CHAPTER 6

Bible and Quran

6a. The Stone age

The Stone age.

6b. Bronze and Iron

Bronze and Iron.

6c. The Bible

The Bible.

6d. The Quran

The Quran.

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Bonus exercise.

CHAPTER 7

Karl Marx

7a. Crusaders

Crusaders.

7b. Martin Luther

Martin Luther.

7c. Karl Marx

Karl Marx.

7d. Stalin and others

Stalin and others.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

Sigmund Freud

8a. Kant, Nietzsche

Kant, Nietzsche.

8b. Sigmund Freud

Sigmund Freud.

8c. Eliade, Foucault

Eliade, Foucault.

8d. Marx, revised

Marx, revised.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.

Part III

Economy models

Don't cry for me Argentina
The truth is I never left you
All through my wild days, my mad existence
I kept my promise, don't keep your distance

CHAPTER 9

Game theory

9a. Basic games

Basic games.

9b. Saddle points

Saddle points.

9c. Minimax theorem

Minimax theorem.

9d. Order and chaos

Order and chaos.

9e. Exercises

Exercises:

Exercise 9.1.

Exercise 9.2.

Exercise 9.3.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

Exercise 9.7.

Exercise 9.8.

Bonus exercise.

CHAPTER 10

Life models

10a. Life models

Life models.

10b. Logistic equation

Logistic equation.

10c. Advanced models

Advanced models.

10d. Thermodynamics

Thermodynamics.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

Basic finance

11a. Money

Money.

11b. Banks

Banks.

11c. Taxes

Taxes.

11d. Welfare

Welfare.

11e. Exercises

Exercises:

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.

Exercise 11.4.

Exercise 11.5.

Exercise 11.6.

Exercise 11.7.

Exercise 11.8.

Bonus exercise.

CHAPTER 12

Peace and war

12a. War games

War games.

12b. Money and war

Money and war.

12c. Famine, disease

Famine, disease.

12d. The Mad doctrine

The Mad doctrine.

12e. Exercises

Exercises:

Exercise 12.1.

Exercise 12.2.

Exercise 12.3.

Exercise 12.4.

Exercise 12.5.

Exercise 12.6.

Exercise 12.7.

Exercise 12.8.

Bonus exercise.

Part IV

Modern finance

Shades of death are all I see
Skeletons of society
Fragments of what used to be
Skeletons of society

CHAPTER 13

Ponzi schemes

13a. Charles Ponzi

Charles Ponzi.

13b. Finance and law

Finance and law.

13c. Inflation

Inflation.

13d. Hyperinflation

Hyperinflation.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

Stock market

14a. Industrial society

Industrial society.

14b. Stock dynamics

Stock dynamics.

14c. Advanced math

Advanced math.

14d. Pros and cons

Pros and cons.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

Tax havens

15a. Story of taxes

Story of taxes.

15b. Layers, federalism

Layers, federalism.

15c. Corporate tax

Corporate tax.

15d. Tax havens

Tax havens.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

Sovereign debt

16a. Silver and gold

Silver and gold.

16b. Bretton Woods

Bretton Woods.

16c. States and banks

States and banks.

16d. Bubbles and crises

Bubbles and crises.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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