The normal and Poisson laws

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Plan

The Gauss integral

Theorem. We have the following formula:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

Proof. The square of the integral is given by:

$$\begin{aligned} I^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dxdy \\ &= \int_0^{2\pi} \int_0^{\infty} re^{-r^2} drdt \\ &= \int_0^{2\pi} \left[-\frac{e^{-r^2}}{2} \right]_0^{\infty} dt \end{aligned}$$

We obtain $I^2 = (2\pi) \times \frac{1}{2} = \pi$, and so $I = \sqrt{\pi}$.

The normal law

Definition. The normal law of parameter 1 is:

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

Remark. The Gauss formula gives with $x = y/\sqrt{2t}$

$$\int_{\mathbb{R}} e^{-y^2/2t} dy = \sqrt{2\pi t}$$

so these laws have indeed mass 1.

Variance

Theorem. We have the following formula, for any $t > 0$:

$$V(g_t) = t$$

Proof. The first moment is 0, and the second moment is:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx) \left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t e^{-x^2/2t} dx \end{aligned}$$

We obtain $M_2 = t$, so the variance is $V = t$.

Fourier transform

Theorem. Assuming that f, g are independent, we have

$$F_{f+g} = F_f F_g$$

where $F_f(x) = \mathbb{E}(e^{ixf})$ is the Fourier transform.

Proof. We have indeed the following computation:

$$\begin{aligned} F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixy} d\mu_{f+g}(y) \\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{ix(y+z)} d\mu_f(y) d\mu_g(z) \\ &= \int_{\mathbb{R}} e^{ixy} d\mu_f(y) \int_{\mathbb{R}} e^{ixz} d\mu_g(z) \end{aligned}$$

Thus, we obtain $F_{f+g}(x) = F_f(x) F_g(x)$, as desired.

Convolution

Theorem. We have the following formula, for any $t > 0$:

$$Fg_t(x) = e^{-tx^2/2}$$

Proof. This follows from the following computation:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-y^2/2t+ixy}dy \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2}dy \\
&= \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-z^2-tx^2/2}dz
\end{aligned}$$

As a consequence, we have the following result:

Theorem. We have $g_s * g_t = g_{s+t}$, for any $s, t > 0$.

CLT

Theorem. Assuming that $f_1, f_2, f_3, \ldots$ are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i \sim g_t$$

Proof. We have the following formula, in terms of moments:

$$F_f(x) = \sum_{k=0}^{\infty} \frac{i^k M_k(f)}{k!} \, x^k$$

Thus, the Fourier transform of the variable in the statement is:

$$F(x) = \left[F_f \left(\frac{x}{\sqrt{n}} \right) \right]^n = \left[1 - \frac{tx^2}{2n} + o(x^2) \right]^n$$

Thus we obtain $F(x) \simeq e^{-tx^2/2} = F_{g_t}(x)$, as desired.

Moments 1/2

Theorem. The moments of the normal law are

$$M_k(g_t) = t^{k/2} \times k!!$$

where $k!! = 1.3.5 \ldots (k-1)$, with $k!! = 0$ when k is odd.

Proof. We have the following computation:

$$\begin{aligned}
M_k &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^k e^{-x^2/2t} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx^{k-1}) \left(-e^{-x^2/2t}\right)' dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t(k-1)x^{k-2} e^{-x^2/2t} dx
\end{aligned}$$

We obtain $M_k = t(k-1)M_{k-2}$, which gives the result.

Moments 2/2

Theorem. The moments of the normal law are

$$M_k(g_t) = t^{k/2}|P_2(k)|$$

where $P_2(k)$ is the set of pairings of $\{1, \ldots, k\}$.

Proof. By pairing 1 with one of $2, \ldots, k$, we obtain

$$|P_2(k)| = (k-1)|P_2(k-2)|$$

which gives $|P_2(k)| = k!!$, and so the result.

Variation. We have the moment formula

$$M_k(g_t) = \sum_{\pi \in P_2(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks.

Spheres 1/3

Goal. Understand the laws of the coordinates $x_i : S^{N-1} \to \mathbb{R}$, called "hyperspherical", and their $N \to \infty$ behavior.

At $N = 2$ the coordinates are $\cos t, \sin t$, and we have:

Theorem. We have the following formula

$$\int_0^{\pi/2} \cos^p t \sin^q t \, dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

where $\varepsilon(p) = 1$ when p is even, and $\varepsilon(p) = 0$ when p is odd.

Proof. Partial integration, and double recurrence.

Spheres 2/3

Theorem. The integration over the sphere is given by

$$\int_{S^{N-1}} x_{i_1} \ldots x_{i_k} \, dx = \frac{(N-1)!! l_1!! \ldots l_N!!}{(N + \sum l_i - 1)!!}$$

where l_a is the number of occurrences of a inside $i_1, \ldots, i_k$.

Proof. In spherical coordinates the integral is as follows:

$$I = \frac{2^N}{V} \int_0^{\pi/2} \ldots \int_0^{\pi/2} x_1^{l_1} \ldots x_N^{l_N} J \, dt_1 \ldots dt_{N-1}$$

The normalization constant in front of the integral is

$$\frac{2^N}{V} = \frac{2^N}{N\pi^{N/2}} \cdot \Gamma\left(\frac{N}{2} + 1\right) = \left(\frac{2}{\pi}\right)^{[N/2]} (N-1)!!$$

and the integral can be computed by using the $N = 2$ formula.

Spheres 3/3

Theorem. The moments of the hyperspherical variables are

$$\int_{S^{N-1}} x_i^k dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $x_i/\sqrt{N}$ become normal with $N \to \infty$.

Proof. The formula in the statement follows from the previous result. With $N \to \infty$ we obtain, as desired:

$$\int_{S^{N-1}} x_i^k dx \simeq N^{k/2} k!! = N^{k/2} M_k(g_1)$$

Remark. The previous result shows as well that the rescaled variables $x_i/\sqrt{N}$ become independent with $N \to \infty$.

Rotations

Theorem. We have the integration formula

$$\int_{O_N} U_{ij}^k dU = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $U_{ij}/\sqrt{N}$ become normal with $N \to \infty$.

Proof. This follows from the previous result, and from the fact that we have an embedding as follows, for any j,

$$C(S^{N-1}) \subset C(O_N) \quad , \quad x_i \to U_{ij}$$

which makes correspond the respective integration functionals.

Comment. The rescaled variables $U_{ij}/\sqrt{N}$ can be shown to become independent with $N \to \infty$. We will be back to this.

Calculus

Theorem. The following formulae define the same number:
(1) $\pi = L/2$, where L is the length of the unit circle.
(2) $\pi = A$, where A is the area of the unit circle.

Theorem. The following formulae define the same number:
(1) $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.
(2) $e = \sum_{k=0}^{\infty}\frac{1}{k!}$.
(3) $e = f(1)$, where $f' = f$, $f(0) = 1$.

Theorem. We have the formula $e^{\pi i} = -1$.

Some magics

Theorem. The probability for a permutation $\sigma \in S_N$ to be a derangement is, in the $N \to \infty$ limit:

$$P = \frac{1}{e}$$

Proof. We must be outside the union $F = \bigcup_i F_i$, where:

$$F_i = \left\{ \sigma \in S_N \middle| \sigma(i) = i \right\}$$

The inclusion-exclusion principle gives:

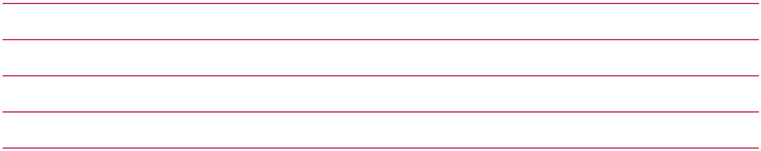
$$F^c = N! - \sum_i |F_i| + \sum_{i<j} |F_i \cap F_j| - \sum_{i<j<k} |F_i \cap F_j \cap F_k| + \ldots$$

We obtain $P = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots \simeq \frac{1}{e}$, as claimed.

What about pi?

In order to discuss this, take a needle of lenght 1:

Throw it (many times) on a grid of 1-spaced lines:



The probablity for the needle to intersect the grid is then:

$$P = \frac{2}{\pi}$$

The proof is quite tricky, and needs a correct modelling.

Fixed points

Theorem. The number of fixed points of permutations,

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

follows with $N \to \infty$ the following law:

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

Proof. We already know that the formula holds at 0. The same method, inclusion-exclusion, gives, more generally:

$$\lim_{N\to\infty} \mathbb{P}(\chi = k) = \frac{1}{e} \cdot \frac{1}{k!}$$

Thus, we obtain the law in the statement.

Poisson laws

Definition. The Poisson law of parameter 1 is:

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

More generally, the Poisson law of parameter $t > 0$ is:

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \, \delta_k$$

Remark. These laws have indeed mass 1.

Truncation

Theorem. The number of truncated fixed points of permutations,

$$\chi_t(\sigma) = \#\left\{i \in \{1, \ldots, [tN]\} \middle| \sigma(i) = i\right\}$$

follows with $N \to \infty$ the Poisson law p_t, for any $t \in (0, 1]$.

Proof. We already know that the formula holds at $t = 1$. The same method, inclusion-exclusion, gives, more generally:

$$\lim_{N\to\infty} \mathbb{P}(\chi = k) = \frac{1}{e^t} \cdot \frac{t^k}{k!}$$

Thus, we obtain with $N \to \infty$ the Poisson law p_t, as claimed.

Theory 1/2

Theorem. We have the following formula, for any $s, t > 0$:

$$p_s * p_t = p_{s+t}$$

Proof. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula:

$$\begin{aligned} p_s * p_t &= e^{-s} \sum_k \frac{s^k}{k!} \delta_k * e^{-t} \sum_l \frac{t^l}{l!} \delta_l \\ &= e^{-s-t} \sum_n \delta_n \sum_{k+l=n} \frac{s^k t^l}{k! l!} \\ &= e^{-s-t} \sum_n \frac{(s+t)^n}{n!} \delta_n \end{aligned}$$

Thus, we obtain the Poisson law p_{s+t}, as claimed.

Theory 2/2

Theorem. The Poisson laws appear as exponentials

$$p_t = \sum_k \frac{t^k(\delta_1 - \delta_0)^{*k}}{k!}$$

with respect to the convolution of measures $*$.

Proof. By using the binomial formula, the measure at right is:

$$\begin{aligned} \mu &= \sum_k t^k \sum_{p+q=k} (-1)^q \frac{\delta_p}{p!q!} \\ &= \sum_p \frac{t^p \delta_p}{p!} \sum_q \frac{(-t)^q}{q!} \end{aligned}$$

Thus, we obtain the Poisson law p_t, as claimed.

Fourier

<u>Theorem</u>. The Fourier transform of p_t is given by:

$$F_{p_t}(x) = \exp\left((e^{ix} - 1)t\right)$$

<u>Proof</u>. We have $F_f(x) = \mathbb{E}(e^{ixf})$, and we obtain:

$$\begin{aligned} F_{p_t}(x) &= e^{-t} \sum_k \frac{t^k}{k!} e^{ikx} \\ &= e^{-t} \sum_k \frac{(e^{ix} t)^k}{k!} \\ &= \exp(-t) \exp(e^{ix} t) \end{aligned}$$

Thus, we obtain the formula in the statement.

PLT

Theorem. We have the following convergence, in moments:

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

Proof. We have the following computation:

$$\begin{aligned}
F_{\delta_t}(x)=e^{itx} \quad&\implies\quad F_{\mu_n}(x)=\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\\
&\implies\quad F_{\mu_n^{*n}}(x)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n\\
&\implies\quad F_{\mu_n^{*n}}(x)=\left(1+\frac{(e^{ix}-1)t}{n}\right)^n\\
&\implies\quad F(x)=\exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Thus, we obtain the Fourier transform of p_t.

Moments 1/2

Theorem. The moments of p_1 are the Bell numbers,

$$M_k(p_1) = |P(k)|$$

where $P(k)$ is the set of partitions of $\{1, \ldots, k\}$.

Proof. The moments of p_1 are given by:

$$M_k = \frac{1}{e} \sum_s \frac{s^k}{s!}$$

A direct computation gives the following formula:

$$M_{k+1} = \sum_r \binom{k}{r} M_{k-r}$$

Thus, we have the same recurrence as for the Bell numbers.

Moments 2/2

Theorem. The moments of p_t are given by

$$M_k(p_t) = \sum_{\pi \in P(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks.

Proof. The moments of p_t are given by:

$$M_k = e^{-t} \sum_s \frac{t^s s^k}{s!}$$

We are therefore led into Stirling numbers.

Summary

We have seen that the normal laws g_t and the Poisson laws p_t have many common features:

(1) They appear via limiting theorems, CLT/PLT.

(2) Their moments are related to partitions, $P_2(k)/P(k)$.

(3) Interesting connections with groups, O_N/S_N.

Thanks

Next lecture: integration over compact groups.