Operator algebras and free probability

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Plan

1. C^*-algebras
2. Von Neumann algebras
3. Free probability
4. R-transform, CLT

Linear operators

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, which:

(1) Is complete with respect to $||.||$ (Banach algebra).

(2) Has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Proof. Complex algebra is clear, given $\{T_n\}$ Cauchy we can set $Tx = \lim_{n\to\infty} T_n x$, the involution comes from $\varphi(x) =< Tx, y >$ which is linear, and $||TT^*|| = ||T||^2$ is by double inequality.

Operator algebras

Definition. A C^*-algebra is an algebra $A \subset B(H)$, which:

(1) Is norm closed: $T_n \in A,\ T_n \to T \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Definition. A von Neumann algebra is an algebra $A \subset B(H)$, which:

(1) Is weakly closed: $T_n \in A,\ T_n x \to Tx, \forall x \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Examples. The commutative C^*-algebras $C(X)$, and von Neumann algebras $L^\infty(X)$, acting by multiplication on $L^2(X)$.

Spectral theory

Definition. The spectrum of an element $a \in A$ is the set

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1} \right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

Definition. The spectral radius $\rho(a)$ of an element $a \in A$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. Let A be a C^*-algebra.

(1) The spectrum of a norm 1 element is on the unit disk.

(2) The spectrum of a unitary ($a^* = a^{-1}$) is on the unit circle.

(3) The spectrum of a self-adjoint element ($a = a^*$) is real.

(4) ρ of a normal element ($aa^* = a^*a$) equals its norm.

Proof

(1) Clear from $(1-a)^{-1} = 1 + a + a^2 + \ldots$ for $||a|| < 1$.

(2) Follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

(3) Follows from (2), by using $f(z) = (z+it)/(z-it)$.

(4) By (1) we have $\rho(a) \leq ||a||$. Given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

When $a = a^*$ we are done. In general, we can use $||aa^*|| = ||a||^2$.

Gelfand theorem

Theorem. The commutative C^*-algebras are the algebras of the form $C(X)$, with X being a compact space.

Proof. If X is compact, $C(X)$ is indeed a C^*-algebra. Conversely, given A commutative, consider the space of characters

$$X = \{\chi : A \to \mathbb{C}\}$$

with topology making continuous each $ev_a : \chi \to \chi(a)$. Then X is compact, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

GNS theorem

Theorem. Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. In the commutative case, $A = C(X)$, we have indeed:

$$A \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

In general the idea is similar, by constructing an integration

$$\varphi : A \to \mathbb{C}$$

then defining a space $H = L^2(A)$, and using $a \to (b \to ab)$.

Von Neumann algebras

Definition. A von Neumann algebra is a $*$-algebra of operators $A \subset B(H)$ which is closed under the weak topology:

$$T_n \in A, T_n x \to Tx \implies T \in A$$

Examples. The usual C^*-algebras, in finite dimensions. Also, the algebras $L^\infty(X) \subset B(L^2(X))$, which are commutative.

Theorem. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Proof. Basic functional analysis and operator theory. The full statement involves as well a multiplicity, in regards with H.

Basic theory

Theorem. For a $*$-algebra of operators $A \subset B(H)$, the following conditions are equivalent:

(1) A is weakly closed, i.e. is a von Neumann algebra.

(2) A is equal to its algebraic bicommutant, $A = A''$.

This is von Neumann's "bicommutant theorem". As a consequence, the von Neumann algebras appear as commutants, $A = P'$.

Comments. Von Neumann $\implies$ C^*. Conversely, the von Neumann algebras are the C^*-algebras having separable predual. Also,

$$L^\infty(X) = C(\widehat{X})$$

by Gelfand, with $\widehat{X}$ being the Stone-Čech compactification of X.

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

Reduction theory

Theorem. When writing the center of the algebra as

$$Z(A) = L^\infty(X)$$

wihh X measured space, the algebra decomposes as

$$A = \int_X A_x \, dx$$

with the summands being "factors", $Z(A_x) = \mathbb{C}$.

Example. In finite dimensions the algebra must be

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

and this is its decomposition as a sum of factors.

Factors

Theorem. The factors, $Z(A) = \mathbb{C}$, fall into 3 classes:

(1) Type I. These are the usual matrix algebras $M_N(\mathbb{C})$ (type I_N), and the algebra $B(H)$, with H separable (type I_∞).

(2) Type II. These are the ∞D factors having a trace $tr : A \to \mathbb{C}$ (type II_1) and their tensor products with $B(H)$ (type II_∞).

(3) Type III. These fall into several classes, III_λ with $\lambda \in [0, 1]$, and appear from II_1 factors, via crossed product type constructions.

Proof. This is heavy, due to Murray and von Neumann, and then Connes, based on ideas of Tomita, Takesaki and others.

$\implies$ The II_1 factors are the "building blocks" of the theory.

The factor R

Theorem 1. The following limiting von Neumann algebra,

$$R = \lim_{k\to\infty} M_{N_k}(\mathbb{C})$$

is a II_1 factor, independent of the limiting procedure.

Theorem 2. R is the unique "hyperfinite" II_1 factor.

Theorem 3. R is the unique "building block" for the whole hyperfinite von Neumann algebra theory.

These results, building on what has been said before, are heavy, due to Murray-von Neumann, Connes, and Haagerup.

NC probability

<u>Definition</u>. Let A be a $*$-algebra, given with a trace tr.

(1) The elements $a \in A$ are called random variables.

(2) The moments of $a \in A$ are the numbers $M_k(a) = tr(a^k)$.

(3) The law of $a \in A$ is the functional $\mu_a : P \to tr(P(a))$.

Here $k = \circ \bullet \bullet \circ \ldots$ is a colored integer, and the powers a^k are defined by $a^\emptyset = 1, a^\circ = a, a^\bullet = a^*$ and multiplicativity.

The law is uniquely determined by the moments, because

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

for any $P \in \mathbb{C} < X, X^* >$, with the above conventions.

Spectral measures

Theorem. Assume that A is a C^*-algebra, that $tr : A \to \mathbb{C}$ is positive, $x \geq 0 \implies tr(x) \geq 0$, and that a is self-adjoint:

$$a = a^*$$

(1) μ_a is a real probability measure, or rather the integration with respect to such a measure, satisfying $supp(\mu_a) \subset \sigma(a)$.

(2) Assuming that tr is faithful, $x > 0 \implies tr(x) > 0$, the support of the law is the whole spectrum, $supp(\mu_a) = \sigma(a)$.

Moreover, these results extend to the normal case, $aa^* = a^*a$.

Proof. This is standard, coming from the Riesz theorem.

Random matrices

Definition. A random matrix algebra is a von Neumann algebra

$$A = M_N(\mathbb{C}) \otimes L^\infty(X)$$

endowed with its canonical unital trace, $tr = tr_N \otimes \int_X$.

Theorem. The matrices $M \in A$ having i.i.d. normal entries, up to the constraint $M = M^*$, follow with $N \to \infty$ the semicircle law:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. The Wick formula gives with $N \to \infty$ the Catalan numbers, which are the moments of the semicircle law.

Free probability

Definition. Let A be a $*$-algebra, given with a unital trace $tr : A \to \mathbb{C}$. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Examples. Two $*$-algebras B, C are independent inside their tensor product $B \otimes C$, and free inside their free product $B * C$.

Definition. Two elements $b, c \in A$ are called independent/free when the $*$-algebras that they generate

$$B =< b > \quad , \quad C =< c >$$

in the general $*$-algebra sense, or the C^*-algebra sense, or the von Neumann algebra sense, are independent/free.

Group algebras 1/2

Definition. Given a discrete group Γ, we endow the algebra $\mathbb{C}[\Gamma]$ with the involution $g^* = g^{-1}$, we consider the representation

$$\mathbb{C}[\Gamma] \subset B(l^2(\Gamma)) \quad , \quad g(\delta_h) = \delta_{gh}$$

and by closing we obtain operator algebras $C^*(\Gamma)$ and $L(\Gamma)$. These algebras have a faithful trace, given by:

$$tr(g) = \delta_{g1} \quad , \quad \forall g \in \Gamma$$

Properties. When Γ is abelian, we obtain the algebras $C(\widehat{\Gamma})$ and $L^\infty(\widehat{\Gamma})$. Also, $L(\Gamma)$ is a II_1 factor when Γ has infinite conjugacy classes. If in addition Γ is amenable, we have $L(\Gamma) \simeq R$.

Group algebras 2/2

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma\times\Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma*\Lambda)$.

Proof. This follows either from the product formulae

$$C^*(\Gamma\times\Lambda)=C^*(\Gamma)\otimes C^*(\Lambda)$$

$$C^*(\Gamma*\Lambda)=C^*(\Gamma)*C^*(\Lambda)$$

or by checking the independence/freeness on group elements.

Free convolution

Definition. The classical additive and multiplicative convolutions are constructed as follows, with a, b being independent:

$$\mu_a * \mu_b = \mu_{a+b} \quad , \quad \mu_a \times \mu_b = \mu_{ab}$$

Similarly, the free additive and multiplicative convolutions are constructed as follows, with a, b being free:

$$\mu_a \boxplus \mu_b = \mu_{a+b} \quad , \quad \mu_a \boxtimes \mu_b = \mu_{ab}$$

Remark. These operations are indeed well-defined, because the above compositions depend only on μ_a and μ_b.

R-transform 1/2

Theorem. Given a real probability measure μ, consider its Cauchy transform, and define its R-transform as follows:

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu\left(R_\mu(\xi) + \frac{1}{\xi} \right) = \xi$$

This transform linearizes then the free convolution operation:

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu$$

Remark. This is similar to the fact that the log of the Fourier transform $F_{\mu_a}(\xi) = \mathbb{E}(e^{i\xi a})$ linearizes the usual convolution $*$.

R-transform 2/2

<u>Proof</u>. We use the monoid algebra $C^*(\mathbb{N} * \mathbb{N})$. We have freeness here, a bit as for group algebras, and the point is that the variables of type $S^* + f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and $f \in \mathbb{C}[X]$, model in moments all the distributions $\mu : \mathbb{C}[X] \to \mathbb{C}$.

Now let $f, g \in \mathbb{C}[X]$ and consider the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N} * \mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$. These variables are free, and by using a 45° argument, their sum has the same law as $S^* + (f + g)(S)$.

Thus the operation $\mu \to f$ linearizes the free convolution. We are left with a computation inside $C^*(\mathbb{N})$, which is elementary, and whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ, as in the statement.

S-transform

Theorem. Given a real probability measure μ, consider its moment generating function, or Stieltjes transform,

$$f(z) = 1 + M_1 z + M_2 z^2 + M_3 z^3 + \ldots$$

set $\psi(z) = f(z) - 1$, then invert, $\psi(\chi(z)) = z$, and then set:

$$S(z) = (1 + z^{-1})\chi(z)$$

Then $\log S$ linearizes the free multiplicative convolution:

$$S_{\mu \boxtimes \nu} = S_\mu S_\nu$$

Remark. The operation $\boxtimes$ is well-defined for real measures, because $\mu_a \boxtimes \mu_b = \mu_{\sqrt{a}b\sqrt{a}}$, with a, b self-adjoint and free.

CLT

Theorem. Assuming that $x_1, x_2, x_3, \ldots \in A$ are i/f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim g_t/\gamma_t$$

where g_t/γ_t are the normal/Wigner semicircle laws.

Proof. In the classical case this follows from the linearization property of $\log F$, namely $F_{\mu*\nu} = F_\mu F_\nu$, and from:

$$Fg_t(\xi) = e^{-t\xi^2/2}$$

In the free case, this follows from the linearization property of R, namely $R_{\mu\boxplus\nu} = R_\mu + R_\nu$, and from $R_{\gamma_t}(\xi) = t\xi$.

Wigner matrices

Theorem. Given a family of Wigner random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. normal entries, up to the constraint $M_i = M_i^*$, the following happen:

(1) Each M_i follows a semicircle law γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. Here (1) is Wigner's theorem and (2) is Voiculescu's theorem. Both can be proved with the moment method.

Summary

We have seen that:

(1) Classical and free probability are twin sisters.

(2) The free CLT makes appear the Wigner law.

(3) The Wigner matrices are asymptotically free.

Thanks

Next lecture: further limiting theorems.