

# QUANTUM GROUPS AND FUSS-CATALAN ALGEBRAS

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**Abstract.** The categories of representations of compact quantum groups of automorphisms of certain inclusions of finite dimensional  $\mathbb{C}^*$ -algebras are shown to be isomorphic to the categories of Fuss-Catalan diagrams.

## INTRODUCTION

Let  $(D, \tau)$  be a finite dimensional  $\mathbb{C}^*$ -algebra together with a trace. In [14] Wang constructs an algebra  $A_{aut}(D, \tau)$ : the biggest Hopf  $\mathbb{C}^*$ -algebra co-acting on  $(D, \tau)$ .

From the point of view of noncommutative geometry the algebra  $D$  corresponds to a noncommutative finite space  $\widehat{D}$  and the trace  $\tau$  corresponds to a mesure  $\widehat{\tau}$  on  $\widehat{D}$ . Thus  $A_{aut}(D, \tau)$  corresponds to the “quantum symmetry group”  $G_{aut}(\widehat{D}, \widehat{\tau})$ .

If  $D = \mathbb{C}^n$  with  $n = 1, 2, 3$  then  $A_{aut}(D, \tau)$  is just the algebra of functions on the  $n$ -th symmetric group. But if  $\dim(D) \geq 4$  then  $A_{aut}(D, \tau)$  is infinite-dimensional ([14]).

The corepresentations of  $A_{aut}(D, \tau)$  are studied in [1]: under a suitable assumption on the trace  $\tau$ , the algebras of symmetries of the fundamental corepresentation (i.e. the one on  $D$ ) are shown to be isomorphic to the Temperley-Lieb algebras.

Let  $B \subset D$  be an inclusion of finite dimensional  $\mathbb{C}^*$ -algebras and let  $\varphi$  be a state on  $D$ . Following Wang one can construct a Hopf algebra  $A_{aut}(B \subset D, \varphi)$ : the biggest Hopf  $\mathbb{C}^*$ -algebra co-acting on  $D$  such that  $B$  and  $\varphi$  are left invariant.

The main result in this paper is that, under suitable assumptions on  $\varphi$ , the category of finite dimensional corepresentations of  $A_{aut}(B \subset D, \varphi)$  is isomorphic to the completion of the category of Fuss-Catalan diagrams. (These are certain colored Temperley-Lieb diagrams, discovered by Bisch and Jones in connection with intermediate subfactors [5].) The proof (§1 – §4) uses [1], [5], [16] and reconstruction techniques.

The Fuss-Catalan diagrams were recently shown to appear in several contexts, related to subfactors, planar algebras and integrable lattice models. See e.g. Bisch and Jones [5], [6], Di Francesco [7], Landau [12] and the references there in. In the last section of the paper (§5) we discuss the relation between  $A_{aut}(B \subset D, \varphi)$  and subfactors.

## 1. PRELIMINARIES

The Fuss-Catalan category, as well as other categories to be used in this paper, is a tensor  $\mathbb{C}^*$ -category having  $(\mathbb{N}, +)$  as monoid of objects. For simplifying writing, such

a tensor category will be called a  $\mathbb{N}$ -algebra. If  $C$  is a  $\mathbb{N}$ -algebra we use the notations

$$C(m, n) = \text{Hom}_C(m, n) \quad C(m) = \text{End}_C(m)$$

As a first class of examples, associated to any object  $O$  in a tensor  $\mathbb{C}^*$ -category is the  $\mathbb{N}$ -algebra  $\mathbb{N}O$  given by

$$\mathbb{N}O(m, n) = \text{Hom}(O^{\otimes m}, O^{\otimes n})$$

Fix  $\delta > 0$ . The  $\mathbb{N}$ -algebra  $TL^2$  is defined as follows. The space  $TL^2(m, n)$  consists of linear combinations of Temperley-Lieb diagrams between  $2m$  points and  $2n$  points

$$TL^2(m, n) = \left\{ \sum \alpha \begin{array}{c} \cdots \cdots \leftarrow 2m \text{ points} \\ \mathfrak{W} \leftarrow m+n \text{ strings} \\ \cdots \leftarrow 2n \text{ points} \end{array} \right\}$$

(strings join points and don't cross) and the operations  $\circ$ ,  $\otimes$  and  $*$  are induced by vertical and horizontal concatenation and upside-down turning of diagrams. With the following rule: erasing a circle is the same as multiplying by  $\delta$ .

$$A \circ B = \begin{array}{c} B \\ A \end{array} \quad A \otimes B = AB \quad A^* = \nabla \quad \bigcirc = \delta$$

Consider the following two elements  $u \in TL^2(0, 1)$  and  $m \in TL^2(2, 1)$ :

$$u = \delta^{-\frac{1}{2}} \cap \quad m = \delta^{\frac{1}{2}} \mid \cup \mid$$

**Theorem 1.** *The following relations*

- (i)  $mm^* = \delta^2 1$
- (ii)  $u^*u = 1$
- (iii)  $m(m \otimes 1) = m(1 \otimes m)$
- (iv)  $m(1 \otimes u) = m(u \otimes 1) = 1$
- (v)  $(m \otimes 1)(1 \otimes m^*) = (1 \otimes m)(m^* \otimes 1) = m^*m$

are a presentation of  $TL^2$  by  $u \in TL^2(0, 1)$  and  $m \in TL^2(2, 1)$ .

The result says that  $u$  and  $m$  satisfy the relations and that if  $C$  is a  $\mathbb{N}$ -algebra and  $v \in C(0, 1)$  and  $n \in C(2, 1)$  satisfy the relations then there exists a  $\mathbb{N}$ -algebra morphism

$$TL^2 \rightarrow C \quad u \mapsto v \quad m \mapsto n$$

This is proved in [1]. Actually in [1] the ‘‘index’’  $\delta^2$  is an integer and  $u$  and  $m$  are certain explicit operators, but these extra structures are not used.

Let  $D$  be a finite dimensional  $\mathbb{C}^*$ -algebra with a state  $\varphi$  on it. We have a scalar product  $\langle x, y \rangle = \varphi(y^*x)$  on  $D$ , so  $D$  is an object in the category of finite dimensional Hilbert spaces. Consider the unit  $u$  and the multiplication  $m$  of  $D$ .

$$u \in \mathbb{N}D(0, 1) \quad m \in \mathbb{N}D(2, 1)$$

The relations in theorem 1 are satisfied if and only if the first one, namely  $mm^* = \delta^2 1$ , is satisfied. If  $D = \oplus M_{n_\beta}$  is a decomposition of  $D$ , we must have  $\text{Tr}(Q_\beta^{-1}) = \delta^2$  for any

block  $Q_\beta$  of the unique  $Q \in D$  such that  $\varphi = Tr(Q)$ . This can be cheked by direct computation; see [1] for the case  $\varphi = \text{trace}$ .

A linear form  $\varphi$  such that  $mm^* = \delta^2 1$ , where the adjoint of the multiplication is taken with respect to the scalar product associated to  $\varphi$ , will be called a  $\delta$ -form.

One can deduce from theorem 1 that if  $\varphi$  is a  $\delta$ -form then category of corepresentations of the Hopf  $\mathbb{C}^*$ -algebra  $A_{aut}(D, \varphi)$  is the completion of  $TL^2$ . The case  $\varphi = \text{trace}$  is studied in [1]; for the general case, see §4 below.

## 2. THE FUSS-CATALAN CATEGORY

A Fuss-Catalan diagram is a planar diagram formed by an upper row of  $4m$  points, a lower row of  $4n$  points and by  $2m + 2n$  non-crossing strings joining them. Both rows of points are colored from left to right in the following standard way

white, black, black, white, white, black, black,...

and strings have to join pairs of points having the same color.

Fix  $\beta > 0$  and  $\omega > 0$ . The  $\mathbb{N}$ -algebra  $FC$  is defined as follows. The spaces  $FC(m, n)$  consist of linear combinations of Fuss-Catalan diagrams

$$FC(m, n) = \left\{ \sum \alpha \begin{array}{c} \text{w,b,b,w,w,b,b,w,...} \\ \mathfrak{W} \\ \text{w,b,b,w,w,b,b,w,...} \end{array} \left. \begin{array}{l} \leftarrow 4m \text{ colored points} \\ m+n \text{ white strings} \\ \text{and} \\ m+n \text{ black strings} \\ \leftarrow 4n \text{ colored points} \end{array} \right\}$$

and the operations  $\circ$ ,  $\otimes$  and  $*$  are induced by vertical and horizontal concatenation and upside-down turning of diagrams. With the following rule: erasing a black/white circle is the same as multiplying by  $\beta/\omega$ .

$$A \circ B = \begin{array}{c} B \\ A \end{array} \quad A \otimes B = AB \quad A^* = \nabla \quad \text{black} \rightarrow \bigcirc = \beta \quad \text{white} \rightarrow \bigcirc = \omega$$

Let  $\delta = \beta\omega$ . The following bicolored analogues of the elements  $u$  and  $m$  in §1

$$u = \delta^{-\frac{1}{2}} \bigcap \quad m = \delta^{\frac{1}{2}} \parallel \bigcup \parallel$$

generate in  $FC$  a  $\mathbb{N}$ -subalgera which is isomorphic to  $TL^2$ .

Consider also the black and white Jones projections.

$$e = \omega^{-1} \left| \begin{array}{c} \bigcup \\ \bigcap \end{array} \right| \quad f = \beta^{-1} \parallel \bigcup \parallel$$

We have  $f = \beta^{-2}(1 \otimes me)m^*$ , so we won't need  $f$  for presenting  $FC$ .

For simplifying writing we identify  $x$  and  $x \otimes 1$ , for any  $x$ .

**Theorem 2.** *The following relations, with  $f = \beta^{-2}(1 \otimes me)m^*$  and  $\delta = \beta\omega$*

- (1) *the relations in theorem 1*
- (2)  *$e = e^2 = e^*$ ,  $f = f^*$  and  $(1 \otimes f)f = f(1 \otimes f)$*

$$(3) \quad eu = u$$

$$(4) \quad mem^* = m(1 \otimes e)m^* = \beta^2 1$$

$$(5) \quad mm(e \otimes e \otimes e) = emm(e \otimes 1 \otimes e)$$

are a presentation of  $FC$  by  $m \in FC(2, 1)$ ,  $u \in FC(0, 1)$  and  $e \in FC(1)$ .

*Proof.* As for any presentation result, we have to prove two assertions.

(I) The elements  $m, u, e$  satisfy the relations (1–5) and generate the  $\mathbb{N}$ -algebra  $FC$ .

(II) If  $M, U$  and  $E$  in a  $\mathbb{N}$ -algebra  $C$  satisfy the relations (1–5), then there exists a morphism of  $\mathbb{N}$ -algebras  $FC \rightarrow C$  sending  $m \mapsto M$ ,  $u \mapsto U$  and  $e \mapsto E$ .

The proof will be based on the results in the paper of Bisch and Jones [5], plus some graphic computations for (I) and some purely algebraic computations for (II).

(I) First, the relations (1–5) are easily verified by drawing pictures.

Let us show that the  $\mathbb{N}$ -subalgebra  $C = \langle m, u, e \rangle$  of  $FC$  is equal to  $FC$ . First,  $C$  contains the infinite sequence of black and white Jones projections

$$p_1 = e = \omega^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

$$p_2 = f = \beta^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

$$p_3 = 1 \otimes e = \omega^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

$$p_4 = 1 \otimes f = \beta^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

...

as well as the infinite sequence of bicolored Jones projections

$$e_1 = uu^* = \delta^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

$$e_2 = \delta^{-2} m^* m = \delta^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

$$e_3 = 1 \otimes uu^* = \delta^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

$$e_4 = \delta^{-2} (1 \otimes m^* m) = \delta^{-1} \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right| \left| \begin{array}{c} \cup \\ \cap \end{array} \right|$$

...

which by the result of Bisch and Jones ([5]) generate the diagonal  $\mathbb{N}$ -algebra  $\Delta FC$ . (If  $X$  is a  $\mathbb{N}$ -algebra, its diagonal  $\Delta X$  is defined by  $\Delta X(m) = X(m)$  and  $\Delta X(m, n) = \emptyset$  if  $m \neq n$ .) Thus we have inclusions

$$\Delta FC \subset C \subset FC$$

so we can use the following standard argument. First, we have  $\Delta FC = \Delta C$ . Second, the existence of semicircles shows that the objects of  $C$  and  $FC$  are selfdual and by Frobenius reciprocity we get

$$\dim(C(m, n)) = \dim\left(C\left(\frac{m+n}{2}\right)\right) = \dim\left(FC\left(\frac{m+n}{2}\right)\right) = \dim(FC(m, n))$$

for  $m+n$  even. By tensoring with  $u$  and  $u^*$  we get embeddings

$$C(m, n) \subset C(m, n+1) \quad FC(m, n) \subset FC(m, n+1)$$

and this shows that the dimension equalities hold for any  $m$  and  $n$ . Together with  $\Delta FC \subset C \subset FC$ , this shows that  $C = FC$ .

(II) Assume that  $M, U$  and  $E$  in a  $\mathbb{N}$ -algebra  $C$  satisfy the relations (1–5). We have to construct a morphism  $FC \rightarrow C$  sending

$$m \mapsto M, u \mapsto U, e \mapsto E$$

This will be done in two steps. First, we restrict attention to diagonals: we would like to construct a morphism  $\Delta FC \rightarrow \Delta C$  sending

$$m^*m \mapsto M^*M, uu^* \mapsto UU^*, e \mapsto E$$

By constructing the corresponding Jones projections  $E_i$  and  $P_i$ , we must send

$$e_i \mapsto E_i, p_i \mapsto P_i \quad (i = 1, 2, 3, \dots)$$

The presentation result for  $\Delta FC$  of Bisch and Jones ([5]) reduces this to an algebraic computation. More precisely, it is proved in [5] that the following relations

- (a)  $e_i^2 = e_i$ ,  $e_i e_j = e_j e_i$  if  $|i-j| \geq 2$  and  $e_i e_{i\pm 1} e_i = \delta^{-2} e_i$
- (b)  $p_i^2 = p_i$  and  $p_i p_j = p_j p_i$
- (c)  $e_i p_i = p_i e_i = e_i$  and  $p_i e_j = e_j p_i$  if  $|i-j| \geq 2$
- (d)  $e_{2i\pm 1} p_{2i} e_{2i\pm 1} = \beta^{-2} e_{2i\pm 1}$  and  $e_{2i} p_{2i\pm 1} e_{2i} = \omega^{-2} e_{2i}$
- (e)  $p_{2i} e_{2i\pm 1} p_{2i} = \beta^{-2} p_{2i\pm 1} p_{2i}$  and  $p_{2i\pm 1} e_{2i} p_{2i\pm 1} = \omega^{-2} p_{2i} p_{2i\pm 1}$

are a presentation of  $\Delta FC$ . So it remains to verify that

$$(1-5) \implies (a-e)$$

where  $m, u$  are  $e$  are abstract objects and we are no longer allowed to draw pictures.

First, by using  $e_{n+2} = 1 \otimes e_n$  and  $p_{n+2} = 1 \otimes p_n$  these relations reduce to:

- ( $\alpha$ )  $e_i^2 = e_i$  for  $i = 1, 2$ ,  $e_1 e_2 e_1 = \delta^{-2} e_1$  and  $e_2 e_1 e_2 = \delta^{-2} e_2$ .
- ( $\beta$ )  $p_i^2 = p_i$  for  $i = 1, 2$  and  $[p_1, p_2] = [1 \otimes p_1, p_2] = [1 \otimes p_2, p_2] = 0$
- ( $\gamma$ )  $[e_2, 1 \otimes p_2] = [p_2, 1 \otimes e_2] = 0$  and  $e_i p_i = p_i e_i = e_i$  for  $i = 1, 2$
- ( $\delta 1$ )  $e_1 p_2 e_1 = \beta^{-2} e_1$  and  $(1 \otimes e_1) p_2 (1 \otimes e_1) = \beta^{-2} (1 \otimes e_1)$
- ( $\delta 2$ )  $e_2 p_1 e_2 = e_2 (1 \otimes p_1) e_2 = \omega^{-2} e_2$
- ( $\epsilon 1$ )  $\beta^2 p_2 e_1 p_2 = \omega^2 p_1 e_2 p_1 = p_1 p_2$
- ( $\epsilon 2$ )  $\beta^2 p_2 (1 \otimes e_1) p_2 = \omega^2 (1 \otimes p_1) e_2 (1 \otimes p_1) = (1 \otimes p_1) p_2$

With  $e_1 = uu^*$ ,  $e_2 = \delta^{-2} m^*m$ ,  $p_1 = e$  and  $p_2 = f$  one can see that most of them are trivial. What is left can be reformulated in the following way.

- (x)  $em^*me = \beta^2 f^*e$
- (y)  $(1 \otimes e)m^*m(1 \otimes e) = \beta^2 f^*(1 \otimes e)$
- (z)  $f^* = f^*f$
- (t)  $[e, f] = [1 \otimes e, f] = [m^*m, 1 \otimes f] = [f, 1 \otimes m^*m] = 0$

By multiplying the relation (5) by  $u$  and by  $1 \otimes 1 \otimes u$  to the right we get the following useful formula, to be used many times in what follows.

$$m(e \otimes e) = em(1 \otimes e) = eme$$

Let us verify (x–t). First, we have

$$\beta^2 f^*e = m(e \otimes e)(1 \otimes m^*)$$

and by replacing  $m(e \otimes e)$  with  $eme$  we get  $em^*me$ , so (x) is true. We have

$$(1 \otimes e)m^*m(1 \otimes e) = m(1 \otimes (em(1 \otimes e))^*)$$

and by replacing  $em(1 \otimes e)$  with  $eme$  we get  $\beta^2 f^*(1 \otimes e)$ , so (y) is true. We have

$$f^*f = \beta^{-4}m(1 \otimes em^*me)m^*$$

and by replacing  $em^*me$  with  $eme(1 \otimes m^*)$ , then  $eme$  with  $m(e \otimes e)$  we get  $f^*$ , so (z) is true. The first two commutators are zero, because  $fe$  and  $f(1 \otimes e)$  are selfadjoint. Same for the others, because of the formulae

$$\begin{aligned} mm^*(1 \otimes ff^*) &= \beta^{-4}(1 \otimes 1 \otimes me)m^*m^*mm(1 \otimes 1 \otimes em^*) \\ (1 \otimes m^*m)ff^* &= \beta^{-4}(1 \otimes m^*me)m^*m(1 \otimes em^*m) \end{aligned}$$

The conclusion is that we constructed a certain  $\mathbb{N}$ -algebra morphism

$$\Delta J : \Delta FC \rightarrow \Delta C$$

that we have to extend now to a morphism  $J : FC \rightarrow C$  sending  $u \mapsto U$  and  $m \mapsto M$ . We will use a standard argument (see [11]). For  $w$  bigger than  $k$  and  $l$  we define

$$\begin{aligned} \phi : FC(l, k) &\rightarrow FC(w) & x &\mapsto (u^{\otimes(w-k)} \otimes 1_k) x ((u^*)^{\otimes(w-l)} \otimes 1_l) \\ \theta : FC(w) &\rightarrow FC(l, k) & x &\mapsto ((u^*)^{\otimes(w-k)} \otimes 1_k) x (u^{\otimes(w-l)} \otimes 1_l) \end{aligned}$$

where  $1_k = 1^{\otimes k}$  and where the convention  $x = x \otimes 1$  is no longer used. We define  $\Phi$  and  $\Theta$  in  $C$  by similar formulae. We have  $\theta\phi = \Theta\Phi = Id$ . We define a map  $J$  by

$$\begin{array}{ccc} FC(l, k) & \xrightarrow{J} & C(l, k) \\ \phi \downarrow & & \uparrow \Theta \\ FC(w) & \xrightarrow{\Delta J} & C(w) \end{array}$$

As  $J(a)$  doesn't depend on the choice of  $w$ , these  $J$ 's are the components of a map  $J : FC \rightarrow C$ . This map  $J$  extends  $\Delta J$  and sends  $u \mapsto U$  and  $m \mapsto M$ . It remains to prove that  $J$  is a morphism. We have

$$Im(\phi) = \{x \in FC(w) \mid x = ((uu^*)^{\otimes(w-k)} \otimes 1_k) x ((uu^*)^{\otimes(w-l)} \otimes 1_l)\}$$

as well as a similar description of  $Im(\Phi)$ , so  $J$  sends  $Im(\phi)$  to  $Im(\Phi)$ . On the other hand we have  $\Theta\Phi = Id$ , so  $\Phi\Theta = Id$  on  $Im(\Phi)$ . Thus

$$\begin{array}{ccc} FC(l, k) & \xrightarrow{J} & C(l, k) \\ \phi \downarrow & & \downarrow \Phi \\ FC(w) & \xrightarrow{\Delta J} & C(w) \end{array}$$

commutes, so  $J$  is multiplicative:

$$J(ab) = \Theta(\Delta J\phi(a)\Delta J\phi(b)) = \Theta(\Phi J(a)\Phi J(b)) = \Theta\Phi(J(a)J(b)) = J(a)J(b)$$

It remains to prove that  $J(a \otimes b) = J(a) \otimes J(b)$ . We have  $a \otimes b = (a \otimes 1_s)(1_t \otimes b)$  for certain  $s$  and  $t$ , so it is enough to prove it for pairs  $(a, b)$  of the form  $(1_t, b)$  or  $(a, 1_s)$ . For  $(a, 1_s)$  this is clear, so it remains to prove that the set

$$B = \{b \in FC \mid J(1_t \otimes b) = 1_t \otimes J(b), \forall t \in \mathbb{N}\}$$

is equal  $FC$ . First,  $\Delta J$  being a  $\mathbb{N}$ -algebra morphism, we have  $\Delta FC \subset B$ . On the other hand, computation gives  $J(1_t \otimes u \otimes 1_s) = 1_t \otimes U \otimes 1_s$ . Also,  $J$  being involutive and multiplicative,  $B$  is stable by involution and multiplication. We conclude that  $B$  contains the compositions of elements of  $\Delta FC$  with  $1_t \otimes u \otimes 1_s$ 's and  $1_t \otimes u^* \otimes 1_s$ 's. But any  $b$  in  $FC$  is equal to  $\theta\phi(b)$ , so it is of this form and we are done.  $\square$

### 3. INCLUSIONS OF FINITE DIMENSIONAL $\mathbb{C}^*$ -ALGEBRAS

Let  $B \subset D$  be an inclusion of finite dimensional  $\mathbb{C}^*$ -algebras and let  $\varphi$  be a state on  $D$ . We have the scalar product  $\langle x, y \rangle = \varphi(y^*x)$  on  $D$ . The multiplication  $m$  of  $D$ , the unit  $u$  of  $D$  and the orthogonal projection  $e$  from  $D$  onto  $B$

$$m : D \otimes D \rightarrow D \quad u : \mathbb{C} \rightarrow D \quad e : D \rightarrow D$$

can be regarded as elements of the  $\mathbb{N}$ -algebra  $\mathbb{N}D$  given by  $\mathbb{N}D(m, n) = \mathcal{L}(D^{\otimes m}, D^{\otimes n})$ .

We say that  $\varphi$  is a  $(\beta, \omega)$ -form on  $B \subset D$  if it is a  $\beta\omega$ -form on  $D$ , if its restriction  $\varphi|_B$  is a  $\beta$ -form on  $B$  and if  $e$  is a  $B - B$  bimodule morphism. (For  $\delta$ -forms, see §1.)

As a first example, if  $\phi$  is a  $\beta$ -form on  $B$  and  $\psi$  is a  $\omega$ -form on  $W$  then  $\phi \otimes \psi$  is a  $(\beta, \omega)$ -form on  $B \subset B \otimes W$ . In particular a  $\delta$ -form on  $D$  is a  $(1, \delta)$ -form on  $\mathbb{C} \subset D$ .

**Theorem 3.** *If  $\varphi$  is a  $(\beta, \omega)$ -form on  $B \subset D$  then  $\langle m, u, e \rangle = FC$ .*

*Proof.* We prove that  $m, u, e$  satisfy the relations (1–5). The formulae  $e = e^2 = e^*$  and (3) are true, (1) is equivalent to the fact that  $\varphi$  is a  $\beta\omega$ -form and (5) says that

$$e(b)e(c)e(d) = e(e(b)ce(d))$$

for any  $b, c, d$  in  $B$ , i.e. that  $e$  is a morphism of  $B - B$  bimodules.

Let  $\{b_{-i}\}_{i \in \mathbb{N}}$  be an orthonormal basis of  $B$  and let  $\{b_j\}_{j \in \mathbb{N}}$  be an orthonormal basis of  $B^\perp$ . We denote by  $\{b_n\}_{n \in \mathbb{Z}}$  the orthonormal basis  $\{b_{-i}, b_j\}_{i, j \in \mathbb{N}}$  of  $D$ . We have

$$m^*(b) = \sum_{k, s \in \mathbb{Z}} b_k \otimes b_s \langle b, b_k b_s \rangle = \sum_{k, s \in \mathbb{Z}} b_k \otimes b_s \langle b_k^* b, b_s \rangle = \sum_{k \in \mathbb{Z}} b_k \otimes b_k^* b$$

so  $\varphi$  is a  $\delta$ -form if and only if  $\sum b_k b_k^* = \delta^2 1$ . On the other hand, we get

$$mem^*(b) = m \left( \sum_{k \in \mathbb{Z}} e(b_k) \otimes b_k^* b \right) = \left( \sum_{k \in \mathbb{Z}} e(b_k) b_k^* \right) b = \left( \sum_{i \in \mathbb{N}} b_{-i} b_{-i}^* \right) b$$

so the formula  $mem^* = \beta^2 1$  in (4) is equivalent to the fact that  $\varphi|_N$  is a  $\beta$ -form on  $B$ .

It remains to check the following three formulae, with  $f = \beta^{-2}(1 \otimes me)m^*$ .

$$f = f^* \quad (1 \otimes f)f = f(1 \otimes f) \quad m(1 \otimes e)m^* = \beta^2 1 \quad (\star)$$

By using the fact that  $e$  is a bimodule morphism we get succesively that

$$\sigma(B) = B \quad e* = *e$$

where  $\sigma : D \rightarrow D$  is such that  $\varphi(ab) = \varphi(b\sigma(a))$ . By using the above formula for  $m^*$  we get

$$f(x \otimes y) = \beta^{-2}(1 \otimes me)m^*(x \otimes y) = \beta^{-2} \sum_{k \in \mathbb{Z}} b_k \otimes e(b_k^* b_m) b_n$$

This allows us to prove the first  $(\star)$  formula, because we have

$$\langle f(b_m \otimes b_n), b_M \otimes b_N \rangle = \beta^{-2} \varphi(b_N^* e(b_M^* b_m) b_n) = \langle b_m \otimes b_n, f(b_M \otimes b_N) \rangle$$

for any  $m, n, M, N$ . The second  $(\star)$  formula follows from

$$\begin{aligned} \langle (1 \otimes f)f(x \otimes y \otimes z), b_k \otimes b_s \otimes w \rangle &= \beta^{-4} \langle e(b_s^* a y) z, w \rangle \\ \langle f(1 \otimes f)(x \otimes y \otimes z), b_k \otimes b_s \otimes w \rangle &= \beta^{-4} \sum_{t \in \mathbb{Z}} \langle a b_t, b_s \rangle \langle e(b_t^* y) z, w \rangle \end{aligned}$$

with  $a = e(b_k^* x)$ , for any  $x, y, z, w, k, s$ . For the third  $(\star)$  formula, we have

$$m^*(b) = \sum_{k, s \in \mathbb{Z}} b_k \otimes b_s \langle b, b_k b_s \rangle = \sum_{k, s \in \mathbb{Z}} b_k \langle b\sigma(b_s^*), b_k \rangle \otimes b_s = \sum_{s \in \mathbb{Z}} b\sigma(b_s^*) \otimes b_s$$

and this gives  $m(1 \otimes e)m^*(b) = bq$  with  $q$  given by

$$q = \sum_{s \in \mathbb{Z}} \sigma(b_s^*) e(b_s) = \sum_{i \in \mathbb{N}} \sigma(b_{-i}^*) b_{-i} = m_B m_B^*(1) = \beta^2 1$$

where  $m_B$  is the multiplication of  $N$ , which was computed in a similar way.

Thus  $m, u, e$  satisfy the relations (1–5), so theorem 2 applies and gives a certain  $\mathbb{N}$ -algebra surjective morphism  $J : FC \rightarrow \langle m, u, e \rangle$ .

It remains to prove that  $J$  is faithful. For, consider the maps

$$\phi_n : FC(n) \rightarrow FC(n-1) \quad x \mapsto (1^{\otimes(n-1)} \otimes v^*)(x \otimes 1)(1^{\otimes(n-1)} \otimes v)$$

$$\psi_n : C(n) \rightarrow C(n-1) \quad x \mapsto (1^{\otimes(n-1)} \otimes J(v)^*)(x \otimes 1)(1^{\otimes(n-1)} \otimes J(v))$$

where  $v = m^*u \in FC(0, 2)$ . These make the following diagram commutative

$$\begin{array}{ccc} FC(n) & \xrightarrow{J} & C(n) \\ \phi_n \downarrow & & \downarrow \psi_n \\ FC(n-1) & \xrightarrow{J} & C(n-1) \end{array}$$



and by gluing such diagrams we get a factorisation by  $J$  of the composition on the left of conditional expectations, which is the Markov trace. By positivity  $J$  is faithful on  $\Delta FC$ , then by Frobenius reciprocity faithfulness has to hold on the whole  $FC$ .  $\square$

#### 4. QUANTUM AUTOMORPHISM GROUPS OF INCLUSIONS

Let  $B \subset D$  be an inclusion of finite dimensional  $\mathbb{C}^*$ -algebras and let  $\varphi$  be a state on  $D$ . Following Wang ([14]) we define the universal  $\mathbb{C}^*$ -algebra  $A_{aut}(B \subset D, \varphi)$  generated by the coefficients  $v_{ij}$  of a unitary matrix  $v$  subject to the conditions

$$m \in Hom(v^{\otimes 2}, v) \quad u \in Hom(1, v) \quad e \in End(v)$$

where  $m : D \otimes D \rightarrow D$  is the multiplication,  $u : \mathbb{C} \rightarrow D$  is the unit and  $e : D \rightarrow D$  is the projection onto  $B$ , with respect to the scalar product  $\langle x, y \rangle = \varphi(y^*x)$ .

This definition has to be understood as follows. Let  $n = dim(D)$  and fix a vector space isomorphism  $D \simeq \mathbb{C}^n$ . Let  $F$  be the free  $*$ -algebra on  $n^2$  variables  $\{v_{ij}\}_{i,j=1,\dots,n}$  and let  $v = (v_{ij}) \in M_n \otimes F$ . For any  $k \in \mathbb{N}$  define  $v^{\otimes k}$  to be

$$v^{\otimes k} = v_{1,k+1}v_{2,k+1} \dots v_{k,k+1} \in M_n^{\otimes k} \otimes F$$

If  $a, b \in n$  and  $t \in \mathcal{L}(M_n^{\otimes a}, M_n^{\otimes b})$ , the collection of relations between  $v_{ij}$ 's and their adjoints obtained by identifying coefficients in the formula

$$(t \otimes id)v^{\otimes a} = v^{\otimes b}(t \otimes id)$$

can be called “the relation  $t \in Hom(v^{\otimes a}, v^{\otimes b})$ ”. With this definition, let  $J \subset F$  be the two-sided  $*$ -ideal generated by the relations  $m \in Hom(v^{\otimes 2}, v)$ ,  $u \in Hom(1, v)$  and  $e \in End(v)$ , together with the relations obtained by identifying coefficients in

$$vv^* = v^*v = 1$$

The  $*$ -algebra  $F/J$  being generated by the coefficients of a unitary matrix, its enveloping  $\mathbb{C}^*$ -algebra of  $F/J$  is well-defined. We call it  $A_{aut}(B \subset D, \varphi)$ .

By universality we can construct a  $\mathbb{C}^*$ -algebra morphism

$$\Delta : A_{aut}(B \subset D, \varphi) \rightarrow A_{aut}(B \subset D, \varphi) \otimes A_{aut}(B \subset D, \varphi)$$

sending  $v_{ij} \mapsto \sum_k v_{ik} \otimes v_{kj}$  for any  $i$  and  $j$  (the tensor product being the “min” tensor product). We have  $(id \otimes \Delta)v = v_{12}v_{13}$ , so the comultiplication relation

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$$

holds on the generating set  $\{v_{ij}\}_{i,j=1,\dots,n}$ . It follows that the comultiplication relation holds on the whole  $A_{aut}(B \subset D, \varphi)$ . Summing up, we have constructed a pair

$$(A_{aut}(B \subset D, \varphi), v)$$

consisting of a unital Hopf  $\mathbb{C}^*$ -algebra together with a generating corepresentation, i.e. a compact matrix pseudogroup in the sense of Woronowicz ([15]).

The matrix  $v$  is a corepresentation of  $A_{aut}(B \subset D, \varphi)$  on the Hilbert space  $D$ . The three “Hom” conditions translate into the fact that  $v$  corresponds to a coaction of

$A_{aut}(B \subset D, \varphi)$  on the  $\mathbb{C}^*$ -algebra  $D$ , which leaves  $\varphi$  and  $B$  invariant. See Wang ([14]) and [1] for details and comments, in the case  $B = \mathbb{C}$ .

See §3 in [3] for a general construction of such universal Hopf  $\mathbb{C}^*$ -algebras.

We recall from Woronowicz ([16]) that the completion of a tensor  $\mathbb{C}^*$ -category is by definition the smallest semisimple tensor  $\mathbb{C}^*$ -category containing it.

**Theorem 4.** *If  $\varphi$  is a  $(\beta, \omega)$ -form on  $B \subset D$  then the tensor  $\mathbb{C}^*$ -category of finite dimensional corepresentations of  $A_{aut}(B \subset D, \varphi)$  is the completion of  $FC$ .*

*Proof.* The unital Hopf  $\mathbb{C}^*$ -algebra  $A_{aut}(B \subset D, \varphi)$  being presented by the relations corresponding to  $m$ ,  $u$  and  $e$ , its tensor  $\mathbb{C}^*$ -category of corepresentations has to be completion of the tensor  $\mathbb{C}^*$ -category  $\langle m, u, e \rangle$  generated by  $m$ ,  $u$  and  $e$ . (This is a direct consequence of tannakian duality [16], cf. theorem 3.1 in [3].)

On the other hand, the linear form  $\varphi$  being a  $(\beta, \omega)$ -form, theorem 3 applies and gives an isomorphism  $\langle m, u, e \rangle \simeq FC$ .  $\square$

In the case  $B = \mathbb{C}$  and  $\varphi = \text{trace}$ , studied in [1], we have  $FC = TL^2$ .

Note the following corollary of theorem 4: if  $\varphi$  is a  $(\beta, \omega)$ -form then  $\mathbb{N}v = FC$ . (This follows from the basic facts about completion in [16].)

Note also, as an even weaker version of theorem 4, the dimension equalities

$$\dim(\text{Hom}(v^{\otimes m}, v^{\otimes n})) = \dim(FC(m, n))$$

for any  $m$  and  $n$ . Together with standard reconstruction tricks (see e.g. [3]) and with the results in [5], these equalities could be used for classifying the irreducible corepresentations of  $A_{aut}(B \subset D, \varphi)$  and for finding their fusion rules.

## 5. FIXED POINT SUBFACTORS

If  $D$  is a finite dimensional  $\mathbb{C}^*$ -algebra then there exists a unique  $\delta$ -trace on it: the canonical trace  $\tau_D$ . This is by definition the restriction to  $D$  of the unique unital trace of matrices, via the left regular representation. We have  $\delta = \sqrt{\dim(D)}$ . See [1].

In [4] we construct inclusions of fixed point von Neumann algebras of the form

$$(P \otimes B)^K \subset (P \otimes D)^K$$

where  $P$  is a  $II_1$ -factor,  $B \subset D$  is an inclusion of finite dimensional  $\mathbb{C}^*$ -algebras with a trace  $\tau$  and  $K$  is a compact quantum group acting minimally on  $P$  and acting on  $D$  such that  $B$  and  $\tau$  are left invariant. (See [4] for technical details, in terms of unital Hopf  $\mathbb{C}^*$ -algebras.) We also show that if  $K$  acts on  $(C, \tau)$  and acts minimally on a  $II_1$ -factor  $P$  then we have the following implications

$$Z((P \otimes C)^K) = \mathbb{C} \iff Z(C)^K = \mathbb{C} \implies \tau = \tau_C$$

and we deduce from this that we have the following sequence of implications

$$\left\{ \begin{array}{l} (P \otimes B)^K \subset (P \otimes D)^K \\ \text{is a subfactor} \end{array} \right\} \iff \left\{ \begin{array}{l} Z(D)^K = \mathbb{C} \\ Z(B)^K = \mathbb{C} \end{array} \right\} \implies \left\{ \begin{array}{l} \tau = \tau_D \\ \tau|_B = \tau_B \end{array} \right\}$$

which can be glued to the following sequence of elementary implications

$$\left\{ \begin{array}{l} \tau = \tau_D \\ \tau|_B = \tau_B \end{array} \right\} \implies \{ \tau_{D|B} = \tau_B \} \implies \left\{ \text{Ind}(B \subset D) = \frac{\dim(D)}{\dim(B)} \in \mathbb{N} \right\}$$

The following question is raised in [4]: is  $\tau_{D|B} = \tau_B$  the only restriction on  $B \subset D$ ?

**Theorem 5.** *For an inclusion  $B \subset D$  the following are equivalent*

- there exist subfactors of the form  $(P \otimes B)^K \subset (P \otimes D)^K$
- $B \subset D$  commutes with the canonical traces of  $B$  and  $D$ .

*Proof.* The canonical trace  $\tau_D$  is a  $\delta$ -form, with  $\delta = \sqrt{\dim(D)}$ . Its restriction to  $B$  is the canonical trace of  $B$ , so it is a  $\beta$ -form, with  $\beta = \sqrt{\dim(B)}$ . Also  $\tau_D$  being a trace, the projection  $e$  has to be a  $B - B$  bimodule morphism. Thus  $\tau_D$  is a  $(\beta, \omega)$ -form, with

$$\beta = \sqrt{\dim(B)} \qquad \omega = \sqrt{\dim(D)/\dim(B)}$$

Thus theorem 4 applies to  $\tau_D$ . In terms of quantum groups, we get that the fundamental representation  $\pi$  of  $K = G_{\text{aut}}(B \subset D, \tau_D)$  satisfies

$$\dim(\text{Hom}(\pi^{\otimes m}, \pi^{\otimes n})) = \dim(\text{FC}(m, n))$$

for any  $m$  and  $n$ . With  $m = 0$  and  $n = 1$  we get

$$\dim(\text{Hom}(1, \pi)) = \dim(\text{FC}(0, 1)) = 1$$

Together with the canonical isomorphism  $D^K \simeq \text{Hom}(1, \pi)$ , this shows that the action of  $K$  on  $D$  is ergodic. In particular  $Z(D)^K = Z(B)^K = \mathbb{C}$ , so by the above, if  $P$  is a  $II_1$ -factor with minimal action of  $K$  (the existence of such a  $P$  is shown by Ueda in [13]) then  $(P \otimes B)^K \subset (P \otimes D)^K$  is a subfactor and we are done.  $\square$

Note that by [4] the standard invariant of  $P^K \subset (P \otimes D)^K$  is the Popa system associated to  $\pi$ , which by theorem 4 is the Fuss-Catalan Popa system. Equivalently,

$$P^K \subset (P \otimes B)^K \subset (P \otimes D)^K$$

is isomorphic to a free composition of  $A_\infty$  subfactors.

In [2] and [4] standard invariant of subfactors are associated to actions of compact quantum groups on objects of the form  $(\mathbb{C} \subset M_n, \varphi)$  and  $(B \subset D, \tau)$ . This gives evidence for the existence of a general construction, starting with objects of the form  $(B \subset D, \varphi)$ , subject to certain conditions. The results in this paper suggest that there should be only one condition on  $(B \subset D, \varphi)$ , namely “ $\varphi$  is a  $(\beta, \omega)$ -form on  $B \subset D$ ”.

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