A guide to free geometry

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Abstract. This is an introduction to free manifolds, and to analysis on them. The space $\mathbb{R}^N$ has no free analogue, but the unit sphere does have a free analogue, $S^{N-1}_{\mathbb{R},+}$. More generally, we can talk about submanifolds $X \subset S^{N-1}_{\mathbb{R},+}$, which under suitable assumptions have a Laplace operator Δ. We discuss here the basics of free manifolds, then the Laplace equation $\Delta f = 0$ in this setting, and then various free analogues of the main PDE of physics. The mathematics is quite interesting, suggesting the existence of a free electrodynamics theory, conjecturally related to questions in QCD.

Preface

As you surely know, the main question in theoretical physics is that of improving the Standard Model for elementary particles, which dates back to the 1970s. Although there have been many discoveries recently in quantum physics, sometimes accompanied by new engineering feats, the truth remains that our basic knowledge of quantum theory goes back to that old model. And as long as we remain unable to improve that model, our flagship quantum technologies, such as nuclear power, will basically remain stuck.

You probably know too that theoretical physicists are not alone in struggling with this question, because large branches of mathematics are trying to solve this problem too. Indeed, this is certainly true for many people doing PDE or probability, who often get involved, openly, into such questions. As for pure mathematics, that is not as pure as it might seem, because its main architects from the 70s and 80s, such as Atiyah, Connes, Jones and others, were having precisely these Standard Model questions in mind.

The aim of the present book is to present one of the many speculations that can be made, in connection with such questions. Importantly, while not yet really connected to physics, these speculations are quite fresh, going back to the 2010s and early 2020s, and so are a sort of a "start-up" operation, whose potential remains to be determined.

The starting point is the start of quantum mechanics, as we know it from Heisenberg and others. As you zoom down, to the level of protons, electrons and neutrons, things become noncommutative. And this leads to the natural idea that, maybe, if we zoom further down, things might perhaps drastically simplify, and become free.

At the first glance, this might sound like a worthless, wild speculation. However, there is in fact increasing evidence for this. To start with, linguistically at least, it is known since 1973 that quarks are subject to "asymptotic freedom", and whether that famous freedom is the same as mathematical freeness, remains to be determined.

More concretely now, Connes and collaborators have done a lot of work on the Standard Model in their noncommutative geometry formulation, and one of the features of their formalism is that it allows the construction of a "free gauge group" of the Standard Model. Via some standard twisting results, acting on the QED part is S_4^+, and acting on the QCD part is S_9^+. This is quite interesting, suggesting that QED and QCD, suitably twisted, might be some sort of Yang-Mills theories based on S_4^+, S_9^+, respectively.

Another approach, with the theory here going back to work of Yang-Baxter, Faddeev and the Leningrad School, then Drinfeld-Jimbo, and especially Jones and others, is via statistical mechanics and lattice models. Again, this leads to quantum groups, which are traditionally deformed with the help of a parameter $q \in \mathbb{C}$, but which can be as well undeformed, and rather free, depending on which precise model you are looking at.

Yet another approach, and facet of the problem, which is the one that we will describe in this book, is via some sort of "reverse engineering". Indeed, let us temporarily forget about physics. Mathematically then, a free sphere $S^{N-1}_{\mathbb{R},+}$ is not hard to construct, and afterwards you can simply go ahead with mathematics, developed without thinking much: free manifolds, free Laplace operator, free harmonic functions, free PDE. In short, free everything, and the question which appears at the end, coming from free PDE, is whether that new mathematics corresponds to some sort of "free physics", and then, importantly, whether that free physics is true physics, at very small scales, or not.

This sounds quite reasonable, and we will have here a look, at all this. The conjecture at the end will be that there should be a kind of "free electrodynamics" theory, very related to QCD. However, a bit as before with the above-mentioned other approaches, this remains just a facet of the problem. Further advancing, and then putting all the pieces of the puzzle together, remains of course an open problem.

Many thanks go to my colleagues who contributed to the theory discussed here. Thanks as well to my sister Valeria, who's a mathematician like me, but doing hard-line PDE, and I will certainly find a way to talk about her exciting work with Luis, in this book. Finally, many thanks to my PDE colleagues in Cergy, and cats at home, nothing better as work environment, than being surrounded by various apex predators.

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Contents

Part I

Free space

Plain talking
Take us so far
Broken down cars
Like strung out old stars

CHAPTER 1

Free spheres

1a. Free tori

Welcome to freeness. We will be interested in this book in developing free geometry and analysis, with the hope that all this might be related to physics, at very small scales, quarks or below. Before anything, all this is well-known to be complicated business, and technically, it is an open problem. So, we will use a trick, developing first as much free geometry and analysis as we can, hard work done in the dark, a bit like miners working in a mine, and only afterwards, towards the end of the book, we will go to the surface, and look at all this under the light of true physics, see if we have some diamonds or not. With diamond meaning free PDE having an interesting physical meaning.

In short, expect a lot of mathematics, at least to start with, correct as mathematics usually comes, but not necessarily very logical, also as mathematics usually comes.

Helping with writing, however, will be my cat assistant, who knows some physics. Usually cats won't tell, at that level of wisdom you admire this world as it was created, with bigger animals eating smaller animals, evolution and so on. However, I have my own tricks, and although I'm very slow, and with a lame diet by his standards, cat ranks me somewhere higher than dogs and bears, and is sometimes willing to help.

And good news, cat is here, so let's ask him how to get started:

Cat 1.1. *Normally for high speed physics and freeness, you need to be fast and free yourself. But yes, do some math, and start with what you know.*

Thanks cat, I was kind of expecting this, but the advice at the end is really helpful. I was twisting my mind with looking for a free analogue of $\mathbb{R}^N$, for developing afterwards free geometry and analysis inside, sort of a nice program, as any mathematician would do. But, as cat says, let's better relax, and start with what we know.

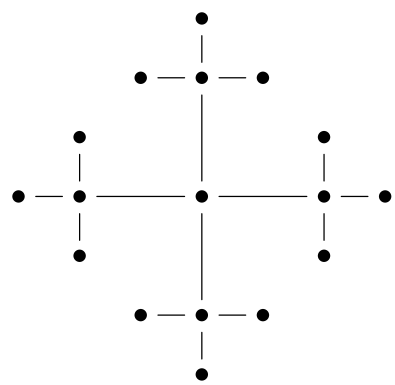
So, what's free? The simplest free object in mathematics is the free group F_N:

Definition 1.2. *The free group F_N is the infinite group*

$$F_N = \left\langle g_1, \ldots, g_N \middle| \emptyset \right\rangle$$

generated by N variables $g_1, \ldots, g_N$, with no relations between them.

This might look a bit abstract, but no worries, F_N has some interesting mathematics, coming right away, if you have some knowledge in discrete groups, and know how to look for interesting questions. For instance if you want to draw the Cayley graph of F_N, whose vertices are the elements of F_N, with edges $h-k$ drawn when $h=g_i^{\pm 1}k$ for some i, you will end up with an interesting picture, which at $N=2$ looks like this:



And this type of graph certainly has interesting mathematics. One good question for instance is that of computing the number of length $2k$ loops based at the root. Another question, which is in fact equivalent, via moments, is that of computing the Kesten measure of F_N, which is that of the following variable in the group algebra of F_N:

$$\chi=g_1+\ldots+g_N$$

All this looks very good, we most likely have here our first object of free geometry, the above graph, regarded as some sort of "manifold", and mathematically speaking, this manifold is as good and interesting as manifolds can get. However, before going ahead with loops and Kesten, let's ask the cat, who's still around. Not that I need help with math, but sometimes a piece of recognition from a fellow physics colleague, for a bright idea like this, can bring pleasure. To my surprise, however, cat answers:

Cat 1.3. *You got it wrong with your math, that graph is not continuous, even by alien standards. It's the dual of F_N which is a free manifold.*

Thanks cat, and interesting remark, indeed. In fact, I was too quick in developing free geometry, and forgot to think at classical geometry first. Here, if there is an interesting formula in relation with free groups and manifolds, this is the following formula, with $\mathbb{T}_N=\mathbb{T}^N$ being the usual torus, and with $\mathbb{Z}^N$ being the free abelian group:

$$\mathbb{T}_N=\widehat{\mathbb{Z}^N}$$

Thus, getting back now to our free group F_N, which is the free analogue of $\mathbb{Z}^N$, it is its dual $\widehat{F_N}$ which is a free manifold, and more specifically the free analogue of $\mathbb{T}^N$. Which is a nice finding, so let us formulate our conclusions as follows:

Definition 1.4. *The free torus* $\mathbb{T}_N^+$ *is the dual of the free group* F_N*,*

$$\mathbb{T}_N^+ = \widehat{F_N}$$

in analogy with the fact that the usual torus $\mathbb{T}_N = \mathbb{T}^N$ *appears as*

$$\mathbb{T}_N = \widehat{\mathbb{Z}^N}$$

with on the right the group $\mathbb{Z}^N$ *being the free abelian group.*

It is of course possible to formulate things more precisely, and we will be back to this in a moment, but before that, isn't this a bit too abstract? But the point here is that no, at the level of questions to be solved, these remain the same, as for instance the computation of the Kesten measure, which is now a "function" on the free torus:

$$\chi \in C(\mathbb{T}_N^+)$$

In fact, this function is the main character of $\mathbb{T}_N^+$, regarded as a compact quantum group, and so our Kesten problem suddenly becomes something very conceptual, namely the computation of the law of the main character of $\mathbb{T}_N^+$. Which is very nice.

Before getting into details regarding all this, recall that $\mathbb{R}^N$ is as interesting as $\mathbb{C}^N$. So, let us formulate as well the real version of Definition 1.4, as follows:

Definition 1.5. *The free real torus, or free cube,* T_N^+ *is the dual*

$$T_N^+ = \widehat{L_N}$$

of the group $L_N = F_N / < g_i^2 = 1 >$*, in analogy with the fact that the usual cube is*

$$T_N = \widehat{\mathbb{Z}_2^N}$$

with on the right the group $\mathbb{Z}_2^N$ *being the free real abelian group.*

Here the "real" at the end stands for the fact that the generators must satisfy the real reflection condition $g^2 = 1$. As for the fact that "real torus = cube", as stated, this needs some thinking, and in the hope that, after such thinking, you will agree with me that there is indeed a standard torus inside $\mathbb{R}^N$, and that is the unit cube.

As before with the free complex torus $\mathbb{T}_N^+$, there is some mathematics to be done with the free real torus T_N^+, for instance in relation with the law of $\chi = g_1 + \ldots + g_N$.

Summarizing, all this sounds good, we have a beginning of free geometry, both real and complex, worth developing, by knowing at least what the torus of each theory is. In practice now, at the level of details, in order to talk about $\mathbb{T}_N^+ = \widehat{F_N}$ and $T_N^+ = \widehat{L_N}$ we need an extension of the usual Pontrjagin duality theory for the abelian groups, and this is best done via operator algebras, and the related notion of compact quantum group.

1b. Quantum spaces

In view of the above, in order to fully understand what happens, let us start with operator algebras. You have probably already heard about infinite matrices, operators and operator algebras, from Heisenberg, Schrödinger, Dirac and others. As a starting point for this, we need a complex Hilbert space H, with the main example in mind being the space $H = L^2(\mathbb{R}^3)$ of the wave functions of the electron. So, let us formulate:

Definition 1.6. *A Hilbert space is a complex vector space H, given with a scalar product $< x, y >$, satisfying the following conditions:*

1. $< x, y >$ *is linear in x, and antilinear in y.*
2. $\overline{< x, y >} =< y, x >$, *for any x, y.*
3. $< x, x >> 0$, *for any $x \neq 0$.*
4. *H is complete with respect to the norm $||x|| = \sqrt{< x, x >}$.*

This looks nice and correct, with the remark that (4) assumes that you know about Cauchy-Schwarz, but thinking well, I'm using here mathematicians' convention for scalar products, linear at left, and aren't we supposed to do as Dirac and other physicists do, with the scalar products linear at right. And making a decision here does not seem to be an easy question, shall we trade the usefulness of Dirac's bras and kets $< x|$ and $|y >$ for mathematical simplicity, I mean what's simple and linear must come first.

I'm afraid I will have to disturb again the cat. And cat says:

Cat 1.7. *Bras and kets are made to interact, and love each other, and that vertical bar is a bad idea, preventing the physics to happen.*

Interesting remark, so if I understand well $< x|y >$ being a bad idea, and I fully agree with this because that vertical bar $|$ slows down computations anyway, we are left with $< x, y >$, and free to choose the linearity as we like. So, Definition 1.6 is correct.

Moving ahead, we need to talk about operators. Again, you might have heard of these from Heisenberg, Schrödinger, Dirac and others, and with the theory being quite complicated to read and digest, because these operators, while fortunately self-adjoint, are unfortunately unbounded. However, cat who's still around, declares:

Cat 1.8. *Self-adjoint and unbounded operators are nice, but not fast enough. For fast physics, you need non-self-adjoint, bounded operators.*

Thanks cat, this sounds good, and again agrees with my mathematical intuition, the bounded operators are the simplest, and who cares about self-adjointness, and I would be even happier not to get into that, I prefer these bounded operators to be arbitrary.

So, bounded operators. These are in fact quite tricky to study, even when taken arbitrary, and after some work, we can formulate, as a first theorem for our book:

THEOREM 1.9. *The linear operators $T : H \to H$ which are bounded, meaning that*

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, having the following properties:

1. *$B(H)$ is complete with respect to $||.||$, so we have a Banach algebra.*
2. *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

In addition, the norm and involution are related by the formula $||TT^*|| = ||T||^2$.

PROOF. The fact that we have an algebra is clear, and the completness comes from the fact that, assuming that $\{T_n\} \subset B(H)$ is Cauchy, then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Regarding $T \to T^*$, this comes from the fact that $\varphi(x) =< Tx, y >$ being a linear form $\varphi : H \to \mathbb{C}$, we must have $\varphi(x) =< x, T^*y >$, for a certain vector $T^*y \in H$. Thus we have a well-defined involution $T \to T^*$, which stays inside $B(H)$, because:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y > \\ &= ||T^*|| \end{aligned}$$

Regarding now the last assertion, observe first that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, so we are done. □

Observe that when H comes with an orthonormal basis $\{e_i\}_{i\in I}$, the linear map $T \to M$ given by $M_{ij} =< Te_j, e_i >$ produces an embedding as follows:

$$B(H) \subset M_I(\mathbb{C})$$

Moreover, in this picture the operation $T \to T^*$ takes a very simple form, namely:

$$(M^*)_{ij} = \overline{M}_{ji}$$

However, with examples like Schrödinger's wave function space $H = L^2(\mathbb{R}^3)$ in mind, it is better in general not to use bases, and accept Theorem 1.9 as stated.

Moving ahead, the conditions found in Theorem 1.9 suggest formulating:

Definition 1.10. *A C^*-algebra is a complex algebra A, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra.*
2. *An involution $a \to a^*$, satisfying $||aa^*|| = ||a||^2$.*

As basic examples, we have $B(H)$ itself, as well as any norm closed $*$-subalgebra $A \subset B(H)$. It is possible to prove that any C^*-algebra appears in this way, but we will not need in what follows this deep result, called GNS theorem after Gelfand, Naimark, Segal. So, let us simply agree that, by definition, the C^*-algebras A are some sort of "generalized operator algebras", and their elements $a \in A$ can be thought of as being some kind of "generalized operators", on some Hilbert space which is not present.

In practice, this vague idea is all that we need. Indeed, by taking some inspiration from linear algebra, we can emulate spectral theory in our setting, as follows:

Theorem 1.11. *Given $a \in A$, define its spectrum as being the set*

$$\sigma(a) = \left\{\lambda \in \mathbb{C}\Big| a - \lambda \notin A^{-1}\right\}$$

and its spectral radius $\rho(a)$ as the radius of the smallest centered disk containing $\sigma(a)$.

1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

Proof. Our first claim is that for any polynomial $f \in \mathbb{C}[X]$, and more generally for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$, we have:

$$\sigma(f(a)) = f(\sigma(a))$$

This indeed something well-known for the usual matrices. In the general case, assume first that we have a polynomial, $f \in \mathbb{C}[X]$. If we pick an arbitrary number $\lambda \in \mathbb{C}$, and write $f(X) - \lambda = c(X - r_1) \ldots (X - r_k)$, we have then, as desired:

$$\begin{aligned}
\lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\
&\iff c(a - r_1) \ldots (a - r_k) \in A^{-1} \\
&\iff a - r_1, \ldots, a - r_k \in A^{-1} \\
&\iff r_1, \ldots, r_k \notin \sigma(a) \\
&\iff \lambda \notin f(\sigma(a))
\end{aligned}$$

Assume now that we are in the general case, $f \in \mathbb{C}(X)$. We pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $F = P - \lambda Q$. By using the above finding, we obtain, as desired:

$$\begin{aligned}\lambda \in \sigma(f(a)) &\iff F(a) \notin A^{-1}\\ &\iff 0 \in \sigma(F(a))\\ &\iff 0 \in F(\sigma(a))\\ &\iff \exists \mu \in \sigma(a), F(\mu) = 0\\ &\iff \lambda \in f(\sigma(a))\end{aligned}$$

Regarding now the assertions in the statement, these basically follows from this:

(1) This comes from the following formula, valid when $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

(2) Assuming $a^* = a^{-1}$, if we denote by D the unit disk, we have, by using (1):

$$||a|| = 1 \implies \sigma(a) \subset D$$

$$||a^{-1}|| = 1 \implies \sigma(a^{-1}) \subset D$$

On the other hand, by using the rational function $f(z) = z^{-1}$, we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Now by putting everything together we obtain, as desired:

$$\sigma(a) \subset D \cap D^{-1} = \mathbb{T}$$

(3) This follows from (2), by using the rational function $f(z) = (z + it)/(z - it)$. Indeed, for $t >> 0$ we have the following computation:

$$\left(\frac{a+it}{a-it}\right)^* = \frac{a-it}{a+it} = \left(\frac{a+it}{a-it}\right)^{-1}$$

Thus the element $f(a)$ is a unitary, and by using (2) its spectrum is contained in $\mathbb{T}$. We conclude from this that we have:

$$f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$$

But this shows that we have $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, as desired.

(4) We already know that we have $\rho(a) \leq ||a||$, for any $a \in A$. For the reverse inequality, when a is normal, we fix a number $\rho > \rho(a)$. We have then:

$$\begin{aligned}
\int_{|z|=\rho} \frac{z^n}{z-a}\,dz &= \int_{|z|=\rho} \sum_{k=0}^{\infty} z^{n-k-1} a^k\,dz \\
&= \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k \\
&= a^{n-1}
\end{aligned}$$

By applying the norm and taking n-th roots we obtain from this formula:

$$\rho \geq \lim_{n\to\infty} ||a^n||^{1/n}$$

When $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of type $n = 2^k$, by using the C^*-algebra condition $||aa^*|| = ||a||^2$, and by taking n-th roots we get, as desired:

$$\rho(a) \geq ||a||$$

In the general normal case now, $aa^* = a^*a$, we have $a^n(a^n)^* = (aa^*)^n$, and by using this, along with the result for self-adjoints, applied to aa^*, we obtain:

$$\begin{aligned}
\rho(a) &\geq \lim_{n\to\infty} ||a^n||^{1/n} \\
&= \sqrt{\lim_{n\to\infty} ||a^n(a^n)^*||^{1/n}} \\
&= \sqrt{\lim_{n\to\infty} ||(aa^*)^n||^{1/n}} \\
&= \sqrt{\rho(aa^*)} \\
&= \sqrt{||a||^2} \\
&= ||a||
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Generally speaking, Theorem 1.11 is all that you need for doing further operator algebras, only military grade weapons there. As a main application, we have:

THEOREM 1.12 (Gelfand). *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a commutative C^*-algebra, with structure as follows:*

(1) *The norm is the usual sup norm, $||f|| = \sup_{x\in X} |f(x)|$.*
(2) *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

Conversely, any commutative C^*-algebra is of the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.

PROOF. Given a commutative C^*-algebra A, we can define indeed X to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras:

$$ev : A \to C(X)$$

We first prove that ev is involutive. We use the following formula:

$$a = \frac{a + a^*}{2} - i \cdot \frac{i(a - a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$. So, claim proved. Also, since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The Gelfand theorem suggests formulating the following definition:

DEFINITION 1.13. *Given a C^*-algebra A, not necessarily commutative, we write*

$$A = C(X)$$

and call the abstract object X a "compact quantum space".

This might look quite revolutionary, but in practice, this definition changes nothing to what we have been doing so far, namely studying the C^*-algebras. So, we will keep studying the C^*-algebras, but by using the above fancy quantum space terminology. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

We will be back to all this later, including with a modification, the idea being that the above definition is in fact quite naive, and needs a fix. More on this later.

Let us discuss now the other basic result regarding the C^*-algebras, namely the GNS representation theorem. We will need some more spectral theory, as follows:

PROPOSITION 1.14. *For a normal element $a \in A$, the following are equivalent:*

(1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
(2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
(3) *$a = cc^*$, for some $c \in A$.*

PROOF. This is something very standard, as follows:

(1) $\implies$ (2) Since our element a is normal the algebra $< a >$ that is generates is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can set $c = b$.

(3) $\implies$ (1) We can proceed here by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$. But this contradicts the elementary fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, which can be checked by explicit inversion. □

Here is now the GNS representation theorem, along with the idea of the proof:

THEOREM 1.15 (GNS theorem). *Let A be a C^*-algebra.*

1. *A appears as a closed $*$-subalgebra $A \subset B(H)$, for some Hilbert space H.*
2. *When A is separable (usually the case), H can be chosen to be separable.*
3. *When A is finite dimensional, H can be chosen to be finite dimensional.*

PROOF. Let us first discuss the commutative case, $A = C(X)$. Our claim here is that if we pick a probability measure on X, we have an embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

Indeed, given a function $f \in C(X)$, consider the operator $T_f(g) = fg$, acting on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 dx} \leq ||f||_\infty ||g||_2$$

Thus, $f \to T_f$ provides us with a C^*-algebra embedding $C(X) \subset B(H)$, as claimed. In general now, assuming that a linear form $\varphi : A \to \mathbb{C}$ has some suitable positivity properties, making it analogous to the integration functionals $\int_X : A \to \mathbb{C}$ from the commutative case, we can define a scalar product on A, by the following formula:

$$< a, b >= \varphi(ab^*)$$

By completing we obtain a Hilbert space H, and we have an embedding as follows:

$$A \subset B(H) \quad , \quad a \to (b \to ab)$$

Thus we obtain the assertion (1), and a careful examination of the construction $A \to H$, outlined above, shows that the assertions (2,3) are in fact proved as well. □

Taking a break now from all this, mathematics endlessly building and self-replicating, once started, like some sort of monster, shall we perhaps think a bit at the physical meaning of all this. I am particularly concerned by the fact that our quantum spaces are compact, if there is one good space for math and physics, that is $\mathbb{R}^N$, which is obviously not compact, so shouldn't be our quantum spaces not compact either.

This does not look like an obvious question, so time to ask the cat. And cat says:

CAT 1.16. *The strong force is confined, expect mathematical freeness to be confined too. As strange as this might sound, linguistically speaking.*

Thanks cat, but this sounds a bit too deep, to the point that I cannot tell if it's a joke or not. In any case, I take it as an encouragement, so we'll go for confinement and compactness, as a continuation of the above, and may the strong force be with us.

So, getting back now to our operator algebra machinery, what's next? Actually, now that we have our definition for the quantum spaces, good time to get back towards Definitions 1.4 and 1.5. In order to understand what that free tori are, we will need:

THEOREM 1.17. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^*=g^{-1}$.*

1. *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
2. *When Γ is abelian, we have an isomorphism $C^*(\Gamma)\simeq C(G)$, where $G=\widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi:\Gamma\to\mathbb{T}$.*

PROOF. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma]\subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H=l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi:\mathbb{C}[\Gamma]\subset B(H)\quad,\quad\pi(g)(h)=gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h)=gh$ that $g\to\pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g\in\Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A=C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A=C(X)$, with:

$$X=Spec(A)$$

But the spectrum $X=Spec(A)$, consisting of the characters $\chi:C^*(\Gamma)\to\mathbb{C}$, can be identified with the Pontrjagin dual $G=\widehat{\Gamma}$, and this gives the result. $\square$

The above result suggests the following definition:

DEFINITION 1.18. *Given a discrete group Γ, the compact quantum space G given by*

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Good news, this definition is exactly what we need, in order to understand the meaning of Definitions 1.4 and 1.5. To be more precise, we have the following result:

THEOREM 1.19. *The basic tori are all group duals, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} \qquad = \qquad \begin{array}{ccc} \widehat{L_N} & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N \end{array}$$

where $F_N = \mathbb{Z}^{*N}$ is the free group on N generators, and $L_N = \mathbb{Z}_2^{*N}$ is its real version.

PROOF. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 1.17 (2), this gives the result. □

Moving ahead, now that we have our formalism, we can start developing free geometry. As a first objective, we would like to better understand the relation between the classical and free tori. In order to discuss this, let us introduce the following notion:

DEFINITION 1.20. *Given a compact quantum space X, its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I = < [a,b] >$$

In this situation, we also say that X appears as a "liberation" of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space.

In relation now with our tori, we have the following result:

THEOREM 1.21. *We have inclusions between the various tori, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and the free tori on top appear as liberations of the tori on the bottom.

PROOF. This is indeed clear from definitions, because commutativity of a group algebra means precisely that the group in question is abelian. □

As a conclusion now to all this, we have a beginning of free geometry, both real and complex, by knowing at least what the torus of each theory is. And with our construction being definitely the good one, for the simple reason that the main problems in the analysis of the free groups correspond in this way the main questions in our free geometry.

1c. Free spheres

In order to extend now the free geometries that we have, real and complex, let us begin with the spheres. Following [11], we have the following notions:

DEFINITION 1.22. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

Here the fact that these algebras are indeed well-defined comes from the following estimate, which shows that the biggest C^*-norms on these $*$-algebras are bounded:

$$||x_i||^2=||x_ix_i^*||\leq\left|\left|\sum_ix_ix_i^*\right|\right|=1$$

As a first result now, regarding the above free spheres, we have:

THEOREM 1.23. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc}S^{N-1}_{\mathbb{R},+}&\longrightarrow&S^{N-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb{R}}&\longrightarrow&S^{N-1}_{\mathbb{C}}\end{array}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{array}{ccc} C(S^{N-1}_{\mathbb{R},+}) & \leftarrow & C(S^{N-1}_{\mathbb{C},+}) \\ \downarrow & & \downarrow \\ C(S^{N-1}_{\mathbb{R}}) & \leftarrow & C(S^{N-1}_{\mathbb{C}}) \end{array}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

It is enough to establish the second isomorphism. So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_{\mathbb{C}}$ satisfy the defining relations for A, we have a quotient map of as follows:

$$A \to C(S^{N-1}_{\mathbb{C}})$$

Conversely, let us write $A = C(S)$, by using the Gelfand theorem. The variables $x_1,\ldots,x_N$ become in this way true coordinates, providing us with an embedding $S \subset \mathbb{C}^N$. Also, the quadratic relations become $\sum_i |x_i|^2 = 1$, so we have $S \subset S^{N-1}_{\mathbb{C}}$. Thus, we have a quotient map $C(S^{N-1}_{\mathbb{C}}) \to A$, as desired, and this gives all the results. □

1d. Algebraic manifolds

By using the free spheres constructed above, we can now formulate:

Definition 1.24. *A real algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X) = C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N) = 0\Big\rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i \in \mathbb{C} < x_1,\ldots,x_N >$$

We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinates $x_1,\ldots,x_N$.

As a basic example here, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_{\mathbb{C}}, S^{N-1}_{\mathbb{R}}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem:

Theorem 1.25. *If $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class} = \left\{ x \in S^{N-1}_{\mathbb{C}} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_{\mathbb{C}}$. Now since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

Definition 1.26. *We agree to identify two real algebraic submanifolds $X, Y \subset S^{N-1}_{\mathbb{C},+}$ when we have a $*$-algebra isomorphism between $*$-algebras of coordinates*

$$f : \mathcal{C}(Y) \to \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

We will see later the reasons for making this convention, coming from amenability. Now back to the tori, as constructed before, we can see that these are examples of algebraic manifolds, in the sense of Definition 1.24. In fact, we have the following result:

Theorem 1.27. *The four main quantum spheres produce the main quantum tori*

$$\begin{matrix} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

via the formula $T = S \cap \mathbb{T}_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb{C},+}$.

Proof. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb{T}_N^+ = S^{N-1}_{\mathbb{C},+} \cap \mathbb{T}_N^+$. But this is something trivial, because we have $\mathbb{T}_N^+ \subset S^{N-1}_{\mathbb{C},+}$.

(2) Free real case. Here the formula in the statement reads $T_N^+ = S^{N-1}_{\mathbb{R},+} \cap \mathbb{T}_N^+$. But this is clear as well, the real version of $\mathbb{T}_N^+$ being T_N^+.

(3) Classical complex case. Here the formula in the statement reads $\mathbb{T}_N = S^{N-1}_{\mathbb{C}} \cap \mathbb{T}_N^+$. But this is clear as well, the classical version of $\mathbb{T}_N^+$ being $\mathbb{T}_N$.

(4) Classical real case. Here the formula in the statement reads $T_N = S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). $\square$

1e. Exercises

Exercises:

EXERCISE 1.28.

EXERCISE 1.29.

EXERCISE 1.30.

EXERCISE 1.31.

EXERCISE 1.32.

EXERCISE 1.33.

Bonus exercise.

CHAPTER 2

Free rotations

2a. Quantum groups

In order to better understand the structure of $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}$, we need to talk about free rotations. Following Woronowicz [99], let us start with:

Definition 2.1. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras as follows,

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to\mathbb{C}\quad,\quad S:A\to A^{opp}$$

called comultiplication, counit and antipode.

Obviously, this is something tricky, and we will see details in a moment, the idea being that these are the axioms which best fit with what we want to do, in this book. Let us also mention, technically, that $\otimes$ in the above can be any topological tensor product, and with the choice of $\otimes$ being irrelevant, but more on this later. Also, A^{opp} is the opposite algebra, with multiplication $a\cdot b=ba$, and more on this later too.

We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. With this convention, we have the following key result, from Woronowicz [99]:

Proposition 2.2. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi)=(g,h)\to\varphi(gh)\quad,\quad\varepsilon(\varphi)=\varphi(1)\quad,\quad S(\varphi)=g\to\varphi(g^{-1})$$

(2) *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g)=g\otimes g\quad,\quad\varepsilon(g)=1\quad,\quad S(g)=g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. This is something very standard, the idea being as follows:

(1) Given $G\subset U_N$, we can set $A=C(G)$, which is a Woronowicz algebra, together with the matrix $u=(u_{ij})$ formed by coordinates of G, given by:

$$g=\begin{pmatrix}u_{11}(g)&\ldots&u_{1N}(g)\\ \vdots&&\vdots\\ u_{N1}(g)&\ldots&u_{NN}(g)\end{pmatrix}$$

Conversely, if (A,u) is a commutative Woronowicz algebra, by using the Gelfand theorem we can write $A=C(X)$, with X being a certain compact space. The coordinates u_{ij} give then an embedding $X\subset M_N(\mathbb C)$, and since the matrix $u=(u_{ij})$ is unitary we actually obtain an embedding $X\subset U_N$, and finally by using the maps Δ,ε,S we conclude that our compact subspace $X\subset U_N$ is in fact a compact Lie group, as desired.

(2) Consider a finitely generated group $F_N\to\Gamma$. We can set $A=C^*(\Gamma)$, which is by definition the completion of the complex group algebra $\mathbb C[\Gamma]$, with involution given by $g^*=g^{-1}$, for any $g\in\Gamma$, with respect to the biggest C^*-norm, and we obtain a Woronowicz algebra, together with the diagonal matrix formed by the generators of Γ:

$$u=\begin{pmatrix}g_1&&0\\&\ddots&\\0&&g_N\end{pmatrix}$$

Conversely, if (A,u) is a cocommutative Woronowicz algebra, the Peter-Weyl theory of Woronowicz, to be explained below, shows that the irreducible corepresentations of A are all 1-dimensional, and form a group Γ, and so we have $A=C^*(\Gamma)$, as desired. Thus, theorem proved, modulo a representation theory discussion, to come soon. □

In general now, the structural maps Δ,ε,S have the following properties:

Proposition 2.3. *Let (A,u) be a Woronowicz algebra.*

(1) *Δ,ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$(\Delta\otimes id)\Delta = (id\otimes\Delta)\Delta$$
$$(\varepsilon\otimes id)\Delta = (id\otimes\varepsilon)\Delta=id$$

(2) *S satisfies the antipode axiom, on the $*$-subalgebra generated by entries of u:*

$$m(S\otimes id)\Delta=m(id\otimes S)\Delta=\varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2=id$.*

Proof. Observe first that the result holds in the case where A is commutative. Indeed, by using Proposition 2.2 (1) we can write:

$$\Delta=m^t\quad,\quad\varepsilon=u^t\quad,\quad S=i^t$$

The 3 conditions in the statement come then by transposition from the basic 3 group theory conditions satisfied by m,u,i, which are as follows, with $\delta(g)=(g,g)$:

$$m(m\times id)=m(id\times m)$$
$$m(id\times u)=m(u\times id)=id$$
$$m(id\times i)\delta=m(i\times id)\delta=1$$

Observe also that the result holds as well in the case where A is cocommutative, by using Proposition 2.2 (1). In the general case now, the proof goes as follows:

(1) We have the following computation:

$$(\Delta\otimes id)\Delta(u_{ij})=\sum_l\Delta(u_{il})\otimes u_{lj}=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

We have as well the following computation, which gives the first formula:

$$(id\otimes\Delta)\Delta(u_{ij})=\sum_k u_{ik}\otimes\Delta(u_{kj})=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

On the other hand, we have the following computation:

$$(id\otimes\varepsilon)\Delta(u_{ij})=\sum_k u_{ik}\otimes\varepsilon(u_{kj})=u_{ij}$$

We have as well the following computation, which gives the second formula:

$$(\varepsilon\otimes id)\Delta(u_{ij})=\sum_k\varepsilon(u_{ik})\otimes u_{kj}=u_{ij}$$

(2) By using the fact that the matrix $u=(u_{ij})$ is unitary, we obtain:

$$\begin{aligned}m(id\otimes S)\Delta(u_{ij})&=\sum_k u_{ik}S(u_{kj})\\&=\sum_k u_{ik}u_{jk}^*\\&=(uu^*)_{ij}\\&=\delta_{ij}\end{aligned}$$

We have as well the following computation, which gives the result:

$$\begin{aligned}m(S\otimes id)\Delta(u_{ij})&=\sum_k S(u_{ik})u_{kj}\\&=\sum_k u_{ki}^*u_{kj}\\&=(u^*u)_{ij}\\&=\delta_{ij}\end{aligned}$$

(3) Finally, the formula $S^2=id$ holds as well on generators, and so in general too. $\square$

Let us record as well the following technical result:

PROPOSITION 2.4. *Given a Woronowicz algebra (A,u), we have $u^t=\bar{u}^{-1}$, so u is biunitary, in the sense that it is unitary, with unitary transpose.*

PROOF. We have the following computation, based on the fact that u is unitary:

$$\begin{aligned}(uu^*)_{ij}=\delta_{ij}\quad&\implies\quad\sum_kS(u_{ik}u_{jk}^*)=\delta_{ij}\\&\implies\quad\sum_ku_{kj}u_{ki}^*=\delta_{ij}\\&\implies\quad(u^t\bar{u})_{ji}=\delta_{ij}\end{aligned}$$

Similarly, we have the following computation, once agan using the unitarity of u:

$$\begin{aligned}(u^*u)_{ij}=\delta_{ij}\quad&\implies\quad\sum_kS(u_{ki}^*u_{kj})=\delta_{ij}\\&\implies\quad\sum_ku_{jk}^*u_{ik}=\delta_{ij}\\&\implies\quad(\bar{u}u^t)_{ji}=\delta_{ij}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, the Woronowicz algebras appear to have nice properties. In view of Proposition 2.2 and Proposition 2.3, we can formulate the following definition:

DEFINITION 2.5. *Given a Woronowicz algebra A, we formally write*

$$A=C(G)=C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is commutative and cocommutative, G and Γ are usual abelian groups, dual to each other. In general, we still agree to write $G=\widehat{\Gamma},\Gamma=\widehat{G}$, but in a formal sense. As a final piece of general theory now, let us complement Definition 2.1 with:

DEFINITION 2.6. *Given two Woronowicz algebras (A,u) and (B,v), we write*

$$A\simeq B$$

and identify the corresponding quantum groups, when we have an isomorphism

$$<u_{ij}>\simeq<v_{ij}>$$

of $*$-algebras, mapping standard coordinates to standard coordinates.

With this convention, which is in tune with our conventions for algebraic manifolds from chapter 1, and more on this later, any compact or discrete quantum group corresponds to a unique Woronowicz algebra, up to equivalence. Also, we can see now why in

Definition 2.1 the choice of the exact topological tensor product $\otimes$ is irrelevant. Indeed, no matter what tensor product $\otimes$ we use there, we end up with the same Woronowicz algebra, and the same compact and discrete quantum groups, up to equivalence.

In practice, we will use in what follows the simplest such tensor product $\otimes$, which is the maximal one, obtained as completion of the usual algebraic tensor product with respect to the biggest C^*-norm. With the remark that this product is something rather abstract, and so can be treated, in practice, as a usual algebraic tensor product.

Moving ahead now, let us call corepresentation of A any unitary matrix $v\in M_n(\mathcal{A})$, where $\mathcal{A}=<u_{ij}>$, satisfying the same conditions are those satisfied by u, namely:

$$\Delta(v_{ij})=\sum_k v_{ik}\otimes v_{kj}\quad,\quad\varepsilon(v_{ij})=\delta_{ij}\quad,\quad S(v_{ij})=v_{ji}^*$$

These corepresentations can be then thought of as corresponding to the finite dimensional unitary smooth representations of the underlying compact quantum group G. Following Woronowicz [99], we have the following key result:

THEOREM 2.7. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a=(\tau\otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v)=\left\{\xi\in\mathbb{C}^n\Big|v\xi=\xi\right\}$$

The point now is that when $\varphi\in A^*$ is faithful, by using a standard positivity trick, one can prove that we have $P_\varphi=P$. Assume indeed $P_\varphi\xi=\xi$, and let us set:

$$a=\sum_i\left(\sum_j v_{ij}\xi_j-\xi_i\right)\left(\sum_k v_{ik}\xi_k-\xi_i\right)^*$$

We must prove that we have $a=0$. Since v is biunitary, we have:

$$\begin{aligned}
a &= \sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar{\xi}_k-\frac{1}{N}\bar{\xi}_i\right)\right)\\
&= \sum_{ijk}v_{ij}v_{ik}^*\xi_j\bar{\xi}_k-\frac{1}{N}v_{ij}\xi_j\bar{\xi}_i-\frac{1}{N}v_{ik}^*\xi_i\bar{\xi}_k+\frac{1}{N^2}\xi_i\bar{\xi}_i\\
&= \sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar{\xi}_i-\sum_{ik}v_{ik}^*\xi_i\bar{\xi}_k+\sum_i|\xi_i|^2\\
&= ||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&= 2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $P_\varphi\xi=\xi$, we obtain from this:

$$\begin{aligned}
\varphi(a) &= 2\varphi(||\xi||^2-Re(<v\xi,\xi>))\\
&= 2(||\xi||^2-Re(<P_\varphi\xi,\xi>))\\
&= 2(||\xi||^2-||\xi||^2)\\
&= 0
\end{aligned}$$

Now since φ is faithful, this gives $a=0$, and so $v\xi=\xi$. Thus $\int_\varphi$ is independent of φ, and is given on coefficients $a=(\tau\otimes id)v$ by the following formula:

$$\left(id\otimes\int_\varphi\right)v=P$$

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

Consider the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

We have then the following result, also due to Woronowicz [99]:

THEOREM 2.8. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. *$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [99], the idea being as follows:

(1) Given a corepresentation $v \in M_n(A)$, consider its interwiner algebra:

$$End(v) = \left\{ T \in M_n(\mathbb{C}) \Big| Tv = vT \right\}$$

It is elementary to see that this is a finite dimensional C^*-algebra, and we conclude from this that we have a decomposition as follows:

$$End(v) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1 = p_1 + \ldots + p_k$$

But this decomposition allows us to define subcorepresentations $v_i \subset v$, which are irreducible, so we obtain, as desired, a decomposition $v = v_1 + \ldots + v_k$.

(2) To any corepresentation $v \in M_n(A)$ we associate its space of coefficients, given by $C(v) = span(v_{ij})$. The construction $v \to C(v)$ is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

Now given an arbitrary corepresentation $v \in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v) \subset \mathcal{A}$, and so we must have, for certain positive integers $k_1, \ldots, k_p$, an inclusion of vector spaces, as follows:

$$C(v) \subset C(u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v \subset u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}$$

Thus, by using (1), we are led to the conclusion in the statement.

(3) By using (1) and (2), we obtain a linear space decomposition as follows:

$$\mathcal{A}=\sum_{v\in Irr(A)}C(v)=\sum_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$$

In order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C(v)\perp C(w)$$

As a first observation, which follows from an elementary computation, for any two corepresentations v,w we have a Frobenius type isomorphism, as follows:

$$Hom(v,w)\simeq Fix(\bar{v}\otimes w)$$

Now let us set $P_{ia,jb}=\int_G v_{ij}w_{ab}^*$. According to Theorem 2.7, the matrix P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(\bar{v},\bar{w})=\{0\}$$

Thus we have $P=0$, and so $C(v)\perp C(w)$, which gives the result.

(4) The algebra $\mathcal{A}_{central}$ contains indeed all the characters, because we have:

$$\Sigma\Delta(\chi_v)=\sum_{ij}v_{ji}\otimes v_{ij}=\Delta(\chi_v)$$

The fact that the characters span $\mathcal{A}_{central}$, and form an orthogonal basis of it, follow from (3). Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl}=\int_G v_{ij}v_{kl}^*$$

We know from Theorem 2.7 that these integrals form the orthogonal projection onto $Fix(v\otimes\bar{v})\simeq End(\bar{v})=\mathbb{C}1$. By using this fact, we obtain the following formula:

$$\int_G\chi_v\chi_v^*=\sum_{ij}\int_G v_{ii}v_{jj}^*=\sum_i\frac{1}{N}=1$$

Thus the characters have indeed norm 1, and we are done. □

We refer to Woronowicz [99] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A=C^*(\Gamma)$ for some discrete group Γ, as mentioned in Proposition 2.2.

At a more technical level now, we have a number of more advanced results, from Woronowicz [99], [100] and other papers, that must be known as well. We will present them quickly, and for details you check my book [7]. First we have:

THEOREM 2.9. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some standard functional analysis. □

Yet another important result is Tannakian duality, as follows:

THEOREM 2.10. *The following operations are inverse to each other:*

1. *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
2. *The construction $C \to A$, which associates to a tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. This is something quite deep, the idea being as follows:

(1) We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

(2) We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, after some elementary algebra we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [**100**]. □

2b. Free rotations

Good news, with the above general theory in hand, we can go back now to our free geometry program, as developed in chapter 1, and substantially build on that. Indeed, the point is that we can talk now about free rotations. Following Wang [**89**], we have:

THEOREM 2.11. *The following constructions produce compact quantum groups,*

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)\\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)\end{aligned}$$

which appear respectively as liberations of the groups O_N and U_N.

PROOF. This first assertion follows from the elementary fact that if a matrix $u=(u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Indeed, the biunitarity of u^Δ can be checked by a direct computation. Regarding now the matrix $u^\varepsilon=1_N$, this is clearly biunitary. Also, regarding the matrix u^S, there is nothing to prove here either, because its unitarity its clear too. And finally, observe that if u has self-adjoint entries, then so do the above matrices $u^\Delta,u^\varepsilon,u^S$.

Thus our claim is proved, and we can define morphisms Δ,ε,S as in Definition 2.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, this follows exactly as for the free spheres, by adapting the sphere proof from chapter 1. □

The basic properties of O_N^+,U_N^+ can be summarized as follows:

THEOREM 2.12. *The quantum groups O_N^+,U_N^+ have the following properties:*

1. *The closed subgroups $G\subset U_N^+$ are exactly the $N\times N$ compact quantum groups. As for the closed subgroups $G\subset O_N^+$, these are those satisfying $u=\bar{u}$.*
2. *We have liberation embeddings $O_N\subset O_N^+$ and $U_N\subset U_N^+$, obtained by dividing the algebras $C(O_N^+),C(U_N^+)$ by their respective commutator ideals.*
3. *We have as well embeddings $\widehat{L}_N\subset O_N^+$ and $\widehat{F}_N\subset U_N^+$, where L_N is the free product of N copies of $\mathbb{Z}_2$, and where F_N is the free group on N generators.*

PROOF. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 2.1, the formula $S(u_{ij})=u_{ji}^*$ shows that the matrix $\bar{u}$ must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N),C(U_N)$, similar to those in Theorem 2.11, but with the commutativity between the standard coordinates and their adjoints added:

$$\begin{aligned} C(O_N) &= C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)\\ C(U_N) &= C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)\end{aligned}$$

Thus, we are led to the conclusion in the statement.

(3) This follows indeed from (1) and from Proposition 2.2, with the remark that with $u = diag(g_1, \ldots, g_N)$, the condition $u = \bar{u}$ is equivalent to $g_i^2 = 1$, for any i. □

The last assertion in Theorem 2.12 suggests the following construction:

PROPOSITION 2.13. *Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

PROOF. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

With this notion in hand, Theorem 2.12 (3) reformulates as follows:

THEOREM 2.14. *The diagonal tori of the basic unitary groups are the basic tori:*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array} \quad \rightarrow \quad \begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

In particular, the basic unitary groups are all distinct.

PROOF. This is something clear and well-known in the classical case, and in the free case, this is a reformulation of Theorem 2.12 (3), which tells us that the diagonal tori of O_N^+, U_N^+, in the sense of Proposition 2.13, are the group duals $\widehat{L}_N, \widehat{F}_N$. □

There is an obvious relation here with the considerations from chapter 1, that we will analyse later on. As a second result now regarding our free quantum groups, relating them this time to the free spheres constructed in chapter 1, we have:

THEOREM 2.15. *We have embeddings of algebraic manifolds as follows, obtained in double indices by rescaling the coordinates, $x_{ij} = u_{ij}/\sqrt{N}$:*

$$\begin{matrix} O_N^+ & \to & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \to & U_N \end{matrix} \quad \to \quad \begin{matrix} S^{N^2-1}_{\mathbb{R},+} & \to & S^{N^2-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N^2-1}_{\mathbb{R}} & \to & S^{N^2-1}_{\mathbb{C}} \end{matrix}$$

Moreover, the quantum groups appear from the quantum spheres via

$$G = S \cap U_N^+$$

with the intersection being computed inside the free sphere $S^{N^2-1}_{\mathbb{C},+}$.

PROOF. As explained in Theorem 2.12, the biunitarity of the matrix $u = (u_{ij})$ gives an embedding of algebraic manifolds, as follows:

$$U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$$

Now since the relations defining $O_N, O_N^+, U_N \subset U_N^+$ are the same as those defining $S^{N^2-1}_{\mathbb{R}}, S^{N^2-1}_{\mathbb{R},+}, S^{N^2-1}_{\mathbb{C}} \subset S^{N^2-1}_{\mathbb{C},+}$, this gives the result. □

Summarizing, we have now up and working some free rotation groups, which are closely related to the free spheres and tori constructed in chapter 1.

2c. Quantum isometries

In order to further discuss now the relation with the spheres, which can only come via some sort of "isometric actions", let us start with the following standard fact:

PROPOSITION 2.16. *Given a closed subset $X \subset S^{N-1}_{\mathbb{C}}$, the formula*

$$G(X) = \left\{ U \in U_N \Big| U(X) = X \right\}$$

defines a compact group of unitary matrices, or isometries, called affine isometry group of X. For the spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain in this way the groups O_N, U_N.

PROOF. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X \subset \mathbb{C}^N$, but we are not interested here in such general spaces. □

Observe that in the case of the real and complex spheres, the affine isometry group $G(X)$ leaves invariant the Riemannian metric, because this metric is equivalent to the one inherited from $\mathbb{C}^N$, which is preserved by our isometries $U \in U_N$.

Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X, with of course some extra care in relation with the complex structure, as for the complex sphere $X = S^{N-1}_\mathbb{C}$ to produce $G(X) = U_N$ instead of $G(X) = O_{2N}$. But, such things won't really work for the free spheres, and so are to be avoided.

The point now is that we have the following quantum analogue of Proposition 2.16, which is a perfect analogue, save for the fact that X is now assumed to be algebraic, for some technical reasons, which allows us to talk about quantum isometry groups:

THEOREM 2.17. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

defines a morphism of C^*-algebras $\Phi : C(X) \to C(X) \otimes C(G)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

PROOF. Assume indeed that our manifold $X \subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

In order to prove the result, consider the following variables:

$$X_i = \sum_j x_j \otimes u_{ji} \in C(X) \otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G = G^+(X)$ appears as:

$$C(G) = C(U_N^+) \Big/ \Big\langle f_\alpha(X_1, \ldots, X_N) = 0 \Big\rangle$$

In order to prove this, pick one of the defining polynomials, and write it as follows:

$$f_\alpha(x_1, \ldots, x_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \cdot x_{i_1^r} \ldots x_{i_{s_r}^r}$$

With $X_i = \sum_j x_j \otimes u_{ji}$ as above, we have the following formula:

$$f_\alpha(X_1, \ldots, X_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \sum_{j_1^r \ldots j_{s_r}^r} x_{j_1^r} \ldots x_{j_{s_r}^r} \otimes u_{j_1^r i_1^r} \ldots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f_\alpha(X_1, \ldots, X_N) = 0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a closed subspace $G \subset U_N^+$, with a universal map, as follows:

$$\Phi : C(X) \to C(X) \otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider as well the following elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f_\alpha(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f_\alpha(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f_\alpha(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. □

We can now formulate a result about spheres and rotations, as follows:

THEOREM 2.18. *The quantum isometry groups of the basic spheres are*

$$\begin{matrix}
S^{N-1}_{\mathbb{R},+}&\longrightarrow&S^{N-1}_{\mathbb{C},+}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb{R}}&\longrightarrow&S^{N-1}_{\mathbb{C}}
\end{matrix}
\qquad\to\qquad
\begin{matrix}
O_N^+&\longrightarrow&U_N^+\\
\uparrow&&\uparrow\\
O_N&\longrightarrow&U_N
\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

PROOF. We have 4 results to be proved, the idea being as follows:

$\underline{S^{N-1}_{\mathbb{C},+}}$. Let us first construct an action $U_N^+\curvearrowright S^{N-1}_{\mathbb{C},+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

By using the biunitarity of u, we have the following computation:

$$\sum_i X_iX_i^*=\sum_{ijk}x_jx_k^*\otimes u_{ji}u_{ki}^*=\sum_j x_jx_j^*\otimes 1=1\otimes 1$$

Once again by using the biunitarity of u, we have as well:

$$\sum_i X_i^*X_i=\sum_{ijk}x_j^*x_k\otimes u_{ji}^*u_{ki}=\sum_j x_j^*x_j\otimes 1=1\otimes 1$$

Thus we have an action $U_N^+\curvearrowright S^{N-1}_{\mathbb{C},+}$, which gives $G^+(S^{N-1}_{\mathbb{C},+})=U_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R},+}}$. Let us first construct an action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$. We already know that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, so we just have to check that these variables are self-adjoint. But this is clear from $u=\bar{u}$, as follows:

$$X_i^*=\sum_j x_j^*\otimes u_{ji}^*=\sum_j x_j\otimes u_{ji}=X_i$$

Conversely, assume that we have an action $G\curvearrowright S^{N-1}_{\mathbb{R},+}$, with $G\subset U_N^+$. The variables $X_i=\sum_j x_j\otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u=\bar{u}$. Thus our quantum group must satisfy $G\subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{C}}}$. The fact that we have an action $U_N\curvearrowright S^{N-1}_{\mathbb{C}}$ is clear. Conversely, assume that we have an action $G\curvearrowright S^{N-1}_{\mathbb{C}}$, with $G\subset U_N^+$. We must prove that this implies $G\subset U_N$, and we will use a standard trick of Bhowmick-Goswami [11]. We have:

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_ix_k)=\sum_{jl}x_jx_l\otimes u_{ji}u_{lk}$$

$$\Phi(x_kx_i)=\sum_{jl}x_lx_j\otimes u_{lk}u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_ix_k)=\sum_{j<l}x_jx_l\otimes(u_{ji}u_{lk}+u_{li}u_{jk})+\sum_j x_j^2\otimes u_{ji}u_{jk}$$

$$\Phi(x_ix_k)=\sum_{j<l}x_jx_l\otimes(u_{lk}u_{ji}+u_{jk}u_{li})+\sum_j x_j^2\otimes u_{jk}u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji}u_{lk}+u_{li}u_{jk}=u_{lk}u_{ji}+u_{jk}u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^*u_{ij}^*+u_{kj}^*u_{il}^*=u_{ij}^*u_{kl}^*+u_{il}^*u_{kj}^*$$

$$u_{ij}u_{kl}+u_{il}u_{kj}=u_{kl}u_{ij}+u_{kj}u_{il}$$

$$u_{ji}u_{lk}+u_{jk}u_{li}=u_{lk}u_{ji}+u_{li}u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li}u_{jk}=u_{jk}u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_ix_k^*)=\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*$$

$$\Phi(x_k^*x_i)=\sum_{jl}x_l^*x_j\otimes u_{lk}^*u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*=\sum_{jl}x_jx_l^*\otimes u_{lk}^*u_{ji}$$

Thus we have $u_{ji}u_{lk}^*=u_{lk}^*u_{ji}$, and so $G\subset U_N$, as claimed.

$\underline{S^{N-1}_\mathbb{R}}$. The fact that we have an action $O_N\curvearrowright S^{N-1}_\mathbb{R}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned}G\curvearrowright S^{N-1}_\mathbb{R}\quad&\implies\quad G\curvearrowright S^{N-1}_{\mathbb{R},+},S^{N-1}_\mathbb{C}\\&\implies\quad G\subset O_N^+,U_N\\&\implies\quad G\subset O_N^+\cap U_N=O_N\end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_\mathbb{R})=O_N$, as desired. □

2d. Haar integration

Let us discuss now the correspondence $U\to S$. In the classical case the situation is very simple, because the sphere $S=S^{N-1}$ appears by rotating the point $x=(1,0,\ldots,0)$ by the isometries in $U=U_N$. Moreover, the stabilizer of this action is the subgroup $U_{N-1}\subset U_N$ acting on the last $N-1$ coordinates, and so the sphere $S=S^{N-1}$ appears from the corresponding rotation group $U=U_N$ as an homogeneous space, as follows:

$$S^{N-1}=U_N/U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U\to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1})\subset C(U_N)\quad,\quad x_i\to u_{1i}$$

In general now, the straightforward homogeneous space interpretation of S as above fails. However, we can have some theory going by using the functional analytic viewpoint, with an embedding $x_i\to u_{1i}$ as above. Let us start with the following result:

Theorem 2.19. *For the basic spheres, we have a diagram as follows,*

$$\begin{array}{ccc}
C(S) & \xrightarrow{\ \ \Phi\ \ } & C(S)\otimes C(U)\\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha\otimes id}\\
C(U) & \xrightarrow{\ \ \Delta\ \ } & C(U)\otimes C(U)
\end{array}$$

where on top $\Phi(x_i)=\sum_j x_j\otimes u_{ji}$*, and on the left* $\alpha(x_i)=u_{1i}$*.*

Proof. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows, on these coordinates:

$$\begin{array}{ccc}
x_i & \longrightarrow & \sum_j x_j\otimes u_{ji}\\
\downarrow & & \downarrow\\
u_{1i} & \longrightarrow & \sum_j u_{1j}\otimes u_{ji}
\end{array}$$

Thus by linearity and multiplicativity, the whole the diagram commutes. $\square$

The point now is that, by further building on the above result, we obtain the desired correspondence $U\to S$, and some useful integration results as well.

At the level of the fine structure of the free spheres $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}$ now, we have some obvious formal eigenspaces for the Laplace operator, and a Weingarten integration formula as well, both coming from the representation theory of O_N^+,U_N^+. Moreover, it is possible to get beyond this, with a full construction of a Laplace operator.

Regarding other possible invariants, orientability does not work, the Dirac operator does not exist, smoothness does not work either, and in what regards K-theory, with our free objects we are a bit too far away from the traditional "reasonable" range of K-theory, usually requiring amenability, or at least some form of K-amenability.

However, after some thinking, maybe including some physical thoughts too, in connection with what is smoothness and is that wished or not, in the present situation, all this is normal. So, no worries, and as we will soon discover, we will get away with the tools that we have, namely Laplace operator and the Weingarten formula, which are not that bad, technically speaking, for all the problems that we will choose to solve.

2e. Exercises

Exercises:

EXERCISE 2.20.

EXERCISE 2.21.

EXERCISE 2.22.

EXERCISE 2.23.

EXERCISE 2.24.

EXERCISE 2.25.

Bonus exercise.

CHAPTER 3

Fine structure

3a. Diagrams, easiness

We have so far a beginning of free geometry, in the real case with a triple of basic objects $(S^{N-1}_{\mathbb{R},+}, O_N^+, T_N^+)$, and in the complex case with objects $(S^{N-1}_{\mathbb{C},+}, U_N^+, \mathbb{T}_N^+)$. This is not bad, and our purpose in what follows will be that of expanding these two collections of objects, from 3 items each, to 10, 100, 1000, or as many as we can, and the more the merrier, in the name of pure mathematics, where new objects are always welcome.

This being said, what to start with? Leaving aside the tori, which are just duals of discrete groups, and as old as modern mathematics, we face a choice between spheres S, and rotation groups U. As a first observation, these two types of objects are closely related, because in the classical case, given a sphere S, we can recover U as being its isometry group, and conversely, given a group U, we can recover S just by rotating a point. And, as seen in chapter 2, the situation is quite similar in the free case.

This being said, spheres S are not the same thing as rotation groups U, and we will have to make a choice. Normally spheres S look a bit more important, but on the other hand physics, or even mathematics, tell us that no matter what we want to do, of advanced type, about either S or U, we will always end up in struggling with U.

So, we will go for U, and our goal in this chapter will be that of better understanding O_N^+, U_N^+, and also look for more free quantum groups, as many as we can find. And regarding spheres S and other such manifolds, we will leave this for later. Sounds good, doesn't it? Before getting into this, however, let us check with physics and cat:

Cat 3.1. *Gauge invariance gives you everything. But don't forget to do some manifolds too, all our kittens learn that, and it's good learning.*

Thanks cat, this is a pleasure to hear, and in tune with my mathematical intuition. Getting started now, we would like to have a better understanding of the liberation operations that we have, $O_N \to O_N^+$ and $U_N \to U_N^+$, and also have more examples of liberation operations of the same type, $G_N \to G_N^+$. And then, once we will have enough theory and examples, look for classification results for the free quantum groups $\{G_N^+\}$.

Let us start with the construction of more examples, which is certainly a very exciting business, and leave the abstractions for later. Following Wang [89], we first have:

PROPOSITION 3.2. *Consider the symmetric group S_N, viewed as permutation group of the N coordinate axes of $\mathbb{R}^N$. The coordinate functions on $S_N \subset O_N$ are given by*

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

and the matrix $u = (u_{ij})$ that these functions form is magic, in the sense that its entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column.

PROOF. The action of S_N on the standard basis $e_1, \ldots, e_N \in \mathbb{R}^N$ being given by $\sigma : e_j \to e_{\sigma(j)}$, this gives the formula of u_{ij} in the statement. As for the fact that the matrix $u = (u_{ij})$ that these functions form is magic, this is clear. □

With a bit more effort, we obtain the following nice characterization of S_N:

PROPOSITION 3.3. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

PROOF. The universal algebra A in the statement being commutative, by the Gelfand theorem it must be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates form a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 entry on each row and each column, and so $X = S_N$. Thus we have proved the first assertion, and the second assertion is clear as well. □

Still following Wang [89], we can now liberate S_N, as follows:

THEOREM 3.4. *The following universal C^*-algebra, with magic meaning as usual formed by projections ($p^2 = p^* = p$), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = \text{magic}\right)$$

is a Woronowicz algebra, with comultiplication, counit and antipode given by:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

Thus the space S_N^+ is a compact quantum group, called quantum permutation group.

PROOF. As a first observation, the universal C^*-algebra in the statement is indeed well-defined, because the conditions $p^2 = p^* = p$ satisfied by the coordinates give:

$$||u_{ij}|| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps Δ, ε, S given by the formulae in the statement. Consider the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}$$

Our claim is that, since u is magic, so are these three matrices. Indeed, regarding u^Δ, its entries are idempotents, as shown by the following computation:

$$(u_{ij}^\Delta)^2 = \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj} = \sum_{kl} \delta_{kl}u_{ik} \otimes \delta_{kl}u_{kj} = u_{ij}^\Delta$$

These elements are self-adjoint as well, as shown by the following computation:

$$(u_{ij}^\Delta)^* = \sum_k u_{ik}^* \otimes u_{kj}^* = \sum_k u_{ik} \otimes u_{kj} = u_{ij}^\Delta$$

The row and column sums for the matrix u^Δ can be computed as follows:

$$\sum_j u_{ij}^\Delta = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1$$

$$\sum_i u_{ij}^\Delta = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1$$

Thus, u^Δ is magic. Regarding now u^ε, u^S, these matrices are magic too, and this for obvious reasons. Thus, all our three matrices $u^\Delta, u^\varepsilon, u^S$ are magic, so we can define Δ, ε, S by the formulae in the statement, by using the universality property of $C(S_N^+)$. ☐

Our first task now is to make sure that Theorem 3.4 produces indeed a new quantum group, which does not collapse to S_N. Still following Wang [89], we have:

THEOREM 3.5. *We have an embedding $S_N \subset S_N^+$, given at the algebra level by:*

$$u_{ij} \to \chi\left(\sigma \in S_N \Big| \sigma(j) = i\right)$$

This is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. The fact that we have indeed an embedding as above follows from Proposition 3.3. Observe that in fact more is true, because our results above give:

$$C(S_N) = C(S_N^+) \Big/ \Big\langle ab = ba \Big\rangle$$

Thus, the inclusion $S_N \subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N=2$. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+=S_2$.

Case $N=3$. By using the same argument as in the $N=2$ case, and the symmetries of the problem, it is enough to check that u_{11},u_{22} commute. But this follows from:

$$\begin{aligned}u_{11}u_{22}&=u_{11}u_{22}(u_{11}+u_{12}+u_{13})\\&=u_{11}u_{22}u_{11}+u_{11}u_{22}u_{13}\\&=u_{11}u_{22}u_{11}+u_{11}(1-u_{21}-u_{23})u_{13}\\&=u_{11}u_{22}u_{11}\end{aligned}$$

Indeed, by applying the involution to this formula, we obtain that we have as well $u_{22}u_{11}=u_{11}u_{22}u_{11}$. Thus, we obtain $u_{11}u_{22}=u_{22}u_{11}$, as desired.

Case $N=4$. Consider the following matrix, with p,q being projections:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and we can choose $p,q\in B(H)$ as for the algebra $<p,q>$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N\geq 5$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well. □

With the above results in hand, we can introduce as well quantum reflections:

Theorem 3.6. *The following constructions produce compact quantum groups,*

$$\begin{aligned}C(H_N^+)&=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u_{ij}=u_{ij}^*,\ (u_{ij}^2)=\text{magic}\right)\\C(K_N^+)&=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|[u_{ij},u_{ij}^*]=0,\ (u_{ij}u_{ij}^*)=\text{magic}\right)\end{aligned}$$

which appear as liberations of the reflection groups $H_N=\mathbb{Z}_2\wr S_N$ and $K_N=\mathbb{T}\wr S_N$.

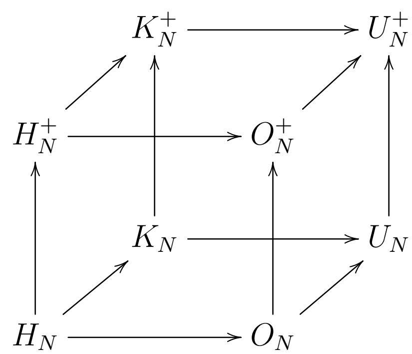
PROOF. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, and with the second assertion being trivial. Let us also mention that, in analogy with $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$, we have decomposition results as follows:

$$H_N^+ = \mathbb{Z}_2 \wr_* S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_* S_N^+$$

To be more precise, here $\wr_*$ is a free wreath product, and these formulae can be established a bit as in the classical case. For more on all this, we refer to [8]. □

All the above is very nice, and as a conclusion to all this, let us record the following result, which collects and refines the various liberation statements formulated above:

THEOREM 3.7. *The quantum unitary and reflection groups are as follows,*



and in this diagram, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

PROOF. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For some further details on all this, we refer to [8]. □

As a comment here, observe that the symmetric group S_N and its free analogue S_N^+, while certainly being very interesting objects, had not made the cut for appearing in the above almighty cube, called "standard cube" in quantum algebra. However, this is something quite natural, because S_N and S_N^+ are objects on their own, neither real or complex, and for practical purposes, like ours with our cube, these quantum groups must be replaced with H_N, H_N^+ in the real case, and with K_N, K_N^+ in the free case.

Actually I'm not quite sure about this, time to ask the cat. Who says:

CAT 3.8. *Do not worry, the high speed world is projective anyway, and it is better to use reflections instead of permutations.*

Thanks cat, not that I really understand what you say, but it fits with my purposes and cube, which looks really cool. But I will keep this in mind, and discuss later the relation between affine and projective geometry, in the free setting, that is promised.

With this done, let us get now into the second question that we were having, namely the conceptual understanding of the various liberation operations $G_N \to G_N^+$. In order to discuss this, we will need Tannakian duality, and Brauer type theorems. Let us start with Tannakian duality. This is a rather abstract statement, as follows:

THEOREM 3.9. *The following operations are inverse to each other:*

(1) *The construction $G \to C$, which associates to a closed subgroup $G \subset_u U_N^+$ the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
(2) *The construction $C \to G$, associating to a tensor category C the closed subgroup $G \subset_u U_N^+$ coming from the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. We have indeed a construction $G \to C_G$, whose output is a subcategory of the tensor C^*-category of finite dimensional Hilbert spaces, as follows:

$$(C_G)_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

We have as well a construction $C \to G_C$, obtained by setting:

$$C(G_C) = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C_{kl} \right\rangle$$

Regarding now the bijection claim, some elementary algebra shows that $C = C_{G_C}$ implies $G = G_{C_G}$, and that $C \subset C_{G_C}$ is automatic. Thus we are left with proving:

$$C_{G_C} \subset C$$

But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. □

The above result is something quite abstract, yet powerful. We will see applications of it in a moment, in the form of Brauer theorems for S_N, O_N, U_N and S_N^+, O_N^+, U_N^+, and other quantum groups. In order to formulate these Brauer theorems, let us start with:

DEFINITION 3.10. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A collection of sets*

$$D = \bigsqcup_{k,l} D(k,l)$$

with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:

(1) *Stability under the horizontal concatenation, $(\pi, \sigma) \to [\pi\sigma]$.*
(2) *Stability under the vertical concatenation, $(\pi, \sigma) \to [^\sigma_\pi]$.*
(3) *Stability under the upside-down turning, $\pi \to \pi^*$.*
(4) *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
(5) *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

As a basic example, we have the category of all partitions P itself. Other basic examples are the category of pairings P_2, and the categories NC, NC_2 of noncrossing partitions, and pairings. We have as well the category $\mathcal{P}_2$ of pairings which are "matching", in the sense that they connect $\circ-\circ$, $\bullet-\bullet$ on the vertical, and $\circ-\bullet$ on the horizontal, and its subcategory $\mathcal{NC}_2\subset\mathcal{P}_2$ consisting of the noncrossing matching pairings.

There are many other examples, and we will be back to this, gradually, in what follows. Regarding now the relation with the Tannakian categories, this comes from:

PROPOSITION 3.11. *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

As for the composition and involution axioms, their proof is similar. □

In relation now with quantum groups, we have the following result:

THEOREM 3.12. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and so a closed subgroup $G_N\subset_u U_N^+$.

Proof. Let call C_{kl} the spaces on the right. By using the axioms in Definition 3.10, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 3.11, we see that $C = (C_{kl})$ is a Tannakian category. Thus Theorem 3.9 applies, and gives the result. $\square$

We can now formulate a key definition, as follows:

Definition 3.13. *A compact quantum group G_N is called easy when we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for any colored integers k, l, for a certain category of partitions $D \subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool. In relation now with the orthogonal, unitary and symmetric quantum groups, here is the result:

Theorem 3.14. *The basic quantum permutation and rotation groups,*

$$\begin{array}{ccccc} S_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N & \longrightarrow & U_N \end{array}$$

are all easy, the corresponding categories of partitions being as follows,

$$\begin{array}{ccccc} NC & \longleftarrow & NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P & \longleftarrow & P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

with 2 standing for pairings, NC for noncrossing, and calligraphic for matching.

Proof. This is something quite fundamental, the proof being as follows:

(1) The quantum group U_N^+ is defined via the following relations:

$$u^* = u^{-1} \quad , \quad u^t = \bar{u}^{-1}$$

But, by doing some elementary computations, these relations tell us precisely that the following two operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi = {}^{\sqcap}_{\circ\bullet} \ , \ {}^{\sqcap}_{\bullet\circ}$$

Thus, the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = < {}^{\sqcap}_{\circ\bullet} \ , \ {}^{\sqcap}_{\bullet\circ} > = \mathcal{NC}_2$$

(2) The subgroup $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi =$$  ,

We conclude that the Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$D =< \mathcal{NC}_2,$  ,  $>= NC_2$

(3) The subgroup $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0 \quad , \quad [u_{ij}, \bar{u}_{kl}] = 0$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi =$$  , 

Thus the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$D =< \mathcal{NC}_2,$  ,  $>= \mathcal{P}_2$

(4) In order to deal now with O_N, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us that O_N is indeed easy, coming from:

$$D =< NC_2, \mathcal{P}_2 >= P_2$$

(5) We know that the subgroup $S_N^+ \subset O_N^+$ appears as follows:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle u = \text{magic} \Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y \in P(2,1)$$

Given a corepresentation u, we have the following formulae:

$$(T_Y u^{\otimes 2})_{i,jk} = \sum_{lm} (T_Y)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

$$(uT_Y)_{i,jk} = \sum_l u_{il} (T_Y)_{l,jk} = \delta_{jk} u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i,j,k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle T_Y \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D = < Y > = NC$$

(6) Finally, in order to deal with S_N, we can use the following formula:

$$S_N = S_N^+ \cap O_N$$

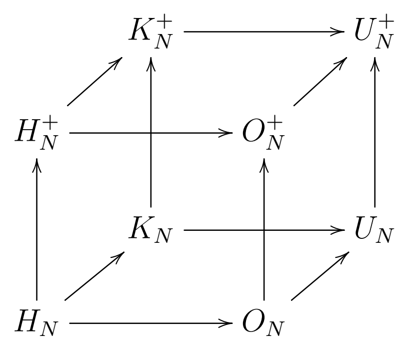
At the categorical level, this tells us that S_N is indeed easy, coming from:

$$D = < NC, P_2 > = P$$

Thus, we are led to the conclusions in the statement. □

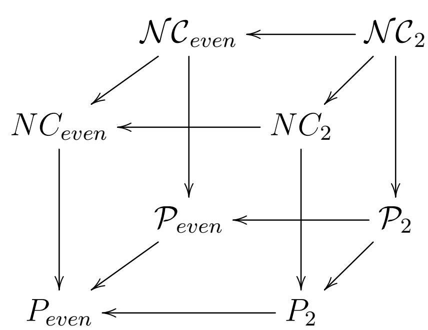
Moving ahead, we can upgrade what we have into a cube result, as follows:

THEOREM 3.15. *The basic quantum unitary and reflection groups,*



are all easy, and the corresponding categories of partitions form an intersection diagram.

PROOF. The precise claim is that the categories are as follows, with P_{even} being the category of partitions having even blocks, and with $\mathcal{P}_{even}(k,l) \subset P_{even}(k,l)$ consisting of the partitions satisfying $\# \circ = \# \bullet$ in each block, when flattening the partition:



But this is something that we already know for the right face, from Theorem 3.14, and in what regards the left face, the proof here is similar, by using the results for S_N, S_N^+ from that same Theorem 3.14. As for the last assertion, this is something trivial. □

The above results are something quite deep, and we will see in what follows countless applications of them. As a first such application, rather philosophical, we have:

THEOREM 3.16. *The constructions $G_N \to G_N^+$ with $G = O, U, S, H, K$ are easy quantum group liberations, in the sense that they come from the construction*

$$D \to D \cap NC$$

at the level of the associated categories of partitions.

PROOF. This is clear indeed from Theorem 3.14 and Theorem 3.15, and from the following trivial equalities, connecting the categories found there:

$$NC_2 = P_2 \cap NC \quad , \quad \mathcal{NC}_2 = \mathcal{P}_2 \cap NC$$

$$NC = P \cap NC$$

$$NC_{even} = P_{even} \cap NC \quad , \quad \mathcal{NC}_{even} = \mathcal{P}_{even} \cap NC$$

Thus, we are led to the conclusion in the statement. □

The above result is quite nice, because the various constructions $G_N \to G_N^+$ that we made so far, although natural, were something quite ad-hoc. Now all this is no longer ad-hoc, and the next time that we will have to liberate a subgroup $G_N \subset U_N$, we know what the recipe is, namely check if G_N is easy, and if so, simply define $G_N^+ \subset U_N^+$ as being the easy quantum group coming from the category $D = D_G \cap NC$.

3b. Uniformity, characters

In general, the study of the free quantum groups, in the "easy" sense explained above, is something quite complex. In order to cut a bit from complexity, we will use:

PROPOSITION 3.17. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D \subset P$, the following conditions are equivalent:*

1. *$G_{N-1} = G_N \cap U_{N-1}^+$, via the embedding $U_{N-1}^+ \subset U_N^+$ given by $u \to diag(u, 1)$.*
2. *$G_{N-1} = G_N \cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+ \subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

PROOF. We use the general easiness theory, as explained above:

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N \subset G_N$, which makes everything S_N-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given a subgroup $K \subset U_{N-1}^+$, with fundamental corepresentation u, consider the $N \times N$ matrix $v = diag(u, 1)$. Our claim is that for any $\pi \in P(k)$ we have:

$$\xi_\pi \in Fix(v^{\otimes k}) \iff \xi_{\pi'} \in Fix(v^{\otimes k'}), \forall \pi' \in P(k'), \pi' \subset \pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}\xi_\pi\in Fix(v^{\otimes k}) &\iff (v^{\otimes k}\xi_\pi)_{i_1\ldots i_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\ &\iff \sum_j(v^{\otimes k})_{i_1\ldots i_k,j_1\ldots j_k}(\xi_\pi)_{j_1\ldots j_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\ &\iff \sum_j\delta_\pi(j_1,\ldots,j_k)v_{i_1j_1}\ldots v_{i_kj_k}=\delta_\pi(i_1,\ldots,i_k),\forall i\end{aligned}$$

Now let us recall that our corepresentation has the special form $v=diag(u,1)$. We conclude from this that for any index $a\in\{1,\ldots,k\}$, we must have:

$$i_a=N\implies j_a=N$$

With this observation in hand, if we denote by i',j' the multi-indices obtained from i,j obtained by erasing all the above $i_a=j_a=N$ values, and by $k'\leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'}\delta_\pi(j_1,\ldots,j_k)(v^{\otimes k'})_{i'j'}=\delta_\pi(i_1,\ldots,i_k),\forall i$$

Here the index j is by definition obtained from j' by filling with N values. In order to finish now, we have two cases, depending on i, as follows:

<u>Case 1</u>. Assume that the index set $\{a|i_a=N\}$ corresponds to a certain subpartition $\pi'\subset\pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'}\delta_\pi(j'_1,\ldots,j'_{k'})(v^{\otimes k'})_{i'j'}=\delta_\pi(i'_1,\ldots,i'_{k'})$$

<u>Case 2</u>. Assume now the opposite, namely that the set $\{a|i_a=N\}$ does not correspond to a subpartition $\pi'\subset\pi$. In this case the indices mix, and our formula reads:

$$0=0$$

Thus, we are led to $\xi_{\pi'}\in Fix(v^{\otimes k'})$, for any subpartition $\pi'\subset\pi$, as claimed. Thus our claim is proved, and with this in hand, the result follows from Tannakian duality. □

Based on the above result, let us formulate the following definition:

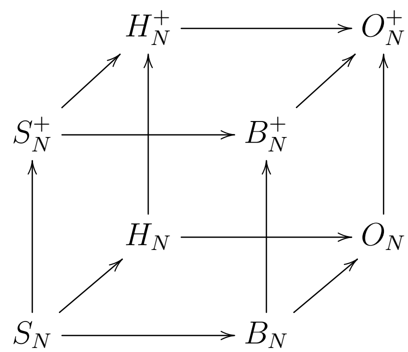
DEFINITION 3.18. *An easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, is called uniform when we have, for any $N\in\mathbb N$:*

$$G_{N-1}=G_N\cap U_{N-1}^+$$

Equivalently, D must be stable under the operation which consists in removing blocks.

For classification purposes the uniformity axiom is something very natural and useful, substantially cutting from complexity, and we have the following result:

THEOREM 3.19. *The classical and free uniform orthogonal easy quantum groups are*



with B_N, B_N^+ being the classical and quantum bistochastic groups.

PROOF. There are several things to be proved, the idea being as follows:

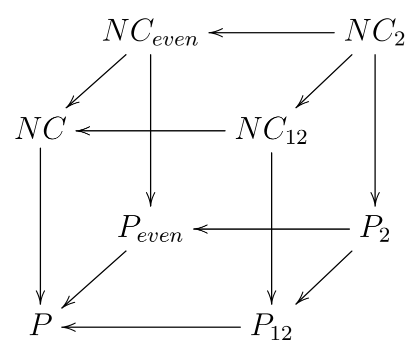
(1) We first recall that the bistochastic group $B_N \subset O_N$ consists of the orthogonal matrices whose entries sum up to 1 on each row, or equivalently, sum up to 1 on each column. Thus, if we denote by $\xi \in \mathbb{C}^N$ the all-one vector, we have:

$$B_N = \left\{ U \in O_N \Big| U\xi = \xi \right\}$$

Based on this, we can construct a free analogue of B_N as follows, and the fact that we obtain indeed a quantum group follows exactly as for O_N^+, U_N^+:

$$C(B_N^+) = C(O_N^+) \Big/ \left\langle u\xi = \xi \right\rangle$$

(2) Since the relation $u\xi = \xi$ reads $T_| \in Fix(u)$, with $| \in P(0,1)$ being the singleton partition, we conclude that B_N, B_N^+ are easy, coming from the categories P_{12}, NC_{12} of singletons and pairings, and noncrossing singletons and pairings. Thus, all the quantum groups in the statement are easy, the corresponding categories of partitions being:



(3) Regarding now the classification, consider an easy quantum group $S_N \subset G_N \subset O_N$. This must come from a category $P_2 \subset D \subset P$, and if we assume $G = (G_N)$ to be uniform,

then D is uniquely determined by the subset $L\subset\mathbb N$ consisting of the sizes of the blocks of the partitions in D. Our claim is that the admissible sets are as follows:

– $L=\{2\}$, producing O_N.

– $L=\{1,2\}$, producing B_N.

– $L=\{2,4,6,\ldots\}$, producing H_N.

– $L=\{1,2,3,\ldots\}$, producing S_N.

(4) Indeed, in one sense, this follows from our easiness results for O_N,B_N,H_N,S_N. In the other sense now, assume that $L\subset\mathbb N$ is such that the set P_L consisting of partitions whose sizes of the blocks belong to L is a category of partitions. We know from the axioms of the categories of partitions that the semicircle $\cap$ must be in the category, so we have $2\in L$. We claim that the following conditions must be satisfied as well:

$$k,l\in L,\ k>l\implies k-l\in L$$

$$k\in L,\ k\geq 2\implies 2k-2\in L$$

(5) Indeed, we will prove that both conditions follow from the axioms of the categories of partitions. Let us denote by $b_k\in P(0,k)$ the one-block partition:

$$b_k=\begin{Bmatrix}\sqcap\!\sqcap&\ldots&\sqcap\\ 1\ 2&\ldots&k\end{Bmatrix}$$

For $k>l$, we can write b_{k-l} in the following way:

$$b_{k-l}=\begin{Bmatrix}\sqcap\!\sqcap&\ldots&\ldots&\ldots&\ldots&\sqcap\\ 1\ 2&\ldots&l&l+1&\ldots&k\\ \sqcup\!\sqcup&\ldots&\sqcup&|&\ldots&|\\ &&&1&\ldots&k-l\end{Bmatrix}$$

In other words, we have the following formula:

$$b_{k-l}=(b_l^*\otimes|^{\otimes k-l})b_k$$

Since all the terms of this composition are in P_L, we have $b_{k-l}\in P_L$, and this proves our first claim. As for the second claim, this can be proved in a similar way, by capping two adjacent k-blocks with a 2-block, in the middle.

(6) With these conditions in hand, we can conclude in the following way:

<u>Case 1</u>. Assume $1\in L$. By using the first condition with $l=1$ we get:

$$k\in L\implies k-1\in L$$

This condition shows that we must have $L = \{1, 2, \ldots, m\}$, for a certain number $m \in \{1, 2, \ldots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} m \in L &\implies 2m - 2 \in L \\ &\implies 2m - 2 \leq m \\ &\implies m \in \{1, 2, \infty\} \end{aligned}$$

The case $m = 1$ being excluded by the condition $2 \in L$, we reach to one of the two sets producing the groups S_N, B_N.

<u>Case 2</u>. Assume $1 \notin L$. By using the first condition with $l = 2$ we get:

$$k \in L \implies k - 2 \in L$$

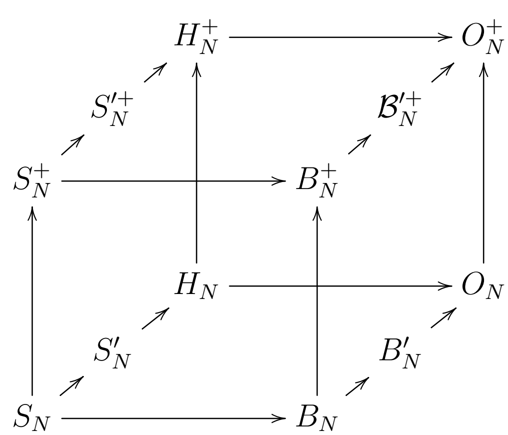
This condition shows that we must have $L = \{2, 4, \ldots, 2p\}$, for a certain number $p \in \{1, 2, \ldots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} 2p \in L &\implies 4p - 2 \in L \\ &\implies 4p - 2 \leq 2p \\ &\implies p \in \{1, \infty\} \end{aligned}$$

Thus L must be one of the two sets producing O_N, H_N, and we are done. In the free case, $S_N^+ \subset G_N \subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time $O_N^+, B_N^+, H_N^+, S_N^+$. □

When removing the uniformity axiom things become more complicated, as follows:

Theorem 3.20. *The classical and free orthogonal easy quantum groups are*



with $S'_N = S_N \times \mathbb{Z}_2$, $B'_N = B_N \times \mathbb{Z}_2$, and with $S_N'^+, \mathcal{B}_N'^+$ being their liberations, where $\mathcal{B}_N'^+$ stands for the two possible such liberations, $B_N'^+ \subset B_N''^+$.

Proof. The idea here is that of jointly classifying the "classical" categories of partitions $P_2 \subset D \subset P$, and the "free" ones $NC_2 \subset D \subset NC$:

(1) At the classical level this leads, via a study which is quite similar to that from the proof of Theorem 3.19, to 2 more groups, namely S_N', B_N'.

(2) At the free level we obtain 3 more quantum groups, $S_N'^+, B_N'^+, B_N''^+$, with the inclusion $B_N'^+ \subset B_N''^+$, which is something a bit surprising, being best thought of as coming from an inclusion $B_N' \subset B_N''$, which happens to be an isomorphism. □

It is possible to obtain similar results in the general unitary case, first with a quite simple statement, regarding the uniform case, and then with something more complicated, regarding the non-uniform case. We refer here to the paper of Tarrago-Weber [80].

Importantly, the uniformity assumption has some interesting analytic consequences, making the link with the Bercovici-Pata bijection [19]. In order to discuss this, we first need to know how to integrate on the easy quantum groups, and we have here:

THEOREM 3.21. *Assuming that a closed subgroup $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1} \ldots u_{i_kj_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where $\delta \in \{0,1\}$ are the usual Kronecker type symbols, and where the Weingarten matrix $W_{kN} = G_{kN}^{-1}$ is the inverse of the Gram matrix $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$.

PROOF. We know from the general theory in chapter 1 that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

In order to prove the result, consider the following linear map:

$$E(x) = \sum_{\pi \in D(k)} < x, \xi_\pi > \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse on $Fix(u^{\otimes k})$ of the restriction of E. But this restriction is the linear map given by G_{kN}, and so W is the linear map given by W_{kN}, and this gives the result. □

In relation now with characters, we have the following moment formula:

PROPOSITION 3.22. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11} + \ldots + u_{ss})^k = Tr(W_{kN}G_{ks})$$

where G_{kN} and $W_{kN} = G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

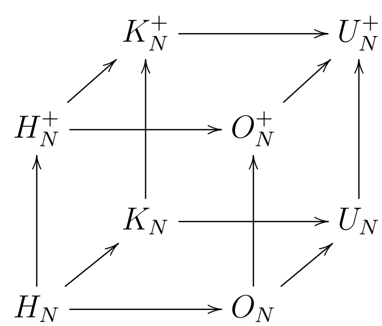
Proof. We have indeed the following computation:

$$\begin{aligned}
\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s \ldots \sum_{i_k=1}^s \int u_{i_1i_1}\ldots u_{i_ki_k}\\
&= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s \delta_\pi(i)\delta_\sigma(i)\\
&= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\
&= Tr(W_{kN}G_{ks})
\end{aligned}$$

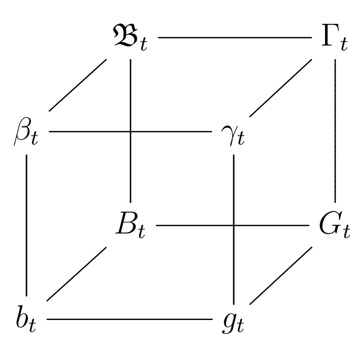
Thus, we have obtained the formula in the statement. □

With the above general theory in hand, we can now formulate our character results for the main examples of uniform easy quantum groups, as follows:

Theorem 3.23. *For the main quantum rotation and reflection groups,*

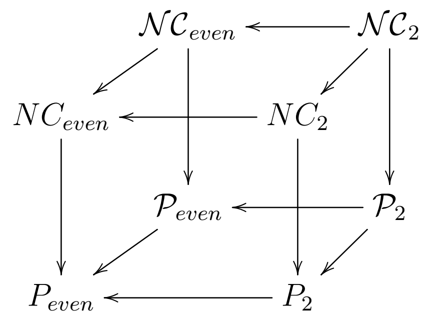


the corresponding truncated characters follow with $N \to \infty$ the laws



which are the main limiting laws in classical and free probability.

PROOF. We know from Theorem 3.15 that the above quantum groups are all easy, coming from the following categories of partitions:



Now by using Proposition 3.22, we obtain the following formula:

$$\lim_{N\to\infty}\int_{G_N}\chi_t^k=\sum_{\pi\in D(k)}t^{|\pi|}$$

But this gives the laws in the statement, via some standard calculus. □

3c. Temperley-Lieb

All the above is sweet, and there are many other things that can be said, along the same lines, about the liberation operations $G_N\to G_N^+$, using easiness and partitions. This being said, we are rather interested in free quantum groups, so we do not need partitions with crossings, and this leads us to a quite puzzling question, as follows:

QUESTION 3.24. *Among the many objects which are in bijection with the noncrossing partitions, which are the most adapted to the study of the free quantum groups?*

To be more precise here, in order to give you a taste on what this question is about, you have surely heard for instance about the Catalan numbers:

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

These Catalan numbers count the partitions in $NC(k)$, but they count as well a zillion other interesting things, just ask and any expert in combinatorics will probably get you stuck for 1 hour in the coffee room, in explaining you all this, and our problem is, among these zillion things, what are the best for the study of free quantum groups.

This does not look obvious, and so time to ask the cat. And cat says:

CAT 3.25. *You're getting old, double the strings as to have Temperley-Lieb diagrams, as in the heyday of free quantum group theory.*

Thanks cat, and yes indeed, age does not help much with knowledge and memory, in fact Question 3.24 is something that I already thought about, some 30 years ago, when developing the basic theory of free quantum groups. Following Temperley-Lieb, who by the way were first-class physicists, and then Jones, who was a first-class physicist too, and many others, including myself when younger, not to forget cat of course, we will of course go for this, doubling strings and using Temperley-Lieb diagrams.

Let us start with the following result, which is well-known:

PROPOSITION 3.26. *We have a bijection $NC(k) \simeq NC_2(2k)$, as follows:*

1. *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

PROOF. The fact that the above two operations are indeed inverse to each other is clear, by drawing pictures, and computing the corresponding compositions. □

With the above result in hand, we can axiomatize the free quantum groups, in terms of Temperley-Lieb diagrams NC_2, and say many interesting things about them, based on the work of Jones and others on subfactor theory and planar algebras [64].

We can compute representations and their fusion rules, Cayley graphs, growth exponents, laws of characters and more, by using diagrams, and more specifically Temperley-Lieb diagrams NC_2, which are quite often the most adapted, to our questions.

As a basic example for what can be done here, regarding O_N^+, we have:

THEOREM 3.27. *The irreducible representations of O_N^+ with $N \geq 2$ can be labelled by positive integers, r_k with $k \in \mathbb{N}$, the fusion rules for these representations are*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

and the dimensions are $\dim r_k = (q^{k+1} - q^{-k-1})/(q - q^{-1})$, *with* $q + q^{-1} = N$.

PROOF. The idea is to skilfully recycle the well-known proof for SU_2. Our claim is that we can construct, by recurrence on $k \in \mathbb{N}$, a sequence $r_0, r_1, r_2, \ldots$ of irreducible, self-adjoint and distinct representations of O_N^+, satisfying:

$$r_0 = 1 \quad , \quad r_1 = u \quad , \quad r_{k-1} \otimes r_1 = r_{k-2} + r_k$$

In order to do so, we can use the formula $r_{k-2} \otimes r_1 = r_{k-3} + r_{k-1}$ and Frobenius duality, and we conclude there exists a certain representation r_k such that:

$$r_{k-1} \otimes r_1 = r_{k-2} + r_k$$

As a first observation, r_k is self-adjoint, because its character is a certain polynomial with integer coefficients in χ, which is self-adjoint. In order to prove now that r_k is irreducible, and non-equivalent to $r_0,\ldots,r_{k-1}$, let us split as before $u^{\otimes k}$, as follows:

$$u^{\otimes k}=c_kr_k+c_{k-2}r_{k-2}+c_{k-4}r_{k-4}+\ldots$$

The point now is that we have the following equalities and inequalities:

$$C_k=\sum_i c_i^2\leq\dim(End(u^{\otimes k}))\leq|NC_2(k,k)|=C_k$$

Indeed, the equality at left is clear as before, then comes a standard inequality, then an inequality coming from easiness, then a standard equality. Thus, we have equality, so r_k is irreducible, and non-equivalent to $r_{k-2},r_{k-4},\ldots$ Moreover, r_k is not equivalent to $r_{k-1},r_{k-3},\ldots$ either, by using the same argument as for SU_2, and the end of the proof is exactly as for SU_2. As for dimensions, by recurrence we obtain, with $q+q^{-1}=N$:

$$\dim r_k=q^k+q^{k-2}+\ldots+q^{-k+2}+q^{-k}$$

But this gives the dimension formula in the statement, and we are done. $\square$

It is possible to use similar methods for the other main examples of free quantum groups, and do many other things, in relation with the Temperley-Lieb algebra.

3d. Meander determinants

We discuss now, following Di Francesco [40] and others, the computation of the Gram determinants for the free quantum groups, which is a very interesting question, related to many things. But let us start with S_N and other classical groups. We will need:

Definition 3.28. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi,\sigma)=\begin{cases}1&\text{if }\pi=\sigma\\-\sum_{\pi\leq\tau<\sigma}\mu(\pi,\tau)&\text{if }\pi<\sigma\\0&\text{if }\pi\not\leq\sigma\end{cases}$$

with the construction being performed by recurrence.

As an illustration here, for $P(2)=\{||,\sqcap\}$, we have by definition:

$$\mu(||,||)=\mu(\sqcap,\sqcap)=1$$

Also, $||<\sqcap$, with no intermediate partition in between, so we obtain:

$$\mu(||,\sqcap)=-\mu(||,||)=-1$$

Finally, we have $\sqcap\not\leq||$, and so we have as well the following formula:

$$\mu(\sqcap,||)=0$$

We will need the Möbius inversion formula, which can be formulated as follows:

THEOREM 3.29. *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1 & \text{if } \pi\leq\sigma\\ 0 & \text{if } \pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.

PROOF. This is well-known, coming from the fact that A_k is upper triangular. Indeed, when inverting, we are led into the recurrence for μ, from Definition 3.28. □

As an illustration, for $P(2)$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

PROPOSITION 3.30. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G_k=A_kL_k$, where

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1) & \text{if } \sigma\leq\pi\\ 0 & \text{otherwise}\end{cases}$$

and where A_k is the adjacency matrix of $P(k)$.

PROOF. We have indeed the following computation:

$$\begin{aligned}G_k(\pi,\sigma)&=N^{|\pi\vee\sigma|}\\&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According now to the definition of A_k,L_k, this formula reads:

$$\begin{aligned}G_k(\pi,\sigma)&=\sum_{\tau\geq\pi}L_k(\tau,\sigma)\\&=\sum_\tau A_k(\pi,\tau)L_k(\tau,\sigma)\\&=(A_kL_k)(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix}N^2&N\\ N&N\end{pmatrix}=\begin{pmatrix}1&1\\ 0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\ N&N\end{pmatrix}$$

We are led in this way to the following formula, due to Lindstöm:

THEOREM 3.31. *The determinant of the Gram matrix G_k is given by*

$$\det(G_k)=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N<k$ we obtain 0.

PROOF. If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, A_k is upper triangular, and L_k is lower triangular. Thus, we have:

$$\begin{aligned}\det(G_k)&=\det(A_k)\det(L_k)\\ &=\det(L_k)\\ &=\prod_\pi L_k(\pi,\pi)\\ &=\prod_\pi N(N-1)\ldots(N-|\pi|+1)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us discuss as well the case of the orthogonal group O_N. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. We have then:

THEOREM 3.32. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN})=\prod_{|\lambda|=k/2}f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda)=\prod_{(i,j)\in\lambda}(N+2j-i-1)$.

PROOF. For the group O_N the Gram matrix is diagonalizable, as follows:

$$G_{kN}=\sum_{|\lambda|=k/2}f_N(\lambda)P_{2\lambda}$$

Here $1=\sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda})=f^{2\lambda}$, this gives the formula in the statement. □

In order to deal now with O_N^+,S_N^+, we will need the following fact:

Proposition 3.33. *The Gram matrices of $NC_2(2k) \simeq NC(k)$ are related by*

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1}G_{k,n^2}\Delta_{kn}^{-1})(\pi',\sigma')$$

where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}.

Proof. In the context of the bijection from Proposition 3.26, we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi\vee\sigma|} = n^{k+2|\pi'\vee\sigma'|-|\pi'|-|\sigma'|}$$

Thus, we are led to the formula in the statement. □

Now back to O_N^+, S_N^+, let us begin with some examples. We first have:

Proposition 3.34. *The first Gram matrices and determinants for O_N^+ are*

$$\det\begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix} = N^2(N^2-1)$$

$$\det\begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix} = N^5(N^2-1)^4(N^2-2)$$

with the matrices being written by using the lexicographic order on $NC_2(2k)$.

Proof. The formula at $k=2$, where $NC_2(4) = \{\sqcap\sqcap, \bigcap\!\!\!\!\cap\}$, is clear from definitions. At $k=3$ however, things are tricky. The partitions here are as follows:

$$NC(3) = \{|||, \sqcap|, \sqcap\!\!\!|\,, |\sqcap, \sqcap\!\sqcap\}$$

The Gram matrix and its determinant are, according to Theorem 3.31:

$$\det\begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^2 & N & N & N \\ N^2 & N & N^2 & N & N \\ N^2 & N & N & N^2 & N \\ N & N & N & N & N \end{pmatrix} = N^5(N-1)^4(N-2)$$

By using Proposition 3.33, the Gram determinant of $NC_2(6)$ is given by:

$$\begin{aligned} \det(G_{6N}) &= \frac{1}{N^2\sqrt{N}} \times N^{10}(N^2-1)^4(N^2-2) \times \frac{1}{N^2\sqrt{N}} \\ &= N^5(N^2-1)^4(N^2-2) \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In general, such tricks won't work, because $NC(k)$ is strictly smaller than $P(k)$ at $k\geq 4$. However, following Di Francesco [40], we have the following result:

THEOREM 3.35. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb Z$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb Z$.

PROOF. This is something quite technical, obtained by using a decomposition as follows of the Gram matrix G_{kN}, with the matrix T_{kN} being lower triangular:

$$G_{kN}=T_{kN}T_{kN}^t$$

Thus, a bit as in the proof of the Lindstöm formula, we obtain the result, but the problem lies however in the construction of T_{kN}, which is non-trivial. See [40]. □

With this in hand, we have as well a similar formula for S_N^+, obtained from Theorem 3.35 via Proposition 3.33. For the other free quantum groups, the computations can be done as well. For more on all this, we refer to [40] and related papers.

3e. Exercises

Exercises:

EXERCISE 3.36.

EXERCISE 3.37.

EXERCISE 3.38.

EXERCISE 3.39.

EXERCISE 3.40.

EXERCISE 3.41.

Bonus exercise.

CHAPTER 4

Free space

4a. Projective space

We discuss in this chapter several things that can be done, going beyond the sphere setting. First we will discuss free projective geometry, which is by definition compact, and so can be developed in full generality, without norm restrictions. Then, at the end of the chapter, we will go back to the affine setting, with some further results.

As a first topic that we would like to discuss, which historically speaking, was at the beginning of everything, we have the following remarkable isomorphism:

$$PO_N^+=PU_N^+$$

In order to get started, let us first discuss the classical case, and more specifically the precise relation between the orthogonal group O_N, and the unitary group U_N. Contrary to the passage $\mathbb R^N\to\mathbb C^N$, or to the passage $S^{N-1}_\mathbb R\to S^{N-1}_\mathbb C$, which are both elementary, the passage $O_N\to U_N$ cannot be understood directly. In order to understand this passage we must pass through the corresponding Lie algebras, a follows:

THEOREM 4.1. *The passage $O_N\to U_N$ appears via Lie algebra complexification,*

$$O_N\to\mathfrak o_N\to\mathfrak u_n\to U_N$$

with the Lie algebra $\mathfrak u_N$ being a complexification of the Lie algebra $\mathfrak o_N$.

PROOF. This is something rather philosophical, and advanced as well, that we will not really need here, the idea being as follows:

(1) The unitary and orthogonal groups U_N,O_N are both Lie groups, in the sense that they are smooth manifolds. The corresponding Lie algebras $\mathfrak u_N,\mathfrak o_N$, which are by definition the respective tangent spaces at 1, can be computed by differentiating the equations defining U_N,O_N, with the conclusion being as follows:

$$\mathfrak u_N=\left\{A\in M_N(\mathbb C)\Big|A^*=-A\right\}$$

$$\mathfrak o_N=\left\{B\in M_N(\mathbb R)\Big|B^t=-B\right\}$$

(2) This was for the correspondences $U_N\to\mathfrak u_N$ and $O_N\to\mathfrak o_N$. In the other sense, the correspondences $\mathfrak u_N\to U_N$ and $\mathfrak o_N\to O_N$ appear by exponentiation, the result here

stating that, around 1, the unitary matrices can be written as $U = e^A$, with $A \in \mathfrak{u}_N$, and the orthogonal matrices can be written as $U = e^B$, with $B \in \mathfrak{o}_N$.

(3) In view of all this, in order to understand the passage $O_N \to U_N$ it is enough to understand the passage $\mathfrak{o}_N \to \mathfrak{u}_N$. But, in view of the above formulae for $\mathfrak{o}_N, \mathfrak{u}_N$, this is basically an elementary linear algebra problem. Indeed, let us pick an arbitrary matrix $A \in M_N(\mathbb{C})$, and write it as follows, with $B, C \in M_N(\mathbb{R})$:

$$A = B + iC$$

In terms of B, C, the equation $A^* = -A$ defining the Lie algebra $\mathfrak{u}_N$ reads:

$$B^t = -B \quad , \quad C^t = C$$

(4) As a first observation, we must have $B \in \mathfrak{o}_N$. Regarding now C, let us decompose this matrix as follows, with D being its diagonal, and C' being the reminder:

$$C = D + C'$$

The matrix C' being symmetric with 0 on the diagonal, by swithcing all the signs below the main diagonal we obtain a certain matrix $C'_- \in \mathfrak{o}_N$. Thus, we have decomposed $A \in \mathfrak{u}_N$ as follows, with $B, C' \in \mathfrak{o}_N$, and with $D \in M_N(\mathbb{R})$ being diagonal:

$$A = B + iD + iC'_-$$

(5) As a conclusion now, we have shown that we have a direct sum decomposition of real linear spaces as follows, with $\Delta \subset M_N(\mathbb{R})$ being the diagonal matrices:

$$\mathfrak{u}_N \simeq \mathfrak{o}_N \oplus \Delta \oplus \mathfrak{o}_N$$

Thus, we can stop our study here, and say that we have reached the conclusion in the statement, namely that $\mathfrak{u}_N$ appears as a "complexification" of $\mathfrak{o}_N$. $\square$

As before with many other things, that we will not really need in what follows, this was just an introduction to the subject. More can be found in any Lie group book. In the free case now, the situation is much simpler, and we have:

THEOREM 4.2. *The passage $O_N^+ \to U_N^+$ appears via free complexification,*

$$U_N^+ = \widetilde{O_N^+}$$

where the free complexification of a pair (G, u) is the pair $(\widetilde{G}, \widetilde{u})$ with

$$C(\widetilde{G}) = < zu_{ij} > \subset C(\mathbb{T}) * C(G) \quad , \quad \widetilde{u} = zu$$

where $z \in C(\mathbb{T})$ is the standard generator, given by $x \to x$ for any $x \in \mathbb{T}$.

PROOF. We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of U_N^+:

$$O_N^+ \subset \widetilde{O_N^+} \subset U_N^+$$

We must prove that the embedding on the right is an isomorphism, and there are several ways of doing this, all instructive, as follows:

(1) If we denote by v, u the fundamental corepresentations of O_N^+, U_N^+, we have:

$$Fix(v^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in NC_2(k)\right)$$

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in \mathcal{NC}_2(k)\right)$$

Moreover, the above vectors ξ_π are known to be linearly independent at $N \geq 2$, and so the above results provide us with bases, and we obtain:

$$\dim(Fix(v^{\otimes k})) = |NC_2(k)| \quad , \quad \dim(Fix(u^{\otimes k})) = |\mathcal{NC}_2(k)|$$

Now since integrating the character of a corepresentation amounts in counting the fixed points, the above two formulae can be rewritten as follows:

$$\int_{O_N^+} \chi_v^k = |NC_2(k)| \quad , \quad \int_{U_N^+} \chi_u^k = |\mathcal{NC}_2(k)|$$

But this shows, via standard free probability theory, that χ_v must follow the Winger semicircle law γ_1, and that χ_u must follow the Voiculescu circular law Γ_1:

$$\chi_v \sim \gamma_1 \quad , \quad \chi_u \sim \Gamma_1$$

On the other hand, by [87], when freely multiplying a semicircular variable by a Haar unitary we obtain a circular variable. Thus, the main character of $\widetilde{O_N^+}$ is circular:

$$\chi_{zv} \sim \Gamma_1$$

Now by forgetting about circular variables and free probability, the conclusion is that the inclusion $\widetilde{O_N^+} \subset U_N^+$ preserves the law of the main character:

$$law(\chi_{zv}) = law(u)$$

Thus by Peter-Weyl we obtain that the inclusion $\widetilde{O_N^+} \subset U_N^+$ must be an isomorphism, modulo the usual equivalence relation for quantum groups.

(2) A version of the above proof, not using any prior free probability knowledge, makes use of the easiness property of O_N^+, U_N^+ only, namely:

$$Hom(v^{\otimes k}, v^{\otimes l}) = span\left(\xi_\pi \Big| \pi \in NC_2(k,l)\right)$$

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\xi_\pi \Big| \pi \in \mathcal{NC}_2(k,l)\right)$$

Indeed, let us look at the following inclusions of quantum groups:

$$O_N^+ \subset \widetilde{O_N^+} \subset U_N^+$$

At the level of the associated Hom spaces we obtain reverse inclusions, as follows:

$$Hom(v^{\otimes k}, v^{\otimes l}) \supset Hom((zv)^{\otimes k}, (zv)^{\otimes l}) \supset Hom(u^{\otimes k}, u^{\otimes l})$$

The spaces on the left and on the right are known from easiness, the result being that these spaces are as follows:

$$span\left(T_\pi\Big|\pi \in NC_2(k,l)\right) \supset span\left(T_\pi\Big|\pi \in \mathcal{NC}_2(k,l)\right)$$

Regarding the spaces in the middle, these are obtained from those on the left by "coloring", so we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O_N^+} \subset U_N^+$ is an isomorphism, modulo the usual equivalence relation. □

As an interesting consequence of the above result, we have:

THEOREM 4.3. *We have an identification as follows,*

$$PO_N^+ = PU_N^+$$

modulo the usual equivalence relation for compact quantum groups.

PROOF. As before, we have several proofs for this result, as follows:

(1) This follows from Theorem 4.2, because we have:

$$PU_N^+ = P\widetilde{O_N^+} = PO_N^+$$

(2) We can deduce this as well directly. With notations as before, we have:

$$Hom\left((v \otimes v)^k, (v \otimes v)^l\right) = span\left(T_\pi\Big|\pi \in NC_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

$$Hom\left((u \otimes \bar{u})^k, (u \otimes \bar{u})^l\right) = span\left(T_\pi\Big|\pi \in \mathcal{NC}_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusion $PO_N^+ \subset PU_N^+$ preserves the corresponding Tannakian categories, and so must be an isomorphism. □

As a conclusion, the passage $O_N^+ \to U_N^+$ is something much simpler than the passage $O_N \to U_N$, with this ultimately coming from the fact that the combinatorics of O_N^+, U_N^+ is something much simpler than the combinatorics of O_N, U_N. In addition, all this leads as well to the interesting conclusion that the free projective geometry does not fall into real and complex, but is rather unique and "scalarless". We will be back to this.

Let us discuss now the projective spaces. Our starting point is the following functional analytic description of the real and complex projective spaces $P^{N-1}_\mathbb{R}, P^{N-1}_\mathbb{C}$:

PROPOSITION 4.4. *We have presentation results as follows,*

$$C(P^{N-1}_{\mathbb{R}}) = C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=\bar{p}=p^t=p^2,Tr(p)=1\right)$$
$$C(P^{N-1}_{\mathbb{C}}) = C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

for the algebras of continuous functions on the real and complex projective spaces.

PROOF. We use the fact that the projective spaces $P^{N-1}_{\mathbb{R}}, P^{N-1}_{\mathbb{C}}$ can be respectively identified with the spaces of rank one projections in $M_N(\mathbb{R}), M_N(\mathbb{C})$. With this picture in mind, it is clear that we have arrows $\leftarrow$. In order to construct now arrows $\to$, consider the universal algebras on the right, A_R, A_C. These algebras being both commutative, by the Gelfand theorem we can write, with X_R, X_C being certain compact spaces:

$$A_R = C(X_R) \quad , \quad A_C = C(X_C)$$

Now by using the coordinate functions p_{ij}, we conclude that X_R, X_C are certain spaces of rank one projections in $M_N(\mathbb{R}), M_N(\mathbb{C})$. In other words, we have embeddings:

$$X_R \subset P^{N-1}_{\mathbb{R}} \quad , \quad X_C \subset P^{N-1}_{\mathbb{C}}$$

By transposing we obtain arrows $\to$, as desired. □

The above result suggests the following definition:

DEFINITION 4.5. *Associated to any $N \in \mathbb{N}$ is the following universal algebra,*

$$C(P^{N-1}_+) = C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

whose abstract spectrum is called "free projective space".

Observe that, according to our presentation results for the real and complex projective spaces $P^{N-1}_{\mathbb{R}}$ and $P^{N-1}_{\mathbb{C}}$, we have embeddings of compact quantum spaces, as follows:

$$P^{N-1}_{\mathbb{R}} \subset P^{N-1}_{\mathbb{C}} \subset P^{N-1}_+$$

Our first goal will be that of explaining why, in analogy with the uniqueness of the quantum group $PO_N^+ = PU_N^+$, the free projective space P^{N-1}_+ is unique, and scalarless.

Let us first discuss the relation with the spheres. Given a closed subset $X \subset S^{N-1}_{\mathbb{R},+}$, its projective version is by definition the quotient space $X \to PX$ determined by the fact that $C(PX) \subset C(X)$ is the subalgebra generated by the following variables:

$$p_{ij} = x_i x_j$$

In order to discuss the relation with the spheres, it is convenient to neglect the material regarding the complex and hybrid cases, the projective versions of such spheres bringing nothing new. Thus, we are left with the 3 real spheres, and we have:

Theorem 4.6. *The projective versions of the 3 real spheres are as follows,*

$$\begin{array}{ccccc} S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \\ \downarrow & & \downarrow & & \downarrow \\ P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{+} \end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

Proof. The assertion at left is true by definition. For the assertion at right, we have to prove that the variables $p_{ij} = z_i z_j$ over the free sphere $S^{N-1}_{\mathbb{R},+}$ satisfy the defining relations for $C(P^{N-1}_+)$, from Definition 4.5, namely:

$$p = p^* = p^2 \quad , \quad Tr(p) = 1$$

We first have the following computation:

$$(p^*)_{ij} = p^*_{ji} = (z_j z_i)^* = z_i z_j = p_{ij}$$

We have as well the following computation:

$$(p^2)_{ij} = \sum_k p_{ik} p_{kj} = \sum_k z_i z_k^2 z_j = z_i z_j = p_{ij}$$

Finally, we have as well the following computation:

$$Tr(p) = \sum_k p_{kk} = \sum_k z_k^2 = 1$$

Regarding now $PS^{N-1}_{\mathbb{R},*} = P^{N-1}_{\mathbb{C}}$, the inclusion "$\subset$" follows from $abcd = cbad = cbda$. In the other sense now, the point is that we have a matrix model, as follows:

$$\pi : C(S^{N-1}_{\mathbb{R},*}) \to M_2(C(S^{N-1}_{\mathbb{C}})) \quad , \quad x_i \to \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

But this gives the missing inclusion "$\supset$", and we are done. See [11]. □

In addition to the above result, let us mention that, as already discussed above, passing to the complex case brings nothing new. This is because the projective version of the free complex sphere is equal to the free projective space constructed above:

$$PS^{N-1}_{\mathbb{C},+} = P^{N-1}_+$$

And the same goes for the "hybrid" spheres. For details on all this, we refer to [8].

Following [12], we can axiomatize our various projective spaces, as follows:

Definition 4.7. *A monomial projective space is a closed subset $P\subset P^{N-1}_+$ obtained via relations of type*

$$p_{i_1i_2}\ldots p_{i_{k-1}i_k}=p_{i_{\sigma(1)}i_{\sigma(2)}}\ldots p_{i_{\sigma(k-1)}i_{\sigma(k)}},\ \forall(i_1,\ldots,i_k)\in\{1,\ldots,N\}^k$$

with σ ranging over a certain subset of the infinite symmetric group

$$S_\infty=\bigcup_{k\in2\mathbb N}S_k$$

which is stable under the operation $\sigma\to|\sigma|$.

Here the stability under the operation $\sigma\to|\sigma|$ means that if the above relation associated to σ holds, then the following relation, associated to $|\sigma|$, must hold as well:

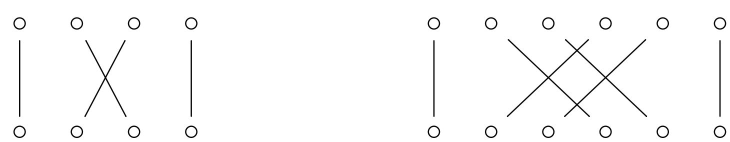
$$p_{i_0i_1}\ldots p_{i_ki_{k+1}}=p_{i_0i_{\sigma(1)}}p_{i_{\sigma(2)}i_{\sigma(3)}}\ldots p_{i_{\sigma(k-2)}i_{\sigma(k-1)}}p_{i_{\sigma(k)}i_{k+1}}$$

As an illustration, the basic projective spaces are all monomial:

Proposition 4.8. *The 3 projective spaces are all monomial, with the permutations*

producing respectively the spaces $P^{N-1}_{\mathbb R},P^{N-1}_{\mathbb C}$, and with no relation needed for P^{N-1}_+.

Proof. We must divide the algebra $C(P^{N-1}_+)$ by the relations associated to the diagrams in the statement, as well as those associated to their shifted versions, given by:



(1) The basic crossing, and its shifted version, produce the following relations:

$$p_{ab}=p_{ba}$$

$$p_{ab}p_{cd}=p_{ac}p_{bd}$$

Now by using these relations several times, we obtain the following formula:

$$p_{ab}p_{cd}=p_{ac}p_{bd}=p_{ca}p_{db}=p_{cd}p_{ab}$$

Thus, the space produced by the basic crossing is classical, $P\subset P^{N-1}_{\mathbb C}$. By using one more time the relations $p_{ab}=p_{ba}$ we conclude that we have $P=P^{N-1}_{\mathbb R}$, as claimed.

(2) The fattened crossing, and its shifted version, produce the following relations:

$$p_{ab}p_{cd}=p_{cd}p_{ab}$$

$$p_{ab}p_{cd}p_{ef}=p_{ad}p_{eb}p_{cf}$$

The first relations tell us that the projective space must be classical, $P \subset P^{N-1}_\mathbb{C}$. Now observe that with $p_{ij} = z_i \bar{z}_j$, the second relations read:

$$z_a \bar{z}_b z_c \bar{z}_d z_e \bar{z}_f = z_a \bar{z}_d z_e \bar{z}_b z_c \bar{z}_f$$

Since these relations are automatic, we have $P = P^{N-1}_\mathbb{C}$, and we are done. □

Following [12], we can now formulate our classification result, as follows:

THEOREM 4.9. *The basic projective spaces, namely*

$$P^{N-1}_\mathbb{R} \subset P^{N-1}_\mathbb{C} \subset P^{N-1}_+$$

are the only monomial ones.

PROOF. We follow the proof from the affine case. Let $\mathcal{R}_\sigma$ be the collection of relations associated to a permutation $\sigma \in S_k$ with $k \in 2\mathbb{N}$, as in Definition 4.7. We fix a monomial projective space $P \subset P^{N-1}_+$, and we associate to it subsets $G_k \subset S_k$, as follows:

$$G_k = \begin{cases} \{\sigma \in S_k | \mathcal{R}_\sigma \text{ hold over } P\} & (k \text{ even}) \\ \{\sigma \in S_k | \mathcal{R}_{|\sigma} \text{ hold over } P\} & (k \text{ odd}) \end{cases}$$

As in the affine case, we obtain in this way a filtered group $G = (G_k)$, which is stable under removing outer strings, and under removing neighboring strings. Thus the computations from the affine case apply, and show that we have only 3 possible situations, corresponding to the 3 projective spaces in Proposition 4.8. See [12]. □

Let us discuss now similar results for the projective quantum groups. Given a closed subgroup $G \subset O_N^+$, its projective version $G \to PG$ is by definition given by the fact that $C(PG) \subset C(G)$ is the subalgebra generated by the following variables:

$$w_{ij,ab} = u_{ia} u_{jb}$$

In the classical case we recover in this way the usual projective version:

$$PG = G/(G \cap \mathbb{Z}_2^N)$$

We have the following key result:

THEOREM 4.10. *The quantum group O_N^* is the unique intermediate easy quantum group $O_N \subset G \subset O_N^+$. Moreover, in the non-easy case, the following happen:*

(1) *The group inclusion $\mathbb{T}O_N \subset U_N$ is maximal.*
(2) *The group inclusion $PO_N \subset PU_N$ is maximal.*
(3) *The quantum group inclusion $O_N \subset O_N^*$ is maximal.*

PROOF. The first assertion comes by classifying the categories of pairings, and then:

(1) This can be obtained by using standard Lie group methods.

(2) This follows from (1), by taking projective versions.

(3) This follows from (2), via standard algebraic lifting results. □

Our claim now is that, under suitable assumptions, PU_N is the only intermediate object $PO_N \subset G \subset PO_N^+$. In order to formulate a precise statement here, we will need:

Definition 4.11. *A projective category of pairings is a collection of subsets*

$$NC_2(2k,2l) \subset E(k,l) \subset P_2(2k,2l)$$

stable under the usual categorical operations, and satisfying $\sigma \in E \implies |\sigma| \in E$.

As basic examples for this notion, we have the following projective categories of pairings, where P_2^* is the category of matching pairings:

$$NC_2 \subset P_2^* \subset P_2$$

This follows indeed from definitions. Now with the above notion in hand, we can formulate the following projective analogue of the notion of easiness:

Definition 4.12. *An intermediate compact quantum group*

$$PO_N \subset H \subset PO_N^+$$

is called projectively easy when its Tannakian category

$$span(NC_2(2k,2l)) \subset Hom(v^{\otimes k}, v^{\otimes l}) \subset span(P_2(2k,2l))$$

comes via via the following formula, using the standard $\pi \to T_\pi$ *construction,*

$$Hom(v^{\otimes k}, v^{\otimes l}) = span(E(k,l))$$

for a certain projective category of pairings $E = (E(k,l))$.

Thus, we have a projective notion of easiness. Observe that, given an easy quantum group $O_N \subset G \subset O_N^+$, its projective version $PO_N \subset PG \subset PO_N^+$ is projectively easy in our sense. In particular the basic projective quantum groups $PO_N \subset PU_N \subset PO_N^+$ are all projectively easy in our sense, coming from the categories $NC_2 \subset P_2^* \subset P_2$.

We have in fact the following general result, from [**12**]:

Theorem 4.13. *We have a bijective correspondence between the affine and projective categories of partitions, given by the operation*

$$G \to PG$$

at the level of the corresponding affine and projective easy quantum groups.

Proof. The construction of correspondence $D \to E$ is clear, simply by setting:

$$E(k,l) = D(2k,2l)$$

Indeed, due to the axioms in Definition 4.11, the conditions in Definition 4.12 are satisfied. Conversely, given $E = (E(k,l))$ as in Definition 4.12, we can set:

$$D(k,l) = \begin{cases} E(k,l) & (k,l \text{ even}) \\ \{\sigma : |\sigma \in E(k+1,l+1)\} & (k,l \text{ odd}) \end{cases}$$

Our claim is that $D = (D(k,l))$ is a category of partitions. Indeed:

(1) The composition action is clear. Indeed, when looking at the numbers of legs involved, in the even case this is clear, and in the odd case, this follows from:

$$\begin{aligned} |\sigma, |\sigma' \in E \quad &\implies \quad |^{\sigma}_{\tau} \in E \\ &\implies \quad {}^{\sigma}_{\tau} \in D \end{aligned}$$

(2) For the tensor product axiom, we have 4 cases to be investigated, depending on the parity of the number of legs of σ, τ, as follows:

– The even/even case is clear.

– The odd/even case follows from the following computation:

$$\begin{aligned} |\sigma, \tau \in E \quad &\implies \quad |\sigma\tau \in E \\ &\implies \quad \sigma\tau \in D \end{aligned}$$

– Regarding now the even/odd case, this can be solved as follows:

$$\begin{aligned} \sigma, |\tau \in E \quad &\implies \quad |\sigma|, |\tau \in E \\ &\implies \quad |\sigma||\tau \in E \\ &\implies \quad |\sigma\tau \in E \\ &\implies \quad \sigma\tau \in D \end{aligned}$$

– As for the remaining odd/odd case, here the computation is as follows:

$$\begin{aligned} |\sigma, |\tau \in E \quad &\implies \quad ||\sigma|, |\tau \in E \\ &\implies \quad ||\sigma||\tau \in E \\ &\implies \quad \sigma\tau \in E \\ &\implies \quad \sigma\tau \in D \end{aligned}$$

(3) Finally, the conjugation axiom is clear from definitions. It is also clear that both compositions $D \to E \to D$ and $E \to D \to E$ are the identities, as claimed. As for the quantum group assertion, this is clear as well from definitions. □

Now back to uniqueness issues, we have here the following result, also from [12]:

Theorem 4.14. *We have the following results:*

(1) *O_N^* is the only intermediate easy quantum group $O_N \subset G \subset O_N^+$.*
(2) *PU_N is the only intermediate projectively easy quantum group $PO_N \subset G \subset PO_N^+$.*

Proof. The idea here is as follows:

(1) The assertion regarding $O_N \subset O_N^* \subset O_N^+$ is from [14], and this is something that we already know, explained in chapter 3.

(2) The assertion regarding $PO_N \subset PU_N \subset PO_N^+$ follows from the classification result in (1), and from the duality in Theorem 4.13. □

Summarizing, we have analogues of the various affine classification results, with the remark that everything becomes simpler in the projective setting.

Our next goal will be that of finding projective versions of the quantum isometry group results that we have in the affine setting. We use the following action formalism, which is quite similar to the affine action formalism introduced in chapter 2:

DEFINITION 4.15. *Consider a closed subgroup of the free orthogonal group, $G \subset O_N^+$, and a closed subset of the free real sphere, $X \subset S^{N-1}_{\mathbb{R},+}$.*

(1) *We write $G \curvearrowright X$ when we have a morphism of C^*-algebras, as follows:*

$$\Phi : C(X) \to C(X) \otimes C(G)$$

$$\Phi(z_i) = \sum_a z_a \otimes u_{ai}$$

(2) *We write $PG \curvearrowright PX$ when we have a morphism of C^*-algebras, as follows:*

$$\Phi : C(PX) \to C(PX) \otimes C(PG)$$

$$\Phi(z_i z_j) = \sum_a z_a z_b \otimes u_{ai} u_{bj}$$

Observe that the above morphisms Φ, if they exist, are automatically coaction maps. Observe also that an affine action $G \curvearrowright X$ produces a projective action $PG \curvearrowright PX$. Let us also mention that given an algebraic subset $X \subset S^{N-1}_{\mathbb{R},+}$, it is routine to prove that there exist indeed universal quantum groups $G \subset O_N^+$ acting as (1), and as in (2). We have the following result, from [11] and related papers, with respect to the above notions:

THEOREM 4.16. *The quantum isometry groups of basic spheres and projective spaces,*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \\
\downarrow & & \downarrow & & \downarrow \\
P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{+}
\end{array}$$

are the following affine and projective quantum groups,

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
\downarrow & & \downarrow & & \downarrow \\
PO_N & \longrightarrow & PU_N & \longrightarrow & PO_N^+
\end{array}$$

with respect to the affine and projective action notions introduced above.

PROOF. The fact that the 3 quantum groups on top act affinely on the corresponding 3 spheres is known since [11], and is elementary, explained before. By restriction, the 3 quantum groups on the bottom follow to act on the corresponding 3 projective spaces. We must prove now that all these actions are universal. At right there is nothing to prove, so we are left with studying the actions on $S^{N-1}_\mathbb{R},S^{N-1}_{\mathbb{R},*}$ and on $P^{N-1}_\mathbb{R},P^{N-1}_\mathbb{C}$.

$\underline{P^{N-1}_\mathbb{R}}$. Consider the following projective coordinates:

$$p_{ij}=z_iz_j\quad,\quad w_{ij,ab}=u_{ai}u_{bj}$$

In terms of these projective coordinates, the coaction map is given by:

$$\Phi(p_{ij})=\sum_{ab}p_{ab}\otimes w_{ij,ab}$$

Thus, we have the following formulae:

$$\begin{aligned}\Phi(p_{ij})&=\sum_{a<b}p_{ab}\otimes(w_{ij,ab}+w_{ij,ba})+\sum_a p_{aa}\otimes w_{ij,aa}\\ \Phi(p_{ji})&=\sum_{a<b}p_{ab}\otimes(w_{ji,ab}+w_{ji,ba})+\sum_a p_{aa}\otimes w_{ji,aa}\end{aligned}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{ab}=z_az_b$ for $a\leq b$, we conclude that we must have:

$$w_{ij,ab}+w_{ij,ba}=w_{ji,ab}+w_{ji,ba}$$

Let us apply now the antipode to this formula. For this purpose, observe that:

$$\begin{aligned}S(w_{ij,ab})&=S(u_{ai}u_{bj})\\ &=S(u_{bj})S(u_{ai})\\ &=u_{jb}u_{ia}\\ &=w_{ba,ji}\end{aligned}$$

Thus by applying the antipode we obtain:

$$w_{ba,ji}+w_{ab,ji}=w_{ba,ij}+w_{ab,ij}$$

By relabelling, we obtain the following formula:

$$w_{ji,ba}+w_{ij,ba}=w_{ji,ab}+w_{ij,ab}$$

Now by comparing with the original relation, we obtain:

$$w_{ij,ab}=w_{ji,ba}$$

But, with $w_{ij,ab}=u_{ai}u_{bj}$, this formula reads:

$$u_{ai}u_{bj}=u_{bj}u_{ai}$$

Thus $G\subset O_N$, and it follows that we have $PG\subset PO_N$, as claimed.

$\underline{P^{N-1}_\mathbb{C}}$. Consider a coaction map, written as follows, with $p_{ab}=z_a\bar{z}_b$:

$$\Phi(p_{ij})=\sum_{ab}p_{ab}\otimes u_{ai}u_{bj}$$

The idea here will be that of using the following formula:

$$p_{ab}p_{cd}=p_{ad}p_{cb}$$

We have the following formulae:

$$\begin{aligned}\Phi(p_{ij}p_{kl})&=\sum_{abcd}p_{ab}p_{cd}\otimes u_{ai}u_{bj}u_{ck}u_{dl}\\ \Phi(p_{il}p_{kj})&=\sum_{abcd}p_{ad}p_{cb}\otimes u_{ai}u_{dl}u_{ck}u_{bj}\end{aligned}$$

The terms at left being equal, and the last terms at right being equal too, we deduce that, with $[a,b,c]=abc-cba$, we must have the following formula:

$$\sum_{abcd}u_{ai}[u_{bj},u_{ck},u_{dl}]\otimes p_{ab}p_{cd}=0$$

Now since the quantities $p_{ab}p_{cd}=z_a\bar{z}_bz_c\bar{z}_d$ at right depend only on the numbers $|\{a,c\}|,|\{b,d\}|\in\{1,2\}$, and this dependence produces the only possible linear relations between the variables $p_{ab}p_{cd}$, we are led to $2\times 2=4$ equations, as follows:

(1) $u_{ai}[u_{bj},u_{ak},u_{bl}]=0$, $\forall a,b$.

(2) $u_{ai}[u_{bj},u_{ak},u_{dl}]+u_{ai}[u_{dj},u_{ak},u_{bl}]=0$, $\forall a$, $\forall b\neq d$.

(3) $u_{ai}[u_{bj},u_{ck},u_{bl}]+u_{ci}[u_{bj},u_{ak},u_{bl}]=0$, $\forall a\neq c$, $\forall b$.

(4) $u_{ai}[u_{bj},u_{ck},u_{dl}]+u_{ai}[u_{dj},u_{ck},u_{bl}]+u_{ci}[u_{bj},u_{ak},u_{dl}]+u_{ci}[u_{dj},u_{ak},u_{bl}]=0$, $\forall a\neq c,b\neq d$.

We will need in fact only the first two formulae. Since (1) corresponds to (2) at $b=d$, we conclude that (1,2) are equivalent to (2), with no restriction on the indices. By multiplying now this formula to the left by u_{ai}, and then summing over i, we obtain:

$$[u_{bj},u_{ak},u_{dl}]+[u_{dj},u_{ak},u_{bl}]=0$$

We use now the antipode/relabel trick from [11]. By applying the antipode we obtain:

$$[u_{ld},u_{ka},u_{jb}]+[u_{lb},u_{ka},u_{jd}]=0$$

By relabelling we obtain the following formula:

$$[u_{dl},u_{ak},u_{bj}]+[u_{dj},u_{ak},u_{bl}]=0$$

Now by comparing with the original relation, we obtain:

$$[u_{bj},u_{ak},u_{dl}]=[u_{dj},u_{ak},u_{bl}]=0$$

Thus $G\subset O_N^*$, and it follows that we have $PG\subset PU_N$, as desired. □

The above results can be probably improved. As an example, let us say that a closed subgroup $G\subset U_N^+$ acts projectively on PX when we have a coaction map as follows:

$$\Phi(z_iz_j)=\sum_{ab}z_az_b\otimes u_{ai}u_{bj}^*$$

The above proof can be adapted, by putting $*$ signs where needed, and Theorem 4.16 still holds, in this setting. However, establishing general universality results, involving arbitrary subgroups $H\subset PO_N^+$, looks like a quite non-trivial question.

4b. Grassmannians

In order to develop free projective geometry, a first piece of work is that of developing a theory of free Grassmannians, free flag manifolds, and free Stiefel manifolds. To be more precise, the definition of the free Grassmannians is straightforward, as follows, and the definition of the free flag manifolds and free Stiefel manifolds is very similar:

$$C(Gr_{LN}^+)=C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=L\right)$$

Most of the arguments from the affine case carry over in the projective setting. We will be back to this later, with nmore details, in Part II of the present book.

We would like to end this discussion with something refreshing, namely a preliminary study of the free analogue of $P^2_\mathbb{R}$. We recall that the projective space $P^{N-1}_\mathbb{R}$ is the space of lines in $\mathbb{R}^N$ passing through the origin, the basic examples being as follows:

(1) At $N=2$ each such a line, in $\mathbb{R}^2$ passing through the origin, corresponds to 2 opposite points on the unit circle $\mathbb{T}\subset\mathbb{R}^2$. Thus, $P^1_\mathbb{R}$ corresponds to the upper semicircle of $\mathbb{T}$, with the endpoints identified, and so we obtain a circle, $P^1_\mathbb{R}=\mathbb{T}$.

(2) At $N=3$ the situation is similar, with $P^2_\mathbb{R}$ corresponding to the upper hemisphere of the sphere $S^2_\mathbb{R}\subset\mathbb{R}^3$, with the points on the equator identified via $x=-x$. Topologically speaking, we can deform if we want the upper hemisphere into a square, with the equator becoming the boundary of this square, and in this picture, the $x=-x$ identification corresponds to the "identify opposite edges, with opposite orientations" folding method for the square, leading to a space $P^2_\mathbb{R}$ which is obviously not embeddable into $\mathbb{R}^3$.

In what follows we will be interested in the free analogue P^2_+ of this projective space $P^2_\mathbb{R}$. Our main motivation comes from the fact that, according to the work of Bhowmick-D'Andrea-Dabrowski [20], later on continued with Das [21], the quantum isometry group $PO_3^+=PU_3^+$ of the free projective space P^2_+ acts on the quark part of the Standard Model spectral triple, in Chamseddine-Connes formulation [26], [27].

We recall that the free projective space is defined by the following formula:

$$C(P^{N-1}_+)=C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p=p^*=p^2,Tr(p)=1\right)$$

Let us first discuss, as a warm-up, the 2D case. Here the above matrix of projective coordinates is as follows, with $a=a^*$, $b=b^*$, $a+b=1$:

$$p=\begin{pmatrix}a&c\\ c^*&b\end{pmatrix}$$

We have the following computation:

$$p^2=\begin{pmatrix}a&c\\ c^*&b\end{pmatrix}\begin{pmatrix}a&c\\ c^*&b\end{pmatrix}=\begin{pmatrix}a^2+cc^*&ac+cb\\ c^*a+bc^*&c^*c+b^2\end{pmatrix}$$

Thus, the equations to be satisfied are as follows:

$$a^2+cc^*=a$$

$$b^2+c^*c=b$$

$$ac+cb=c$$

$$c^*a+bc^*=c^*$$

The 4th equation is the conjugate of the 3rd equation, so we remove it. By using $a+b=1$, the remaining equations can be written as:

$$cc^*=c^*c=ab$$

$$ac+ca=0$$

We have several explicit models for this, using the spheres $S^1_{\mathbb R,+}$ and $S^1_{\mathbb C,+}$, as well as the first row spaces of O_2^+ and U_2^+, which ultimately lead us to SU_2 and $\bar{S}U_2$. These models are known to be all equivalent under Haar, and the question is whether they are identical. Thus, we must do computations as above in all models, and compare. These are all interesting questions, whose precise answers are not known, so far.

In the 3D case now, that of projective space P^2_+, that we are mainly interested in here, the matrix of coordinates is as follows, with r,s,t self-adjoint, $r+s+t=1$:

$$p=\begin{pmatrix}r&a&b\\ a^*&s&c\\ b^*&c^*&t\end{pmatrix}$$

The square of this matrix is given by:

$$p^2=\begin{pmatrix}r&a&b\\ a^*&s&c\\ b^*&c^*&t\end{pmatrix}\begin{pmatrix}r&a&b\\ a^*&s&c\\ b^*&c^*&t\end{pmatrix}$$

We obtain the following formula:

$$p^2=\begin{pmatrix}r^2+aa^*+bb^*&ra+as+bc^*&rb+ac+bt\\a^*r+sa^*+cb^*&a^*a+s^2+cc^*&a^*b+sc+ct\\b^*r+c^*a^*+tb^*&b^*a+c^*s+tc^*&b^*b+c^*c+t^2\end{pmatrix}$$

On the diagonal, the equations for $p^2=p$ are as follows:

$$aa^*+bb^*=r-r^2$$

$$a^*a+cc^*=s-s^2$$

$$b^*b+c^*c=t-t^2$$

On the off-diagonal upper part, the equations for $p^2=p$ are as follows:

$$ra+as+bc^*=a$$

$$rb+ac+bt=b$$

$$a^*b+sc+ct=c$$

On the off-diagonal lower part, the equations for $p^2=p$ are those above, conjugated. Thus, we have 6 equations. The first problem is that of using $r+s+t=1$, in order to make these equations look better. Again, many interesting questions here.

4c. Lifting questions

The are many interesting lifting questions, between affine and projective geometry, with all sorts of half-liberations involved when lifting, and also within affine geometry itself, in connection with the free analogue of the stereographic projection.

So, what is $\mathbb{R}^N_+$? There are several approaches to this problem, and in each case we are looking for a triple (A,Δ,h) consisting of an operator algebra A, typically a non-unital C^*-algebra, then a comultiplication Δ, understood to come accompanied by maps ε,S too, and then a Haar integration functional h. As a starting point, we have:

1. Products. Using $\mathbb{R}^N=(\mathbb{R})^N$. At the algebra level we have $C_0(\mathbb{R}^N)=C_0(\mathbb{R})^{\otimes N}$, and this suggests setting $C_0(\mathbb{R}^N_+)=C_0(\mathbb{R})^{*N}$. Thus we have a well-defined algebra A, and we have a comultiplication Δ too. The problem is with the Haar integration h. Our belief is that this problem can be solved by using suitable $N\times N$ matrix models, with our algebra A appearing on the diagonal. This looks quite tricky.

2. Polar coordinates. Using $[0,\infty)\times S^{N-1}_\mathbb{R}\to\mathbb{R}^N$. At the algebra level we have $C_0(\mathbb{R}^N)\subset C_0[0,\infty)\otimes C(S^{N-1}_\mathbb{R})$, and the very first question is that of understanding what the subalgebra $C_0(\mathbb{R}^N)$ exactly is. Since the quotient map $[0,\infty)\times S^{N-1}_\mathbb{R}\to\mathbb{R}^N$, given by $(r,x)\to rx$, has the property $0x=0y$ for any x,y, this suggests that $C_0(\mathbb{R}^N)\subset C_0[0,\infty)\otimes C(S^{N-1}_\mathbb{R})$ consists of functions such that $f(0,x)$ does not depend on x. It is not very clear what this means, algebrically. Once this difficulty solved, we can probably

go ahead and construct something similar in the free case, $C_0(\mathbb{R}^N_+) \subset C_0[0,\infty) * C(S^{N-1}_{\mathbb{R},+})$, then look for a comultiplication Δ, and a Haar functional h.

2b. An alternative approach here would be by using $\mathbb{R}^N - \{0\} = (0,\infty) \times S^{N-1}_{\mathbb{R}}$. Here we have at the algebra level $C_0(\mathbb{R}^N - \{0\}) = C_0(0,\infty) \otimes C(S^{N-1}_{\mathbb{R}})$, so at least we have a clearly defined algebra, that we can generalize right away to the free settting, in the form of something of type $C_0(\mathbb{R}^N_+ - \{0\}) = C_0(0,\infty) * C(S^{N-1}_{\mathbb{R},+})$. However, we cannot really investigate the Δ problem in this setting, so we run once again into a difficulty, namely constructing the correct lifts $C_0(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N_+)$. This being said, the question of investigating the Haar functional h seems to make sense, even in this "$-\{0\}$" setting, meaning without solving the lifting problem. This is actually quite unclear.

2c. Yet another alternative approach would be by using $P^{N-1}_{\mathbb{R}}$ instead of $S^{N-1}_{\mathbb{R}}$. The first question here is that of understanding the precise relation between the spaces $\mathbb{R} \times P^{N-1}_{\mathbb{R}}$ and $\mathbb{R}^N$, which is probably something well-known, but looks quite geometric and tricky. Assuming this geometric problem solved, we can probably have $C_0(\mathbb{R}^N)$ constructed afterwards in terms of $C_0(\mathbb{R}) \otimes C(P^{N-1}_{\mathbb{R}})$, and then at the free level, we can have $C_0(\mathbb{R}^N_+)$ constructed in terms of $C_0(\mathbb{R}) * C(P^{N-1}_{\mathbb{R},+})$, and then look for Δ, and for h.

2d. In fact, in modern terms, we are looking for a "free suspension of the free sphere".

3. Compactification. Using $\mathbb{R}^N = S^N_{\mathbb{R}} - \{\infty\}$. To be more precise, we want to use the fact that $S^N_{\mathbb{R}}$ appears as the 1-point compactification of $\mathbb{R}^N$, with the isomorphism being the standard stereographic projection map. This might look like a weird idea, because it is not group-theoretical at all, the main feature of the stereographic projection being the fact that it is conformal, preserving angles, and so useful in geometry, but not in group theory. This being said, this is an idea to be explored too, especially since the formula for h should be not that complicated, and here are some preliminary computations:

Let us start with some abstract considerations. The 1-point compactification of $\mathbb{R}^N$ is indeed the sphere $S^N_{\mathbb{R}}$, and for precise formulae and everything, to be given later, the best is to say that the 1-point compactification of $\mathbb{R}^N = \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ is the unit sphere $S^N_{\mathbb{R}} \subset \mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty = (1, 0, \ldots, 0)$. Also, we make the convention that the coordinates on $\mathbb{R}^{N+1}$ are denoted $x_0, \ldots, x_N$.

In functional analysis terms, we have a diagram as follows, with all horizontal maps being inclusions, with the bar on $C_0(\mathbb{R}^N)$ standing for unitization, and with the 0 subscript to $C(S^N_\mathbb{R})$ standing for taking the ideal generated by the first coordinate x_0:

$$\begin{array}{ccccc}
C_0(\mathbb{R}^N) & \longrightarrow & \bar{C}_0(\mathbb{R}^N) & \longrightarrow & C_b(\mathbb{R}^N)\\
\| & & \| & & \\
C(S^N_\mathbb{R})_0 & \longrightarrow & C(S^N_\mathbb{R}) & &
\end{array}$$

In view of our motivations, this is not bad, because in the free case we can normally talk as well about the ideal $C(S^N_{\mathbb{R},+})_0 \subset C(S^N_{\mathbb{R},+})$ generated by the first coordinate x_0. The problem is whether we can declare this ideal to be $C_0(\mathbb{R}^N_+)$, with a Δ and h.

In order to comment on this, let us do some computations, in the classical case. We first need the precise formulae of the isomorphism $\mathbb{R}^N \simeq S^N_\mathbb{R} - \{\infty\}$, obtained in practice by identifying $\mathbb{R}^N = \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ with the unit sphere $S^N_\mathbb{R} \subset \mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty = (1, 0, \ldots, 0)$, via the stereographic projection. That is, we need the precise formulae of two inverse maps, as follows:

$$\Phi : \mathbb{R}^N \to S^N_\mathbb{R} - \{\infty\}$$

$$\Psi : S^N_\mathbb{R} - \{\infty\} \to \mathbb{R}^N$$

In one sense we must have $\Phi(v) = t(0, v) + (1-t)(1, 0)$, with $t \in (0, 1)$ being such that $||\Phi(v)|| = 1$. The equation here is $(1-t)^2 + t^2||v||^2 = 1$, which simplifies to $t^2(1+||v||^2) = 2t$, with solution $t = \frac{2}{1+||v||^2}$, and so the formula of Φ is as follows:

$$\Phi(v) = (1, 0) + \frac{2}{1 + ||v||^2}(-1, v)$$

In the other sense we must have $(0, \Psi(c, x)) = \alpha(c, x) + (1 - \alpha)(1, 0)$ for a certain $\alpha \in \mathbb{R}$, and from $\alpha c + 1 - \alpha = 0$ we get $\alpha = \frac{1}{1-c}$, so the formula of Ψ is as follows:

$$\Psi(c, x) = \frac{x}{1 - c}$$

Here, as before, and in what follows too, we use $\mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R}^N$, with the coordinate of $\mathbb{R}$ denoted x_0, and with the coordinates of $\mathbb{R}^N$ denoted $x_1, \ldots, x_N$.

Let us discuss now the Δ problematics. We can transport the group structure of $\mathbb{R}^N$ to a group structure on $S^N_\mathbb{R} - \{\infty\}$, as follows:

$$p \cdot q = \Phi(\Psi(p) + \Psi(q))$$

In view of the above formulae of Φ,Ψ, the multiplication on $S^N_\mathbb{R}-\{\infty\}$ that we obtain is given by the following formula:

$$\begin{aligned}(c,x)\cdot(d,y) &= \Phi(\Psi(c,x)+\Psi(d,y))\\ &= \Phi\left(\frac{x}{1-c}+\frac{y}{1-d}\right)\\ &= (1,0)+\frac{2}{1+t}\left(-1,\frac{x}{1-c}+\frac{y}{1-d}\right)\end{aligned}$$

Here the parameter t is given by the following formula:

$$t=\left|\left|\frac{x}{1-c}+\frac{y}{1-d}\right|\right|^2$$

Now by transposing, we obtain a comultiplication map as follows, with $C(S^N_\mathbb{R})_0\subset C(S^N_\mathbb{R})$ being the ideal generated by the first coordinate x_0:

$$\Delta:C(S^N_\mathbb{R})_0\to C(S^N_\mathbb{R})_0\otimes C(S^N_\mathbb{R})_0$$

$$f\to\Big[(c,x),(d,y)\to f((c,x)\cdot(d,y))\Big]$$

The problem is that of slowly working out the details of this map Δ, on various products of coordinates and so on, and see if we can get a decent formula for Δ out of this, and then if this formula has a free generalization or not.

Let us discuss now the Haar problematics, which is the point where we wanted to get, where things might get simpler. As before with Δ, we can transport the Haar integration over $\mathbb{R}^N$ into an integration over $S^N_\mathbb{R}-\{\infty\}$, according to the following formula:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{\mathbb{R}^N}f(\Phi(v))dv$$

In practice, according to the above formula of Φ, the precise formula is:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{\mathbb{R}^N}f\left((1,0)+\frac{2}{1+||v||^2}(-1,v)\right)dv$$

Passed the details of this formula, which might look quite complicated, the transport of the Haar integration over $\mathbb{R}^N$ into an integration over $S^N_\mathbb{R}-\{\infty\}$ looks like something quite simple. Indeed, the measure on $S^N_\mathbb{R}-\{\infty\}$ should not be that far from the usual Haar measure of $S^N_\mathbb{R}$, with just a density added on the x_0 direction, and this because both measures, the transported one on $S^N_\mathbb{R}-\{\infty\}$, and the Haar one on $S^N_\mathbb{R}$, are invariant under the action of O_N, acting on the coordinates $x_1,\ldots,x_N$.

In short, we should have a formula as follows, with on the right the integration being the usual Haar one on $S^N_\mathbb{R}$, and with $\varphi:[-1,1]\to(0,\infty)$ being a certain density:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{S^N_\mathbb{R}}f(x)\varphi(x_0)dx$$

Assuming all this understood, and φ explicitely computed, the extension to the free case would be probably quite routine, our conjecture being that the integration on $\mathbb{R}^N_+$, in a "free stereographic picture", should be just a modification of the usual Weingarten formula for $S^N_{\mathbb{R},+}$, via a horizontal density $\psi:[-1,1]\to(0,\infty)$, appearing as the free version of $\varphi:[-1,1]\to(0,\infty)$, in the sense of the Bercovici-Pata bijection.

4d. Sums of squares

Another way of "escaping" from spheres, in the affine setting, is via various sums of squares, chosen to be more complicated than those defining the spheres. In order to discuss this, let us first study the compact hypersurfaces $X\subset\mathbb{R}^N_+$. These hypersurfaces fit into the C^*-algebra formalism, their definition being as follows:

Definition 4.17. *A real compact hypersurface in N variables, denoted $X_f\subset\mathbb{R}^N_+$, is the abstract spectrum of a universal C^*-algebra of the following type,*

$$C(X_f)=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,f(x_1,\ldots,x_N)=0\right)$$

with the noncommutative polynomial $f\in\mathbb{R}<x_1,\ldots,x_N>$ being such the maximal C^*-norm on the complex $*$-algebra $\mathbb{C}<x_1,\ldots,x_N>/(f)$ is bounded.

As a first result here, coming from the Gelfand theorem, we have:

Theorem 4.18. *In order for X_f to exist, the real algebraic manifold*

$$X_f^\times=\left\{x\in\mathbb{R}^N\Big|f(x_1,\ldots,x_N)=0\right\}$$

must be compact. In addition, in this case we have $||x_i||_\times\leq||x_i||$, for any i.

Proof. Assuming that X_f exists, the Gelfand theorem shows that the algebra of continuous functions on the manifold $X_f^\times$ in the statement appears as:

$$C(X_f^\times)=C(X_f)\Big/\Big\langle[x_i,x_j]=0\Big\rangle$$

Thus we have an embedding of compact quantum spaces $X_f^\times\subset X_f$, and the norm estimate is clear as well, because such embeddings increase the norms. □

Let us first discuss the quadratic case. We have here:

THEOREM 4.19. *Given a quadratic polynomial $f \in \mathbb{R} < x_1, \ldots, x_N >$, written as*

$$f = \sum_{ij} A_{ij} x_i x_j + \sum_i B_i x_i + C$$

the following conditions are equivalent:

(1) X_f *exists.*
(2) $X_f^\times$ *is compact.*
(3) *The symmetric matrix* $Q = \frac{A+A^t}{2}$ *is positive or negative.*

PROOF. The implication (1) $\implies$ (2) being known from Theorem 4.18, and the implication (2) $\iff$ (3) being well-known, we are left with proving (3) $\implies$ (1). As a first remark here, by applying the adjoint, our manifold X_f is defined by:

$$\begin{cases}\sum_{ij} A_{ij} x_i x_j + \sum_i B_i x_i + C = 0\\ \sum_{ij} A_{ij} x_j x_i + \sum_i B_i x_i + C = 0\end{cases}$$

In terms of $P = \frac{A-A^t}{2}$ and $Q = \frac{A+A^t}{2}$, these equations can be written as:

$$\begin{cases}\sum_{ij} P_{ij} x_i x_j = 0\\ \sum_{ij} Q_{ij} x_i x_j + \sum_i B_i x_i + C = 0\end{cases}$$

Let us first examine the second equation. When regarding x as a column vector, and B as a row vector, this equation becomes an equality of 1×1 matrices, as follows:

$$x^t Q x + Bx + C = 0$$

Now let us assume that Q is positive or negative. Up to a sign change, we can assume $Q > 0$. We can write $Q = UDU^t$, with $D = diag(d_i)$ and $d_i > 0$, and with $U \in O_N$. In terms of the vector $y = U^t x$, and with $E = BU$, our equation becomes:

$$y^t D y + Ey + C = 0$$

By reverting back to sums and indices, this equation reads:

$$\sum_i d_i y_i^2 + \sum_i e_i y_i + C = 0$$

Now by making squares, this equation takes the following form:

$$\sum_i d_i \left(y_i + \frac{e_i}{2d_i} \right)^2 = c$$

By positivity, we deduce that we have the following estimate:

$$\left\| y_i + \frac{e_i}{2d_i} \right\|^2 \leq \frac{|c|}{d_i}$$

Thus our hypersurface X_f is well-defined, and we are done. □

We have in fact the following result:

THEOREM 4.20. *Up to linear changes of coordinates, the free compact quadrics in* $\mathbb{R}^N_+$ *are the empty set, the point, the standard free sphere* $S^{N-1}_{\mathbb{R},+}$*, defined by*

$$\sum_i x_i^2 = 1$$

and some intermediate spheres $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$*, which can be explicitly characterized. Moreover, for all these free quadrics, we have* $||x_i|| = ||x_i||_\times$*, for any* i.

PROOF. We use the computations from the proof of Theorem 4.19. The first equation there, making appear the matrix $P = \frac{A-A^t}{2}$, is as follows:

$$\sum_{ij} P_{ij} x_i x_j = 0$$

As for the second equation, up to a linear change of the coordinates, this reads:

$$\sum_i z_i^2 = c$$

At $c < 0$ we obtain the empty set. At $c = 0$ we must have $z = 0$, and depending on whether the first equation is satisfied or not, we obtain either a point, or the empty set. At $c > 0$ now, we can assume by rescaling $c = 1$, and our second equation reads:

$$X_f \subset S^{N-1}_{\mathbb{R},+}$$

As a conclusion, the solutions here are certain subspaces $S \subset S^{N-1}_{\mathbb{R},+}$ which appear via equations of type $\sum_{ij} P_{ij} x_i x_j = 0$, with $P \in M_N(\mathbb{R})$ being antisymmetric, and with $x_1, \ldots, x_N$ appearing via $z_1, \ldots, z_N$ via a linear change of variables. Now observe that when redoing the above computation with $X_f^\times$ at the place of X_f, we obtain $X_f = S^{N-1}_{\mathbb{R}}$, and this, because the equations $\sum_{ij} P_{ij} x_i x_j = 0$ are trivial for commuting variables. We conclude that our subspaces $S \subset S^{N-1}_{\mathbb{R},+}$ must satisfy:

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$$

Thus, we are left with investigating which such subspaces can indeed be solutions. Observe that both the extreme cases can appear as solutions, as shown by:

$$\begin{aligned} X_{2x^2+y^2+\frac{3}{2}xy+\frac{1}{2}yx} &= S^1_{\mathbb{R}} \\ X_{2x^2+y^2+xy+yx} &= S^1_{\mathbb{R},+} \end{aligned}$$

Finally, the last assertion is clear for the empty set and for the point, and for the remaining hypersurfaces, this follows from $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$. □

Here is now yet another version of Theorem 4.19:

THEOREM 4.21. *Given M real linear functions $L_1,\ldots,L_M$ in N noncommuting variables $x_1,\ldots,x_N$, the following are equivalent:*

(1) *$\sum_k L_k(x_1,\ldots,x_N)^2=1$ defines a compact hypersurface in $\mathbb R^N$.*
(2) *$\sum_k L_k(x_1,\ldots,x_N)^2=1$ defines a compact quantum hypersurface.*
(3) *The matrix formed by the coefficients of $L_1,\ldots,L_M$ has rank N.*

PROOF. The equivalence (1) $\iff$ (2) follows from the equivalence (1) $\iff$ (2) in Theorem 4.19, because the surfaces under investigation are quadrics. As for the equivalence (2) $\iff$ (3), this is well-known. More precisely, our equation read:

$$\begin{aligned}1&=\sum_k L_k(x_1,\ldots,x_N)^2\\&=\sum_k\sum_i L_{ki}x_i\sum_j L_{kj}x_j\\&=\sum_{ij}(L^tL)_{ij}x_ix_j\end{aligned}$$

Thus, in the context of Theorem 4.19, the underlying square matrix $A\in M_N(\mathbb R)$ is given by $A=L^tL$. It follows that we have $Q=A=L^tL$, and so the condition $Q>0$ is equivalent to L^tL being invertible, and so to L to have rank N, as claimed. □

In order to construct more examples, we will need the following basic fact:

PROPOSITION 4.22. *In a C^*-algebra we have*

$$||x||^p\leq 1\implies ||x||\leq 1$$

for any self-adjoint element x.

PROOF. With $n\in\mathbb N$ being such that $2^n\geq p$, we have:

$$||x||^{2^n}=||x^2||^{2^{n-1}}=\ldots=||x^{2^n}||\leq||x^p||\cdot||x^{2^n-p}||\leq 1\cdot||x||^{2^n-p}$$

Thus, we obtain $||x||^p\leq 1$, and so $||x||\leq 1$, as desired. □

As an application, we have the following construction:

PROPOSITION 4.23. *Given integers $p_i\in\mathbb N$, the equation*

$$\sum_i x_i^{2p_i}=1$$

defines a noncommutative hypersurface.

PROOF. This follows indeed from Proposition 4.22, by positivity. □

More generally, we have the following result, covering our various examples, so far:

THEOREM 4.24. *Given M real linear functions $L_1,\ldots,L_M$ in N noncommuting variables $x_1,\ldots,x_N$, and exponents $p_1,\ldots,p_M\in\mathbb N$, the equation*

$$\sum_k L_k(x_1,\ldots,x_N)^{2p_i}=1$$

defines a quantum hypersurface, provided that the $M\times N$ matrix formed by the coefficients of $L_1,\ldots,L_M$ has rank N.

PROOF. By positivity, imposing the above equation leads to:

$$||L_k(x_1,\ldots,x_N)||\leq 1\quad,\quad\forall k$$

We are therefore left with the problem of uniformly bounding the norms $||x_i||$, and normally we can proceed here exactly as in the classical case. □

More generally now, we have the following result:

THEOREM 4.25. *General construction of hypersurfaces, via equations of type*

$$\sum_k L_kL_k^*=1$$

with $L_k\in\mathbb R<x_1,\ldots,x_N>$, improving the construction from Theorem 4.24.

PROOF. This does not look obvious at all. As usual, there are some norm estimates to be worked out too, in relation with the basic inequality $||x_i||_\times\leq||x_i||$. □

Going beyond the above looks like a non-trivial question.

4e. Exercises

Exercises:

EXERCISE 4.26.

EXERCISE 4.27.

EXERCISE 4.28.

EXERCISE 4.29.

EXERCISE 4.30.

EXERCISE 4.31.

Bonus exercise.

Part II

Free manifolds

In the midnight hour
She cried more, more, more
With a rebel yell
She cried more, more, more

CHAPTER 5

Free manifolds

5a. Quotient spaces

Let us begin with some generalities regarding the quotient spaces, and more general homogeneous spaces. Regarding the quotients, we have the following construction:

PROPOSITION 5.1. *Given a quantum subgroup $H\subset G$, with associated quotient map $\rho:C(G)\to C(H)$, if we define the quotient space $X=G/H$ by setting*

$$C(X)=\left\{f\in C(G)\Big|(\rho\otimes id)\Delta f=1\otimes f\right\}$$

then we have a coaction map as follows,

$$\Phi:C(X)\to C(X)\otimes C(G)$$

obtained as the restriction of the comultiplication of $C(G)$. In the classical case, we obtain in this way the usual quotient space $X=G/H$.

PROOF. Observe that the linear subspace $C(X)\subset C(G)$ defined in the statement is indeed a subalgebra, because it is defined via a relation of type $\varphi(f)=\psi(f)$, with both φ,ψ being morphisms of algebras. Observe also that in the classical case we obtain the algebra of continuous functions on the quotient space $X=G/H$, because:

$$\begin{aligned}(\rho\otimes id)\Delta f=1\otimes f&\iff (\rho\otimes id)\Delta f(h,g)=(1\otimes f)(h,g),\forall h\in H,\forall g\in G\\&\iff f(hg)=f(g),\forall h\in H,\forall g\in G\\&\iff f(hg)=f(kg),\forall h,k\in H,\forall g\in G\end{aligned}$$

Regarding now the construction of Φ, observe that for $f\in C(X)$ we have:

$$\begin{aligned}(\rho\otimes id\otimes id)(\Delta\otimes id)\Delta f&=(\rho\otimes id\otimes id)(id\otimes\Delta)\Delta f\\&=(id\otimes\Delta)(\rho\otimes id)\Delta f\\&=(id\otimes\Delta)(1\otimes f)\\&=1\otimes\Delta f\end{aligned}$$

Thus the condition $f\in C(X)$ implies $\Delta f\in C(X)\otimes C(G)$, and this gives the existence of Φ. Finally, the other assertions are all clear. □

As an illustration, in the group dual case we have:

Proposition 5.2. *Assume that $G=\widehat{\Gamma}$ is a discrete group dual.*

1. *The quantum subgroups of G are $H=\widehat{\Lambda}$, with $\Gamma\to\Lambda$ being a quotient group.*
2. *For such a quantum subgroup $\widehat{\Lambda}\subset\widehat{\Gamma}$, we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, where:*

$$\Theta=\ker(\Gamma\to\Lambda)$$

Proof. This is well-known, the idea being as follows:

(1) In one sense, this is clear. Conversely, since the algebra $C(G)=C^*(\Gamma)$ is cocommutative, so are all its quotients, and this gives the result.

(2) Consider a quotient map $r:\Gamma\to\Lambda$, and denote by $\rho:C^*(\Gamma)\to C^*(\Lambda)$ its extension. Consider a group algebra element, written as follows:

$$f=\sum_{g\in\Gamma}\lambda_g\cdot g\in C^*(\Gamma)$$

We have then the following computation:

$$\begin{aligned}
f\in C(\widehat{\Gamma}/\widehat{\Lambda})&\iff(\rho\otimes id)\Delta(f)=1\otimes f\\
&\iff\sum_{g\in\Gamma}\lambda_g\cdot r(g)\otimes g=\sum_{g\in\Gamma}\lambda_g\cdot 1\otimes g\\
&\iff\lambda_g\cdot r(g)=\lambda_g\cdot 1,\forall g\in\Gamma\\
&\iff supp(f)\subset\ker(r)
\end{aligned}$$

But this means that we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, with $\Theta=\ker(\Gamma\to\Lambda)$, as claimed. □

Given two compact quantum spaces X,Y, we say that X is a quotient space of Y when we have an embedding of C^*-algebras $\alpha:C(X)\subset C(Y)$. We have:

Definition 5.3. *We call a quotient space $G\to X$ homogeneous when*

$$\Delta(C(X))\subset C(X)\otimes C(G)$$

where $\Delta:C(G)\to C(G)\otimes C(G)$ is the comultiplication map.

In other words, an homogeneous quotient space $G\to X$ is a quantum space coming from a subalgebra $C(X)\subset C(G)$, which is stable under the comultiplication. The relation with the quotient spaces from Proposition 5.1 is as follows:

Theorem 5.4. *The following results hold:*

1. *The quotient spaces $X=G/H$ are homogeneous.*
2. *In the classical case, any homogeneous space is of type G/H.*
3. *In general, there are homogeneous spaces which are not of type G/H.*

Proof. Once again these results are well-known, the proof being as follows:

(1) This is clear from Proposition 5.1.

(2) Consider a quotient map $p : G \to X$. The invariance condition in the statement tells us that we must have an action $G \curvearrowright X$, given by:

$$g(p(g')) = p(gg')$$

Thus, we have the following implication:

$$p(g') = p(g'') \implies p(gg') = p(gg''),\ \forall g \in G$$

Now observe that the following subset $H \subset G$ is a subgroup:

$$H = \left\{ g \in G \Big| p(g) = p(1) \right\}$$

Indeed, $g, h \in H$ implies that we have:

$$p(gh) = p(g) = p(1)$$

Thus we have $gh \in H$, and the other axioms are satisfied as well. Our claim now is that we have an identification $X = G/H$, obtained as follows:

$$p(g) \to Hg$$

Indeed, the map $p(g) \to Hg$ is well-defined and bijective, because $p(g) = p(g')$ is equivalent to $p(g^{-1}g') = p(1)$, and so to $Hg = Hg'$, as desired.

(3) Given a discrete group Γ and an arbitrary subgroup $\Theta \subset \Gamma$, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is homogeneous. Now by using Proposition 5.2, we can see that if $\Theta \subset \Gamma$ is not normal, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is not of the form G/H. $\square$

With the above formalism in hand, let us try now to understand the general properties of the homogeneous spaces $G \to X$, in the sense of Theorem 5.4. We have:

Proposition 5.5. *Assume that a quotient space $G \to X$ is homogeneous.*

(1) *We have a coaction map as follows, obtained as restriction of Δ:*

$$\Phi : C(X) \to C(X) \otimes C(G)$$

(2) *We have the following implication:*

$$\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$$

(3) *We have as well the following formula:*

$$\left(id \otimes \int_G \right) \Phi f = \int_G f$$

(4) *The restriction of $\int_G$ is the unique unital form satisfying:*

$$(\tau \otimes id)\Phi = \tau(.)1$$

PROOF. These results are all elementary, the proof being as follows:

(1) This is clear from definitions, because Δ itself is a coaction.

(2) Assume that $f \in C(G)$ satisfies $\Delta(f) = f \otimes 1$. By applying the counit we obtain:

$$(\varepsilon \otimes id)\Delta f = (\varepsilon \otimes id)(f \otimes 1)$$

We conclude from this that we have $f = \varepsilon(f)1$, as desired.

(3) The formula in the statement, $(id \otimes \int_G)\Phi f = \int_G f$, follows indeed from the left invariance property of the Haar functional of $C(G)$, namely:

$$\left(id \otimes \int_G\right)\Delta f = \int_G f$$

(4) We use here the right invariance of the Haar functional of $C(G)$, namely:

$$\left(\int_G \otimes id\right)\Delta f = \int_G f$$

Indeed, we obtain from this that $tr = (\int_G)_{|C(X)}$ is G-invariant, in the sense that:

$$(tr \otimes id)\Phi f = tr(f)1$$

Conversely, assuming that $\tau : C(X) \to \mathbb{C}$ satisfies $(\tau \otimes id)\Phi f = \tau(f)1$, we have:

$$\begin{aligned}\left(\tau \otimes \int_G\right)\Phi(f) &= \int_G (\tau \otimes id)\Phi(f)\\ &= \int_G (\tau(f)1)\\ &= \tau(f)\end{aligned}$$

On the other hand, we can compute the same quantity as follows:

$$\begin{aligned}\left(\tau \otimes \int_G\right)\Phi(f) &= \tau\left(id \otimes \int_G\right)\Phi(f)\\ &= \tau(tr(f)1)\\ &= tr(f)\end{aligned}$$

Thus we have $\tau(f) = tr(f)$ for any $f \in C(X)$, and this finishes the proof. □

Summarizing, we have a notion of noncommutative homogeneous space, which perfectly covers the classical case. In general, however, the group dual case shows that our formalism is more general than that of the quotient spaces G/H.

We discuss now an extra issue, of analytic nature. The point indeed is that for one of the most basic examples of actions, namely $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, the associated morphism $\alpha : C(X) \to C(G)$ is not injective. The same is true for other basic actions, in the free setting. In order to include such examples, we must relax our axioms:

Definition 5.6. *An extended homogeneous space over a compact quantum group G consists of a morphism of C^*-algebras, and a coaction map, as follows,*

$$\alpha : C(X) \to C(G)$$
$$\Phi : C(X) \to C(X) \otimes C(G)$$

such that the following diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \Phi\ } & C(X) \otimes C(G) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha \otimes id} \\ C(G) & \xrightarrow{\ \Delta\ } & C(G) \otimes C(G) \end{array}$$

and such that the following diagram commutes as well,

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \Phi\ } & C(X) \otimes C(G) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle id \otimes \int} \\ C(G) & \xrightarrow{\ \int(.)1\ } & C(X) \end{array}$$

where $\int$ is the Haar integration over G. We write then $G \to X$.

As a first observation, when the morphism α is injective we obtain an homogeneous space in the previous sense. The examples with α not injective, which motivate the above formalism, include the standard action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, and the standard action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$. Here are a few general remarks on the above axioms:

Proposition 5.7. *Assume that we have morphisms of C^*-algebras*

$$\alpha : C(X) \to C(G)$$
$$\Phi : C(X) \to C(X) \otimes C(G)$$

satisfying the coassociativity condition $(\alpha \otimes id)\Phi = \Delta\alpha$.

1. *If α is injective on a dense $*$-subalgebra $A \subset C(X)$, and $\Phi(A) \subset A \otimes C(G)$, then Φ is automatically a coaction map, and is unique.*
2. *The ergodicity type condition $(id \otimes \int)\Phi = \int \alpha(.)1$ is equivalent to the existence of a linear form $\lambda : C(X) \to \mathbb{C}$ such that $(id \otimes \int)\Phi = \lambda(.)1$.*

Proof. This is something elementary, the idea being as follows:

(1) Assuming that we have a dense $*$-subalgebra $A \subset C(X)$ as in the statement, satisiying $\Phi(A) \subset A \otimes C(G)$, the restriction $\Phi_{|A}$ is given by:

$$\Phi_{|A} = (\alpha_{|A} \otimes id)^{-1}\Delta\alpha_{|A}$$

This restriction and is therefore coassociative, and unique. By continuity, the morphism Φ itself follows to be coassociative and unique, as desired.

(2) Assuming $(id\otimes\int)\Phi=\lambda(.)1$, we have:

$$\left(\alpha\otimes\int\right)\Phi=\lambda(.)1$$

On the other hand, we have as well the following formula:

$$\left(\alpha\otimes\int\right)\Phi=\left(id\otimes\int\right)\Delta\alpha=\int\alpha(.)1$$

Thus we obtain $\lambda=\int\alpha$, as claimed. □

Given an extended homogeneous space $G\to X$ in our sense, with associated map $\alpha:C(X)\to C(G)$, we can consider the image of this latter map:

$$\alpha:C(X)\to C(Y)\subset C(G)$$

Equivalently, at the level of the associated noncommutative spaces, we can factorize the corresponding quotient map $G\to Y\subset X$. With these conventions, we have:

PROPOSITION 5.8. *Consider an extended homogeneous space $G\to X$.*

(1) $\Phi(f)=f\otimes 1\implies f\in\mathbb{C}1$.
(2) *$tr=\int\alpha$ is the unique unital G-invariant form on $C(X)$.*
(3) *The image space obtained by factorizing, $G\to Y$, is homogeneous.*

PROOF. We have several assertions to be proved, the idea being as follows:

(1) This follows indeed from $(id\otimes\int)\Phi(f)=\int\alpha(f)1$, which gives $f=\int\alpha(f)1$.

(2) The fact that $tr=\int\alpha$ is indeed G-invariant can be checked as follows:

$$\begin{aligned}(tr\otimes id)\Phi f&=(\textstyle\int\alpha\otimes id)\Phi f\\&=(\textstyle\int\otimes id)\Delta\alpha f\\&=\textstyle\int\alpha(f)1\\&=tr(f)1\end{aligned}$$

As for the uniqueness assertion, this follows as before.

(3) The condition $(\alpha\otimes id)\Phi=\Delta\alpha$, together with the fact that i is injective, allows us to factorize Δ into a morphism Ψ, as follows:

$$\begin{array}{ccc}
C(X) & \xrightarrow{\ \ \Phi\ \ } & C(X)\otimes C(G)\\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id}\\
C(Y) & \xrightarrow{\ \ \Psi\ \ } & C(Y)\otimes C(G)\\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i\otimes id}\\
C(G) & \xrightarrow{\ \ \Delta\ \ } & C(G)\otimes C(G)
\end{array}$$

Thus the image space $G\to Y$ is indeed homogeneous, and we are done. □

Finally, we have the following result:

THEOREM 5.9. *Let $G\to X$ be an extended homogeneous space, and construct quotients $X\to X'$, $G\to G'$ by performing the GNS construction with respect to $\int\alpha,\int$. Then α factorizes into an inclusion $\alpha':C(X')\to C(G')$, and we have an homogeneous space.*

PROOF. We factorize $G\to Y\subset X$ as above. By performing the GNS construction with respect to $\int i\alpha,\int i,\int$, we obtain a diagram as follows:

$$\begin{array}{ccccc}
C(X) & \xrightarrow{\ \ p\ \ } & C(X') & & \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} & \searrow{\scriptstyle tr'} & \\
C(Y) & \xrightarrow{\ \ q\ \ } & C(Y') & & \mathbb C\\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} & \nearrow{\scriptstyle\int'} & \\
C(G) & \xrightarrow{\ \ r\ \ } & C(G') & &
\end{array}$$

Indeed, with $tr=\int\alpha$, the GNS quotient maps p,q,r are defined respectively by:

$$\begin{aligned}
\ker p &= \left\{f\in C(X)\Big|tr(f^*f)=0\right\}\\
\ker q &= \left\{f\in C(Y)\Big|\textstyle\int(f^*f)=0\right\}\\
\ker r &= \left\{f\in C(G)\Big|\textstyle\int(f^*f)=0\right\}
\end{aligned}$$

Next, we can define factorizations i', α' as above. Observe that i' is injective, and that α' is surjective. Our claim now is that α' is injective as well. Indeed:

$$\begin{aligned}
\alpha' p(f) = 0 &\implies q\alpha(f) = 0\\
&\implies \int \alpha(f^*f) = 0\\
&\implies tr(f^*f) = 0\\
&\implies p(f) = 0
\end{aligned}$$

We conclude that we have $X' = Y'$, and this gives the result. □

5b. Partial isometries

Our task now will be that of finding a suitable collection of "free homogeneous spaces", generalizing at the same time the free spheres S, and the free unitary groups U. This can be done at several levels of generality, and central here is the construction of the free spaces of partial isometries, which can be done in fact for any easy quantum group. In order to explain this, let us start with the classical case. We have here:

Definition 5.10. *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L \right\}$$

$$U_{MN}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L \right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L = M = N$ we obtain the groups O_N, U_N:

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of O_{1N}^1 are the isometries $T : E \to \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$, and so our results here are:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1} \quad , \quad U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of O_{1N}^1 are the isometries $T : \mathbb{R} \to F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely determined by $T(1) \in \mathbb{R}^M$, which must belong to $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$, and so our results here are:

$$O_{M1}^1 = S_{\mathbb{R}}^{M-1} \quad , \quad U_{M1}^1 = S_{\mathbb{C}}^{M-1}$$

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 5.11. *We have identifications of compact spaces*

$$O_{MN}^L\simeq\left\{U\in M_{M\times N}(\mathbb R)\Big|UU^t=\text{projection of trace }L\right\}$$

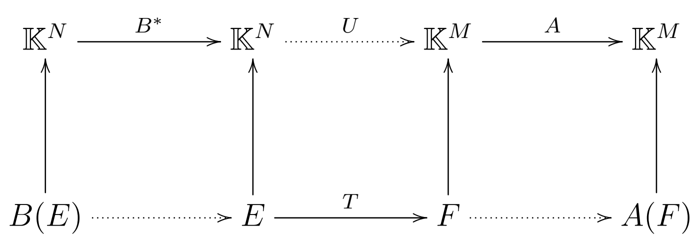
$$U_{MN}^L\simeq\left\{U\in M_{M\times N}(\mathbb C)\Big|UU^*=\text{projection of trace }L\right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T:E\to F$ with their corresponding extensions $U:\mathbb R^N\to\mathbb R^M$, $U:\mathbb C^N\to\mathbb C^M$, obtained by setting $U_{E^\perp}=0$. Then, we can identify these latter maps U with the corresponding rectangular matrices. $\square$

As an illustration, at $L=M=N$ we recover in this way the usual matrix description of O_N,U_N. Also, at $L=M=1$ we obtain the usual description of $S^{N-1}_{\mathbb R},S^{N-1}_{\mathbb C}$, as row spaces over the corresponding groups O_N,U_N. Finally, at $L=N=1$ we obtain the usual description of $S^{N-1}_{\mathbb R},S^{N-1}_{\mathbb C}$, as column spaces over the corresponding groups O_N,U_N.

Now back to the general case, observe that the isometries $T:E\to F$, or rather their extensions $U:\mathbb K^N\to\mathbb K^M$, with $\mathbb K=\mathbb R,\mathbb C$, obtained by setting $U_{E^\perp}=0$, can be composed with the isometries of $\mathbb K^M,\mathbb K^N$, according to the following scheme:



With the identifications in Proposition 5.11 made, the precise statement here is:

Proposition 5.12. *We have action maps as follows, which are both transitive,*

$$O_M\times O_N\curvearrowright O_{MN}^L\quad,\quad(A,B)U=AUB^t$$

$$U_M\times U_N\curvearrowright U_{MN}^L\quad,\quad(A,B)U=AUB^*$$

whose stabilizers are respectively $O_L\times O_{M-L}\times O_{N-L}$ *and* $U_L\times U_{M-L}\times U_{N-L}$.

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

Since $(A,B)\in G$ satisfy $AU=UB$, their components must be of the following form:

$$A=\begin{pmatrix}x&*\\0&a\end{pmatrix}\quad,\quad B=\begin{pmatrix}x&0\\ *&b\end{pmatrix}$$

Now since A,B are unitaries, these matrices follow to be block-diagonal, and so:

$$G=\left\{(A,B)\Big|A=\begin{pmatrix}x&0\\0&a\end{pmatrix},B=\begin{pmatrix}x&0\\0&b\end{pmatrix}\right\}$$

The stabilizer of U is parametrized by triples (x,a,b) belonging to $O_L\times O_{M-L}\times O_{N-L}$ and $U_L\times U_{M-L}\times U_{N-L}$, and we are led to the conclusion in the statement. □

Finally, let us work out the quotient space description of O_{MN}^L,U_{MN}^L. We have here:

THEOREM 5.13. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned}O_{MN}^L&=&(O_M\times O_N)/(O_L\times O_{M-L}\times O_{N-L})\\ U_{MN}^L&=&(U_M\times U_N)/(U_L\times U_{M-L}\times U_{N-L})\end{aligned}$$

with the quotient maps being given by $(A,B)\to AUB^*$, *where* $U=(\begin{smallmatrix}1&0\\0&0\end{smallmatrix})$.

PROOF. This is just a reformulation of Proposition 5.12, by taking into account the fact that the fixed point used in the proof there was $U=(\begin{smallmatrix}1&0\\0&0\end{smallmatrix})$. □

Once again, the basic examples here come from the cases $L=M=N$ and $L=M=1$. At $L=M=N$ the quotient spaces at right are respectively:

$$O_N\quad,\quad U_N$$

At $L=M=1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1}\quad,\quad U_N/U_{N-1}$$

In fact, in the general $L=M$ case we obtain the following spaces:

$$O_{MN}^M=O_N/O_{N-M}\quad,\quad U_{MN}^M=U_N/U_{N-M}$$

Similarly, the examples coming from the cases $L=M=N$ and $L=N=1$ are particular cases of the general $L=N$ case, where we obtain the following spaces:

$$O_{MN}^N=O_N/O_{M-N}\quad,\quad U_{MN}^N=U_N/U_{M-N}$$

Summarizing, we have here some basic homogeneous spaces, unifying the spheres with the rotation groups. The point now is that we can liberate these spaces, as follows:

DEFINITION 5.14. *Associated to any integers* $L\leq M,N$ *are the algebras*

$$\begin{aligned}C(O_{MN}^{L+})&=&C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace }L\right)\\ C(U_{MN}^{L+})&=&C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace }L\right)\end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij} u_{ij}u_{ij}^* = \sum_{ij} u_{ij}^*u_{ij} = L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+} \\ \uparrow & & \uparrow \\ O_{MN}^{L} & \longrightarrow & U_{MN}^{L} \end{array}$$

At the level of basic examples now, at $L=M=1$ and at $L=N=1$ we obtain the following diagrams, showing that our formalism covers indeed the free spheres:

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array} \qquad \begin{array}{ccc} S^{M-1}_{\mathbb{R},+} & \longrightarrow & S^{M-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{M-1}_{\mathbb{R}} & \longrightarrow & S^{M-1}_{\mathbb{C}} \end{array}$$

We have as well the following result, in relation with the free rotation groups:

Proposition 5.15. *At $L=M=N$ we obtain the diagram*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

consisting of the groups O_N, U_N, and their liberations.

Proof. We recall that the various quantum groups in the statement are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$\begin{aligned} C(O_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u}, uu^t=u^tu=1\right) \\ C(U_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1, \bar{u}u^t=u^t\bar{u}=1\right) \end{aligned}$$

On the other hand, according to Proposition 5.11 and to Definition 5.14, we have the following presentation results:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u}, uu^t=\text{projection of trace } N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*, \bar{u}u^t=\text{projections of trace } N\right)\end{aligned}$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. We use as well the following formulae:

$$Tr(uu^*)=Tr(u^t\bar{u})\quad,\quad Tr(\bar{u}u^t)=Tr(u^*u)$$

We therefore obtain the following formulae:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t, u^tu=\text{projections of trace } N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*, u^*u, \bar{u}u^t, u^t\bar{u}=\text{projections of trace } N\right)\end{aligned}$$

Now observe that, in tensor product notation, the conditions at right are all of the form $(tr\otimes id)p=1$. Thus, p must be follows, for the above conditions:

$$p=uu^*, u^*u, \bar{u}u^t, u^t\bar{u}$$

We therefore obtain that, for any faithful state φ, we have $(tr\otimes\varphi)(1-p)=0$. It follows from this that the following projections must be all equal to the identity:

$$p=uu^*, u^*u, \bar{u}u^t, u^t\bar{u}$$

But this leads to the conclusion in the statement. □

Regarding now the homogeneous space structure of $O_{MN}^{L\times}, U_{MN}^{L\times}$, the situation here is a bit more complicated in the free case than in the classical case, due to a number of algebraic and analytic issues. We first have the following result:

PROPOSITION 5.16. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times\times U_N^\times\curvearrowright U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$.*
(2) *We have a map $U_M^\times\times U_N^\times\to U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$.*

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

PROOF. In the classical case, consider the following action and quotient maps:

$$U_M\times U_N\curvearrowright U_{MN}^L\quad,\quad U_M\times U_N\to U_{MN}^L$$

The transposes of these two maps are as follows, where $J=\begin{pmatrix}1&0\\0&0\end{pmatrix}$:

$$\begin{aligned}\varphi &\to ((U,A,B)\to\varphi(AUB^*))\\ \varphi &\to ((A,B)\to\varphi(AJB^*))\end{aligned}$$

But with $\varphi=u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u=u$, let us set:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

We have then the following computation:

$$\begin{aligned}(UU^*U)_{ij}&=\sum_{pq}\sum_{klmnst}u_{kl}u_{mn}^*u_{st}\otimes a_{ki}a_{mq}^*a_{sq}\otimes b_{lp}^*b_{np}b_{tj}^*\\&=\sum_{klmt}u_{kl}u_{ml}^*u_{mt}\otimes a_{ki}\otimes b_{tj}^*\\&=\sum_{kt}u_{kt}\otimes a_{ki}\otimes b_{tj}^*\\&=U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}U_{ij}U_{ij}^*&=\sum_{ij}\sum_{klst}u_{kl}u_{st}^*\otimes a_{ki}a_{si}^*\otimes b_{lj}^*b_{tj}\\&=\sum_{kl}u_{kl}u_{kl}^*\otimes 1\otimes 1\\&=L\end{aligned}$$

(2) Assuming $uu^*u=u$, let us set:

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We have then the following computation:

$$\begin{aligned}(VV^*V)_{ij}&=\sum_{pq}\sum_{x,y,z\leq L}a_{xi}a_{yq}^*a_{zq}\otimes b_{xp}^*b_{yp}b_{zj}^*\\&=\sum_{x\leq L}a_{xi}\otimes b_{xj}^*\\&=V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}V_{ij}V_{ij}^*&=\sum_{ij}\sum_{r,s\leq L}a_{ri}a_{si}^*\otimes b_{rj}^*b_{sj}\\&=\sum_{l\leq L}1\\&=L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. □

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X=G/H$, the associated action and quotient maps are given by:

$$\begin{cases}a:X\times G\to X & : \quad (Hg,h)\to Hgh\\ p:G\to X & : \quad g\to Hg\end{cases}$$

Thus we have $a(p(g),h)=p(gh)$. In our context, a similar result holds:

THEOREM 5.17. *With $G=G_M\times G_N$ and $X=G_{MN}^L$, where $G_N=O_N^\times,U_N^\times$, we have*

$$\begin{array}{ccc} G\times G & \xrightarrow{\ m\ } & G\\ \downarrow{\scriptstyle p\times id} & & \downarrow{\scriptstyle p}\\ X\times G & \xrightarrow{\ a\ } & X\end{array}$$

where a,p are the action map and the map constructed in Proposition 5.16.

PROOF. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ,α are morphisms of algebras induced by a,p:

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \Phi\ } & C(X\times G)\\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha\otimes id}\\ C(G) & \xrightarrow{\ \Delta\ } & C(G\times G)\end{array}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned}(\alpha\otimes id)\Phi(u_{ij}) &= (\alpha\otimes id)\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\\ &= \sum_{kl}\sum_{r\leq L}a_{rk}\otimes b_{rl}^*\otimes a_{ki}\otimes b_{lj}^*\end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned}\Delta\pi(u_{ij}) &= F_{23}(\Delta\otimes\Delta)\sum_{r\leq L}a_{ri}\otimes b_{rj}^*\\ &= F_{23}\left(\sum_{r\leq L}\sum_{kl}a_{rk}\otimes a_{ki}\otimes b_{rl}^*\otimes b_{lj}^*\right)\end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. $\square$

5c. Partial permutations

Let us discuss now some discrete extensions of the above constructions. We have:

DEFINITION 5.18. *Associated to a partial permutation, $\sigma : I \simeq J$ with $I \subset \{1,\ldots,N\}$ and $J \subset \{1,\ldots,M\}$, is the real/complex partial isometry*

$$T_\sigma : span\left(e_i\Big|i\in I\right)\to span\left(e_j\Big|j\in J\right)$$

given on the standard basis elements by $T_\sigma(e_i) = e_{\sigma(i)}$.

Let S_{MN}^L be the set of partial permutations $\sigma : I \simeq J$ as above, with range $I \subset \{1,\ldots,N\}$ and target $J \subset \{1,\ldots,M\}$, and with $L = |I| = |J|$. We have:

PROPOSITION 5.19. *The space of partial permutations signed by elements of $\mathbb{Z}_s$,*

$$H_{MN}^{sL} = \left\{T(e_i) = w_i e_{\sigma(i)}\Big|\sigma\in S_{MN}^L, w_i\in\mathbb{Z}_s\right\}$$

is isomorphic to the quotient space

$$(H_M^s\times H_N^s)/(H_L^s\times H_{M-L}^s\times H_{N-L}^s)$$

via a standard isomorphism.

PROOF. This follows by adapting the computations in the proof of Proposition 5.12 and Theorem 5.13. Indeed, we have an action map as follows, which is transitive:

$$H_M^s\times H_N^s\to H_{MN}^{sL}\quad,\quad (A,B)U = AUB^*$$

Consider now the following point:

$$U = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

The stabilizer of this point follows to be the following group:

$$H_L^s\times H_{M-L}^s\times H_{N-L}^s$$

To be more precise, this group is embedded via:

$$(x,a,b)\to\left[\begin{pmatrix}x&0\\0&a\end{pmatrix},\begin{pmatrix}x&0\\0&b\end{pmatrix}\right]$$

But this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$. The result here is as follows:

PROPOSITION 5.20. *The compact quantum space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+}) = C(U_{MN}^{L+})\Big/\left\langle u_{ij}u_{ij}^* = u_{ij}^*u_{ij} = p_{ij} = \text{projections}, u_{ij}^s = p_{ij}\right\rangle$$

has an action map, and is the target of a quotient map, as in Theorem 5.17.

PROOF. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\quad,\quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We use the fact that the standard coordinates a_{ij},b_{ij} on the quantum groups H_M^{s+},H_N^{s+} satisfy the following relations, for any $x\neq y$ on the same row or column of a,b:

$$xy=xy^*=0$$

We obtain, by using these relations, the following formula:

$$U_{ij}U_{ij}^*=\sum_{klmn}u_{kl}u_{mn}^*\otimes a_{ki}a_{mi}^*\otimes b_{lj}^*b_{mj}=\sum_{kl}u_{kl}u_{kl}^*\otimes a_{ki}a_{ki}^*\otimes b_{lj}^*b_{lj}$$

On the other hand, we have as well the following formula:

$$V_{ij}V_{ij}^*=\sum_{r,t\leq L}a_{ri}a_{ti}^*\otimes b_{rj}^*b_{tj}=\sum_{r\leq L}a_{ri}a_{ri}^*\otimes b_{rj}^*b_{rj}$$

In terms of the projections $x_{ij}=a_{ij}a_{ij}^*$, $y_{ij}=b_{ij}b_{ij}^*$, $p_{ij}=u_{ij}u_{ij}^*$, we have:

$$U_{ij}U_{ij}^*=\sum_{kl}p_{kl}\otimes x_{ki}\otimes y_{lj}\quad,\quad V_{ij}V_{ij}^*=\sum_{r\leq L}x_{ri}\otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij},V_{ij}^*V_{ij}$ are given by the same formulae. Finally, once again by using the relations of type $xy=xy^*=0$, we have:

$$U_{ij}^s=\sum_{k_rl_r}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i}\ldots a_{k_si}\otimes b_{l_1j}^*\ldots b_{l_sj}^*=\sum_{kl}u_{kl}^s\otimes a_{ki}^s\otimes(b_{lj}^*)^s$$

On the other hand, we have as well the following formula:

$$V_{ij}^s=\sum_{r_l\leq L}a_{r_1i}\ldots a_{r_si}\otimes b_{r_1j}^*\ldots b_{r_sj}^*=\sum_{r\leq L}a_{ri}^s\otimes(b_{rj}^*)^s$$

Thus the conditions of type $u_{ij}^s=p_{ij}$ are satisfied as well, and we are done. □

Let us discuss now the general case. We have the following result:

PROPOSITION 5.21. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with $s=(e_1,\ldots,e_s)$ ranging over all the colored integers, and with $\pi,\sigma\in D(0,s)$.

PROOF. According to the various constructions above, the relations defining the quantum space G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D:

$$\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=\delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned}\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s} &= \sum_{i_1\ldots i_s}\delta_\pi(i)\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}\\ &= \sum_{i_1\ldots i_s}\delta_\pi(i)\delta_\sigma(i)\\ &= L^{|\pi\vee\sigma|}\end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. □

In the general case now, where $G=(G_N)$ is an arbitary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

THEOREM 5.22. *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with π,σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 5.17.

PROOF. We proceed as in the proof of Proposition 5.20. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\quad,\quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Regarding the variables U_{ij}, the computation here goes as follows:

$$\begin{aligned}&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\ =&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\otimes\delta_\pi(i)\delta_\sigma(j)a_{k_1i_1}^{e_1}\ldots a_{k_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\ =&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\\ =&\ L^{|\pi\vee\sigma|}\end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{l_1i_1}^{e_1}\ldots a_{l_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)\\
=&\ L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 5.20, and the commutation of the diagram in Theorem 5.17 is then trivial. □

5d. Integration results

Let us discuss now the integration over the various noncommutative spaces constructed so far, and notably over the spaces G_{MN}^L, which are quite general. We first have:

Definition 5.23. *The integration functional of G_{MN}^L is the composition*

$$\int_{G_{MN}^L}:C(G_{MN}^L)\to C(G_M\times G_N)\to\mathbb C$$

of the representation $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$ with the Haar functional of $G_M\times G_N$.

Observe that in the case $L=M=N$ we obtain the integration over G_N. Also, at $L=M=1$, or at $L=N=1$, we obtain the integration over the sphere.

In the general case now, we first have the following result:

Proposition 5.24. *The integration functional of G_{MN}^L has the invariance property*

$$\left(\int_{G_{MN}^L}\otimes id\right)\Phi(x)=\int_{G_{MN}^L}x$$

with respect to the coaction map, namely:

$$\Phi(u_{ij})=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

Proof. We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L}\otimes id\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned}
X &= \left(\int_{G_{MN}^L}\otimes id\right)\sum_{k_xl_x}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{k_xl_x}\sum_{r_x\leq L}a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*\\
&= \sum_{r_x\leq L}\sum_{k_x}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\otimes\sum_{l_x}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*
\end{aligned}$$

By using now the invariance property of the Haar functionals of G_M,G_N, we obtain:

$$\begin{aligned}
X &= \sum_{r_x\leq L}\left(\int_{G_M}\otimes id\right)\Delta(a_{r_1i_1}\ldots a_{r_si_s})\otimes\left(\int_{G_N}\otimes id\right)\Delta(b_{r_1j_1}^*\ldots b_{r_sj_s}^*)\\
&= \sum_{r_x\leq L}\int_{G_M}a_{r_1i_1}\ldots a_{r_si_s}\int_{G_N}b_{r_1j_1}^*\ldots b_{r_sj_s}^*\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)\sum_{r_x\leq L}a_{r_1i_1}\ldots a_{r_si_s}\otimes b_{r_1j_1}^*\ldots b_{r_sj_s}^*
\end{aligned}$$

But this gives the formula in the statement, and we are done. □

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

THEOREM 5.25. *We have the Weingarten type formula*

$$\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where the matrices on the right are given by $W_{sM}=G_{sM}^{-1}$, with $G_{sM}(\pi,\sigma)=M^{|\pi\vee\sigma|}$.

PROOF. We make use of the usual quantum group Weingarten formula, that we know from chapters 2-3. By using this formula for G_M,G_N, we obtain:

$$\begin{aligned}
\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s} &= \sum_{l_1\ldots l_s\leq L}\int_{G_M}a_{l_1i_1}\ldots a_{l_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{l_1\ldots l_s\leq L}\sum_{\pi\sigma}\delta_\pi(l)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\left(\sum_{l_1\ldots l_s\leq L}\delta_\pi(l)\delta_\tau(l)\right)\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)
\end{aligned}$$

The coefficient being $L^{|\pi\vee\tau|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of the integration, as follows:

THEOREM 5.26. *The integration of G_{MN}^L is the unique positive unital trace*

$$C(G_{MN}^L)\to\mathbb{C}$$

which is invariant under the action of the quantum group $G_M\times G_N$.

PROOF. We use a standard method, from [11], the point being to show that we have the following ergodicity formula:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)=\int_{G_{MN}^L}x$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

By using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned}
X &= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\int_{G_M}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\sum_{\pi\sigma}\delta_\pi(k)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\tau(l)u_{k_1l_1}\ldots u_{k_sl_s}
\end{aligned}$$

By using now the summation formula in Theorem 5.25, we obtain:

$$X=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

Now by comparing with the Weingarten formula for G_{MN}^L, this proves our claim. Assume now that $\tau:C(G_{MN}^L)\to\mathbb{C}$ satisfies the invariance condition. We have:

$$\begin{aligned}
\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \left(\tau\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau\otimes id)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau(x)1)\\
&= \tau(x)
\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\begin{aligned}\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \tau(tr(x)1)\\ &= tr(x)\end{aligned}$$

Thus we obtain $\tau=tr$, and this finishes the proof. □

As a main application of the above results, we have:

PROPOSITION 5.27. *For a sum of coordinates of the following type,*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

with the coordinates not overlapping on rows and columns, we have

$$\int_{G_{MN}^L}\chi_E^s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $K=|E|$ is the cardinality of the indexing set.

PROOF. With $K=|E|$, we can write $E=\{(\alpha(i),\beta(i))\}$, for certain embeddings:

$$\alpha:\{1,\ldots,K\}\subset\{1,\ldots,M\}$$

$$\beta:\{1,\ldots,K\}\subset\{1,\ldots,N\}$$

In terms of these maps α,β, the moment in the statement is given by:

$$M_s=\int_{G_{MN}^L}\left(\sum_{i\leq K}u_{\alpha(i)\beta(i)}\right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned}&M_s\\ =\ &\int_{G_{MN}^L}\sum_{i_1\ldots i_s\leq K}u_{\alpha(i_1)\beta(i_1)}\ldots u_{\alpha(i_s)\beta(i_s)}\\ =\ &\sum_{i_1\ldots i_s\leq K}\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(\alpha(i_1),\ldots,\alpha(i_s))\delta_\tau(\beta(i_1),\ldots,\beta(i_s))W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\\ =\ &\sum_{\pi\sigma\tau\nu}\left(\sum_{i_1\ldots i_s\leq K}\delta_\pi(i)\delta_\tau(i)\right)L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\end{aligned}$$

But, as explained before, in the proof of Theorem 5.25, the coefficient on the left in the last formula is $C=K^{|\pi\vee\tau|}$. We therefore obtain the formula in the statement. □

At a more concrete level now, we have the following conceptual result, making the link with the Bercovici-Pata bijection [19]:

Theorem 5.28. *In the context of the liberation operations*

$$O_{MN}^L\to O_{MN}^{L+}\quad,\quad U_{MN}^L\to U_{MN}^{L+}\quad,\quad H_{MN}^{sL}\to H_{MN}^{sL+}$$

the laws of the sums of non-overlapping coordinates,

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

are in Bercovici-Pata bijection, in the

$$|E|=\kappa N,L=\lambda N,M=\mu N$$

regime and $N\to\infty$ limit.

Proof. This follows indeed from the formula in Proposition 5.27. □

5e. Exercises

Exercises:

Exercise 5.29.

Exercise 5.30.

Exercise 5.31.

Exercise 5.32.

Exercise 5.33.

Exercise 5.34.

Bonus exercise.

CHAPTER 6

Affine spaces

6a. Affine spaces

We discuss now an abstract extension of the constructions of manifolds that we have so far. The idea will be that of looking at certain classes of algebraic manifolds $X\subset S^{N-1}_{\mathbb C,+}$, which are homogeneous spaces, of a certain special type. We have:

DEFINITION 6.1. *An affine homogeneous space over a closed subgroup $G\subset U_N^+$ is a closed subset $X\subset S^{N-1}_{\mathbb C,+}$, such that there exists an index set $I\subset\{1,\ldots,N\}$ such that*

$$\alpha(x_i)=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\quad,\quad\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

define morphisms of C^*-algebras, satisfying the following condition,

$$\left(id\otimes\int_G\right)\Phi=\int_G\alpha(.)1$$

called ergodicity condition for the action.

As a basic example, $O_N^+\to S^{N-1}_{\mathbb R,+}$ is indeed affine in our sense, with $I=\{1\}$. The same goes for $U_N^+\to S^{N-1}_{\mathbb C,+}$, which is affine as well, also with $I=\{1\}$. Observe also that the $1/\sqrt{|I|}$ constant appearing above is the correct one, because:

$$\begin{aligned}\sum_i\left(\sum_{j\in I}u_{ji}\right)\left(\sum_{k\in I}u_{ki}\right)^*&=\sum_i\sum_{j,k\in I}u_{ji}u_{ki}^*\\&=\sum_{j,k\in I}(uu^*)_{jk}\\&=|I|\end{aligned}$$

As a first general result about such spaces, we have:

PROPOSITION 6.2. *Consider an affine homogeneous space X, as above.*

1. *The coaction condition $(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$ is satisfied.*
2. *We have as well the formula $(\alpha\otimes id)\Phi=\Delta\alpha$.*

PROOF. The coaction condition is clear. For the second formula, we first have:

$$\begin{aligned}(\alpha\otimes id)\Phi(x_i) &= \sum_k\alpha(x_k)\otimes u_{ki}\\ &= \frac{1}{\sqrt{|I|}}\sum_k\sum_{j\in I}u_{jk}\otimes u_{ki}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\Delta\alpha(x_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\Delta(u_{ji})\\ &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\end{aligned}$$

Thus, by linearity, multiplicativity and continuity, we obtain the result. □

As a second result regarding such spaces, which closes the discussion in the case where α is injective, which is something that happens in many cases, we have:

THEOREM 6.3. *When α is injective we must have $X=X_{G,I}^{min}$, where:*

$$C(X_{G,I}^{min})=\left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)$$

Moreover, $X_{G,I}^{min}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

PROOF. The first assertion is clear from definitions. Regarding now the second assertion, consider the variables in the statement:

$$X_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\in C(G)$$

In order to prove that we have $X_{G,I}^{min}\subset S^{N-1}_{\mathbb{C},+}$, observe first that we have:

$$\begin{aligned}\sum_i X_iX_i^* &= \frac{1}{|I|}\sum_i\sum_{j,k\in I}u_{ji}u_{ki}^*\\ &= \frac{1}{|I|}\sum_{j,k\in I}(uu^*)_{jk}\\ &= 1\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\sum_i X_i^*X_i &= \frac{1}{|I|}\sum_i\sum_{j,k\in I}u_{ji}^*u_{ki}\\ &= \frac{1}{|I|}\sum_{j,k\in I}(\bar{u}u^t)_{jk}\\ &= 1\end{aligned}$$

Thus $X_{G,I}^{min}\subset S^{N-1}_{\mathbb{C},+}$. Finally, observe that we have:

$$\begin{aligned}\Delta(X_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\\ &= \sum_k X_k\otimes u_{ki}\end{aligned}$$

Thus we have indeed a coaction map, given by $\Phi=\Delta$. As for the ergodicity condition, namely $(id\otimes\int_G)\Delta=\int_G(.)1$, this holds as well, by definition of the integration functional $\int_G$. Thus, our axioms for affine homogeneous spaces are indeed satisfied. □

Now back to the general case, we have the following key result:

Proposition 6.4. *The ergodicity condition, namely*

$$\left(id\otimes\int_G\right)\Phi=\int_G\alpha(.)1$$

is equivalent to the condition

$$(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\quad,\quad\forall k,\forall i_1,\ldots,i_k$$

where P is the matrix formed by the Peter-Weyl integrals of exponent k,

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

and where $(x^{\otimes k})_{i_1\ldots i_k}=x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}$.

Proof. We have the following computation:

$$\begin{aligned}\left(id\otimes\int_G\right)\Phi(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}(x^{\otimes k})_{j_1\ldots j_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\int_G\alpha(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a consequence, we have the following result:

THEOREM 6.5. *We must have $X\subset X_{G,I}^{max}$, as subsets of $S^{N-1}_{\mathbb C,+}$, where:*

$$C(X_{G,I}^{max})=C(S^{N-1}_{\mathbb C,+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle$$

Moreover, $X_{G,I}^{max}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

PROOF. Let us first prove that we have an action $G\curvearrowright X_{G,I}^{max}$. We must show here that the variables $X_i=\sum_jx_j\otimes u_{ji}$ satisfy the defining relations for $X_{G,I}^{max}$. We have:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{l_1\ldots l_k}P_{i_1\ldots i_k,l_1\ldots l_k}(X^{\otimes k})_{l_1\ldots l_k}\\ &= \sum_{l_1\ldots l_k}P_{i_1\ldots i_k,l_1\ldots l_k}\sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes u_{j_1l_1}^{e_1}\ldots u_{j_kl_k}^{e_k}\\ &= \sum_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes(u^{\otimes k}P^t)_{j_1\ldots j_k,i_1\ldots i_k}\end{aligned}$$

Since by Peter-Weyl the transpose of $P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$ is the orthogonal projection onto $Fix(u^{\otimes k})$, we have $u^{\otimes k}P^t=P^t$. We therefore obtain:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

Thus we have an action $G\curvearrowright X_{G,I}^{max}$, and since this action is ergodic by Proposition 6.4, we have an affine homogeneous space, as claimed. □

We can now merge the results that we have, and we obtain:

THEOREM 6.6. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following C^*-subalgebra and the following quotient C^*-algebra,*

$$C(X_{G,I}^{min}) = \left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)$$
$$C(X_{G,I}^{max}) = C(S^{N-1}_{\mathbb C,+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle$$

then we have maps as follows,

$$G\to X_{G,I}^{min}\subset X_{G,I}^{max}\subset S^{N-1}_{\mathbb C,+}$$

the space $G\to X_{G,I}^{max}$ is affine homogeneous, and any affine homogeneous space $G\to X$ appears as an intermediate space $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$.

PROOF. This follows indeed from the various results that we have, namely Theorem 6.3 and Theorem 6.5, regarding the minimal and maximal constructions. □

At the level of the general theory, based on Definition 6.1, we will need as well:

THEOREM 6.7. *Assuming that $G\to X$ is an affine homogeneous space, with index set $I\subset\{1,\ldots,N\}$, the Haar integration functional $\int_X=\int_G\alpha$ is given by*

$$\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_{\pi,\sigma\in D}K_I(\pi)\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)$$

where $\{\xi_\pi|\pi\in D\}$ is a basis of $Fix(u^{\otimes k})$, $W_{kN}=G_{kN}^{-1}$ with $G_{kN}(\pi,\sigma)=<\xi_\pi,\xi_\sigma>$ is the associated Weingarten matrix, and $K_I(\pi)=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}(\xi_\pi)_{j_1\ldots j_k}$.

PROOF. By using the Weingarten formula for the quantum group G, in its abstract form, coming from Peter-Weyl theory, we have:

$$\begin{aligned}\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k} &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

But this gives the formula in the statement, and we are done. □

With this discussed, let us go back now to the "minimal vs maximal" discussion, in analogy with the group algebras. Here is a natural example of an intermediate space $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$, which will be of interest for us, in what follows:

THEOREM 6.8. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following quotient algebra*

$$C(X_{G,I}^{med})=C(S^{N-1}_{\mathbb C,+})\Big/\left\langle\sum_{j_1\ldots j_k}\xi_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\xi_{j_1\ldots j_k}\Big|\forall k,\forall\xi\in Fix(u^{\otimes k})\right\rangle$$

we obtain in this way an affine homogeneous space $G\to X_{G,I}$.

PROOF. We know from Theorem 6.5 that $X_{G,I}^{max}\subset S^{N-1}_{\mathbb C,+}$ is constructed by imposing to the standard coordinates the conditions $Px^{\otimes k}=P^I$, where:

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

$$P^I_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}$$

According to the Weingarten integration formula for G, we have:

$$\begin{aligned}(Px^{\otimes k})_{i_1\ldots i_k}&=\sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ P^I_{i_1\ldots i_k}&=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

Thus $X_{G,I}^{med}\subset X_{G,I}^{max}$, and the other assertions are standard as well. □

We can now put everything together, as follows:

THEOREM 6.9. *Given a closed subgroup $G\subset U_N^+$, and a subset $I\subset\{1,\ldots,N\}$, the affine homogeneous spaces over G, with index set I, have the following properties:*

1. *These are exactly the intermediate subspaces $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$ on which G acts affinely, with the action being ergodic.*
2. *For the minimal and maximal spaces $X_{G,I}^{min}$ and $X_{G,I}^{max}$, as well as for the intermediate space $X_{G,I}^{med}$ constructed above, these conditions are satisfied.*
3. *By performing the GNS construction with respect to the Haar integration functional $\int_X=\int_G\alpha$ we obtain the minimal space $X_{G,I}^{min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them $X_{G,I}$.

PROOF. This follows indeed by combining the various results above. □

6b. Basic examples

Let us discuss now some basic examples of affine homogeneous spaces, namely those coming from the classical groups, and those coming from group duals. We will need:

PROPOSITION 6.10. *Assuming that a closed subset $X\subset S^{N-1}_{\mathbb C,+}$ is affine homogeneous over a classical group, $G\subset U_N$, then X itself must be classical, $X\subset S^{N-1}_{\mathbb C}$.*

PROOF. We use the well-known fact that, since the standard coordinates $u_{ij}\in C(G)$ commute, the corepresentation $u^{\circ\circ\bullet\bullet}=u^{\otimes 2}\otimes\bar{u}^{\otimes 2}$ has the following fixed vector:

$$\xi=\sum_{ij}e_i\otimes e_j\otimes e_i\otimes e_j$$

With $k=\circ\circ\bullet\bullet$ and with this vector ξ, the ergodicity formula reads:

$$\begin{aligned}\sum_{ij}x_ix_jx_i^*x_j^* &= \frac{1}{\sqrt{|I|^4}}\sum_{i,j\in I}1\\ &= 1\end{aligned}$$

By using this formula, along with $\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$, we obtain:

$$\begin{aligned}&\sum_{ij}(x_ix_j-x_jx_i)(x_j^*x_i^*-x_i^*x_j^*)\\ =&\sum_{ij}x_ix_jx_j^*x_i^*-x_ix_jx_i^*x_j^*-x_jx_ix_j^*x_i^*+x_jx_ix_i^*x_j^*\\ =&\ 1-1-1+1\\ =&\ 0\end{aligned}$$

We conclude that for any i,j the following commutator vanishes:

$$[x_i,x_j]=0$$

By using now this commutation relation, plus once again the relations defining the free sphere $S^{N-1}_{\mathbb C,+}$, we have as well the following computation:

$$\begin{aligned}&\sum_{ij}(x_ix_j^*-x_j^*x_i)(x_jx_i^*-x_i^*x_j)\\ =&\sum_{ij}x_ix_j^*x_jx_i^*-x_ix_j^*x_i^*x_j-x_j^*x_ix_jx_i^*+x_j^*x_ix_i^*x_j\\ =&\sum_{ij}x_ix_j^*x_jx_i^*-x_ix_i^*x_j^*x_j-x_j^*x_jx_ix_i^*+x_j^*x_ix_i^*x_j\\ =&\ 1-1-1+1\\ =&\ 0\end{aligned}$$

Thus we have $[x_i,x_j^*]=0$ as well, and so $X\subset S^{N-1}_{\mathbb C}$, as claimed. □

We can now formulate the result in the classical case, as follows:

THEOREM 6.11. *In the classical case, $G \subset U_N$, there is only one affine homogeneous space, for each index set $I = \{1, \ldots, N\}$, namely the quotient space*

$$X = G/(G \cap C_N^I)$$

where $C_N^I \subset U_N$ is the group of unitaries fixing the following vector,

$$\xi_I = \frac{1}{\sqrt{|I|}}(\delta_{i \in I})_i$$

which generalizes the complex bistochastic group, $C_N \subset U_N$.

PROOF. Consider an affine homogeneous space $G \to X$. We know from Proposition 6.10 that X is classical. We will first prove that we have $X = X_{G,I}^{min}$, and then we will prove that $X_{G,I}^{min}$ equals the quotient space in the statement.

(1) We use the well-known fact that the functional $E = (id \otimes \int_G)\Phi$ is the projection onto the fixed point algebra of the action, given by:

$$C(X)^\Phi = \left\{ f \in C(X) \middle| \Phi(f) = f \otimes 1 \right\}$$

Thus our ergodicity condition, namely $E = \int_G \alpha(.)1$, shows that we must have:

$$C(X)^\Phi = \mathbb{C}1$$

But in the classical case the condition $\Phi(f) = f \otimes 1$ reformulates as:

$$f(gx) = f(x) \quad , \quad \forall g \in G, x \in X$$

Thus, we recover in this way the usual ergodicity condition, stating that whenever a function $f \in C(X)$ is constant on the orbits of the action, it must be constant. Now observe that for an affine action, the orbits are closed. Thus an affine action which is ergodic must be transitive, and we deduce from this that we have:

$$X = X_{G,I}^{min}$$

(2) We know that the inclusion $C(X) \subset C(G)$ comes via:

$$x_i = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ji}$$

Thus, the quotient map $p : G \to X \subset S_{\mathbb{C}}^{N-1}$ is given by the following formula:

$$p(g) = \left(\frac{1}{\sqrt{|I|}} \sum_{j \in I} g_{ji} \right)_i$$

In particular, the image of the unit matrix $1 \in G$ is the following vector:

$$\begin{aligned} p(1) &= \left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}\delta_{ji}\right)_i \\ &= \left(\frac{1}{\sqrt{|I|}}\delta_{i\in I}\right)_i \\ &= \xi_I \end{aligned}$$

But this gives the quotient space result in the statement.

(3) Finally, regarding the last assertion, stating that our group $C_N^I \subset U_N$ generalizes the complex bishochastic group $C_N \subset U_N$, this is more of a comment, coming from definitions. Indeed, C_N consists by definition of the unitary matrices $g \in U_N$ which are bistochastic, meaning having the same sums on rows and columns. But this bistochasticity condition is equivalent to the following condition, with ξ being the all-1 vector:

$$g\xi = \xi$$

Thus, our group $C_N^I \subset U_N$ generalizes indeed the group $C_N \subset U_N$, as claimed. □

Let us discuss now the group dual case. For simplifying, we will discuss the case of the "diagonal" embeddings only. Given a finitely generated discrete group $\Gamma =< g_1, \ldots, g_N >$, we can consider the following "diagonal" embedding:

$$\widehat{\Gamma} \subset U_N^+ \quad , \quad u_{ij} = \delta_{ij} g_i$$

With this convention, we have the following result:

Theorem 6.12. *In the group dual case, $G = \widehat{\Gamma}$ with $\Gamma =< g_1, \ldots, g_N >$, we have*

$$X = \widehat{\Gamma}_I \quad : \quad \Gamma_I =< g_i | i \in I >\subset \Gamma$$

for any affine homogeneous space X, when identifying full and reduced group algebras.

Proof. Assume indeed that we have an affine homogeneous space $G \to X$. In terms of the rescaled coordinates $h_i = \sqrt{|I|}x_i$, our axioms for α, Φ read:

$$\alpha(h_i) = \delta_{i\in I} g_i \quad , \quad \Phi(h_i) = h_i \otimes g_i$$

As for the ergodicity condition, this translates as follows:

$$\begin{aligned}
&\left(id\otimes\int_G\right)\Phi(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p})=\int_G\alpha(h_{i_1}^{e_p}\ldots h_{i_p}^{e_p})\\
\iff\ &\left(id\otimes\int_G\right)(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}\otimes g_{i_1}^{e_1}\ldots g_{i_p}^{e_p})=\int_G\delta_{i_1\in I}\ldots\delta_{i_p\in I}g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}\\
\iff\ &\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}\delta_{i_1\in I}\ldots\delta_{i_p\in I}\\
\iff\ &\left[g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\implies h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{i_1\in I}\ldots\delta_{i_p\in I}\right]
\end{aligned}$$

Now observe that from $g_ig_i^*=g_i^*g_i=1$ we obtain in this way:

$$h_ih_i^*=h_i^*h_i=\delta_{i\in I}$$

Thus the elements h_i vanish for $i\notin I$, and are unitaries for $i\in I$. We conclude that we have $X=\widehat{\Lambda}$, where $\Lambda=<h_i|i\in I>$ is the group generated by these unitaries. In order to finish now the proof, our claim is that for indices $i_x\in I$ we have:

$$g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\iff h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=1$$

Indeed, $\implies$ comes from the ergodicity condition, as processed above, and $\Longleftarrow$ comes from the existence of the morphism α, which is given by $\alpha(h_i)=g_i$, for $i\in I$. $\square$

Let us go back now to the general case, and discuss a number of further axiomatization issues, based on the examples that we have. We will need the following result:

Proposition 6.13. *The closed subspace $C_N^{I+}\subset U_N^+$ defined via*

$$C(C_N^{I+})=C(U_N^+)\Big/\left<u\xi_I=\xi_I\right>$$

where $\xi_I=\frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$, is a compact quantum group.

Proof. We must check Woronowicz's axioms, and the proof goes as follows:

(1) Let us set $U_{ij}=\sum_ku_{ik}\otimes u_{kj}$. We have then:

$$\begin{aligned}
(U\xi_I)_i&=\frac{1}{\sqrt{|I|}}\sum_{j\in I}U_{ij}\\
&=\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_ku_{ik}\otimes u_{kj}\\
&=\sum_ku_{ik}\otimes(u\xi_I)_k
\end{aligned}$$

Since the vector ξ_I is by definition fixed by u, we obtain:

$$\begin{aligned}(U\xi_I)_i &= \sum_k u_{ik}\otimes(\xi_I)_k\\ &= \frac{1}{\sqrt{|I|}}\sum_{k\in I}u_{ik}\otimes 1\\ &= (u\xi_I)_i\otimes 1\\ &= (\xi_I)_i\otimes 1\end{aligned}$$

Thus we can define indeed a comultiplication map, by $\Delta(u_{ij})=U_{ij}$.

(2) In order to construct the counit map, $\varepsilon(u_{ij})=\delta_{ij}$, we must prove that the identity matrix $1=(\delta_{ij})_{ij}$ satisfies $1\xi_I=\xi_I$. But this is clear.

(3) In order to construct the antipode, $S(u_{ij})=u_{ji}^*$, we must prove that the adjoint matrix $u^*=(u_{ji}^*)_{ij}$ satisfies $u^*\xi_I=\xi_I$. But this is clear from $u\xi_I=\xi_I$. □

Based on the computations that we have so far, we can formulate:

THEOREM 6.14. *Given a closed quantum subgroup $G\subset U_N^+$ and a set $I\subset\{1,\ldots,N\}$, we have a quotient map and an inclusion map as follows:*

$$G/(G\cap C_N^{I+})\to X_{G,I}^{min}\subset X_{G,I}^{max}$$

These maps are both isomorphisms in the classical case. In general, they are both proper.

PROOF. Consider the quantum group $H=G\cap C_N^{I+}$, which is by definition such that at the level of the corresponding algebras, we have:

$$C(H)=C(G)\Big/\left<u\xi_I=\xi_I\right>$$

In order to construct a quotient map $G/H\to X_{G,I}^{min}$, we must check that the defining relations for $C(G/H)$ hold for the standard generators $x_i\in C(X_{G,I}^{min})$. But if we denote by $\rho:C(G)\to C(H)$ the quotient map, then we have, as desired:

$$\begin{aligned}(id\otimes\rho)\Delta x_i &= (id\otimes\rho)\left(\frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{ik}\otimes u_{kj}\right)\\ &= \sum_k u_{ik}\otimes(\xi_I)_k\\ &= x_i\otimes 1\end{aligned}$$

In the classical case, Theorem 6.11 shows that both the maps in the statement are isomorphisms. For the group duals, however, these maps are not isomorphisms, in general. This follows indeed from Theorem 6.12, and from some basic computations. □

We discuss now a number of further examples. We will need:

Proposition 6.15. *Given a compact matrix quantum group $G=(G,u)$, the pair*

$$G^t=(G,u^t)$$

where $(u^t)_{ij}=u_{ji}$, is a compact matrix quantum group as well.

Proof. The construction of the comultiplication is as follows, where Σ is the flip:

$$\begin{aligned}\Delta^t[(u^t)_{ij}]=\sum_k(u^t)_{ik}\otimes(u^t)_{kj}\quad&\iff\quad\Delta^t(u_{ji})=\sum_k u_{ki}\otimes u_{jk}\\&\iff\quad\Delta^t=\Sigma\Delta\end{aligned}$$

As for the corresponding counit and antipode, these can be simply taken to be (ε,S), and the axioms of Woronowicz are then satisfied. □

We will need as well the following result, which is standard too:

Proposition 6.16. *Given closed subgroups $G\subset U_N^+$ and $H\subset U_M^+$, with fundamental corepresentations $u=(u_{ij})$ and $v=(v_{ab})$, their product is a closed subgroup*

$$G\times H\subset U_{NM}^+$$

with fundamental corepresentation $w_{ia,jb}=u_{ij}\otimes v_{ab}$.

Proof. Our claim is that the corresponding structural maps are as follows:

$$\Delta(\alpha\otimes\beta)=\Delta(\alpha)_{13}\Delta(\beta)_{24}$$

$$\varepsilon(\alpha\otimes\beta)=\varepsilon(\alpha)\varepsilon(\beta)$$

$$S(\alpha\otimes\beta)=S(\alpha)S(\beta)$$

Indeed, the verification for the comultiplication goes as follows:

$$\begin{aligned}\Delta(w_{ia,jb})&=\Delta(u_{ij})_{13}\Delta(v_{ab})_{24}\\&=\sum_{kc}u_{ik}\otimes v_{ac}\otimes u_{kj}\otimes v_{cb}\\&=\sum_{kc}w_{ia,kc}\otimes w_{kc,jb}\end{aligned}$$

For the counit, we have the following computation:

$$\begin{aligned}\varepsilon(w_{ia,jb})&=\varepsilon(u_{ij})\varepsilon(v_{ab})\\&=\delta_{ij}\delta_{ab}\\&=\delta_{ia,jb}\end{aligned}$$

As for the antipode, here we have the following computation:

$$\begin{aligned} S(w_{ia,jb}) &= S(u_{ij})S(v_{ab}) \\ &= v_{ba}^* u_{ji}^* \\ &= (u_{ji}v_{ba})^* \\ &= w_{jb,ia}^* \end{aligned}$$

We refer to Wang's paper [89] for more details regarding this construction. □

We will need one more ingredient, which is a definition, as follows:

Definition 6.17. *We call a closed quantum subgroup $G \subset U_N^+$ self-transpose when we have an automorphism $T : C(G) \to C(G)$ given by $T(u_{ij}) = u_{ji}$.*

With the above notions and general theory in hand, let us go back to the affine homogeneous spaces. As a first result here, any closed subgroup $G \subset U_N^+$ appears as an affine homogeneous space over an appropriate quantum group, as follows:

Theorem 6.18. *Given a closed subgroup $G \subset U_N^+$, we have an identification*

$$X_{\mathcal{G},I}^{min} \simeq G$$

given at the level of standard coordinates by $x_{ij} = \frac{1}{\sqrt{N}} u_{ij}$, where:

1. *$\mathcal{G} = G^t \times G \subset U_{N^2}^+$, with coordinates $w_{ia,jb} = u_{ji} \otimes u_{ab}$.*
2. *$I \subset \{1, \ldots, N\}^2$ is the diagonal set, $I = \{(k,k) | k = 1, \ldots, N\}$.*

In the self-transpose case we can choose as well $\mathcal{G} = G \times G$, with $w_{ia,jb} = u_{ij} \otimes u_{ab}$.

Proof. As a first observation, our closed subgroup $G \subset U_N^+$ appears as an algebraic submanifold of the free complex sphere on N^2 variables, as follows:

$$G \subset S_{\mathbb{C},+}^{N^2-1} \quad , \quad x_{ij} = \frac{1}{\sqrt{N}} u_{ij}$$

Let us construct now the affine homogeneous space structure. Our claim is that, with $\mathcal{G} = G^t \times G$ and $I = \{(k,k)\}$ as in the statement, the structural maps are:

$$\alpha = \Delta \quad , \quad \Phi = (\Sigma \otimes id)\Delta^{(2)}$$

Indeed, in what regards $\alpha = \Delta$, this is given by the following formula:

$$\alpha(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} = \sum_k w_{kk,ij}$$

Thus, by dividing by $\sqrt{N}$, we obtain the usual affine homogeneous space formula:

$$\alpha(x_{ij}) = \frac{1}{\sqrt{|I|}} \sum_k w_{kk,ij}$$

Regarding now $\Phi = (\Sigma \otimes id)\Delta^{(2)}$, the formula here is as follows:

$$\begin{aligned}\Phi(u_{ij}) &= (\Sigma \otimes id)\sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}\\ &= \sum_{kl} u_{kl} \otimes u_{ik} \otimes u_{lj}\\ &= \sum_{kl} u_{kl} \otimes w_{kl,ij}\end{aligned}$$

Thus, by dividing by $\sqrt{N}$, we obtain the usual affine homogeneous space formula:

$$\Phi(x_{ij}) = \sum_{kl} x_{kl} \otimes w_{kl,ij}$$

The ergodicity condition being clear as well, this gives the first assertion. Regarding now the second assertion, assume that we are in the self-transpose case, and so that we have an automorphism $T : C(G) \to C(G)$ given by $T(u_{ij}) = u_{ji}$. With $w_{ia,jb} = u_{ij} \otimes u_{ab}$, the modified map $\alpha = (T \otimes id)\Delta$ is then given by the following formula:

$$\begin{aligned}\alpha(u_{ij}) &= (T \otimes id)\sum_{k} u_{ik} \otimes u_{kj}\\ &= \sum_{k} u_{ki} \otimes u_{kj}\\ &= \sum_{k} w_{kk,ij}\end{aligned}$$

As for the modified map $\Phi = (id \otimes T \otimes id)(\Sigma \otimes id)\Delta^{(2)}$, this is given by:

$$\begin{aligned}\Phi(u_{ij}) &= (id \otimes T \otimes id)\sum_{kl} u_{kl} \otimes u_{ik} \otimes u_{lj}\\ &= \sum_{kl} u_{kl} \otimes u_{ki} \otimes u_{lj}\\ &= \sum_{kl} u_{kl} \otimes w_{kl,ij}\end{aligned}$$

Thus we have the correct affine homogeneous space formulae, and once again the ergodicity condition being clear as well, this gives the result. □

Let us discuss now the generalization of the above result. We have:

Definition 6.19. *Given a closed subgroup $G \subset U_N^+$ and an integer $M \leq N$ we set*

$$C(G_{MN}) = \left\langle u_{ij} \Big| i \in \{1, \ldots, M\}, j \in \{1, \ldots, N\} \right\rangle \subset C(G)$$

and we call row space of G the underlying quotient space $G \to G_{MN}$.

As a basic example here, at $M=N$ we obtain the quantum group G itself. Also, at $M=1$ we obtain the space whose coordinates are those on the first row of coordinates on G. Finally, in the case of the basic quantum unitary and reflection groups, these are particular cases of the partial isometry spaces discussed in chapter 5.

Given $G_N\subset U_N^+$ and an integer $M\leq N$, we can consider the quantum group $G_M=G_N\cap U_M^+$, with the intersection taken inside U_N^+, and with $U_M^+\subset U_N^+$ given by:

$$u=diag(v,1_{N-M})$$

Observe that we have a quotient map $C(G_N)\to C(G_M)$, given by $u_{ij}\to v_{ij}$. With these conventions, we have the following extension of Theorem 6.18:

THEOREM 6.20. *Given a closed subgroup $G_N\subset U_N^+$, we have an identification*

$$X_{\mathcal{G},I}^{min}\simeq G_{MN}$$

given at the level of standard coordinates by $x_{ij}=\frac{1}{\sqrt{M}}u_{ij}$, where:

(1) *$\mathcal{G}=G_M^t\times G_N\subset U_{NM}^+$, where $G_M=G_N\cap U_M^+$, with coordinates as follows:*

$$w_{ia,jb}=u_{ji}\otimes v_{ab}$$

(2) *$I\subset\{1,\ldots,M\}\times\{1,\ldots,N\}$ is the diagonal set, namely:*

$$I=\left\{(k,k)\Big|k=1,\ldots,M\right\}$$

In the self-transpose case we can choose as well $\mathcal{G}=G_M\times G_N$, with $w_{ia,jb}=u_{ij}\otimes v_{ab}$.

PROOF. Consider the row space $X=G_{MN}$ constructed in Definition 6.19, with its standard row space coordinates, namely:

$$x_{ij}=\frac{1}{\sqrt{M}}u_{ij}$$

In order to prove the result, we have to show that this space coincides with the space $X_{\mathcal{G},I}^{min}$ constructed in the statement, with its standard coordinates. For this purpose, consider the following composition of morphisms, where in the middle we have the comultiplication, and at left and right we have the canonical maps:

$$C(X)\subset C(G_N)\to C(G_N)\otimes C(G_N)\to C(G_M)\otimes C(G_N)$$

The standard coordinates are then mapped as follows:

$$\begin{aligned}
x_{ij} &= \frac{1}{\sqrt{M}}u_{ij}\\
&\to \frac{1}{\sqrt{M}}\sum_k u_{ik}\otimes u_{kj}\\
&\to \frac{1}{\sqrt{M}}\sum_{k\leq M} u_{ik}\otimes v_{kj}\\
&= \frac{1}{\sqrt{M}}\sum_{k\leq M} w_{kk,ij}
\end{aligned}$$

Thus we obtain the standard coordinates on the space $X_{\mathcal{G},I}^{min}$, as claimed. Finally, the last assertion is standard as well, by suitably modifying the above morphism. □

6c. Integration results

In the easy case, we have the following result:

PROPOSITION 6.21. *When $G\subset U_N^+$ is easy, coming from a category of partitions D, the space $X_{G,I}\subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the relations*

$$\sum_{i_1\ldots i_k}\delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=|I|^{|\pi|-k/2},\quad\forall k,\forall\pi\in D(k)$$

where $D(k)=D(0,k)$, and where $|.|$ denotes the number of blocks.

PROOF. We know by easiness that $Fix(u^{\otimes k})$ is spanned by the vectors $\xi_\pi=T_\pi$, with $\pi\in D(k)$. But these latter vectors are given by:

$$\xi_\pi=\sum_{i_1\ldots i_k}\delta_\pi(i_1\ldots i_k)e_{i_1}\otimes\ldots\otimes e_{i_k}$$

We deduce that $X_{G,I}\subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the following relations:

$$\sum_{i_1\ldots i_k}\delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\delta_\pi(j_1\ldots j_k),\quad\forall k,\forall\pi\in D(k)$$

Now since the sum on the right equals $|I|^{|\pi|}$, this gives the result. □

More generally, it is interesting to work out what happens when G is a product of easy quantum groups, and the index set I appears as follows, for a certain set J:

$$I=\{(c,\ldots,c)|c\in J\}$$

The result here, in its most general form, is as follows:

THEOREM 6.22. *For a product of easy quantum groups $G = G_{N_1}^{(1)} \times \ldots \times G_{N_s}^{(s)}$, and with $I = \{(c,\ldots,c)|c \in J\}$, the space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ appears via the relations*

$$\sum_{i_1\ldots i_k} \delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k} = |J|^{|\pi_1\vee\ldots\vee\pi_s|-k/2}$$

for any $k \in \mathbb{N}$ and any partition of type $\pi \in D^{(1)}(k) \times \ldots \times D^{(s)}(k)$, where $D^{(r)} \subset P$ is the category of partitions associated to $G_{N_r}^{(r)} \subset U_{N_r}^+$, and where

$$\pi_1 \vee \ldots \vee \pi_s \in P(k)$$

is the partition obtained by superposing $\pi_1, \ldots, \pi_s$.

PROOF. Since we are in a direct product situation, a basis for $Fix(u^{\otimes k})$ is provided by the vectors $\rho_\pi = \xi_{\pi_1} \otimes \ldots \otimes \xi_{\pi_s}$ associated to the following partitions:

$$\pi = (\pi_1, \ldots, \pi_s) \in D^{(1)}(k) \times \ldots \times D^{(s)}(k)$$

We conclude that the space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$ appears by imposing the following relations to the standard coordinates:

$$\sum_{i_1\ldots i_k} \delta_\pi(i_1\ldots i_k)x_{i_1}^{e_1}\ldots x_{i_k}^{e_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1\ldots j_k\in I} \delta_\pi(j_1\ldots j_k),\ \forall k, \forall \pi \in D^{(1)}(k) \times \ldots \times D^{(s)}(k)$$

Since the conditions $j_1, \ldots, j_k \in I$ read $j_1 = (l_1, \ldots, l_1), \ldots, j_k = (l_k, \ldots, l_k)$, for certain elements $l_1, \ldots l_k \in J$, the sums on the right are given by:

$$\begin{aligned}
\sum_{j_1\ldots j_k\in I} \delta_\pi(j_1\ldots j_k) &= \sum_{l_1\ldots l_k\in J} \delta_\pi(l_1,\ldots,l_1,\ldots\ldots,l_k,\ldots,l_k)\\
&= \sum_{l_1\ldots l_k\in J} \delta_{\pi_1}(l_1\ldots l_k)\ldots\delta_{\pi_s}(l_1\ldots l_k)\\
&= \sum_{l_1\ldots l_k\in J} \delta_{\pi_1\vee\ldots\vee\pi_s}(l_1\ldots l_k)
\end{aligned}$$

Now since the sum on the right equals $|J|^{|\pi_1\vee\ldots\vee\pi_s|}$, this gives the result. □

We can now discuss probabilistic aspects. We first have:

PROPOSITION 6.23. *The moments of the variable*

$$\chi_T = \sum_{i\leq T} x_{i\ldots i}$$

are given by the following formula,

$$\int_X \chi_T^k \simeq \frac{1}{\sqrt{M^k}} \sum_{\pi\in D^{(1)}(k)\cap\ldots\cap D^{(s)}(k)} \left(\frac{TM}{N}\right)^{|\pi|}$$

in the $N_i \to \infty$ limit, $\forall i$, where $M = |I|$, and $N = N_1 \ldots N_s$.

PROOF. We have the following formula:

$$\pi(x_{i_1\ldots i_s})=\frac{1}{\sqrt{M}}\sum_{c\in J}u_{i_1c}\otimes\ldots\otimes u_{i_sc}$$

For the variable in the statement, we therefore obtain:

$$\pi(\chi_T)=\frac{1}{\sqrt{M}}\sum_{i\leq T}\sum_{c\in J}u_{ic}\otimes\ldots\otimes u_{ic}$$

Now by raising to the power k and integrating, we obtain:

$$\begin{aligned}\int_X\chi_T^k &= \frac{1}{\sqrt{M^k}}\sum_{i_1\ldots i_k\leq T}\sum_{c_1\ldots c_k\in J}\int_{G^{(1)}}u_{i_1c_1}\ldots u_{i_kc_k}\ldots\ldots\int_{G^{(s)}}u_{i_1c_1}\ldots u_{i_kc_k}\\ &= \frac{1}{\sqrt{M^k}}\sum_{ic}\sum_{\pi\sigma}\delta_{\pi_1}(i)\delta_{\sigma_1}(c)W_{kN_1}^{(1)}(\pi_1,\sigma_1)\ldots\delta_{\pi_s}(i)\delta_{\sigma_s}(c)W_{kN_s}^{(s)}(\pi_s,\sigma_s)\\ &= \frac{1}{\sqrt{M^k}}\sum_{\pi\sigma}T^{|\pi_1\vee\ldots\vee\pi_s|}M^{|\sigma_1\vee\ldots\vee\sigma_s|}W_{kN_1}^{(1)}(\pi_1,\sigma_1)\ldots W_{kN_s}^{(s)}(\pi_s,\sigma_s)\end{aligned}$$

We use now the standard fact that the Weingarten functions are concentrated on the diagonal. Thus in the limit we must have $\pi_i=\sigma_i$ for any i, and we obtain:

$$\begin{aligned}\int_X\chi_T^k &\simeq \frac{1}{\sqrt{M^k}}\sum_{\pi}T^{|\pi_1\vee\ldots\vee\pi_s|}M^{|\pi_1\vee\ldots\vee\pi_s|}N_1^{-|\pi_1|}\ldots N_s^{-|\pi_s|}\\ &\simeq \frac{1}{\sqrt{M^k}}\sum_{\pi\in D^{(1)}\cap\ldots\cap D^{(s)}}T^{|\pi|}M^{|\pi|}(N_1\ldots N_s)^{-|\pi|}\\ &= \frac{1}{\sqrt{M^k}}\sum_{\pi\in D^{(1)}\cap\ldots\cap D^{(s)}}\left(\frac{TM}{N}\right)^{|\pi|}\end{aligned}$$

But this gives the formula in the statement, and we are done. $\square$

As a consequence, we have the following result:

THEOREM 6.24. *In the context of a liberation operation for quantum groups*

$$G^{(i)}\to G^{(i)+}$$

the laws of the variables $\sqrt{M}\chi_T$ are in Bercovici-Pata bijection, in the $N_i\to\infty$ limit.

PROOF. Assume indeed that we have easy quantum groups $G^{(1)},\ldots,G^{(s)}$, with free versions $G^{(1)+},\ldots,G^{(s)+}$. At the level of the categories of partitions, we have:

$$\bigcap_i\left(D^{(i)}\cap NC\right)=\left(\bigcap_i D^{(i)}\right)\cap NC$$

Since the intersection of Hom-spaces is the Hom-space for the generated quantum group, we deduce that at the quantum group level, we have:

$$< G^{(1)+},\ldots,G^{(s)+} >=< G^{(1)},\ldots,G^{(s)} >^+$$

Thus the result follows from Proposition 6.23, and from the Bercovici-Pata bijection result for truncated characters for this latter liberation operation. ☐

6d. Tannakian duality

As a starting point here, we have the following simple fact:

Proposition 6.25. *Any affine homogeneous space $X_{G,I}\subset S^{N-1}_{\mathbb C,+}$ is algebraic, with*

$$\sum_{i_1\ldots i_k}\xi_{i_1\ldots i_k}x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{b_1\ldots b_k\in I}\xi_{b_1\ldots b_k}\quad\forall k,\forall\xi\in Fix(u^{\otimes k})$$

as defining relations. Moreover, we can use vectors ξ belonging to a basis of $Fix(u^{\otimes k})$.

Proof. This follows indeed from the various results above. ☐

In order to reach to a more categorical description of $X_{G,I}$, the idea will be that of using Frobenius duality. We use colored indices, and we denote by $k\to\bar{k}$ the operation on the colored indices which consists in reversing the index, and switching all the colors. Also, we agree to identify the linear maps $T:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$ with the corresponding rectangular matrices $T\in M_{N^l\times N^k}(\mathbb C)$, written $T=(T_{i_1\ldots i_l,j_1\ldots j_k})$. With these conventions, the precise formulation of Frobenius duality that we will need is as follows:

Proposition 6.26. *We have an isomorphism of complex vector spaces*

$$T\in Hom(u^{\otimes k},u^{\otimes l})\ \leftrightarrow\ \xi\in Fix(u^{\otimes l}\otimes u^{\otimes\bar{k}})$$

given by the following formulae,

$$T_{i_1\ldots i_l,j_1\ldots j_k}=\xi_{i_1\ldots i_lj_k\ldots j_1}\quad,\quad \xi_{i_i\ldots i_lj_1\ldots j_k}=T_{i_1\ldots i_l,j_k\ldots j_1}$$

and called Frobenius duality.

Proof. This is a well-known result, which follows from the general theory in [99]. To be more precise, given integers $K,L\in\mathbb N$, consider the following standard isomorphism, which in matrix notation makes $T=(T_{IJ})\in M_{L\times K}(\mathbb C)$ correspond to $\xi=(\xi_{IJ})$:

$$T\in\mathcal L(\mathbb C^{\otimes K},\mathbb C^{\otimes L})\ \leftrightarrow\ \xi\in\mathbb C^{\otimes L+K}$$

Given now two arbitrary corepresentations $v\in M_K(C(G))$ and $w\in M_L(C(G))$, the abstract Frobenius duality result established by Woronowicz in [99] states that the above isomorphism restricts into an isomorphism of vector spaces, as follows:

$$T\in Hom(v,w)\ \leftrightarrow\ \xi\in Fix(w\otimes\bar{v})$$

In our case, we can apply this result with $v=u^{\otimes k}$ and $w=u^{\otimes l}$. Since, according to our conventions, we have $\bar{v}=u^{\otimes\bar{k}}$, this gives the isomorphism in the statement. ☐

With the above result in hand, we can enhance the construction of $X_{G,I}$, as follows:

THEOREM 6.27. *Any affine homogeneous space $X_{G,I}$ is algebraic, with*

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}(x_{j_1}^{f_1}\ldots x_{j_k}^{f_k})^*=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}$$

for any k,l, and any $T\in Hom(u^{\otimes k},u^{\otimes l})$, as defining relations.

PROOF. We must prove that the relations in the statement are satisfied, over $X_{G,I}$. We know from Proposition 6.25 that, with $k\to l\bar{k}$, the following relation holds:

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}\xi_{i_1\ldots i_lj_k\ldots j_1}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}x_{j_k}^{\bar{f}_k}\ldots x_{j_1}^{\bar{f}_1}=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}\xi_{b_1\ldots b_lc_k\ldots c_1}$$

In terms of the matrix $T_{i_1\ldots i_l,j_1\ldots j_k}=\xi_{i_1\ldots i_lj_k\ldots j_1}$ from Proposition 6.26, we obtain:

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}x_{j_k}^{\bar{f}_k}\ldots x_{j_1}^{\bar{f}_1}=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}$$

But this gives the formula in the statement, and we are done. □

The above results suggest the following notion:

DEFINITION 6.28. *Given a submanifold $X\subset S^{N-1}_{\mathbb{C},+}$ and a subset $I\subset\{1,\ldots,N\}$, we say that X is I-affine when $C(X)$ is presented by relations of type*

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}(x_{j_1}^{f_1}\ldots x_{j_k}^{f_k})^*=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}$$

with the operators T belonging to certain linear spaces

$$F(k,l)\subset M_{N^l\times N^k}(\mathbb{C})$$

which altogether form a tensor category $F=(F(k,l))$.

According to Theorem 6.27, any affine homogeneous space $X_{G,I}$ is an I-affine manifold, with the corresponding tensor category being the one associated to the quantum group $G\subset U_N^+$ which produces it, formed by the following linear spaces:

$$F(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

Let us study now the quantum isometry groups $G^+(X)$ of the manifolds $X\subset S^{N-1}_{\mathbb{C},+}$ which are I-affine, in the above sense. We have here the following result:

PROPOSITION 6.29. *For an I-affine manifold $X\subset S^{N-1}_{\mathbb{C},+}$ we have*

$$G\subset G^+(X)$$

where $G\subset U_N^+$ is the Tannakian dual of the associated tensor category F.

PROOF. We recall from chapter 5 that the relations defining $G^+(X)$ are those expressing the vanishing of the following quantities:

$$P(X_1,\ldots,X_N)=\sum_r\alpha_r\sum_{j_1^r\ldots j_{s(r)}^r}u_{i_1^rj_1^r}\ldots u_{i_{s(r)}^rj_{s(r)}^r}\otimes x_{j_1^r}\ldots x_{j_{s(r)}^r}$$

In the case of an I-affine manifold, the defining relations are those from Definition 6.28, with the corresponding polynomials P being indexed by the elements of F. But the vanishing of the associated relations $P(X_1,\ldots,X_N)=0$ corresponds precisely to the Tannakian relations defining $G\subset U_N^+$, and so we obtain $G\subset G^+(X)$, as claimed. □

We have now all the needed ingredients, and we can prove:

THEOREM 6.30. *Assuming that an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$ is I-affine, with associated tensor category F, the following happen:*

(1) *We have an inclusion $G\subset G^+(X)$, where G is the Tannakian dual of F.*
(2) *X is an affine homogeneous space, $X=X_{G,I}$, over this quantum group G.*

PROOF. In the context of Definition 6.28, the tensor category F there gives rise, by the Tannakian duality of Woronowicz [100], to a quantum group $G\subset U_N^+$. What is left is to construct the affine space morphisms α,Φ, and the proof here goes as follows:

(1) Construction of α. We want to construct a morphism, as follows:

$$\alpha:C(X)\to C(G)\quad,\quad x_i\to X_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ij}$$

In view of Definition 6.28, we must therefore prove that we have:

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}X_{i_1}^{e_1}\ldots X_{i_l}^{e_l}(X_{j_1}^{f_1}\ldots X_{j_k}^{f_k})^*=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}$$

By replacing the variables X_i by their above values, we want to prove that:

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}\sum_{r_1\ldots r_l\in I}\sum_{s_1\ldots s_k\in I}T_{i_1\ldots i_l,j_1\ldots j_k}u_{i_1r_1}^{e_1}\ldots u_{i_lr_l}^{e_l}(u_{j_1s_1}^{f_1}\ldots u_{j_ks_k}^{f_k})^*=\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}$$

Now observe that from the relation $T\in Hom(u^{\otimes k},u^{\otimes l})$ we obtain:

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}u_{i_1r_1}^{e_1}\ldots u_{i_lr_l}^{e_l}(u_{j_1s_1}^{f_1}\ldots u_{j_ks_k}^{f_k})^*=T_{r_1\ldots r_l,s_1\ldots s_k}$$

Thus, by summing over indices $r_i\in I$ and $s_i\in I$, we obtain the desired formula.

(2) Construction of Φ. We want to construct a morphism, as follows:

$$\Phi:C(X)\to C(G)\otimes C(X)\quad,\quad x_i\to X_i=\sum_j u_{ij}\otimes x_j$$

But this is precisely the coaction map constructed in Proposition 6.29.

(3) Proof of the ergodicity. If we go back to the general theory developed before, we see that the ergodicity condition is equivalent to a number of Tannakian conditions, which are automatic in our case. Thus, the ergodicity condition is automatic. □

The above result, based on the notion of I-affine manifold, remains quite theoretical. In order to further advance, let us formulate:

DEFINITION 6.31. *Given a submanifold $X\subset S^{N-1}_{\mathbb C,+}$ and a subset $I\subset\{1,\ldots,N\}$, we let $F_{X,I}(k,l)\subset M_{N^l\times N^k}(\mathbb C)$ be the linear space of linear maps T such that*

$$\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}(x_{j_1}^{f_1}\ldots x_{j_k}^{f_k})^*=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}$$

holds over X. We say that X is I-saturated when

$$F_{X,I}=(F_{X,l}(k,l))$$

is a tensor category, and the collection of the above relations presents $C(X)$.

Observe that any I-saturated manifold is automatically I-affine. The known results seem to suggest that the converse of this fact should hold. We do not have a proof of this fact, but we would like to present a few observations on this subject. First, we have:

PROPOSITION 6.32. *The linear spaces $F_{X,I}(k,l)\subset M_{N^l\times N^k}(\mathbb C)$ are as follows:*

1. *They contain the units.*
2. *They are stable by conjugation.*
3. *They satisfy the Frobenius duality condition.*

PROOF. All these assertions are elementary, as follows:

(1) Consider indeed the unit map. The associated relation is:

$$\sum_{i_1\ldots i_k}x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k})^*=1$$

But this relation holds indeed, due to the defining relations for $S^{N-1}_{\mathbb C,+}$.

(2) We have indeed the following sequence of equivalences:

$$\begin{aligned}
&T^*\in F_{X,I}(l,k)\\
\iff&\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T^*_{j_1\ldots j_k,i_1\ldots i_l}x_{j_1}^{f_1}\ldots x_{j_k}^{f_k}(x_{i_1}^{e_1}\ldots x_{i_l}^{e_l})^*=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots i_l\in I}\sum_{c_1\ldots c_k\in I}T^*_{c_1\ldots c_k,b_1\ldots b_l}\\
\iff&\sum_{i_1\ldots i_l}\sum_{j_1\ldots j_k}T_{i_1\ldots i_l,j_1\ldots j_k}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}(x_{j_1}^{f_1}\ldots x_{j_k}^{f_k})^*=\frac{1}{\sqrt{|I|^{k+l}}}\sum_{b_1\ldots b_l\in I}\sum_{c_1\ldots c_k\in I}T_{b_1\ldots b_l,c_1\ldots c_k}\\
\iff&T\in F_{X,I}(k,l)
\end{aligned}$$

(3) We have indeed a correspondence $T\in F_{X,I}(k,l)\ \leftrightarrow\ \xi\in F_{X,I}(\emptyset,l\bar k)$, given by the usual formulae for the Frobenius isomorphism. □

Based on the above result, we can now formulate our observations, as follows:

THEOREM 6.33. *Given a closed subgroup $G \subset U_N^+$, and an index set $I \subset \{1, \ldots, N\}$, consider the corresponding affine homogeneous space $X_{G,I} \subset S^{N-1}_{\mathbb{C},+}$.*

(1) *$X_{G,I}$ is I-saturated precisely when the collection of spaces $F_{X,I} = (F_{X,I}(k,l))$ is stable under compositions, and under tensor products.*
(2) *We have $F_{X,I} = F$ precisely when we have*

$$\sum_{j_1\ldots j_l\in I}\Big(\sum_{i_1\ldots i_l}\xi_{i_1\ldots i_l}u_{i_1j_1}^{e_1}\ldots u_{i_lj_l}^{e_l}-\xi_{j_1\ldots j_l}\Big)=0$$
$$\implies \sum_{i_1\ldots i_l}\xi_{i_1\ldots i_l}u_{i_1j_1}^{e_1}\ldots u_{i_lj_l}^{e_l}-\xi_{j_1\ldots j_l}=0$$

for any choice of the indices $j_1, \ldots, j_l$.

PROOF. We use the fact, from Theorem 6.27, that with $F(k,l) = Hom(u^{\otimes k}, u^{\otimes l})$, we have inclusions of vector spaces $F(k,l) \subset F_{X,I}(k,l)$. Moreover, once again by Theorem 6.27, the relations coming from the elements of the category formed by the spaces $F(k,l)$ present $X_{G,I}$. Thus, the relations coming from the elements of $F_{X,I}$ present $X_{G,I}$ as well. With this observation in hand, our assertions follow from Proposition 6.32:

(1) According to Proposition 6.32 (1,2) the unit and conjugation axioms are satisfied, so the spaces $F_{X,I}(k,l)$ form a tensor category precisely when the remaining axioms, namely the composition and the tensor product one, are satisfied. Now by assuming that these two axioms are satisfied, X follows to be I-saturated, by the above observation.

(2) Since we already have inclusions in one sense, the equality $F_{X,I} = F$ from the statement means that we must have inclusions in the other sense, as follows:

$$F_{X,I}(k,l) \subset F(k,l)$$

By using now Proposition 6.32 (3), it is enough to discuss the case $k = 0$. And here, assuming that we have $\xi \in F_{X,L}(0,l)$, the following condition must be satisfied:

$$\sum_{i_1\ldots i_l}\xi_{i_1\ldots i_l}x_{i_1}^{e_1}\ldots x_{i_l}^{e_l}=\sum_{j_1\ldots j_l\in I}\xi_{j_1\ldots j_l}$$

By applying now the morphism $\alpha : C(X_{G,I}) \to C(G)$, we deduce that we have:

$$\sum_{i_1\ldots i_l}\xi_{i_1\ldots i_l}\sum_{j_1\ldots j_l\in I}u_{i_1j_1}^{e_1}\ldots u_{i_lj_l}^{e_l}=\sum_{j_1\ldots j_l\in I}\xi_{j_1\ldots j_l}$$

Now recall that $F(0,l) = Fix(u^{\otimes l})$ consists of the vectors ξ satisfying:

$$\sum_{i_1\ldots i_l}\xi_{i_1\ldots i_l}u_{i_1j_1}^{e_1}\ldots u_{i_lj_l}^{e_l}=\xi_{j_1\ldots j_l},\forall j_1,\ldots,j_l$$

We are therefore led to the conclusion in the statement. □

6e. Exercises

Exercises:

Exercise 6.34.

Exercise 6.35.

Exercise 6.36.

Exercise 6.37.

Exercise 6.38.

Exercise 6.39.

Bonus exercise.

CHAPTER 7

Projective freeness

7a. Projective spaces

This chapter is an introduction to projective geometry, in our sense. As a first topic that we would like to discuss, we have the following remarkable isomorphism:

$$PO_N^+ = PU_N^+$$

In order to get started, let us first discuss the classical case. We have here:

THEOREM 7.1. *The passage $O_N \to U_N$ appears via Lie algebra complexification,*

$$O_N \to \mathfrak{o}_N \to \mathfrak{u}_n \to U_N$$

with the Lie algebra $\mathfrak{u}_N$ being a complexification of the Lie algebra $\mathfrak{o}_N$.

PROOF. This is something rather philosophical, and advanced as well, that we will not really need here, the idea being as follows:

(1) The unitary and orthogonal groups U_N, O_N are both Lie groups, in the sense that they are smooth manifolds. The corresponding Lie algebras $\mathfrak{u}_N, \mathfrak{o}_N$, which are by definition the respective tangent spaces at 1, can be computed by differentiating the equations defining U_N, O_N, with the conclusion being as follows:

$$\mathfrak{u}_N = \left\{A \in M_N(\mathbb{C})\Big|A^* = -A\right\}$$

$$\mathfrak{o}_N = \left\{B \in M_N(\mathbb{R})\Big|B^t = -B\right\}$$

(2) This was for the correspondences $U_N \to \mathfrak{u}_N$ and $O_N \to \mathfrak{o}_N$. In the other sense, the correspondences $\mathfrak{u}_N \to U_N$ and $\mathfrak{o}_N \to O_N$ appear by exponentiation, the result here stating that, around 1, the unitary matrices can be written as $U = e^A$, with $A \in \mathfrak{u}_N$, and the orthogonal matrices can be written as $U = e^B$, with $B \in \mathfrak{o}_N$.

(3) In view of all this, in order to understand the passage $O_N \to U_N$ it is enough to understand the passage $\mathfrak{o}_N \to \mathfrak{u}_N$. But, in view of the above formulae for $\mathfrak{o}_N, \mathfrak{u}_N$, this is basically an elementary linear algebra problem. Indeed, let us pick an arbitrary matrix $A \in M_N(\mathbb{C})$, and write it as follows, with $B, C \in M_N(\mathbb{R})$:

$$A = B + iC$$

In terms of B,C, the equation $A^*=-A$ defining the Lie algebra $\mathfrak u_N$ reads:

$$B^t=-B\quad,\quad C^t=C$$

(4) As a first observation, we must have $B\in\mathfrak o_N$. Regarding now C, let us decompose this matrix as follows, with D being its diagonal, and C' being the reminder:

$$C=D+C'$$

The matrix C' being symmetric with 0 on the diagonal, by swithcing all the signs below the main diagonal we obtain a certain matrix $C'_-\in\mathfrak o_N$. Thus, we have decomposed $A\in\mathfrak u_N$ as follows, with $B,C'\in\mathfrak o_N$, and with $D\in M_N(\mathbb R)$ being diagonal:

$$A=B+iD+iC'_-$$

(5) As a conclusion now, we have shown that we have a direct sum decomposition of real linear spaces as follows, with $\Delta\subset M_N(\mathbb R)$ being the diagonal matrices:

$$\mathfrak u_N\simeq\mathfrak o_N\oplus\Delta\oplus\mathfrak o_N$$

Thus, we can stop our study here, and say that we have reached the conclusion in the statement, namely that $\mathfrak u_N$ appears as a "complexification" of $\mathfrak o_N$. □

In the free case now, the situation is much simpler, and we have:

THEOREM 7.2. *The passage $O_N^+\to U_N^+$ appears via free complexification,*

$$U_N^+=\widetilde{O_N^+}$$

where the free complexification of a pair (G,u) is the pair $(\widetilde{G},\widetilde{u})$ with

$$C(\widetilde{G})=<zu_{ij}>\subset C(\mathbb T)*C(G)\quad,\quad\widetilde{u}=zu$$

where $z\in C(\mathbb T)$ is the standard generator, given by $x\to x$ for any $x\in\mathbb T$.

PROOF. We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of U_N^+:

$$O_N^+\subset\widetilde{O_N^+}\subset U_N^+$$

We must prove that the embedding on the right is an isomorphism, and there are several ways of doing this, all instructive, as follows:

(1) If we denote by v,u the fundamental corepresentations of O_N^+,U_N^+, we have:

$$Fix(v^{\otimes k})=span\left(\xi_\pi\Big|\pi\in NC_2(k)\right)$$

$$Fix(u^{\otimes k})=span\left(\xi_\pi\Big|\pi\in\mathcal{NC}_2(k)\right)$$

Moreover, the above vectors ξ_π are known to be linearly independent at $N\geq2$, and so the above results provide us with bases, and we obtain:

$$\dim(Fix(v^{\otimes k}))=|NC_2(k)|\quad,\quad\dim(Fix(u^{\otimes k}))=|\mathcal{NC}_2(k)|$$

Now since integrating the character of a corepresentation amounts in counting the fixed points, the above two formulae can be rewritten as follows:

$$\int_{O_N^+}\chi_v^k=|NC_2(k)|\quad,\quad\int_{U_N^+}\chi_u^k=|\mathcal{NC}_2(k)|$$

But this shows, via standard free probability theory, that χ_v must follow the Winger semicircle law γ_1, and that χ_u must follow the Voiculescu circular law Γ_1:

$$\chi_v\sim\gamma_1\quad,\quad\chi_u\sim\Gamma_1$$

On the other hand, by [87], when freely multiplying a semicircular variable by a Haar unitary we obtain a circular variable. Thus, the main character of $\widetilde{O_N^+}$ is circular:

$$\chi_{zv}\sim\Gamma_1$$

Now by forgetting about circular variables and free probability, the conclusion is that the inclusion $\widetilde{O_N^+}\subset U_N^+$ preserves the law of the main character:

$$law(\chi_{zv})=law(u)$$

Thus by Peter-Weyl we obtain that the inclusion $\widetilde{O_N^+}\subset U_N^+$ must be an isomorphism, modulo the usual equivalence relation for quantum groups.

(2) A version of the above proof, not using any prior free probability knowledge, makes use of the easiness property of O_N^+,U_N^+ only, namely:

$$Hom(v^{\otimes k},v^{\otimes l})=span\left(\xi_\pi\Big|\pi\in NC_2(k,l)\right)$$

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\xi_\pi\Big|\pi\in \mathcal{NC}_2(k,l)\right)$$

Indeed, let us look at the following inclusions of quantum groups:

$$O_N^+\subset\widetilde{O_N^+}\subset U_N^+$$

At the level of the associated Hom spaces we obtain reverse inclusions, as follows:

$$Hom(v^{\otimes k},v^{\otimes l})\supset Hom((zv)^{\otimes k},(zv)^{\otimes l})\supset Hom(u^{\otimes k},u^{\otimes l})$$

The spaces on the left and on the right are known from easiness, the result being that these spaces are as follows:

$$span\left(T_\pi\Big|\pi\in NC_2(k,l)\right)\supset span\left(T_\pi\Big|\pi\in \mathcal{NC}_2(k,l)\right)$$

Regarding the spaces in the middle, these are obtained from those on the left by "coloring", so we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O_N^+}\subset U_N^+$ is an isomorphism, modulo the usual equivalence relation. □

As an interesting consequence of the above result, we have:

THEOREM 7.3. *We have an identification as follows,*

$$PO_N^+ = PU_N^+$$

modulo the usual equivalence relation for compact quantum groups.

PROOF. As before, we have several proofs for this result, as follows:

(1) This follows from Theorem 7.2, because we have:

$$PU_N^+ = P\widetilde{O_N^+} = PO_N^+$$

(2) We can deduce this as well directly. With notations as before, we have:

$$Hom\left((v \otimes v)^k, (v \otimes v)^l\right) = span\left(T_\pi \Big| \pi \in NC_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

$$Hom\left((u \otimes \bar{u})^k, (u \otimes \bar{u})^l\right) = span\left(T_\pi \Big| \pi \in \mathcal{NC}_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusion $PO_N^+ \subset PU_N^+$ preserves the corresponding Tannakian categories, and so must be an isomorphism. □

As a conclusion, the passage $O_N^+ \to U_N^+$ is something much simpler than the passage $O_N \to U_N$, with this ultimately coming from the fact that the combinatorics of O_N^+, U_N^+ is something much simpler than the combinatorics of O_N, U_N. In addition, all this leads as well to the interesting conclusion that the free projective geometry does not fall into real and complex, but is rather unique and "scalarless". We will be back to this.

Let us discuss now the projective spaces. We first have:

PROPOSITION 7.4. *We have presentation results as follows,*

$$\begin{aligned} C(P^{N-1}_{\mathbb{R}}) &= C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N} \Big| p = \bar{p} = p^t = p^2, Tr(p) = 1\right) \\ C(P^{N-1}_{\mathbb{C}}) &= C^*_{comm}\left((p_{ij})_{i,j=1,\ldots,N} \Big| p = p^* = p^2, Tr(p) = 1\right) \end{aligned}$$

for the algebras of continuous functions on the real and complex projective spaces.

PROOF. We use the fact that the projective spaces $P^{N-1}_{\mathbb{R}}, P^{N-1}_{\mathbb{C}}$ can be respectively identified with the spaces of rank one projections in $M_N(\mathbb{R}), M_N(\mathbb{C})$. With this picture in mind, it is clear that we have arrows $\leftarrow$. In order to construct now arrows $\to$, consider the universal algebras on the right, A_R, A_C. These algebras being both commutative, by the Gelfand theorem we can write, with X_R, X_C being certain compact spaces:

$$A_R = C(X_R) \quad , \quad A_C = C(X_C)$$

Now by using the coordinate functions p_{ij}, we conclude that X_R, X_C are certain spaces of rank one projections in $M_N(\mathbb{R}), M_N(\mathbb{C})$. In other words, we have embeddings:

$$X_R \subset P^{N-1}_{\mathbb{R}} \quad , \quad X_C \subset P^{N-1}_{\mathbb{C}}$$

By transposing we obtain arrows $\to$, as desired. □

The above result suggests the following definition:

Definition 7.5. *Associated to any $N \in \mathbb{N}$ is the following universal algebra,*

$$C(P^{N-1}_+) = C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big| p = p^* = p^2, Tr(p) = 1\right)$$

whose abstract spectrum is called "free projective space".

Observe that, according to our presentation results for the real and complex projective spaces $P^{N-1}_\mathbb{R}$ and $P^{N-1}_\mathbb{C}$, we have embeddings of compact quantum spaces, as follows:

$$P^{N-1}_\mathbb{R} \subset P^{N-1}_\mathbb{C} \subset P^{N-1}_+$$

Let us first discuss the relation with the spheres. Given a closed subset $X \subset S^{N-1}_{\mathbb{R},+}$, its projective version is by definition the quotient space $X \to PX$ determined by the fact that $C(PX) \subset C(X)$ is the subalgebra generated by the following variables:

$$p_{ij} = x_i x_j$$

In order to discuss the relation with the spheres, it is convenient to neglect the material regarding the complex and hybrid cases, the projective versions of such spheres bringing nothing new. Thus, we are left with the 3 real spheres, and we have:

Proposition 7.6. *The projective versions of the* 3 *real spheres are as follows,*

$$\begin{array}{ccccc}
S^{N-1}_\mathbb{R} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \\
\downarrow & & \downarrow & & \downarrow \\
P^{N-1}_\mathbb{R} & \longrightarrow & P^{N-1}_\mathbb{C} & \longrightarrow & P^{N-1}_+
\end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

Proof. The assertion at left is true by definition. For the assertion at right, we have to prove that the variables $p_{ij} = z_i z_j$ over the free sphere $S^{N-1}_{\mathbb{R},+}$ satisfy the defining relations for $C(P^{N-1}_+)$, from Definition 7.5, namely:

$$p = p^* = p^2 \quad , \quad Tr(p) = 1$$

We first have the following computation:

$$(p^*)_{ij} = p^*_{ji} = (z_j z_i)^* = z_i z_j = p_{ij}$$

We have as well the following computation:

$$(p^2)_{ij} = \sum_k p_{ik} p_{kj} = \sum_k z_i z_k^2 z_j = z_i z_j = p_{ij}$$

Finally, we have as well the following computation:

$$Tr(p)=\sum_k p_{kk}=\sum_k z_k^2=1$$

Regarding now $PS^{N-1}_{\mathbb{R},*}=P^{N-1}_{\mathbb{C}}$, the inclusion "$\subset$" follows from $abcd=cbad=cbda$. In the other sense now, the point is that we have a matrix model, as follows:

$$\pi:C(S^{N-1}_{\mathbb{R},*})\to M_2(C(S^{N-1}_{\mathbb{C}}))\quad,\quad x_i\to\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

But this gives the missing inclusion "$\supset$", and we are done. See [11]. □

In addition to the above result, let us mention that, as already discussed above, passing to the complex case brings nothing new. This is because the projective version of the free complex sphere is equal to the free projective space constructed above:

$$PS^{N-1}_{\mathbb{C},+}=P^{N-1}_+$$

And the same goes for the "hybrid" spheres. For details on all this, we refer to chapters 5-6. In what regards now the tori, we have here the following result:

Proposition 7.7. *The projective versions of the* 3 *real tori are as follows,*

$$\begin{array}{ccccc}T_N&\longrightarrow&T_N^*&\longrightarrow&T_N^+\\ \downarrow&&\downarrow&&\downarrow\\ PT_N&\longrightarrow&P\mathbb{T}_N&\longrightarrow&PT_N^+\end{array}$$

modulo the standard equivalence relation for the quantum algebraic manifolds.

Proof. This follows indeed by using the same arguments as for the spheres. □

In what regards the orthogonal groups, we have here the following result:

Proposition 7.8. *The projective versions of the* 3 *orthogonal groups are*

$$\begin{array}{ccccc}O_N&\longrightarrow&O_N^*&\longrightarrow&O_N^+\\ \downarrow&&\downarrow&&\downarrow\\ PO_N&\longrightarrow&PU_N&\longrightarrow&PO_N^+\end{array}$$

modulo the standard equivalence relation for the compact quantum groups.

Proof. This follows by using the same arguments as for spheres, or tori. □

Finally, in what regards the reflection groups, we have here the following result:

PROPOSITION 7.9. *The projective versions of the* 3 *reflection groups are*

$$\begin{array}{ccccc} H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+ \\ \downarrow & & \downarrow & & \downarrow \\ PH_N & \longrightarrow & PK_N & \longrightarrow & PH_N^+ \end{array}$$

modulo the standard equivalence relation for the compact quantum groups.

PROOF. This follows indeed by using the same arguments as before. □

As a conclusion to this, in the projective geometry setting, we have 3 projective quadruplets, whose construction and main properties can be summarized as follows:

THEOREM 7.10. *We have projective quadruplets* (P, PT, PU, PK) *as follows,*

(1) *A classical real quadruplet, as follows,*

$$\begin{array}{ccc} P^{N-1}_{\mathbb{R}} & - & PT_N \\ | & \times & | \\ PO_N & - & PH_N \end{array}$$

(2) *A classical complex quadruplet, as follows,*

$$\begin{array}{ccc} P^{N-1}_{\mathbb{C}} & - & P\mathbb{T}_N \\ | & \times & | \\ PU_N & - & PK_N \end{array}$$

(3) *A free quadruplet, as follows,*

$$\begin{array}{ccc} P^{N-1}_{+} & - & PT_N^+ \\ | & \times & | \\ PO_N^+ & - & PH_N^+ \end{array}$$

which appear as projective versions of the main 3 *real quadruplets.*

Proof. This follows indeed from the results that already have. To be more precise, the details, that we will we need in what comes next, are as follows:

(1) Consider the classical affine real quadruplet, which is as follows:

$$\begin{matrix} S^{N-1}_{\mathbb{R}} & - & T_N \\ | & \times & | \\ O_N & - & H_N \end{matrix}$$

The projective version of this quadruplet is then the quadruplet in (1).

(2) Consider the half-classical affine real quadruplet, which is as follows:

$$\begin{matrix} S^{N-1}_{\mathbb{R},*} & - & T_N^* \\ | & \times & | \\ O_N^* & - & H_N^* \end{matrix}$$

The projective version of this quadruplet is then the quadruplet in (2).

(3) Consider the free affine real quadruplet, which is as follows:

$$\begin{matrix} S^{N-1}_{\mathbb{R},+} & - & T_N^+ \\ | & \times & | \\ O_N^+ & - & H_N^+ \end{matrix}$$

The projective version of this quadruplet is then the quadruplet in (3). □

7b. The threefold way

Getting back now to our general projective geometry program, we would like to have axiomatization and classification results for such quadruplets. In order to do this, following [12], we can axiomatize our various projective spaces, as follows:

Definition 7.11. *A monomial projective space is a closed subset $P\subset P^{N-1}_+$ obtained via relations of type*

$$p_{i_1i_2}\ldots p_{i_{k-1}i_k}=p_{i_{\sigma(1)}i_{\sigma(2)}}\ldots p_{i_{\sigma(k-1)}i_{\sigma(k)}},\ \forall(i_1,\ldots,i_k)\in\{1,\ldots,N\}^k$$

with σ ranging over a certain subset of $\bigcup_{k\in2\mathbb{N}}S_k$, which is stable under $\sigma\to|\sigma|$.

Observe the similarity with the corresponding monomiality notion for the spheres, from before. The only subtlety in the projective case is the stability under the operation $\sigma \to |\sigma|$, which in practice means that if the above relation associated to σ holds, then the following relation, associated to $|\sigma|$, must hold as well:

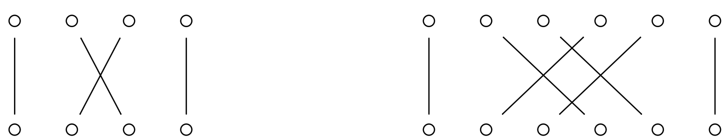
$$p_{i_0i_1}\ldots p_{i_ki_{k+1}} = p_{i_0i_{\sigma(1)}}p_{i_{\sigma(2)}i_{\sigma(3)}}\ldots p_{i_{\sigma(k-2)}i_{\sigma(k-1)}}p_{i_{\sigma(k)}i_{k+1}}$$

As an illustration, the basic projective spaces are all monomial:

PROPOSITION 7.12. *The 3 projective spaces are all monomial, with the permutations*

producing respectively the spaces $P^{N-1}_\mathbb{R}, P^{N-1}_\mathbb{C}$, and with no relation needed for P^{N-1}_+.

PROOF. We must divide the algebra $C(P^{N-1}_+)$ by the relations associated to the diagrams in the statement, as well as those associated to their shifted versions, given by:



(1) The basic crossing, and its shifted version, produce the following relations:

$$p_{ab} = p_{ba}$$

$$p_{ab}p_{cd} = p_{ac}p_{bd}$$

Now by using these relations several times, we obtain the following formula:

$$p_{ab}p_{cd} = p_{ac}p_{bd} = p_{ca}p_{db} = p_{cd}p_{ab}$$

Thus, the space produced by the basic crossing is classical, $P \subset P^{N-1}_\mathbb{C}$. By using one more time the relations $p_{ab} = p_{ba}$ we conclude that we have $P = P^{N-1}_\mathbb{R}$, as claimed.

(2) The fattened crossing, and its shifted version, produce the following relations:

$$p_{ab}p_{cd} = p_{cd}p_{ab}$$

$$p_{ab}p_{cd}p_{ef} = p_{ad}p_{eb}p_{cf}$$

The first relations tell us that the projective space must be classical, $P \subset P^{N-1}_\mathbb{C}$. Now observe that with $p_{ij} = z_i\bar{z}_j$, the second relations read:

$$z_a\bar{z}_bz_c\bar{z}_dz_e\bar{z}_f = z_a\bar{z}_dz_e\bar{z}_bz_c\bar{z}_f$$

Since these relations are automatic, we have $P = P^{N-1}_\mathbb{C}$, and we are done. ☐

Following [12], we can now formulate our classification result, as follows:

THEOREM 7.13. *The basic projective spaces, namely*

$$P^{N-1}_\mathbb{R}\subset P^{N-1}_\mathbb{C}\subset P^{N-1}_+$$

are the only monomial ones.

PROOF. We follow the proof from the affine case. Let $\mathcal{R}_\sigma$ be the collection of relations associated to a permutation $\sigma\in S_k$ with $k\in 2\mathbb{N}$, as in Definition 7.11. We fix a monomial projective space $P\subset P^{N-1}_+$, and we associate to it subsets $G_k\subset S_k$, as follows:

$$G_k=\begin{cases}\{\sigma\in S_k|\mathcal{R}_\sigma\text{ hold over }P\}&(k\text{ even})\\ \{\sigma\in S_k|\mathcal{R}_{|\sigma}\text{ hold over }P\}&(k\text{ odd})\end{cases}$$

As in the affine case, we obtain in this way a filtered group $G=(G_k)$, which is stable under removing outer strings, and under removing neighboring strings. Thus the computations in chapter 13 apply, and show that we have only 3 possible situations, corresponding to the 3 projective spaces in Proposition 7.12. □

Let us discuss now similar results for the projective quantum groups. Given a closed subgroup $G\subset O_N^+$, its projective version $G\to PG$ is by definition given by the fact that $C(PG)\subset C(G)$ is the subalgebra generated by the following variables:

$$w_{ij,ab}=u_{ia}u_{jb}$$

In the classical case we recover in this way the usual projective version:

$$PG=G/(G\cap\mathbb{Z}_2^N)$$

We have the following key result:

THEOREM 7.14. *The quantum group O_N^* is the unique intermediate easy quantum group $O_N\subset G\subset O_N^+$. Moreover, in the non-easy case, the following happen:*

(1) *The group inclusion $\mathbb{T}O_N\subset U_N$ is maximal.*
(2) *The group inclusion $PO_N\subset PU_N$ is maximal.*
(3) *The quantum group inclusion $O_N\subset O_N^*$ is maximal.*

PROOF. This is something that we discussed before, the idea being that the first assertion comes by classifying the categories of pairings, and then:

(1) This can be obtained by using standard Lie group methods.

(2) This follows from (1), by taking projective versions.

(3) This follows from (2), via standard algebraic lifting results. □

Our claim now is that, under suitable assumptions, PU_N is the only intermediate object $PO_N\subset G\subset PO_N^+$. In order to formulate a precise statement here, we first recall the following notion, that we have already heavily used in this book:

Definition 7.15. *A collection of sets $D=\bigsqcup_{k,l}D(k,l)$ with*

$$D(k,l)\subset P(k,l)$$

is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*

The above definition is something inspired from the axioms of Tannakian categories, and going hand in hand with it is the following definition:

Definition 7.16. *An intermediate compact quantum group*

$$O_N\subset G\subset O_N^+$$

is called easy when the corresponding Tannakian category

$$span(NC_2(k,l))\subset Hom(u^{\otimes k},u^{\otimes l})\subset span(P_2(k,l))$$

comes via the following formula, using the standard $\pi\to T_\pi$ construction,

$$Hom(u^{\otimes k},u^{\otimes l})=span(D(k,l))$$

from a certain collection of sets of pairings $D=(D(k,l))$.

As a key remark here, by "saturating" the sets $D(k,l)$, we can assume that the collection $D=(D(k,l))$ is a category of pairings, in the sense that it is stable under vertical and horizontal concatenation, upside-down turning, and contains the semicircle.

In the projective case now, following [12], let us formulate:

Definition 7.17. *A projective category of pairings is a collection of subsets*

$$NC_2(2k,2l)\subset E(k,l)\subset P_2(2k,2l)$$

stable under the usual categorical operations, and satisfying $\sigma\in E\implies|\sigma|\in E$.

As basic examples here, we have the following projective categories of pairings, where P_2^* is the category of matching pairings:

$$NC_2\subset P_2^*\subset P_2$$

This follows indeed from definitions. Now with the above notion in hand, we can formulate the following projective analogue of the notion of easiness:

Definition 7.18. *An intermediate compact quantum group*

$$PO_N\subset H\subset PO_N^+$$

is called projectively easy when its Tannakian category

$$span(NC_2(2k,2l))\subset Hom(v^{\otimes k},v^{\otimes l})\subset span(P_2(2k,2l))$$

comes via via the following formula, using the standard $\pi\to T_\pi$ construction,

$$Hom(v^{\otimes k},v^{\otimes l})=span(E(k,l))$$

for a certain projective category of pairings $E=(E(k,l))$.

Thus, we have a projective notion of easiness. Observe that, given an easy quantum group $O_N\subset G\subset O_N^+$, its projective version $PO_N\subset PG\subset PO_N^+$ is projectively easy in our sense. In particular the basic projective quantum groups $PO_N\subset PU_N\subset PO_N^+$ are all projectively easy in our sense, coming from the categories $NC_2\subset P_2^*\subset P_2$.

We have in fact the following general result, from [12]:

Theorem 7.19. *We have a bijective correspondence between the affine and projective categories of partitions, given by the operation*

$$G\to PG$$

at the level of the corresponding affine and projective easy quantum groups.

Proof. The construction of correspondence $D\to E$ is clear, simply by setting:

$$E(k,l)=D(2k,2l)$$

Indeed, due to the axioms in Definition 7.15, the conditions in Definition 7.17 are satisfied. Conversely, given $E=(E(k,l))$ as in Definition 7.17, we can set:

$$D(k,l)=\begin{cases}E(k,l)&(k,l\text{ even})\\ \{\sigma:|\sigma\in E(k+1,l+1)\}&(k,l\text{ odd})\end{cases}$$

Our claim is that $D=(D(k,l))$ is a category of partitions. Indeed:

(1) The composition action is clear. Indeed, when looking at the numbers of legs involved, in the even case this is clear, and in the odd case, this follows from:

$$\begin{aligned}|\sigma,|\sigma'\in E\quad&\implies\quad |^\sigma_\tau\in E\\ &\implies\quad {}^\sigma_\tau\in D\end{aligned}$$

(2) For the tensor product axiom, we have 4 cases to be investigated, depending on the parity of the number of legs of σ,τ, as follows:

– The even/even case is clear.

– The odd/even case follows from the following computation:

$$\begin{aligned}|\sigma,\tau\in E &\implies |\sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

– Regarding now the even/odd case, this can be solved as follows:

$$\begin{aligned}\sigma,|\tau\in E &\implies |\sigma|,|\tau\in E\\ &\implies |\sigma||\tau\in E\\ &\implies |\sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

– As for the remaining odd/odd case, here the computation is as follows:

$$\begin{aligned}|\sigma,|\tau\in E &\implies ||\sigma|,|\tau\in E\\ &\implies ||\sigma||\tau\in E\\ &\implies \sigma\tau\in E\\ &\implies \sigma\tau\in D\end{aligned}$$

(3) Finally, the conjugation axiom is clear from definitions. It is also clear that both compositions $D\to E\to D$ and $E\to D\to E$ are the identities, as claimed. As for the quantum group assertion, this is clear as well from definitions. □

Now back to uniqueness issues, we have here the following result, also from [12]:

THEOREM 7.20. *We have the following results:*

(1) O_N^* *is the only intermediate easy quantum group, as follows:*

$$O_N\subset G\subset O_N^+$$

(2) PU_N *is the only intermediate projectively easy quantum group, as follows:*

$$PO_N\subset G\subset PO_N^+$$

PROOF. The idea here is as follows:

(1) The assertion regarding $O_N\subset O_N^*\subset O_N^+$ is well-known, and this is something that we already know, explained in the above.

(2) The assertion regarding $PO_N\subset PU_N\subset PO_N^+$ follows from the classification result in (1), and from the duality in Theorem 7.19. □

Summarizing, we have analogues of the various affine classification results, with the remark that everything becomes simpler in the projective setting.

7c. Projective geometry

We have so far projective analogues of the various affine classification results. In view of this, our next goal will be that of finding projective versions of the quantum isometry group results that we have in the affine setting. We use the following action formalism, which is quite similar to the affine action formalism introduced in chapter 2:

Definition 7.21. *Consider a closed subgroup of the free orthogonal group, $G\subset O_N^+$, and a closed subset of the free real sphere, $X\subset S^{N-1}_{\mathbb R,+}$.*

(1) *We write $G\curvearrowright X$ when we have a morphism of C^*-algebras, as follows:*

$$\Phi:C(X)\to C(X)\otimes C(G)$$
$$\Phi(z_i)=\sum_a z_a\otimes u_{ai}$$

(2) *We write $PG\curvearrowright PX$ when we have a morphism of C^*-algebras, as follows:*

$$\Phi:C(PX)\to C(PX)\otimes C(PG)$$
$$\Phi(z_iz_j)=\sum_a z_az_b\otimes u_{ai}u_{bj}$$

Observe that the above morphisms Φ, if they exist, are automatically coaction maps. Observe also that an affine action $G\curvearrowright X$ produces a projective action $PG\curvearrowright PX$. Let us also mention that given an algebraic subset $X\subset S^{N-1}_{\mathbb R,+}$, it is routine to prove that there exist indeed universal quantum groups $G\subset O_N^+$ acting as (1), and as in (2). We have the following result, from [11] and related papers, with respect to the above notions:

Theorem 7.22. *The quantum isometry groups of basic spheres and projective spaces,*

$$\begin{matrix}
S^{N-1}_{\mathbb R}&\longrightarrow&S^{N-1}_{\mathbb R,*}&\longrightarrow&S^{N-1}_{\mathbb R,+}\\
\downarrow&&\downarrow&&\downarrow\\
P^{N-1}_{\mathbb R}&\longrightarrow&P^{N-1}_{\mathbb C}&\longrightarrow&P^{N-1}_+
\end{matrix}$$

are the following affine and projective quantum groups,

$$\begin{matrix}
O_N&\longrightarrow&O_N^*&\longrightarrow&O_N^+\\
\downarrow&&\downarrow&&\downarrow\\
PO_N&\longrightarrow&PU_N&\longrightarrow&PO_N^+
\end{matrix}$$

with respect to the affine and projective action notions introduced above.

PROOF. The fact that the 3 quantum groups on top act affinely on the corresponding 3 spheres is known since [11], and is elementary, explained before. By restriction, the 3 quantum groups on the bottom follow to act on the corresponding 3 projective spaces. We must prove now that all these actions are universal. At right there is nothing to prove, so we are left with studying the actions on $S^{N-1}_\mathbb{R},S^{N-1}_{\mathbb{R},*}$ and on $P^{N-1}_\mathbb{R},P^{N-1}_\mathbb{C}$.

$\underline{P^{N-1}_\mathbb{R}}$. Consider the following projective coordinates:

$$p_{ij}=z_iz_j\quad,\quad w_{ij,ab}=u_{ai}u_{bj}$$

In terms of these projective coordinates, the coaction map is given by:

$$\Phi(p_{ij})=\sum_{ab}p_{ab}\otimes w_{ij,ab}$$

Thus, we have the following formulae:

$$\begin{aligned}\Phi(p_{ij})&=\sum_{a<b}p_{ab}\otimes(w_{ij,ab}+w_{ij,ba})+\sum_a p_{aa}\otimes w_{ij,aa}\\ \Phi(p_{ji})&=\sum_{a<b}p_{ab}\otimes(w_{ji,ab}+w_{ji,ba})+\sum_a p_{aa}\otimes w_{ji,aa}\end{aligned}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{ab}=z_az_b$ for $a\leq b$, we conclude that we must have:

$$w_{ij,ab}+w_{ij,ba}=w_{ji,ab}+w_{ji,ba}$$

Let us apply now the antipode to this formula. For this purpose, observe that:

$$\begin{aligned}S(w_{ij,ab})&=S(u_{ai}u_{bj})\\ &=S(u_{bj})S(u_{ai})\\ &=u_{jb}u_{ia}\\ &=w_{ba,ji}\end{aligned}$$

Thus by applying the antipode we obtain:

$$w_{ba,ji}+w_{ab,ji}=w_{ba,ij}+w_{ab,ij}$$

By relabelling, we obtain the following formula:

$$w_{ji,ba}+w_{ij,ba}=w_{ji,ab}+w_{ij,ab}$$

Now by comparing with the original relation, we obtain:

$$w_{ij,ab}=w_{ji,ba}$$

But, with $w_{ij,ab}=u_{ai}u_{bj}$, this formula reads:

$$u_{ai}u_{bj}=u_{bj}u_{ai}$$

Thus $G\subset O_N$, and it follows that we have $PG\subset PO_N$, as claimed.

$\underline{P^{N-1}_\mathbb{C}}$. Consider a coaction map, written as follows, with $p_{ab}=z_a\bar{z}_b$:

$$\Phi(p_{ij})=\sum_{ab}p_{ab}\otimes u_{ai}u_{bj}$$

The idea here will be that of using the following formula:

$$p_{ab}p_{cd}=p_{ad}p_{cb}$$

We have the following formulae:

$$\Phi(p_{ij}p_{kl}) = \sum_{abcd}p_{ab}p_{cd}\otimes u_{ai}u_{bj}u_{ck}u_{dl}$$
$$\Phi(p_{il}p_{kj}) = \sum_{abcd}p_{ad}p_{cb}\otimes u_{ai}u_{dl}u_{ck}u_{bj}$$

The terms at left being equal, and the last terms at right being equal too, we deduce that, with $[a,b,c]=abc-cba$, we must have the following formula:

$$\sum_{abcd}u_{ai}[u_{bj},u_{ck},u_{dl}]\otimes p_{ab}p_{cd}=0$$

Now since the quantities $p_{ab}p_{cd}=z_a\bar{z}_bz_c\bar{z}_d$ at right depend only on the numbers $|\{a,c\}|,|\{b,d\}|\in\{1,2\}$, and this dependence produces the only possible linear relations between the variables $p_{ab}p_{cd}$, we are led to $2\times 2=4$ equations, as follows:

(1) $u_{ai}[u_{bj},u_{ak},u_{bl}]=0$, $\forall a,b$.

(2) $u_{ai}[u_{bj},u_{ak},u_{dl}]+u_{ai}[u_{dj},u_{ak},u_{bl}]=0$, $\forall a$, $\forall b\neq d$.

(3) $u_{ai}[u_{bj},u_{ck},u_{bl}]+u_{ci}[u_{bj},u_{ak},u_{bl}]=0$, $\forall a\neq c$, $\forall b$.

(4) $u_{ai}[u_{bj},u_{ck},u_{dl}]+u_{ai}[u_{dj},u_{ck},u_{bl}]+u_{ci}[u_{bj},u_{ak},u_{dl}]+u_{ci}[u_{dj},u_{ak},u_{bl}]=0$, $\forall a\neq c,b\neq d$.

We will need in fact only the first two formulae. Since (1) corresponds to (2) at $b=d$, we conclude that (1,2) are equivalent to (2), with no restriction on the indices. By multiplying now this formula to the left by u_{ai}, and then summing over i, we obtain:

$$[u_{bj},u_{ak},u_{dl}]+[u_{dj},u_{ak},u_{bl}]=0$$

We use now a standard antipode/relabel trick. By applying the antipode we obtain:

$$[u_{ld},u_{ka},u_{jb}]+[u_{lb},u_{ka},u_{jd}]=0$$

By relabeling we obtain the following formula:

$$[u_{dl},u_{ak},u_{bj}]+[u_{dj},u_{ak},u_{bl}]=0$$

Now by comparing with the original relation, we obtain:

$$[u_{bj},u_{ak},u_{dl}]=[u_{dj},u_{ak},u_{bl}]=0$$

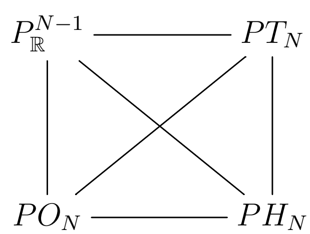
Thus $G\subset O_N^*$, and it follows that we have $PG\subset PU_N$, as desired. □

The above results can be probably improved. As an example, let us say that a closed subgroup $G\subset U_N^+$ acts projectively on PX when we have a coaction map as follows:

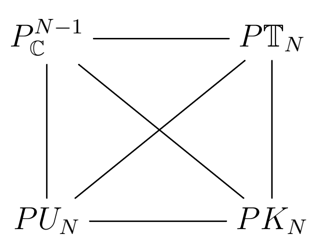
$$\Phi(z_iz_j)=\sum_{ab}z_az_b\otimes u_{ai}u_{bj}^*$$

The above proof can be adapted, by putting $*$ signs where needed, and Theorem 7.22 still holds, in this setting. However, establishing general universality results, involving arbitrary subgroups $H\subset PO_N^+$, looks like a quite non-trivial question.

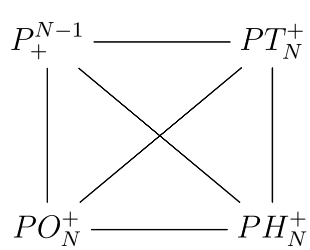
Let us discuss now the axiomatization question for the projective quadruplets of type (P,PT,PU,PK). We recall that we first have a classical real quadruplet, as follows:



We have then a classical complex quadruplet, which can be thought of as well as being a real half-classical quadruplet, which is as follows:

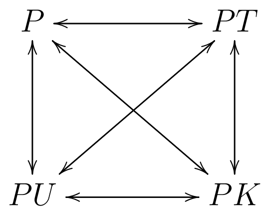


Finally, we have a free quadruplet, which can be thought of as being the same time real and complex, which is as follows:



The question is that of axiomatizing these quadruplets.

To be more precise, in analogy with what happens in the affine case, the problem is that of establishing correspondences as follows:



Modulo this problem, which is for the moment open, things are potentially quite nice, because we seem to have only 3 geometries, namely real, complex and free. Generally speaking, we are led in this way into several questions:

(1) We first need functoriality results for the operations $<,>$ and $\cap$, in relation with taking the projective version, and taking affine lifts, as to deduce most of our 7 axioms, in their obvious projective formulation, from the affine ones.

(2) Then, we need quantum isometry results in the projective setting, for the projective spaces themselves, and for the projective tori, either established ad-hoc, or by using the affine results. For the projective spaces, this was done above.

(3) We need as well some further functoriality results, in order to axiomatize the intermediate objects that we are dealing, the problem here being whether we want to use projective objects, or projective versions, perhaps saturated, of affine objects.

(4) Modulo this, things are quite clear, with the final result being the fact that we have only 3 projective geometries. Technically, the proof should be using the fact that, in the easy setting, $PO_N \subset PU_N \subset PO_N^+$ are the only possible unitary groups.

7d. Small dimensions

We would like to end this chapter with something refreshing, namely a preliminary study of the free analogue of $P^2_\mathbb{R}$. We recall that the projective space $P^{N-1}_\mathbb{R}$ is the space of lines in $\mathbb{R}^N$ passing through the origin, the basic examples being as follows:

(1) At $N = 2$ each such a line, in $\mathbb{R}^2$ passing through the origin, corresponds to 2 opposite points on the unit circle $\mathbb{T} \subset \mathbb{R}^2$. Thus, $P^1_\mathbb{R}$ corresponds to the upper semicircle of $\mathbb{T}$, with the endpoints identified, and so we obtain a circle, $P^1_\mathbb{R} = \mathbb{T}$.

(2) At $N = 3$ the situation is similar, with $P^2_\mathbb{R}$ corresponding to the upper hemisphere of the sphere $S^2_\mathbb{R} \subset \mathbb{R}^3$, with the points on the equator identified via $x = -x$. Topologically speaking, we can deform if we want the upper hemisphere into a square, with the equator becoming the boundary of this square, and in this picture, the $x = -x$ identification

corresponds to the "identify opposite edges, with opposite orientations" folding method for the square, leading to a space $P^2_\mathbb{R}$ which is obviously not embeddable into $\mathbb{R}^3$.

We recall that the free projective space is defined by the following formula:

$$C(P^{N-1}_+) = C^*\left((p_{ij})_{i,j=1,\ldots,N}\Big|p = p^* = p^2, Tr(p) = 1\right)$$

Let us first discuss, as a warm-up, the 2D case. Here the above matrix of projective coordinates is as follows, with $a = a^*$, $b = b^*$, $a + b = 1$:

$$p = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$$

We have the following computation:

$$p^2 = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix} = \begin{pmatrix} a^2 + cc^* & ac + cb \\ c^*a + bc^* & c^*c + b^2 \end{pmatrix}$$

Thus, the equations to be satisfied are as follows:

$$a^2 + cc^* = a$$

$$b^2 + c^*c = b$$

$$ac + cb = c$$

$$c^*a + bc^* = c^*$$

The 4th equation is the conjugate of the 3rd equation, so we remove it. By using $a + b = 1$, the remaining equations can be written as:

$$cc^* = c^*c = ab$$

$$ac + ca = 0$$

We have several explicit models for this, using the spheres $S^1_{\mathbb{R},+}$ and $S^1_{\mathbb{C},+}$, as well as the first row spaces of O_2^+ and U_2^+, which ultimately lead us to SU_2 and $\bar{S}U_2$. These models are known to be all equivalent under Haar, and the question is whether they are identical. Thus, we must do computations as above in all models, and compare. These are all interesting questions, whose precise answers are not known, so far.

In the 3D case now, that of projective space P^2_+, that we are mainly interested in here, the matrix of coordinates is as follows, with r, s, t self-adjoint, $r + s + t = 1$:

$$p = \begin{pmatrix} r & a & b \\ a^* & s & c \\ b^* & c^* & t \end{pmatrix}$$

The square of this matrix is given by:

$$p^2=\begin{pmatrix}r&a&b\\a^*&s&c\\b^*&c^*&t\end{pmatrix}\begin{pmatrix}r&a&b\\a^*&s&c\\b^*&c^*&t\end{pmatrix}$$

We obtain the following formula:

$$p^2=\begin{pmatrix}r^2+aa^*+bb^*&ra+as+bc^*&rb+ac+bt\\a^*r+sa^*+cb^*&a^*a+s^2+cc^*&a^*b+sc+ct\\b^*r+c^*a^*+tb^*&b^*a+c^*s+tc^*&b^*b+c^*c+t^2\end{pmatrix}$$

On the diagonal, the equations for $p^2=p$ are as follows:

$$aa^*+bb^*=r-r^2$$

$$a^*a+cc^*=s-s^2$$

$$b^*b+c^*c=t-t^2$$

On the off-diagonal upper part, the equations for $p^2=p$ are as follows:

$$ra+as+bc^*=a$$

$$rb+ac+bt=b$$

$$a^*b+sc+ct=c$$

On the off-diagonal lower part, the equations for $p^2=p$ are those above, conjugated. Thus, we have 6 equations. The first problem is that of using $r+s+t=1$, in order to make these equations look better. Again, many interesting questions here.

7e. Exercises

Exercises:

Exercise 7.23.

Exercise 7.24.

Exercise 7.25.

Exercise 7.26.

Exercise 7.27.

Exercise 7.28.

Bonus exercise.

CHAPTER 8

Matrix models

8a. Matrix models

You can model everything with random matrices, the saying in analysis goes. We have already seen an instance of this phenomenon in chapter 9, when talking about half-liberation. To be more precise, for certain manifolds $X\subset S^{N-1}_{\mathbb C,*}$, we have constructed embeddings of algebras of the following type, with Y being a certain classical manifold, and $T_1,\ldots,T_N\in M_2(C(Y))$ being certain suitable antidiagonal 2×2 matrices:

$$\pi:C(X)\subset M_2(C(Y))\quad,\quad x_i\to T_i$$

These models, which are quite powerful, were used afterwards in order to establish several non-trivial results on the original half-classical manifolds $X\subset S^{N-1}_{\mathbb C,*}$. Indeed, some knowledge and patience helping, any computation inside the target algebra $M_2(C(Y))$ can only be fun and doable, and produce results about $X\subset S^{N-1}_{\mathbb C,*}$ itself.

We discuss here, following [10], modeling questions for general manifolds $X\subset S^{N-1}_{\mathbb C,+}$, by using the same idea, suitably modified and generalized, as to cover most of the manifolds that we are interested in. Let us start with a broad definition, as follows:

Definition 8.1. *A model for a real algebraic manifold $X\subset S^{N-1}_{\mathbb C,+}$ is a morphism of C^*-algebras of the following type,*

$$\pi:C(X)\to B$$

with B being a C^*-algebra, called target of the model. We say that the model is faithful if π is faithful, in the usual sense.

Obviously, this is something too broad, because we can simply take $B=C(X)$, and we have in this way our faithful model, which is of course something unuseful:

$$id:C(X)\to C(X)$$

Thus, we must suitably restrict the class of target algebras B that we use, to algebras that we know well. However, this is something quite tricky, because if we want our model to be faithful, we cannot use simple algebras like the algebras $M_2(C(Y))$ used in the half-classical setting. In short, we are running into some difficulties here, of functional analytic nature, and a systematic discussion of all this is needed.

As a first objective, let us try to understand if an arbitrary manifold $X \subset S^{N-1}_{\mathbb{C},+}$ can be modelled by using familiar variables such as usual matrices, or operators. The answer here is yes, when using operators on a separable Hilbert space, with this coming from the GNS representation theorem, that we know from chapter 1, which is as follows:

THEOREM 8.2. *Any C^*-algebra A appears as closed $*$-algebra of operators on a Hilbert space, $A \subset B(H)$, in the following way:*

1. *In the commutative case, where $A = C(X)$, we can set $H = L^2(X)$, with respect to some probability measure on X, and use the embedding $g \to (g \to fg)$.*
2. *In general, we can set $H = L^2(A)$, with respect to some faithful positive trace $tr : A \to \mathbb{C}$, and then use a similar embedding, $a \to (b \to ab)$.*

PROOF. This is something that we already know, from chapter 1, coming from basic measure theory and functional analysis, the idea being as follows:

(1) In the commutative case, where $A = C(X)$ by the Gelfand theorem, we can pick a probability measure on X, and then we have an embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

(2) In general, assuming that a linear form $\varphi : A \to \mathbb{C}$ has suitable positivity properties, we can define a scalar product on A, by the following formula:

$$< a, b >= \varphi(ab^*)$$

By completing we obtain a Hilbert space H, and we have a representation as follows, called GNS representation of our algebra with respect to the linear form φ:

$$A \to B(H) \quad , \quad a \to (b \to ab)$$

Moreover, when $\varphi : A \to \mathbb{C}$ has suitable faithfulness properties, making it analogous to the integration functionals $\int_X : A \to \mathbb{C}$ from the commutative case, with respect to faithful probability measures on X, this representation is faithful, as desired. ☐

Now back to our questions, the above result tells us that we have:

THEOREM 8.3. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, coming via*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

we have a morphism of C^*-algebras as follows,

$$\pi : C(X) \to B(H) \quad , \quad x_i \to T_i$$

whenever the operators $T_i \in B(H)$ satisfy the following relations:

$$\sum_i T_i T_i^* = \sum_i T_i^* T_i = 1 \quad , \quad f_\alpha(T_1, \ldots, T_N) = 0$$

Moreover, we can always find a Hilbert space H and operators $\{T_i\}$ such that π is faithful.

Proof. Here the first assertion is more of an empty statement, explaining the definition of the algebra $C(X)$, via generators and relations, and the second assertion is something non-trivial, coming as a consequence of the GNS theorem. □

In practice now, all this is a bit too general, and not very useful. We need a good family of target algebras B, that we understand well. And here, we can use:

Definition 8.4. *A random matrix C^*-algebra is an algebra of type*

$$B = M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

The terminology here comes from the fact that, in practice, the space T usually comes with a probability measure on it, which makes the elements of B "random matrices". Observe that we can write our random matrix algebra as follows:

$$B = M_K(\mathbb{C}) \otimes C(T)$$

Thus, the random matrix algebras appear by definition as tensor products of the simplest types of C^*-algebras that we know, namely the full matrix algebras, $M_K(\mathbb{C})$ with $K \in \mathbb{N}$, and the commutative algebras, $C(T)$, with T being a compact space. Getting back now to our modelling questions for manifolds, we can formulate:

Definition 8.5. *A matrix model for a noncommutative algebraic manifold*

$$X \subset S^{N-1}_{\mathbb{C},+}$$

is a morphism of C^*-algebras of the following type,

$$\pi : C(X) \to M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

As a first observation, when X happens to be classical, we can take $K = 1$ and $T = X$, and we have a faithful model for our manifold, namely:

$$id : C(X) \to M_1(C(X))$$

In general, we cannot use $K = 1$, and the smallest value $K \in \mathbb{N}$ doing the job, if any, will correspond somehow to the "degree of noncommutativity" of our manifold.

Before getting into this, we would like to clarify a few abstract issues. As mentioned above, the algebras of type $B = M_K(C(T))$ are called random matrix C^*-algebras. The reason for this is the fact that most of the interesting compact spaces T come by definition with a natural probability measure of them. Thus, B is a subalgebra of the bigger algebra $B'' = M_K(L^\infty(T))$, usually known as a "random matrix algebra".

This perspective is quite interesting for us, because most of our examples of manifolds $X \subset X^{N-1}_{\mathbb{C},+}$ appear as homogeneous spaces, and so are measured spaces too. Thus, we can further ask for our models $C(X) \to M_K(C(T))$ to extend into models of the following type, which can be of help in connection with integration problems:

$$L^\infty(X) \to M_K(L^\infty(T))$$

In short, time now to talk about L^∞-functions, in the noncommutative setting.

8b. Operator algebras

In order to discuss all this, we will need some basic von Neumann algebra theory, coming as a complement to the basic C^*-algebra theory from chapter 1. Let us start with a key result in functional analysis, as follows:

PROPOSITION 8.6. *For an operator algebra $A \subset B(H)$, the following are equivalent:*

(1) *A is closed under the weak operator topology, making each of the linear maps $T \to < Tx, y >$ continuous.*
(2) *A is closed under the strong operator topology, making each of the linear maps $T \to Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

PROOF. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S \subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A \subset B(H)$, strongly closed implies weakly closed.

(2) But this latter fact is something standard, which can be proved via an amplification trick. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator $T \in \bar{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x \in K$:

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)x} \\ &\implies \pi(T)x \in \overline{\pi(A)x}^{||\cdot||} \end{aligned}$$

Now observe that the last formula tells us that for any $x = (x_1, \ldots, x_n)$, and any $\varepsilon > 0$, we can find $S \in A$ such that the following holds, for any i:

$$||Sx_i - Tx_i|| < \varepsilon$$

Thus T belongs to the strong operator closure of A, as desired. □

In the above the terminology, while standard, is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution, we agree in what follows to call the norm topology "strong", and the weak and strong operator topologies "weak", whenever these two topologies coincide. With this convention, the algebras from Proposition 8.6 are those which are weakly closed, and we can formulate:

DEFINITION 8.7. *A von Neumann algebra is a $*$-algebra of operators*

$$A \subset B(H)$$

which is closed under the weak topology.

As basic examples, we have the algebra $B(H)$ itself, then the singly generated von Neumann algebras, $A = < T >$, with $T \in B(H)$, and then the multiply generated von Neumann algebras, namely $A = < T_i >$, with $T_i \in B(H)$. At the level of the general results, we first have the bicommutant theorem of von Neumann, as follows:

THEOREM 8.8. *For a $*$-algebra $A \subset B(H)$, the following are equivalent:*

(1) *A is weakly closed, so it is a von Neumann algebra.*
(2) *A equals its algebraic bicommutant A'', taken inside $B(H)$.*

PROOF. Since the commutants are automatically weakly closed, it is enough to show that weakly closed implies $A = A''$. For this purpose, we will prove something a bit more general, stating that given a $*$-algebra of operators $A \subset B(H)$, the following holds, with A'' being the bicommutant inside $B(H)$, and with $\bar{A}$ being the weak closure:

$$A'' = \bar{A}$$

We prove this equality by double inclusion, as follows:

"$\supset$" Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S \subset S''$, valid for any set $S \subset B(H)$. In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \to T$ in this topology, we have:

$$\begin{aligned}
T_i \in A'' &\implies ST_i = T_iS,\ \forall S \in A' \\
&\implies ST = TS,\ \forall S \in A' \\
&\implies T \in A
\end{aligned}$$

Thus our claim is proved, and together with Proposition 8.6, which allows us to pass from the strong to the weak operator topology, this gives the desired inclusion:

$$\bar{A} \subset A''$$

"$\subset$" Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 8.5. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \Big| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \ldots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \ldots & S_{nn} \end{pmatrix} \Big| S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any $x \in K$, we conclude that any $T \in A''$ belongs to the strong operator closure of A. By using now Proposition 8.5, which allows us to pass from the strong to the weak operator closure, we conclude that we have $A'' \subset \bar{A}$, as desired. ☐

In order to develop now some general theory, let us start by investigating the finite dimensional case. Here the ambient operator algebra is $B(H) = M_N(\mathbb{C})$, and any subspace $A \subset B(H)$ is automatically closed, for all 3 topologies from Proposition 8.6. Thus, we are left with the question of investigating the $*$-algebras of usual matrices $A \subset M_N(\mathbb{C})$. But this is a purely algebraic question, whose answer is as follows:

THEOREM 8.9. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $n_1, \ldots, n_k \in \mathbb{N}$ satisfying $n_1 + \ldots + n_k = N$, we have an obvious embedding of $*$-algebras, via matrix blocks, as follows:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \to UMU^*$$

(2) In the other sense now, consider an arbitrary $*$-algebra of the $N \times N$ matrices, $A \subset M_N(\mathbb{C})$. Let us first look at the center of this algebra, which given by:

$$Z(A) = A \cap A'$$

It is elementary to prove that this center, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \ldots, e_k \in \mathbb{C}^k$, and let $p_1, \ldots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \ldots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \ldots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2 = p_i^* = p_i$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \ldots + p_k = 1$ and the projection equations $p_i^2 = p_i^* = p_i$, we conclude that we have the needed generation property, namely:

$$A_1 + \ldots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \ldots + p_k = 1$, and from the projection equations $p_i^2 = p_i^* = p_i$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i = rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{n_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A = A_1 \oplus \ldots \oplus A_k \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Moreover, a careful look at the isomorphisms established in Step 3 shows that at the global level, of the algebra A itself, the above isomorphism comes by twisting the standard multimatrix embedding $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$, discussed in the beginning of the proof, (1) above, by a certain unitary $U \in U_N$. Thus, we obtain the result. □

As an application of Theorem 8.9, clarifying the relation with linear algebra, or operator theory in finite dimensions, we have the following result:

Proposition 8.10. *Given an operator $T\in B(H)$ in finite dimensions, $H=\mathbb C^N$, the von Neumann algebra $A=<T>$ that it generates inside $B(H)=M_N(\mathbb C)$ is*

$$A=M_{r_1}(\mathbb C)\oplus\ldots\oplus M_{r_k}(\mathbb C)$$

with the sizes of the blocks $r_1,\ldots,r_k\in\mathbb N$ coming from the spectral theory of the associated matrix $M\in M_N(\mathbb C)$. In the normal case $TT^*=T^*T$, this decomposition comes from

$$T=UDU^*$$

with $D\in M_N(\mathbb C)$ diagonal, and with $U\in U_N$ unitary.

Proof. This is something standard, by using the basic linear algebra theory and spectral theory for the usual matrices $M\in M_N(\mathbb C)$. □

Let us get now to infinite dimensions, with Proposition 8.10 as our main source of inspiration. We have here the following result:

Theorem 8.11. *Given an operator $T\in B(H)$ which is normal,*

$$TT^*=T^*T$$

the von Neumann algebra $A=<T>$ that it generates inside $B(H)$ is

$$<T>=L^\infty(\sigma(T))$$

with $\sigma(T)$ being its spectrum, formed of numbers $\lambda\in\mathbb C$ such that $T-\lambda$ is not invertible.

Proof. This is something standard as well, by using the spectral theory for the normal operators $T\in B(H)$, coming from chapter 1. □

More generally, along the same lines, we have the following result, dealing this time with commuting families of normal operators:

Theorem 8.12. *Given operators $T_i\in B(H)$ which are normal, and which commute, the von Neumann algebra $A=<T_i>$ that these operators generates inside $B(H)$ is*

$$<T_i>=L^\infty(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

Proof. This is again routine, by using this time the spectral theory for the families of commuting normal operators $T_i\in B(H)$. See for instance Blackadar. □

As an interesting abstract consequence of this, we have:

Theorem 8.13. *The commutative von Neumann algebras are the algebras of type*

$$A=L^\infty(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X, we can realize the commutative algebra $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H. But this is something that we already know, coming from the multiplicity operators $T_f(g) = fg$ from the proof of the GNS theorem, the representation being as follows:

$$L^\infty(X) \subset B(L^2(X))$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X, and an identification $A = L^\infty(X)$. But this follows from Theorem 8.12, because we can write our algebra as follows:

$$A =< T_i >$$

To be more precise, A being commutative, any element $T \in A$ is normal. Thus, we can pick a basis $\{T_i\} \subset A$, and then we have $A =< T_i >$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 8.12 applies, and gives the result. □

Moving ahead now, we can combine Proposition 8.8 with Theorem 8.13, and by building along the lines of Theorem 8.9, but this time in infinite dimensions, we are led to the following statement, due to Murray-von Neumann and Connes:

THEOREM 8.14. *Given a von Neumann algebra $A \subset B(H)$, if we write its center as*

$$Z(A) = L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x having trivial center:

$$A = \int_X A_x \, dx$$

Moreover, the factors, $Z(A) = \mathbb{C}$, can be basically classified in terms of the II_1 *factors, which are those satisfying* $\dim A = \infty$*, and having a faithful trace $tr : A \to \mathbb{C}$.*

PROOF. This is something that we know to hold in finite dimensions, as a consequence of Theorem 8.9. In general, this is something heavy, the idea being as follows:

(1) This is von Neumann's reduction theory main result, whose statement is already quite hard to understand, and whose proof uses advanced functional analysis.

(2) This is heavy, due to Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions. □

All this is certainly quite advanced, taking substantial time to be fully understood. For general reading on von Neumann algebras we recommend the book of Blackadar, but be aware tough that, while being at the same time well-written, condensed and reasonably thick, that book is only an introduction to Theorem 8.14. So, if we want to learn the full theory, with the complete proof of Theorem 8.14, you will have to go, every now and then, through the original papers of Murray-von Neumann and Connes.

Now back to work, and our noncommutative geometry questions, as a first application of the above, we can extend our noncommutative space setting, as follows:

THEOREM 8.15. *Consider the category of "noncommutative measure spaces", having as objects the pairs (A,tr) consisting of a von Neumann algebra with a faithful trace, and with the arrows reversed, which amounts in writing $A=L^\infty(X)$ and $tr=\int_X$.*

1. *The category of usual measured spaces embeds into this category, and we obtain in this way the objects whose associated von Neumann algebra is commutative.*
2. *Each C^*-algebra given with a trace produces as well a noncommutative measure space, by performing the GNS construction, and taking the weak closure.*
3. *In what regards the finitely generated group duals, or more generally the compact matrix quantum groups, the corresponding identification is injective.*
4. *Even more generally, for noncommutative algebraic manifolds having an integratiuon functional, like the spheres, the identification is injective.*

PROOF. This is clear indeed from the basic properties of the GNS construction, from Theorem 8.2, and from the general theory from Theorem 8.14. □

Before getting back to matrix models, we would like to formulate the following result, in relation with our axiomatization questions discussed in the above:

THEOREM 8.16. *In the context of noncommutative geometries coming from quadruplets (S,T,U,K), we have von Neumann algebras, with traces, as follows,*

$$\begin{array}{ccc} L^\infty(S) & \longleftrightarrow & L^\infty(T) \\ \updownarrow & \times & \updownarrow \\ L^\infty(U) & \longleftrightarrow & L^\infty(K) \end{array}$$

with $L^\infty(S)\subset L^\infty(U)$ being obtained by taking the first row algebra.

PROOF. This follows indeed from the various results that we already have, from above, by using the general formalism from Theorem 8.15. □

This statement, which is quite interesting, philosophically speaking, raises the question of axiomatizing, or rather re-axiomatizing, the quadruplets (S,T,U,K) that we are interested in directly in terms of the associated von Neumann algebras, as above. Indeed, in view of our general quantum mechanics motivations, we are after all mostly interested in integrating over our quantum manifolds, and so with this is mind, the von Neumann algebra formalism seems to be the one which is best adapted to our questions.

However, this is wrong. The above result is something theoretical, because it assumes the existence of Haar measures on our spaces S,T,U,K, which itself is something coming as a theorem. Thus, while all this is nice, the good way of doing things is with C^*-algebras, as we did before. And the von Neumann algebras from Theorem 8.16 remain something more advanced and specialized, coming afterwards.

As a side comment here, and for ending with some physics, the question "does the algebra or the Hilbert space come first" is a well-known one in quantum mechanics, basically leading to 2 different schools of thought. We obviously adhere here to the "algebra comes first" school. But let us not get here into this, perhaps enough controversies discussed, so far in this book. For more on this, get to know about the Bohr vs Einstein debate, which is the mother of all debates, in quantum mechanics.

8c. Matrix truncations

In relation now with the modelling questions that we are interested in here, with all the above operator algebra material digested, we can now go ahead with our program, and discuss von Neumann algebraic extensions. We have the following result:

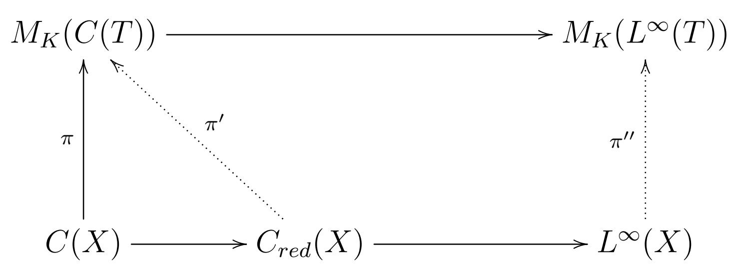
THEOREM 8.17. *Given a matrix model $\pi : C(X) \to M_K(C(T))$, with both X, T being assumed to have integration functionals, the following are equivalent:*

1. *π is stationary, in the sense that $\int_X = (tr \otimes \int_T)\pi$.*
2. *π produces an inclusion $\pi' : C_{red}(X) \subset M_K(X(T))$.*
3. *π produces an inclusion $\pi'' : L^\infty(X) \subset M_K(L^\infty(T))$.*

Moreover, in the quantum group case, these conditions imply that π is faithful.

PROOF. This is standard functional analysis, as follows:

(1) Consider the following diagram, with all the solid arrows being by definition the canonical maps between the algebras concerned:



(2) With this picture in hand, the implications (1) $\iff$ (2) $\iff$ (3) between the conditions (1,2,3) in the statement are all clear, coming from the basic properties of the GNS construction, and of the von Neumann algebras, explained in the above.

(3) As for the last assertion, this is something more subtle, coming from the fact that if $L^\infty(G)$ is of type I, as required by (3), then G must be coamenable. $\square$

The above result raises a number of interesting questions, notably in what regards the extension of the last assertion, to the case of more general homogeneous spaces.

Before going further, we would like to record as well the following key result regarding the matrix models, valid so far in the quantum group case only:

THEOREM 8.18. *Consider a matrix model $\pi : C(G) \to M_K(C(T))$ for a closed subgroup $G \subset U_N^+$, with T being assumed to be a compact probability space.*

1. *There exists a smallest subgroup $G' \subset G$, producing a factorization of type:*
$$\pi : C(G) \to C(G') \to M_K(C(T))$$
The algebra $C(G')$ is called Hopf image of π.
2. *When π is inner faithful, in the sense that $G = G'$, we have the formula*
$$\int_G = \lim_{k\to\infty} \sum_{r=1}^k \varphi^{*r}$$
where $\varphi = (tr \otimes \int_T)\pi$, and $\phi * \psi = (\phi \otimes \psi)\Delta$.

PROOF. All this is well-known, but quite specialized, the idea being as follows:

(1) This follows by dividing the algebra $C(G)$ by a suitable ideal, namely the Hopf ideal generated by the kernel of the matrix model map $\pi : C(G) \to M_K(C(T))$.

(2) This follows by suitably adapting Woronowicz's proof for the existence and formula of the Haar integration functional from [99], to the matrix model situation. □

The above result is quite important, for a number of reasons. Indeed, as a main application of it, while the existence of a faithful matrix model $\pi : C(G) \subset M_K(C(T))$ forces the C^*-algebra $C(G)$ to be of type I, and so G to be coamenable, as already mentioned in the proof of Theorem 8.17, there is no known restriction coming from the existence of an inner faithful model $\pi : C(G) \to M_K(C(T))$.

In the general manifold setting, talking about such things is in general not possible, unless our manifold X has some extra special structure, as for instance being an homogeneous space, in the spirit of the various such spaces discussed in chapters 5-6. However, in practice, such a theory has not been developed yet.

Let us go back now to our basic notion of a matrix model, from Definition 8.5, and develop some more general theory, in that setting. We first have:

PROPOSITION 8.19. *A 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ must come from a map*
$$p : T \to X_{class} \subset X$$
and π is faithful precisely when $X = X_{class}$, and when p is surjective.

PROOF. According to our conventions, a 1×1 model for a manifold $X\subset S^{N-1}_{\mathbb{C},+}$ is simply a morphism of algebras $\pi:C(X)\to C(T)$. Now since $C(T)$ is commutative, this morphism must factorize through the abelianization of $C(X)$, as follows:

$$\pi:C(X)\to C(X_{class})\to C(T)$$

Thus, our morphism π must come by transposition from a map p, as claimed. □

Following [10], in order to generalize the above trivial fact, we can use:

DEFINITION 8.20. *Let $X\subset S^{N-1}_{\mathbb{C},+}$. We define a closed subspace $X^{(K)}\subset X$ by*

$$C(X^{(K)})=C(X)/J_K$$

where J_K is the common null space of matrix representations of $C(X)$, of size $L\leq K$,

$$J_K=\bigcap_{L\leq K}\ \bigcap_{\pi:C(X)\to M_L(\mathbb{C})}\ker(\pi)$$

and we call $X^{(K)}$ the "part of X which is realizable with $K\times K$ models".

As a basic example here, the first such space, at $K=1$, is the classical version:

$$X^{(1)}=X_{class}$$

Observe that we have embeddings of quantum spaces, as follows:

$$X^{(1)}\subset X^{(2)}\subset X^{(3)}\ldots\ldots\subset X$$

As a first result now on these spaces, we have the following well-known fact:

THEOREM 8.21. *The increasing union of compact quantum spaces*

$$X^{(\infty)}=\bigcup_{K\geq 1}X^{(K)}$$

equals X precisely when the algebra $C(X)$ is residually finite dimensional.

PROOF. This is something well-known. We refer to Chirvasitu for a discussion on this topic, in the context of the quantum groups, and to [10] for more. □

Getting back now to the case $K<\infty$, we first have, following [10]:

PROPOSITION 8.22. *Consider an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$.*

1. *Given a closed subspace $Y\subset X\subset S^{N-1}_{\mathbb{C},+}$, we have $Y\subset X^{(K)}$ precisely when any irreducible representation of $C(Y)$ has dimension $\leq K$.*
2. *In particular, we have $X^{(K)}=X$ precisely when any irreducible representation of $C(X)$ has dimension $\leq K$.*

Proof. This follows from general C^*-algebra theory, as follows:

(1) If any irreducible representation of $C(Y)$ has dimension $\leq K$, then we have $Y \subset X^{(K)}$, because the irreducible representations of a C^*-algebra separate its points. Conversely, assuming $Y \subset X^{(K)}$, it is enough to show that any irreducible representation of the algebra $C(X^{(K)})$ has dimension $\leq K$. But this is once again well-known.

(2) This follows indeed from (1). ☐

The connection with the previous considerations comes from:

Theorem 8.23. *If $X \subset S^{N-1}_{\mathbb{C},+}$ has a faithful matrix model*

$$C(X) \to M_K(C(T))$$

then we have $X = X^{(K)}$.

Proof. This follows from the above and from the standard representation theory for the C^*-algebras. For full details on all this, we refer as before to [10]. ☐

We can now discuss the universal $K \times K$-matrix model, constructed as follows:

Theorem 8.24. *Given $X \subset S^{N-1}_{\mathbb{C},+}$ algebraic, the category of its $K \times K$ matrix models, with $K \geq 1$ being fixed, has a universal object as follows:*

$$\pi_K : C(X) \to M_K(C(T_K))$$

That is, given a model $\rho : C(X) \to M_K(C(T))$, we have a diagram of type

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \pi\ } & M_K(C(T_K)) \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & M_K(C(T)) & \end{array}$$

where the map on the right is unique, and arises from a continuous map $T \to T_K$.

Proof. Consider the universal commutative C^*-algebra generated by elements $x_{ij}(a)$, with $1 \leq i, j \leq K$ and $a \in \mathcal{O}(X)$, subject to the following relations:

$$x_{ij}(a + \lambda b) = x_{ij}(a) + \lambda x_{ij}(b)$$

$$x_{ij}(ab) = \sum_k x_{ik}(a) x_{kj}(b)$$

$$x_{ij}(1) = \delta_{ij}$$

$$x_{ij}(a)^* = x_{ji}(a^*)$$

This algebra is indeed well-defined because of the following relations:

$$\sum_l \sum_k x_{ik}(z_l^*) x_{ki}(z_l) = 1$$

Now let T_K be the spectrum of this algebra. Since X is algebraic, we have:

$$\pi : C(X) \to M_K(C(T_K)) \quad , \quad \pi(z_k) = (x_{ij}(z_k))$$

By construction of T_K and π, we have the universal matrix model. See [10]. □

Still following [10], as an illustration for the above, we have:

PROPOSITION 8.25. *Let $X \subset S^{N-1}_{\mathbb{C},+}$ with X algebraic and $X_{class} \neq \emptyset$, and let*

$$\pi : C(X) \to M_K(C(T_K))$$

be the universal matrix model. Then we have

$$C(X^{(K)}) = C(X)/Ker(\pi)$$

and hence $X = X^{(K)}$ if and only if X has a faithful $K \times K$-matrix model.

PROOF. We have to prove that $Ker(\pi) = J_K$, the latter ideal being the intersection of the kernels of all matrix representations as follows, with $L \leq K$:

$$C(X) \to M_L(\mathbb{C})$$

For $a \notin Ker(\pi)$, we see that $a \notin J_K$ by evaluating at an appropriate element of T_K. Conversely, assume that we are given $a \in Ker(\pi)$. Let $\rho : C(X) \to M_L(\mathbb{C})$ be a representation with $L \leq K$, and let $\varepsilon : C(X) \to \mathbb{C}$ be a representation. We can extend ρ to a representation $\rho' : C(X) \to M_K(\mathbb{C})$ by letting, for any $b \in C(X)$:

$$\rho'(b) = \begin{pmatrix} \rho(b) & 0 \\ 0 & \varepsilon(b) I_{K-L} \end{pmatrix}$$

The universal property of the universal matrix model yields that $\rho'(a) = 0$, since $\pi(a) = 0$. Thus $\rho(a) = 0$. We therefore have $a \in J_K$, and $Ker(\pi) \subset J_K$, and the first statement is proved. The last statement follows from the first one. See [10]. □

Next, we have the following result, also from [10]:

PROPOSITION 8.26. *Let $X \subset S^{N-1}_{\mathbb{C},+}$ be algebraic, and satisfying:*

$$X_{class} \neq \emptyset$$

Then $X^{(K)}$ is algebraic as well.

PROOF. We keep the notations above, and consider the following map:

$$\pi_0 : \mathcal{O}(X) \to M_K(C(T_K)) \quad , \quad z_l \to (x_{ij}(z_l))$$

This induces a $*$-algebra map, as follows:

$$\tilde{\pi_0} : C^*(\mathcal{O}(X)/Ker(\pi_0)) \to M_K(C(T_K))$$

We need to show that $\tilde{\pi}_0$ is injective. For this purpose, observe that the universal model factorizes as follows, where p is canonical surjection:

$$\pi : C(X) \xrightarrow{p} C^*(\mathcal{O}(X)/Ker(\pi_0)) \xrightarrow{\tilde{\pi}_0} M_K(C(T_K))$$

We therefore obtain $Ker(\pi)=Ker(p)$, and we conclude that:

$$C(X^{(K)})=C(X)/Ker(p)=C^*(\mathcal{O}(X)/Ker(\pi_0))$$

Thus $X^{(K)}$ is indeed algebraic. Since $\mathcal{O}(X)/Ker(\pi_0)$ is isomorphic to a $*$-subalgebra of $M_K(C(T_K))$, it satisfies the standard Amitsur-Levitski polynomial identity:

$$S_{2K}(x_1,\ldots,x_{2K})=0$$

By density, so does $C^*(\mathcal{O}(X)/Ker(\pi_0))$. Thus any irreducible representation of the algebra $C^*(\mathcal{O}(X)/Ker(\pi_0))$ has dimension $\leq K$. Consider now an element as follows:

$$a\in C^*(\mathcal{O}(X)/Ker(\pi_0))$$

Assuming $a\neq 0$ we can, by the same reasoning as in the previous proof, find a representation as follows, such that $\rho(a)\neq 0$:

$$\rho:C^*(\mathcal{O}(X)/Ker(\pi_0))\to M_K(\mathbb{C})$$

Indeed, a given algebra map $\varepsilon:C(X)\to\mathbb{C}$ induces an algebra map as follows:

$$C(T_K)\to\mathbb{C}\quad,\quad x_{ij}(a)\to\delta_{ij}\varepsilon(a)$$

But this map enables us to extend representations, as before. By construction the universal model space yields an algebra map as follows:

$$M_K(C(T_K))\to M_K(\mathbb{C})$$

The composition with $\tilde{\pi}_0 p=\pi$ is then ρp, so $\tilde{\pi}_0(a)\neq 0$, and $\tilde{\pi}_0$ is injective. □

Summarizing, we have proved the following result:

THEOREM 8.27. *Let $X\subset S^{N-1}_{\mathbb{C},+}$ be algebraic, satisfying $X_{class}\neq\emptyset$. Then we have an increasing sequence of algebraic submanifolds*

$$X_{class}=X^{(1)}\subset X^{(2)}\subset X^{(3)}\subset\ldots\ldots\subset X$$

where $X^{(K)}$ is given by the fact that

$$C(X^{(K)})\subset M_K(C(T_K))$$

is obtained by factorizing the universal matrix model.

PROOF. This follows indeed from the above results. See [10]. □

There are many other things that can be said about the above matrix truncations $X^{(K)}$, and we refer here to [10] and related papers. However, the main problem remains that of suitably fine-tuning this theory, as to make it compatible with the theory of matrix models for the Woronowicz algebras, which itself is something quite advanced, and rather satisfactory. To be more precise here, the situation is as follows:

(1) As a first observation, when taking as input a quantum group, $X=G$, the above truncation procedure does not produce a quantum group at $K\geq 2$, because the

compultiplication Δ does not factorize. Thus, Theorem 8.27 as stated remains something a bit orthogonal to what is known about the matrix models for quantum groups.

(2) Conversely, as already said before, the main results on the matrix models for quantum groups regard the notion of inner faithfulness from Theorem 8.18. And such results cannot extend to general manifolds $X\subset S^{N-1}_{\mathbb{C},+}$, unless we are dealing with special classes of homogeneous spaces, in the spirit of those discussed in chapters 5-6.

Summarizing, many things to be done. The main problem is probably that of talking about inner faithful models for affine homogeneous spaces, but the general theory here is unknown, at least so far. Finally, let us mention that, in the quantum group setting, the known theory of matrix models was heavily inspired by the work of Jones [**62**], [**63**], [**64**], in connection with general problems in statistical mechanics, and in what regards the extension of this to the case of more general homogeneous spaces, or other algebraic manifolds, the motivations remain a bit too advanced to be fully understood.

8d. Half-liberation

As a nice illustration for the above modeling theory, let us discuss now the half-liberation operation, which is connected to $X^{(2)}$, as a continuation of the material from above. We first restrict the attention to the real case. Let us start with:

Definition 8.28. *The half-classical version of a manifold $X\subset S^{N-1}_{\mathbb{R},+}$ is given by:*

$$C(X^*)=C(X)\Big/\left\langle abc=cba\Big|\forall a,b,c\in\{x_i\}\right\rangle$$

We say that X is half-classical when $X=X^*$.

Observe the obvious similarity with the construction of the classical version. In fact, philosophically, this definition is some sort of "next level" definition for the classical version, assuming that you managed, via some sort of yoga, to be as familiar with half-commutation, $abc=cba$, as you are with usual commutation, $ab=ba$.

In order to understand now the structure of X^*, we can use an old matrix model method, which goes back to Bichon and Dubois-Violette, and then to Bichon [**23**]. This is based on the following observation, that we already met in the above:

Proposition 8.29. *For any $z\in\mathbb{C}^N$, the matrices*

$$X_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

are self-adjoint, and half-commute.

PROOF. The matrices X_i are indeed self-adjoint, and their products are given by:

$$X_iX_j = \begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}\begin{pmatrix}0&z_j\\ \bar{z}_j&0\end{pmatrix} = \begin{pmatrix}z_i\bar{z}_j&0\\ 0&\bar{z}_iz_j\end{pmatrix}$$

Also, we have as well the following formula:

$$X_iX_jX_k = \begin{pmatrix}z_i\bar{z}_j&0\\ 0&\bar{z}_iz_j\end{pmatrix}\begin{pmatrix}0&z_k\\ \bar{z}_k&0\end{pmatrix} = \begin{pmatrix}0&z_i\bar{z}_jz_k\\ \bar{z}_iz_j\bar{z}_k&0\end{pmatrix}$$

Now since this latter quantity is symmetric in i,k, we obtain from this that we have the half-commutation formula $X_iX_jX_k = X_kX_jX_i$, as desired. □

The idea now will be that of using the matrices in Proposition 8.29 in order to model the coordinates of arbitrary half-classical manifolds. In order to connect the algebra of the classical coordinates z_i to that of the noncommutative coordinates X_i, we will need:

DEFINITION 8.30. *Given a noncommutative polynomial $f \in \mathbb{R} < x_1,\ldots,x_N >$ in N variables, we define a usual polynomial in $2N$ variables*

$$f^\circ \in \mathbb{R}[z_1,\ldots,z_N,\bar{z}_1,\ldots,\bar{z}_N]$$

according to the formula

$$f = x_{i_1}x_{i_2}x_{i_3}x_{i_4}\ldots \implies f^\circ = z_{i_1}\bar{z}_{i_2}z_{i_3}\bar{z}_{i_4}\ldots$$

in the monomial case, and then by extending this correspondence, by linearity.

As a basic example here, the polynomial defining the free real sphere $S^{N-1}_{\mathbb{R},+}$ produces in this way the polynomial defining the complex sphere $S^{N-1}_{\mathbb{C}}$:

$$f = x_1^2 + \ldots + x_N^2 \implies f^\circ = |z_1|^2 + \ldots + |z_N|^2$$

Also, given a polynomial $f \in \mathbb{R} < x_1,\ldots,x_N >$, we can decompose it into its even and odd parts, $f = g + h$, by putting into g/h the monomials of even/odd length. Observe that with $z = (z_1,\ldots,z_N)$, these odd and even parts are given by:

$$g(z) = \frac{f(z)+f(-z)}{2} \quad , \quad h(z) = \frac{f(z)-f(-z)}{2}$$

With these conventions, we have the following result:

PROPOSITION 8.31. *Given a manifold X, coming from a family of noncommutative polynomials $\{f_\alpha\} \subset \mathbb{R} < x_1,\ldots,x_N >$, we have a morphism algebras*

$$\pi : C(X) \to M_2(\mathbb{C}) \quad , \quad \pi(x_i) = \begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

precisely when $z = (z_1,\ldots,z_N) \in \mathbb{C}^N$ belongs to the real algebraic manifold

$$Y = \left\{z \in \mathbb{C}^N \middle| g_\alpha^\circ(z_1,\ldots,z_N) = h_\alpha^\circ(z_1,\ldots,z_N) = 0, \forall\alpha\right\}$$

where $f_\alpha = g_\alpha + h_\alpha$ is the even/odd decomposition of f_α.

Proof. Let X_i be the matrices in the statement. In order for $x_i \to X_i$ to define a morphism of algebras, these matrices must satisfy the equations defining X. Thus, the space Y in the statement consists of the points $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ satisfying:

$$f_\alpha(X_1, \ldots, X_N) = 0 \quad , \quad \forall\alpha$$

Now observe that the matrices X_i in the statement multiply as follows:

$$X_{i_1}X_{j_1}\ldots X_{i_k}X_{j_k} = \begin{pmatrix} z_{i_1}\bar{z}_{j_1}\ldots z_{i_k}\bar{z}_{j_k} & 0 \\ 0 & \bar{z}_{i_1}z_{j_1}\ldots\bar{z}_{i_k}z_{j_k} \end{pmatrix}$$

$$X_{i_1}X_{j_1}\ldots X_{i_k}X_{j_k}X_{i_{k+1}} = \begin{pmatrix} 0 & z_{i_1}\bar{z}_{j_1}\ldots z_{i_k}\bar{z}_{j_k}z_{i_{k+1}} \\ \bar{z}_{i_1}z_{j_1}\ldots\bar{z}_{i_k}z_{j_k}\bar{z}_{i_{k+1}} & 0 \end{pmatrix}$$

We therefore obtain, in terms of the even/odd decomposition $f_\alpha = g_\alpha + h_\alpha$:

$$f_\alpha(X_1, \ldots, X_N) = \begin{pmatrix} g_\alpha^\circ(z_1, \ldots, z_N) & h_\alpha^\circ(z_1, \ldots, z_N) \\ \overline{h_\alpha^\circ(z_1, \ldots, z_N)} & \overline{g_\alpha^\circ(z_1, \ldots, z_N)} \end{pmatrix}$$

Thus, we obtain the equations for Y from the statement. □

As a first consequence, of theoretical interest, a necessary condition for X to exist is that the manifold $Y \subset \mathbb{C}^N$ constructed above must be compact, and we will be back to this later. In order to discuss now modelling questions, we will need as well:

Definition 8.32. *Assuming that we are given a manifold Z, appearing via*

$$C(Z) = C^*\left(z_1, \ldots, z_N \Big| f_\alpha(z_1, \ldots, z_N) = 0\right)$$

we define the projective version of Z to be the quotient space $Z \to PZ$ corresponding to the subalgebra $C(PZ) \subset C(Z)$ generated by the variables $x_{ij} = z_i z_j^*$.

The relation with the half-classical manifolds comes from the fact that the projective version of a half-classical manifold is classical. Indeed, from $abc = cba$ we obtain:

$$\begin{aligned} ab \cdot cd &= (abc)d \\ &= (cba)d \\ &= c(bad) \\ &= c(dab) \\ &= cd \cdot ab \end{aligned}$$

Finally, let us call as before "matrix model" any morphism of unital C^*-algebras $f : A \to B$, with target algebra $B = M_K(C(Y))$, with $K \in \mathbb{N}$, and Y being a compact space. With these conventions, following Bichon [23], we have the following result:

THEOREM 8.33. *Given a half-classical manifold X which is symmetric, in the sense that all its defining polynomials f_α are even, its universal 2×2 antidiagonal model,*

$$\pi : C(X) \to M_2(C(Y))$$

where Y is the manifold constructed in Proposition 8.31, is faithful. In addition, the construction $X \to Y$ is such that X exists precisely when Y is compact.

PROOF. We can proceed as in [23]. Indeed, the universal model π in the statement induces, at the level of projective versions, a certain representation:

$$C(PX) \to M_2(C(PY))$$

By using the multiplication formulae from the proof of Proposition 8.31, the image of this representation consists of diagonal matrices, and the upper left components of these matrices are the standard coordinates of PY. Thus, we have an isomorphism:

$$PX \simeq PY$$

We can conclude then by using a grading trick. See [23]. □

As a first observation, this result shows that when X is symmetric, we have $X^* \subset X^{(2)}$. Going beyond this observation is an interesting problem.

In what follows, we will rather need a more detailed version of the above result. For this purpose, we can use the following definition:

DEFINITION 8.34. *Associated to any compact manifold $Y \subset \mathbb{C}^N$ is the real compact half-classical manifold $[Y]$, having as coordinates the following variables,*

$$X_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

where $z_1, \ldots, z_N$ are the standard coordinates on Y. In other words, $[Y]$ is given by the fact that $C([Y]) \subset M_2(C(Y))$ is the algebra generated by these matrices.

Here the fact that the manifold $[Y]$ is indeed half-classical follows from the results above. As for the fact that $[Y]$ is indeed algebraic, this follows from Theorem 8.33. Now with this notion in hand, we can reformulate Theorem 8.33, as follows:

THEOREM 8.35. *The symmetric half-classical manifolds X appear as follows:*

(1) *We have $X = [Y]$, for a certain conjugation-invariant subspace $Y \subset \mathbb{C}^N$.*
(2) *$PX = P[Y]$, and X is maximal with this property.*
(3) *In addition, we have an embedding $C([X]) \subset C(X) \rtimes \mathbb{Z}_2$.*

PROOF. This follows from Theorem 8.33, with the embedding in (3) being constructed as in [23], by $x_i = z_i \otimes \tau$, where τ is the standard generator of $\mathbb{Z}_2$. See [23]. □

And this is all, on this subject. In the unitary case things are a bit more complicated, and in connection with this, there are also some higher analogues of the above developed, using $K\times K$ matrix models. We refer to [**10**], [**23**] for more on these topics.

8e. Exercises

Exercises:

EXERCISE 8.36.

EXERCISE 8.37.

EXERCISE 8.38.

EXERCISE 8.39.

EXERCISE 8.40.

EXERCISE 8.41.

Bonus exercise.

Part III

Free equations

The time has come
To say fair's fair
To pay the rent
To pay our share

CHAPTER 9

Laplace operator

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

Bonus exercise.

CHAPTER 10

Harmonic functions

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

Bonus exercise.

CHAPTER 11

Free equations

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

Bonus exercise.

CHAPTER 12

Analytic aspects

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

Bonus exercise.

Part IV

Free physics

It has to start somewhere
It has to start sometime
What better place than here
What better time than now

CHAPTER 13

Free mechanics

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Bonus exercise.

CHAPTER 14

Electrodynamics

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Bonus exercise.

CHAPTER 15

Twisted electrodynamics

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

EXERCISE 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

Bonus exercise.

CHAPTER 16

Free electrodynamics

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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Index