Introduction to free probability

Teo Banica

Classical probability, Weingarten integration, Free probability, Limiting theorems, Quantum algebra, Random matrices

08/20

Foreword

This is an introduction to classical and free probability, from a quantum algebra and random matrix perspective.

We discuss the foundations, insisting on limiting theorems, and then we discuss a number of more specialized aspects.

These lecture notes consist of slides written in the Summer 2020. Presentations available at my Youtube channel.

Contents 1/2

Contents 2/2

The normal and Poisson laws

Teo Banica

"Introduction to free probability", 1/6

07/20

Plan

1. The normal law
2. Advanced theory
3. The Poisson law
4. Advanced aspects

The Gauss integral

Theorem. We have the following formula:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

Proof. The square of the integral is given by:

$$\begin{aligned} I^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dxdy \\ &= \int_0^{2\pi} \int_0^{\infty} re^{-r^2} drdt \\ &= \int_0^{2\pi} \left[-\frac{e^{-r^2}}{2} \right]_0^{\infty} dt \end{aligned}$$

We obtain $I^2 = (2\pi) \times \frac{1}{2} = \pi$, and so $I = \sqrt{\pi}$.

The normal law

Definition. The normal law of parameter 1 is:

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

Remark. The Gauss formula gives with $x = y/\sqrt{2t}$

$$\int_{\mathbb{R}} e^{-y^2/2t} dy = \sqrt{2\pi t}$$

so these laws have indeed mass 1.

Variance

Theorem. We have the following formula, for any $t > 0$:

$$V(g_t) = t$$

Proof. The first moment is 0, and the second moment is:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^2 e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx) \left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t e^{-x^2/2t} dx \end{aligned}$$

We obtain $M_2 = t$, so the variance is $V = t$.

Fourier transform

Theorem. Assuming that f, g are independent, we have

$$F_{f+g} = F_f F_g$$

where $F_f(x) = \mathbb{E}(e^{ixf})$ is the Fourier transform.

Proof. We have indeed the following computation:

$$\begin{aligned} F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixy} d\mu_{f+g}(y) \\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{ix(y+z)} d\mu_f(y) d\mu_g(z) \\ &= \int_{\mathbb{R}} e^{ixy} d\mu_f(y) \int_{\mathbb{R}} e^{ixz} d\mu_g(z) \end{aligned}$$

Thus, we obtain $F_{f+g}(x) = F_f(x) F_g(x)$, as desired.

Convolution

Theorem. We have the following formula, for any $t > 0$:

$$F g_t(x) = e^{-tx^2/2}$$

Proof. This follows from the following computation:

$$\begin{aligned} F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2/2t+ixy} dy \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2} dy \\ &= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-z^2-tx^2/2} dz \end{aligned}$$

As a consequence, we have the following result:

Theorem. We have $g_s * g_t = g_{s+t}$, for any $s, t > 0$.

CLT

<u>Theorem</u>. Assuming that $f_1, f_2, f_3, \ldots$ are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i \sim g_t$$

<u>Proof</u>. We have the following formula, in terms of moments:

$$F_f(x) = \sum_{k=0}^{\infty} \frac{i^k M_k(f)}{k!} x^k$$

Thus, the Fourier transform of the variable in the statement is:

$$F(x) = \left[F_f \left(\frac{x}{\sqrt{n}} \right) \right]^n = \left[1 - \frac{tx^2}{2n} + o(x^2) \right]^n$$

Thus we obtain $F(x) \simeq e^{-tx^2/2} = F_{g_t}(x)$, as desired.

Moments 1/2

Theorem. The moments of the normal law are

$$M_k(g_t) = t^{k/2} \times k!!$$

where $k!! = 1.3.5\dots(k-1)$, with $k!! = 0$ when k is odd.

Proof. We have the following computation:

$$\begin{aligned} M_k &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^k e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx^{k-1}) \left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t(k-1)x^{k-2} e^{-x^2/2t} dx \end{aligned}$$

We obtain $M_k = t(k-1)M_{k-2}$, which gives the result.

Moments 2/2

Theorem. The moments of the normal law are

$$M_k(g_t) = t^{k/2}|P_2(k)|$$

where $P_2(k)$ is the set of pairings of $\{1, \ldots, k\}$.

Proof. By pairing 1 with one of $2, \ldots, k$, we obtain

$$|P_2(k)| = (k-1)|P_2(k-2)|$$

which gives $|P_2(k)| = k!!$, and so the result.

Variation. We have the moment formula

$$M_k(g_t) = \sum_{\pi \in P_2(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks.

Spheres 1/3

Goal. Understand the laws of the coordinates $x_i : S^{N-1} \to \mathbb{R}$, called "hyperspherical", and their $N \to \infty$ behavior.

At $N = 2$ the coordinates are $\cos t, \sin t$, and we have:

Theorem. We have the following formula

$$\int_0^{\pi/2} \cos^p t \sin^q t \, dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)} \frac{p!!q!!}{(p+q+1)!!}$$

where $\varepsilon(p) = 1$ when p is even, and $\varepsilon(p) = 0$ when p is odd.

Proof. Partial integration, and double recurrence.

Spheres 2/3

Theorem. The integration over the sphere is given by

$$\int_{S^{N-1}} x_{i_1} \dots x_{i_k} \, dx = \frac{(N-1)!! l_1!! \dots l_N!!}{(N + \sum l_i - 1)!!}$$

where l_a is the number of occurrences of a inside $i_1, \dots, i_k$.

Proof. In spherical coordinates the integral is as follows:

$$I = \frac{2^N}{V} \int_0^{\pi/2} \dots \int_0^{\pi/2} x_1^{l_1} \dots x_N^{l_N} J \, dt_1 \dots dt_{N-1}$$

The normalization constant in front of the integral is

$$\frac{2^N}{V} = \frac{2^N}{N \pi^{N/2}} \cdot \Gamma\left(\frac{N}{2} + 1\right) = \left(\frac{2}{\pi}\right)^{[N/2]} (N-1)!!$$

and the integral can be computed by using the $N = 2$ formula.

Spheres 3/3

Theorem. The moments of the hyperspherical variables are

$$\int_{S^{N-1}} x_i^k\,dx = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $x_i/\sqrt{N}$ become normal with $N \to \infty$.

Proof. The formula in the statement follows from the previous result. With $N \to \infty$ we obtain, as desired:

$$\int_{S^{N-1}} x_i^k\,dx \simeq N^{k/2}k!! = N^{k/2}M_k(g_1)$$

Remark. The previous result shows as well that the rescaled variables $x_i/\sqrt{N}$ become independent with $N \to \infty$.

Rotations

Theorem. We have the integration formula

$$\int_{O_N} U_{ij}^k dU = \frac{(N-1)!!k!!}{(N+k-1)!!}$$

and the variables $U_{ij}/\sqrt{N}$ become normal with $N \to \infty$.

Proof. This follows from the previous result, and from the fact that we have an embedding as follows, for any j,

$$C(S^{N-1}) \subset C(O_N) \quad , \quad x_i \to U_{ij}$$

which makes correspond the respective integration functionals.

Comment. The rescaled variables $U_{ij}/\sqrt{N}$ can be shown to become independent with $N \to \infty$. We will be back to this.

Calculus

Theorem. The following formulae define the same number:
(1) $\pi = L/2$, where L is the length of the unit circle.
(2) $\pi = A$, where A is the area of the unit circle.

Theorem. The following formulae define the same number:
(1) $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.
(2) $e = \sum_{k=0}^{\infty} \frac{1}{k!}$.
(3) $e = f(1)$, where $f' = f$, $f(0) = 1$.

Theorem. We have the formula $e^{\pi i} = -1$.

Some magics

Theorem. The probability for a permutation $\sigma \in S_N$ to be a derangement is, in the $N \to \infty$ limit:

$$P = \frac{1}{e}$$

Proof. We must be outside the union $F = \bigcup_i F_i$, where:

$$F_i = \left\{ \sigma \in S_N \middle| \sigma(i) = i \right\}$$

The inclusion-exclusion principle gives:

$$F^c = N! - \sum_i |F_i| + \sum_{i<j} |F_i \cap F_j| - \sum_{i<j<k} |F_i \cap F_j \cap F_k| + \ldots$$

We obtain $P = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots \simeq \frac{1}{e}$, as claimed.

What about pi?

In order to discuss this, take a needle of lenght 1:

Throw it (many times) on a grid of 1-spaced lines:

The probablity for the needle to intersect the grid is then:

$$P = \frac{2}{\pi}$$

The proof is quite tricky, and needs a correct modelling.

Fixed points

Theorem. The number of fixed points of permutations,

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

follows with $N \to \infty$ the following law:

$$p_1 = \frac{1}{e}\sum_k \frac{\delta_k}{k!}$$

Proof. We already know that the formula holds at 0. The same method, inclusion-exclusion, gives, more generally:

$$\lim_{N\to\infty} \mathbb{P}(\chi = k) = \frac{1}{e} \cdot \frac{1}{k!}$$

Thus, we obtain the law in the statement.

Poisson laws

Definition. The Poisson law of parameter 1 is:

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

More generally, the Poisson law of parameter $t > 0$ is:

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

Remark. These laws have indeed mass 1.

Truncation

Theorem. The number of truncated fixed points of permutations,

$$\chi_t(\sigma) = \#\left\{ i \in \{1, \ldots, [tN]\} \middle| \sigma(i) = i \right\}$$

follows with $N \to \infty$ the Poisson law p_t, for any $t \in (0, 1]$.

Proof. We already know that the formula holds at $t = 1$. The same method, inclusion-exclusion, gives, more generally:

$$\lim_{N \to \infty} \mathbb{P}(\chi = k) = \frac{1}{e^t} \cdot \frac{t^k}{k!}$$

Thus, we obtain with $N \to \infty$ the Poisson law p_t, as claimed.

Theory 1/2

Theorem. We have the following formula, for any $s,t>0$:

$$p_s * p_t = p_{s+t}$$

Proof. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula:

$$\begin{aligned} p_s * p_t &= e^{-s}\sum_k \frac{s^k}{k!}\,\delta_k * e^{-t}\sum_l \frac{t^l}{l!}\,\delta_l \\ &= e^{-s-t}\sum_n \delta_n \sum_{k+l=n}\frac{s^k t^l}{k!\,l!} \\ &= e^{-s-t}\sum_n \frac{(s+t)^n}{n!}\,\delta_n \end{aligned}$$

Thus, we obtain the Poisson law p_{s+t}, as claimed.

Theory 2/2

Theorem. The Poisson laws appear as exponentials

$$p_t = \sum_k \frac{t^k(\delta_1 - \delta_0)^{*k}}{k!}$$

with respect to the convolution of measures $*$.

Proof. By using the binomial formula, the measure at right is:

$$\begin{aligned} \mu &= \sum_k t^k \sum_{p+q=k} (-1)^q \frac{\delta_p}{p!q!} \\ &= \sum_p \frac{t^p \delta_p}{p!} \sum_q \frac{(-t)^q}{q!} \end{aligned}$$

Thus, we obtain the Poisson law p_t, as claimed.

Fourier

Theorem. The Fourier transform of p_t is given by:

$$F_{p_t}(x) = \exp\left((e^{ix} - 1)t\right)$$

Proof. We have $F_f(x) = \mathbb{E}(e^{ixf})$, and we obtain:

$$\begin{aligned} F_{p_t}(x) &= e^{-t}\sum_k \frac{t^k}{k!}\, e^{ikx} \\ &= e^{-t}\sum_k \frac{(e^{ix}t)^k}{k!} \\ &= \exp(-t)\exp(e^{ix}t) \end{aligned}$$

Thus, we obtain the formula in the statement.

PLT

Theorem. We have the following convergence, in moments:

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

Proof. We have the following computation:

$$\begin{aligned}
F_{\delta_t}(x)=e^{itx} \quad&\implies\quad F_{\mu_n}(x)=\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\\
&\implies\quad F_{\mu_n^{*n}}(x)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n\\
&\implies\quad F_{\mu_n^{*n}}(x)=\left(1+\frac{(e^{ix}-1)t}{n}\right)^n\\
&\implies\quad F(x)=\exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Thus, we obtain the Fourier transform of p_t.

Moments 1/2

Theorem. The moments of p_1 are the Bell numbers,

$$M_k(p_1) = |P(k)|$$

where $P(k)$ is the set of partitions of $\{1, \ldots, k\}$.

Proof. The moments of p_1 are given by:

$$M_k = \frac{1}{e} \sum_s \frac{s^k}{s!}$$

A direct computation gives the following formula:

$$M_{k+1} = \sum_r \binom{k}{r} M_{k-r}$$

Thus, we have the same recurrence as for the Bell numbers.

Moments 2/2

Theorem. The moments of p_t are given by

$$M_k(p_t) = \sum_{\pi \in P(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks.

Proof. The moments of p_t are given by:

$$M_k = e^{-t} \sum_s \frac{t^s s^k}{s!}$$

We are therefore led into Stirling numbers.

Summary

We have seen that the normal laws g_t and the Poisson laws p_t have many common features:

(1) They appear via limiting theorems, CLT/PLT.

(2) Their moments are related to partitions, $P_2(k)/P(k)$.

(3) Interesting connections with groups, O_N/S_N.

Thanks

Next lecture: integration over compact groups.

Groups and Weingarten integration

Teo Banica

"Introduction to free probability", 2/6

07/20

Plan

1. Group theory

2. Reflection groups

3. Compact groups

4. Weingarten integration

Finite groups

We are interested in the closed subgroups $G \subset U_N$. These can be finite or continuous. In the finite case, we have:

Theorem 1. Any finite group appears as a subgroup $G \subset S_N$.

Proof. This is Cayley's theorem. We have indeed a group embedding $G \subset S_{|G|}$ given by $\sigma_g(h) = gh$.

Theorem 2. Any finite group appears as a subgroup $G \subset U_N$.

Proof. It is enough to do this for $G = S_N$. But we have

$$S_N \subset O_N \subset U_N$$

by making permutations $\sigma \in S_N$ act on the coordinate axes of $\mathbb{R}^N$.

Basic examples

(1) Cyclic groups $\mathbb{Z}_N$. These appear as subgroups $\mathbb{Z}_N \subset O_N$, by cyclically permuting the coordinate axes of $\mathbb{R}^N$.

(2) Dihedral groups D_N. We have $D_N \subset O_N$, because the "1" points on the N coordinate axes form a regular N-gon.

(3) Permutation groups S_N. Here $S_N \subset O_N$ comes via permutation matrices, permuting the coordinate axes of $\mathbb{R}^N$.

(4) Hyperoctahedral groups H_N. Here $H_N \subset O_N$ is by definition the symmetry group of the N-hypercube.

(5) Complex reflection groups K_N. Here $K_N \subset U_N$ consists by definition of permutation-like matrices with entries in $\mathbb{T}$.

Observe that we have $\mathbb{Z}_N \subset D_N \subset S_N \subset H_N \subset K_N$.

Compact groups

In the continuous case now, there are many examples of closed subgroups $G \subset U_N$. Besides K_N, the main examples are:

(1) O_N, U_N. The orthogonal and unitary groups.

(2) SO_N, SU_N. The subgroups defined by $\det U = 1$.

(3) $Sp_N \subset U_N$. The symplectic group, defined for N even.

These are all smooth. In fact, the closed subgroups $G \subset U_N$ are exactly the compact Lie groups. We will be back to them later.

Characters

Definition. Given a closed subgroup $G \subset U_N$, the variable

$$\chi(U) = Tr(U)$$

with respect to the uniform measure, is called main character.

We will see later that the computation of $law(\chi)$ is the "main problem" regarding G. For the moment, let us record:

Theorem. For the symmetric group $S_N \subset U_N$ we have

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

and $law(\chi) = p_1$ in the $N \to \infty$ limit.

Truncated characters

Definition. Given a closed subgroup $G \subset U_N$, the variable

$$\chi_t(U) = \sum_{i=1}^{[tN]} U_{ii}$$

is called truncated character, of parameter $t \in (0,1]$.

Theorem. For the symmetric group $S_N \subset U_N$ we have

$$\chi(\sigma) = \# \left\{ i \in \{1, \ldots, [tN]\} \middle| \sigma(i) = i \right\}$$

and $law(\chi_t) = p_t$ in the $N \to \infty$ limit.

Proofs

(1) By using inclusion-exclusion. Indeed, we obtain

$$\mathbb{P}(\chi = 0) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \simeq \frac{1}{e}$$

and the computation of $\mathbb{P}(\chi = k)$, and of $\mathbb{P}(\chi_t = k)$, is similar.

(2) Via the moment method. We can use indeed the formula

$$\int_{S_N} U_{i_1j_1} \ldots U_{i_kj_k} \, dU = \begin{cases} \frac{(N-|\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

where $\ker i$ is the partition of $\{1, \ldots, k\}$ whose blocks collect the equal indices of i, and where $|.|$ is the number of blocks.

Cyclic groups

Theorem. For the cyclic group $\mathbb{Z}_N \subset O_N$ we have

$$\chi(g) = N\delta_{g0}$$

and the corresponding distribution is a Bernoulli law:

$$law(\chi) = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}\delta_N$$

Proof. The cyclic matrices have 0 on the diagonal, and so trace 0, expect for the identity, having 1 on the diagonal, and trace N.

Remark. The truncated characters and the asymptotics are not interesting. We do not have convolution semigroups.

Dihedral groups

Theorem. For the dihedral group $D_N \subset S_N$ we have:

$$law(\chi) = \begin{cases} \left(1 - \frac{1}{2N}\right)\delta_0 + \frac{1}{2N}\delta_N & (N\ even) \\ \\ \left(\frac{1}{2} - \frac{1}{2N}\right)\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{2N}\delta_N & (N\ odd) \end{cases}$$

Proof. The dihedral group D_N consists of:

(1) N symmetries, having $0, 1$ fixed points, depending on N.

(2) N rotations, having 0 fixed points, except for the identity.

Remark. The truncations and asymptotics are not interesting.

Reflections

Definition. The hyperoctahedral group $H_N \subset O_N$ is:

(1) The symmetry group of the unit hypercube $\square_N \subset \mathbb{R}^N$.

(2) The group of symmetries of the N coordinate axes of $\mathbb{R}^N$.

(3) The group of permutation-like matrices with ± 1 entries.

Theorem. The laws of truncated characters for H_N are

$$\mathrm{law}(\chi_t) \simeq e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

for any $t \in (0, 1]$, in the $N \to \infty$ limit.

Limiting laws

Remark. The limiting truncated character law for H_N is

$$b_t = e^{-t} \sum_{k \in \mathbb{Z}} \delta_k \, f_k(t/2)$$

where f_k is the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

Theorem. The Bessel laws b_t have the semigroup property

$$b_s * b_t = b_{s+t}$$

with respect to the usual convolution of real measures.

Bessel laws

Theorem. The following limit converges, for any measure ν,

$$p_t^\nu = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\nu\right)^{*n}$$

and the limiting measure p_t^ν is called compound Poisson law.

Definition. The measures $b_t^s = p_t^{\varepsilon_s}$, with ε_s being the uniform measure on the s-roots of unity, are called Bessel laws.

Examples. At $s = 1$ we obtain the Poisson law p_t. At $s = 2$ we obtain the real Bessel law b_t. At $s = \infty$ we obtain a law B_t.

Complex reflections

Theorem. For the complex reflection group $H_N^s \subset U_N$, consisting of permutation-like matrices with entries in $\mathbb{Z}_s$,

$$H_N^s = \mathbb{Z}_s \wr S_N$$

the truncated characters become Bessel with $N \to \infty$:

$$\mathrm{law}(\chi_t) \simeq b_t^s$$

Examples. At $s = 1$ we obtain the Poisson result for S_N. At $s = 2$ we obtain the real Bessel result for H_N. At $s = \infty$ we obtain the complex Bessel law B_t, for the full reflection group $K_N = \mathbb{T} \wr S_N$.

Compact groups

In the continuous case, the main examples of closed subgroups $G \subset U_N$ are as follows:

(1) O_N, U_N. The orthogonal and unitary groups.

(2) SO_N, SU_N. The subgroups defined by $\det U = 1$.

(3) $Sp_N \subset U_N$. The symplectic group, defined for N even.

Observe that these groups are all smooth. In fact, the closed subgroups $G \subset U_N$ are exactly the compact Lie groups.

SU_2

Theorem. The group $SU_2 = \{U \in U_2 | \det U = 1\}$ is given by

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}$$

and the main character follows the Wigner semicircle law:

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

Proof. The matrices $U \in SU_2$ can be computed using $U^* = U^{-1}$ and the inversion formula for 2×2 matrices, with $\det U = 1$.

With $a = x + iy$, $b = z + it$ we must have $x^2 + y^2 + z^2 + t^2 = 1$, and so $SU_2 = S^3$, and $\chi = 2Re(a)$ follows to be semicircular.

SO_3

Theorem. The elements of $SO_3 = \{U \in O_3 | \det U = 1\}$ are

$$U = \begin{pmatrix} x^2+y^2-z^2-t^2 & 2(yz-xt) & 2(xz+yt) \\ 2(xt+yz) & x^2+z^2-y^2-t^2 & 2(zt-xy) \\ 2(yt-xz) & 2(xy+zt) & x^2+t^2-y^2-z^2 \end{pmatrix}$$

with $x, y, z, t \in \mathbb{R}$ satisfying $x^2+y^2+z^2+t^2 = 1$, and the main character of SO_3 follows the Marchenko-Pastur law:

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

Proof. This follows from the result for SU_2, by using the double cover map $SU_2 \to SO_3$. Indeed, we obtain the above formula for $U \in SO_3$, as well as the fact that χ is squared-semicircular.

Theory 1/3

Theorem. Any closed subgroup $G \subset U_N$ has a Haar measure,

$$\mu(gE) = \mu(Eg) = \mu(E)$$

constructed by starting with any measure ν, and setting:

$$\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \nu^{*k}$$

Equivalently, for any representation $\rho : G \to U_n$, the quantities

$$P_{ij} = \int_G \rho_{ij}(U) dU$$

must be such that $P = (P_{ij})$ is the projection onto $Fix(\rho)$.

Theory 2/3

Theorem. The finite dimensional smooth unitary representations of $G \subset U_N$ are subject to the following Peter-Weyl results:

(1) Any such representation decomposes as a sum of irreducible representations, which are unique up to equivalence.

(2) The irreducibles appear inside the tensor products between the fundamental representation $\rho : G \to U_N$ and its adjoint.

(3) We have a decomposition $C^\infty(G) = \oplus_{r \in Irr(A)} B(H_r)$, as linear spaces, with the summands being pairwise orthogonal.

(4) The characters of the irreducible representations of G form an orthonormal basis of the algebra $C(G)_{central}$.

Theory 3/3

Theorem. Given a closed subgroup $G \subset U_N$, consider the character of the fundamental representation $\rho : G \to U_N$:

$$\chi = Tr(\rho)$$

The moments of $\chi : G \to \mathbb{C}$ are given then by the formula

$$\int_G \chi^k = \dim(Fix(\rho^{\otimes k}))$$

where $k \in \mathbb{N}$ in the case $G \subset O_N$, and $k \in \mathbb{N} \times \mathbb{N}$ in general.

Remark. This does not apply to individual coordinates $U \to U_{ij}$, or truncated characters χ_t, or other more complicated variables.

Integration formula

Theorem. The Haar integration over $G \subset_\rho U_N$ is given by

$$\int_G U_{i_1j_1}^{s_1} \dots U_{i_kj_k}^{s_k} dU = \sum_{\pi,\sigma \in D_k} \delta_\pi(i)\delta_\sigma(j)W_k(\pi,\sigma)$$

where D_k is a basis of $Fix(\rho^{\otimes k})$, $\delta_\pi(i) = <\pi, e_{i_1} \otimes \dots \otimes e_{i_k}>$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\pi,\sigma) = <\pi,\sigma>$.

Proof. The integrals in the statement form the projection P onto $Fix(\rho^{\otimes k}) = span(D_k)$. Consider the following linear map:

$$E(x) = \sum_{\pi \in D_k} < x, \pi > \pi$$

By linear algebra we have $P = WE$, where W is the inverse on $span(D_k)$ of the restriction of E, and this gives the result.

Tannakian duality

Definition. The Tannakian category of a closed subgroup $G \subset_\rho U_N$ is the following collection $C = (C(k,l))$ of vector spaces:

$$C(k,l) = Hom(\rho^{\otimes k}, \rho^{\otimes l})$$

Definition. The closed subgroup $G \subset U_N$ associated to an abstract Tannakian category $C = (C(k,l))$ is constructed as follows:

$$G = \left\{ U \in U_N \middle| T \in Hom(U^{\otimes k}, U^{\otimes l}), \forall T \in C(k,l) \right\}$$

Theorem. These operations produce a bijection $G \leftrightarrow C$, between compact Lie groups G, and Tannakian categories C.

Easiness

Definition. A closed subgroup $G \subset_\rho U_N$ is called easy when

$$Hom(\rho^{\otimes k}, \rho^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Examples

(1) The basic unitary and reflection groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{matrix} K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & O_N \end{matrix} \quad : \quad \begin{matrix} \mathcal{P}_{even} & \longleftarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow \\ P_{even} & \longleftarrow & P_2 \end{matrix}$$

(2) In relation with reflection groups, S_N is easy as well, coming from P itself. In fact all groups $H_N^s = \mathbb{Z}_s \wr S_N$ are easy.

(3) The symplectic group $Sp_N \subset U_N$, defined for N even, is not exactly easy, but rather "super-easy".

(4) The remaining groups, namely SO_N, SU_N, and H_N^{sd} as well, which all involve the determinant, are not easy.

Weingarten formula

Theorem. For an easy group $G_N \subset U_N$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_{G_N} U_{i_1 j_1}^{s_1} \ldots U_{i_k j_k}^{s_k} dU = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. The vectors associated to partitions are given by:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Thus the Gram matrix and Kronecker symbols are those above.

Applications

Theorem. The truncated characters χ_t for the main unitary and reflection groups are as follows, in the $N \to \infty$ limit,

$$\begin{matrix} K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & O_N \end{matrix} \qquad \sim \qquad \begin{matrix} B_t & \cdots & G_t \\ \vdots & & \vdots \\ b_t & \cdots & g_t \end{matrix}$$

and we have independence results as well, with $N \to \infty$.

Comment 1. In the discrete case we have more generally results for all groups H_N^s, but this is something that we already know.

Comment 2. In the continuous case, similar techniques apply to other easy groups (B_N, C_N) or super-easy (Sp_N).

Summary

We have seen that:

(1) The Peter-Weyl theory gives a theoretical formula for $\int_G$.

(2) In the easy case this is the Weingarten formula, very concrete.

(3) Easiness checks are non-trivial: Tannaka, Brauer, Schur-Weyl..

(4) Once we have easiness, everything follows from Weingarten.

Thanks

Next lecture: free probability.

Operator algebras and free probability

Teo Banica

"Introduction to free probability", 3/6

08/20

Plan

1. C^*-algebras
2. Von Neumann algebras
3. Free probability
4. R-transform, CLT

Linear operators

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, which:

(1) Is complete with respect to $||.||$ (Banach algebra).

(2) Has an involution $T \to T^*$, $< Tx, y > = < x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Proof. Complex algebra is clear, given $\{T_n\}$ Cauchy we can set $Tx = \lim_{n \to \infty} T_n x$, the involution comes from $\varphi(x) = < Tx, y >$ which is linear, and $||TT^*|| = ||T||^2$ is by double inequality.

Operator algebras

Definition. A C^*-algebra is an algebra $A \subset B(H)$, which:

(1) Is norm closed: $T_n \in A,\ T_n \to T \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Definition. A von Neumann algebra is an algebra $A \subset B(H)$, which:

(1) Is weakly closed: $T_n \in A,\ T_n x \to Tx, \forall x \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Examples. The commutative C^*-algebras $C(X)$, and von Neumann algebras $L^\infty(X)$, acting by multiplication on $L^2(X)$.

Spectral theory

Definition. The spectrum of an element $a \in A$ is the set

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1} \right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

Definition. The spectral radius $\rho(a)$ of an element $a \in A$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. Let A be a C^*-algebra.

(1) The spectrum of a norm 1 element is on the unit disk.

(2) The spectrum of a unitary ($a^* = a^{-1}$) is on the unit circle.

(3) The spectrum of a self-adjoint element ($a = a^*$) is real.

(4) ρ of a normal element ($aa^* = a^*a$) equals its norm.

Proof

(1) Clear from $(1-a)^{-1} = 1 + a + a^2 + \ldots$ for $||a|| < 1$.

(2) Follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

(3) Follows from (2), by using $f(z) = (z + it)/(z - it)$.

(4) By (1) we have $\rho(a) \leq ||a||$. Given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

When $a = a^*$ we are done. In general, we can use $||aa^*|| = ||a||^2$.

Gelfand theorem

Theorem. The commutative C^*-algebras are the algebras of the form $C(X)$, with X being a compact space.

Proof. If X is compact, $C(X)$ is indeed a C^*-algebra. Conversely, given A commutative, consider the space of characters

$$X = \{\chi : A \to \mathbb{C}\}$$

with topology making continuous each $ev_a : \chi \to \chi(a)$. Then X is compact, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

GNS theorem

Theorem. Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. In the commutative case, $A = C(X)$, we have indeed:

$$A \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

In general the idea is similar, by constructing an integration

$$\varphi : A \to \mathbb{C}$$

then defining a space $H = L^2(A)$, and using $a \to (b \to ab)$.

Von Neumann algebras

Definition. A von Neumann algebra is a $*$-algebra of operators $A \subset B(H)$ which is closed under the weak topology:

$$T_n \in A, T_n x \to Tx \implies T \in A$$

Examples. The usual C^*-algebras, in finite dimensions. Also, the algebras $L^\infty(X) \subset B(L^2(X))$, which are commutative.

Theorem. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Proof. Basic functional analysis and operator theory. The full statement involves as well a multiplicity, in regards with H.

Basic theory

Theorem. For a $*$-algebra of operators $A \subset B(H)$, the following conditions are equivalent:

(1) A is weakly closed, i.e. is a von Neumann algebra.

(2) A is equal to its algebraic bicommutant, $A = A''$.

This is von Neumann's "bicommutant theorem". As a consequence, the von Neumann algebras appear as commutants, $A = P'$.

Comments. Von Neumann $\implies$ C^*. Conversely, the von Neumann algebras are the C^*-algebras having separable predual. Also,

$$L^\infty(X) = C(\widehat{X})$$

by Gelfand, with $\widehat{X}$ being the Stone-Čech compactification of X.

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

Reduction theory

Theorem. When writing the center of the algebra as

$$Z(A) = L^{\infty}(X)$$

wihh X measured space, the algebra decomposes as

$$A = \int_X A_x \, dx$$

with the summands being "factors", $Z(A_x) = \mathbb{C}$.

Example. In finite dimensions the algebra must be

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

and this is its decomposition as a sum of factors.

Factors

Theorem. The factors, $Z(A) = \mathbb{C}$, fall into 3 classes:

(1) Type I. These are the usual matrix algebras $M_N(\mathbb{C})$ (type I_N), and the algebra $B(H)$, with H separable (type I_∞).

(2) Type II. These are the ∞D factors having a trace $tr : A \to \mathbb{C}$ (type II_1) and their tensor products with $B(H)$ (type II_∞).

(3) Type III. These fall into several classes, III_λ with $\lambda \in [0, 1]$, and appear from II_1 factors, via crossed product type constructions.

Proof. This is heavy, due to Murray and von Neumann, and then Connes, based on ideas of Tomita, Takesaki and others.

$\implies$ The II_1 factors are the "building blocks" of the theory.

The factor R

Theorem 1. The following limiting von Neumann algebra,

$$R = \lim_{k \to \infty} M_{N_k}(\mathbb{C})$$

is a II_1 factor, independent of the limiting procedure.

Theorem 2. R is the unique "hyperfinite" II_1 factor.

Theorem 3. R is the unique "building block" for the whole hyperfinite von Neumann algebra theory.

These results, building on what has been said before, are heavy, due to Murray-von Neumann, Connes, and Haagerup.

NC probability

Definition. Let A be a $*$-algebra, given with a trace tr.

(1) The elements $a \in A$ are called random variables.

(2) The moments of $a \in A$ are the numbers $M_k(a) = tr(a^k)$.

(3) The law of $a \in A$ is the functional $\mu_a : P \to tr(P(a))$.

Here $k = \circ \bullet \bullet \circ \ldots$ is a colored integer, and the powers a^k are defined by $a^\emptyset = 1, a^\circ = a, a^\bullet = a^*$ and multiplicativity.

The law is uniquely determined by the moments, because

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

for any $P \in \mathbb{C} < X, X^* >$, with the above conventions.

Spectral measures

Theorem. Assume that A is a C^*-algebra, that $tr : A \to \mathbb{C}$ is positive, $x \geq 0 \implies tr(x) \geq 0$, and that a is self-adjoint:

$$a = a^*$$

(1) μ_a is a real probability measure, or rather the integration with respect to such a measure, satisfying $supp(\mu_a) \subset \sigma(a)$.

(2) Assuming that tr is faithful, $x > 0 \implies tr(x) > 0$, the support of the law is the whole spectrum, $supp(\mu_a) = \sigma(a)$.

Moreover, these results extend to the normal case, $aa^* = a^*a$.

Proof. This is standard, coming from the Riesz theorem.

Random matrices

Definition. A random matrix algebra is a von Neumann algebra

$$A = M_N(\mathbb{C}) \otimes L^\infty(X)$$

endowed with its canonical unital trace, $tr = tr_N \otimes \int_X$.

Theorem. The matrices $M \in A$ having i.i.d. normal entries, up to the constraint $M = M^*$, follow with $N \to \infty$ the semicircle law:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. The Wick formula gives with $N \to \infty$ the Catalan numbers, which are the moments of the semicircle law.

Free probability

Definition. Let A be a $*$-algebra, given with a unital trace $tr : A \to \mathbb{C}$. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Examples. Two $*$-algebras B, C are independent inside their tensor product $B \otimes C$, and free inside their free product $B * C$.

Definition. Two elements $b, c \in A$ are called independent/free when the $*$-algebras that they generate

$$B = < b > \quad , \quad C = < c >$$

in the general $*$-algebra sense, or the C^*-algebra sense, or the von Neumann algebra sense, are independent/free.

Group algebras 1/2

Definition. Given a discrete group Γ, we endow the algebra $\mathbb{C}[\Gamma]$ with the involution $g^* = g^{-1}$, we consider the representation

$$\mathbb{C}[\Gamma] \subset B(l^2(\Gamma)) \quad , \quad g(\delta_h) = \delta_{gh}$$

and by closing we obtain operator algebras $C^*(\Gamma)$ and $L(\Gamma)$. These algebras have a faithful trace, given by:

$$tr(g) = \delta_{g1} \quad , \quad \forall g \in \Gamma$$

Properties. When Γ is abelian, we obtain the algebras $C(\widehat{\Gamma})$ and $L^\infty(\widehat{\Gamma})$. Also, $L(\Gamma)$ is a II_1 factor when Γ has infinite conjugacy classes. If in addition Γ is amenable, we have $L(\Gamma) \simeq R$.

Group algebras 2/2

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma\times\Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma*\Lambda)$.

Proof. This follows either from the product formulae

$$C^*(\Gamma\times\Lambda)=C^*(\Gamma)\otimes C^*(\Lambda)$$

$$C^*(\Gamma*\Lambda)=C^*(\Gamma)*C^*(\Lambda)$$

or by checking the independence/freeness on group elements.

Free convolution

Definition. The classical additive and multiplicative convolutions are constructed as follows, with a, b being independent:

$$\mu_a * \mu_b = \mu_{a+b} \quad , \quad \mu_a \times \mu_b = \mu_{ab}$$

Similarly, the free additive and multiplicative convolutions are constructed as follows, with a, b being free:

$$\mu_a \boxplus \mu_b = \mu_{a+b} \quad , \quad \mu_a \boxtimes \mu_b = \mu_{ab}$$

Remark. These operations are indeed well-defined, because the above compositions depend only on μ_a and μ_b.

R-transform 1/2

Theorem. Given a real probability measure μ, consider its Cauchy transform, and define its R-transform as follows:

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

This transform linearizes then the free convolution operation:

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu$$

Remark. This is similar to the fact that the log of the Fourier transform $F_{\mu_a}(\xi) = \mathbb{E}(e^{i\xi a})$ linearizes the usual convolution $*$.

R-transform 2/2

Proof. We use the monoid algebra $C^*(\mathbb{N} * \mathbb{N})$. We have freeness here, a bit as for group algebras, and the point is that the variables of type $S^* + f(S)$, with $S \in C^*(\mathbb{N})$ being the shift, and $f \in \mathbb{C}[X]$, model in moments all the distributions $\mu : \mathbb{C}[X] \to \mathbb{C}$.

Now let $f, g \in \mathbb{C}[X]$ and consider the variables $S^* + f(S)$ and $T^* + g(T)$, where $S, T \in C^*(\mathbb{N} * \mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N} * \mathbb{N}$. These variables are free, and by using a 45° argument, their sum has the same law as $S^* + (f + g)(S)$.

Thus the operation $\mu \to f$ linearizes the free convolution. We are left with a computation inside $C^*(\mathbb{N})$, which is elementary, and whose conclusion is that $R_\mu = f$ can be recaptured from μ via the Cauchy transform G_μ, as in the statement.

S-transform

Theorem. Given a real probability measure μ, consider its moment generating function, or Stieltjes transform,

$$f(z) = 1 + M_1 z + M_2 z^2 + M_3 z^3 + \dots$$

set $\psi(z) = f(z) - 1$, then invert, $\psi(\chi(z)) = z$, and then set:

$$S(z) = (1 + z^{-1})\chi(z)$$

Then $\log S$ linearizes the free multiplicative convolution:

$$S_{\mu \boxtimes \nu} = S_\mu S_\nu$$

Remark. The operation $\boxtimes$ is well-defined for real measures, because $\mu_a \boxtimes \mu_b = \mu_{\sqrt{a} b \sqrt{a}}$, with a, b self-adjoint and free.

CLT

Theorem. Assuming that $x_1, x_2, x_3, \ldots \in A$ are i/f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim g_t / \gamma_t$$

where g_t / γ_t are the normal/Wigner semicircle laws.

Proof. In the classical case this follows from the linearization property of $\log F$, namely $F_{\mu * \nu} = F_\mu F_\nu$, and from:

$$F g_t(\xi) = e^{-t\xi^2/2}$$

In the free case, this follows from the linearization property of R, namely $R_{\mu \boxplus \nu} = R_\mu + R_\nu$, and from $R_{\gamma_t}(\xi) = t\xi$.

Wigner matrices

Theorem. Given a family of Wigner random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. normal entries, up to the constraint $M_i = M_i^*$, the following happen:

(1) Each M_i follows a semicircle law γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. Here (1) is Wigner's theorem and (2) is Voiculescu's theorem. Both can be proved with the moment method.

Summary

We have seen that:

(1) Classical and free probability are twin sisters.

(2) The free CLT makes appear the Wigner law.

(3) The Wigner matrices are asymptotically free.

Thanks

Next lecture: further limiting theorems.

Classical and free limiting theorems

Teo Banica

"Introduction to free probability", 4/6

08/20

Plan

1. CLT, CCLT
2. PLT
3. CPLT
4. Cumulants, BP

CLT

Theorem. Assuming that $x_1, x_2, x_3, \ldots$ are self-adjoint, i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim g_t$$

where $g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$ is the normal law of parameter t.

Proof. The Fourier transform of the variable in the statement is:

$$F(\xi) = \left[F_x \left(\frac{\xi}{\sqrt{n}} \right) \right]^n = \left[1 - \frac{t\xi^2}{2n} + O(n^{-2}) \right]^n$$

Thus we obtain $F(\xi) \simeq e^{-t\xi^2/2} = F_{g_t}(\xi)$, as desired.

Free CLT

Theorem. Assuming that $x_1, x_2, x_3, \ldots$ are self-adjoint, f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where $\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$ is the Wigner law of parameter t.

Proof. The R-transform of the variable in the statement is:

$$R(\xi) = n \times R_x\left(\frac{\xi}{n}\right) = n\left[\frac{t\xi}{n} + O(n^{-2})\right]$$

Thus we obtain $R(\xi) \simeq t\xi = R_{\gamma_t}(\xi)$, as desired.

Wigner matrices

Theorem. Given a family of Wigner random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, up to the constraint $M_i = M_i^*$, the following happen:

(1) Each M_i follows a semicircle law γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. Here (1) is Wigner's theorem and (2) is Voiculescu's theorem. Both can be proved via the moment method.

CCLT

Theorem. If $x_1, x_2, x_3, \ldots$ have real and imaginary parts which are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim G_t$$

where G_t is the complex Gaussian law of parameter t,

$$G_t \sim \frac{1}{\sqrt{2}}(a + ib)$$

with a, b being self-adjoint and independent, each following g_t.

Proof. Follows from the CLT, by taking real and imaginary parts.

Free CCLT

Theorem. If $x_1, x_2, x_3, \ldots$ have real and imaginary parts which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \Gamma_t$$

where Γ_t is the Voiculescu circular law of parameter t,

$$\Gamma_t \sim \frac{1}{\sqrt{2}}(a + ib)$$

with a, b being self-adjoint and free, each following γ_t.

Proof. Follows from the CLT, by taking real and imaginary parts.

Gaussian matrices

Theorem. Given a family of Gaussian random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, the following happen:

(1) Each M_i follows a circular law Γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. This follows from the Wigner + Voiculescu theorem, by taking real and imaginary parts. Also, moment method.

PLT

Theorem. The following limit converges, for any $t > 0$,

$$\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}$$

in moments, the limiting measure being

$$p_t=\frac{1}{e^t}\sum_{k=0}^{\infty}\frac{t^k\delta_k}{k!}$$

which is the Poisson law of parameter t.

Proof

With $\mu = \left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1$, we have the following computation:

$$
\begin{aligned}
F_{\delta_z}(x) = e^{izx} \quad &\Longrightarrow \quad F_\mu(x) = \left(1 - \frac{t}{n}\right) + \frac{t}{n}e^{ix} \\
&\Longrightarrow \quad F_{\mu^{*n}}(x) = \left(\left(1 - \frac{t}{n}\right) + \frac{t}{n}e^{ix}\right)^n \\
&\Longrightarrow \quad F_{\mu^{*n}}(x) = \left(1 + \frac{(e^{ix} - 1)t}{n}\right)^n \\
&\Longrightarrow \quad F(x) = \exp\left((e^{ix} - 1)t\right)
\end{aligned}
$$

Thus, we obtain the Fourier transform of p_t, as desired.

Free PLT

Theorem. The following limit converges, for any $t > 0$,

$$\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n}$$

and we obtain the Marchenko-Pastur law of parameter t,

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter t.

Proof

The Cauchy transform of $\mu = \left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1$ is:

$$G_\mu(\xi) = \left(1 - \frac{t}{n}\right)\frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

Thus the equation for $R = R_{\mu^{\boxplus n}}(y) = nR_\mu(y)$ is:

$$\left(1 - \frac{t}{n}\right)\frac{1}{y^{-1} + R/n} + \frac{t}{n} \cdot \frac{1}{y^{-1} + R/n - 1} = y$$

By multiplying by n/y, this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With $n \to \infty$ we obtain $t + yR = \frac{t}{1-y}$, so $R = \frac{t}{1-y} = R_{\pi_t}$.

Wishart matrices

Theorem. The complex Wishart random matrices,

$$W = GG^*$$

with G being rectangular Gaussian, follow in the $N \to \infty$ limit

$$\pi_t = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}\, dx$$

which is the Marchenko-Pastur law of parameter t.

Proof. This follows via the moment method, with t depending on the precise size of G, and on the variance of its entries.

Summary

We have so far 6 main limiting laws, as follows:

$$\begin{matrix} \pi_t & \cdots\cdots & \gamma_t & \cdots\cdots & \Gamma_t \\ \vdots & & \vdots & & \vdots \\ p_t & \cdots\cdots & g_t & \cdots\cdots & G_t \end{matrix}$$

1. The relations $g_t \leftrightarrow G_t$ and $\gamma_t \leftrightarrow \Gamma_t$ are via $z = a + ib$.
2. Also, $G_t = U \times g_t$, via random matrices, or $l^2(\mathbb{N} * \mathbb{Z})$.
3. Also, $a \sim \gamma_t$ implies $a^2 \sim \pi_t$, via moments/transforms.
4. And $p_t \leftrightarrow \pi_t$, $g_t \leftrightarrow \gamma_t$, $G_t \leftrightarrow \Gamma_t$ are via $\log F \leftrightarrow R$.

CPLT

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$p_\nu = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{1}{n}\nu\right)^{*n}$$

and the measure p_ν is called compound Poisson law. For

$$\nu = \sum_{i=1}^{s} t_i \delta_{z_i}$$

with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$p_\nu = \mathrm{law}\left(\sum_{i=1}^{s} z_i \alpha_i\right)$$

whenever the variables α_i are Poisson (t_i), independent.

Free CPLT

<u>Theorem</u>. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$\pi_\nu = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{1}{n}\nu\right)^{\boxplus n}$$

and the measure π_ν is called compound free Poisson law. For

$$\nu = \sum_{i=1}^{s} t_i \delta_{z_i}$$

with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$\pi_\nu = \mathrm{law}\left(\sum_{i=1}^{s} z_i \alpha_i\right)$$

whenever the variables α_i are free Poisson (t_i), free.

Proofs

(1) In the classical case, with $\nu = t\delta_1$ we obtain the PLT. In general the proof is similar, the Fourier transform being:

$$F_{p_\nu}(y) = \exp\left(\sum_{i=1}^{s} t_i(e^{iyz_i} - 1)\right)$$

(2) In the free case, with $\nu = t\delta_1$ we obtain the free PLT. In general the proof is similar, the R-transform being:

$$R_{\pi_\nu}(y) = \sum_{i=1}^{s} \frac{t_i z_i}{1 - yz_i}$$

Examples

Definition. The Bessel laws b_t^s and free Bessel laws β_t^s with $s\in\mathbb N\cup\{\infty\}$ and $t>0$ are the compound Poisson laws

$$b_t^s=p_{t\varepsilon_s}\quad,\quad\beta_t^s=\pi_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s-roots of unity.

Remark. At $s=1$ we obtain the Poisson laws p_t,π_t.

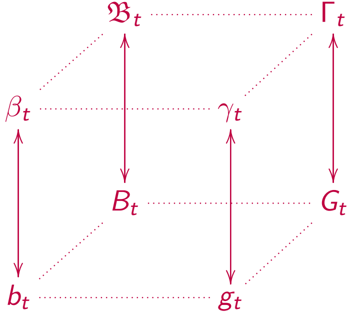
Definition. We use the following conventions, at $s=2,\infty$:

(1) $b_t=b_t^2$ and $\beta_t=\beta_t^2$ are called real Bessel laws.

(2) $B_t=b_t^\infty$ and $\mathfrak B_t=\beta_t^\infty$ are called complex Bessel laws.

Summary

Forgetting about Poisson, we have 8 main limiting laws,



with the vertical arrows being given by $\log F \leftrightarrow R$.

Wishart matrices

Recall the Marchenko-Pastur theorem, stating that for a Wishart matrix, $W = GG^*$ with G rectangular Gaussian, we have

$$W \sim \pi_t$$

in the $N \to \infty$ limit. By performing suitable block modifications of W, or product manipulations, we obtain results of type

$$W' \sim \pi_\nu$$

and in particular free Bessel laws, in the $N \to \infty$ limit. All this is a bit technical, and will be discussed in detail later on.

Bercovici-Pata

Definition. A convolution semigroup of measures

$$\{m_t\}_{t>0} \quad : \quad m_s * m_t = m_{s+t}$$

is in BP bijection with a free convolution semigroup of measures

$$\{\mu_t\}_{t>0} \quad : \quad \mu_s \boxplus \mu_t = \mu_{s+t}$$

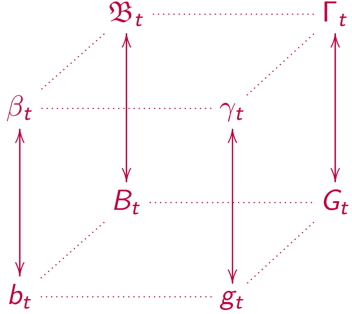
when the corresponding linearization transforms coincide:

$$\log F_{m_t} = R_{\mu_t}$$

Comment. The semigroup framework is quite natural.

Examples

Theorem. The main $4+4$ classical and free limiting laws,



form convolution semigroups, in Bercovici-Pata bijection.

Remark. In what regards the "discrete" case, $\{b_t^s\}_{t>0}$ and $\{\beta_t^s\}_{t>0}$ are in Bercovici-Pata bijection, for any $s \in \mathbb{N} \cup \{\infty\}$.

Cumulants

Definition. Given a random variable a, we write

$$\log F_a(\xi) = \sum_n k_n(a)\xi^n$$

$$R_a(\xi) = \sum_n \kappa_n(a)\xi^n$$

and call $k_n(a), \kappa_n(a)$ the classical and free cumulants of a.

Theorem. The BP bijection $m \leftrightarrow \mu$ is given by the fact that "the classical cumulants of m must equal the free cumulants of μ".

MC formula

Definition. The classical and free cumulants of a,

$$k_n(a) \quad , \quad \kappa_n(a) \quad : \quad n \in \mathbb{N}$$

can be extended into cumulants depending on partitions,

$$k_\pi(a) \quad , \quad \kappa_\pi(a) \quad : \quad \pi \in P$$

by multiplicativity over the blocks of the partitions.

Theorem. We have the classical/free moment-cumulant formulae

$$M_k(a) = \sum_{\pi \in P(k)} k_\pi(a) \quad , \quad M_k(a) = \sum_{\pi \in NC(k)} \kappa_\pi(a)$$

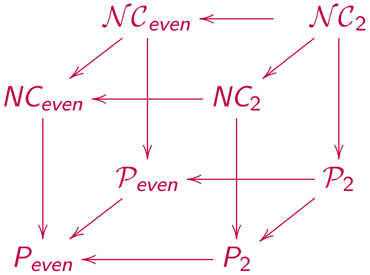
where $k_\pi(a), \kappa_\pi(a)$ are the classical/free cumulants of a.

Moments

Theorem. The moments of the 8 main limiting laws are

$$M_k(\mu) = \sum_{\pi \in D(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks, and D are as follows:



Remark. In the "discrete" case, we have results at $s \in \mathbb{N} \cup \{\infty\}$.

Further results

Theorem. We have the Meixner/free Meixner bijection.

Question. How to unify it with Bercovici-Pata?

Theorem. The normal law g_1 is infinitely $\boxplus$-divisible.

Question. What is the "classical analogue" of g_1?

Summary

We have seen that:

(1) Classical and free probability are twin sisters.

(2) Evidence: limiting theorems, BP bijection.

(3) Beyond this: quantum algebra, random matrices.

Thanks

Next lecture: quantum algebra.

Quantum algebra and free Bessel laws

Teo Banica

"Introduction to free probability", 5/6

08/20

Plan

1. Quantum groups

2. Laws of characters

3. Free Bessel laws

4. Quantum algebra

Quantum groups

Definition. A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.

– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.

– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

Notation. Given a Woronowicz algebra A we write

$$A = C(G) = C^*(\Gamma)$$

and call G, Γ compact and discrete quantum groups.

Basic examples

Example 1. Given a compact Lie group $G \subset U_N$, we have

$$A = C(G) \quad , \quad u_{ij}(g) = g_{ij}$$

with $\Delta = m^T, \varepsilon = u^T, S = i^T$ being the transposes of m, u, i.

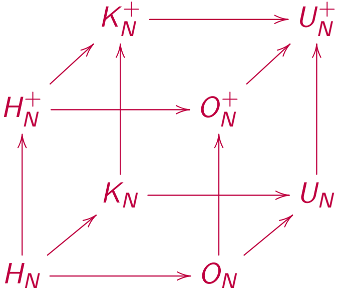
Example 2. Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, we have

$$A = C^*(\Gamma) \quad , \quad u = diag(g_i)$$

with $\Delta(g) = g \otimes g, \varepsilon(g) = 1, S(g) = g^{-1}$ on group elements.

Rotations, reflections

Theorem. We have quantum unitary and reflection groups



obtained by "liberating" the classical unitary/reflection groups.

Theory 1/3

Theorem. Any Woronowicz algebra has a Haar integration,

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

constructed by starting with $\varphi \in A^*$ unital positive, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi^{*k}$$

with the convolution operation being $\phi * \psi = (\phi \otimes \psi)\Delta$.

Examples. For compact Lie groups $G \subset U_N$ this is the usual integration. For group duals $G = \widehat{\Gamma}$ we have $\int_G g = \delta_{g1}$.

Theory 2/3

Definition. A corepresentation of a Woronowicz algebra A is a unitary matrix $v \in M_n(\mathcal{A})$ satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

where $\mathcal{A} =< u_{ij} >$ is the dense $*$-algebra of "smooth elements".

Theorem. The following Peter-Weyl type results hold:

(1) Any corepresentation decomposes as a sum of irreducibles.

(2) The irreducibles appear inside $u^{\otimes k}$, with $k =$ colored integer.

(3) We have $\mathcal{A} = \oplus_{r \in Irr(A)} B(H_r)$, $*$-coalgebra isomorphism, $\perp$.

(4) The characters of irreps form an orthonormal basis of $\mathcal{A}_{central}$.

Theory 3/3

Theorem. Given a corepresentation $v \in M_n(\mathcal{A})$, the integrals

$$P_{ij} = \int_G v_{ij}$$

form altogether the orthogonal projection $P = (P_{ij})$ onto $Fix(v)$.

Theorem. The Haar integration over G is given by

$$\int_G u_{i_1 j_1}^{s_1} \ldots u_{i_k j_k}^{s_k} = \sum_{\pi, \sigma \in D_k} \delta_\pi(i) \delta_\sigma(j) W_k(\pi, \sigma)$$

where D_k is a basis of $Fix(u^{\otimes k})$, $\delta_\pi(i) =< \pi, e_{i_1} \otimes \ldots \otimes e_{i_k} >$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\pi, \sigma) =< \pi, \sigma >$.

Tannaka

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the following collection $C = (C(k, l))$ of vector spaces:

$$C(k, l) = Hom(u^{\otimes k}, u^{\otimes l})$$

Definition. The Woronowicz algebra associated to a Tannakian category $C = (C(k, l))$ is constructed as follows:

$$A = C^* \left((u_{ij})_{i,j=1\ldots N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall T \in C(k, l) \right)$$

Theorem. These operations produce a bijection $A \leftrightarrow C$, between Woronowicz algebras, and Tannakian categories.

Easiness

Definition. A compact quantum group G is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

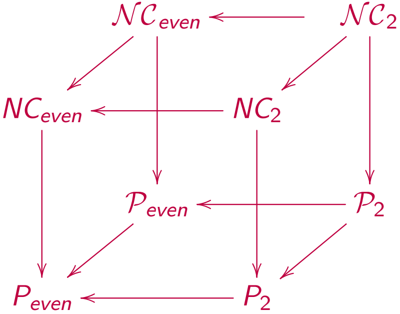
$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Examples. The Brauer theorem says that O_N, U_N are easy, coming from $P_2, \mathcal{P}_2$: the pairings, and the matching pairings.

Basic examples

Theorem. The main unitary/reflection groups are all easy,



being the corresponding categories of partitions $D \subset P$.

Weingarten

Theorem. For an easy quantum group $G\subset U_N^+$, coming from a category of partitions $D=(D(k,l))$, we have

$$\int_G u_{i_1j_1}^{s_1}\ldots u_{i_kj_k}^{s_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

for any colored integer k, where:

(1) $D(k)=D(\emptyset,k)$.

(2) δ are usual Kronecker symbols.

(3) $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Comment. This generalizes the Weingarten formula for easy groups $G\subset U_N$. There are "twisted" and "super-easy" versions of it.

Truncated characters

Theorem. Consider an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$.

(1) The asymptotic moments of the main character are:

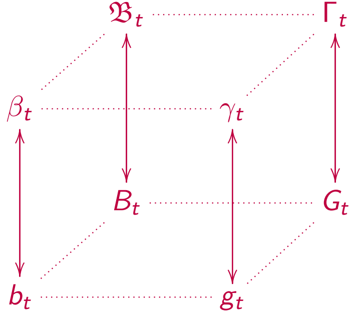
$$\int_{G_N}\left(\sum_i u_{ii}\right)^k \simeq |D(0,k)|$$

(2) The asymptotic moments of the truncated characters are:

$$\int_{G_N}\left(\sum_{i=1}^{[tN]} u_{ii}\right)^k \simeq \sum_{\pi\in D(0,k)} t^{|\pi|}$$

Laws of characters

Theorem. For the main unitary and reflection groups, the truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ follow with $N \to \infty$ the laws



which are the main laws in classical/free probability, modulo the fact that the Poisson laws are replaced by real Bessel laws.

Bessel laws 1/6

Definition. The classical and free Bessel laws are

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

with $s \in \mathbb{N} \cup \{\infty\}$ and $t > 0$. That is, we have

$$b_t^s = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\varepsilon_s\right)^{*n}$$

$$\beta_t^s = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\varepsilon_s\right)^{\boxplus n}$$

with ε_s being the uniform measure on the s-roots of unity.

Bessel laws 2/6

<u>Case $s = 1$</u>. Here the limiting result is the PLT, and we obtain the Poisson law p_t, and the Marchenko-Pastur law π_t.

<u>Case $s = 2$</u>. Here we obtain the real Bessel law b_t, given by

$$b_t = e^{-t} \sum_{k \in \mathbb{Z}} \delta_k \, f_k(t/2)$$

where f_k is the Bessel function of the first kind,

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

and the free real Bessel law β_t, also called Fuss-Catalan law.

<u>Case $s = \infty$</u>. Here we obtain the complex Bessel laws $B_t, \mathfrak{B}_t$.

Bessel laws 3/6

Theorem. The Bessel laws b_t^s and free Bessel laws β_t^s with $s \in \mathbb{N}$ and $t > 0$ can be defined alternatively as

$$b_t^s/\beta_t^s = \mathrm{law}\left(\sum_{r=1}^{s} w^r \alpha_i\right)$$

where $w = e^{2\pi i/s}$, and where α_i are Poisson/free Poisson (t) variables, which are independent/free.

Bessel laws 4/6

Theorem. We have the Fourier transform formula for b_t^s,

$$F_{b_t^s}(y) = \exp\left(t\sum_{r=1}^s (e^{iw^r y} - 1) \right)$$

as well as the following R-transform formula for β_t^s,

$$R_{\beta_t^s}(y) = t\sum_{r=1}^s \frac{w^r}{1 - w^r y}$$

valid for any $s \in \mathbb{N}$ and $t > 0$, where $w = e^{2\pi i/s}$.

Bessel laws 5/6

Theorem. The Bessel laws for a convolution semigroup

$$b_t^s * b_{t'}^s = b_{t+t'}^s$$

and the free Bessel laws form a free convolution semigroup

$$\beta_t^s \boxplus \beta_{t'}^s = \beta_{t+t'}^s$$

and these semigroups are in Bercovici-Pata bijection.

Bessel laws 6/6

Theorem. The free Bessel laws have the following properties,

(1) They appear as free compressions of $\boxtimes$ powers of π:

$$\beta_t^s = \left(\pi^{\boxtimes s}\right)_t$$

(2) Alternatively, we have the following formula:

$$\beta_t^s = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$$

(3) In particular, we have the following formulae:

$$\begin{cases} \beta_1^s = \pi^{\boxtimes s} \\ \beta_t^1 = \pi^{\boxplus t} \end{cases}$$

in terms of the Marchenko-Pastur law $\pi = \pi_1$.

Classical groups

Problem. The laws b_t^s and β_t^s appear as asymptotic laws of truncated characters for the quantum reflection groups:

$$H_N^s = \mathbb{Z}_s \wr S_N \quad , \quad H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

In the classical case, the full series of reflection groups is

$$H_N^{sd} = \left\{ U \in H_N^s \middle| (\det U)^d = 1 \right\}$$

and the character fluctuations are still to be understood.

Quantum groups

<u>Problem</u>. Classify the quantum reflection groups. In the real case, the easy quantum groups $H_N \subset G \subset H_N^+$ are

$$H_N \subset H_N^\Gamma \subset H_N^{(r)} \subset H_N^+$$

with Γ being a uniform real reflection group, and with $r \in \mathbb{N}$ being a parameter. In the purely complex case, namely

$$K_N \subset G \subset K_N^+$$

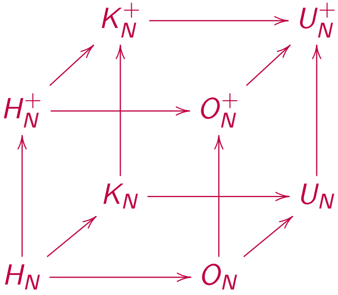
the situation is expected to be quite similar. In the general complex case, the classification of the easy quantum groups

$$H_N \subset G \subset K_N^+$$

is not known yet. In addition to all this, we have the question of computing and understanding the laws of truncated characters.

Noncommutative geometry

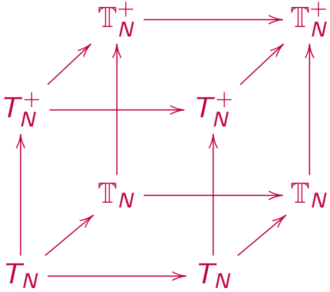
<u>Problem</u>. The main examples of quantum unitary groups and quantum reflection groups, namely



correspond to the 4 main geometries, real/complex, classical/free. In general, the classification problem is open.

Liberation theory

Problem. The main noncommutative tori, appearing as diagonal tori of the main quantum groups, are given by



where $T_N = \mathbb{Z}_2^N$, $\mathbb{T}_N = \mathbb{T}^N$ and $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$, $\mathbb{T}_N^+ = \widehat{F_N}$. At the character level we obtain Meixner/free Meixner instead of BP.

Subfactor theory

Problem. Associated to a finite index subfactor $A_0 \subset A_1$ is the Jones tower $A_0 \subset A_1 \subset A_2 \subset \ldots$, the planar algebra

$$P_k = A_0' \cap A_k$$

and the spectral measure μ, whose moments are given by:

$$M_k = \dim(P_k)$$

For TL subfactors we obtain free Poisson laws, for FC subfactors we obtain free Bessel laws. What is the meaning of $t > 0$ here?

Blowup questions

Problem. For subfactors of index $N \leq 4$, the Jones manipulation

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

on the associated Poincaré series, which is defined as

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

and which is the Stieltjes transform of the spectral measure μ,

$$f(x) = \int_{\mathbb{R}} \frac{1}{1-xz} \, d\mu(x)$$

blows up μ on $\mathbb{T}$, in a nice way. What about in general?

Summary

We have seen that:

(1) The free Bessel laws are central objects in free probability.

(2) And, more generally, in all branches of quantum algebra.

(3) The current research level is right above free Bessel.

Thanks

Next lecture: random matrices.

Wigner and Wishart random matrices

Teo Banica

"Introduction to free probability", 6/6

08/20

Plan

1. Wigner matrices
2. Wishart matrices
3. Block-transposed Wishart
4. Block-modified Wishart

Random matrices

Definition. A random matrix is a matrix as follows:

$$T \in M_N(L^\infty(X))$$

The moments of T are the following numbers, with $k = \circ \bullet \bullet \circ \ldots$ being a colored integer, with the rules $T^\circ = T$, $T^\bullet = T^*$:

$$M_k = \frac{1}{N}\int_X Tr(T^k)$$

The distribution, or law of T is the following abstract functional:

$$\mu : \mathbb{C} < X, X^* > \to \mathbb{C} \quad , \quad P \to \frac{1}{N}\int_X Tr(P(T))$$

Observe that the law is uniquely determined by the moments.

Self-adjoint case

Theorem. In the self-adjoint case, $T = T^*$, the law,

$$\mu : \mathbb{C} < X, X^* > \to \mathbb{C} \quad , \quad P \to \frac{1}{N} \int_X Tr(P(T))$$

when restricted to the usual polynomials

$$\mu : \mathbb{C}[X] \to \mathbb{C} \quad , \quad P \to \frac{1}{N} \int_X Tr(P(T))$$

must come from a probability measure on $\sigma(T) \subset \mathbb{R}$, as:

$$\mu(P) = \int_{\sigma(T)} P(x) d\mu(x)$$

We agree to use the symbol μ for all these notions.

Freeness

Definition. Let A be a $*$-algebra, given with a trace $tr : A \to \mathbb{C}$. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Examples. Two $*$-algebras B, C are independent inside their tensor product $B \otimes C$, and free inside their free product $B * C$.

Definition. Two elements $b, c \in A$ are independent/free when

$$B = < b > \quad , \quad C = < c >$$

are independent/free, in the above sense.

Free CLT

Theorem. If $x_1, x_2, x_3, \dots$ are self-adjoint, f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where $\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$ is the Wigner law of parameter t.

Theorem. If $x_1, x_2, x_3, \dots$ have real and imaginary parts which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \Gamma_t$$

where $\Gamma_t \sim \frac{1}{\sqrt{2}}(a + ib)$ is the Voiculescu law of parameter t.

Wigner matrices

Theorem. Given a family of Wigner random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, up to the constraint $M_i = M_i^*$, the following happen:

(1) Each M_i follows a semicircle law γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. Here (1) is Wigner's theorem and (2) is Voiculescu's theorem. Both can be proved via the moment method.

Gaussian matrices

Theorem. Given a family of Gaussian random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, the following happen:

(1) Each M_i follows a circular law Γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. This follows from the Wigner + Voiculescu theorem on the Wigner matrices, by taking real and imaginary parts.

Poisson laws

Theorem. The following limit converges, for any $t > 0$,

$$\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n}$$

and we obtain the Marchenko-Pastur law of parameter t,

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter t.

Compound Poisson

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$\pi_\nu = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{1}{n}\nu\right)^{\boxplus n}$$

and the measure π_ν is called compound free Poisson law. For $\nu = \sum_{i=1}^s t_i\delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$\pi_\nu = \mathrm{law}\left(\sum_{i=1}^s z_i\alpha_i\right)$$

whenever the variables α_i are free Poisson (t_i), free.

Moments 1/2

Theorem. The moments of the MP law of parameter 1,

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}\,dx$$

are the Catalan numbers C_k, which are given by

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

and which count the NC partitions of $\{1, \ldots, k\}$:

$$C_k = |NC(k)|$$

Moments 2/2

Theorem. The moments of the MP law of parameter t

$$\pi_t = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

with $t > 0$ arbitrary are given by the following formula,

$$M_k = \sum_{\pi \in P(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks.

Wishart matrices

Theorem. The complex Wishart matrices of parameters (N, M),

$$W = \frac{1}{M} GG^*$$

with G being $N \times M$ Gaussian of parameter 1, follow in the

$$M = tN \to \infty$$

limit the Marchenko-Pastur law of parameter $t > 0$:

$$W \sim \pi_t$$

Proof

This follows via the moment method, as follows:

(1) Wick formula.

(2) $M = tN \to \infty$, some combinatorics.

(3) We obtain the Catalan numbers at $t = 1$.

(4) And their t-version, using blocks, in general.

Block transposition

Definition. Consider a complex Wishart matrix of parameters (dn, dm), meaning a $dn \times dn$ random matrix of type

$$W = \frac{1}{dm} GG^*$$

with G being $dn \times dm$ Gaussian of parameter 1. We regard W as being a $d \times d$ matrix with $n \times n$ blocks,

$$W \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$$

and we apply the transposition to all its $n \times n$ blocks:

$$W' = (id \otimes t)W$$

This matrix W' is called block-transposed Wishart matrix.

Limiting law

Theorem. Let W be a complex Wishart matrix of parameters (dn, dm), and consider its block-transposed version:

$$W' = (id \otimes t)W$$

Then with $n, m \in \mathbb{N}$ fixed and with $d \to \infty$, its rescaling mW' follows a free difference of free Poisson laws

$$mW' \sim \pi_s \boxminus \pi_t$$

with parameters as follows:

$$s = \frac{m(n+1)}{2} \quad , \quad t = \frac{m(n-1)}{2}$$

Proof 1/3

We compute the asymptotic moments of mW'.

By applying the Wick formula, then letting $d \to \infty$, and doing some combinatorics, we obtain

$$\lim_{d\to\infty} M_k(mW') = \sum_{\pi\in NC(k)} m^{|\pi|} n^{||\pi||}$$

where $|.|$ denotes as usual the number of blocks, and where $||.||$ denotes the number of blocks having even size.

Proof 2/3

We compute the asymptotic moment generating function of mW'.

By doing some combinatorics, the generating function

$$F(z) = \sum_k M_k z^k$$

of the asymptotic moments that we found, namely

$$M_k = \sum_{\pi \in NC(k)} m^{|\pi|} n^{||\pi||}$$

satisfies the following equation:

$$(F-1)(1-z^2F^2) = mzF(1+nzF)$$

Proof 3/3

We compute the asymptotic R-transform of mW', and conclude.

In terms of the R-transform, the equation that we found reads:

$$zR(1-z^2) = mz(1+nz)$$

Thus the asymptotic R-transform of mW' is given by:

$$R(z) = m\,\frac{1+nz}{1-z^2} = \frac{m}{2}\left(\frac{n+1}{1-z} - \frac{n-1}{1+z}\right)$$

But this is the R-transform of the law $\pi_s \boxminus \pi_t$, with:

$$s = \frac{m(n+1)}{2} \quad , \quad t = \frac{m(n-1)}{2}$$

Support, atoms

Theorem. The $d \to \infty$ limiting law for the block-transposed Wishart matrices of parameters (dn, dm), namely

$$\mu_{m,n} = \pi_s \boxminus \pi_t$$

with $s = \frac{m(n+1)}{2}$, $t = \frac{m(n-1)}{2}$ has the following properties:

(1) It has at most one atom, at 0, of mass $\max\{1 - mn, 0\}$.

(2) It has positive support iff $n \leq m/4 + 1/m$ and $m \geq 2$.

Block modifications

Definition. Given a complex Wishart matrix W of parameters (dn, dm), regarded as a $d \times d$ matrix with $n \times n$ blocks,

$$W \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$$

we can apply to the $n \times n$ blocks any linear transformation

$$\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

and we obtain in this way a matrix as follows,

$$\tilde{W} = (id \otimes \varphi)W$$

called block-modified Wishart matrix.

Limiting laws

<u>Theorem</u>. Consider a (dn, dm) complex Wishart matrix W, let

$$\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

be a self-adjoint linear map, coming from a matrix

$$\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

and consider the block-modified Wishart matrix:

$$\tilde{W} = (id \otimes \varphi)W$$

Then, under suitable "planar" assumptions on φ, we have

$$\delta m\tilde{W} \sim \pi_{mn\rho} \boxtimes \nu$$

with $\rho = law(\Lambda)$, $\nu = law(D)$, $\delta = tr(D)$, where $D = \varphi(1)$.

Basic examples

Theorem. We have the following results:

(1) $tW \sim \pi_t$, where $t = m/n$.

(2) $m(id \otimes t)W \sim \pi_s \boxminus \pi_t$ with $s = \frac{m(n+1)}{2}$, $t = \frac{m(n-1)}{2}$.

(3) $t(id \otimes tr(.)1)W \sim \pi_t$, where $t = mn$.

(4) $m(id \otimes (.)^\delta)W \sim \pi_m$.

Conclusion

The block-modified Wishart matrices cover:

(1) The usual Wishart matrices (MP).

(2) The block-transposed Wishart matrices (A, BN).

(3) The trace-compressed Wishart matrices (CN).

(4) The diagonally compressed Wishart matrices (CN).

Generalizations

There are several extensions of all this:

(1) Arizmendi-Nechita-Vargas.

(2) TB.

(3) Mingo-Popa.

(4) Fukuda-Sniady.

Further results

Further results can be obtained by taking products of Gaussian matrices of longer length. See BBCC and related papers.

Summary

We have seen that:

(1) In what regards the Wigner matrices, free probability is key here, the main result being Wigner + Voiculescu.

(2) The Wishart matrices make the connection with advanced quantum algebra, via their block-modified versions.

Thanks

Thank you for your attention!

References 1/4

[1] O. Arizmendi, I. Nechita and C. Vargas, On the asymptotic distribution of block-modified random matrices, *J. Math. Phys.* **57** (2016), 1–27.

[2] G. Aubrun, Partial transposition of random states and non-centered semicircular distributions, *Random Matrices Theory Appl.* **1** (2012), 125–145.

[3] T. Banica, Block-modified Wishart matrices: the easy case, *Indiana Univ. Math. J.* **69** (2020), 1–34.

[4] T. Banica, S.T. Belinschi, M. Capitaine and B. Collins, Free Bessel laws, *Canad. J. Math.* **63** (2011), 3–37.

[5] T. Banica, J. Bichon and B. Collins, The hyperoctahedral quantum group, *J. Ramanujan Math. Soc.* **22** (2007), 345–384.

[6] T. Banica, S. Curran and R. Speicher, De Finetti theorems for easy quantum groups, Ann. Probab. 40 (2012), 401-435.

[7] T. Banica and I. Nechita, Asymptotic eigenvalue distributions of block-transposed Wishart matrices, *J. Theoret. Probab.* **26** (2013), 855–869.

[8] T. Banica and R. Speicher, Liberation of orthogonal Lie groups, *Adv. Math.* **222** (2009), 1461–1501.

References 2/4

[9] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math.* **149** (1999), 1023–1060.

[10] P. Biane, Some properties of crossings and partitions, *Discrete Math.* **175** (1997), 41–53.

[11] M. Brannan and K. Kirkpatrick, Quantum groups and generalized circular elements, *Pacific J. Math.* **282** (2016), 35–61.

[12] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. of Math.* **38** (1937), 857–872.

[13] B. Collins and I. Nechita, Random quantum channels I: graphical calculus and the Bell state phenomenon, *Comm. Math. Phys.* **297** (2010), 345–370.

[14] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic groups, *Comm. Math. Phys.* **264** (2006), 773–795.

[15] A. Connes, Noncommutative geometry, Academic Press (1994).

[16] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.

References 3/4

[17] P. Diaconis and M. Shahshahani, On the eigenvalues of random matrices, *J. Applied Probab.* **31** (1994), 49–62.

[18] M. Fukuda and P. Śniady, Partial transpose of random quantum states: exact formulas and meanders, *J. Math. Phys.* **54** (2013), 1–31.

[19] A. Guionnet, V.F.R. Jones and D. Shlyakhtenko, Random matrices, free probability, planar algebras and subfactors, *Quanta of maths* **11** (2010), 201–239.

[20] V.F.R. Jones, Planar algebras I, preprint 1999.

[21] V.A. Marchenko and L.A. Pastur, Distribution of eigenvalues in certain sets of random matrices, *Mat. Sb.* **72** (1967), 507–536.

[22] A. Nica and R. Speicher, Lectures on the combinatorics of free probability, Cambridge Univ. Press (2006).

[23] S. Raum and M. Weber, The full classification of orthogonal easy quantum groups, *Comm. Math. Phys.* **341** (2016), 751–779.

[24] R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Math. Ann.* **298** (1994), 611–628.

References 4/4

[25] D.V. Voiculescu, Symmetries of some reduced free product C^*-algebras, in "Operator algebras and their connections with topology and ergodic theory", Springer (1985), 556–588.

[26] D.V. Voiculescu, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66** (1986), 323–346.

[27] D.V. Voiculescu, Multiplication of certain noncommuting random variables, *J. Operator Theory* **18** (1987), 223–235.

[28] D.V. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201–220.

[29] D.V. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory I, *Comm. Math. Phys.* **155** (1993), 71–92.

[30] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, AMS (1992).

[31] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* **62** (1955), 548–564.

[32] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.