Classical and free limiting theorems

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Plan

1. CLT, CCLT
2. PLT
3. CPLT
4. Cumulants, BP

CLT

Theorem. Assuming that $x_1, x_2, x_3, \ldots$ are self-adjoint, i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim g_t$$

where $g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$ is the normal law of parameter t.

Proof. The Fourier transform of the variable in the statement is:

$$F(\xi) = \left[F_x\left(\frac{\xi}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{t\xi^2}{2n} + O(n^{-2})\right]^n$$

Thus we obtain $F(\xi) \simeq e^{-t\xi^2/2} = F_{g_t}(\xi)$, as desired.

Free CLT

Theorem. Assuming that $x_1, x_2, x_3, \ldots$ are self-adjoint, f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where $\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$ is the Wigner law of parameter t.

Proof. The R-transform of the variable in the statement is:

$$R(\xi) = n \times R_x\left(\frac{\xi}{n}\right) = n\left[\frac{t\xi}{n} + O(n^{-2})\right]$$

Thus we obtain $R(\xi) \simeq t\xi = R_{\gamma_t}(\xi)$, as desired.

Wigner matrices

Theorem. Given a family of Wigner random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, up to the constraint $M_i = M_i^*$, the following happen:

(1) Each M_i follows a semicircle law γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. Here (1) is Wigner's theorem and (2) is Voiculescu's theorem. Both can be proved via the moment method.

CCLT

Theorem. If $x_1, x_2, x_3, \ldots$ have real and imaginary parts which are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim G_t$$

where G_t is the complex Gaussian law of parameter t,

$$G_t \sim \frac{1}{\sqrt{2}}(a + ib)$$

with a, b being self-adjoint and independent, each following g_t.

Proof. Follows from the CLT, by taking real and imaginary parts.

Free CCLT

Theorem. If $x_1, x_2, x_3, \ldots$ have real and imaginary parts which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \Gamma_t$$

where Γ_t is the Voiculescu circular law of parameter t,

$$\Gamma_t \sim \frac{1}{\sqrt{2}}(a + ib)$$

with a, b being self-adjoint and free, each following γ_t.

Proof. Follows from the CLT, by taking real and imaginary parts.

Gaussian matrices

Theorem. Given a family of Gaussian random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, the following happen:

(1) Each M_i follows a circular law Γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. This follows from the Wigner + Voiculescu theorem, by taking real and imaginary parts. Also, moment method.

PLT

Theorem. The following limit converges, for any $t > 0$,

$$\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}$$

in moments, the limiting measure being

$$p_t=\frac{1}{e^t}\sum_{k=0}^\infty\frac{t^k\delta_k}{k!}$$

which is the Poisson law of parameter t.

Proof

With $\mu = \left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1$, we have the following computation:

$$
\begin{aligned}
F_{\delta_z}(x) = e^{izx} \quad &\implies \quad F_{\mu}(x) = \left(1 - \frac{t}{n}\right) + \frac{t}{n}e^{ix} \\
&\implies \quad F_{\mu^{*n}}(x) = \left(\left(1 - \frac{t}{n}\right) + \frac{t}{n}e^{ix}\right)^n \\
&\implies \quad F_{\mu^{*n}}(x) = \left(1 + \frac{(e^{ix} - 1)t}{n}\right)^n \\
&\implies \quad F(x) = \exp\left((e^{ix} - 1)t\right)
\end{aligned}
$$

Thus, we obtain the Fourier transform of p_t, as desired.

Free PLT

Theorem. The following limit converges, for any $t > 0$,

$$\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n}$$

and we obtain the Marchenko-Pastur law of parameter t,

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter t.

Proof

The Cauchy transform of $\mu = \left(1 - \frac{t}{n}\right) \delta_0 + \frac{t}{n}\delta_1$ is:

$$G_\mu(\xi) = \left(1 - \frac{t}{n}\right) \frac{1}{\xi} + \frac{t}{n} \cdot \frac{1}{\xi - 1}$$

Thus the equation for $R = R_{\mu^{\boxplus n}}(y) = nR_\mu(y)$ is:

$$\left(1 - \frac{t}{n}\right) \frac{1}{y^{-1} + R/n} + \frac{t}{n} \cdot \frac{1}{y^{-1} + R/n - 1} = y$$

By multiplying by n/y, this equation can be written as:

$$\frac{t + yR}{1 + yR/n} = \frac{t}{1 + yR/n - y}$$

With $n \to \infty$ we obtain $t + yR = \frac{t}{1-y}$, so $R = \frac{t}{1-y} = R_{\pi_t}$.

Wishart matrices

Theorem. The complex Wishart random matrices,

$$W = GG^*$$

with G being rectangular Gaussian, follow in the $N \to \infty$ limit

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

which is the Marchenko-Pastur law of parameter t.

Proof. This follows via the moment method, with t depending on the precise size of G, and on the variance of its entries.

Summary

We have so far 6 main limiting laws, as follows:

$$\begin{array}{ccccc} \pi_t & \cdots & \gamma_t & \cdots & \Gamma_t \\ \vdots & & \vdots & & \vdots \\ p_t & \cdots & g_t & \cdots & G_t \end{array}$$

1. The relations $g_t \leftrightarrow G_t$ and $\gamma_t \leftrightarrow \Gamma_t$ are via $z = a + ib$.
2. Also, $G_t = U \times g_t$, via random matrices, or $l^2(\mathbb{N} * \mathbb{Z})$.
3. Also, $a \sim \gamma_t$ implies $a^2 \sim \pi_t$, via moments/transforms.
4. And $p_t \leftrightarrow \pi_t$, $g_t \leftrightarrow \gamma_t$, $G_t \leftrightarrow \Gamma_t$ are via $\log F \leftrightarrow R$.

CPLT

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$p_\nu = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{1}{n}\nu\right)^{*n}$$

and the measure p_ν is called compound Poisson law. For

$$\nu = \sum_{i=1}^{s} t_i\delta_{z_i}$$

with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$p_\nu = \mathrm{law}\left(\sum_{i=1}^{s} z_i\alpha_i\right)$$

whenever the variables α_i are Poisson (t_i), independent.

Free CPLT

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$\pi_\nu = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right) \delta_0 + \frac{1}{n}\nu \right)^{\boxplus n}$$

and the measure π_ν is called compound free Poisson law. For

$$\nu = \sum_{i=1}^{s} t_i \delta_{z_i}$$

with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$\pi_\nu = \mathrm{law}\left(\sum_{i=1}^{s} z_i \alpha_i \right)$$

whenever the variables α_i are free Poisson (t_i), free.

Proofs

(1) In the classical case, with $\nu = t\delta_1$ we obtain the PLT. In general the proof is similar, the Fourier transform being:

$$F_{p_\nu}(y) = \exp\left(\sum_{i=1}^{s} t_i(e^{iyz_i} - 1)\right)$$

(2) In the free case, with $\nu = t\delta_1$ we obtain the free PLT. In general the proof is similar, the R-transform being:

$$R_{\pi_\nu}(y) = \sum_{i=1}^{s} \frac{t_i z_i}{1 - yz_i}$$

Examples

Definition. The Bessel laws b_t^s and free Bessel laws β_t^s with $s \in \mathbb{N} \cup \{\infty\}$ and $t > 0$ are the compound Poisson laws

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s-roots of unity.

Remark. At $s = 1$ we obtain the Poisson laws p_t, π_t.

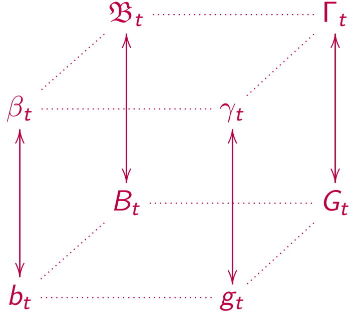
Definition. We use the following conventions, at $s = 2, \infty$:

(1) $b_t = b_t^2$ and $\beta_t = \beta_t^2$ are called real Bessel laws.

(2) $B_t = b_t^\infty$ and $\mathfrak{B}_t = \beta_t^\infty$ are called complex Bessel laws.

Summary

Forgetting about Poisson, we have 8 main limiting laws,



with the vertical arrows being given by $\log F \leftrightarrow R$.

Wishart matrices

Recall the Marchenko-Pastur theorem, stating that for a Wishart matrix, $W = GG^*$ with G rectangular Gaussian, we have

$$W \sim \pi_t$$

in the $N \to \infty$ limit. By performing suitable block modifications of W, or product manipulations, we obtain results of type

$$W' \sim \pi_\nu$$

and in particular free Bessel laws, in the $N \to \infty$ limit. All this is a bit technical, and will be discussed in detail later on.

Bercovici-Pata

Definition. A convolution semigroup of measures

$$\{m_t\}_{t>0} \quad : \quad m_s * m_t = m_{s+t}$$

is in BP bijection with a free convolution semigroup of measures

$$\{\mu_t\}_{t>0} \quad : \quad \mu_s \boxplus \mu_t = \mu_{s+t}$$

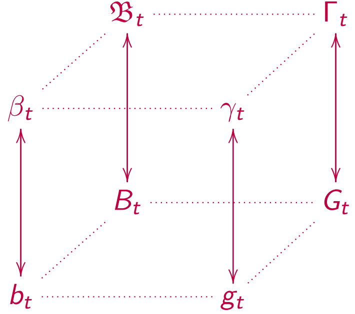
when the corresponding linearization transforms coincide:

$$\log F_{m_t} = R_{\mu_t}$$

Comment. The semigroup framework is quite natural.

Examples

Theorem. The main $4+4$ classical and free limiting laws,



form convolution semigroups, in Bercovici-Pata bijection.

Remark. In what regards the "discrete" case, $\{b_t^s\}_{t>0}$ and $\{\beta_t^s\}_{t>0}$ are in Bercovici-Pata bijection, for any $s \in \mathbb{N} \cup \{\infty\}$.

Cumulants

Definition. Given a random variable a, we write

$$\log F_a(\xi) = \sum_n k_n(a)\xi^n$$

$$R_a(\xi) = \sum_n \kappa_n(a)\xi^n$$

and call $k_n(a), \kappa_n(a)$ the classical and free cumulants of a.

Theorem. The BP bijection $m \leftrightarrow \mu$ is given by the fact that "the classical cumulants of m must equal the free cumulants of μ".

MC formula

Definition. The classical and free cumulants of a,

$$k_n(a) \quad , \quad \kappa_n(a) \quad : \quad n \in \mathbb{N}$$

can be extended into cumulants depending on partitions,

$$k_\pi(a) \quad , \quad \kappa_\pi(a) \quad : \quad \pi \in P$$

by multiplicativity over the blocks of the partitions.

Theorem. We have the classical/free moment-cumulant formulae

$$M_k(a) = \sum_{\pi \in P(k)} k_\pi(a) \quad , \quad M_k(a) = \sum_{\pi \in NC(k)} \kappa_\pi(a)$$

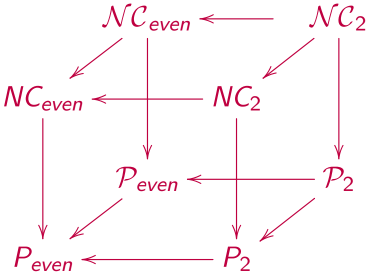
where $k_\pi(a), \kappa_\pi(a)$ are the classical/free cumulants of a.

Moments

<u>Theorem</u>. The moments of the 8 main limiting laws are

$$M_k(\mu) = \sum_{\pi \in D(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks, and D are as follows:



<u>Remark</u>. In the "discrete" case, we have results at $s \in \mathbb{N} \cup \{\infty\}$.

Further results

Theorem. We have the Meixner/free Meixner bijection.

Question. How to unify it with Bercovici-Pata?

Theorem. The normal law g_1 is infinitely $\boxplus$-divisible.

Question. What is the "classical analogue" of g_1?

Summary

We have seen that:

(1) Classical and free probability are twin sisters.

(2) Evidence: limiting theorems, BP bijection.

(3) Beyond this: quantum algebra, random matrices.

Thanks

Next lecture: quantum algebra.