Wigner and Wishart random matrices

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Plan

1. Wigner matrices
2. Wishart matrices
3. Block-transposed Wishart
4. Block-modified Wishart

Random matrices

Definition. A random matrix is a matrix as follows:

$$T \in M_N(L^\infty(X))$$

The moments of T are the following numbers, with $k = \circ\bullet\bullet\circ\ldots$ being a colored integer, with the rules $T^\circ = T$, $T^\bullet = T^*$:

$$M_k = \frac{1}{N}\int_X Tr(T^k)$$

The distribution, or law of T is the following abstract functional:

$$\mu : \mathbb{C} < X, X^* > \to \mathbb{C} \quad , \quad P \to \frac{1}{N}\int_X Tr(P(T))$$

Observe that the law is uniquely determined by the moments.

Self-adjoint case

Theorem. In the self-adjoint case, $T = T^*$, the law,

$$\mu : \mathbb{C} < X, X^* > \to \mathbb{C} \quad , \quad P \to \frac{1}{N} \int_X Tr(P(T))$$

when restricted to the usual polynomials

$$\mu : \mathbb{C}[X] \to \mathbb{C} \quad , \quad P \to \frac{1}{N} \int_X Tr(P(T))$$

must come from a probability measure on $\sigma(T) \subset \mathbb{R}$, as:

$$\mu(P) = \int_{\sigma(T)} P(x) d\mu(x)$$

We agree to use the symbol μ for all these notions.

Freeness

Definition. Let A be a $*$-algebra, given with a trace $tr : A \to \mathbb{C}$. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Examples. Two $*$-algebras B, C are independent inside their tensor product $B \otimes C$, and free inside their free product $B * C$.

Definition. Two elements $b, c \in A$ are independent/free when

$$B = < b > \quad , \quad C = < c >$$

are independent/free, in the above sense.

Free CLT

Theorem. If $x_1, x_2, x_3, \ldots$ are self-adjoint, f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where $\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$ is the Wigner law of parameter t.

Theorem. If $x_1, x_2, x_3, \ldots$ have real and imaginary parts which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \Gamma_t$$

where $\Gamma_t \sim \frac{1}{\sqrt{2}}(a + ib)$ is the Voiculescu law of parameter t.

Wigner matrices

Theorem. Given a family of Wigner random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, up to the constraint $M_i = M_i^*$, the following happen:

(1) Each M_i follows a semicircle law γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. Here (1) is Wigner's theorem and (2) is Voiculescu's theorem. Both can be proved via the moment method.

Gaussian matrices

Theorem. Given a family of Gaussian random matrices

$$M_i \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, the following happen:

(1) Each M_i follows a circular law Γ_t, with $N \to \infty$.

(2) These matrices M_i become free, with $N \to \infty$.

Proof. This follows from the Wigner + Voiculescu theorem on the Wigner matrices, by taking real and imaginary parts.

Poisson laws

Theorem. The following limit converges, for any $t > 0$,

$$\lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{\boxplus n}$$

and we obtain the Marchenko-Pastur law of parameter t,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter t.

Compound Poisson

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$\pi_\nu = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right) \delta_0 + \frac{1}{n}\nu \right)^{\boxplus n}$$

and the measure π_ν is called compound free Poisson law. For $\nu = \sum_{i=1}^s t_i \delta_{z_i}$ with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$\pi_\nu = \mathrm{law} \left(\sum_{i=1}^s z_i \alpha_i \right)$$

whenever the variables α_i are free Poisson (t_i), free.

Moments 1/2

Theorem. The moments of the MP law of parameter 1,

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

are the Catalan numbers C_k, which are given by

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

and which count the NC partitions of $\{1,\ldots,k\}$:

$$C_k = |NC(k)|$$

Moments 2/2

Theorem. The moments of the MP law of parameter t

$$\pi_t = \max(1-t,0)\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

with $t > 0$ arbitrary are given by the following formula,

$$M_k = \sum_{\pi \in P(k)} t^{|\pi|}$$

where $|.|$ is the number of blocks.

Wishart matrices

Theorem. The complex Wishart matrices of parameters (N, M),

$$W = \frac{1}{M} GG^*$$

with G being $N \times M$ Gaussian of parameter 1, follow in the

$$M = tN \to \infty$$

limit the Marchenko-Pastur law of parameter $t > 0$:

$$W \sim \pi_t$$

Proof

This follows via the moment method, as follows:

(1) Wick formula.

(2) $M = tN \to \infty$, some combinatorics.

(3) We obtain the Catalan numbers at $t = 1$.

(4) And their t-version, using blocks, in general.

Block transposition

Definition. Consider a complex Wishart matrix of parameters (dn, dm), meaning a $dn \times dn$ random matrix of type

$$W = \frac{1}{dm} GG^*$$

with G being $dn \times dm$ Gaussian of parameter 1. We regard W as being a $d \times d$ matrix with $n \times n$ blocks,

$$W \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$$

and we apply the transposition to all its $n \times n$ blocks:

$$W' = (id \otimes t)W$$

This matrix W' is called block-transposed Wishart matrix.

Limiting law

Theorem. Let W be a complex Wishart matrix of parameters (dn, dm), and consider its block-transposed version:

$$W' = (id \otimes t)W$$

Then with $n, m \in \mathbb{N}$ fixed and with $d \to \infty$, its rescaling mW' follows a free difference of free Poisson laws

$$mW' \sim \pi_s \boxminus \pi_t$$

with parameters as follows:

$$s = \frac{m(n+1)}{2} \quad , \quad t = \frac{m(n-1)}{2}$$

Proof 1/3

We compute the asymptotic moments of mW'.

By applying the Wick formula, then letting $d \to \infty$, and doing some combinatorics, we obtain

$$\lim_{d\to\infty} M_k(mW') = \sum_{\pi\in NC(k)} m^{|\pi|} n^{||\pi||}$$

where $|.|$ denotes as usual the number of blocks, and where $||.||$ denotes the number of blocks having even size.

Proof 2/3

We compute the asymptotic moment generating function of mW'.

By doing some combinatorics, the generating function

$$F(z) = \sum_k M_k z^k$$

of the asymptotic moments that we found, namely

$$M_k = \sum_{\pi \in NC(k)} m^{|\pi|} n^{||\pi||}$$

satisfies the following equation:

$$(F-1)(1-z^2F^2) = mzF(1+nzF)$$

Proof 3/3

We compute the asymptotic R-transform of mW', and conclude.

In terms of the R-transform, the equation that we found reads:

$$zR(1-z^2) = mz(1+nz)$$

Thus the asymptotic R-transform of mW' is given by:

$$R(z) = m\,\frac{1+nz}{1-z^2} = \frac{m}{2}\left(\frac{n+1}{1-z} - \frac{n-1}{1+z}\right)$$

But this is the R-transform of the law $\pi_s \boxminus \pi_t$, with:

$$s = \frac{m(n+1)}{2} \quad , \quad t = \frac{m(n-1)}{2}$$

Support, atoms

Theorem. The $d \to \infty$ limiting law for the block-transposed Wishart matrices of parameters (dn, dm), namely

$$\mu_{m,n} = \pi_s \boxminus \pi_t$$

with $s = \frac{m(n+1)}{2}$, $t = \frac{m(n-1)}{2}$ has the following properties:

(1) It has at most one atom, at 0, of mass $\max\{1 - mn, 0\}$.

(2) It has positive support iff $n \leq m/4 + 1/m$ and $m \geq 2$.

Block modifications

Definition. Given a complex Wishart matrix W of parameters (dn, dm), regarded as a $d \times d$ matrix with $n \times n$ blocks,

$$W \in M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$$

we can apply to the $n \times n$ blocks any linear transformation

$$\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

and we obtain in this way a matrix as follows,

$$\tilde{W} = (id \otimes \varphi)W$$

called block-modified Wishart matrix.

Limiting laws

Theorem. Consider a (dn, dm) complex Wishart matrix W, let

$$\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

be a self-adjoint linear map, coming from a matrix

$$\Lambda \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$$

and consider the block-modified Wishart matrix:

$$\tilde{W} = (id \otimes \varphi)W$$

Then, under suitable "planar" assumptions on φ, we have

$$\delta m \tilde{W} \sim \pi_{mn\rho} \boxtimes \nu$$

with $\rho = law(\Lambda)$, $\nu = law(D)$, $\delta = tr(D)$, where $D = \varphi(1)$.

Basic examples

Theorem. We have the following results:

(1) $tW \sim \pi_t$, where $t = m/n$.

(2) $m(id \otimes t)W \sim \pi_s \boxminus \pi_t$ with $s = \frac{m(n+1)}{2}$, $t = \frac{m(n-1)}{2}$.

(3) $t(id \otimes tr(.)1)W \sim \pi_t$, where $t = mn$.

(4) $m(id \otimes (.)^\delta)W \sim \pi_m$.

Conclusion

The block-modified Wishart matrices cover:

(1) The usual Wishart matrices (MP).

(2) The block-transposed Wishart matrices (A, BN).

(3) The trace-compressed Wishart matrices (CN).

(4) The diagonally compressed Wishart matrices (CN).

Generalizations

There are several extensions of all this:

(1) Arizmendi-Nechita-Vargas.

(2) TB.

(3) Mingo-Popa.

(4) Fukuda-Sniady.

Further results

Further results can be obtained by taking products of Gaussian matrices of longer length. See BBCC and related papers.

Summary

We have seen that:

(1) In what regards the Wigner matrices, free probability is key here, the main result being Wigner + Voiculescu.

(2) The Wishart matrices make the connection with advanced quantum algebra, via their block-modified versions.

Thanks

Thank you for your attention!