

# Forces and vectors

Teo Banica

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CERGY-PONTOISE, F-95000  
CERGY-PONTOISE, FRANCE. [teo.banica@gmail.com](mailto:teo.banica@gmail.com)

2010 *Mathematics Subject Classification.* 97M50

*Key words and phrases.* Force, Vector

ABSTRACT. This is a joint introduction to basic mechanics and vector mathematics. We discuss as well, at the end, more tricky forces, and their mathematical modeling.

## Preface

How many dimensions do we live in? Good question, and if you ask about this a kid, a good physics graduate, a bad physics graduate, Pac-Man, a Green little man from Mars, an artist, an old senator, an atomic bomb, or a black hole, the answer might differ.

For us, mathematicians, this question is of particular importance, because depending on the number of dimensions  $N \in \mathbb{N}$ , and I'm using here  $\mathbb{N}$  as something approximate, we will have to adapt our mathematics, as for that to be something truly useful.

You would say, for most of basic math,  $N = 1$ , or perhaps  $N = 2$ , motivated by angles and trigonometry, which are certainly something very useful, will do. And this is indeed the case, with the bulk of basic math being indeed developed by using  $N = 1, 2$ . And with the story with  $N = 1, 2$  being not over here, because, a bit as Pac-Man teaches us, there are many levels of knowledge here, and you can even spend your whole life, as a mathematician, working on  $N = 1, 2$ . In fact, for the story, and agreeing this time with old senator's answer, we even have colleagues in math spending their lives on  $N = 0$ .

But let us get now to what the kid says,  $N = 3$ . Very reasonable answer, and at the mathematical level, things start getting quite complicated here. A good knowledge of the vector formalism, which was not really needed at  $N = 2$ , say due to the complex numbers, which do the job at  $N = 2$ , while technically counting as  $N = 1$ , is now required.

Among others, vectors in 3 dimensions are subject to the vector product  $x \times y$ , best understood and computed by using some determinants and the right-hand rule. And with this, we are most likely into some complicated mixture of linear algebra, and physics.

Things however do not stop here, because when asking a physics graduate, the answer will most likely be a certain  $N \in \{4, 5, \dots, \infty\}$ , which can vary with the graduate in question. To be more precise,  $N = 4$  or higher is something well-established, coming from Einstein, who taught us that space  $\mathbb{R}^3$  and time  $\mathbb{R}$  are related, and so that we have to look at spacetime, which is a curved version of  $\mathbb{R}^4$ . Then, the quite plausible possibility of  $N = \infty$ , when looking at very small scales, came from quantum mechanics, as developed by Heisenberg and others. And finally,  $4 < N < \infty$  is more complicated and modern physics business, typically with the bigger the  $N < \infty$ , the fancier the theory.

So, this was for the story of dimensionality of mathematics, in relation with physics, and although I never personally checked with the Green little men from Mars, I am pretty much sure that they use  $N \in \mathbb{N} \cup \{\infty\}$  too. As for the artists, atomic bombs and black holes, their respective answers when asked were “love”, “hey” and “yumm”, most likely some advanced versions of our usual  $N = \infty$  from mathematics.

We will discuss here, in this book, such things, with a joint introduction to basic mechanics, and physics and forces in general, and vector mathematics. We will discuss as well, at the end, some more tricky forces, and their mathematical modeling.

Many thanks to my cats, for precious help with the preparation of the present book, projecting from meow to various  $N \leq \infty$  values being a quite easy task, for them.

*Cergy, December 2024*

*Teo Banica*

## Contents

|                          |          |
|--------------------------|----------|
| Preface                  | 3        |
| <b>Part I. The force</b> | <b>9</b> |
| Chapter 1.               | 11       |
| 1a.                      | 11       |
| 1b.                      | 11       |
| 1c.                      | 11       |
| 1d.                      | 11       |
| 1e. Exercises            | 11       |
| Chapter 2.               | 13       |
| 2a.                      | 13       |
| 2b.                      | 13       |
| 2c.                      | 13       |
| 2d.                      | 13       |
| 2e. Exercises            | 13       |
| Chapter 3.               | 15       |
| 3a.                      | 15       |
| 3b.                      | 15       |
| 3c.                      | 15       |
| 3d.                      | 15       |
| 3e. Exercises            | 15       |
| Chapter 4.               | 17       |
| 4a.                      | 17       |
| 4b.                      | 17       |
| 4c.                      | 17       |
| 4d.                      | 17       |
| 4e. Exercises            | 17       |

|                                 |    |
|---------------------------------|----|
| <b>Part II. Vector calculus</b> | 19 |
| Chapter 5.                      | 21 |
| 5a.                             | 21 |
| 5b.                             | 21 |
| 5c.                             | 21 |
| 5d.                             | 21 |
| 5e. Exercises                   | 21 |
| Chapter 6.                      | 23 |
| 6a.                             | 23 |
| 6b.                             | 23 |
| 6c.                             | 23 |
| 6d.                             | 23 |
| 6e. Exercises                   | 23 |
| Chapter 7.                      | 25 |
| 7a.                             | 25 |
| 7b.                             | 25 |
| 7c.                             | 25 |
| 7d.                             | 25 |
| 7e. Exercises                   | 25 |
| Chapter 8.                      | 27 |
| 8a.                             | 27 |
| 8b.                             | 27 |
| 8c.                             | 27 |
| 8d.                             | 27 |
| 8e. Exercises                   | 27 |
| <b>Part III. Force, revised</b> | 29 |
| Chapter 9.                      | 31 |
| 9a.                             | 31 |
| 9b.                             | 31 |
| 9c.                             | 31 |
| 9d.                             | 31 |
| 9e. Exercises                   | 31 |

CONTENTS

7

|                                |           |
|--------------------------------|-----------|
| Chapter 10.                    | 33        |
| 10a.                           | 33        |
| 10b.                           | 33        |
| 10c.                           | 33        |
| 10d.                           | 33        |
| 10e. Exercises                 | 33        |
| Chapter 11.                    | 35        |
| 11a.                           | 35        |
| 11b.                           | 35        |
| 11c.                           | 35        |
| 11d.                           | 35        |
| 11e. Exercises                 | 35        |
| Chapter 12.                    | 37        |
| 12a.                           | 37        |
| 12b.                           | 37        |
| 12c.                           | 37        |
| 12d.                           | 37        |
| 12e. Exercises                 | 37        |
| <b>Part IV. Further beasts</b> | <b>39</b> |
| Chapter 13.                    | 41        |
| 13a.                           | 41        |
| 13b.                           | 41        |
| 13c.                           | 41        |
| 13d.                           | 41        |
| 13e. Exercises                 | 41        |
| Chapter 14.                    | 43        |
| 14a.                           | 43        |
| 14b.                           | 43        |
| 14c.                           | 43        |
| 14d.                           | 43        |
| 14e. Exercises                 | 43        |
| Chapter 15.                    | 45        |

|                |    |
|----------------|----|
| 15a.           | 45 |
| 15b.           | 45 |
| 15c.           | 45 |
| 15d.           | 45 |
| 15e. Exercises | 45 |
| Chapter 16.    | 47 |
| 16a.           | 47 |
| 16b.           | 47 |
| 16c.           | 47 |
| 16d.           | 47 |
| 16e. Exercises | 47 |
| Bibliography   | 49 |



Part I

**The force**

*Hot and cold emotion  
Confusing my brain  
I could not decide  
Between pleasure and pain*

## CHAPTER 1

1a.

1b.

1c.

1d.

### 1e. Exercises

Exercises:

EXERCISE 1.1.

EXERCISE 1.2.

EXERCISE 1.3.

EXERCISE 1.4.

EXERCISE 1.5.

EXERCISE 1.6.

EXERCISE 1.7.

EXERCISE 1.8.

Bonus exercise.



## CHAPTER 2

**2a.**

**2b.**

**2c.**

**2d.**

### **2e. Exercises**

Exercises:

EXERCISE 2.1.

EXERCISE 2.2.

EXERCISE 2.3.

EXERCISE 2.4.

EXERCISE 2.5.

EXERCISE 2.6.

EXERCISE 2.7.

EXERCISE 2.8.

Bonus exercise.



## CHAPTER 3

**3a.**

**3b.**

**3c.**

**3d.**

### **3e. Exercises**

Exercises:

EXERCISE 3.1.

EXERCISE 3.2.

EXERCISE 3.3.

EXERCISE 3.4.

EXERCISE 3.5.

EXERCISE 3.6.

EXERCISE 3.7.

EXERCISE 3.8.

Bonus exercise.





## CHAPTER 4

4a.

4b.

4c.

4d.

### 4e. Exercises

Exercises:

EXERCISE 4.1.

EXERCISE 4.2.

EXERCISE 4.3.

EXERCISE 4.4.

EXERCISE 4.5.

EXERCISE 4.6.

EXERCISE 4.7.

EXERCISE 4.8.

Bonus exercise.



Part II

Vector calculus

*One night in Bangkok makes a hard man humble  
Not much between despair and ecstasy  
One night in Bangkok and the tough guys tumble  
Can't be too careful with your company*

## CHAPTER 5

5a.

5b.

5c.

5d.

### 5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

EXERCISE 5.7.

EXERCISE 5.8.

Bonus exercise.



## CHAPTER 6

**6a.**

**6b.**

**6c.**

**6d.**

### **6e. Exercises**

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

EXERCISE 6.6.

EXERCISE 6.7.

EXERCISE 6.8.

Bonus exercise.





## CHAPTER 7

7a.

7b.

7c.

7d.

### 7e. Exercises

Exercises:

EXERCISE 7.1.

EXERCISE 7.2.

EXERCISE 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

EXERCISE 7.7.

EXERCISE 7.8.

Bonus exercise.



## CHAPTER 8

8a.

8b.

8c.

8d.

### 8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.



## Part III

# Force, revised

*In between*  
*What I find is pleasing and I'm feeling fine*  
*Love is so confusing*  
*There's no peace of mind*

## CHAPTER 9

**9a.**

**9b.**

**9c.**

**9d.**

### **9e. Exercises**

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.





## CHAPTER 10

**10a.**

**10b.**

**10c.**

**10d.**

**10e. Exercises**

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

EXERCISE 10.8.

Bonus exercise.



## CHAPTER 11

**11a.**

**11b.**

**11c.**

**11d.**

**11e. Exercises**

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.



## CHAPTER 12

**12a.**

**12b.**

**12c.**

**12d.**

**12e. Exercises**

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.



## Part IV

# Further beasts

*Don't give up  
Cause somewhere  
There's a place  
Where we belong*



## CHAPTER 13

**13a.**

**13b.**

**13c.**

**13d.**

**13e. Exercises**

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Bonus exercise.



## CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.



## CHAPTER 15

**15a.**

**15b.**

**15c.**

**15d.**

**15e. Exercises**

Exercises:

EXERCISE 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

EXERCISE 15.7.

EXERCISE 15.8.

Bonus exercise.



## CHAPTER 16

**16a.**

**16b.**

**16c.**

**16d.**

**16e. Exercises**

Congratulations for having read this book, and no exercises for this final chapter.





## Bibliography

- [1] V.I. Arnold, Ordinary differential equations, Springer (1973).
- [2] V.I. Arnold, Lectures on partial differential equations, Springer (1997).
- [3] V.I. Arnold, Catastrophe theory, Springer (1984).
- [4] T. Banica, Calculus and applications (2024).
- [5] T. Banica, Introduction to modern physics (2024).
- [6] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).
- [7] S.J. Blundell and K.M. Blundell, Concepts in thermal physics, Oxford Univ. Press (2006).
- [8] S.M. Carroll, Spacetime and geometry, Cambridge Univ. Press (2004).
- [9] P.A.M. Dirac, Principles of quantum mechanics, Oxford Univ. Press (1930).
- [10] S. Dodelson, Modern cosmology, Academic Press (2003).
- [11] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).
- [12] A. Einstein, Relativity: the special and the general theory, Dover (1916).
- [13] L.C. Evans, Partial differential equations, AMS (1998).
- [14] E. Fermi, Thermodynamics, Dover (1937).
- [15] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics I: mainly mechanics, radiation and heat, Caltech (1963).
- [16] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics II: mainly electromagnetism and matter, Caltech (1964).
- [17] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics III: quantum mechanics, Caltech (1966).
- [18] H. Goldstein, C. Safko and J. Poole, Classical mechanics, Addison-Wesley (1980).
- [19] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).
- [20] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).
- [21] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).
- [22] K. Huang, Introduction to statistical physics, CRC Press (2001).
- [23] K. Huang, Fundamental forces of nature, World Scientific (2007).
- [24] T. Kibble and F.H. Berkshire, Classical mechanics, Imperial College Press (1966).

- [25] C. Kittel, Introduction to solid state physics, Wiley (1953).
- [26] M. Kumar, Quantum: Einstein, Bohr, and the great debate about the nature of reality, Norton (2009).
- [27] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).
- [28] P. Lax, Functional analysis, Wiley (2002).
- [29] P. Lax and M.S. Terrell, Calculus with applications, Springer (2013).
- [30] P. Lax and M.S. Terrell, Multivariable calculus with applications, Springer (2018).
- [31] M.A. Nielsen and I.L. Chuang, Quantum computation and quantum information, Cambridge Univ. Press (2000).
- [32] R.K. Pathria and P.D. Beale, Statistical mechanics, Elsevier (1972).
- [33] W. Rudin, Principles of mathematical analysis, McGraw-Hill (1964).
- [34] W. Rudin, Real and complex analysis, McGraw-Hill (1966).
- [35] W. Rudin, Functional analysis, McGraw-Hill (1973).
- [36] B. Ryden, Introduction to cosmology, Cambridge Univ. Press (2002).
- [37] B. Ryden and B.M. Peterson, Foundations of astrophysics, Cambridge Univ. Press (2010).
- [38] D.V. Schroeder, An introduction to thermal physics, Oxford Univ. Press (1999).
- [39] R. Shankar, Fundamentals of physics I: mechanics, relativity, and thermodynamics, Yale Univ. Press (2014).
- [40] R. Shankar, Fundamentals of physics II: electromagnetism, optics, and quantum mechanics, Yale Univ. Press (2016).
- [41] A.M. Steane, Thermodynamics, Oxford Univ. Press (2016).
- [42] J.R. Taylor, Classical mechanics, Univ. Science Books (2003).
- [43] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).
- [44] J. von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton Univ. Press (1944).
- [45] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).
- [46] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).
- [47] S. Weinberg, Lectures on astrophysics, Cambridge Univ. Press (2019).
- [48] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).
- [49] H. Weyl, The classical groups: their invariants and representations, Princeton Univ. Press (1939).
- [50] H. Weyl, Space, time, matter, Princeton Univ. Press (1918).