# Geometry and trigonometry

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

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Abstract. This is an introduction to plane geometry, angles and trigonometry, starting from zero or almost. We first discuss basic plane geometry, with the main results regarding the triangles explained. Then we get into trigonometry, with the basic properties of the sine, cosine and tangent discussed. We then go on a more advanced discussion, using affine and polar coordinates, then complex numbers, and with a look into trilinear coordinates too. Finally, we get into calculus methods, with an even more advanced study of the trigonometric functions, and with some applications discussed too.

# Preface

Measuring angles is an art, mastered by artists, as well as craftsmen, scientists and engineers, requiring you to know quite a deal of advanced mathematics, that you can hopefully learn from this book. But, before anything, why measuring angles?

Leaving arts aside, where drawing obviously requires some good knowledge of angles and perspective, unless of course you are interested in doing some low-skill work, and sell that as modern art, angles appear naturally in any question related to building, or understanding all sorts of objects, devices and phenomena, typically at big scales.

Let us take for instance, talking big scales, the question of understanding the movements of the Sun, Moon, other planets, and stars, around our Earth. With this being not that philosophical as a question as it might seem, because when sailing at sea, or even walking on unknown land, the Sun, Moon and so on can be very useful in showing you the way. Well, in relation with this, with measuring distances being barred by the big scale of our objects, you are left with observing angles, and then hopefully produce from these angles, via some tricky math computations, the direction that you need.

So, this was for the main principle of angles and trigonometry, big things can only be observed, and used, via angles. As for the applications of this principle, no need of course to go to the astronomical scales evoked above, these abound in various big scale questions from real life, and engineering. Measuring land, or even smaller things, like trees, or building various things, such as bridges, roads, big houses and so on, all this will lead you into angles and trigonometry, exactly as our ship capitain above.

As a concrete illustration, you certainly know about that amazing pyramids built by the ancient Egyptians. Well, that pyramids were built by using an advanced knowledge of trigonometry, available at that time, and which dissapeared in the present modern ages. Or at least this is how one hypothesis about the pyramids goes, and looking around, at the trigonometry knowledge of my mathematics and engineering students, I am pretty much convinced that this is indeed the true explanation for the pyramids question.

Getting now to the present book, this will be an introduction to all this, geometry, angles and trigonometry, starting from zero or almost, meaning basic knowledge of numbers and fractions, and with emphasis on various formulae useful for science and engineering, along the lines evoked above. The book is organized in 4 parts, as follows:

Part I - We discuss here basic plane geometry, with the theorem of Thales, and then with the main results regarding the triangles explained.

Part II - Here we discuss basic trigonometry, with the definition and basic properties of the sine and cosine, and of the other trigonometric functions.

Part III - Here we go on a more advanced discussion, using affine and polar coordinates, then complex numbers, and with a look into trilinear coordinates too.

Part IV - We get here into calculus methods, with an even more advanced study of the trigonometric functions, and with some applications discussed too.

As mentioned, the presentation will be elementary, starting from zero or almost. However, with the book format being chosen long, 400 pages, we will not hesitate to deviate from time to time from the standard paths, and talk about other things too.

Many thanks to my math professors, and now that I am a professor myself, to my students. Thanks as well to my cats, for their teachings regarding the angle of attack, which is a more advanced notion, that we will hopefully discuss too, in this book.

*Cergy, October 2025*
*Teo Banica*

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# Part I

# Plane geometry

*Buffalo Soldier, dreadlock Rasta*
*Fighting on arrival, fighting for survival*
*Driven from the mainland*
*To the heart of the Caribbean*

CHAPTER 1

# Parallel lines

## 1a. Parallel lines

Welcome to plane geometry. At the beginner level, which is ours for the moment, this will be a story of points and lines. So, let us try to understand this first, what can be said about points and lines, and in what regards more complicated things like angles, triangles, and of course, trigonometry, we will leave them for later.

So, points and lines. Here is a basic observation, to start with, and we will call this "axiom" instead of "theorem", as the statements which are true and useful are usually called, in mathematics, for reasons that will become clear in a moment:

Axiom 1.1. *Any two distinct points $P \neq Q$ determine a line, denoted $PQ$.*

Obviously, our axiom holds, and looks like something very useful. Need to draw anything, for various engineering purposes, at your job, or in your garage? The rule will be your main weapon, used exactly as in Axiom 1.1, that is, put the rule on the points $P \neq Q$ that your line must unite, and then draw that line $PQ$.

Actually, in relation with this, drawing lines in the real life, for various engineering purposes, we are rather used in practice to draw segments $PQ$:

$$P \text{———} Q \qquad \leftsquigarrow \textit{ segment}$$

This being said, you certainly know from real life that it never hurts to "enhance" your segment, by extending it a bit on both sides, because who knows when you will need that two extra bits, and matter of not getting back to the rule, at that time:

$$\text{——} P \text{————} Q \text{——} \qquad \leftsquigarrow \textit{ better segment}$$

But now in theory, will having that segment extended to infinity hurt? Certainly not, so this is why our lines $PQ$ in mathematics will be infinite, as above:

$$\text{————} P \text{————} Q \text{————} \qquad \leftsquigarrow \textit{ line}$$

Very good all this, so at least we know one thing, why lines instead of segments. And with this being an instance of a general principle that we will heavily use, throughout this book, as mathematicians, namely use our friend $\infty$, whenever appropriate.

Getting now to point, as already announced, why is Axiom 1.1 an axiom, instead of being a theorem? You would probably argue that this theorem can be proved by using a rule, as indicated above. However, this does not stand for a mathematical proof.

So, this is how things are, you will have to trust me here. And for further making my case, let me mention that my theoretical physics friends agree with me, on the grounds that, when looking with a good microscope at your rule, that rule is certainly bent.

In fact, still talking nature and physics, and jokes left aside now, the situation with the lines in the real life is something quite complicated, as follows:

FACT 1.2. *The lines in the real life are something quite tricky, due to:*

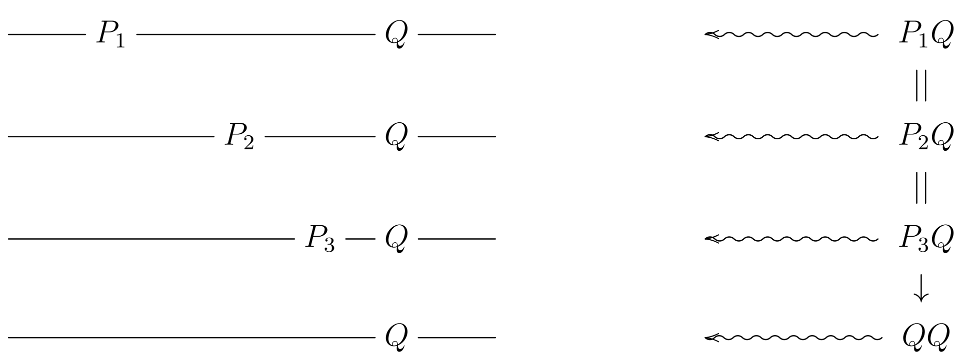
(1) *Certain obviously plane surfaces, such as the Earth, being in fact round.*
(2) *Spacetime itself having, according to relativity theory, a certain curvature.*
(3) *And so on, with true lines, planes, spaces being something rather mathematical.*

So, this is the situation, and you get my point I hope, there is a bit of curvature and geometry in the real life, and due to this, if we want our theory to have something to do with this, it is certainly safer to have Axiom 1.1 stated as such, as an axiom.

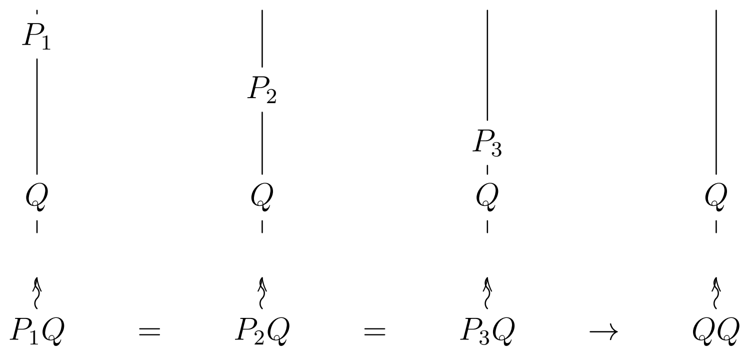
Getting now to more discussion, still around Axiom 1.1, an interesting question appears in connection with our one and only assumption there, namely:

$$P \neq Q$$

Indeed, given a point $Q$ in the plane, we can come up with a sequence of points $P_n \to Q$ horizontally, and in this case the lines $P_nQ$ will all coincide with the horizontal at $Q$. But then, based on this, we could formally say that the $n \to \infty$ limit of these lines, which makes sense to be denoted $QQ$, is also, by definition, the horizontal at $Q$:




However, is this really a good idea, or not. The point indeed is that, when doing exactly the same trick with a series of points $P_n \to Q$ vertically, we will obtain in this way, as our limiting line $QQ$, the vertical at $Q$, as shown by the following picture:




Which does not sound very good, so forget about this. However, since we seem to have some sort of valuable idea here, who knows, let us formulate:

JOB 1.3. *Develop later some kind of analysis theory, generalizing plane geometry, where lines of type $QQ$ make sense too, say as some sort of tangents.*

As a further comment now, still on Axiom 1.1, it is of course understood that the two points $P \neq Q$ appearing there, and the line $PQ$ uniting them, lie in the given plane that we are interested in, in this Part I of the present book. However, Axiom 1.1 obviously holds too in space, and most likely, in higher dimensional spaces too.

So, the question which appears now is, on which type of spaces does Axiom 1.1 hold? And this is a quite interesting question, because if we take a sphere for instance, any two points $P \neq Q$ can be certainly united by a segment, which is by definition the shortest segment, on the sphere, uniting them. And, if we prolong this segment, in the obvious way, what we get is a circle uniting $P, Q$, that we can call line, and denote $P, Q$.

However, not so quick. There is in fact a bug with this, because if we take $P$ to be the North Pole, and $Q$ to be the South Pole, any meridian on the globe will do, as $PQ$. So, as a conclusion, Axiom 1.1 does not really hold on a sphere, but not by much.

Anyway, as before, we seem to have an idea here, so let us formulate:

JOB 1.4. *Develop as well later some advanced geometry theory, generalizing plane geometry, where certain lines $PQ$ can take multiple values.*

Still talking spheres, one idea in order to fix what we have in the above is by restricting the attention to the Northern hemisphere, because with this done, any two points $P \neq Q$ will certainly determine a line, in agreement with what Axiom 1.1 says.

However, now that we know this, what about the Southern hemisphere, can we include it too in our formalism, via some mathematical tricks? And in answer here, yes this can be done, say by identifying on the sphere the pairs of opposite points. Indeed, in this way what we have regarding the Northern hemisphere, that is, Axiom 1.1, will still hold, and as a bonus, we will have this axiom holding on the Southern hemisphere too.

As before with other things, we seem to have a good idea here, so let us record:

JOB 1.5. *Develop later some sort of alternative plane geometry, inspired by what happens here on Earth, but with the opposite points being identified.*

And with this, done I guess with the discussion regarding Axiom 1.1, or at least we have learned enough about it, at least so far. More about this, later in this book.

Moving ahead now, as a natural question, do any two lines $K \neq L$ determine a point? Normally yes, because assuming $P, Q \in K \cap L$ we would have $K = L = PQ$, contradiction. However, it might happen that these distinct lines $K \neq L$ are parallel, $K||L$, in which case we have $K \cap L = \emptyset$. In order to further discuss this, let us formulate:

DEFINITION 1.6. *We say that two lines are parallel, $K||L$, when they do not cross,*

$$K \cap L = \emptyset$$

*or when they coincide, $K = L$. Otherwise, we say that $K, L$ cross, and write $K \not\parallel L$.*

Here we have tricked a bit, by agreeing to call parallel the pairs of identical lines too, and this for simplifying most of our mathematics, in what follows, trust me here.

Very good, and now with Axiom 1.1 and Definition 1.6, we are potentially ready for doing some geometry. However, this is not exactly true, and we will need as well:

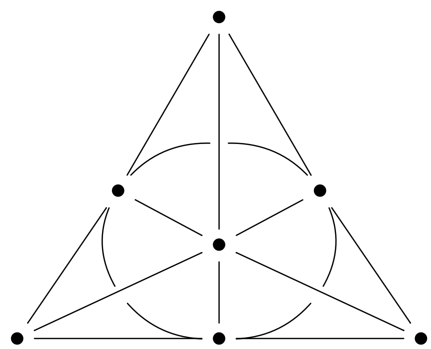
AXIOM 1.7. *Given a point not lying on a line, $P \notin L$, we can draw through $P$ a unique parallel to $L$. That is, we can find a line $K$ satisfying $P \in K$, $K||L$.*

To be more precise, this is again something which obviously holds, but cannot be established, as a theorem. I mean just try, and you will see that you will fail. As before with Axiom 1.1, we will leave as an exercise some further meditating on all this.

Before leaving this preliminary discussion, however, two more comments regarding the two axioms that we have. We have seen in the above that Axiom 1.1 is in fact something quite tricky, and the same can be said about Axiom 1.7, notably with:

COMMENT 1.8. *The axiom of parallels can fail for certain quite natural planes, such as the one evoked in Job 1.5, where any two lines obviously cross.*

Which sounds quite interesting, hope you agree with me. We will talk about such things later in this book, and in the meantime, as a further piece of advertisement for all this, have a look at the following magical configuration, called Fano plane:



Here the circle in the middle is by definition a line, and with this convention, any two points determine a line, and any two lines determine a point. And isn't this beautiful. We will be back to this, with full explanations and comments, later in this book.

As a second comment now on our axioms, which is of key importance too, we have:

COMMENT 1.9. *Assuming the real numbers and some vector mathematics known, we can say that the plane is* $\mathbb{R}^2$*, and with this, the following formula proves Axiom 1.1:*

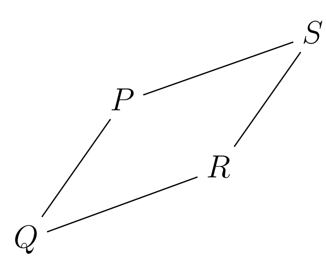
$$PQ=\left\{\lambda P+(1-\lambda)Q\Big|\lambda\in\mathbb{R}\right\}$$

*As for Axiom 1.7, this is easy to establish too, by assuming* $L=QR$ *and setting:*

$$K=\left\{P+\lambda(R-Q)\Big|\lambda\in\mathbb{R}\right\}$$

*However, we will not do so in this book, at least to start with, due to various reasons, including those coming from Fact 1.2.*

To be more precise here, and more on this later when talking vectors, the above formula proving Axiom 1.1 is something quite obvious. As for the second formula, proving Axiom 1.7, this comes from the following configuration, involving a parallelogram:



Indeed, by standard vector calculus, to be discussed later in this book, we must have $P+R=Q+S$, and so $S=P+R-Q$, which leads to the following formula for the line $K=PS$ that we are looking for, which coincides with the formula given above:

$$K=\left\{(1-\lambda)P+\lambda(P+R-Q)\Big|\lambda\in\mathbb{R}\right\}$$

Getting now to Comment 1.9 as stated, in relation with the last assertion there, in the hope that you get my point. Indeed, as explained in Fact 1.2, true mathematics and physics can be far more complicated than what can be said about $\mathbb{R}^2$, so it is safer, at least to start with, to develop geometry based on Axiom 1.1 and Axiom 1.7 only.

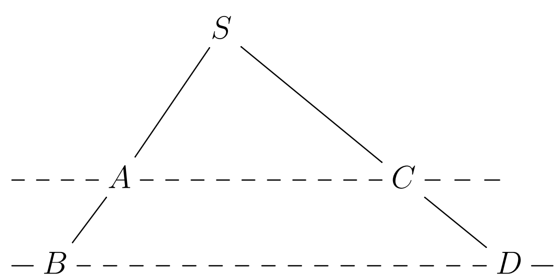
By the way, the approach that we will be using, based on Axiom 1.1 and Axiom 1.7, will be something quite beautiful, old style and everything, as we will soon discover. As for vectors and modernity, do not worry, we will be back to them, later in this book.

## 1b. Thales theorem

Ready for some math? Here we go, and many things can be said here, especially about parallel lines, which are the main objects of basic geometry, as for instance:

Claim 1.10 (Thales). *Proportions are kept, along parallel lines.*

To be more precise, consider a configuration as follows, consisting of two parallel lines, and of two extra lines, which are crossing, and crossing these parallel lines too:




The claim of Thales is then that the following equality holds:

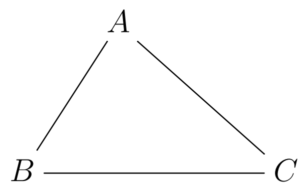
$$\frac{SA}{SB}=\frac{SC}{SD}$$

Moreover, in addition to this, we have some further claims, such as the fact that $AC/BD$ equals the above number, too. And there is more that can be said, along the same lines, this time involving configurations of three parallel lines, and so on.

In what follows the idea will be that of proving the main claim of Thales, which is the equality above, and then deducing from this all sorts of other useful statements, that can be made. But, getting to the point now, how to prove that main claim of Thales?

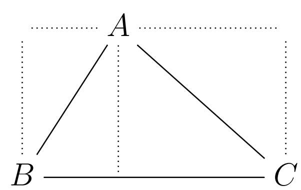
Well, this is actually not obvious, and we will have to trick. Let us start with the following fact, which itself is something quite obvious, and very useful too:

THEOREM 1.11. *The area of a triangle, with a side drawn horizontally,*




*is half the product of that side, and of the height.*

PROOF. This is clear by completing the picture into a rectangle, as follows:




Indeed, the area of the rectangle is easy to compute, given by:

$$area(\square) = side \times height$$

On the other hand, as it is clear on the above picture, our rectangle appears to be made from two triangles equal to $ABC$, via some cutting and pasting. Thus:

$$area(\square) = 2 \times area(ABC)$$

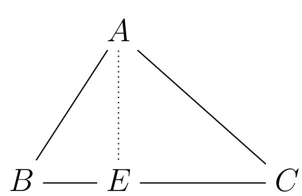
We conclude from this that the area of the triangle is given by:

$$area(ABC) = \frac{1}{2} \times side \times height$$

Thus, we are led to the conclusion in the statement. □

In practice now, it is better to use an equivalent statement, as follows:

THEOREM 1.12. *The area of a triangle, with an altitude drawn,*




*is given by the following formula,*

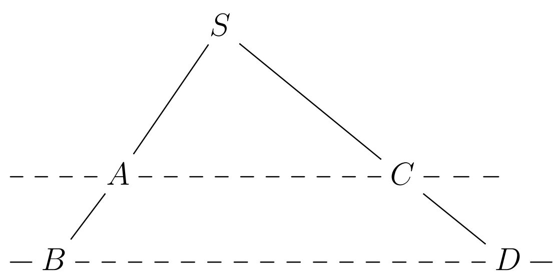
$$area(ABC) = \frac{AE \times BC}{2}$$

*and this no matter how our triangle is oriented, in the plane.*

Proof. This follows indeed from Theorem 1.11, by rotating what we found there, or simply by arguing that the method used there in the proof, with constructing that rectangle, works in any direction, with no need for our triangle to lie on the horizontal. □

Good news, we can now prove the Thales theorem, as follows:

Theorem 1.13 (Thales). *Proportions are kept, along parallel lines. That is, given a configuration as follows, consisting of two parallel lines, and of two extra lines,*




*the following equality holds:*

$$\frac{SA}{SB}=\frac{SC}{SD}$$

*Moreover, the converse of this holds too, in the sense that, in the context of a picture as above, if this equality is satisfied, then the lines AC and BD must be parallel.*

Proof. We can prove indeed the main assertion via the following computation, based on the area formula in Theorem 1.12, used multiple times:

$$\begin{aligned}\frac{SA}{SB}&=\frac{area(CSA)}{area(CSB)}\\&=\frac{area(CSA)}{area(CSA)+area(CAB)}\\&=\frac{area(CSA)}{area(CSA)+area(CAD)}\\&=\frac{area(ASC)}{area(ASD)}\\&=\frac{SC}{SD}\end{aligned}$$

As for the converse, which is actually something quite theoretical, and not that useful in practice, we will leave the proof here as an instructive exercise. □

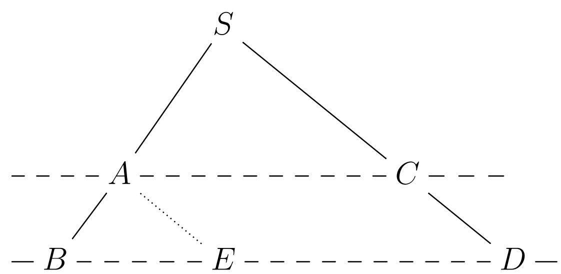
As already mentioned before, there are many other useful versions of the Thales theorem, which are all good to know. Let us start our discussion here with:

Theorem 1.14 (Thales 2). *In the context of the Thales theorem, we have:*

$$\frac{SA}{SB}=\frac{AC}{BD}$$

*However, the converse of this does not necessarily hold.*

Proof. In order to prove the formula in the statement, instead of getting lost into some new area computations, let us draw a tricky parallel, as follows:



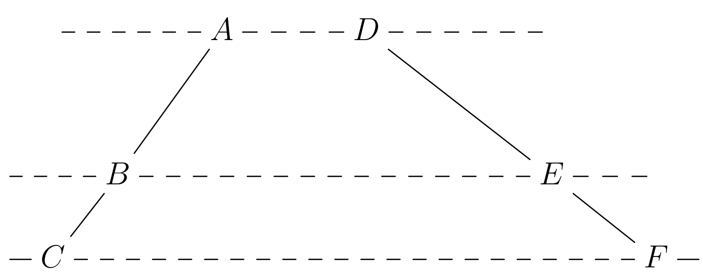

By using Theorem 1.13, we have the following computation, as desired:

$$\frac{SA}{SB}=\frac{DE}{DB}=\frac{AC}{DB}$$

As for the converse, as before this is something quite theoretical, and not that useful in practice, we will leave the proof here as an instructive exercise. □

As a third Thales theorem now, which is something beautiful too, we have:

Theorem 1.15 (Thales 3). *Given a configuration as follows, consisting of three parallel lines, and of two extra lines, which can cross or not,*




*the following equality holds:*

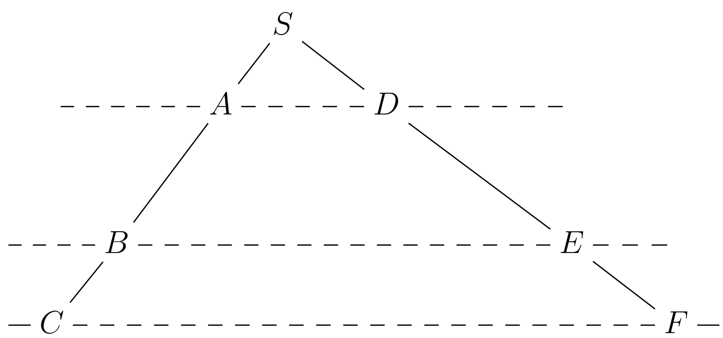
$$\frac{AB}{BC}=\frac{DE}{EF}$$

*That is, once again, the proportions are kept, along parallel lines.*

Proof. We have two cases here, as follows:

(1) When the two extra lines are parallel, the result is clear, because we have plenty of parallelograms there, and the fractions in question are plainly equal.

(2) When the two lines cross, let us call $S$ their intersection:




Now by using Theorem 1.13 several times, we obtain:

$$\begin{aligned}\frac{AB}{BC} &= \frac{SB-SA}{SC-SB}\\ &= \frac{1-\frac{SA}{SB}}{\frac{SC}{SB}-1}\\ &= \frac{1-\frac{SD}{SE}}{\frac{SF}{SE}-1}\\ &= \frac{SE-SD}{SF-SE}\\ &= \frac{DE}{EF}\end{aligned}$$

Thus, we are led to the formula in the statement. □

Very nice all this, we now master the Thales theorem, in its various formulations, the overall conclusion being that, everything that is clear on pictures, regarding proportions and parallel lines, is true indeed, and we have mathematical proof for that.

As a supplementary conclusion now, still about parallel lines, coming from the proof of Thales 2, which was something quite tricky, with that parallel drawn, we have:

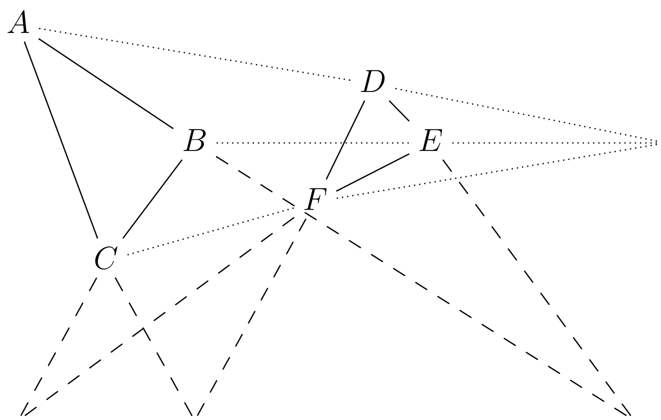
CONCLUSION 1.16. *Many things can be done with parallel lines, with a suitably drawn such line hopefully solving, by some kind of miracle, your plane geometry problem.*

Which is something good to know. We will see more illustrations for this general principle in the next chapter, when getting more in detail into triangle geometry.

## 1c. Desargues theorem

Moving ahead now, many other things can be said about points and lines, and sometimes parallel lines, as a continuation of the Thales theorem. As a basic statement here, due to Desargues, we have the following fact, that we will prove in what follows:

Fact 1.17 (Desargues). *Two triangles are in perspective centrally if and only if they are in perspective axially. That is, in the context of a configuration of type*




*the lines $AD, BE, CF$ cross, so that $ABC, DEF$ are in central perspective, if and only if $AB\cap DE, AC\cap DF, BC\cap EF$ are collinear, so that $ABC, DEF$ are in axial perspective.*

Obviously, this is something that can be very useful for various technical computations and drawings, and more on this later. Getting now to the proof of the result, this is something quite tricky. So, with a bit of imagination, we first have it in one sense:

Theorem 1.18. *The Desargues claim holds in one sense: central perspectivity implies axial perspectivity.*

Proof. The trick here is to pass in 3D, as follows:

(1) Assume first that we are in 3D, with our triangles $ABC$ and $DEF$ lying in distinct planes, say $ABC\subset P$ and $DEF\subset Q$. Assuming central perspectivity, the lines $AD, BE$ cross, so the points $A, B, D, E$ are coplanar. But this tells us that the lines $AB, DE$ cross, and that, in addition, their crossing point lies on the intersection of the planes $P, Q$:

$$(AB\cap DE)\in P\cap Q$$

But a similar argument, again using central perspectivity, shows that we have also:

$$(AC\cap DF)\in P\cap Q\quad,\quad (BC\cap EF)\in P\cap Q$$

Now since the intersection $P\cap Q$ is a certain line in space, we obtain the result.

(2) Thus, almost there, with the theorem proved when the triangles $ABC$ and $DEF$ are both in 3D, in generic position, and the rest is just a matter of finishing. Indeed, when $ABC$ and $DEF$ are still in 3D, but this time lying in the same plane, the result follows too, by perturbing a bit our configuration, as to make it generic. And with this we are done indeed, because we are now in 2D, exactly as in the setting of the theorem. □

In order to prove now to converse, there are several methods and tricks available, and we will choose here to use something quite conceptual. So, temporarily forgetting about Desargues, we have the following result, which is something having its own interest:

THEOREM 1.19. *We have a duality between points and lines, obtained by fixing a circle in the plane, say of center $O$ and radius $r > 0$, and doing the following,*

(1) *Given a point $P$, construct $Q$ on the line $OP$, as to have $OP \cdot OQ = r^2$,*
(2) *Draw the perpendicular at $Q$ on the line $OQ$. This is the dual line $p$,*

*and this duality $P \leftrightarrow p$ transforms collinear points into concurrent lines.*

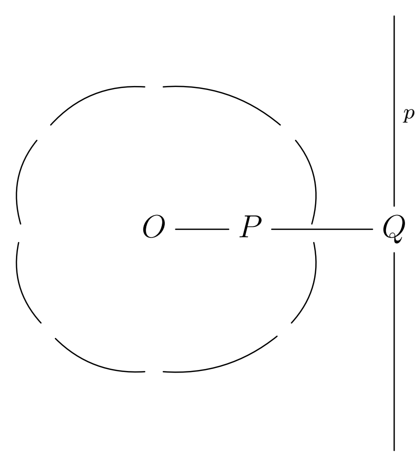
PROOF. Here the fact that we have a duality is something quite self-explanatory, and the statement at the end is something which holds too, the idea being as follows:

(1) We can certainly construct the correspondence $P \to p$ in the statement, which maps points $P \neq O$ to lines $p$ not containing $O$, and which is clearly injective.

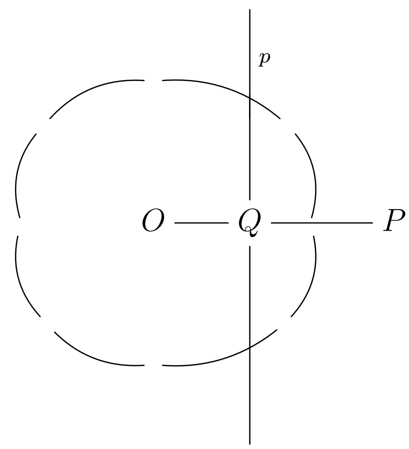
(2) Conversely, given a line $p$ not containing $O$, we can project $O$ on this line, to a point $Q$, and then construct $P \in OQ$ by the formula in the statement, $OP \cdot OQ = r^2$.

(3) We conclude from this that we have indeed a bijection $P \to p$ as in the statement, which maps points $P \neq O$ to lines $p$ not containing $O$.

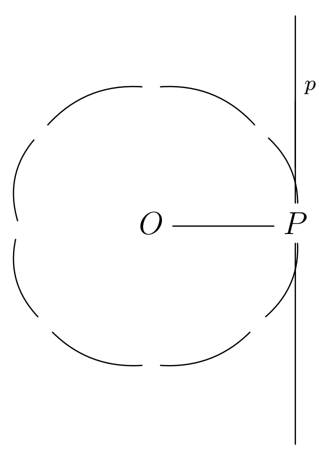
(4) Before getting further, let us make a few simple observations. As a first remark, when the point $P$ is inside the circle, its dual line $p$ is outside of it, as follows:



Conversely, when $P$ is outside the circle, its dual line $p$ crosses the circle:




Finally, when $P$ is on the circle, $p$ is the tangent to the circle, there at $P$:




(5) Getting now to the last assertion, the idea here is to prove that we have the following implication, with $P_n \leftrightarrow p$ and $L \leftrightarrow l$ being instances of our duality:

$$P_1,\ldots,P_n \in l \implies L \in p_1,\ldots,p_n$$

But here, we can assume $n=1$. Thus, we must prove that the following happens:

$$P \in l \implies L \in p$$

(6) In order to prove now this latter fact, given a point $P$, construct its dual line $P \leftrightarrow p$ via a point $Q$ as in the statement, satisfying the following formula:

$$OP \cdot OQ = r^2$$

Now assuming $P \in l$, as above, let us construct the dual point $L \leftrightarrow l$, by projecting $O$ on the line $l$, into a point $R \in l$, and then requiring that $L \in OR$ must satisfy:

$$OL \cdot OR = r^2$$

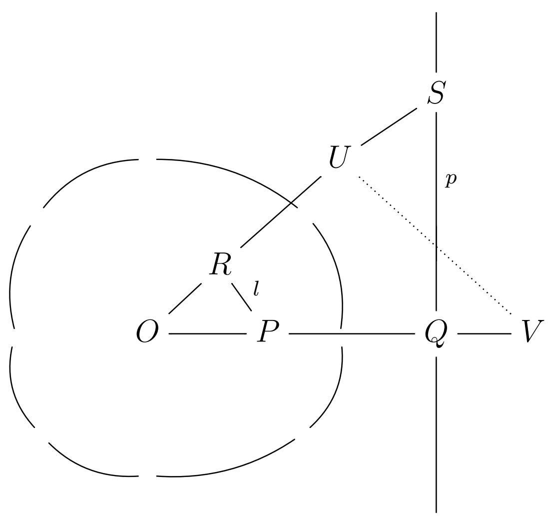
With these constructions made, we want to prove that the following happens:

$$L \in p$$

(7) But this is best seen by considering the following intersection point:

$$S = p \cap OR$$

Indeed, let us draw a picture of $P, Q, R, S$, including, for reasons that will become clear in a moment, points $U \in OS$ and $V \in OQ$ obtained by symmetrizing $Q, S$:




(8) Now since both the lines $RP$ and $UV$ are orthogonal on $OS$, these lines must be parallel, and by using the Thales theorem, we obtain the following formula:

$$\frac{OP}{OR} = \frac{OV}{OU} = \frac{OS}{OQ}$$

(9) But this latter formula can be written as follows, using $OP \cdot OQ = r^2$:

$$OS \cdot OR = OP \cdot OQ = r^2$$

Now by comparing with $OL \cdot OR = r^2$, we conclude that we have:

$$L = S$$

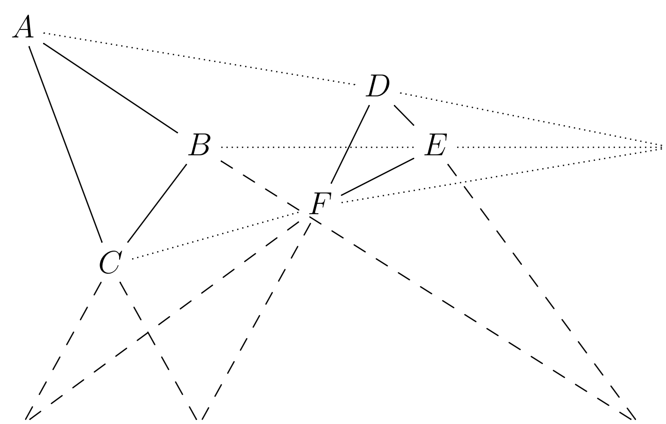
Now since $S \in p$ by definition, we have $L \in p$, which proves our claim in (6). $\square$

Many other interesting things can be said about the duality from Theorem 1.19, and we will be back to this on a regular basis, in what follows.

Getting back now to Desargues, with this technology in hand, the point is that the Desargues configuration is self-dual, so we obtain, as needed:

THEOREM 1.20. *The Desargues claim holds in the other sense too: axial perspectivity implies central perspectivity.*

PROOF. As already mentioned, there are several methods and tricks available, in order to prove this, but the simplest is to argue that this is a trivial consequence of Theorem 1.18 and Theorem 1.19. Indeed, let us look at the Desargues configuration, namely:




Let us look now at the dual Desargues configuration, involving triangles $abc$ and $def$. We have then the following things happening, both coming from Theorem 1.19:

– The original triangles $ABC, DEF$ are in central perspective precisely when the dual triangles $abc, def$ are in axial perspective.

– The original triangles $ABC, DEF$ are in axial perspective precisely when the dual triangles $abc, def$ are in central perspective.

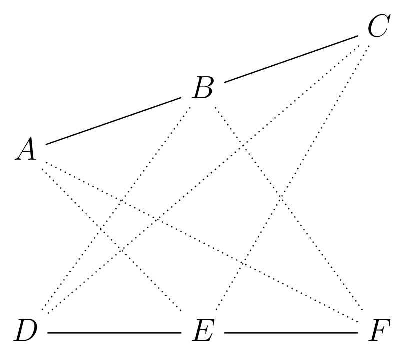
But with this, we are done, because Theorem 1.18 applied to the dual triangles $abc, def$ gives the present result, for the original triangles $ABC, DEF$. □

Summarizing, done with the Desargues theorem, and we have learned many interesting things, including the duality between points and lines, on this occasion.

## 1d. Pappus theorem

Next, we have the following fact, going back in time, to Pappus:

Fact 1.21 (Pappus). *Given a configuration as follows,*




*the three middle points are collinear.*

However, as before with Desargues, or rather with the tricky implication of Desargues, proving such things will need some preparations. So, we must be patient.

Getting to work now on this, and temporarily forgetting about Pappus, the point is that we have the following useful result, which is something having its own interest:

Theorem 1.22. *We can talk about the cross ratio of four collinear points $A, B, C, D$, as being the following quantity, signed according to our usual sign conventions,*

$$(A, B, C, D) = \frac{AC \cdot BD}{BC \cdot AD}$$

*and with this notion in hand, points in central perspective have the same cross ratio:*

$$(A, B, C, D) = (A', B', C', D')$$

*Moreover, the converse of this fact holds too.*

Proof. As before with Theorem 1.19, there is a lot of mathematics hidden here, and with the formula in the statement coming by drawing a suitable parallel line, and computing both $(A, B, C, D), (A', B', C', D')$ in terms of the new points which appear:

(1) To start with, the notion of cross ratio, as constructed in the statement, is something very natural. Observe first that we can write the cross ratio as follows:

$$(A, B, C, D) = \frac{AC}{BC} \cdot \frac{BD}{AD}$$

On the other hand, we can write as well the cross ratio as follows:

$$(A, B, C, D) = \frac{AC}{AD} \cdot \frac{BD}{BC}$$

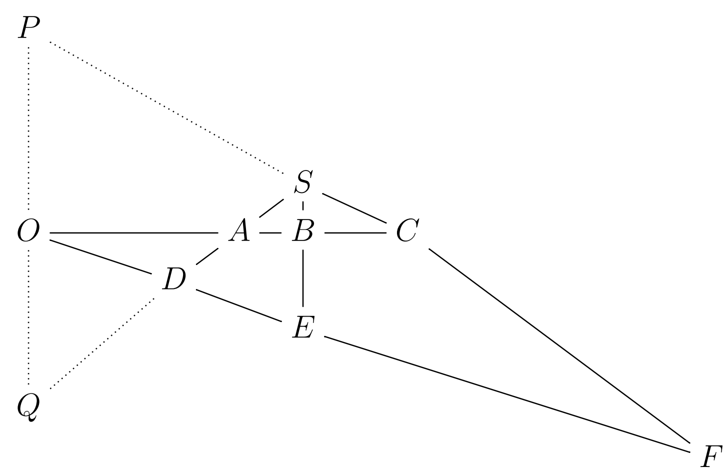
But are these quantities really the same? Sure yes, the theory of fractions says, but go see that geometrically, and have it all the time in mind, when working with the cross ratio, that ain't no easy task, which takes a lot of practice. Welcome to geometry.

(2) Next, many other things can be said, as for instance being the fact that $A, B, C, D$ are somehow "nicely positioned" on their line when their cross ratio is $-1$:

$$(A, B, C, D) = -1$$

Again, try getting familiar with this, by working out some examples, doing some computations and so on. All this is first-class geometry, that you should know.

(3) Getting now to what our statement says, in relation with points in central perspective, consider first the following picture, with the points $A, B, C, D, E, F$ and $S, O$ being as indicated, and with a parallel line to $SE$ drawn on the left, as indicated:




(4) We have then the following equality, obtained by using the Thales theorem:

$$\begin{aligned}(O, B, C, A) &= \frac{OC}{BC} \cdot \frac{BA}{OA} \\ &= \frac{PO}{SB} \cdot \frac{SB}{OQ} \\ &= \frac{PO}{OQ}\end{aligned}$$

On the other hand, again by using the Thales theorem, we have as well:

$$\begin{aligned}(O, E, F, D) &= \frac{OF}{EF} \cdot \frac{ED}{OD} \\ &= \frac{PO}{SE} \cdot \frac{SE}{OQ} \\ &= \frac{PO}{OQ}\end{aligned}$$

We conclude that in the context of the above configuration, we have:

$$(O, B, C, A) = (O, E, F, D)$$

(5) But this gives the equality in statement, by suitably generalizing what we found, somewhat by "blowing up" the point $O$ on the left into a pair of distinct points. To be more precise, let us turn now to the precise equality to be proved, namely:

$$(A,B,C,D)=(A',B',C',D')$$

Here the points $A,B,C,D$ and $A',B',C',D'$ are assumed to be in perspectivity, say with respect to a center of perspectivity $S$. Consider as well the following intersection:

$$O=ABCD\cap A'B'C'D'$$

(6) We have the following formula, coming from the definition of the cross ratio:

$$\begin{aligned}(A,B,C,D) &= \frac{AC\cdot BD}{BC\cdot AD}\\ &= \frac{AC\cdot OD}{OC\cdot AD}\cdot\frac{OC\cdot BD}{BC\cdot OD}\\ &= \frac{OC\cdot BD}{BC\cdot OD}\Big/\frac{OC\cdot AD}{AC\cdot OD}\\ &= \frac{(O,B,C,D)}{(O,A,C,D)}\end{aligned}$$

On the other hand, we have as well the following computation, nearly identical:

$$\begin{aligned}(A',B',C',D') &= \frac{A'C'\cdot B'D'}{B'C'\cdot A'D'}\\ &= \frac{A'C'\cdot O'D'}{O'C'\cdot A'D'}\cdot\frac{O'C'\cdot B'D'}{B'C'\cdot O'D'}\\ &= \frac{O'C'\cdot B'D'}{B'C'\cdot O'D'}\Big/\frac{O'C'\cdot A'D'}{A'C'\cdot O'D'}\\ &= \frac{(O',B',C',D')}{(O',A',C',D')}\end{aligned}$$

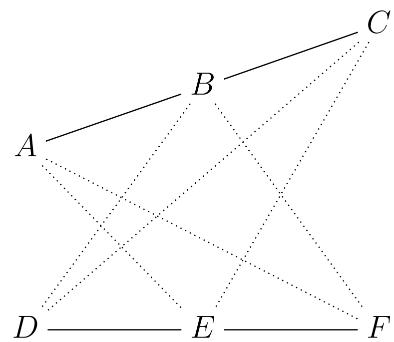
(7) But with these formulae in hand, by using (4) twice, we obtain:

$$\begin{aligned}(A,B,C,D) &= \frac{(O,B,C,D)}{(O,A,C,D)}\\ &= \frac{(O',B',C',D')}{(O',A',C',D')}\\ &= (A',B',C',D')\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Good news, we can now prove the Pappus theorem, as follows:

THEOREM 1.23 (Pappus). *Given a hexagon $AFBDCE$ with both the odd and the even vertices being collinear,*



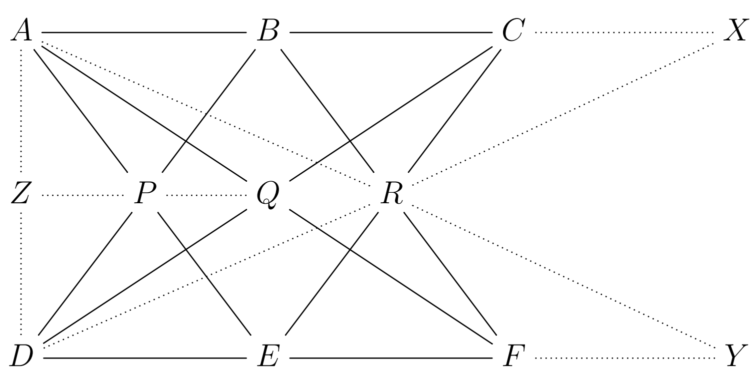

*the pairs of opposite sides cross into three collinear points.*

PROOF. Observe first the fancier formulation of the statement, with respect to what we had before in Fact 1.21, but this is of course more for fun, of perhaps for some deeper reasons too, these mysterious hexagons sort of rule in plane geometry, and more on this later in this book too, on several occasions. In practice now, what we have to prove remains as in Fact 1.21, and the idea is that can be proved by refining the picture, by adding some extra points, and using the cross ratio technology from Theorem 1.22:

(1) Consider indeed the Pappus configuration in the statement, let us call $P, Q, R$ the crossing points appearing there, and construct points $X, Y, Z$ as follows:

$$X = AC \cap DR \quad , \quad Y = AR \cap DF \quad , \quad Z = AD \cap PQ$$

We obtain in this way an enlarged configuration, which looks as follows:




(2) We have then the following equalities, with the first one coming from Theorem 1.22, via the central perspective coming from the point $R$, and with the second one being something trivial, valid for any cross ratio, coming from definitions:

$$(A, C, B, X) = (Y, E, F, D) = (D, F, E, Y)$$

(3) But with this equality, we can conclude. Indeed, let us see what happens to the configurations $ACBX$ and $DFEY$, when projected respectively from the points $D, A$, on the line $PQ$. Via these projections, we have the following correspondences:

$$ACB \to ZQP \quad , \quad DFE \to ZQP$$

(4) Now remember the cross ratio formula found in (2), namely:

$$(A, C, B, X) = (D, F, E, Y)$$

In view of this, and by applying again Theorem 1.22, this time in reverse form, we conclude that the images of $X, Y$ via the above projections must coincide:

$$(DX \cap AY) \in PQ$$

But, according to our conventions above, $DX \cap AY = R$, so we obtain, as desired:

$$R \in PQ$$

(5) Thus, result proved. As a further comment, observe that there is a relation with Desargues too. Finally, note that the Pappus configuration is self-dual. □

Summarizing, Pappus theorem proved, and we have learned many interesting things on this occassion, notably in relation with the notion of cross ratio from Theorem 1.22. As a continuation now of that material, let us formulate the following definition:

DEFINITION 1.24. *Four collinear points $A, B, C, D$,*

$$A \text{ ———— } D \text{ —— } B \text{ ———————— } C$$

*are called in harmonic ratio when their cross ratio equals $-1$:*

$$(A, B, C, D) = -1$$

*In this case, we also say that $D$ is the harmonic conjugate of $C$, with respect to $A, B$.*

In other words, our points $A, B, C, D$ are in harmonic ratio when the following happens, with our usual conventions for the signed lengths of segments:

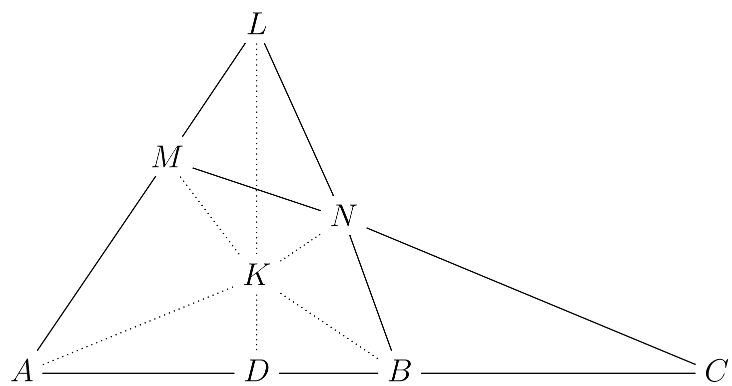
$$\frac{AC \cdot BD}{BC \cdot AD} = -1$$

As an example, the points in the above picture, with $B$ being the middle of $AC$, and $D$ lying $2/3 - 1/3$ on $AB$, are indeed in harmonic ratio, with this coming from:

$$(A, B, C, D) = \frac{AC}{BC} \cdot \frac{BD}{AD} = 2 \times \left(-\frac{1}{2}\right) = -1$$

In general, being in harmonic ratio means somehow that $A, B, C, D$ are "nicely distributed", and this notion appears in relation with many questions, from the real life. In practice now, here is how we can construct such points, in harmonic ratio:

Theorem 1.25. *Given collinear points $A, B, C$, the harmonic conjugate $D$ of $C$ with respect to $A, B$ can be constructed by using the following configuration:*

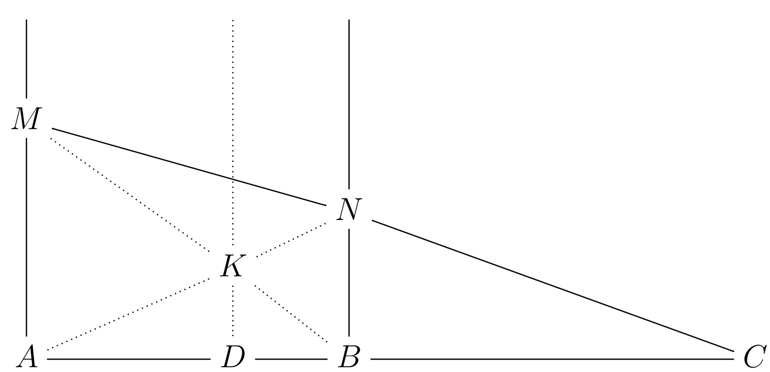



*That is, take any line passing through $C$, take an arbitrary point $L$ too, not lying on the line $ABC$, then construct points $M, N$ as above, then $K$, and finally $D$.*

Proof. This is something very standard, the idea being as follows:

(1) To start with, the point $D$ constructed above does not depend on the choice of the line passing through $C$, with this coming from Desargues. Also, $D$ does not depend either on the choice of the point $L$, with this coming from Desargues too.

(2) Now in view of this, in order to prove the result, we can choose the line passing through $C$, or the point $L$, or both, as we want to. And here, the most convenient is to keep the line passing through $C$ arbitrary, but to choose $L$ high up, towards $\infty$, above the segment $AB$. With this choice, our picture becomes as follows, with parallel lines:




(3) But in this situation, we can easily compute the cross ratio. Indeed, by using the Thales theorem several times, we have the following computation:

$$\frac{AD}{DB} = \frac{AK}{KN} = \frac{MA}{NB} = \frac{CA}{CB}$$

Thus, the cross ratio is given by the following formula:

$$(A,B,C,D)=\frac{AC}{BC}\cdot\frac{BD}{AD}=\frac{AC}{BC}\cdot\frac{BC}{CA}=-1$$

We are therefore led to the conclusion in the statement. □

And with this, end of this preliminary chapter, on lines, points, and related topics. We have learned many interesting things, all good to know, and we will be back to this on many occasions, in what follows, with applications, generalizations, and more.

## 1e. Exercises

This was a quite elementary chapter, on the foundations of geometry, although some things were a bit philosophical, and probably new to you. As exercises, we have:

Exercise 1.26. *Learn a bit about modern geometry, and tangent spaces.*

Exercise 1.27. *Learn also a bit about projective spaces, and projective geometry.*

Exercise 1.28. *Work out some further versions, or consequences, of Thales.*

Exercise 1.29. *Try finding a $2D$ proof for the first implication in Desargues' theorem.*

Exercise 1.30. *Learn more about the duality between points and lines.*

Exercise 1.31. *Learn more about the cross ratio of four collinear points.*

Exercise 1.32. *Try finding the precise relation between Desargues and Pappus.*

Exercise 1.33. *Learn also a bit about the theorems of Pascal and Brianchon.*

As bonus exercise, in case you know a bit about vectors, review what we said above in terms of vectors. We will be back to this, but only later in this book.

CHAPTER 2

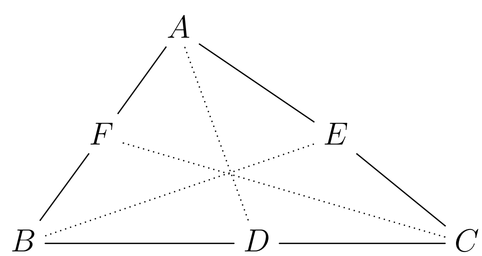
# Triangles

## 2a. Barycenter

Welcome to triangle geometry, which is on the route of what we want to do in this book, namely angles and trigonometry. In fact, you can sense this right away, with "triangle" obviously coming from "three angles". And with the point being that, while angles taken alone are quite hard to investigate, angles coming in triplets, that is, in the form of triangles, are relatively easy to get into, via Thales and other techniques.

But let us start our study of triangles with the most important triangle result of them all, which is actually unrelated to angles. This is the barycenter theorem, as follows:

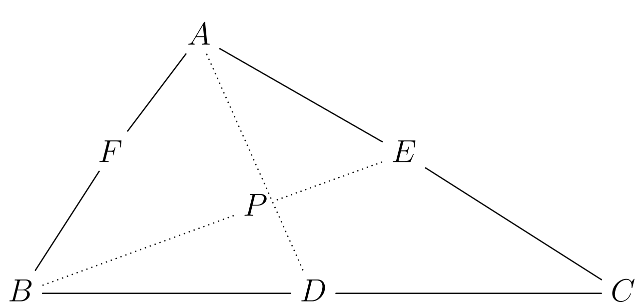
THEOREM 2.1 (Barycenter). *Given a triangle $ABC$, its medians cross,*



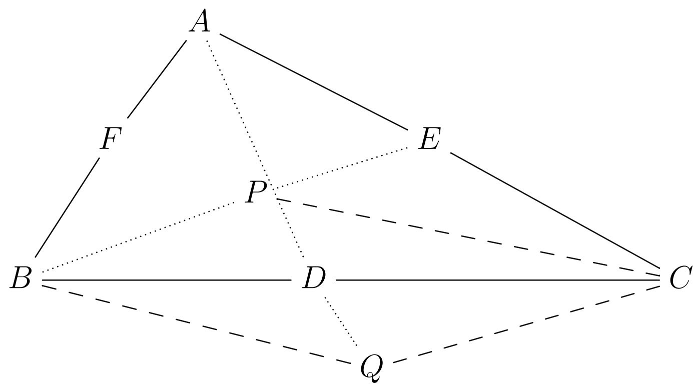

*at a point called barycenter, lying at $1/3 - 2/3$ on each median.*

PROOF. The idea is that we can get this from Thales, via some tricks:

(1) Let us draw indeed the medians $AD$ and $BE$, and call $P$ their intersection:



(2) Now comes the trick. Let us symmetrize $P$ with respect to $D$, into a point $Q$:




(3) Since $BD = DC$ and $PD = DQ$, by Thales, the figure $BPCQ$ is a parallelogram. In particular we have $BP||CQ$, and again by Thales, we obtain from this:

$$AE = EC \implies AP = PQ$$

On the other hand, remember that $D$ was the midpoint of $PQ$. Thus, we obtain:

$$AP = 2PD$$

(4) Summarizing, we have proved that when intersecting two medians, the intersection point lies at $1/3 - 2/3$ on one of the two medians. But, by symmetry, this intersection point must lie as well at $1/3 - 2/3$ on the other median, that we have intersected.

(5) But with this established, we are done. Indeed, if we consider, on each of the 3 medians, the point lying at $1/3 - 2/3$ on that median, then by (4) these 3 points will coincide. Thus, we are led to the conclusion in the statement. □

The barycenter has many interesting properties, the most important of which, in relation with intuition and physics, can be summarized as follows:

Fact 2.2. *The gravity center of a triangle $ABC$ is as follows:*

(1) *In the 0-dimensional case, that is, when putting equal weigths at the vertices $A, B, C$, and computing the center, this is the barycenter.*
(2) *In the 1-dimensional case, that is, with the sides $AB, BC, AC$ have weigths proportional with their length, this is, in general, different from the barycenter.*
(3) *In the 2-dimensional case, that is, with the triangle $ABC$ itself, as an area, having a weight, uniformly distributed, this is again the barycenter.*

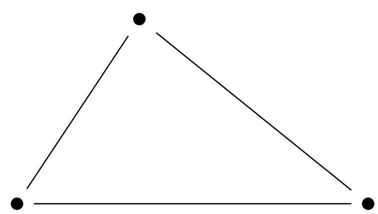
All this looks quite interesting, so let us try now to have some understanding of this. But, we are faced right away with the following question: how to compute, in practice, the barycenter of a configuration of weights, say as in (1), (2), (3) above?

Not an easy question, but based on everyday experience, let us formulate:

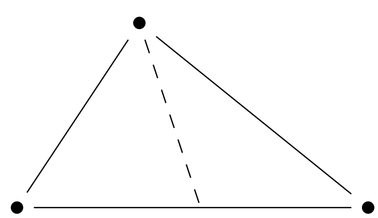
Method 2.3. *In order to compute the barycenter of a plane object:*

(1) *We can come up with a blade, put it under the object, and find the correct angle, such as the object lies in equilibrium, on the blade.*
(2) *In this case, with the object lying in equilibrium, we can say, mathematically, that the barycenter lies on the line of the blade edge.*
(3) *And by doing twice this procedure, we can exactly locate the barycenter, as being the interesection of the two lines that we obtain.*

So, let us see how this method works, in relation with Fact 2.2 (1). Consider, as indicated there, a triangle, with equal weights installed at the vertices:



Now let us come with the blade, as indicated in Method 2.3 (1), and try to find the correct angle, as for the upper vertex to lie on the blade, and for the whole triangle to be in equilibrium. Our claim is that, in order for this to happen, the blade must be precisely positioned on the median emanating from the upper vertex, as follows:



Indeed, in this configuration, we have equilibrium, because the upper weight, which is on the blade, will not matter, and the left and right weigths, being equally distanced from the blade, as you can see by drawing two perpendiculars, which will obviously be equal, will cancel each other's effect. Moreover, we can also see that if we move the blade a bit to the right, the triangle will obviously fall to the left, and that if we move the blade a bit to the left, the triangle will obviously fall to the right. Thus, claim proved.

But with this claim proved, we are done, our conclusion being as follows:

Conclusion 2.4. *When computing the physical barycenter as in Method 2.3, a triangle having equal weights installed at the vertices must have its barycenter on each of the three medians. Thus, these three medians cross, at the physical barycenter.*

Which is very nice, not only we have a proof now for what is said Fact 2.2 (1), equality of the mathematical and physical barycenters, but as a bonus, we have as well an alternative proof for Theorem 2.1, using an old-fashioned blade, instead of math.

Getting now to Fact 2.2 (2), that we would like to understand next, that is a negative result, with a degenerate triangle being a counterexample there, as follows:

$$AB \text{ ———————— } C$$

Indeed, the usual barycenter of this degenerate triangle, appearing as in Theorem 2.1, or as in Fact 2.2 (1), obviously lies at $1/3 - 2/3$ on the segment, as follows:

$$AB \text{ ——— } P \text{ ————— } C$$

However, in the context of Fact 2.2 (2), the side $AB$, which is zero, does not matter, and the sides $AC, BC$ both have their centers at the middle of the segment. Thus, the center of gravity of our degenerate triangle is in this case the middle of the segment:

$$AB \text{ ———— } P_1 \text{ ———— } C$$

Getting now to the context of Fact 2.2 (3), this is something a bit more tricky to understand, with a limit involved, and in the end we obtain the usual barycenter:

$$AB \text{ ——— } P_2 \text{ ————— } C$$

We will leave some thinking here as an instructive exercise, and this because we will have to come back to Fact 2.2 (3) in a moment, anyway. So, degenerate triangles studied, and as a conclusion to this discussion, around Fact 2.2 (2), let us formulate:

Conclusion 2.5. *The centers of a degenerate triangle are as follows, with the subscripts* $0, 1, 2$ *standing for the dimensionality of the problem, in the sense of Fact 2.2,*

$$AB \text{ ——— } P_{0,2} \text{ — } P_1 \text{ ———— } C$$

*and with* $P_{0,2}$ *being the usual, mathematical barycenter, the one from Theorem 2.1.*

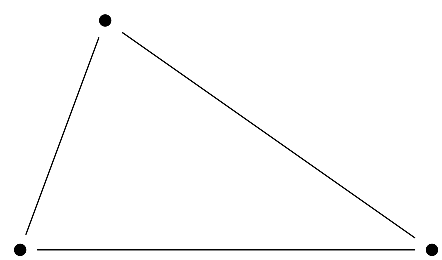
Getting now to the arbitrary triangles, what we have in Method 2.3 does not apply well to the solid triangles, and their discretizations. So, we must come up with something new. And, a bit of thinking here, again inspired from everyday experience with various objects, leads to the following method, standing as a complement to Method 2.3:

Method 2.6. *In order to compute the barycenter of a plane object:*

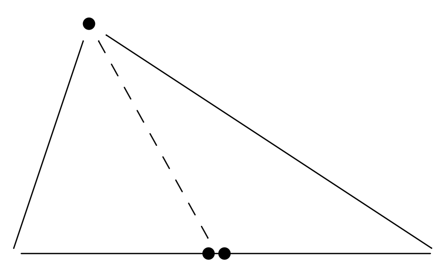
(1) *We can discretize the object, by approximating chunks of mass* $\varepsilon$ *with point weigths of mass* $\varepsilon$*, positioned anywhere inside the corresponding chunk.*
(2) *For discrete objects, we can use the rule that a two-point configuration* $a - b$ *can be replaced with* $a + b$ *lying at* $\frac{b}{a+b} - \frac{a}{a+b}$ *on the segment, and recursivity.*
(3) *Thus, we have an algorithm for computing the barycenter of our discretization. And by taking the limit* $\varepsilon \to 0$*, we reach to the barycenter of the initial object.*

To be more precise here, passed some standard discretization talk, done in (1) and (3), the main point lies in the rule in (2), which itself is something very intuitive, say coming from Method 2.3. Indeed, ignoring the rest of the configuration, a blade passing through the point at $\frac{b}{a+b}-\frac{a}{a+b}$ on the segment will certainly have our $a-b$ configuration lying in equilibrium, so in practice we can replace if we want this $a-b$ configuration by a point mass of $a+b$ positioned there, at $\frac{b}{a+b}-\frac{a}{a+b}$ on the segment, as indicated above.

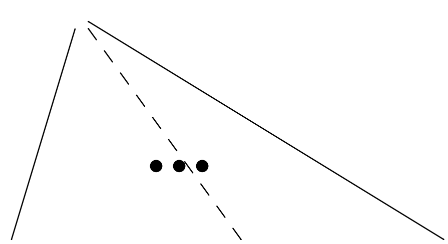
As an illustration, let us reprove Fact 2.2 (1) by using this new method. Consider, as indicated in Fact 2.2 (1), a triangle, with equal weights installed at the vertices:



By using the rule in Method 2.6 (2), we can merge the lower weights, as follows:



But then, by using again this rule, we can further merge our weights, as follows:

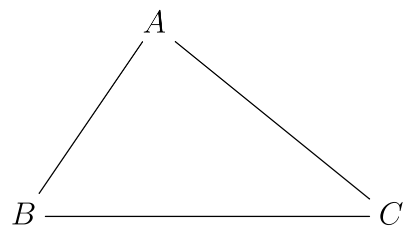


Thus, Fact 2.2 (1) proved again, our conclusion being as follows:

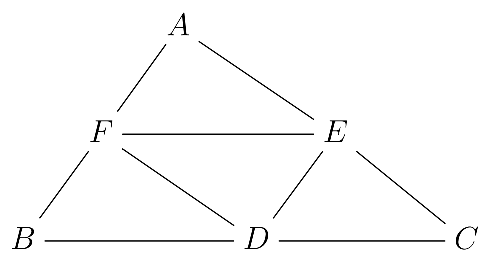
Conclusion 2.7. *When computing the physical barycenter as in Method 2.6, a triangle having equal weights installed at the vertices must have its barycenter lying at* $1/3-2/3$ *on each median. Thus, these three medians cross, at the physical barycenter.*

Which is again nice, not only we have now a new proof for what is said Fact 2.2 (1), equality of the mathematical and physical barycenters, but as a bonus, we have as well a full alternative proof for Theorem 2.1, including the $1/3-2/3$ claim there.

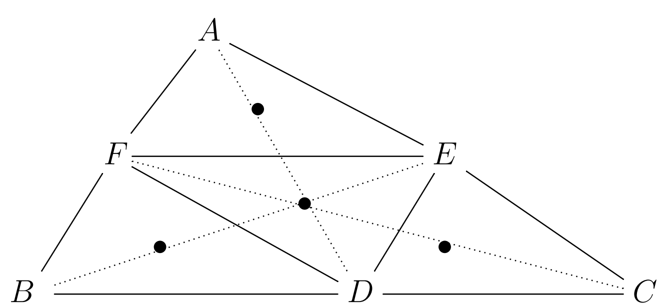
Getting now to what Fact 2.2 (3) says, consider as indicated there a solid triangle, with uniformly distributed weight, all across its surface, of total mass 1:




Now let us discretize this triangle, as in Method 2.6 (1). An easy way of doing so, with $\varepsilon = 1/4$, is by cutting the triangle in 4 obvious equal parts, as follows:




In order to finish our $\varepsilon = 1/4$ discretization, we still have to pick the positions of our 4 point weights, inside the above 4 triangles. As mentioned in Method 2.6 (1), the precise positions of these points will not matter in the end, in relation with our overall $\varepsilon \to 0$ computation, so there are many possible choices here. As a standard choice, however, that we will use here, we have the mathematical barycenters of the above 4 triangles. And with this done, the picture of our $\varepsilon = 1/4$ discretization becomes as follows:




Getting now to the computation of the barycenter of this 4-point configuration, this is clear, because we can see that this 4-point configuration actually consists of a triangle, and its barycenter. Thus, as barycenter, we obtain the point in the middle. Of course, this computation can be done too by using the rules in Method 2.6 (2).

Summarizing, done with $\varepsilon = 1/4$. The next step is $\varepsilon = 1/16$, by cutting each of the small 4 triangles in 4 parts, as before, then $\varepsilon = 1/64$ and so on. We will leave the computations here as an instructive exercise, and as a conclusion to all this, our method works indeed, and we reach in this way to a proof of Fact 2.2 (3):

CONCLUSION 2.8. *When computing the physical barycenter as in Method 2.6, a solid triangle, with uniformly distributed weight, has as barycenter the usual barycenter.*

Very good all this, so we have now a decent knowledge of the barycenter, and time to talk about something else. However, before doing so, let us listen as well to what cat has to say. Cat indeed is constantly meowing, and this since the beginning of this chapter, when I stated Theorem 2.1. So, what is it, cat, found some mice over there?

CAT 2.9. *Yes, with three mice situated at $A, B, C$, a cat situated at*

$$P = \frac{A+B+C}{3}$$

*can catch them all, and with $1/3 - 2/3$ and everything, without much trouble.*

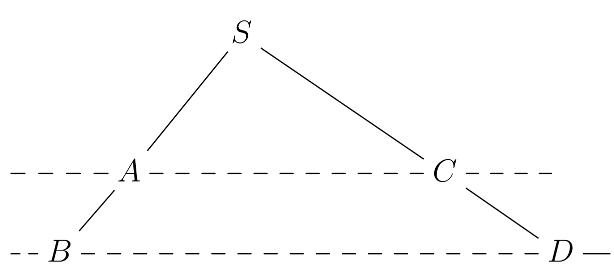
Which sounds like an interesting remark, simplifying what we did in the proof of Theorem 2.1, and perhaps afterwards too, but in practice, go understand what cat is exactly saying, how come he can sum points in the plane, and then divide by 3, just like that. We will leave this for later in this book, and more specifically in Part III, when discussing vectors, which are the good tools for investigating such questions.

## 2b. Angles, basics

Getting now to what we wanted to do in this book, angles and trigonometry, we can certainly talk about angles, in the obvious way, by using triangles and the Thales theorem. Let us record this finding, which is something quite intuitive, as follows:

FACT 2.10. *We can talk about the angle between two crossing lines, and have some basic theory for the angles going, by using triangles, and Thales, in the obvious way.*

To be more precise, let us go back to the configuration from the Thales theorem, from chapter 1, which was as follows, with two parallel lines, and two other lines:




In this situation, we can say that the two triangles $SAC$ and $SBD$ are similar, and with an equivalent formulation of similarity being the fact that the angles are equal:

DEFINITION 2.11. *We say that two triangles are similar, and we write*

$$SAC \sim SBD$$

*when their respective angles are equal.*

The point now is that, in this situation, we can have some mathematics going, for the lengths, coming from the following formula, which is the Thales theorem:

$$\frac{SA}{SB}=\frac{SC}{SD}=\frac{AC}{BD}$$

Many other things can be said here. We will be back to this, and to similar triangles in general, with all sorts of applications, to triangle geometry questions, in a moment.

At the philosophical level now, you might wonder of course what the values of the angles should be, say as real numbers. But this is something quite tricky, that will take us some time to understand. In the lack of something bright, for the moment, let us formulate the following definition, which is quite intuitive, and does the job:

DEFINITION 2.12. *We can talk about the numeric value of angles, as follows:*

(1) *The right angle has value* $90^\circ$.
(2) *We can double angles, in the obvious way.*
(3) *Thus, the half right angle has value* $45^\circ$, *and the flat angle has value* $180^\circ$.
(4) *We can also triple, quadruple and so on, again in the obvious way.*
(5) *Thus, we can talk about arbitrary rational multiples of* $90^\circ$.
(6) *And, with a bit of analysis helping, we can in fact measure any angle.*

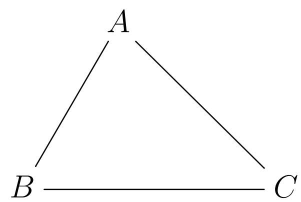
So, this will be our starting definition for the numeric values of the angles. Of course, all this might seem a bit improvised, but do not worry, we will come back later to this, with a better, more advanced definition for these numeric values of the angles.

As another comment, you might wonder what that 90 figure for the right angles stands for. In answer, no one really knows, this is just some convention, old as our modern world and mathematics, say a bit similar to the 10 that we use as numeration basis. Although, with the terrestrial month, based on the movement of the Moon, having about 30 days, we can see here why 10 and its multiples are so important to us, humans.

In any case, comment recorded, and we will come back to this later, with a genius new method for rescaling the angles, independent of astronomy and the Moon, with the right angle $90^\circ$ being destined to be called $\pi/2$, with $\pi = 3.14159\ldots$ being a certain extremely complicated number. And with this being not a joke. More later.

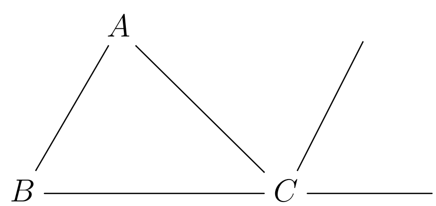
Getting back to work now, theorems and proofs, in relation with the above, here is a key result, which will be our main tool for the study of the angles:

THEOREM 2.13. *In an arbitrary triangle*




*the sum of all three angles is* $180^\circ$.

PROOF. This does not seem obvious to prove, with bare hands, but as usual, in such situations, some tricky parallels can come to the rescue. Let us prolong indeed the segment $BC$ a bit, on the $C$ side, and then draw a parallel at $C$, to the line $AB$, as follows:




We can see that the three angles around $C$, summing up to the flat angle $180^\circ$, are in fact the 3 angles of our triangle. Thus, theorem proved, just like that. □

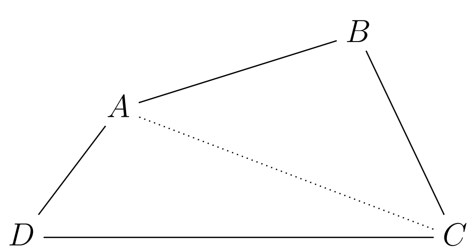
More generally now, we have the following result, dealing with arbitrary polygons:

THEOREM 2.14. *In an arbitrary polygon, the sum of all angles is*

$$\Sigma = (N-2)180^\circ$$

*with* $N = 3, 4, 5, \ldots$ *being the number of vertices.*

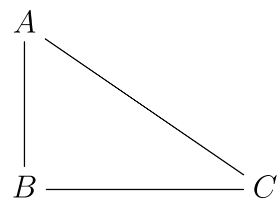
PROOF. This follows indeed by decomposing our polygon having $N$ vertices into $N-2$ triangles, in the obvious way, with the picture at $N = 4$ being as follows:




Thus, by using Theorem 2.13, we are led to the conclusion in the statement. □

Going ahead now with our study of angles, as a continuation of the above, let us first talk about the simplest angle of them all, which is the right angle, denoted $90^\circ$. In relation with it, let us formulate the following definition, making the link with triangles:

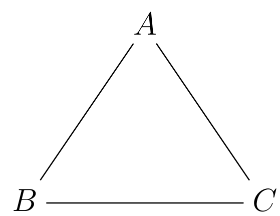
Definition 2.15. *We call right triangle a triangle of type*




*having one of the angles equal to* $90^\circ$*.*

Many things can be said about right triangles, and we will be back to this. As a second important angle now, we have the $60^\circ$ angle, which usually appears via:

Theorem 2.16. *In an equilateral triangle, having all sides equal,*

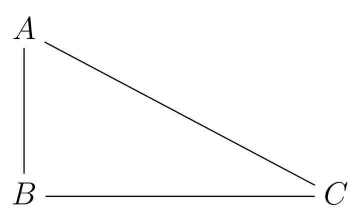



*all angles equal* $60^\circ$*.*

Proof. This is clear indeed from the fact that the sum of angles is $180^\circ$. ☐

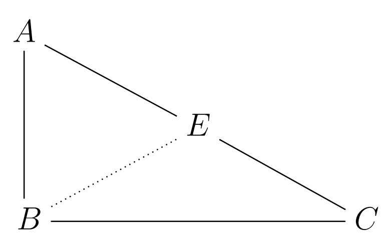
Yet another interesting angle is the $30^\circ$ one. About it, we have:

Theorem 2.17. *In a right triangle having small angles* $30^\circ, 60^\circ$*,*




*we have* $AB = AC/2$*.*

Proof. This is clear indeed by considering the middle $E$ of the side $AC$, which makes appear an equilateral triangle $ABE$, as follows:

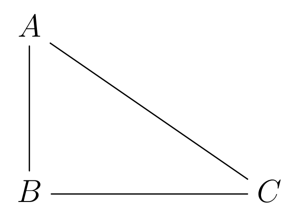



Thus, we are led to the conclusion in the statement. ☐

## 2c. Pythagoras theorem

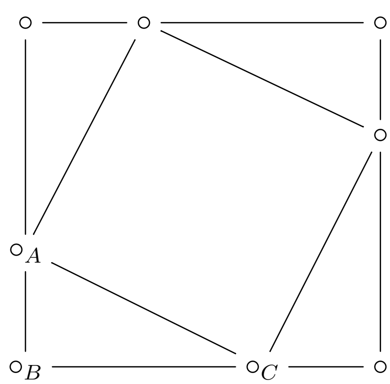
Many other interesting things can be said about the right angle 90°, and about right triangles, in particular with the following key result, due to Pythagoras:

THEOREM 2.18 (Pythagoras). *In a right triangle $ABC$,*




*we have $AB^2 + BC^2 = AC^2$.*

PROOF. This comes indeed from the following picture, consisting of two squares, and four triangles which are identical to our triangle $ABC$, as indicated:




Indeed, let us compute the area $S$ of the outer square. This can be done in two ways. First, since the side of this square is $AB + BC$, we obtain:

$$\begin{aligned} S &= (AB+BC)^2 \\ &= AB^2 + BC^2 + 2 \times AB \times BC \end{aligned}$$

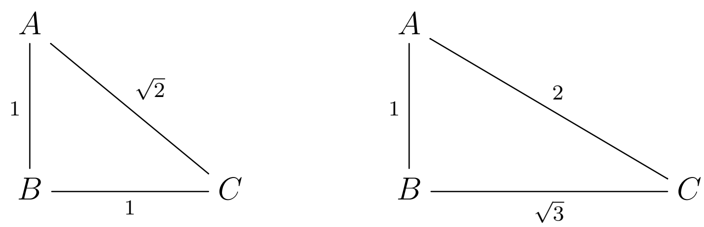
On the other hand, the outer square is made of the smaller square, having side $AC$, and of four identical right triangles, having sizes $AB, BC$. Thus:

$$\begin{aligned} S &= AC^2 + 4 \times \frac{AB \times BC}{2} \\ &= AC^2 + 2 \times AB \times BC \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a basic application of the Pythagoras theorem, we have:

THEOREM 2.19. *The* $45° - 45°$ *and* $30° - 60°$ *right triangles are as follows,*

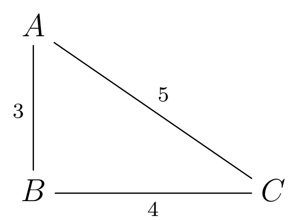



*up to a rescaling of the sides.*

PROOF. These results come indeed from $1 + 1 = 2$, and from $1 + 3 = 4$. □

As another basic application of the Pythagoras theorem, which is something widely useful in practice, and this since the ancient times, we have:

THEOREM 2.20. *A triangle having sides* $3, 4, 5$ *is a right triangle:*




*Thus, for drawing right angles, you only need a loop, with* 12 *knots on it.*

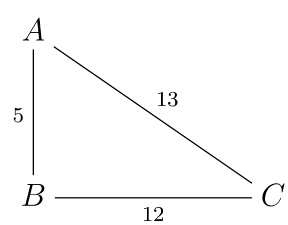
PROOF. This comes indeed from $9 + 16 = 25$, and from the obvious converse of the Pythagoras theorem, and up to you to check the details here. As for the second assertion, and how can that be used in practice, we will leave this as an engineering exercise. □

Still speaking engineering, having 12 knots equally spaced on a loop is certainly possible, and reliable for most tasks, but if we want to improve our tool, it would be desirable to have more knots on out loop. So, here we are looking for integer solutions of:

$$a^2 + b^2 = c^2$$

Which is not exactly obvious, but with a bit of patience, we are led to:

THEOREM 2.21. *A triangle having sides* $5, 12, 13$ *is a right triangle:*




*Thus, for drawing right angles, you only need a loop, with* 30 *knots on it.*

PROOF. Here the first assertion comes from the following equality, and with the comment that this is the simplest possible one, passed $9+16=25$:

$$25+144=169$$

As for the second assertion, we will leave this again as an engineering exercise. As a bonus exercise, try further improving this, say with a solution using 90 knots. $\square$

Along the same lines, at a more advanced level, we have the following result, which fully closes the discussion, regarding the Pythagoras equation over the integers:

THEOREM 2.22. *The Pythagoras equation, namely*

$$a^2+b^2=c^2$$

*can be fully solved over the integers, the solutions being*

$$a=d(m^2-n^2)\quad,\quad b=2dmn\quad,\quad c=d(m^2+n^2)$$

*with $(m,n)=1$, up to exchanging $a,b$.*

PROOF. This is something standard, due to Euclid, the idea being as follows:

(1) Let us try to solve $a^2+b^2=c^2$. If we divide $a,b,c$ by their greatest common divisor $d=(a,b,c)$, the equation is still satisfied. Thus, we can assume $(a,b,c)=1$, and we want to prove that the solutions are as follows, up to exchanging $a,b$:

$$a=m^2-n^2\quad,\quad b=2mn\quad,\quad c=m^2+n^2$$

(2) To start with, in one sense our result is clear, because given any two numbers $m,n$, the above formulae produce a solution to our equation, as shown by:

$$\begin{aligned}(m^2-n^2)^2+(2mn)^2&=m^4+n^4-2m^2n^2+4m^2n^2\\&=m^4+n^4+2m^2n^2\\&=(m^2+n^2)^2\end{aligned}$$

(3) So, we must prove now the converse, stating that if $a,b,c$ satisfying $(a,b,c)=1$ are solutions of $a^2+b^2=c^2$, then we can write them as in (1). For this purpose, the first observation is that, due to $a^2+b^2=c^2$, our assumption $(a,b,c)=1$ implies:

$$(a,b)=(a,c)=(b,c)=1$$

(4) Let us study now the parity of $a,b,c$. Since $(a,b)=1$, one of these two numbers, say $a$, is odd. Now assuming that $b$ is odd too, we would get $a^2+b^2=2(4)$, which is impossible, due to $a^2+b^2=c^2$. Thus $b$ must be even, and as a conclusion to this study, up to exchanging $a,b$, we can assume that the parity of our numbers is as follows:

$$a=\text{odd}\quad,\quad b=\text{even}\quad,\quad c=\text{odd}$$

(5) Now comes the trick. We can rewrite our equation in the following way:

$$\begin{aligned} a^2+b^2=c^2 &\iff b^2=c^2-a^2 \\ &\iff b^2-(c-a)(c+a) \\ &\iff \frac{c+a}{b}=\frac{b}{c-a} \end{aligned}$$

(6) With this done, let us look at the fraction on the left. This is a rational number, so we can write it in reduced form, as follows, with $(m,n)=1$:

$$\frac{c+a}{b}=\frac{m}{n}$$

Now observe that our equation, as reformulated in (5), takes the following form:

$$\frac{c+a}{b}=\frac{m}{n} \quad , \quad \frac{c-a}{b}=\frac{n}{m}$$

Equivalently, our equation, as reformulated in (5), takes the following form:

$$\frac{c}{b}+\frac{a}{b}=\frac{m}{n} \quad , \quad \frac{c}{b}-\frac{a}{b}=\frac{n}{m}$$

But this latter system is equivalent to the following two formulae:

$$\frac{a}{b}=\frac{1}{2}\left(\frac{m}{n}-\frac{m}{n}\right)=\frac{m^2-n^2}{2mn}$$

$$\frac{c}{b}=\frac{1}{2}\left(\frac{m}{n}+\frac{m}{n}\right)=\frac{m^2+n^2}{2mn}$$

(7) Good work that we did, and time to breathe, and see what we have. We have proved so far that if $a,b,c$ satisfying $(a,b,c)=1$ are solutions of $a^2+b^2=c^2$, then up to exchanging $a,b$, we can find numbers $m,n$ satisfying $(m,n)=1$, such that:

$$\frac{a}{b}=\frac{m^2-n^2}{2mn} \quad , \quad \frac{c}{b}=\frac{m^2+n^2}{2mn}$$

Which sounds nice, because due to $(a,b)=(b,c)=1$, as noted in (3), the two fractions on the left are in reduced form. So, if we manage to prove that the two fractions on the right are in reduced form too, this would finish the proof, because we would get:

$$a=m^2-n^2 \quad , \quad b=2mn \quad , \quad c=m^2+n^2$$

(8) So, let us look now at the two fractions on the right, appearing above. As a first observation, due to $(m,n)=1$, the following two fractions are in reduced form:

$$\frac{m^2-n^2}{mn} \quad , \quad \frac{m^2+n^2}{mn}$$

The problem, however, is that the fractions in (7) are the halves of these quantities. So, all we need is a study modulo 2, and with this, normally done.

(9) Getting now to the endgame, from $(m,n)=1$, the case where both $m,n$ are even is excluded. But the case where both $m,n$ are odd is excluded too, due to:

$$\frac{a}{b}=\frac{m^2-n^2}{2mn}$$

Indeed, if $m,n$ were both to be odd, we would have $m^2-n^2=0(4)$ and $2mn=2(4)$, so the fraction on the right, when reduced, would have an even denominator. But this would tell us that $b$ must be even, which contradicts our $b$ odd choice from (4).

(10) Summarizing, one of the numbers $m,n$ must be even, and the other must be odd. But this does the job, because it shows that $m^2-n^2$ and $m^2+n^2$ are both odd, so when dividing the reduced fractions from (7) by 2, these fractions remain still reduced. Thus, as a conclusion to our study, the following two fractions are reduced:

$$\frac{m^2-n^2}{2mn}\quad,\quad\frac{m^2+n^2}{2mn}$$

(11) So, theorem proved. Indeed, as indicated in (7), let us look now at:

$$\frac{a}{b}=\frac{m^2-n^2}{2mn}\quad,\quad\frac{c}{b}=\frac{m^2+n^2}{2mn}$$

Since all fractions appearing here are in reduced form, we obtain from this:

$$a=m^2-n^2\quad,\quad b=2mn\quad,\quad c=m^2+n^2$$

And finally, as indicated in (1), by multiplying $a,b,c$ by an arbitrary number $d$, we obtain the general solutions from the statement, namely:

$$a=d(m^2-n^2)\quad,\quad b=2dmn\quad,\quad c=d(m^2+n^2)$$

(12) At the level of the interesting examples now, there are of course many of them, and we have for instance a solution as follows:

$$9^2+40^2=1681=41^2$$

Thus, and good news here, we have solved as well a quite difficult exercise left, the one at the end of the proof of Theorem 2.21. □

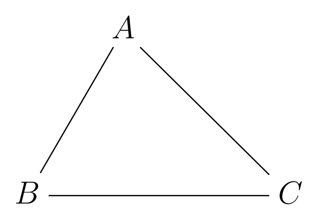
## 2d. Triangle centers

Now that we know about angles, let us go back to triangles. We have the following result, making appear 3 more centers of our triangle, which all have their own importance and interest, and which are in general different from the barycenter:

THEOREM 2.23. *Given a triangle $ABC$, the following happen:*

(1) *The medians cross, at the barycenter $G$.*
(2) *The angle bisectors cross, at the incenter $I$.*
(3) *The perpendicular bisectors cross, at the circumcenter $O$.*
(4) *The altitudes cross, at the orthocenter $H$.*

PROOF. Let us first draw our triangle, with this being always the first thing to be done in geometry, draw a picture, and then thinking and computations afterwards:



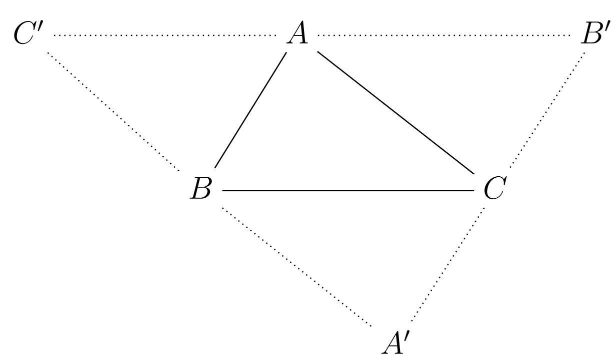

Allowing us the freedom to play with some tricks, as advanced mathematicians, both students and professors, are allowed to, here is how the proof goes:

(1) This is something that we know well, from Theorem 2.1.

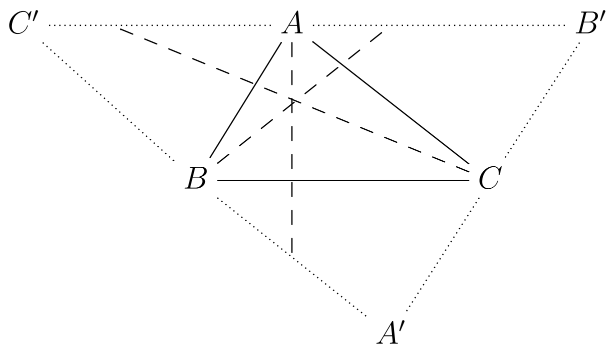
(2) Come with a small circle, inside $ABC$, and then inflate it, as to touch all 3 edges. The center of the circle will be then at equal distance from all 3 edges, so it will lie on all 3 angle bisectors. Thus, we have constructed the incenter, as required.

(3) We can use the same method as for (2). Indeed, come with a big circle, containing $ABC$, and then deflate it, as for it to pass through $A, B, C$. The center of the circle will be then at equal distance from all 3 vertices, so it will lie on all 3 perpendicular bisectors. Thus, we have constructed the circumcenter, as required.

(4) This is something tougher, and I must admit that, when writing this book, I first struggled a bit with this, then ended looking it up on the internet. So, here is the trick. Draw a parallel to $BC$ at $A$, and similarly, parallels to $AB$ and $AC$ at $C$ and $B$. You will get in this way a bigger triangle, upside-down, $A'B'C'$, as follows:



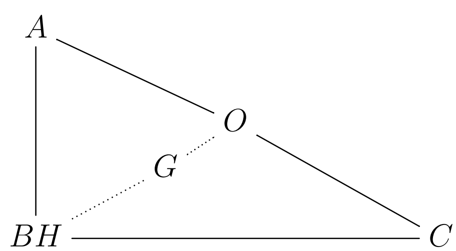
But then, the circumcenter of this bigger triangle $A'B'C'$, that we know to exist from (3), will be the orthocenter of $ABC$, as shown by the following picture:




Thus, we are led to the conclusions in the statement. □

As an illustration, let us work out the case of the right triangles. Here we can say more about the various triangle centers, the result being as follows:

THEOREM 2.24. *The various centers $G, I, O, H$ of a right triangle $ABC$ are subject to the fact that we have $H = B$, and that $O$ is the middle of $AC$,*




*and based on this, we can conclude that the following happen:*

(1) *$O, G, H$ are collinear, with $GH = 2GO$.*
(2) *$I$ does not generally lie on the line $OGH$.*

PROOF. This is indeed something quite self-explanatory, as follows:

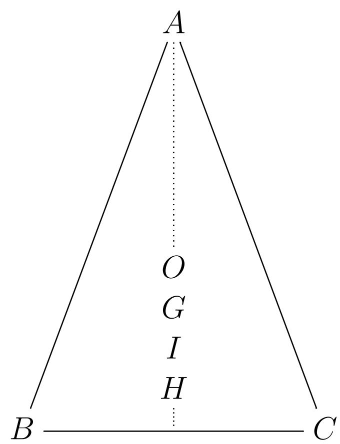
(1) To start with, we have indeed $H = B$, as being the intersection of the altitudes drawn from $A, C$. Also, the middle of $AC$ being at equal distance from $A, B, C$, as we can see by completing $ABC$ into a rectangle, it is the circumcenter $O$.

(2) Regarding now the barycenter $G$, we know from Theorem 2.1 that this lies on the median emanating from $B$, which is the segment $HO$, at $1/3 - 2/3$. Thus, by putting everything together, we have proved the assertion (1) in the statement.

(3) Finally, regarding the incenter $I$, this must lie on the angle bisector drawn from $B$, and unless our right triangle $ABC$ is isosceles, this angle bisector will certainly not coincide with the median $BO$. Thus, we have proved as well the assertion (2). □

As another illustration, let us work out as well the case of the isosceles triangles. Again, here we can say more about the various triangle centers, and we have:

THEOREM 2.25. *In the case of an isosceles triangle, the barycenter $G$, incenter $I$, circumcenter $O$ and orthocenter $H$ all lie on the main median, the picture being*

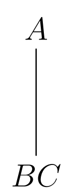



*when the angle at $A$ is small, and with the order of these points being reversed, namely $H, I, G, O$, when the angle at $A$ is big. In all cases, we can conclude that:*

(1) *$O, G, H$ are collinear, with $GH = 2GO$.*
(2) *$I$ lies also on the line $OGH$, but at various positions.*

PROOF. There are several things going on here, the idea being as follows:

(1) Let us start our study with the case of a degenerate isosceles triangle, having null angle at $A$. The picture of this triangle is something very simple, as follows:




Now let us compute the various centers. Regarding the barycenter $G$, this lies at $1/3 - 2/3$ on the main median, no question about this, so we have:

$$AG = 2GB$$

Regarding the incenter $I$, this is the center of the inner circle, which gets squeezed to a null circle as the angle at $A$ gets squeezed to 0, so we have:

$$I = B = C$$

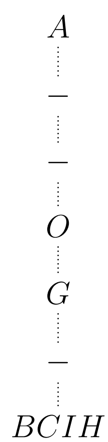
Regarding now the circumcenter $O$, this is the center of the outer circle, passing through $A$ and $B = C$, so this point must be the middle of the main median:

$$AO = OB$$

Finally, regarding the orthocenter $H$, which appears as the intersection of the 3 altitudes, this gets squeezed to $B = C$, when the angle at $A$ gets squeezed to 0:

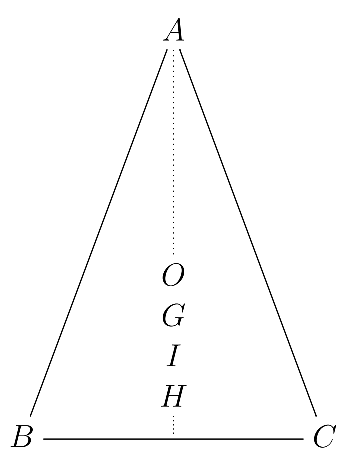
$$H = B = C$$

Summarizing, we have our points, and by putting together all the above, the precise picture is as follows, with the main median being divided here into 6 equal parts:




In particular, as a consequence of this, we have indeed $GH = 2GO$, as stated.

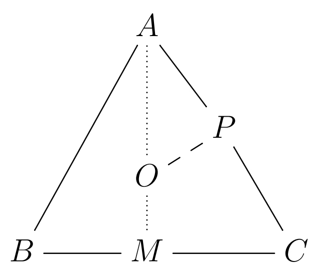
(2) Next, let us see what happens when the angle at $A$ is small. Here $G, I, O, H$ are respectively at the intersection of the main median with the median at $B$, angle bisector at $B$, perpendicular bisector opposed to $B$, and altitude at $B$, and a quick study on how these 4 latter lines are positioned leads to the picture in the statement, namely:



In order to prove now $GH = 2GO$, we must do some computations. And for this purpose, the best is to consider the middle $M$ of the side $BC$, and think of our triangle $ABC$ as appearing from the right triangle $AMC$, by symmetrizing. Indeed, when doing so, we have right away a simple formula for $AG$, in terms of $AMC$, namely:

$$AG = \frac{2}{3} \cdot AM$$

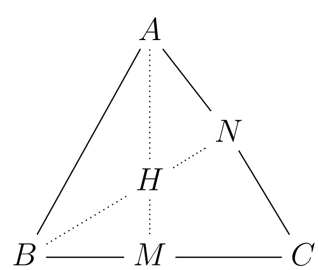
Regarding now the computation of $AO$, consider the middle $P$ of the side $AC$, and the corresponding perpendicular bisector, passing through the circumcenter $O$:




We have then the following computation, using similar triangles:

$$\frac{AO}{AP} = \frac{AC}{AM} \implies AO = \frac{1}{2} \cdot \frac{AC^2}{AM}$$

Finally, regarding the computation of $AH$, let us draw the altitude from $B$:




Again we have some similar triangles appearing, which leads to:

$$\frac{AH}{AN} = \frac{AC}{AM} \quad , \quad \frac{NC}{BC} = \frac{MC}{AC}$$

Thus, we can compute $AH$ in terms of the sides of $AMC$, as follows:

$$\begin{aligned} AH &= AN \cdot \frac{AC}{AM} \\ &= (AC - NC)\frac{AC}{AM} \\ &= \left(AC - \frac{BC \cdot MC}{AC}\right)\frac{AC}{AM} \\ &= \frac{AC^2 - 2MC^2}{AM} \end{aligned}$$

Summarizing, we have established the following formulae:

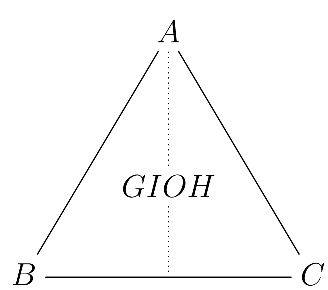
$$AG = \frac{2}{3} \cdot AM \quad , \quad AO = \frac{1}{2} \cdot \frac{AC^2}{AM} \quad , \quad AH = \frac{AC^2 - 2MC^2}{AM}$$

Now observe that we have the following equality, coming from Pythagoras:

$$\begin{aligned} 3AG &= \frac{2AM^2}{AM} \\ &= \frac{2(AC^2 - MC^2)}{AM} \\ &= 2AO + AH \end{aligned}$$

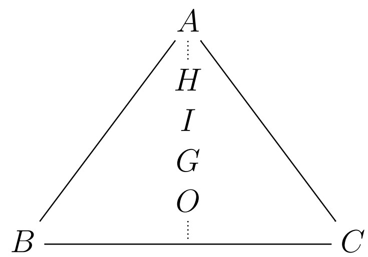
Thus $AH - AG = 2(AG - AO)$, and so $GH = 2GO$, as claimed. Finally, in what regards the incenter $I$, nothing much can be said about it. We will leave some exploration here as an exercise, and come back to it later, in chapter 3, with some formulae.

(3) Next, when further enlarging the angle at $A$, we will meet a new configuration when the triangle $ABC$ gets equilateral, that is, with all its sides equal. In this case $G, I, O, H$ all coincide with the obvious center of the triangle, the picture being as follows:



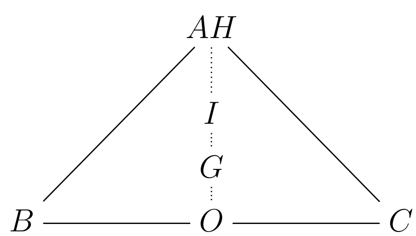

Observe that in this case we still have $GH = 2GO$, as stated, coming as $0 = 0$.

(4) Next, when further enlarging the angle at $A$, while still keeping it smaller than a right angle, a similar study to the one performed in (2) leads to the following picture:



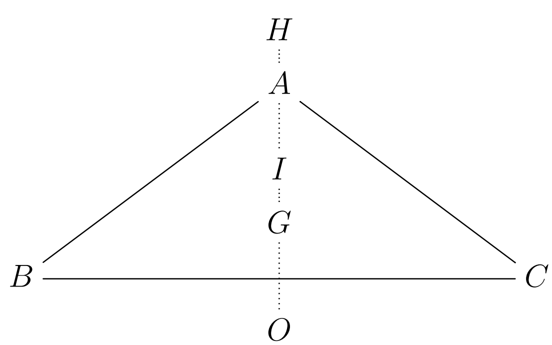

Also as before in (2), the formula $GH = 2GO$ still holds, and in what regards $I$, nothing simple can be said about it, and we will leave all this as an exercise.

(5) Getting now to the case where $A$ becomes a right angle, here the orthocenter becomes $H = A$, and the circumcenter $O$ becomes the middle of $BC$, as follows:



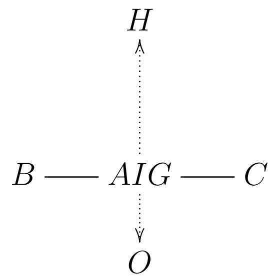

Observe that the formula $GH = 2GO$ still holds in this case, trivially.

(6) Next, when the angle at $A$ becomes obtuse, what happens is that $H$ escapes from the triangle, and the same happens for $O$, with the picture being as follows:




As before in (2) and (4), the formula $GH = 2GO$ still holds, and in what regards $I$, nothing simple can be said about it, and we will leave all this as an exercise.

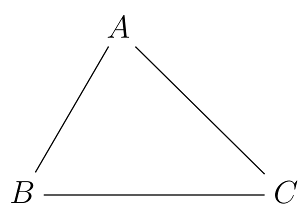
(7) Finally, in the case where the triangle $ABC$ becomes degenerate, with a flat angle at $A$, the picture becomes as follows, with both $H, O$ being sent to the infinity:




To be more precise here, in what regards the vertical alignment, in view of the formula $GH = 2GO$ from (6), we can say that $H$ is sent to infinity twice as fast as $O$, so we are led to the above picture, making it clear that we have $GH = 2GO$ too, in this case. □

The above study was quite interesting, and based on what we have in Theorem 2.24 and Theorem 2.25, we can now formulate the following conjecture:

CONJECTURE 2.26. *For an arbitrary triangle $ABC$,*

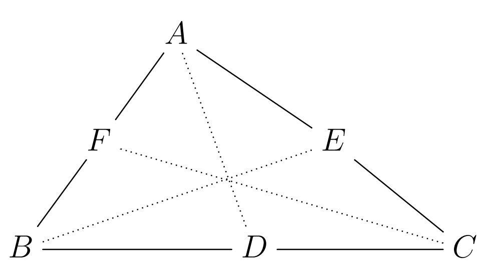



*the points $O, G, H$ are collinear, with $GH = 2GO$.*

We will see in the next chapter that this is something which holds indeed. And also, we will be back to the incenter $I$ too, with some formulae regarding it.

As another question now, also coming from our study, we have:

QUESTION 2.27. *Can we have some general theory going, for the various centers of a triangle, notably with results stating that when drawing lines of type $AD, BE, CF$,*




*these lines cross indeed? Also, what about the various centers of a triangle, that we can obtain in this way, what are the exact relations between them?*

These are all interesting questions, and we will answer them too, in due time.

## 2e. Exercises

This was a quite standard chapter on plane geometry, only fundamental things, which are all good to know, in the real life, and as exercises on this, we have:

EXERCISE 2.28. *Study some more the alternative barycenter of triangles, the* $1D$ *one, constructed by using triangles with weighted edges.*

EXERCISE 2.29. *Clarify what we said in the above about the barycenter, and its computation, in relation with discretization methods.*

EXERCISE 2.30. *Try finding an alternative proof, without any kind of tricks, for the existence of the orthocenter.*

EXERCISE 2.31. *Learn a bit about the story of mankind, and of human numbers, about basis* $10$*, and about* $360^\circ$ *too.*

EXERCISE 2.32. *Look up and learn some other proofs, most likely more complicated, of the Pythagoras theorem.*

EXERCISE 2.33. *Review if needed the arithmetic ingredients used by Euclid for solving the Pythagoras equation, over the integers.*

EXERCISE 2.34. *Find some other interesting solutions of the Pythagoras equation over the integers. Also, learn a bit about the Fermat equation, too.*

EXERCISE 2.35. *Verify that what we said above regarding the centers of isosceles triangles, with* $GH = 2GO$*, holds no matter how acute or obtuse the angle at* $A$ *is.*

As bonus exercise, meditate a bit about angles, and about the best way of assigning them numeric values. We will be back to this, but only later in this book.

CHAPTER 3

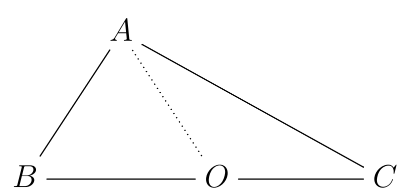
# Triangle centers

## 3a. Euler circle, line

Getting now to more advanced plane geometry, still in relation with the triangles, we know from chapter 2 that associated to any triangle $ABC$ are two remarkable circles, namely the inner one, centered at the incenter $I$, and the outer one, centered at the circumcenter $O$. But, which of these two circles is the most important?

In answer, the point is that we have as well a third circle, called nine-point circle, which for many purposes is actually the most important one. In order to discuss this, we will need to know more about circles, notably with the following basic fact:

THEOREM 3.1. *Any triangle lying on a circle, with two vertices on a diameter,*




*is a right triangle.*

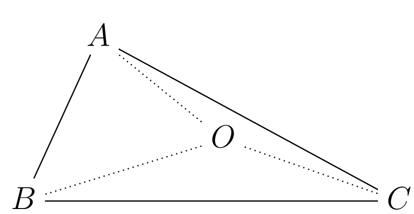
PROOF. This is clear, because we have two isosceles triangles appearing, so at the level of the corresponding angles, the 180° equation for our triangle is as follows:

$$b + (b + c) + c = 180^\circ$$

Thus, we obtain $b + c = 90^\circ$, so the angle at $A$ is indeed $90^\circ$, as claimed. □

More generally now, we have the following result, which is very useful too:

THEOREM 3.2. *Given a triangle $ABC$ lying on a circle,*




*the angle at $A$ does not depend on the position of $A$, and equals half the angle $BOC$.*

PROOF. This follows a bit as before. Indeed, the angles of our triangle $ABC$ are as follows, with $p, q, r$ being the smaller angles on the picture, from left to right:

$$a = p + q \quad , \quad b = p + r \quad , \quad c = r + q$$

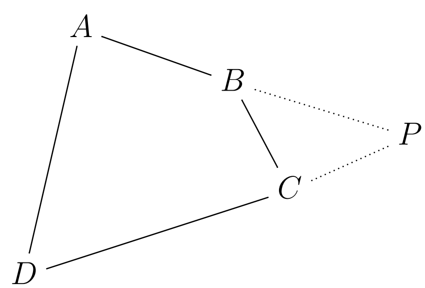
Now let us look at the angle $BOC$. This is given by the following formula:

$$\begin{aligned} BOC &= 180° - 2r \\ &= (a + b + c) - 2r \\ &= (2p + 2q + 2r) - 2r \\ &= 2p + 2q \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a third basic result about circles, which is good to know too, we have:

THEOREM 3.3. *Given a configuration as follows, with $A, B, C, D$ on a circle,*




*we have $PAD \sim PCB$. In particular we have the formula*

$$PA \cdot PB = PC \cdot PD$$

*and this quantity is called the power of $P$ with respect to the circle in question.*

PROOF. There are several things going on here, the idea being as follows:

(1) To start with, by using Theorem 3.2 we can see that the angles $A, B, C, D$ of the quadrilateral $ABCD$ are related by $A + C = 180°$ and $B + D = 180°$. But this shows that the triangles $PAD$, $PCB$ have the same angles, so they are similar, as stated.

(2) Next, from the similarity $PAD \sim PCB$ we obtain the following formula:

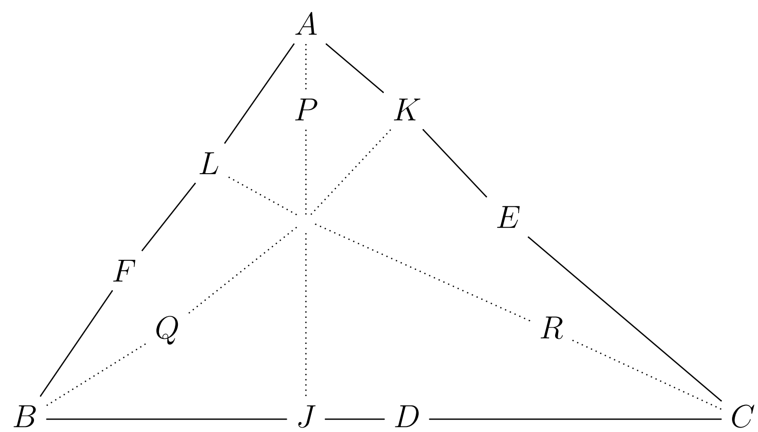
$$\frac{PA}{PD} = \frac{PC}{PB}$$

But this can be written in the form in the statement, $PA \cdot PB = PC \cdot PD$.

(3) Finally, we can forget about $A, B, C, D$, and conclude that given a point $P$ outside a circle, we can talk about the quantity $PA \cdot PB$, which is independent of the secant $PAB$ used. And with this being called power of $P$ with respect to the circle. □

Getting now to what we wanted to do, nine-point circle, let us start with the following key result, due to Euler and others, establishing the existence of this circle:

THEOREM 3.4. *Associated to any triangle $ABC$ we have a nine-point circle,*



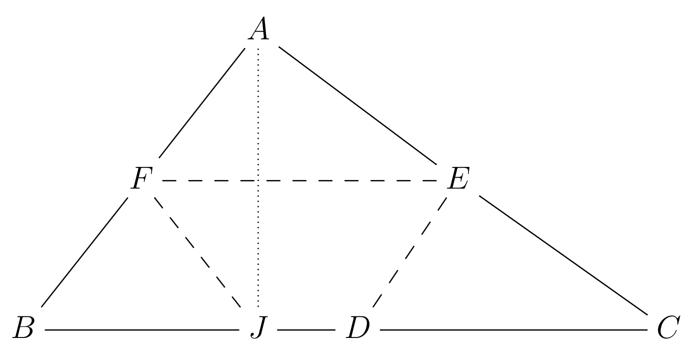

*passing through the following points, pictured above:*

(1) *The midpoints $D, E, F$ of each side.*
(2) *The feet $J, K, L$ of each altitude.*
(3) *The midpoints $P, Q, R$ of each segment vertex - orthocenter.*

PROOF. This is something quite tricky, the idea being as follows:

(1) Consider the circle passing through $D, E, F$. We will prove in what follows that this circle passes through $J$, and by symmetry, this circle will have to pass through $K, L$ too. Then we will prove that this circle passes through $P$, and by symmetry, this circle will have to pass through $Q, R$ too. And so, we will have our nine-point circle.

(2) So, let us first prove that the points $D, E, F, J$ are cocyclic. The simplified picture here, with the triangle $ABC$ and these points $D, E, F, J$, is as follows:




Now let us look at the trapezoid $DEFJ$, highlighted above. As a first observation, by definition of the points $E, F$, as being side midpoints, we have:

$$EF||JD$$

Now let us compute the sides $DE, FJ$ of this trapezoid. In what regards $DE$, this is an easy task, because by definition of the points $D, E$, we have:

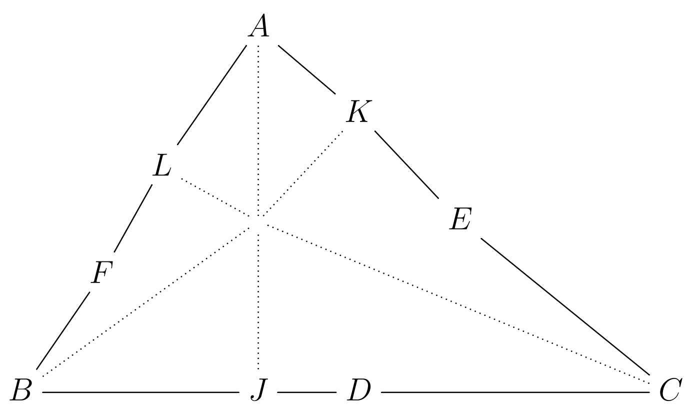
$$DE = \frac{AB}{2}$$

Regarding now $FJ$, observe that $ABJ$ is a right triangle, having $FJ$ as median. We conclude that the length of this median $FJ$ is half of the corresponding side:

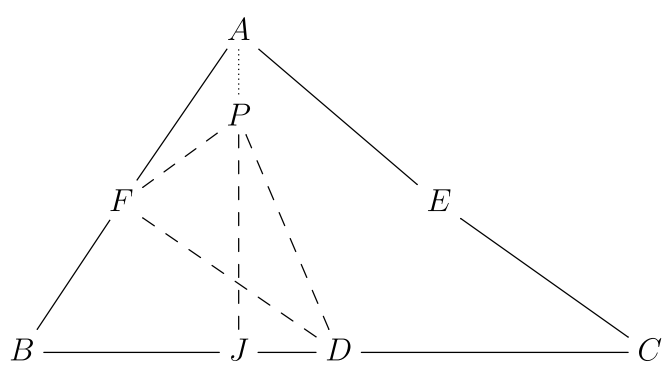
$$FJ = \frac{AB}{2}$$

Now by looking at what we have in the above, we conclude that $DEFJ$ is an isosceles trapezoid. But such an isosceles trapezoid must obviously lie on a circle, by obvious symmetry reasons, so its vertices $D, E, F, J$ are indeed cocyclic, as claimed.

(3) Next, by using the same argument, the circle through $D, E, F$ must pass through $K, L$ too. Thus, we have our six-point circle, passing through the following points:




(4) Now let us prove that this six-point circle passes through $P$. The simplified picture here, with the triangle $ABC$ and with the points $D, E, F, P$, is as follows:




For this purpose, the trick is to look at the triangles $PJD$, $PFD$, highlighted above. Indeed, regarding these triangles, we have the following observations:

– In what regards $PJD$, this is by definition a right triangle. We conclude that $J$ lies on the circle centered at the middle of $PD$, and having radius $PD/2$, and with this latter circle being of course, by uniqueness, the six-point circle constructed in (3).

– In what regards now $PFD$, our claim is that this is a right triangle too. Indeed, by definition of the points $F, D$, as being the midpoints of $AB, BC$, we have:

$$FD||AC$$

On the other hand, since the point $P$ is by definition the middle of the segment $AH$, with $H$ being the orthocenter of the triangle $ABC$, we have:

$$FP||BH$$

But since $BH \perp AC$, by definition of the orthocenter $H$, this shows that we have:

$$FP \perp AC$$

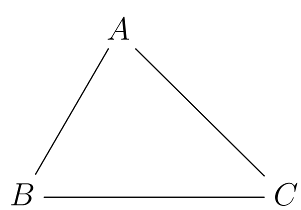
Thus $FD \perp FP$, so our triangle $PFD$ is indeed a right triangle, as claimed.

– But with this, we can finish, because it follows that $P$ lies as well on the circle found before, the one centered at the middle of $PD$, and having radius $PD/2$. Thus, the points $P, J, F, D$ are cocyclic, and so $P$ lies on the six-point circle constructed in (3).

(5) Finally, a similar argument shows that the remaining two points $Q, R$ lie on this six-point circle too. Thus, we have our nine-point circle, as desired. □

As a second result now regarding the nine-point circle, we have:

Theorem 3.5. *The center $N$ of the nine-point circle associated to a triangle $ABC$,*

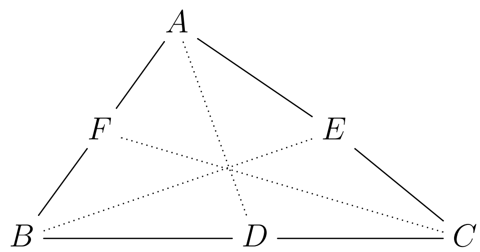



*is on the line $OG$, positioned as follows, with $GO = 2NG$,*

$$N \text{——} G \text{——} \times \text{——} O$$

*and its radius is half of the radius of the circumscribed circle.*

Proof. In order to prove this, consider the configuration producing the barycenter of our triangle $ABC$, that we know well since the beginning of chapter 2, namely:




We can see that the triangles $ABC$ and $DEF$ are similar, and with this similarity coming from the homothety centered at the barycenter $G$ of our triangle $ABC$, of ratio $-1/2$. In other words, we know that this latter homothety has the following property:

$$ABC \to DEF$$

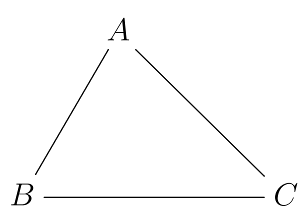
Now by looking at the corresponding circumscribed circles, these must be in correspondence too, via our homothety. In particular, the centers of these circumscribed circles, which are respectively $O$ and $N$, must be positioned as follows, with $GO = 2NG$:

$$N \text{ —— } G \text{ —— } \times \text{ —— } O$$

Thus, we proved the first assertion. As for the radii of these two circles, these must be related as well by our ratio $-1/2$ homothety, which in practice means that the radius of the nine-point circle is half of the radius of the circumscribed circle, as stated. □

As a continuation of the above, we have the following key result:

Theorem 3.6. *Associated to any triangle $ABC$, not equilateral,*




*is its Euler line, notably passing through $H, N, G, O$, with the proportions being*

$$H - \times - \times - N - G - \times - O$$

*that is, with $GH = 2GO$, and with $N$ being midway between $H$ and $O$.*

Proof. This is something which fine-tunes and generalizes a number of things that we knew from before, the idea with this being as follows:

(1) To start with, this is something that we discussed in chapter 2 for the right triangles and isosceles triangles, where the Euler line is the main median, save for the details of $NH = NO$, which will follow from the discussion below, regarding the general case.

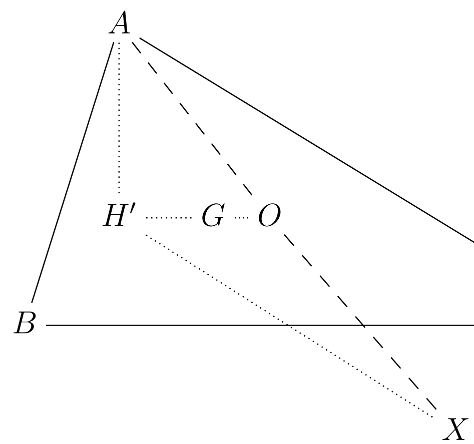
(2) In order to discuss now the general case, let us go back to the proof of Theorem 3.5, and to the homothety used there, centered at the barycenter $G$, and of ratio $-1/2$. Let us denote by $H'$ the preimage of the circumcenter $O$ under this homothety:

$$H' —— \times —— G —— O$$

In view of what we already know from Theorem 3.5, it remains to prove that:

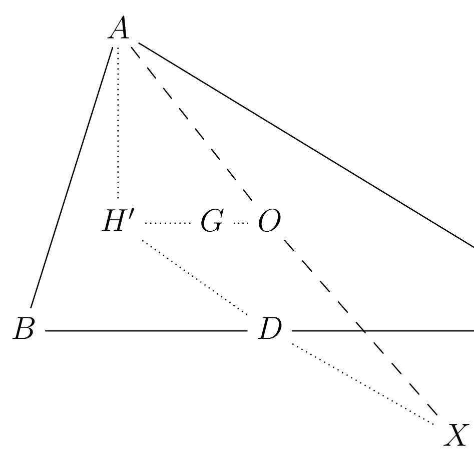
$$H' = H$$

(3) So, let us prove this latter equality. In order to do so, the trick is to consider the point $X$, symmetric of $A$ with respect to $O$, with the picture being as follows:




(4) Now let us look at the triangle $AH'X$, hightlighted above. As a first observation, its barycenter lies on its median $H'O$ at $1/3 - 2/3$, so this barycenter is $G$.

(5) Now with this is hand, we conclude that $AG$ is a median of this triangle $AH'X$ too. But since the barycenter $G$ of the triangle $AH'X$ must be located $1/3 - 2/3$ on this latter median, we conclude that the side midpoint $D$ is the middle of $H'X$:



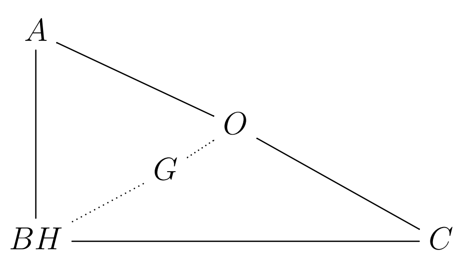
(6) But with this, we can finish. Indeed, it follows that we have $OD||AH'$, and since we have $OD \perp BC$, we conclude that we have $AH' \perp BC$. Similarly, we have $BH' \perp AC$ and $CH' \perp AB$, so the point $H'$ is indeed the orthocenter, $H' = H$, as desired. □

Quite interesting all this, with as a philosophical conclusion, any triangle $ABC$ not coming exactly alone, but rather accompanied by an extra line, and by a circle too. Thus, and anticipating here a bit, what we have is a configuration of total degree 6. And we will see later in this book other magical occurrences of degree 6 configurations.

## 3b. Special triangles

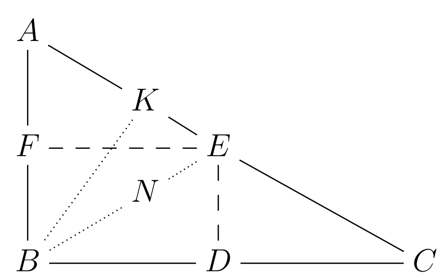
The results established above are quite general, valid for any triangle, and time now to see what happens in practice, for various particular triangles, such as the right triangles, or the isosceles triangles. So, we will do such a study, and then, based on what we find, we will formulate and prove some further results, in the general case.

Let us start with the right triangles. We have already talked about them in chapter 2, with the conclusion that in what regards $O, G, H$ the situation is quite trivial, with their collinearity, and the formula $GH = 2GO$, coming from the following configuration:




Thus, we have the Euler line, coming trivially in this case, as being the main median of our right triangle. It remains now to update our discussion, by talking about the nine-point circle and its center $N$, and the situation here is trivial too, as follows:

THEOREM 3.7. *For a right triangle the nine-point circle is a five-point circle,*




*whose existence is trivial. As for the Euler line, this is the main median, trivially too.*

Proof. There are several things going on here, the idea being as follows:

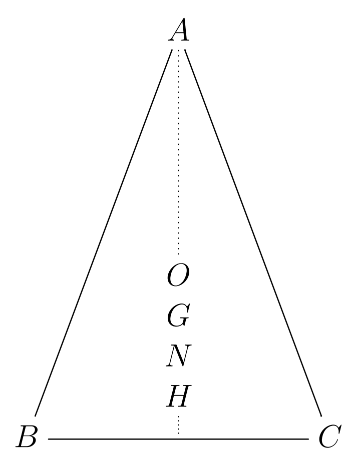
(1) To start with, in our case, the 9 points of the nine-point circle, as considered in Theorem 3.4, are the side midpoints $D, E, F$, the altitude feet $B, K, B$, and the midpoints of the vertex-orthocenter segments $F, B, D$. Thus, all in all, we have in fact only 5 points, namely the vertex $B$, the side midpoints $D, E, F$, and the altitude foot $K$.

(2) In order to prove now that these points are cocyclic, consider the middle $N$ of the segment $BE$. Since the figure $BDEF$ is a rectangle, having center $N$, the points $B, D, E, F$ are indeed cocyclic, around $N$. As for the remaining point $K$, since $BKE$ is a right triangle with median $KN$, we have $KN = NB = NE$, so done with this too.

(3) Summarizing, we have proved that for a right triangle the nine-point circle exists, as a five-point circle, for trivial reasons, and with its center and radius being, again for trivial reasons, those coming from the general theory, exactly as desired. □

Getting now to the isosceles triangles, we know from chapter 2 that things will be less trivial, because we have struggled a bit there, in order to establish the formula $GH = 2GO$. Now by adding the nine-point circle to the discussion there, we have the following result, which is something quite modest, and is formulated a bit informally too:

Theorem 3.8. *For an isosceles triangle the nine-point circle is an eight-point circle, and the Euler line is trivially the main median, as follows,*




*and the fact that the eight-point circle exists indeed, and that we have, on the Euler line,*

$$H - \times - \times - N - G - \times - O$$

*can be proved as well, with some patience, by using similar triangles and Pythagoras.*

Proof. There are several things going on here, the idea being as follows:

(1) To start with, the Euler line is trivially the main median.

(2) Next, as explained in chapter 2, in order to do the computations, the best is to consider the middle point $M$ of the side $BC$, and to think of our original isosceles triangle $ABC$ as appearing from the right triangle $AMC$, by reflection. That is, our main object becomes $AMC$, and we will do the computations in terms of the side lengths of $AMC$.

(3) Still reviewing the material from chapter 2, let us recall from there that, by using the above philosophy, and similar triangles, we are led to the following formulae:

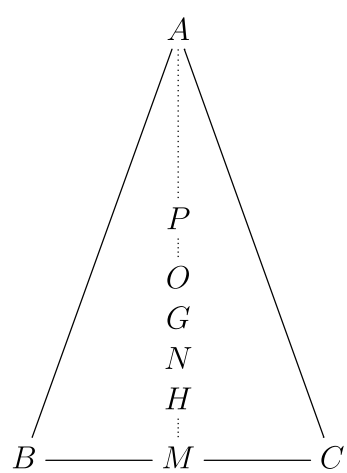
$$AG = \frac{2}{3} \cdot AM \quad , \quad AO = \frac{1}{2} \cdot \frac{AC^2}{AM} \quad , \quad AH = \frac{AC^2 - 2MC^2}{AM}$$

As a consequence, we have the following equality, coming from Pythagoras:

$$\begin{aligned} 3AG &= \frac{2AM^2}{AM} \\ &= \frac{2(AC^2 - MC^2)}{AM} \\ &= 2AO + AH \end{aligned}$$

Thus $AH - AG = 2(AG - AO)$, and so $GH = 2GO$, as we know since chapter 2.

(4) Getting now to new computations, involving the nine-point circle, which in our case is rather an eight-point circle, and its center $N$, there are many things to be done here. So, let us start by showing that everything is fine on the Euler line. If we denote as usual by $P$ the middle of the segment $AH$, the picture of our triangle becomes:




To be more precise, the points $A, P, O, G, H, M$ are as before, and we have added to these the middle $N$ of the segment $PM$, which must be the center of the nine-point circle under construction, due to the fact that this nine-point circle must pass through $P, M$.

(5) So, what is to be proved? Well, as mentioned above, we first want to make sure that things fine on the Euler line, and with $GH = 2GO$ already established in (3), we are

left with proving $ON = NH$. So, let us prove this. We have the following computation, using the various formulae from (3), along with Pythagoras, at the end:

$$\begin{aligned} AN &= \frac{AP+AM}{2} \\ &= \frac{AH}{4}+\frac{AM}{2} \\ &= \frac{AC^2-2MC^2}{4AM}+\frac{AM}{2} \\ &= \frac{AC^2-2MC^2+2AM^2}{4AM} \\ &= \frac{AC^2-2MC^2+2(AC^2-MC^2)}{4AM} \\ &= \frac{3AC^2-4MC^2}{4AM} \end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned} \frac{AO+AH}{2} &= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{AC^2}{AM}+\frac{AC^2-2MC^2}{AM}\right) \\ &= \frac{AC^2+2(AC^2-2MC^2)}{4AM} \\ &= \frac{3AC^2-4MC^2}{4AM} \end{aligned}$$

Thus we have $AN = (AO + AH)/2$, so we obtain $ON = NH$, as desired.

(6) Next, now that we have the center $N$ of our nine-point circle in construction, let us look at its radius. Since this circle must pass through $P$, its radius must be:

$$\begin{aligned} \rho &= AN - AP \\ &= AN - \frac{AH}{2} \\ &= \frac{3AC^2-4MC^2}{4AM} - \frac{AC^2-2MC^2}{2AM} \\ &= \frac{AC^2}{4AM} \\ &= \frac{AO}{2} \end{aligned}$$

Thus, the radius is half of the radius of the circumscribed circle, as desired.

(7) In the order to finish now, we still must establish the existence of the nine-point circle, which in our case is an eight-point circle. But for this purpose, by symmetry, it

is enough to consider the point $N$ constructed above, and prove that its distance to the points $K,E,R$, appearing respectively as the foot of the altitude from $B$, the middle of $AC$, and the middle of $CH$, all equal the number $\rho$ computed above.

(8) Which is not exactly trivial, and we will leave having some fun with all this as an instructive exercise, and with the remark of course that we already know that this happens indeed, as a consequence of Theorems 3.4, 3.5, 3.6. □

## 3c. Heron, Brahmagupta

Moving on, let us talk now about the forgotten triangle center, which is the incenter $I$. As a first question, we must compute the radius of the incircle, and we have here:

THEOREM 3.9. *Given a right triangle $ABC$, the radius of the incircle is:*

$$r=\frac{AB+BC-AC}{2}$$

*For an isosceles triangle, appearing from a right triangle $AMC$ by reflection, we have:*

$$r=\frac{MC(AC-MC)}{AM}$$

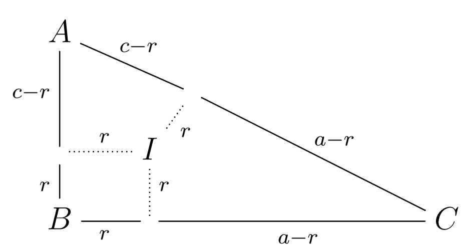
*In the general case, that of an arbitrary triangle, we have the following formula:*

$$r=\frac{2\times area(ABC)}{AB+BC+AC}$$

*Moreover, the distances $AI,BI,CI$ can be computed as well, by using this.*

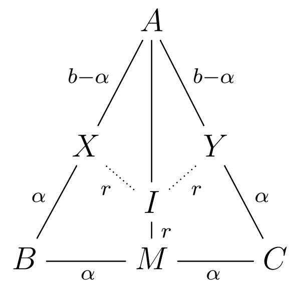
PROOF. There are several things going on here, the idea being as follows:

(1) For a right triangle with sides $a,b,c$, we have a simple configuration, namely:




By looking at $AC$ we have $b=a+c-2r$, and so $r=(a+c-b)/2$, as claimed.

(2) For an isosceles triangle with sides $2\alpha, b, b$, the configuration is as follows:

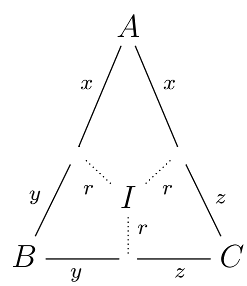



Now observe that by using similar triangles we have the following formula:

$$\frac{IY}{MC} = \frac{AY}{AM}$$

But this formula for the length $r = IY$, together with input coming from the above picture, namely $AY = AC - MC$, gives the formula for $r$ in the statement.

(3) In the case of an arbitary triangle now, the configuration is as follows:




If we denote by $a, b, c$ the sides of our triangle $ABC$, the area is then given by the following formula, which gives the formula for $r$ in the statement:

$$area(ABC) = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2}$$

(4) Next, in order to compute $AI, BI, CI$, we can use Pythagoras, which gives:

$$AI = \sqrt{x^2 + r^2} \quad , \quad BI = \sqrt{y^2 + r^2} \quad , \quad CI = \sqrt{z^2 + r^2}$$

In order to compute now the lengths $x, y, z$, observe that we have:

$$x + y = c \quad , \quad x + z = b \quad , \quad y + z = a$$

Solving this system gives the following formulae for the lengths $x, y, z$:

$$x = \frac{b + c - a}{2} \quad , \quad y = \frac{a + c - b}{2} \quad , \quad z = \frac{a + b - c}{2}$$

Thus, we can compute indeed the lengths $AI, BI, CI$, using our formula for $r$. □

At a more advanced level now, we have the following remarkable result:

**Theorem 3.10** (Heron). *The area of a triangle having sides $a,b,c$ is given by*

$$S=\sqrt{s(s-a)(s-b)(s-c)}$$

*with $s=(a+b+c)/2$ being the semiperimeter. As a consequence, we have*

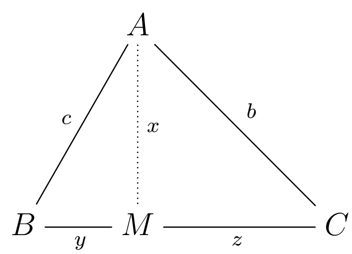
$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

*and the distances $AI,BI,CI$ can be computed as well, by using this.*

Proof. To start with, the Heron product $P$ in the statement is given by:

$$\begin{aligned}16P^2&=16s(s-a)(s-b)(s-c)\\&=(a+b+c)(b+c-a)(a+c-b)(a+b-c)\\&=[(b+c)^2-a^2]\cdot[a^2-(b-c)^2]\\&=(b^2+c^2-a^2+2bc)(a^2-b^2-c^2+2bc)\\&=4b^2c^2-(b^2+c^2-a^2)^2\end{aligned}$$

In order to further process this quantity, we can think of our triangle $ABC$ as appearing by gluing two right triangles, according to the following picture:



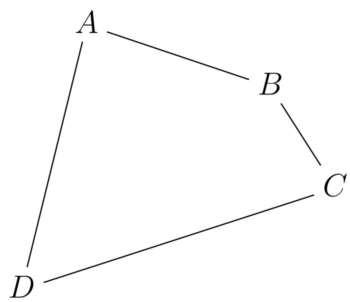

Indeed, we can express our product in terms of $x,y,z$, and we obtain, as desired:

$$\begin{aligned}16P^2&=4b^2c^2-(b^2+c^2-a^2)^2\\&=4(x^2+y^2)(x^2+z^2)-(2x^2-2yz)^2\\&=4(x^2y^2+x^2z^2+2x^2yz)\\&=4x^2(y+z)^2\\&=4(ax)^2\\&=16S^2\end{aligned}$$

Finally, the formula for $r$ follows from this, and from the formula $r=S/s$ from Theorem 3.9, and $AI,BI,CI$ can be computed too, as explained in Theorem 3.9. $\square$

As a generalization of the Heron formula, we have the following result:

THEOREM 3.11 (Brahmagupta). *The area of a quadrilateral lying on a circle*




*having sides* $a, b, c, d$ *is given by the following formula,*

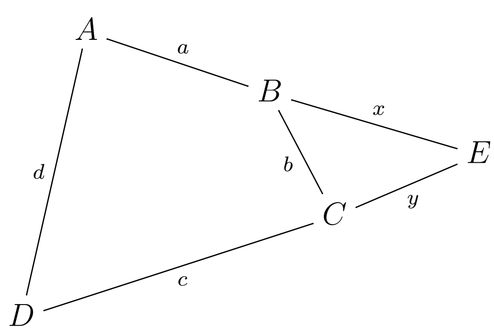
$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

*with* $s = (a+b+c+d)/2$ *being the semiperimeter.*

PROOF. As a first observation, this formula generalizes indeed the Heron formula for triangles, which can be obtained with $d = 0$. In order to prove now the formula, the idea will be that of applying the Heron formula, to certain pair of similar triangles:

(1) To start with, the result for trapezoids comes as a limiting case of the formula for non-trapezoids, so we can assume that the sides $AB, CD$ are not parallel. Alternatively, assuming $AB||CD$, and with the result being clear for rectangles, we can assume that the sides $AD, BC$ are not parallel, which in practice means, by cyclically permuting the vertices $A, B, C, D$, that we can again assume that the sides $AB, CD$ are not parallel.

(2) So, assuming that the sides $AB, CD$ are not parallel, let us prolong these sides $AB$ and $CD$, until they meet at a point $E$, with the picture being as follows:




The point now is that, due to the presence of the circle through the points $A, B, C, D$, we have a pair of similar triangles appearing on this picture, as follows:

$$ADE \sim CBE$$

(3) In relation now with areas, since the proportionality factor between the above two similar triangles is $d/b$, we conclude that we have the following formula:

$$area(ADE)=\frac{d^2}{b^2}\times area(CBE)$$

Now in terms of the original quadrilateral $ABCD$, this formula gives:

$$\begin{aligned}area(ABCD)&=area(ADE)-area(CBE)\\&=\left(\frac{d^2}{b^2}-1\right)\times area(CBE)\end{aligned}$$

(4) Next, let us apply the Heron formula to $CBE$. We obtain:

$$area(CBE)=\frac{1}{4}\sqrt{(x+y+b)(x+y-b)(x-y+b)(y-x+b)}$$

Thus, done or almost, and in order to finish, we have to compute $x,y$, or rather the quantities $x+y,x-y$ appearing above, in terms of $a,b,c,d$, and see what we get.

(5) But this can be done by using $ADE\sim CBE$, which gives:

$$\frac{x}{b}=\frac{y+c}{d}\quad,\quad\frac{y}{b}=\frac{x+a}{d}$$

Indeed, let us write these formulae in the following way:

$$dx=by+bc\quad,\quad dy=bx+ab$$

By making the sum and difference of these equations, we obtain:

$$d(x+y)=b(x+y)+b(c+a)\quad,\quad d(x-y)=b(y-x)+b(c-a)$$

Thus, we have the following formulae for the quantities $x+y,x-y$:

$$x+y=b\cdot\frac{c+a}{d-b}\quad,\quad x-y=b\cdot\frac{c-a}{d+b}$$

(6) Now let us compute the quantities in the Heron formula, from (4). These are as follows, with $s=(a+b+c+d)/2$ being the semiperimeter of our quadrilateral:

$$x+y+b=b\left(\frac{c+a}{d-b}+1\right)=2b\cdot\frac{s-b}{d-b}$$

$$x+y-b=b\left(\frac{c+a}{d-b}-1\right)=2b\cdot\frac{s-d}{d-b}$$

$$x-y+b=b\left(\frac{c-a}{d+b}+1\right)=2b\cdot\frac{s-a}{d+b}$$

$$y-x+b=b\left(\frac{a-c}{d+b}+1\right)=2b\cdot\frac{s-c}{d+b}$$

(7) But with this, we can now finish. Indeed, the Heron formula from (4) takes the following form, in terms of $a,b,c,d$ and of the semiperimeter $s=(a+b+c+d)/2$:

$$\begin{aligned} area(CBE) &= b^2\sqrt{\frac{s-b}{d-b}\cdot\frac{s-d}{d-b}\cdot\frac{s-a}{d+b}\cdot\frac{s-c}{d+b}} \\ &= b^2\sqrt{\frac{(s-a)(s-b)(s-c)(s-d)}{(d-b)^2(d+b)^2}} \\ &= \frac{b^2}{d^2-b^2}\sqrt{(s-a)(s-b)(s-c)(s-d)} \end{aligned}$$

Now by using the formula in (3), we are led to the conclusion in the statement. □

The Brahmagupta formula established above is something quite interesting, and many other things can be said about it, notably with the following claim:

Claim 3.12. *The maximum area of a quadrilateral having sides $a,b,c,d$ is*

$$S=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

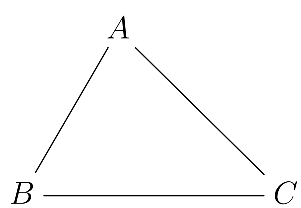
$s=(a+b+c+d)/2$ *being the semiperimeter, achieved when the points are cyclic.*

Obviously, this is something quite intuitive, because when inflating something, that thing tends to become a circle, right. However, in what regards the precise mathematics of this phenomenon, there are surely some things to be done, and we prefer to defer the discussion here for later in this book, in Part II, when doing trigonometry.

## 3d. Feuerbach points

Quite nice all the above, and with the centers $O,G,H,N,I$ all discussed, end of the story, you would say. Well, not at all, because here is an amazing result about this:

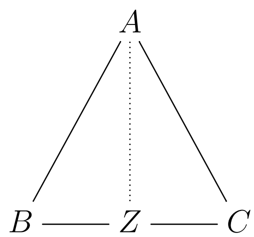
Theorem 3.13 (Feuerbach). *Given an arbitrary triangle $ABC$, not equilateral,*




*the inner and nine-point circles are tangent at a point $Z$, called Feuerbach point.*

Proof. Strange statement that we have here, because $Z$ appears by definition as some sort of "center" of our triangle, but is obviously not a center. So, this is definitely next-level mathematics, with respect to what we know. In what regards now the proof, this is unfortunately next-level too, the idea with all this being as follows:

(1) To start with, for an isosceles triangle, not taken equilateral, the Feuerbach point exists indeed, and is the middle of the main side, as shown by the following picture:

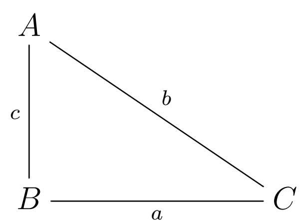



Thus, very good, at least we know one thing, $Z$ exists for isosceles triangles.

(2) The next step is to look at the right triangles, with the idea in mind of proving the following formula for them, which would prove the result:

$$NI = \rho - r$$

In order to do this, consider our right triangle, with edges $a, b, c$, as follows:



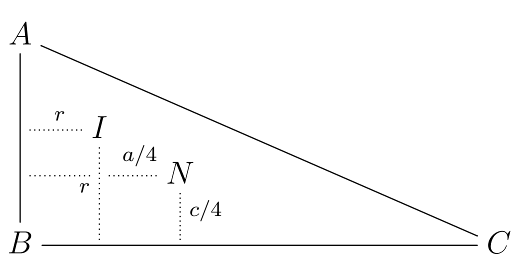

According to the various formulae above, the radii $\rho$ and $r$ are given by:

$$\rho = \frac{b}{4} \quad , \quad r = \frac{a+c-b}{2}$$

Thus, one of the quantities that we are interested in is given by:

$$\rho - r = \frac{b}{4} - \frac{a+c-b}{2} = \frac{3b-2a-2c}{4}$$

(3) Regarding now the other quantity that we are interested in, the length of the segment $IN$, in order to compute it, we can use the following configuration:



Indeed, by using Pythagoras, we have the following formula for the length of $IN$:

$$\begin{aligned} IN &= \sqrt{\left(r-\frac{a}{4}\right)^2+\left(r-\frac{c}{4}\right)^2} \\ &= \sqrt{\left(\frac{a+c-b}{2}-\frac{a}{4}\right)^2+\left(\frac{a+c-b}{2}-\frac{c}{4}\right)^2} \\ &= \frac{1}{4}\sqrt{(a+2c-b)^2+(2a+c-b)^2} \end{aligned}$$

(4) Thus, we are left with proving that the quantities found in (2) and (3) coincide, provided that $a,b,c$ satisfy $a^2+c^2=b^2$, and this can be done indeed, as follows:

$$\begin{aligned} IN &= \frac{1}{4}\sqrt{(a+2c-b)^2+(2a+c-b)^2} \\ &= \frac{1}{4}\sqrt{5a^2+5c^2+8b^2+8ac-12ab-12bc} \\ &= \frac{1}{4}\sqrt{4a^2+4c^2+9b^2+8ac-12ab-12bc} \\ &= \frac{1}{4}\sqrt{(3b-2a-2c)^2} \\ &= \frac{3b-2a-2c}{4} \\ &= \rho-r \end{aligned}$$

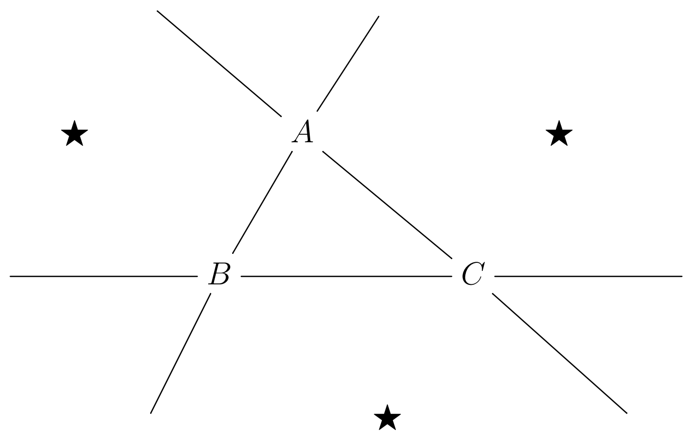
(5) Summarizing, good news, done with the right triangles. Of course, many more things can be said here, for instance with some precise formulae for the position of the Feuerbach point $Z$, in this case, that of the right triangles, and we will leave some study here, based on the various computations and formulae above, as an exercise.

(6) As for the general case, that of the arbitrary triangles $ABC$, this is something fairly complicated, at least with the technology that we presently have.

(7) But, we will be back to this later, when knowing more things, first in Part II after learning some trigonometry, with a number of comments on this, and then later in Part III, after learning vectors, with some further comments on this, in general. □

As a complement of Theorem 3.13, we have as well the following result:

Theorem 3.14. *Given a triangle $ABC$, the nine-point circle is tangent as well to the 3 external tangent circles, appearing according to the following picture,*




*at 3 further points associated to the triangle, called secondary Feuerbach points.*

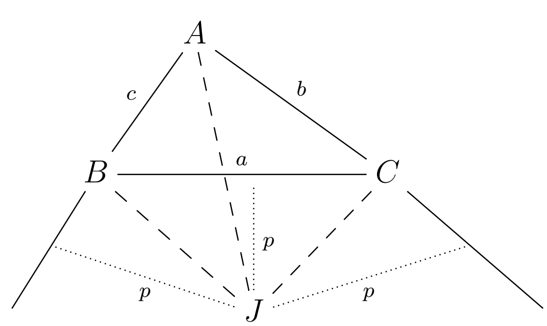
Proof. This is again something quite subtle, the idea being as follows:

(1) To start with, a bit philosophically, a triangle $ABC$ can be thought of as being a collection of 3 lines, and from this perspective, forgetting a bit about the vertices $A, B, C$ where these 3 lines cross, the picture of our triangle is the one above.

(2) But with this interpretation in hand, we can see that the incircle does not come alone, but rather accompanied by 3 more circles, called external tangent circles.

(3) And, in relation with this, coming as a continuation of Theorem 3.13, the present theorem states that the nine-point circle is tangent to these 3 external circles too.

(4) Getting to work now, we first need to know more about the external circles, in analogy with what we know about the incircle. In order to compute the radius $p$ of the external circle opposed to $A$, we can use the following configuration:



Indeed, a bit like before for the incircle, this leads to the following formula:

$$\begin{aligned} area(ABC) &= area(ABJ) + area(ACJ) - area(BCJ) \\ &= \frac{pc}{2} + \frac{pb}{2} - \frac{pa}{2} \end{aligned}$$

(5) As a conclusion to this, and with more standard notations $r_a, r_b, r_c$ for the radii of the 3 external circles, and by including the previous formula of $r$ too, for symmetry reasons, the formulae are as follows, with $S$ being the area of the triangle $ABC$:

$$r = \frac{2S}{a+b+c} \quad , \quad r_a = \frac{2S}{b+c-a} \quad , \quad r_b = \frac{2S}{a+c-b} \quad , \quad r_c = \frac{2S}{a+b-c}$$

(6) Equivalently, by using the Heron formula for the area $S$, the formulae of the various radii are as follows, with $s = (a+b+c)/2$ being the semiperimeter:

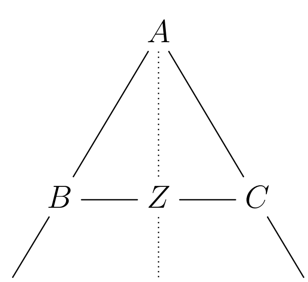
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \quad , \quad r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}$$

$$r_b = \sqrt{\frac{s(s-a)(s-c)}{s-b}} \quad , \quad r_c = \sqrt{\frac{s(s-a)(s-b)}{s-c}}$$

(7) Finally, let us record as well the formulae for right triangles, where some simplifications appear, due to $2S = ac$ and to $a^2 + c^2 = b^2$. These are as follows:

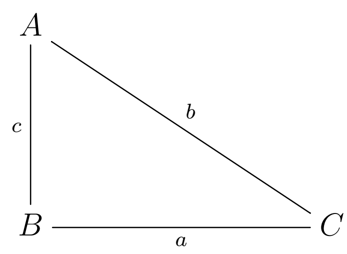
$$r = \frac{a+c-b}{2} \quad , \quad r_a = \frac{a+b-c}{2} \quad , \quad r_b = \frac{a+b+c}{2} \quad , \quad r_c = \frac{b+c-a}{2}$$

(8) In relation now with what the theorem says, let us start our study with the case of the isosceles triangles. Here one of the extra Feuerbach points comes for free, being equal to the previous Feuerbach point $Z$, as shown by the following picture:




(9) Thus, we are left with proving that the nine-point circle is tangent to one of the external circles on the left or right, and this can be done indeed, via some computations, based on the various formulae for the isosceles triangles established before. We will leave this for now as an exercise, and come back later to this, directly in the general case.

(10) As a continuation of this, let us examine now the case of the right triangles:




By symmetry, we have two questions to be solved, namely one regarding the external circle far from $B$, and one regarding one of the 2 external circles next to $B$.

(11) Let us start with the first question, in relation with the external circle far from $B$. By arguing like in the proof of Theorem 3.13, we can get the result as follows:

$$\begin{aligned}
I_bN &= \sqrt{\left(r_b-\frac{a}{4}\right)^2+\left(r_b-\frac{c}{4}\right)^2}\\
&= \sqrt{\left(\frac{a+c+b}{2}-\frac{a}{4}\right)^2+\left(\frac{a+c+b}{2}-\frac{c}{4}\right)^2}\\
&= \frac{1}{4}\sqrt{(a+2c+2b)^2+(2a+c+2b)^2}\\
&= \frac{1}{4}\sqrt{5a^2+5c^2+8b^2+8ac+12ab+12bc}\\
&= \frac{1}{4}\sqrt{4a^2+4c^2+9b^2+8ac+12ab+12bc}\\
&= \frac{1}{4}\sqrt{(3b+2a+2c)^2}\\
&= \frac{3b+2a+2c}{4}\\
&= \frac{b}{4}+\frac{a+c+b}{2}\\
&= \rho+r_b
\end{aligned}$$

(12) As for the second question, say in relation with the external circle on the left, we can get the result here too, via some similar computations, as follows:

$$\begin{aligned}
I_cN &= \sqrt{\left(r_c+\frac{a}{4}\right)^2+\left(r_c-\frac{c}{4}\right)^2}\\
&= \sqrt{\left(\frac{b+c-a}{2}+\frac{a}{4}\right)^2+\left(\frac{b+c-a}{2}-\frac{c}{4}\right)^2}\\
&= \frac{1}{4}\sqrt{(2b+2c-a)^2+(2b+c-2a)^2}\\
&= \frac{1}{4}\sqrt{5a^2+5c^2+8b^2-8ac-12ab+12bc}\\
&= \frac{1}{4}\sqrt{4a^2+4c^2+9b^2-8ac-12ab+12bc}\\
&= \frac{1}{4}\sqrt{(3b-2a+2c)^2}\\
&= \frac{3b-2a+2c}{4}\\
&= \frac{b}{4}+\frac{b+c-a}{2}\\
&= \rho+r_c
\end{aligned}$$

Thus, good news, done with all Feuerbach points for the right triangles.

(13) In the general case now, in what regards the proof of the result, the situation is a bit similar to that in Theorem 3.13, with this being something fairly complicated, at least with the technology that we presently have. We will be back to this later, first in Part II after learning some trigonometry, with a number of comments about this, and then later in Part III, after learning vectors, with some further comments on this, in general. □

The above results are quite interesting, and many other things can be said along these lines, with a summary of what can be done being as follows:

(1) In what regards the nine-point circle and Euler line, several other interesting things can be said in relation with various suitable homotheties centered on the Euler line, different from the one centered at $G$ and of ratio $-1/2$, that we used above.

(2) In what regards the various Feuerbach points, and the tangent circles in general, many other interesting things can be said here, notably with some general questions of Apollonius and others regarding the configurations of tangent circles.

(3) In short, many things to learned, and we will leave some exploration here as an exercise. In what concerns us, we will be back to this quite sporadically, notably with some further constructions of triangle centers, belonging or not to the Euler line.

So long for the various triangle centers. We will be back to them on several occasions, first in the next chapter, with a number of new methods, then later in Part II, after learning trigonometry, and then later in Part III too, after learning vector calculus.

## 3e. Exercises

This was a quite advanced chapter, and as exercises on this, we have:

EXERCISE 3.15. *Learn more about powers of points with respect to circles.*

EXERCISE 3.16. *Learn more, as much as you can, about the nine-point circle.*

EXERCISE 3.17. *Learn also, again as much as you can, about the Euler line.*

EXERCISE 3.18. *In particular, learn about the further homotheties based there.*

EXERCISE 3.19. *Learn also a bit about the other possible triangle centers.*

EXERCISE 3.20. *Prove the nine-point circle for isosceles triangles, via Pythagoras.*

EXERCISE 3.21. *Learn more about the Brahmagupta formula, and related topics.*

EXERCISE 3.22. *Find the secondary Feuerbach points, for the isosceles triangles.*

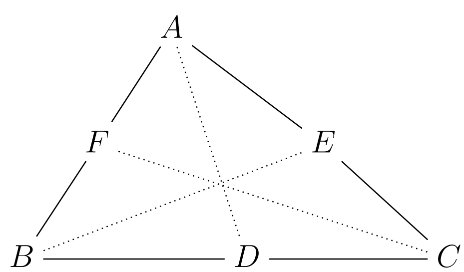
As bonus exercise, you can start reading an advanced plane geometry book.

CHAPTER 4

# Incidence results

## 4a. Menelaus, Ceva

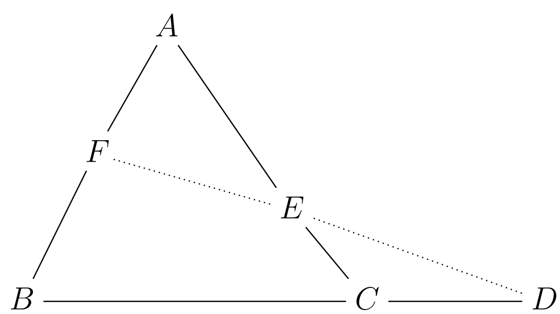
Let us go back now to the basic triangle geometry and centers, as developed at the end of chapter 2. In order to further build on that material, we first need to answer our question there, asking for general crossing results, of the following type:




We will discusss this slowly, with several results on this, and on related topics. Among others, we will see that the other triangle centers usually belong to the Euler line.

First on our list we have the following key result, due to Menelaus:

Theorem 4.1 (Menelaus). *In a configuration of the following type, with a triangle $ABC$ cut by a line $FED$,*




*we have the following formula, with all segments being taken oriented:*

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=-1$$

*Moreover, the converse holds, with this formula guaranteeing that $F,E,D$ are colinear.*

PROOF. This is indeed something very standard, the idea being as follows:

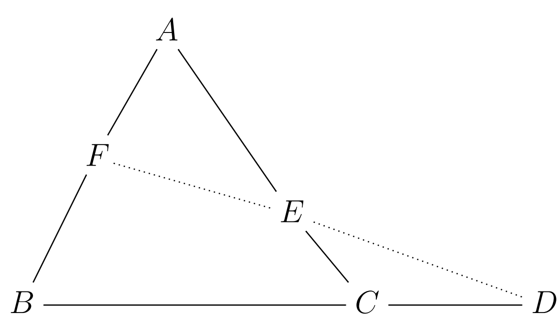
(1) Let us first try to prove the following equality, which is a bit weaker than what the theorem says, with all segments being by definition taken oriented:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

But this is something clear, because by projecting the vertices $A,B,C$ on the line $DEF$, into points $A',B',C'$, we have the following computation:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{AA'}{BB'}\cdot\frac{BB'}{CC'}\cdot\frac{CC'}{AA'}=1$$

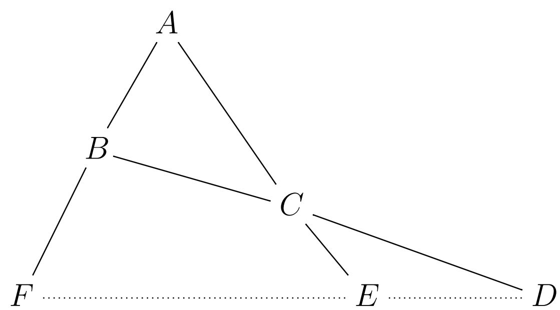
(2) Next, we must see what happens to the above equality, when allowing the segments to be oriented. But here, there are several cases to be considered, depending on whether the line $DEF$ intersects the triangle $ABC$, a bit as in the picture in the statement, or not. Let us first examine the crossing configuration, as in the statement, namely:




In this case, with all the segments being by definition taken oriented, we are led indeed to the formula in the statement, as follows:

$$\begin{aligned}\frac{AF}{FB}\cdot\frac{BD}{DC}\frac{CE}{EA} &= \frac{|AF|}{|FB|}\left(-\frac{|BD|}{|DC|}\right)\cdot\frac{CE}{EA}\\ &= -\frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|}\\ &= -1\end{aligned}$$

(3) Let us examine now the non-crossing configuration, which is as follows:



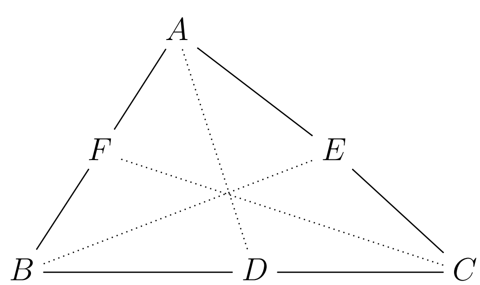

In this case, again with all the segments being by definition taken oriented, we are again led to the formula in the statement, as follows:

$$\begin{aligned}\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA} &= \left(-\frac{|AF|}{|FB|}\right)\left(-\frac{|BD|}{|DC|}\right)\left(-\frac{CE}{EA}\right)\\ &= -\frac{|AF|}{|FB|}\cdot\frac{|BD|}{|DC|}\cdot\frac{|CE|}{|EA|}\\ &= -1\end{aligned}$$

(4) Thus, we have proved the formula in the statement. As for the converse, this follows from the main result, in the obvious way, and as usual with converses of such statements, we will leave the discussion here as an instructive exercise for you. □

We can now answer our original question about crossing lines inside a triangle, drawn from the vertices, with the following remarkable result, about this:

Theorem 4.2 (al-Mutaman, Ceva). *In a configuration of the following type, with a triangle $ABC$ containing inner lines $AD, BE, CF$ which cross,*




*we have the following formula:*

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

*Moreover, the converse holds, with this formula guaranteeing that $AD, BE, CF$ cross.*

PROOF. This is something very standard again, the idea being as follows:

(1) Some history first. This theorem was first discovered by al-Mutaman, king of Zaragoza, and fine intellectual and scientist, in the 11th century, and published in his treatise "Book of perfection". However, a bit later, the Arab civilization in Western Europe started to decay, and they lost al-Andalus to the Spaniards, along with many other things, including the credit for the present theorem going to al-Mutaman.

(2) In fact, the al-Mutaman theorem was completely forgotten, and rediscovered in the 17th century by the Italian mathematician Ceva, who published it, along with his rediscovery of the Menelaus theorem too, in his book "De lineis rectis". Subsequently, Ceva, along with Gergonne, Nagel and others, and more on this in a moment, managed to use this theorem for pushing the boundaries of triangle geometry to a new level.

(3) So, this was for the story, and in practice, as of now, early 21th century, it is still very customary to refer to this as Ceva's theorem, and to the lines $AD, BE, CF$ in the statement as being cevians. Moral of the story I guess, no mercy for decaying civilizations, and with the remark of course that things here are quite cyclic, and never-ending, and one day the credit for the present theorem might well go back to al-Mutaman.

(4) Getting to work now, a first way of proving this result is by using the Menelaus theorem, applied twice. Indeed, if we denote by $O$ the point in the middle, we have the following formula, coming from the line $COF$ cutting the triangle $ABD$:

$$\frac{AF}{FB}\cdot\frac{BC}{CD}\cdot\frac{DO}{OA}=-1$$

On the other hand, again by using the Menelaus theorem, we have as well the following formula, coming this time from the line $BEO$ cutting the triangle $ADC$:

$$\frac{AO}{OD}\cdot\frac{DB}{BC}\cdot\frac{CE}{EA}=-1$$

By multiplying now the above two formulae, we obtain, as desired:

$$\begin{aligned}1&=\frac{AF}{FB}\cdot\frac{BC}{CD}\cdot\frac{DO}{OA}\times\frac{AO}{OD}\cdot\frac{DB}{BC}\cdot\frac{CE}{EA}\\&=\frac{AF}{FB}\cdot\frac{BC}{CD}\times\frac{DB}{BC}\cdot\frac{CE}{EA}\\&=\frac{AF}{FB}\cdot\frac{BC}{DC}\times\frac{BD}{BC}\cdot\frac{CE}{EA}\\&=\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}\end{aligned}$$

(5) An alternative proof, which is more elegant, is by using the same idea as for Menelaus, namely some fractions which cancel. Again by denoting by $O$ the point in the

middle, we have the following formulae for the quotient $AF/FB$, in terms of areas:

$$\frac{AF}{FB}=\frac{AFO}{FBO}=\frac{AFC}{FBC}$$

We deduce from this that we have the following extra formula for $AF/FB$:

$$\frac{AF}{FB}=\frac{AFC-AFO}{FBC-FBO}=\frac{AOC}{BOC}$$

Similarly, we have the following formulae for $BD/DC$, and for $CE/EA$:

$$\frac{BD}{DC}=\frac{AOB}{AOC}\quad,\quad\frac{CE}{EA}=\frac{BOC}{AOB}$$

Now by multiplying all these formulae we obtain, as desired:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{AOC}{BOC}\cdot\frac{AOB}{AOC}\cdot\frac{BOC}{AOB}=1$$

(6) As for the converse, this follows from the main result, in the obvious way, and as usual with such converses, we will leave the discussion here as an exercise. □

Summarizing, question about crossing lines inside a triangle solved, and we will see applications in a moment. Before that, however, let us meditate a bit more on the relation between Menelaus and Ceva. These statements are obviously related, and a natural guess here would be that these are equivalent. So, let us formulate the following question:

Question 4.3. *What is the precise relation between Menelaus and Ceva?*

In answer now, we have seen in the above that Menelaus implies Ceva, so we are left with proving that Ceva implies Menelaus. However, and as surprising as this might seem, this is something non-trivial, which cannot be really done with bare hands, and in order to understand what I am talking about, try a bit, and you will get the point.

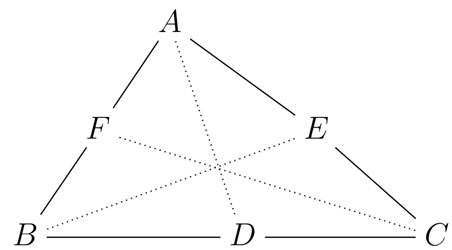
However, with the help of more advanced technology, such as the duality between points and lines from chapter 1, the equivalence can be established indeed, as follows:

Theorem 4.4. *The Menelaus and Ceva configurations are dual, via the usual duality between points and lines, with respect to a circle, and with this bringing:*

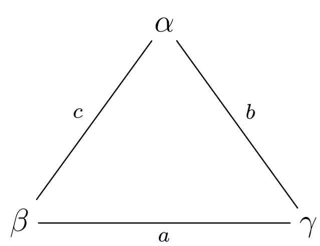
(1) *An extra proof for Ceva, assuming Menelaus known.*
(2) *Or a proof for Menelaus, assuming Ceva known.*

Proof. This is something quite tricky, the idea being as follows:

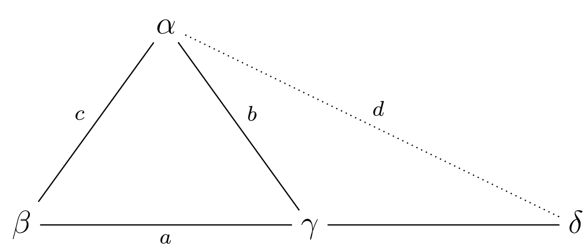
(1) Consider indeed the Ceva configuration, which was as follows:



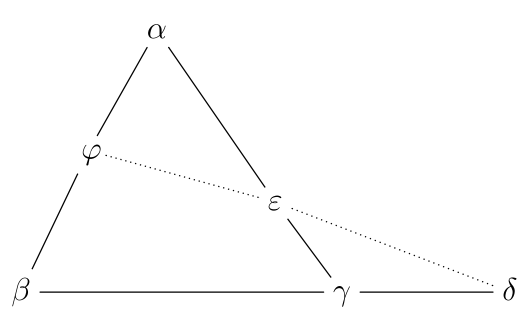

(2) Now let us compute the dual of this configuration, with respect to an arbitrary circle in the plane, not passing through the above points. If we denote by $\alpha,\beta,\gamma$ the duals of $BC,AC,AB$, and by $a,b,c$ the duals of $A,B,C$, then we have a triangle, as follows:




(3) Next, if we denote by $\delta$ the dual of $AD$, and by $d$ that of $D$, the picture becomes:



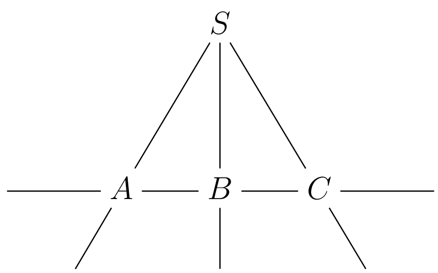

(4) Similarly, we can draw the duals $\varepsilon,\varphi$ of $BE,CF$, and the duals $e,f$ of $E,F$. Now the point is that, according to the properties of the duality, the lines $AD,BE,CF$ cross precisely when their dual points $\delta,\varepsilon,\varphi$ are aligned, leading to the following picture:



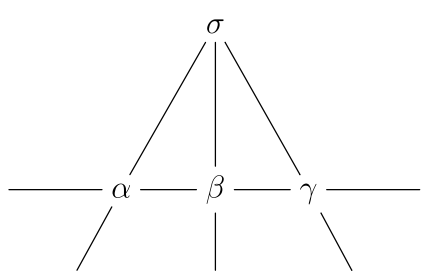
(5) But this is precisely the picture of the Menelaus configuration, and with a bit more work, that we will leave as an exercise, the product of quotients from the Ceva theorem corresponds to the product of quotients from the Menelaus theorem, as desired.

(6) To be more precise here, in what regards the exercise that we left, we are in need of some sort of lemma, stating that certain ratios are preserved by duality. But, with our duality mapping points to lines, and vice versa, how can we can even formulate such a lemma? Not very clear all this, and in the hope that you agree with me.

(7) In answer now, and forgetting about Menelaus and Ceva, and with the aim of formulating our lemma, let us consider configurations of the following type:




What we have here are 4 points and 4 lines, with 3 collinear points, and 3 concurrent lines. Thus, and quite remarkably, the dual configuration should look the same, namely with 4 points and 4 lines, featuring 3 collinear points, and 3 concurrent lines:




But with this, we can now state our lemma, which can only be the following equality:

$$\frac{AB}{BC}=\frac{\alpha\beta}{\beta\gamma}$$

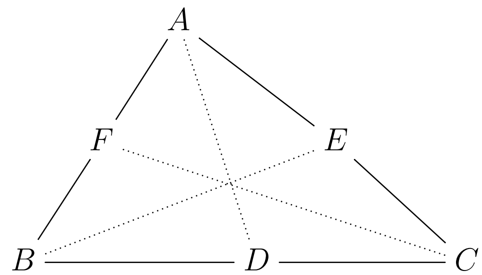
(8) And with this, job done for me I guess, we are now on a large highway leading to the present theorem, and exercise for you to accept the journey, and enjoy it.

(9) As for the consequences of this, mentioned in the statement, these all follow from the discussion in the proof of Theorem 4.2, and we will leave some meditation here as an instructive exercise too. By the way, there are as well some other methods for establishing the equivalence between Menelaus and Ceva, as for instance by using the cross ratio technology from chapter 1, and again exercise for you, to learn more about all this. □

## 4b. Basic applications

As a basic application of the Ceva theorem, we have now a new point of view on the barycenter, and on the incenter, circumcenter and orthocenter too, as follows:

THEOREM 4.5. *The barycenter, incenter, circumcenter, orthocenter theorems*



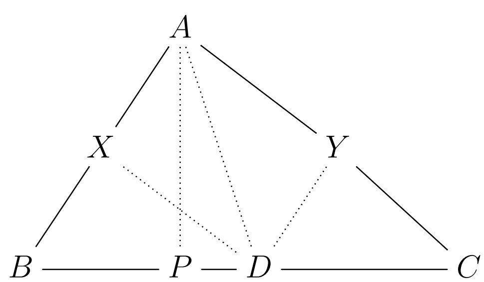

*can all be proved by using the Ceva theorem.*

PROOF. All this is indeed quite standard, the idea being as follows:

(1) In what regards the barycenter, the fact that the medians cross can be seen indeed as coming from the Ceva theorem, via the following trivial computation:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1\times 1\times 1=1$$

(2) Regarding now the incenter, let us draw the angle bisector $AD$, then project $D$ on the sides $AB, AC$, to points $X, Y$, and draw the altitude $AP$ as well:




We have then similar triangles $BDX \sim BAP$ and $CDY \sim CAP$, which give:

$$\frac{BD}{DX}=\frac{AB}{AP}\quad,\quad\frac{CD}{DY}=\frac{AC}{AP}$$

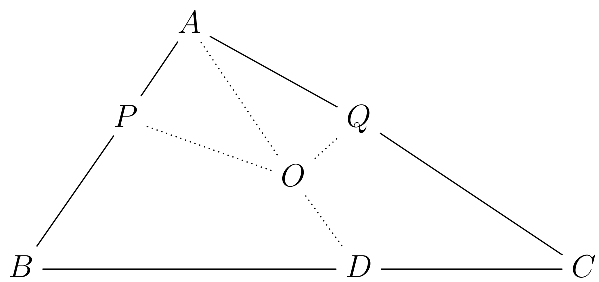
But $DX = DY$, with $D$ lying on the bisector, so by dividing these relations, we get:

$$\frac{BD}{DC}=\frac{AB}{AC}$$

And with this, done, because we can conclude that the angle bisectors cross indeed, as a consequence of the Ceva theorem, via the following computation:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{AC}{BC}\cdot\frac{AB}{AC}\cdot\frac{BC}{AB}=1$$

(3) Regarding now the circumcenter, whose existence is plainly trivial, let us attempt to deduce this via Ceva too. So, let us draw the perpendicular bisectors of the sides $AB, AC$, intersect them at $O$, and then consider the point $D = AO \cap BC$:




In order to compute the ratio $BD/DC$, let us project $B, C$ on the line $AO$, into points $X, Y$. We have then the following formula, coming from Thales:

$$\frac{BD}{DC} = \frac{BX}{CY}$$

On the other hand, due to the various square angles at $P, Q$ and $X, Y$, we have similar triangles $ABX \sim AOP$ and $ACY \sim AOQ$, which give the following formulae:

$$\frac{BX}{AB} = \frac{OP}{AO} \quad , \quad \frac{CY}{AC} = \frac{OQ}{AO}$$

Now by dividing these two latter formulae, we obtain the following formula:

$$\frac{BX}{CY} = \frac{OP}{OQ} \cdot \frac{AB}{AC}$$

Summarizing, we have proved that the ratio $BD/DC$ is given by:

$$\frac{BD}{DC} = \frac{OP}{OQ} \cdot \frac{AB}{AC}$$

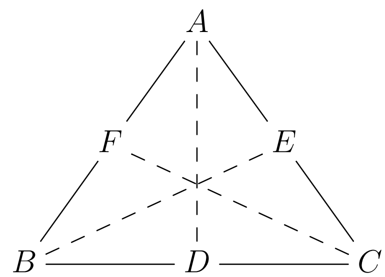
Which does not look very good, so it is better now to cheat, project $O$ on $BC$, to a point $R$, and declare that the circumcenter sort of comes from Ceva too, via:

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \left(\frac{OQ}{OR} \cdot \frac{AC}{BC}\right)\left(\frac{OP}{OQ} \cdot \frac{AB}{AC}\right)\left(\frac{OR}{OP} \cdot \frac{BC}{AB}\right) = 1$$

Nevermind. As a conclusion here, which is something quite interesting, while the existence of the circumcenter was the simplest to establish, among all triangles centers, and with this existence result being actually a plain triviality, in what regards the associated Ceva computations, these are quite complicated. We will be back to this phenomenon later in this book, with more details and comments, when doing trigonometry.

(4) Finally, let us get to the orthocenter, whose existence was the hardest one to establish, among all basic triangle centers, in chapter 2. Consider the configuration producing

the orthocenter, which is as follows, with $AD$, $BE$, $CF$ being the altitudes:




The quantity in the Ceva theorem can be then written as follows:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\left(\frac{AF}{FC}\cdot\frac{FC}{FB}\right)\left(\frac{BD}{DA}\cdot\frac{DA}{DC}\right)\left(\frac{CE}{EB}\cdot\frac{EB}{EA}\right)$$

Now the point is that, by using similar triangles, we have:

$$\frac{AF}{FC}=\frac{EA}{EB}\quad,\quad\frac{BD}{DA}=\frac{FB}{FC}\quad,\quad\frac{CE}{EB}=\frac{DC}{DA}$$

Thus everything simplifies, and we obtain, as desired:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

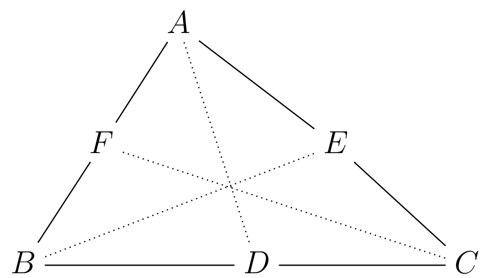
(5) Finally, as previously mentioned in the context of our circumcenter computations in (3), this is not the end of the story with the various triangle centers viewed via Ceva, because there is still a potential trigonometry discussion coming on top of this, which can clarify a number of things. More on this later, when doing trigonometry. □

Getting now to new triangle centers that can be constructed via Ceva, there are potentially plenty of them, because given any numbers $a,b,c$ we can construct points $D,E,F$ on edges, and a new point inside $ABC$, according to the following recipe:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}=1$$

In fact, in relation with this method, we can formulate the following interesting result, making the link with the various barycenter considerations from the beginning of chapter 2, and generally speaking, explaining the physics of the Ceva theorem:

THEOREM 4.6. *Given a triangle $ABC$, we have crossing lines as follows,*




*precisely when it is possible to assign weigths $a,b,c$ to the vertices $A,B,C$, as for each of the lines $AD,BE,CF$ to unite a vertex with the barycenter of the opposite side.*

PROOF. This is a reformulation of the Ceva theorem. Indeed, recall that the Ceva condition, for the lines $AD,BE,CF$ to cross indeed, was as follows:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

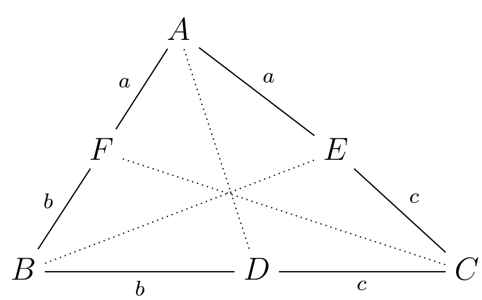
The point now is that, by basic arithmetic, a product of three numbers equaling 1 amounts in saying that the product in question is as follows:

$$\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}=1$$

Thus, the Ceva condition for $AD,BE,CF$ to cross can be reformulated as follows:

$$\frac{AF}{FB}=\frac{a}{b}\quad,\quad\frac{BD}{DC}=\frac{b}{c}\quad,\quad\frac{CE}{EA}=\frac{c}{a}$$

Geometrically, this means that the ratios on the sides must be as follows:

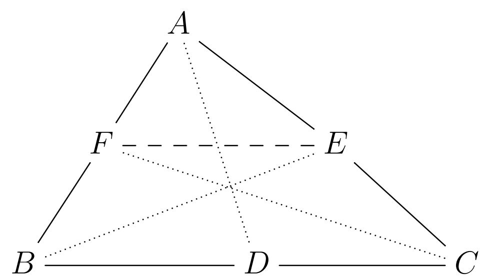



But this shows that, when installing the numbers $a,b,c$ at the vertices $A,B,C$, as weights, each of the lines $AD,BE,CF$ will unite a vertex with the barycenter of the opposite side. Thus, we are led to the conclusion in the statement. □

Moving on, let us further explore the applications of the Ceva theorem, with the aim of constructing new centers of a triangle $ABC$, according to the following recipe:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}=1$$

As a first observation, when one of the ratios in the Ceva product equals 1, say when $BD = DC$, and so when $AD$ is a median, by Thales the Ceva condition reads $FE||BC$, and the fact that the lines $AD, BE, CF$ cross is clear from Thales too, as follows:

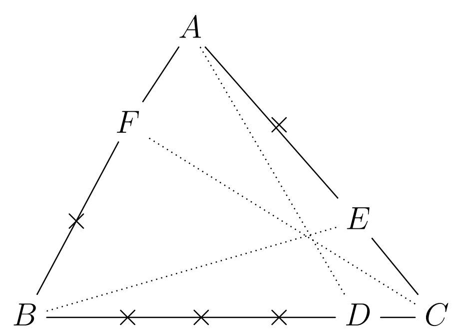



Thus, at the abstract level, of arithmetic nature, the simplest non-trivial application of the Ceva theorem should come from the following identity:

$$\frac{1}{2}\cdot\frac{4}{1}\cdot\frac{1}{2}=1$$

Which sounds quite interesting, so let us record this finding, as follows:

THEOREM 4.7. *Given a triangle $ABC$ with internal lines drawn as follows,*

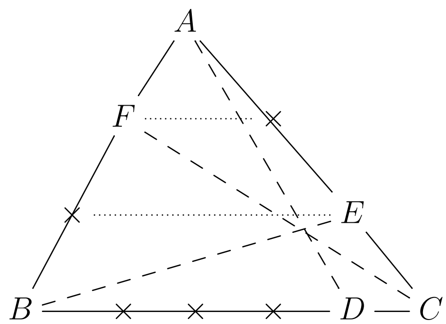



*at $1/3 - 2/3$ twice and $1/5 - 4/5$, as indicated, these lines cross indeed.*

PROOF. As a first question, you might wonder where the above thirds and fifths come from, but the mathematics of Ceva is indeed there, as simple as possible, as follows:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{1}{2}\cdot\frac{4}{1}\cdot\frac{1}{2}=1$$

As a second question now, is this "simplest non-trivial" application of Ceva indeed non-trivial, as advertised? In answer, yes, because such things are indeed not easy to prove with bare hands. The first thought goes to drawing two parallels, as follows:



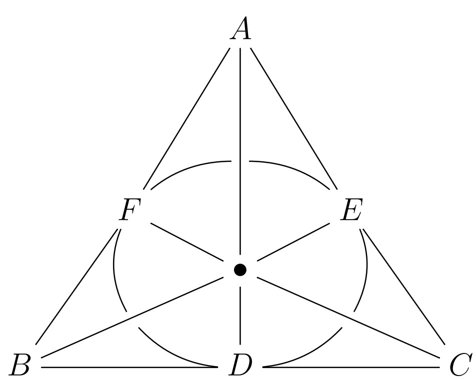

And we will leave some exploration here, based on Thales, as an instructive exercise. Of course, in case you find this exercise, and the present theorem, too easy, all this needs an update, say based on the following formula, which is the next simplest one:

$$\frac{1}{3}\cdot\frac{6}{1}\cdot\frac{1}{2}=1$$

Finally, observe that in the context of the above exercise, it might help to complete the configuration formed by the two parallels into a complete grid, with each of the sides $AB, AC, BC$ divided into $3\times 5=15$ equal parts, and then with lots of parallels drawn. And with this, interestingly, making the link with the discretization considerations from chapter 2, in the context of our barycenter discussion there. Quite nice all this, so many things to be explored here, and with the hope that you will spend some time on this. □

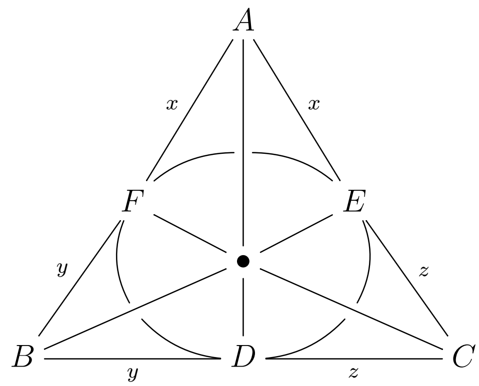
Moving away now from all this quite abstract mathematics, which rather counts as algebra, let us discuss some further triangle centers, obtained via Ceva, and enlarging the list that we already have, from Theorem 4.5. As a first result here, we have:

THEOREM 4.8 (Gergonne). *Given a triangle $ABC$ with its incircle drawn,*




*the lines $AD, BE, CF$ cross indeed, at a point called Gergonne point of $ABC$.*

Proof. The incircle configuration is indeed an invitation to apply the Ceva theorem, because the various lengths appearing on the sides of $ABC$ are as follows:




Thus, the lines $AD, BE, CF$ cross indeed, according to the following computation:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{x}{y}\cdot\frac{y}{z}\cdot\frac{z}{x}=1$$

For the record, a bit as we did before in Theorem 4.5 for the other triangle centers, let us record as well the numerics. Solving for $x, y, z$ in terms of the sides $a, b, c$ gives:

$$x=\frac{b+c-a}{2}\quad,\quad y=\frac{a+c-b}{2}\quad,\quad z=\frac{a+b-c}{2}$$

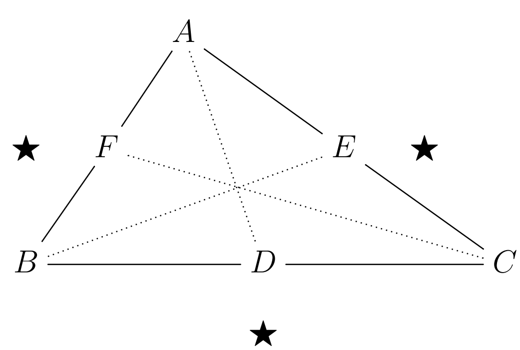
Thus, the detailed Ceva computation for the Gergonne point is as follows:

$$\frac{b+c-a}{a+c-b}\cdot\frac{a+c-b}{a+b-c}\cdot\frac{a+b-c}{b+c-a}=1$$

There are of course many other things that can be said about the Gergonne point, and for more on all this, we refer to any advanced plane geometry book. □

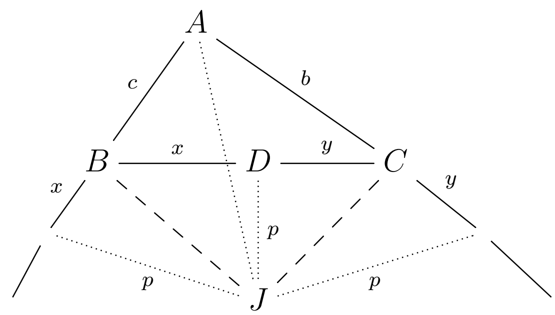
Along the same lines, we have as well the following result:

Theorem 4.9 (Nagel). *Given a triangle $ABC$ with its external circles drawn,*




*the lines $AD, BE, CF$ cross indeed, at a point called Nagel point of $ABC$.*

PROOF. With our usual apologies for the graphics, when it comes to external circles, let us attempt to prove this, in analogy with what we did for Theorem 4.8. The configuration for the external circle opposed to $A$ is something quite simple, as follows:




Since $J$ is equally distant from $AB, AC$ we must have $c+x=b+y$, and together with $x+y=a$ this allows the computation of $x, y$, with the solution being:

$$x=\frac{a+b-c}{2} \quad , \quad y=\frac{a+c-b}{2}$$

Thus, a bit as before in the Gergonne theorem, but with all fractions being now inverted, the present Nagel theorem holds indeed, thanks to the following formula:

$$\frac{a+c-b}{b+c-a}\cdot\frac{a+b-c}{a+c-b}\cdot\frac{b+c-a}{a+b-c}=1$$

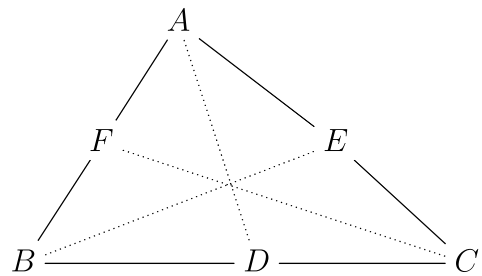
As before with Gergonne, many other things can be said about the Nagel point, and for more on all this, we refer to any advanced plane geometry book. □

So long for triangles and their centers, and there are in fact far more than this, namely dozens, hundreds, thousands, and even tens of thousands, for all levels and tastes. For the story here, we already talked about al-Mutaman, king of Zaragoza, in the above, but many other busy people, as for instance Napoleon, contributed as well to this.

In fact, studying triangles and their centers was a very fashionable business, since the ancient times, and up to not so long ago. In more modern times, however, the goals of mathematicians have slightly deviated towards arithmetic, with the must-do thing here, instead of constructing a new triangle center, being that of joining the list of generalizators of the Legendre symbol, for the quadratic residues. As for the truly modern times, the present ones, here the goal is that of having your own version of quantum field theory.

Back now to concrete mathematics and theorems, let us end this discussion about triangle centers with something a bit philosophical. In order for the point that we obtain via Ceva to deserve the name "triangle center", we must choose our numbers $a, b, c$ carefully, and ideally having something to do with the angles $A, B, C$. We are led in this way to:

Theorem 4.10. *Given a triangle $ABC$, and a function $f$, mapping angles to numbers, we can construct points $D,E,F$ on edges, and a new center of $ABC$,*




*according to the following recipe, based on the Ceva theorem,*

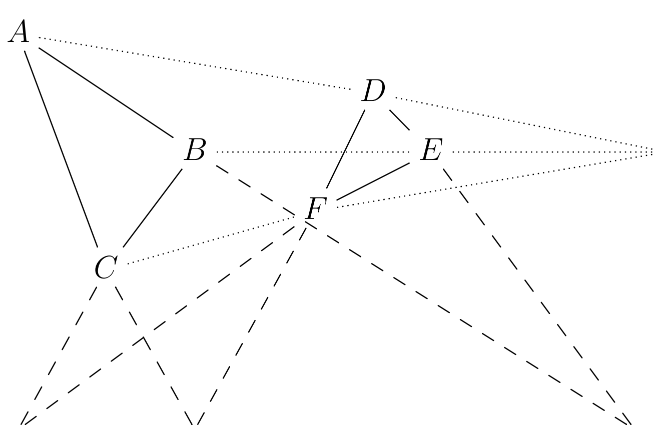
$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{f(A)}{f(B)}\cdot\frac{f(B)}{f(C)}\cdot\frac{f(C)}{f(A)}=1$$

*and with as basic illustration here, the barycenter coming from $f=1$.*

Proof. This is indeed something self-explanatory, and with the further comment that, with a bit of trigonometric know-how, the computations in the proof of Theorem 4.5 show that the incenter, circumcenter and orthocenter appear too in this way, with $f$ being various trigonometric functions. More on this later, when doing trigonometry. □

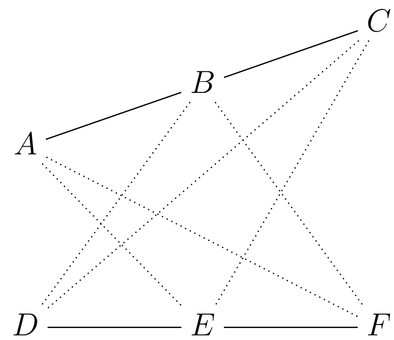
## 4c. Pascal, Brianchon

Switching topics, but still in relation with incidence questions, let us recall from chapter 1 something fundamental, namely the Desargues configuration, as follows:




Obviously, this is something fundamental in relation with the notion of perspectivity, and the applications of this to life and engineering abound. In fact, it was because of this that, passed Thales, we chose this to be our very first theorem in this book.

Along the same lines, and also going back to the material from chapter 1, of fundamental importance too is the Pappus configuration, which is a bit similar, as follows:



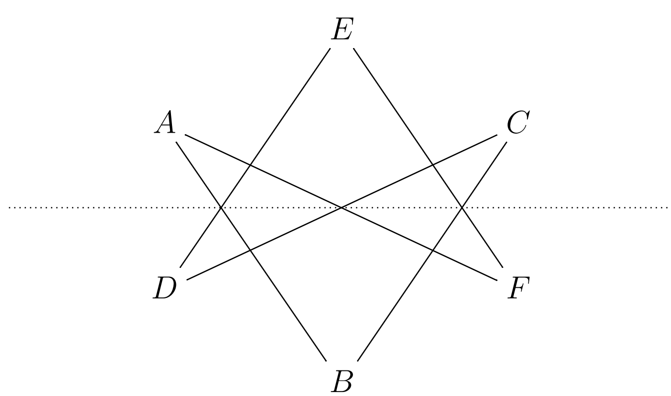

So, these are the two main configurations of plane geometry, perhaps more important than those met in the context of the triangles and their centers, and when attempting to generalize them, as to have some interesting mathematics going, we are led to:

QUESTION 4.11. *How to generalize Desargues and Pappus, say by replacing some of the lines there by more complicated curves, such as circles, ellipses and so on?*

And with this being, and we insist, a very good question. Among others because, as the recent history of mathematics has shown, this was more or less the question at the origin of modern algebraic geometry, and of modern mathematics in general.

In answer now, we have the following key theorem of Pascal, which can be regarded as being a version of the Pappus theorem, with the 2 lines there replaced by a circle:

THEOREM 4.12 (Pascal). *Given a hexagon lying on a circle*




*the pairs of opposite sides intersect in points which are collinear.*

PROOF. Many things can be said here, the idea being as follows:

(1) To start with, as mentioned above, the Pascal theorem and related results are the source of many advanced things in mathematics, all leading to all sorts of possible proofs.

With the technology that we have so far, the best is to prove this by applying three times the Menelaus theorem, a bit as we did for the Ceva theorem.

(2) Let us first fix some notation. We denote by $ABCDEF$ our arbitrary hexagon lying on the circle, and with the actual order of the points on the circle being irrelevant, and with, in practice, a quite nice order here being $ADBFCE$, and this in order to have a nice picture, as above, of the whole configuration, with obvious crossing points.

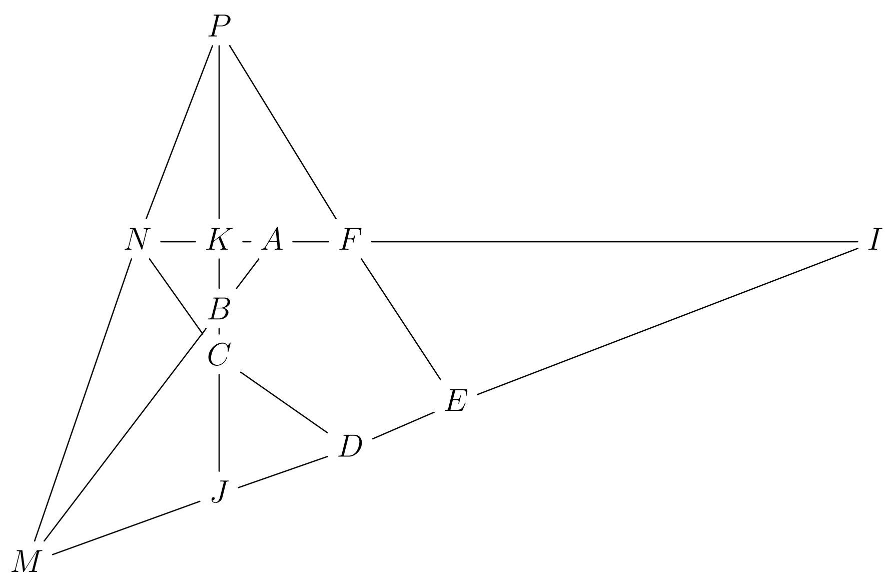
(3) We must prove that the following points are collinear:

$$M = AB \cap DE \quad , \quad N = AF \cap CD \quad , \quad P = BC \cap EF$$

In order to do so, with Menelaus in mind, consider as well the following points:

$$I = AF \cap DE \quad , \quad J = BC \cap DE \quad , \quad K = AF \cap BC$$

The picture becomes then as follows, with the hexagon being now chosen to have its vertices ordered $ABCDEF$, in order to better understand what is going on:




(4) Now let us look at the triangle $IJK$. The points $M, N, P$ that we are interested in lie on its sides, and according to Menelaus, in order to prove that these points $M, N, P$ are indeed collinear, we must prove that the following product equals 1:

$$Z = \frac{MI}{MJ} \cdot \frac{PJ}{PK} \cdot \frac{NK}{NI}$$

(5) But this can be proved by applying Menelaus three times. Indeed, we first have the following formula, coming from the triangle $IJK$ cut by the line $MAB$:

$$\frac{MI}{MJ} = \frac{BK}{BJ} \cdot \frac{AI}{AK}$$

Next, with the same triangle $IJK$, cut this time by the line $NCD$, we obtain:

$$\frac{NK}{NI}=\frac{CK}{CJ}\cdot\frac{DJ}{BI}$$

And finally, with the same triangle $IJK$, cut by the line $PEF$, we obtain:

$$\frac{PJ}{PK}=\frac{EJ}{EI}\cdot\frac{FI}{FK}$$

(6) By putting everything together, we conclude that the product from (4), that we want to prove to be equal to 1, is given by the following formula:

$$Z=\left(\frac{BK}{BJ}\cdot\frac{AI}{AK}\right)\left(\frac{EJ}{EI}\cdot\frac{FI}{FK}\right)\left(\frac{CK}{CJ}\cdot\frac{DJ}{BI}\right)$$

But $Z=1$ is true indeed, due to the following formulae, coming from the notion of power of a point with respect to a circle, that we know well since chapter 3:

$$BK\cdot CK=AK\cdot FK$$

$$AI\cdot FI=EI\cdot DI$$

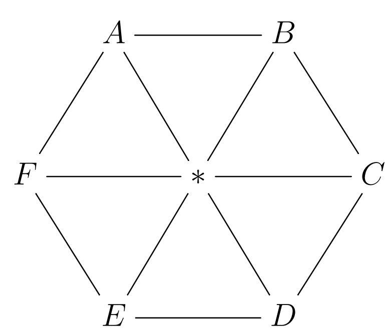
$$EJ\cdot BJ=DJ\cdot CJ$$

(7) Summarizing, Pascal theorem proved, by applying the Menelaus theorem three times, and with some help from the notion of power of a point with respect to a circle. As mentioned in the above, there are many other proofs as well, quite often based on more advanced technology, and we will be back to this, later on this book.

(8) Finally, observe the similarity with the Pappus theorem. We will see later that the Pascal theorem generalizes to the case of conics, and with this generalizing Pappus. □

Along the same lines, we have as well the following result:

Theorem 4.13 (Brianchon). *Given a hexagon circumscribed around on a circle*




*its main diagonals intersect.*

Proof. Long story with this too, the idea being as follows:

(1) To start with, this result is nearly impossible to prove, with bare hands, and ask around kids preparing for Math Olympiads, they will witness for that.

(2) But, this follows by duality from Pascal, because the dual of the Pascal configuration, with respect to the circle in question, on which the hexagon lies, is obviously the Brianchon configuration, and vice versa. Thus, theorem proved, just like that.

(3) Finally, as before with Pascal, we will see later that this extends to conics. □

With this discussed, let us briefly talk now about conics. Normally this is more advanced material, for later in this book, but coming in advance, here is what can be said about them, and about the light that they bring on our various incidence theorems:

Fact 4.14. *The conics, which are the algebraic curves of degree* 2*, appearing by cutting a two-sided cone with a plane, fall into three classes, as follows:*

(1) *Generic conics: the ellipses, and hyperbolas.*
(2) *Limiting cases: the circles, and parabolas.*
(3) *Degenerate conics: the point, the line, the* 2 *lines.*

*Based on this, we can extend the Pascal and Brianchon theorems from circles to all conics, by projecting, and this generalization of Pascal covers Pappus, coming from* 2 *lines.*

Obviously, many things going on here, and we won't attempt to comment more on this, at this stage of things. More later, but you get the point I hope, in what regards the mathematics that we know, it is all about conics. By the way, in what regards the physics that we know, namely planets and comets moving on ellipses, and asteroids moving sometimes on parabolas and hyperbolas, it is all about conics too. More later.

In any case, quite interesting all these results about hexagons, and the relation between them. So let us ask the cat, what he thinks about all this. And cat answers:

Cat 4.15. *In hexagrammum mysticum you will trust.*

Okay, and not that I really understand what cat says, he might be a reincarnation of Pascal's cat, or perhaps of Pascal himself, but the plan for what follows next becomes now clear, keep developing geometry, by keeping an eye on hexagons.

## 4d. Projective geometry

Moving on, still in relation with the various incidence results that we know, namely Desargues, Pappus, Menelaus, Ceva, Pascal, Brianchon, but on a different take, one annoying thing is that, while the parallels are certainly useful for the proofs, usually coming via Thales, the same parallels complicate the statements, with many cases appearing due to them, via various lines which can be parallel or not. So, let us formulate:

DREAM 4.16. *It would be nice to have a more advanced version of plane geometry, where parallel lines are allowed to meet, at infinity.*

So, this would be our dream, and we insist on the word "advanced" in the above, because we are here formulating this dream not as total beginners, but rather after about 100 pages of learning plane geometry, by heavy relying on parallels and Thales.

The question is now, how to make our dream come true? And in answer here, nothing simpler than that, all we have to do is to forget all the mathematics and other science that we know, and return to a modest status of average Instagram user, according to:

FACT 4.17. *Our dream comes true in real life. Indeed, take a picture of some railroad tracks, and look at that picture. Do the railroad tracks cross? Sure they do.*

So, this was for the story, the mathematical wonderland where all lines cross does exist, and is called Instagram. Abstractly now, here are some axioms, to start with:

DEFINITION 4.18. *A projective space is a space consisting of points and lines, subject to the following conditions:*

1. *Each* 2 *points determine a line.*
2. *Each* 2 *lines cross, on a point.*

As a basic example we have the usual projective plane $P^2$, which is best seen as being the space of lines in space passing through the origin. To be more precise, let us call each of these lines in space passing through the origin a point of $P^2$, and let us also call each plane in space passing through the origin a line of $P^2$. Now observe the following:

(1) Each 2 points determine a line. Indeed, 2 points in our sense means 2 lines in space passing through the origin, and these 2 lines obviously determine a plane passing through the origin, namely the plane they belong to, which is a line in our sense.

(2) Each 2 lines cross, on a point. Indeed, 2 lines in our sense means 2 planes passing through the origin, and these 2 planes obviously determine a line in space passing through the origin, namely their intersection, which is a point in our sense.

Thus, what we have is indeed a projective space in the sense of Definition 4.18. More generally, we have the following construction, in arbitrary dimensions:

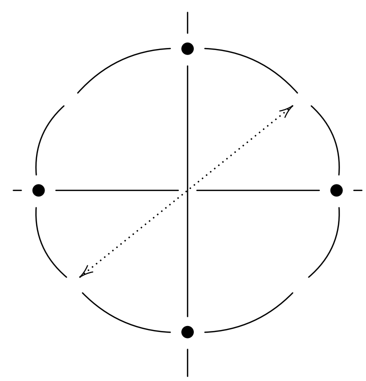
THEOREM 4.19. *We can define the projective space $P^{N-1}$ as being the space of lines in $N$-dimensional space passing through the origin, and in small dimensions:*

1. *$P^1$ is the usual circle.*
2. *$P^2$ is some sort of twisted sphere.*

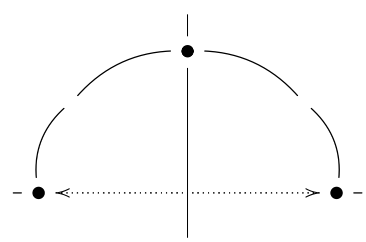
PROOF. We have several assertions here, with all this being of course a bit informal, and self-explanatory, the idea and some further details being as follows:

(1) To start with, the fact that the space $P^{N-1}$ constructed in the statement is indeed a projective space in the sense of Definition 4.18 follows from definitions, exactly as in the discussion preceding the statement, regarding the case $N = 3$.

(2) At $N = 2$ now, a line in the plane passing through the origin corresponds to 2 opposite points on the unit circle $T$, according to the following scheme:

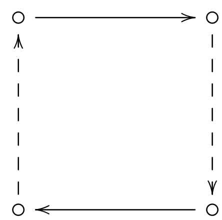


Thus, $P^1$ corresponds to the upper semicircle of $T$, with the endpoints identified, and so we obtain a circle, $P^1 = T$, according to the following scheme:

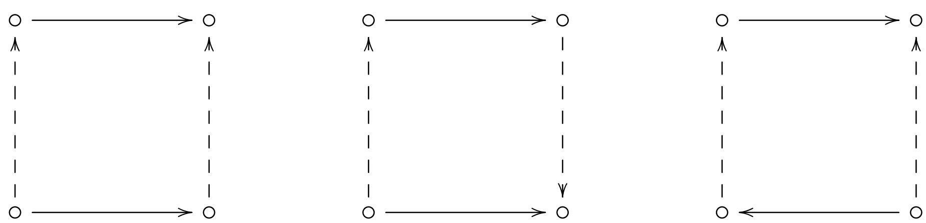


(3) At $N = 3$ now, the space $P^2$ corresponds to the upper hemisphere of the unit sphere $S^2$, with the points on the equator identified via $x = -x$. But, we can deform if we want the hemisphere into a square, with the equator becoming the boundary of this square, and in this picture, the $x = -x$ identification corresponds to a "identify opposite

edges, with opposite orientations" folding method for the square:



(4) Thus, we have our space. In order to understand now what this beast is, let us look first at the other 3 possible methods of folding the square, which are as follows:



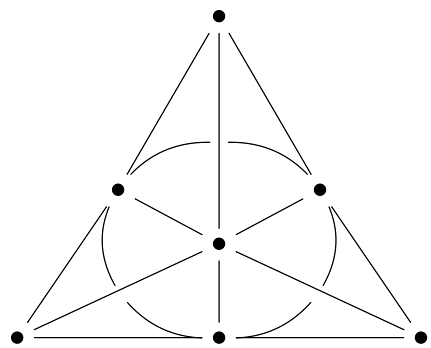
Regarding the first space, the one on the left, things here are quite simple. Indeed, when identifying the solid edges we get a cylinder, and then when further identifying the dotted edges, what we get is some sort of closed cylinder, which is a torus.

(5) Regarding the second space, the one in the middle, things here are more tricky. Indeed, when identifying the solid edges we get again a cylinder, but then when further identifying the dotted edges, we obtain some sort of "impossible" closed cylinder, called Klein bottle. This Klein bottle obviously cannot be drawn in 3 dimensions, but with a bit of imagination, you can see it, in its full splendor, in 4 dimensions.

(6) Finally, regarding the third space, the one on the right, we know by symmetry that this must be the Klein bottle too. But we can see this as well via our standard folding method, namely identifying solid edges first, and dotted edges afterwards. Indeed, we first obtain in this way a Möbius strip, and then, well, the Klein bottle.

(7) With these preliminaries made, and getting back now to the projective space $P^2$, we can see that this is something more complicated, of the same type, reminding the torus and the Klein bottle. So, we will call it "sort of twisted sphere", as in the statement, and exercise for you to imagine how this beast looks like, in 4 dimensions. □

Next, let us mention that Definition 4.18 is something far wider than it might seem. Consider indeed the following configuration of 7 points and 7 lines, called Fano plane:



Here the circle in the middle is by definition a line, and with this convention, the basic axioms in Definition 4.18 are satisfied, in the sense that any two points determine a line, and any two lines determine a point. And isn't this beautiful.

In practice now, projective geometry can help in relation with many questions. We will be back to this later in this book, when discussing more systematically all this.

## 4e. Exercises

This was again a quite advanced chapter, and as exercises on this, we have:

EXERCISE 4.20. *Clarify the duality between Menelaus and Ceva.*

EXERCISE 4.21. *Work some more on the circumcenter, using Ceva.*

EXERCISE 4.22. *Explore some more with weights and barycenters.*

EXERCISE 4.23. *Have some fun with Ceva with various rational ratios.*

EXERCISE 4.24. *Learn more about the Gergonne and Nagel points.*

EXERCISE 4.25. *Learn also about the various other triangle centers.*

EXERCISE 4.26. *Learn more about the theorems of Pascal and Brianchon.*

EXERCISE 4.27. *Learn more about the Fano plane, and the Paley biplane too.*

As bonus exercise, reiterated, start reading an advanced plane geometry book.

# Part II

# Basic trigonometry

*In the clearing stands a boxer*
*And a fighter by his trade*
*And he carries the reminders*
*Of every glove that laid him down*

CHAPTER 5

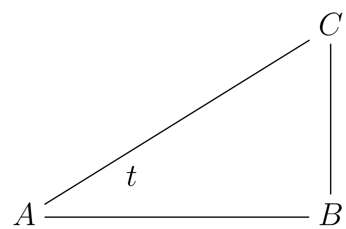
# Sine, cosine

## 5a. Sine, cosine

Time now to go back to the basics, namely angles and Pythagoras, that we learned about some time ago, in chapter 2. As a continuation of that material, let us develop now some trigonometry, among others with the aim of better understanding the triangles, and having a new look at the various more advanced results from chapters 3-4.

In short, welcome to trigonometry, which will be something quite simple in the beginning, and later, will gradually evolve. At the beginning of everything, we have:

Definition 5.1. *Given a right triangle $ABC$,*

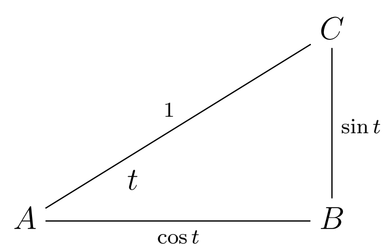



*we define the sine and cosine of the angle at $A$, denoted $t$, by the following formulae:*

$$\sin t = \frac{BC}{AC} \quad , \quad \cos t = \frac{AB}{AC}$$

*We call the sine and cosine basic trigonometric functions.*

As a first observation, the sine and cosine do not depend on the choice of the right triangle $ABC$, having angle $t$ at $A$, and this due to the Thales theorem. In view of this, we can choose our right triangle $ABC$ as to have $AC = 1$, and in this case we have $\sin t = BC$ and $\cos t = AB$. We can encode all this in a single picture, as follows:



As a few basic examples now, for the sine, coming from things that we know well about right triangles, from chapter 2, all consequences of Pythagoras, we have:

$$\sin 0^\circ = 0 \quad , \quad \sin 30^\circ = \frac{1}{2} \quad , \quad \sin 45^\circ = \frac{1}{\sqrt{2}} \quad , \quad \sin 60^\circ = \frac{\sqrt{3}}{2} \quad , \quad \sin 90^\circ = 1$$

Let us record as well the list of corresponding cosines. These are as follows:

$$\cos 0^\circ = 1 \quad , \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad , \quad \cos 45^\circ = \frac{1}{\sqrt{2}} \quad , \quad \cos 60^\circ = \frac{1}{2} \quad , \quad \cos 90^\circ = 0$$

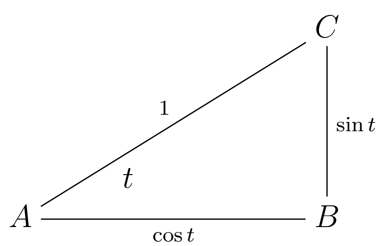
Observe that the numbers in the above two lists are the same, but written backwards in the second list. In fact, we have the following result, regarding this:

THEOREM 5.2. *The sines and cosines are subject to the formulae*

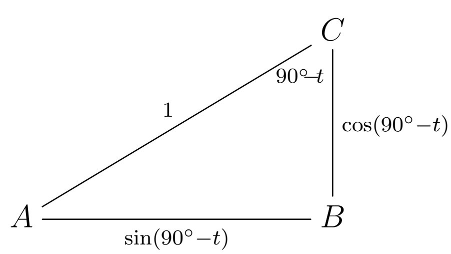
$$\sin(90^\circ - t) = \cos t \quad , \quad \cos(90^\circ - t) = \sin t$$

*valid for any angle* $t \in [0^\circ, 90^\circ]$.

PROOF. In order to understand this, the best is to choose our right triangle $ABC$ with $AC = 1$, as suggested after Definition 5.1, with the picture being as follows:




On the other hand, by focusing now at the angle at $C$, and perhaps twisting a bit our minds too, we have as well the following picture, for the same triangle:




Thus, we are led to the conclusion in the statement. □

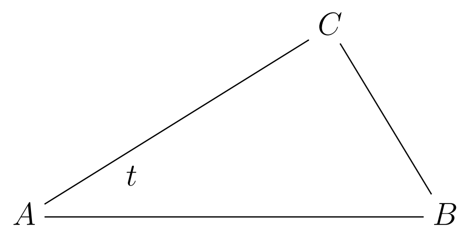
Before going ahead with more trigonometry, with all sorts of properties of the sine and cosine, that we can surely work out, a question that you might have:

QUESTION 5.3. *Why bothering with sine and cosine?*

In answer, good question indeed, and you won't believe me, but when writing this book, at this point that we are now, I asked this myself too, and could not find any simple answer. So, I went into a tour of my Mathematics Department, here at Cergy, desperately asking colleagues about this, with some of them being actually world class geometers, but no one remembered the answer to this question either.

So, what do to. And here, you guessed it right, go back home at full speed, using various driving techniques that I learned as a youngster, in the Bucharest of the early 1990s, good times back then, and ask the cat. And cat looked at me, and declared:

Cat 5.4. *The area of an arbitrary triangle, having an angle $t$ at $A$,*

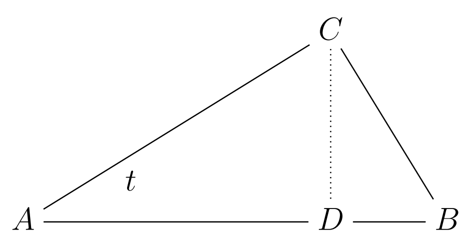



*is given by the following formula, making appear the sine:*

$$area(ABC)=\frac{AB\times AC\times\sin t}{2}$$

*As for the need for cosines, homework for you buddy.*

Thanks cat, quite interesting all this, so let us try to understand the above formula. But, in order to do so, the simplest is to draw an altitude of our triangle, as follows:




Indeed, with this altitude drawn, we have the following computation:

$$\begin{aligned}area(ABC)&=\frac{basis\times height}{2}\\&=\frac{AB\times CD}{2}\\&=\frac{AB\times AC\times\sin t}{2}\end{aligned}$$

Thus, formula proved, so the sine is definitely a good and useful thing, as cat says. As for the cosine, damn cat has assigned this to us as an exercise, so we will have to think about it, and come back to it, in due time. And no late homework, of course.

Moving forward now, still in relation with Cat 5.4, we have the following question:

QUESTION 5.5. *What happens to the cat formula,*

$$area(ABC) = \frac{AB \times AC \times \sin t}{2}$$

*when the angle at $A$ is obtuse, $t > 90°$?*

Which looks like a very good question. In answer now, given a triangle which is obtuse at $A$, we can simply rotate the $AC$ side to the right, as for that obtuse angle to become acute, $t' = 180° - t$, and the area of the triangle obviously remains the same, and this since both the basis and height remain unchanged. Thus, the correct definition for $\sin t$ for obtuse angles should be the one making the following formula work:

$$\frac{AB \times AC \times \sin t}{2} = \frac{AB \times AC \times \sin(180° - t)}{2}$$

Now by simplifying, we are led to the following formula:

$$\sin t = \sin(180° - t)$$

Thus, Question 5.5 answered, with our conclusions being as follows:

THEOREM 5.6. *We can talk about the sine of any angle $t \in [0°, 180°]$, according to*

$$\sin t = \sin(180° - t)$$

*and with this, the cat formula for the area of a triangle, namely*

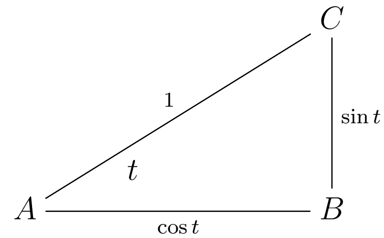
$$area(ABC) = \frac{AB \times AC \times \sin t}{2}$$

*holds for any triangle, without any assumption on it.*

PROOF. This follows indeed from the above discussion. □

Moving ahead now, defining sines as in Definition 5.1 for $t \in [0°, 90°]$, and as above for $t \in [90°, 180°]$ certainly does the job, as explained above, but is not very elegant. So, let us try to improve this. We have here the following obvious speculation:

SPECULATION 5.7. *The sine of any angle $t \in [0°, 180°]$ can be defined geometrically, according to the usual picture*




*with the convention that for $t > 90°$, the triangle is drawn at the left of $A$.*

Which sounds quite good, but when thinking some more, things fine of course with the sine, but what about the cosine? The problem indeed is that, in the case $t > 90°$, when the triangle is drawn at the left of $A$, the lower side $AB$ changes orientation:

$$AB \to BA$$

But, as we know well from chapter 1, from various considerations regarding segments and orientation, this would amount in saying that we are replacing:

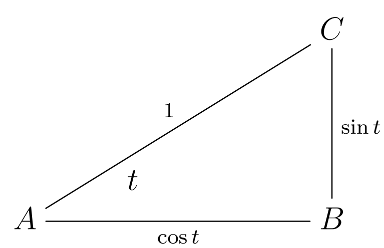
$$AB \to -AB$$

And so, we are led to the following formula for the cosine, in this case:

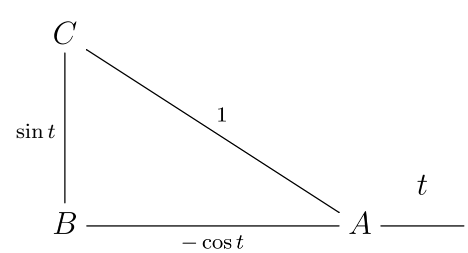
$$\cos t = -\cos(180° - t)$$

Very good all this, so let us update now Theorem 5.6, and by incorporating as well Speculation 5.7, in the form of a grand result, in the following way:

THEOREM 5.8 (update). *We can talk about the sine and cosine of any angle* $t \in [0°, 180°]$*, according to the following picture,*




*which in the case of obtuse angles becomes by definition as follows,*




*and with this, we have the following formulae, valid for any* $t \in [0°, 180°]$*:*

$$\sin t = \sin(180° - t) \quad , \quad \cos t = -\cos(180° - t)$$

*Moreover, the cat formula for the area of a triangle, namely*

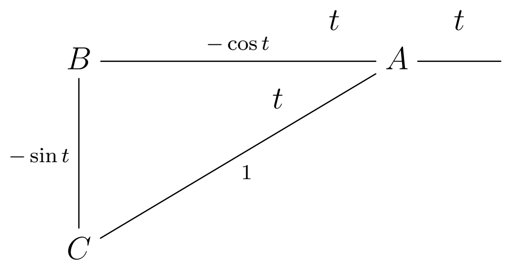
$$area(ABC) = \frac{AB \times AC \times \sin t}{2}$$

*holds for any triangle, without any assumption on it.*

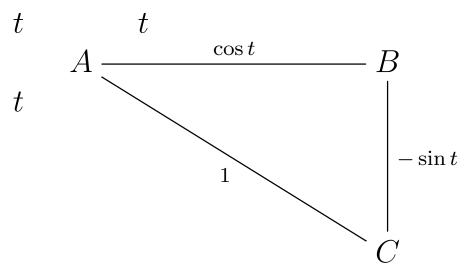
PROOF. This follows indeed by putting together all the above. □

Which sounds quite good, and normally end of the story, but let us be crazy now, and try to talk as well about the sine or cosine of angles $t < 0°$, or $t > 180°$.

Indeed, we know the recipe, namely suitably drawing our right triangle, with attention to positive and negatives. Thus, for $t \in [180°, 270°]$, our picture should be as follows:



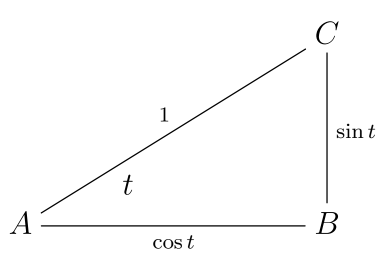

As for the next case, $t \in [270°, 360°]$, here our picture should be as follows:




But with this, we are done, because adding or substracting 360° to our angles won't change the corresponding right triangle, and so won't change the sine and cosine.

Hope you're still with me, after all these wild speculations. Good work that we did, and time now to further improve Theorem 5.8, into something really final, as follows:

THEOREM 5.9 (final update). *We can talk about the sine and cosine of any angle $t \in \mathbb{R}$, according to the following picture,*




*suitably drawn for angles $t < 0°$, or $t > 90°$, with attention to positive and negative lengths, as explained above. With this, all the basic formulae still hold, for any $t \in \mathbb{R}$.*

PROOF. This follows indeed by putting together all the above, and with the basic formulae in question being as follows, and in the hope that I forgot none:

$$\sin(90^\circ - t) = \cos t \quad , \quad \cos(90^\circ - t) = \sin t$$
$$\sin(90^\circ + t) = \cos t \quad , \quad \cos(90^\circ + t) = -\sin t$$

$$\sin(180^\circ - t) = \sin t \quad , \quad \cos(180^\circ - t) = -\cos t$$
$$\sin(180^\circ + t) = -\sin t \quad , \quad \cos(180^\circ + t) = -\cos t$$

$$\sin(270^\circ - t) = -\cos t \quad , \quad \cos(270^\circ - t) = -\sin t$$
$$\sin(270^\circ + t) = -\cos t \quad , \quad \cos(270^\circ + t) = \sin t$$

$$\sin(360^\circ - t) = -\sin t \quad , \quad \cos(360^\circ - t) = \cos t$$
$$\sin(360^\circ + t) = \sin t \quad , \quad \cos(360^\circ + t) = \cos t$$

Plus of course, not to forget about this, and thanks cat for meowing and reminding me this, the cat formula for the area of a triangle, which was as follows:

$$area(ABC) = \frac{AB \times AC \times \sin t}{2}$$

Here actually some discussion is needed, in relation with positives and negatives, and we will leave this as an instructive exercise for you, reader. □

## 5b. Pythagoras, again

In order to study now the sine and cosine, let us first update the numerics that we already have, for very simple angles in $[0^\circ, 90^\circ]$, to more angles, in $[0^\circ, 360^\circ]$. We have here the following statement, which is something straightforward:

THEOREM 5.10. *The sines of the basic angles are as follows,*

$$\sin 0^\circ = 0 \quad , \quad \sin 30^\circ = \frac{1}{2} \quad , \quad \sin 45^\circ = \frac{1}{\sqrt{2}} \quad , \quad \sin 60^\circ = \frac{\sqrt{3}}{2} \quad , \quad \sin 90^\circ = 1$$

$$\sin 120^\circ = \frac{\sqrt{3}}{2} \quad , \quad \sin 135^\circ = \frac{1}{\sqrt{2}} \quad , \quad \sin 150^\circ = \frac{1}{2} \quad , \quad \sin 180^\circ = 0$$

$$\sin 210^\circ = -\frac{1}{2} \quad , \quad \sin 225^\circ = -\frac{1}{\sqrt{2}} \quad , \quad \sin 240^\circ = -\frac{\sqrt{3}}{2} \quad , \quad \sin 270^\circ = -1$$

$$\sin 300^\circ = -\frac{\sqrt{3}}{2} \quad , \quad \sin 315^\circ = -\frac{1}{\sqrt{2}} \quad , \quad \sin 330^\circ = -\frac{1}{2} \quad , \quad \sin 360^\circ = 0$$

*with this coming from the basic geometry of right triangles.*

PROOF. This is indeed self-explanatory, with input coming from chapter 2. By the way, let us record as well the formulae of the corresponding cosines, as follows:

$$\cos 0^\circ = 1 \quad , \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad , \quad \cos 45^\circ = \frac{1}{\sqrt{2}} \quad , \quad \cos 60^\circ = \frac{1}{2} \quad , \quad \cos 90^\circ = 0$$

$$\cos 120^\circ = -\frac{1}{2} \quad , \quad \cos 135^\circ = -\frac{1}{\sqrt{2}} \quad , \quad \cos 150^\circ = -\frac{\sqrt{3}}{2} \quad , \quad \cos 180^\circ = -1$$

$$\cos 210^\circ = -\frac{\sqrt{3}}{2} \quad , \quad \cos 225^\circ = -\frac{1}{\sqrt{2}} \quad , \quad \cos 240^\circ = -\frac{1}{2} \quad , \quad \cos 270^\circ = 0$$

$$\cos 300^\circ = \frac{1}{2} \quad , \quad \cos 315^\circ = \frac{1}{\sqrt{2}} \quad , \quad \cos 330^\circ = \frac{\sqrt{3}}{2} \quad , \quad \cos 360^\circ = 1$$

We will be back to such things, with more angles, when knowing more things. □

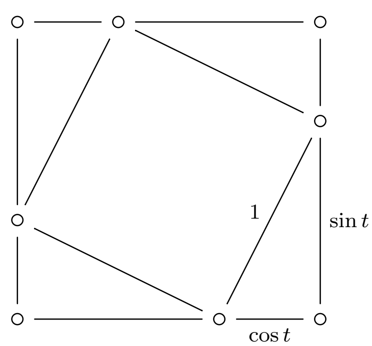
The problem is now, how to get beyond the above formulae? Not an easy question, but do not worry, we will be back to this, in due time. For the moment, as a complement to the above, let us record the following key formula, coming from Pythagoras:

THEOREM 5.11. *The sines and cosines are subject to the formula*

$$\sin^2 t + \cos^2 t = 1$$

*coming from Pythagoras' theorem.*

PROOF. This is something which is certainly true, but for pure mathematical pleasure, let us reproduce the picture leading to Pythagoras, in the trigonometric setting:



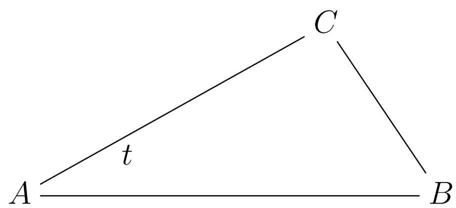

When computing the area of the outer square, we obtain:

$$(\sin t + \cos t)^2 = 1 + 4 \times \frac{\sin t \cos t}{2}$$

Now when expanding we obtain $\sin^2 t + \cos^2 t = 1$, as claimed. □

Next, with our knowledge of the sine and cosine, we can now formulate a technical generalization of the Pythagoras theorem, in the following way:

THEOREM 5.12. *Given an arbitrary triangle, as follows,*

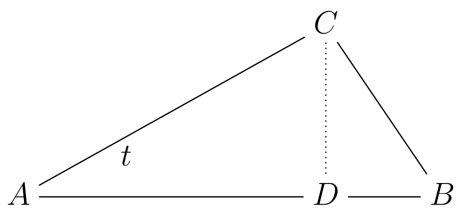



*the length of the side which is away from the vertex $A$ is given by the formula*

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos t$$

*called law of cosines, and with this generalizing Pythagoras.*

PROOF. Let us draw indeed an altitude of our triangle, as follows:




We have then the following computation, coming from Pythagoras, applied twice:

$$\begin{aligned} BC^2 &= CD^2 + BD^2 \\ &= CD^2 + (AB - AD)^2 \\ &= CD^2 + AB^2 + AD^2 - 2AB \cdot AD \\ &= AB^2 + AC^2 - 2AB \cdot AD \\ &= AB^2 + AC^2 - 2AB \cdot AC \cdot \cos t \end{aligned}$$

Finally, the last assertion is clear, because with $\cos t = 0$ we obtain Pythagoras. □

The above result looks quite interesting, for engineering purposes, and we have:

CONCLUSION 5.13. *The law of cosines found above can be effectively used for making money, by computing distances $BC$ over wild land, for various interested customers.*

Which might sound quite interesting, for us humans, but my cat, who is not into making money, seems unfazed. In fact, here is what he has to say, about this:

CAT 5.14. *That law of cosines is ugly, and no match for my law of sines:*

$$area(ABC) = \frac{AB \cdot AC \cdot \sin t}{2}$$

*I would suggest you humans to look into the quantity*

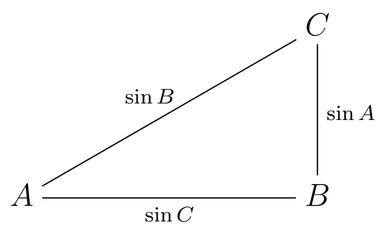
$$< AB, AC >= AB \cdot AC \cdot \cos t$$

*in order to understand what the cosines are good for. And change your diet, too.*

Quite interesting all this, but in practice, this $< AB, AC >$ quantity does not seem to be something very intuitive, at least to my human brain. We will leave this for later.

So, what to do, in this situation? Find some sort of better law of cosines, of course. And for this purpose, in the lack of any bright idea, I would just go to an arbitrary triangle $ABC$, a bit as in Theorem 5.12, and do some further computations there, for all sorts of lengths that can appear, in the hope that one of these computations leads to cosines.

With this idea in mind, let us first examine the right triangles. As a cheap trick here, our usual picture of a right triangle, with the big side having length 1, can be drawn as follows, with $A, B, C$ standing for the angles at the vertices $A, B, C$:



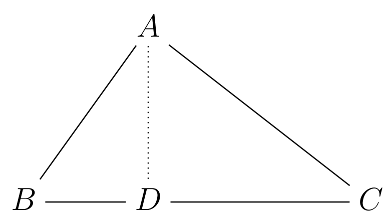

Thus, the lenghts of the sides of a right triangle are proportional to the sines of the opposite angles. Quite remarkably, the same happens in general:

THEOREM 5.15. *Given an arbitrary triangle $ABC$, we have:*

$$[BC - AC - AB] \sim [\sin A - \sin B - \sin C]$$

*That is, the lenghts of the sides are proportional to the sines of the opposite angles.*

PROOF. Let us draw indeed an altitude of our triangle, as follows:



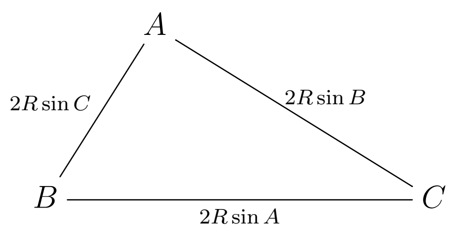

We have then the following computation, for the ratio $AB/AC$:

$$\frac{AB}{AC} = \frac{AD/\sin B}{AD/\sin C} = \frac{\sin C}{\sin B}$$

As for $AB/BC$ and $AC/BC$, these are given by similar formulae, again involving quotients of corresponding sines, and this leads to the conclusion in the statement. ☐

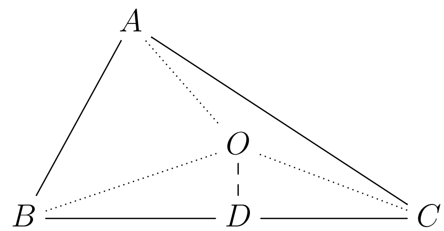
Getting now to our goals, in relation with Cat 5.4 and Cat 5.14, this is quite mixed news, because what we have in Theorem 5.15 is rather a new law of sines. Nevermind. So, now that we know this, let us compute as well the missing factor. We have here:

THEOREM 5.16. *The lengths of sides of an arbitrary triangle $ABC$ are given by*




*with $R$ being the radius of the circumscribed circle.*

PROOF. In order to prove this, let us draw a perpendicular bisector, as follows:




We have then 3 isosceles triangles appearing, say with angles $\alpha, \beta, \gamma$ at the point $O$, satisfying $\alpha+\beta+\gamma = 360°$. The other angles of these isosceles triangles, those coming in pairs, being $90° - \alpha/2, 90° - \beta/2, 90° - \gamma/2$, we conclude, by looking at what happens at each of the vertices of our triangle $ABC$, that we have the following formulae:

$$\alpha = 2A \quad , \quad \beta = 2B \quad , \quad \gamma = 2C$$

But with this we can compute the triangle edges. Indeed, we have:

$$BC = 2BD = 2BO \sin\left(\frac{\alpha}{2}\right) = 2R \sin A$$

Similarly, we have $AB = 2R\sin C$ and $AC = 2R\sin B$, as claimed. □

Very nice all this, we are learning new things, but the cosine problem remains open. As an idea for a solution, we can try to look instead at the lengths of the altitudes, with the rationale being that, these being orthogonal to the sides, the sines might get converted in this way to cosines. But, as bad news, this leads again to sines, as follows:

THEOREM 5.17. *The lengths of altitudes in an arbitrary triangle $ABC$ satisfy*

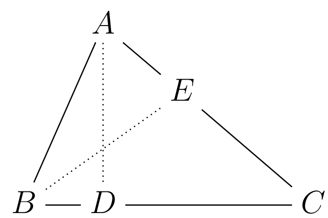
$$[AD - BE - CF] \sim \left[\frac{1}{\sin A} - \frac{1}{\sin B} - \frac{1}{\sin C}\right]$$

*and in fact we have the following formulae for these altitude lengths,*

$$AD = \frac{S}{R\sin A} \quad , \quad BE = \frac{S}{R\sin B} \quad , \quad CF = \frac{S}{R\sin C}$$

*with $R$ being the radius of the circumscribed circle, and $S$ being the area.*

PROOF. In order to prove the first assertion, let us draw two altitudes, as follows:




We have then the following computation for the ratio $AD/BE$, which along with similar formulae for $AD/CF$ and $BE/CF$ leads to the first assertion:

$$\frac{AD}{BE}=\frac{AB\sin B}{AB\sin A}=\frac{1/\sin A}{1/\sin B}$$

As for the second assertion, which reproves and fine-tunes the first assertion, this comes from Theorem 5.16. Indeed, the area of our triangle is given by:

$$S=\frac{AD\cdot BC}{2}=\frac{AD\cdot 2R\sin A}{2}=AD\cdot R\sin A$$

Thus we have $AD=S/(R\sin A)$, and the computation of $BE,CF$ is similar. □

Time perhaps to give up, with our quest for cosines? We have learned many interesting things on the way, no question about this, our conclusions being as follows:

CONCLUSION 5.18. *The sine is definitely a very interesting and useful quantity, for all sorts of questions. As for the true need for cosines, this remains an open question.*

And do not worry, we will come back to this. In fact, as a piece of advertisement for what will come later, when talking truly advanced mathematics, involving higher dimensions, quantum mechanics and so on, the $<AB,AC>=AB\cdot AC\cdot\cos t$ beast that cat was talking about is the useful, and in fact one and only, geometric tool. So, staying for the moment with the sine, but the cosine will strike back, and eventually win.

## 5c. Tangent, cotangent

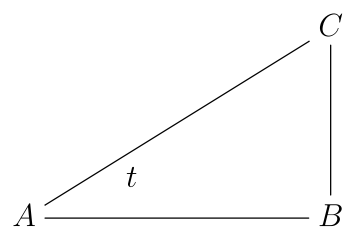
Back now to the basics, it is possible to say many more things about angles and $\sin t$, $\cos t$, and also talk about some supplementary quantities, such as the tangent:

DEFINITION 5.19. *We can talk about the tangent of angles $t\in\mathbb{R}$, as being given by*

$$\tan t=\frac{\sin t}{\cos t}$$

*with $\sin t,\cos t$ being defined as before.*

In more geometric terms, consider an arbitrary right triangle, as follows:




We have then the following computation, for the tangent of $t$:

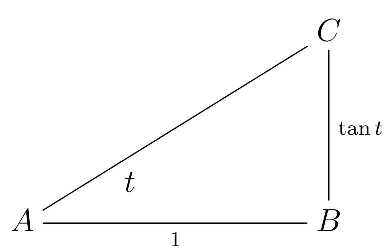
$$\tan t = \frac{\sin t}{\cos t} = \frac{BC}{AC}\Big/\frac{AB}{AC} = \frac{BC}{AB}$$

Thus, the tangent defined above complements the sine and cosine, because we have:

$$\sin t = \frac{BC}{AC} \quad , \quad \cos t = \frac{AB}{AC} \quad , \quad \tan t = \frac{BC}{AB}$$

A similar interpretation works for obtuse right triangles, and even for right triangles with an arbitrary angle $t \in \mathbb{R}$, and we can formulate, in the spirit of Theorem 5.9:

THEOREM 5.20. *We can talk, geometrically, about the tangent of any angle $t \in \mathbb{R}$, according to the following picture,*




*suitably drawn for angles $t < 0°$, or $t > 90°$, with attention to positive and negative lengths, as explained above. With this, all the basic formulae still hold, for any $t \in \mathbb{R}$.*

PROOF. Here the first assertion follows by reasoning as in the proof of Theorem 5.9, or simply follows from Theorem 5.9 itself. As for the second assertion, the basic formulae for the tangent, all coming from what we know, are as follows:

$$\tan(-t) = -\tan t$$

$$\tan(90° - t) = \frac{1}{\tan t} \quad , \quad \cos(90° + t) = -\frac{1}{\tan t}$$

$$\tan(180° - t) = -\tan t \quad , \quad \tan(180° + t) = \tan t$$

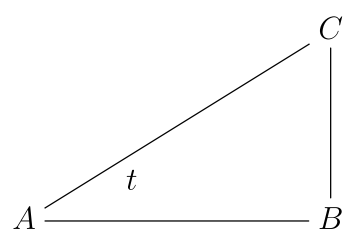
Let us record as well the formulae for the basic angles. These are as follows:

$$\tan 0^\circ = 0 \quad , \quad \tan 30^\circ = \frac{1}{\sqrt{3}} \quad , \quad \sin 45^\circ = 1 \quad , \quad \sin 60^\circ = \sqrt{3}$$

$$\tan 120^\circ = -\sqrt{3} \quad , \quad \tan 135^\circ = -1 \quad , \quad \tan 150^\circ = -\frac{1}{\sqrt{3}} \quad , \quad \tan 180^\circ = 0$$

Thus, we are led to the conclusions in the statement. □

Very nice all this, but are we really done with generalities and definitions? Not yet, because, let us go back to our basic right triangle, with an angle $t$, as follows:




We know from the above that we have the following formulae:

$$\sin t = \frac{BC}{AC} \quad , \quad \cos t = \frac{AB}{AC} \quad , \quad \tan t = \frac{BC}{AB}$$

However, there are still 3 fractions left, in need of a name, so let us formulate the following definition, completing what we already have, regarding sin, cos, tan:

Definition 5.21. *We can talk about the secant, cosecant and cotangent, as being*

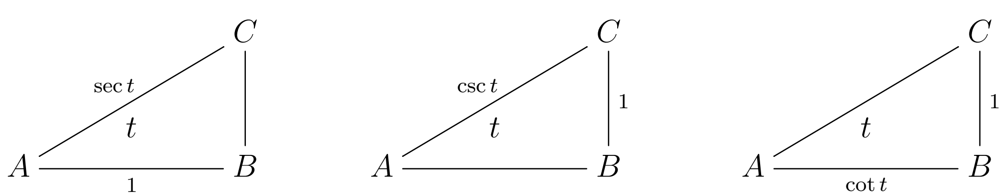
$$\sec t = \frac{AC}{AB} \quad , \quad \csc t = \frac{AC}{BC} \quad , \quad \cot t = \frac{BC}{AB}$$

*in the context of a right triangle, as above, or equivalently, as being*

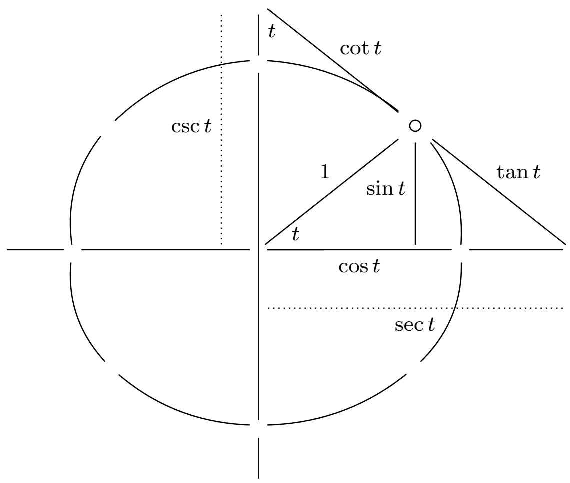
$$\sec t = \frac{1}{\cos t} \quad , \quad \csc t = \frac{1}{\sin t} \quad , \quad \cot t = \frac{1}{\tan t}$$

*in terms of the standard trigonometric functions* $\sin, \cos, \tan$.

In practice, the secant, cosecant and cotangent can be understood as well geometrically, by using right triangles $ABC$ as above, with a suitable side chosen to be 1:



In relation with this, we have as well the following catch-all picture, featuring a circle too, and justifying the use of the words "secant" and "cosecant" in the above:




We will be back to this configuration, with more about it, in chapter 7.

As a last piece of discussion, in relation with Concusion 5.18, we can still ask about the usefulness of our new functions, $\tan,\sec,\csc,\cot$. Wait and see here, with some applications coming next, and in the meantime, in relation with what we already have, we can reformulate Theorem 5.17 in a nicer way, in terms of cosecants, as follows:

THEOREM 5.22. *The lengths of altitudes in an arbitrary triangle $ABC$ satisfy*

$$[AD-BE-CF]\sim[\csc A-\csc B-\csc C]$$

*and in fact we have the following formulae for these altitude lengths,*

$$AD=\delta\csc A\quad,\quad BE=\delta\csc B\quad,\quad CF=\delta\csc C$$

*with $\delta=S/R$, where $S$ is the area, and $R$ is the radius of the circumscribed circle.*

PROOF. This is indeed a reformulation of what we have in Theorem 5.17, by using the function $\csc=1/\sin$, along with the above notation $\delta=S/R$. $\square$

## 5d. Back to centers

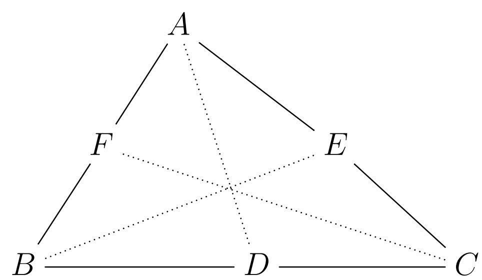
Remember the discussion following the Ceva theorem, from chapter 4? We had some unfinished business there, in what regards the applications, and we promised to get back to this, once we know some trigonometry. So, time to do this, and as a nice surprise, we get into something quite interesting, involving cosecants and cotangents, as follows:

THEOREM 5.23. *The barycenter, incenter and orthocenter theorems can be all deduced from the Ceva theorem, with the computations being respectively as follows,*

$$1 \times 1 \times 1 = 1$$
$$\frac{\csc A}{\csc B} \cdot \frac{\csc B}{\csc C} \cdot \frac{\csc C}{\csc A} = 1$$
$$\frac{\cot A}{\cot B} \cdot \frac{\cot B}{\cot C} \cdot \frac{\cot C}{\cot A} = 1$$

*with $A, B, C$ being the angles of our triangle.*

PROOF. Let us first recall from chapter 3 that the Ceva theorem concerns a configuration as follows, with a triangle $ABC$ containing inner lines $AD, BE, CF$:




The theorem states that $AD, BE, CF$ cross precisely when the following happens:

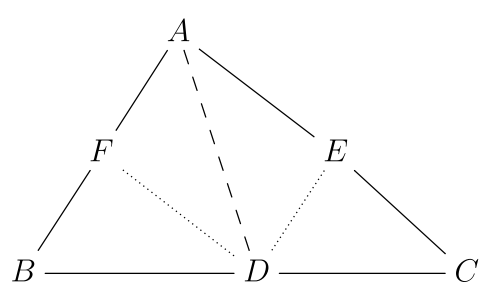
$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$$

Regarding now the barycenter, incenter and orthocenter, the situation is as follows:

(1) In what regards the barycenter, the computation is trivial, as follows:

$$1 \times 1 \times 1 = 1$$

(2) In order to deal now with the incenter, consider indeed a triangle, with an angle bisector drawn, and with two perpendiculars drawn as well, as indicated:



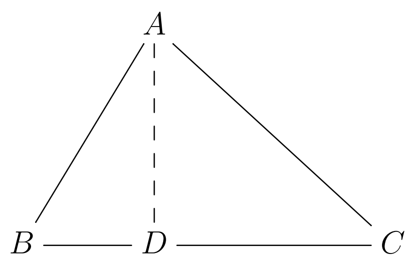

We have then the following computation, using $FD = DE$:

$$\frac{BD}{DC} = \frac{FD \csc B}{DE \csc C} = \frac{\csc B}{\csc C}$$

We conclude that Ceva gives indeed the incenter, with the computation being:

$$\frac{\csc A}{\csc B}\cdot\frac{\csc B}{\csc C}\cdot\frac{\csc C}{\csc A}=1$$

(3) Finally, in order to deal now with the orthocenter, a bit in a similar way, consider indeed a triangle, with an altitude drawn, as follows:




We have then the following computation, coming from definitions:

$$\frac{BD}{DC}=\frac{BD}{AD}\Big/\frac{DC}{AD}=\frac{\cot B}{\cot C}$$

Thus Ceva gives as well the orthocenter, with the computation being as follows:

$$\frac{\cot A}{\cot B}\cdot\frac{\cot B}{\cot C}\cdot\frac{\cot C}{\cot A}=1$$

And with this being something nice, remember the mess with the orthocenter when first proving the theorem, with that trick involved. Gone all that. □

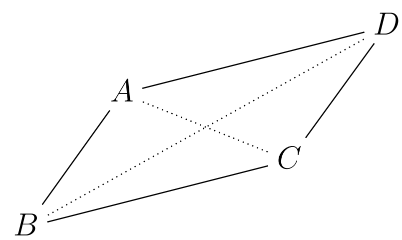
Before going further, with a similar study of the circumcenter, and as a matter of having everything regarding the barycenter, incenter and orthocenter understood, let us do some more computations. The situation here is as follows:

(1) In what regards orthocenter configuration, involving the triangle altitudes, we already have formulae for everything, coming from the right angles appearing in this configuration, which are certainly an invitation to trigonometry.

(2) Regarding the incenter configuration, we have already talked about it in chapter 3, when talking about the Feuerbach points. There are of course many other computations that can be done, using trigonometry, but for now, what we have here will basically do.

Thus, we are left with doing some more computations for the barycenter configuration, involving the triangle medians. And here, we have the following remarkable result:

**Theorem 5.24.** *Given an arbitrary parallelogram $ABCD$,*




*its sides and diagonals are related by the following formula, called parallelogram law,*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2$$

*and this can be used, in the obvious way, in order to compute the triangle medians.*

Proof. There are several things going on here, the idea being as follows:

(1) In the case of a rectangle the parallelogram law is Pythagoras' theorem, and this suggests using the natural generalization of Pythagoras' theorem, which is the law of cosines from Theorem 5.12. Indeed, with $O$ being the middle point of the parallelogram, and with $s, t$ being the angles there of the triangles $OAB$ and $OBC$, we have:

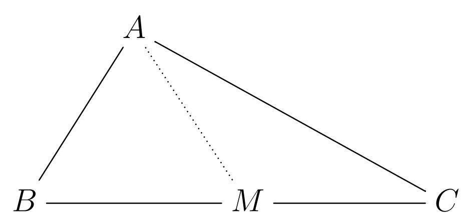
$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cdot \cos s$$

$$BC^2 = OB^2 + OC^2 - 2OB \cdot OC \cdot \cos t$$

But $OA = OC$, and $\cos s = -\cos t$, due to $s + t = 180^\circ$, so by summing we get the following formula, which is exactly the parallelogram law, divided by 2:

$$AB^2 + BC^2 = 2OA^2 + 2OB^2$$

(2) Regarding now the medians, consider a triangle $ABC$, with a median drawn:




By completing to a parallelogram, and using the parallelogram law, we obtain:

$$2AB^2 + 2AC^2 = 4AM^2 + BC^2$$

Thus, we are led to the following formula, for the length of the median:

$$AM = \sqrt{\frac{2AB^2 + 2AC^2 - BC^2}{4}}$$

(3) Finally, as a philosophical comment, the parallelogram law looks as a highly refined version of the law of cosines. Unfortunately, the cosines are gone. Damn. □

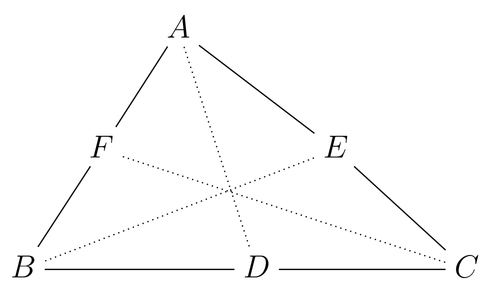
Back now to our regular business in this section, applications of the Ceva theorem, regarding the circumcenter, we have the following result about it:

THEOREM 5.25. *The circumcenter theorem can be deduced too from Ceva, via*

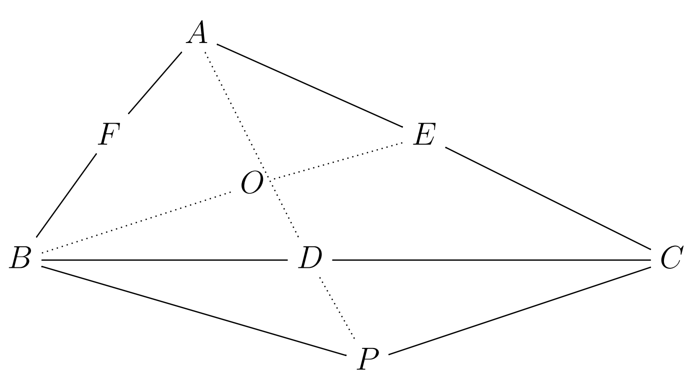
$$\frac{\csc 2A}{\csc 2B}\cdot\frac{\csc 2B}{\csc 2C}\cdot\frac{\csc 2C}{\csc 2A}=1$$

*with $A,B,C$ being the angles of our triangle.*

PROOF. Consider indeed the configuration associated to the circumcenter:




(1) In order to compute the ratio $BD/DC$, let us prolong $AD$ until it meets the outer circle, in a point $P$. Now if we look at the trapezoid $ABPC$, cut by its diagonals $AP$ and $BC$, all 8 angles which appear at vertices equal $B,C$ or $90^\circ-B,90^\circ-C$:




(2) Getting now to the ratio $BD/DC$, if we project the points $B,C$ on the line $AP$, into points $X,Y$, then we have the following computation, coming from similar triangles

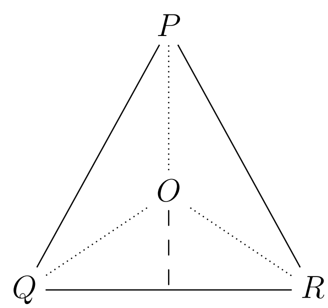
appearing via the observations in (1), and with $R$ being the radius of the outer circle:

$$\begin{aligned}\frac{BD}{DC} &= \frac{BX}{CY}\\ &= \frac{AB\cdot\cos C}{AC\cdot\cos B}\\ &= \frac{2R\sin C\cdot\cos C}{2R\sin B\cdot\cos B}\\ &= \frac{\sin C\cdot\cos C}{\sin B\cdot\cos B}\end{aligned}$$

(3) Now the computation for $AF/FB$ and $CE/EA$ being similar, we conclude that the circumcenter theorem can be indeed deduced from Ceva, via:

$$\frac{\sec A\csc A}{\sec B\csc B}\cdot\frac{\sec B\csc B}{\sec C\csc C}\cdot\frac{\sec C\csc C}{\sec A\csc A}=1$$

(4) However, we can do better. Consider indeed the following configuration, on a circle having radius 1, with the triangle $PQR$ being taken isosceles:




Now let us compute the area of $QOR$. On one hand, since the angle $QOR$ equals $2P$, this area is $\sin 2P/2$. On the other hand, by cutting the triangle $QOR$ into two halves, as indicated above, the area follows to be $2\times\sin P\cos P/2$. Thus, we have:

$$\sin 2P = 2\sin P\cos P$$

(5) But with this, we can go back to the computation in (2), and we get:

$$\frac{BD}{DC}=\frac{\sin 2C}{\sin 2B}$$

Thus, we have our final trigonometry formula for $BD/DC$, and the computation for $AF/FB$ and $CE/EA$ being similar, we are led to the conclusion in the statement. □

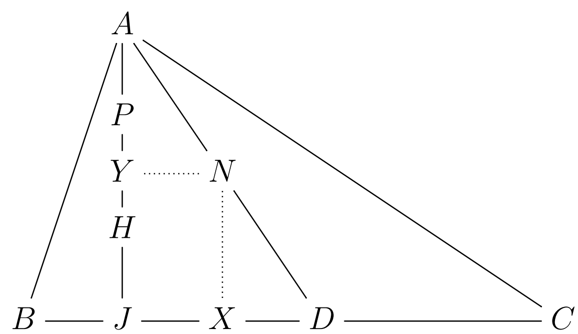
Getting now to the center of the nine-point circle, this is of different nature, involving more complicated trigonometric quantities, the result here being as follows:

THEOREM 5.26. *The center of the nine-point circle comes from Ceva via*

$$\left(\frac{\csc A}{\csc B}\cdot\frac{\cos(C-A)}{\cos(C-B)}\right)\left(\frac{\csc B}{\csc C}\cdot\frac{\cos(A-B)}{\cos(A-C)}\right)\left(\frac{\csc C}{\csc A}\cdot\frac{\cos(B-C)}{\cos(B-A)}\right)=1$$

*with $A,B,C$ being the angles of our triangle.*

PROOF. We have the following configuration, with the altitude $AJ$ drawn, $H$ being the orthocenter, $P$ being the middle of $AH$, and $Y$ being the middle of $PJ$:




By using the various ratios on the altitude, we have the following computation:

$$NX=YJ=\frac{PJ}{2}=\frac{AJ+HJ}{2}$$

Regarding now the components appearing on the right, we first have:

$$AJ=AB\sin B=2R\sin B\sin C$$

As for the other component, this can be computed by drawing the altitude $BK$ and using the similarity of triangles $BHJ\sim BCK$, which gives the following formula:

$$HJ=\frac{BJ\cdot KC}{BK}=\frac{AB\cos B\cdot 2R\sin A\cos C}{AB\sin A}=2R\cos B\cos C$$

As a conclusion, the distance from $N$ to the side $BC$ is given by:

$$NX=R(\cos B\cos C+\sin B\sin C)$$

But now we can compute the ratio $BD/DC$, by using areas, and we get:

$$\frac{BD}{DC}=\frac{area(ABN)}{area(ACN)}=\frac{\sin C}{\sin B}\cdot\frac{\cos A\cos B+\sin A\sin B}{\cos A\cos C+\sin A\sin C}$$

But the quantities on the right are respectively $\cos(A-B)$ and $\cos(A-C)$, as we will soon learn, in the next chapter, so we are led to the conclusion in the statement. □

Finally, regarding the Gergonne and Nagel points, the Ceva type formulae follow from what we have in chapter 4, and are again a bit complicated, as follows:

THEOREM 5.27. *The Gergonne point comes from Ceva via*

$$\frac{sinB+\sin C-\sin A}{\sin A+\sin C-\sin B}\cdot\frac{\sin A+\sin C-\sin B}{\sin A+\sin B-\sin C}\cdot\frac{\sin A+\sin B-\sin C}{\sin B+\sin C-\sin A}=1$$

*and the same holds for the Nagel point, with all fractions inverted.*

PROOF. As explained in chapter 4, the Gergonne point comes indeed, and in fact by definition, from Ceva, according to the following computation:

$$\frac{b+c-a}{a+c-b}\cdot\frac{a+c-b}{a+b-c}\cdot\frac{a+b-c}{b+c-a}=1$$

Now since the triangle sides $a,b,c$ are proportional to $\sin A,\sin B,\sin C$, this leads to the formula in the statement. As for the Nagel point, here the computation was:

$$\frac{a+c-b}{b+c-a}\cdot\frac{a+b-c}{a+c-b}\cdot\frac{b+c-a}{a+b-c}=1$$

Thus, we are again led to the conclusion in the statement. □

Very nice all this, we have now some trigonometric intuition on the various triangle centers constructed in Part I. Actually, in relation with this, we still have the Feuerbach point $Z$ to be discussed. We will do this later, in Part III, using more advanced tools.

## 5e. Exercises

Welcome to trigonometry, good to have you here, and as exercises, we have:

EXERCISE 5.28. *Meditate some more on the need for the sine.*

EXERCISE 5.29. *Meditate also some more on the need for the cosine.*

EXERCISE 5.30. *And meditate too on the need for the tangent.*

EXERCISE 5.31. *Learn more about the secant, and its interpretations.*

EXERCISE 5.32. *Learn more about the cosecant, and its interpretations.*

EXERCISE 5.33. *Learn also more about the cotangent, and its interpretations.*

EXERCISE 5.34. *Write a short essay on* sin *and* cos*, using directly angles* $t\in\mathbb{R}$.

EXERCISE 5.35. *Experiment more with Ceva, using other trigonometric functions.*

As bonus exercise, reiterated, meditate a bit more on the numeric angles.

CHAPTER 6

# Sums, duplication

## 6a. Sums of angles

Getting back now to the basics, sine, cosine and tangent, how these can be computed, and what can be done with them, we have the following key result:

THEOREM 6.1. *The sines and cosines of sums are given by*

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

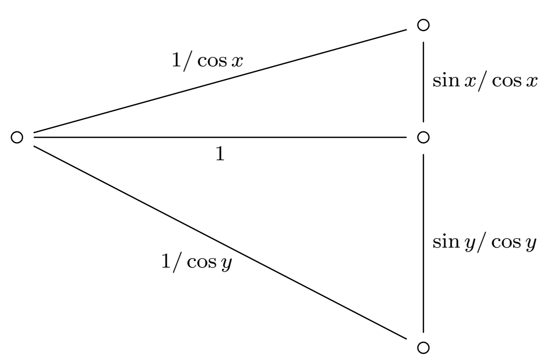
*and these formulae give a formula for the tangent too, namely*

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

*provided of course that the denominator is nonzero.*

PROOF. This is something quite tricky, using the same idea as in the proof of Pythagoras' theorem, that is, computing certain areas, the idea being as follows:

(1) Let us first establish the formula for the sines. In order to do so, consider the following picture, consisting of a length 1 line segment, with angles $x, y$ drawn on each side, and with everything being completed, and lengths computed, as indicated:




Now let us compute the area of the big triangle, or rather the double of that area. We can do this in two ways, either directly, with a formula involving $\sin(x+y)$, or by using the two small triangles, involving functions of $x, y$. We obtain in this way:

$$\frac{1}{\cos x}\cdot\frac{1}{\cos y}\cdot\sin(x+y)=\frac{\sin x}{\cos x}\cdot 1+\frac{\sin y}{\cos y}\cdot 1$$

But this gives the formula for $\sin(x+y)$ from the statement.

(2) Moving ahead, no need of new tricks for cosines, because by using the formula for $\sin(x+y)$ we can deduce a formula for $\cos(x+y)$, as follows:

$$\begin{aligned}\cos(x+y)&=\sin\left(\frac{\pi}{2}-x-y\right)\\&=\sin\left[\left(\frac{\pi}{2}-x\right)+(-y)\right]\\&=\sin\left(\frac{\pi}{2}-x\right)\cos(-y)+\cos\left(\frac{\pi}{2}-x\right)\sin(-y)\\&=\cos x\cos y-\sin x\sin y\end{aligned}$$

(3) Finally, in what regards the tangents, we have, according to the above:

$$\begin{aligned}\tan(x+y)&=\frac{\sin x\cos y+\cos x\sin y}{\cos x\cos y-\sin x\sin y}\\&=\frac{\sin x\cos y/\cos x\cos y+\cos x\sin y/\cos x\cos y}{1-\sin x\sin y/\cos x\cos y}\\&=\frac{\tan x+\tan y}{1-\tan x\tan y}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Let us record as well the formulae for the secondary trigonometric functions:

Proposition 6.2. *The secants and cosecants of sums are given by*

$$\sec(x+y)=\frac{\sec x\sec y}{1-\tan x\tan y}$$

$$\csc(x+y)=\frac{\csc x\csc y}{\cot x+\cot y}$$

*and we have a formula for the cotangent too, namely*

$$\cot(x+y)=\frac{\cot x\cot y-1}{\cot x+\cot y}$$

*provided of course that the denominator is nonzero.*

Proof. This comes from the formulae in Theorem 6.1, as follows:

(1) In what regards the secant, we have the following computation:

$$\begin{aligned}
\sec(x+y) &= \frac{1}{\cos(x+y)}\\
&= \frac{1}{\cos x\cos y-\sin x\sin y}\\
&= \frac{\sec x\sec y}{1-\sin x\sec x\sin y\sec y}\\
&= \frac{\sec x\sec y}{1-\tan x\tan y}
\end{aligned}$$

(2) In what regards the cosecant, we have the following computation:

$$\begin{aligned}
\csc(x+y) &= \frac{1}{\sin(x+y)}\\
&= \frac{1}{\sin x\cos y+\cos x\sin y}\\
&= \frac{\csc x\csc y}{\csc y\cos y+\csc x\cos x}\\
&= \frac{\csc x\csc y}{\cot x+\cot y}
\end{aligned}$$

(3) In what regards the cotangent, we have the following computation:

$$\begin{aligned}
\cot(x+y) &= \frac{1}{\tan(x+y)}\\
&= \frac{1-\tan x\tan y}{\tan x+\tan y}\\
&= \frac{\cot x\cot y-1}{\cot x+\cot y}
\end{aligned}$$

Thus, we are led to the formulae in the statement. □

Getting back to Theorem 6.1 as stated, let us record as well what happens when replacing sums by substractions. The formulae here are as follows:

Theorem 6.3. *The sines and cosines of differences are given by*

$$\sin(x-y)=\sin x\cos y-\cos x\sin y$$

$$\cos(x-y)=\cos x\cos y+\sin x\sin y$$

*and these formulae give a formula for the tangent too, namely*

$$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

*provided of course that the denominator is nonzero.*

PROOF. These are all consequences of what we have in Theorem 6.1, as follows:

(1) Regarding the sine, we have here the following computation:

$$\begin{aligned}\sin(x-y) &= \sin x\cos(-y)+\cos x\sin(-y)\\ &= \sin x\cos y-\cos x\sin y\end{aligned}$$

(2) Regarding the cosine, the computation here is similar, as follows:

$$\begin{aligned}\cos(x-y) &= \cos x\cos(-y)-\sin x\sin(-y)\\ &= \cos x\cos y+\sin x\sin y\end{aligned}$$

(3) Finally, for the tangent we have a similar computation, as follows:

$$\begin{aligned}\tan(x-y) &= \frac{\tan x+\tan(-y)}{1-\tan x\tan(-y)}\\ &= \frac{\tan x-\tan y}{1+\tan x\tan y}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Let us record as well the formulae for the secondary trigonometric functions:

PROPOSITION 6.4. *The secants and cosecants of differences are given by*

$$\sec(x-y)=\frac{\sec x\sec y}{1+\tan x\tan y}$$

$$\csc(x-y)=\frac{\csc x\csc y}{\cot y-\cot x}$$

*and we have a formula for the cotangent too, namely*

$$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}$$

*provided of course that the denominator is nonzero.*

PROOF. These are all consequences of Proposition 6.2, as follows:

(1) Regarding the secant, we have here the following computation:

$$\begin{aligned}\sec(x-y) &= \frac{\sec x\sec(-y)}{1-\tan x\tan(-y)}\\ &= \frac{\sec x\sec y}{1+\tan x\tan y}\end{aligned}$$

(2) Regarding the cosecant, the computation here is similar, as follows:

$$\begin{aligned}\csc(x-y) &= \frac{\csc x\csc(-y)}{\cot x+\cot(-y)}\\ &= -\frac{\csc x\csc y}{\cot x-\cot y}\\ &= \frac{\csc x\csc y}{\cot y-\cot x}\end{aligned}$$

(3) Finally, for the cotangent we have a similar computation, as follows:

$$\begin{aligned}\cot(x-y) &= \frac{\cot x\cot(-y)-1}{\cot x+\cot(-y)}\\ &= \frac{-\cot x\cot y-1}{\cot x-\cot y}\\ &= \frac{\cot x\cot y+1}{\cot y-\cot x}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As illustrations for the above formulae, we can now compute the sine, cosine and tangent of various interesting new angles, appearing as sums and differences, such as:

$$15^\circ = 45^\circ - 30^\circ \quad , \quad 75^\circ = 45^\circ + 30^\circ$$

In fact, thinking well, this is pretty much it, modulo periodicity formulae. So, all in all, with our formulae for sums, we can deal now with all multiples of $15^\circ$.

Let us record our result here, dealing with the main functions, as follows:

THEOREM 6.5. *The sine, cosine and tangent of multiples of* $15^\circ$ *are given by*

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}\ ,\ \sin 30^\circ = \frac{1}{2}\ ,\ \sin 45^\circ = \frac{1}{\sqrt{2}}\ ,\ \sin 60^\circ = \frac{\sqrt{3}}{2}\ ,\ \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}\ ,\ \cos 30^\circ = \frac{\sqrt{3}}{2}\ ,\ \cos 45^\circ = \frac{1}{\sqrt{2}}\ ,\ \cos 60^\circ = \frac{1}{2}\ ,\ \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1}\ ,\ \tan 30^\circ = \frac{1}{\sqrt{3}}\ ,\ \tan 45^\circ = 1\ ,\ \tan 60^\circ = \sqrt{3}\ ,\ \tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

*plus various periodicity formulae.*

Proof. For the quantity $\sin 15^\circ = \cos 75^\circ$, we have the following computation:

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

Also, for the quantity $\cos 15^\circ = \sin 75^\circ$, we have the following computation:

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

As for the other formulae in the statement, these all follow from this. □

For completness, let us record as well the result for the secondary functions:

Theorem 6.6. *The secant, cosecant and cotangent of multiples of* $15^\circ$ *are*

$$\sec 15^\circ = \frac{2\sqrt{2}}{\sqrt{3}+1}\ ,\ \sec 30^\circ = \frac{2}{\sqrt{3}}\ ,\ \sec 45^\circ = \sqrt{2}\ ,\ \sec 60^\circ = 2\ ,\ \sec 75^\circ = \frac{2\sqrt{2}}{\sqrt{3}-1}$$

$$\csc 15^\circ = \frac{2\sqrt{2}}{\sqrt{3}-1}\ ,\ \csc 30^\circ = 2\ ,\ \csc 45^\circ = \sqrt{2}\ ,\ \csc 60^\circ = \frac{2}{\sqrt{3}}\ ,\ \csc 75^\circ = \frac{2\sqrt{2}}{\sqrt{3}+1}$$

$$\cot 15^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1}\ ,\ \cot 30^\circ = \sqrt{3}\ ,\ \cot 45^\circ = 1\ ,\ \cot 60^\circ = \frac{1}{\sqrt{3}}\ ,\ \cot 75^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

*plus various periodicity formulae.*

Proof. This follows indeed by inverting the various fractions in Theorem 6.5. □

As a conclusion to all this, not a big deal, you would say, but wait for it. We will see in what comes next that, with some halving tricks helping, our general formulae for sums can be used in order to compute the trigonometric functions of nearly all angles.

## 6b. Duplication

Time now for more advanced trigonometry. Indeed, by taking $x = y$ in Theorem 6.1 we obtain some interesting formulae for the duplication of angles, as follows:

THEOREM 6.7. *The sines of the doubles of angles are given by*

$$\sin(2t) = 2\sin t\cos t$$

*and the corresponding cosines are given by the following equivalent formulae,*

$$\begin{aligned}\cos(2t) &= \cos^2 t - \sin^2 t\\ &= 2\cos^2 t - 1\\ &= 1 - 2\sin^2 t\end{aligned}$$

*with all these three formulae being useful, in practice.*

PROOF. This is something very standard, the idea being as follows:

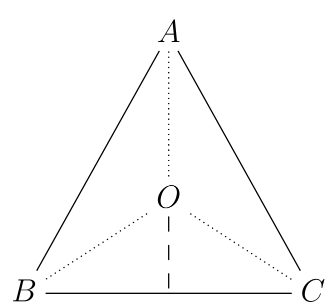
(1) By taking $x = y = t$ in the formulae from Theorem 6.1, we obtain:

$$\sin(2t) = 2\sin t\cos t$$

$$\cos(2t) = \cos^2 t - \sin^2 t$$

As for the extra formulae for $\cos(2t)$, these follow by using $\cos^2 + \sin^2 = 1$.

(2) Alternatively, and as already explained in fact in chapter 5, where we first met the formula for the sine in the statement, consider the following very familiar configuration, lying on a circle, with the triangle in question being taken isosceles:




Let us assume as well that the circle radius is 1, and let us compute the area of $BOC$. On one hand, by using the fact that the angle $BOC$ equals $2A$, this area follows to be $\sin 2A/2$. On the other hand, by cutting this triangle $BOC$ into two halves, as indicated above, the area follows to be $2 \times \sin A\cos A/2$. Thus, we have the following formula:

$$\sin 2A = 2\sin A\cos A$$

(3) Finally, regarding the cosine, by using this and Pythagoras twice, we have:

$$\begin{aligned}
\cos 2A &= \sqrt{1-\sin^2 2A}\\
&= \sqrt{1-4\sin^2 A\cos^2 A}\\
&= \sqrt{1-4\sin^2 A\cos^2 A}\\
&= \sqrt{(\cos^2 A+\sin^2 A)^2-4\sin^2 A\cos^2 A}\\
&= \sqrt{\cos^4 A+\sin^4 A-2\sin^2 A\cos^2 A}\\
&= \sqrt{(\cos^2 A-\sin^2 A)^2}\\
&= \cos^2 A-\sin^2 A
\end{aligned}$$

To be more precise, this computation holds indeed, with some discussion needed at the end, when extracting the square root. Alternatively, we can deduce this formula by using the configuration from (2), and we will leave this as an instructive exercise.

(4) As for the other formulae for the cosine in the statement, these follow from this, and from Pythagoras, as already mentioned in (1). □

Let us record as well the formula for the tangents, which is as follows:

THEOREM 6.8. *The tangents of the doubles of angles are given by*

$$\tan(2t)=\frac{2\tan t}{1-\tan^2 t}$$

*provided as usual that the denominator is nonzero.*

PROOF. This follows indeed by taking $x=y=t$ in the formula for tangents from Theorem 6.1. Equivalently, you can check, as an easy, instructive exercise, that this is indeed what we get, by dividing the sine and cosine computed in Theorem 6.7. □

The point now is that, with this, we can substantially improve our data from Theorem 6.5, by computing the cosines of the halves of the angles there, using the above formula for $\cos(2t)$, and then computing the sines of these angles too, by using Pythagoras, and finally by computing the tangents too, as quotients. As a result here, let us record:

THEOREM 6.9. *The sine, cosine and tangent of* $22.5°$ *are given by*

$$\sin 22.5^\circ=\frac{\sqrt{2-\sqrt{2}}}{2}\quad,\quad \cos 22.5^\circ=\frac{\sqrt{2+\sqrt{2}}}{2}\quad,\quad \tan 22.5^\circ=\sqrt{2}-1$$

*and for the odd multiples of* $22.5°$*, we have similar formulae.*

PROOF. For the cosine we can use $\cos(2t) = 2\cos^2 t - 1$, and we obtain:

$$\begin{aligned}\cos 22.5^\circ &= \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}\\ &= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}\\ &= \frac{\sqrt{2+\sqrt{2}}}{2}\end{aligned}$$

For the sine we can use Pythagoras, $\sin^2 + \cos^2 = 1$, and we obtain:

$$\begin{aligned}\sin 22.5^\circ &= \sqrt{1-\cos^2 22.5^\circ}\\ &= \sqrt{1-\frac{2+\sqrt{2}}{4}}\\ &= \frac{\sqrt{2-\sqrt{2}}}{2}\end{aligned}$$

Finally, by taking the quotient we obtain a formula for the tangent, as follows:

$$\begin{aligned}\tan 22.5^\circ &= \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\\ &= \sqrt{\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}}\\ &= \frac{2-\sqrt{2}}{\sqrt{2}}\\ &= \sqrt{2}-1\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Along the same lines, at a more advanced level, we have as well:

THEOREM 6.10. *The sine, cosine and tangent of* $7.5^\circ$ *are given by*

$$\sin 7.5^\circ = \sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{8}}\ ,\ \cos 7.5^\circ = \sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{8}}\ ,\ \tan 7.5^\circ = \sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}+\sqrt{6}}}$$

*and for the odd multiples of* $7.5^\circ$, *we have similar formulae.*

Proof. For the cosine we can use $\cos(2t) = 2\cos^2 t - 1$, and we obtain:

$$\begin{aligned}
\cos 7.5^\circ &= \sqrt{\frac{1+\cos 15^\circ}{2}} \\
&= \sqrt{\frac{1+\frac{1+\sqrt{3}}{2\sqrt{2}}}{2}} \\
&= \sqrt{\frac{2\sqrt{2}+1+\sqrt{3}}{4\sqrt{2}}} \\
&= \sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{8}}
\end{aligned}$$

For the sine we can use Pythagoras, $\sin^2 + \cos^2 = 1$, and we obtain:

$$\begin{aligned}
\sin 7.5^\circ &= \sqrt{1-\cos^2 7.5^\circ} \\
&= \sqrt{1-\frac{4+\sqrt{2}+\sqrt{6}}{8}} \\
&= \sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{8}}
\end{aligned}$$

Finally, by taking the quotient we obtain the formula for the tangent. As for the last assertion, it is clear that the same method will work for all multiples of 7.5°, with input from Theorem 6.5, and we will leave the computations here as an instructive exercise. □

As a conclusion to all this, we have quite mixed news, as follows:

(1) On one hand the formulae in Theorem 6.7 are definitely something powerful, allowing us in theory to indefinitely halve the angles that we know, and so to virtually obtain, via some limits if needed, all the sines and cosines in this world.

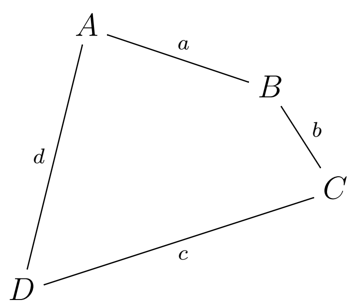
(2) On the other hand, in practice, all this leads us into the question of extracting square roots, which rather belongs to arithmetic. So, all in all, not that much of a total kill, Theorem 6.7 transferring our questions, from trigonometry to arithmetic.

As an application now of the new formulae that we learned, let us go back to the Brahmagupta formula for the area of a cyclic quadrilateral from chapter 3, namely:

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

We have the following generalization of this formula, due to Bretschneider:

THEOREM 6.11 (Bretschneider). *The area of an arbitrary quadrilateral*




*having sides $a,b,c,d$ is given by the following formula,*

$$S=\sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2 t}$$

*with $s=(a+b+c+d)/2$ being the semiperimeter, and $t=(A+C)/2$.*

PROOF. This is something quite tricky, the idea being as follows:

(1) As a first observation, in relation with the apparent lack of symmetry of the formula, we have $A+B+C+D=360^\circ$, so the angles $t=(A+C)/2$ and $s=(B+D)/2$ are related by $s+t=180^\circ$, and so they have the same squared cosines:

$$s+t=180^\circ \implies \cos s=-\cos t \implies \cos^2 s=\cos^2 t$$

Thus, there is in fact no lack of symmetry in the formula. Good.

(2) As a second observation, this generalizes the Brahmagupta formula. Indeed, for a cyclic quadrilateral we have $A+C=B+D=180^\circ$, so $t=90^\circ$, and we get:

$$S=\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

(3) As a third observation, the Bretschneider formula gives the following estimate, which shows that the area of a quadrilateral having sides $a,b,c,d$ is maximized when the quadrilateral is cyclic, confirming a conjecture that we made in chapter 3:

$$S\leq\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

(4) Getting now to the proof of the formula, and coming as bad news, what we did in chapter 3, namely Heron formula, generalized into the Brahmagupta formula, does not help much in relation with our question. So, we will have to redo everything, and with the proof below providing of course a new proof for Heron, and Brahmagupta.

(5) Time to get to work. By using the law of sines twice, we have:

$$S=\frac{ad\sin A}{2}+\frac{bc\sin C}{2}$$

With Brahmagupta in mind, suggesting the use of $S^2$, let us write this as:

$$4S^2=(ad)^2\sin^2 A+(bc)^2\sin^2 C+2abcd\sin A\sin C$$

(6) Next comes the trick. We obviously have to some trigonometry here, in order to get rid of the angles $A, C$, or maybe just replace them by a single one, say $t = (A+C)/2$. And after trying a million things here, as Bretschneider, and probably people like Heron and Brahmagupta too, most likely did, we are led to the following lucky formula, coming from the law of cosines applied twice, to the triangles $ADB$ and $CDB$:

$$a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C$$

(7) Indeed, let us write this latter formula in the following way:

$$a^2 + d^2 - b^2 - c^2 = 2(ad\cos A - bc\cos C)$$

By squaring, we obtain from this the following formula:

$$\frac{(a^2 + d^2 - b^2 - c^2)^2}{4} = (ad)^2\cos^2 A + (bc)^2\cos^2 C - 2abcd\cos A\cos C$$

(8) But this latter formula is very similar to what we have in (5), and does the simplification job for that formula in (5). Indeed, by summing our two formulae, and by using the various trigonometry rules that we learned in this chapter, we obtain:

$$\begin{aligned}
4S^2 + \frac{(a^2 + d^2 - b^2 - c^2)^2}{4} &= (ad)^2 + (bc)^2 - 2abcd\cos(A+C) \\
&= (ad+bc)^2 - 2abcd(1+\cos(A+C)) \\
&= (ad+bc)^2 - 4abcd\cos^2\left(\frac{A+C}{2}\right)
\end{aligned}$$

(9) Now by setting $t = (A+C)/2$, as in the statement, and with $s = (a+b+c+d)/2$ being the semiperimeter, we can finish the computation, as follows:

$$\begin{aligned}
& 16(S^2 + abcd\cos^2 t) \\
= \ & 4(ad+bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 \\
= \ & (2ad + 2bc + a^2 + d^2 - b^2 - c^2)(2ad + 2bc - a^2 - d^2 + b^2 + c^2) \\
= \ & [(a+d)^2 - (b-c)^2] \cdot [(b+c)^2 - (a-d)^2] \\
= \ & (a+d+b-c)(a+d-b+c)(b+c-a+d)(b+c+a-d) \\
= \ & 16(s-a)(s-b)(s-c)(s-d)
\end{aligned}$$

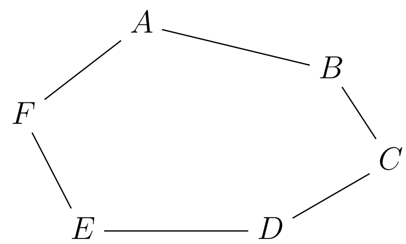
(10) Thus, we have reached to the following formula:

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2 t}$$

But this is exactly the formula in the statement. $\square$

Regarding now the area of arbitrary polygons, let us record here:

THEOREM 6.12. *The area of an arbitrary articulated polygon,*

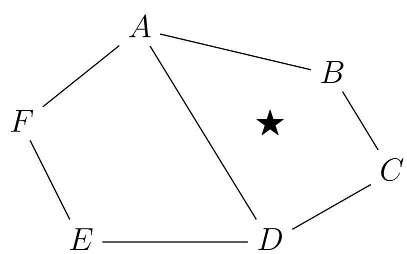



*that is, having edges of fixed lenghts, is maximized when the polygon lies on a circle.*

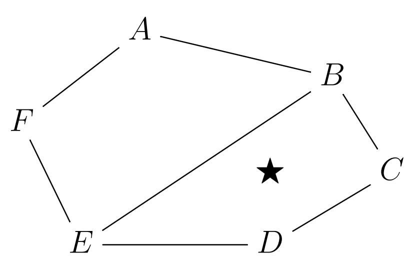
PROOF. This is again something quite tricky, the idea being as follows:

(1) To start with, given an articulated polygon, it is quite clear that there is a way, which is unique, of putting it on a circle. Indeed, the solution comes by starting with a circle which is much bigger than needed, putting inside the polygon, cut at one vertex, and then deflating the circle, until that cut vertex becomes a vertex again.

(2) Next, in what regards the proof, we have seen that the Bretschneider formula does the job for quadrilaterals. But we can use this same formula for dealing with any polygon. Indeed, if we consider the first four vertices $A, B, C, D$, add an edge $AD$, and apply Bretschneider, we conclude that these four vertices $A, B, C, D$ must be cyclic:



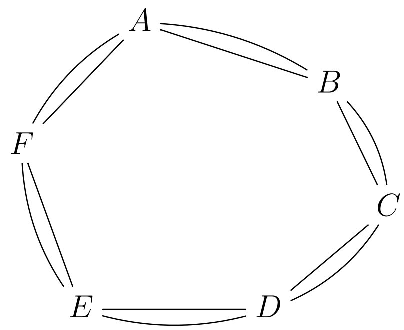

Next, we can do the same trick for $B, C, D, E$, which follow to be cyclic as well:




And so on, with the conclusion being that all vertices must be cyclic, as claimed.

(3) Finally, you might wonder, is the present theorem, which sounds so conceptual and simple, really in need of such a complicated proof, using the Bretschneider formula, which is definitely something quite complicated, as we have seen in the above.

(4) In answer, here is an alternative, physics type proof for the present theorem, based on nothing or almost, and more specifically, based only on the very intuitive fact that, among all shapes of a given perimeter, it is the disk which maximizes the area. So, let us arrange first our articulated polygon on a circle, as explained in (1) above:




(5) Now let us start playing with our articulated polygon, attempting to maximize the area. But this is impossible, because by keeping the above small strips attached, the perimeter will remain that same, namely the perimeter of the original circle, while the area will increase, which is contradictory. Thus theorem proved, just like that.

(6) Amazing all this, isn't it, and do not hesitate to tell this to your fellow students too. And if among them, some math nerd does not agree with the claim in (4), stating that the disk maximizes the area, for a given perimeter, well, tell him that you know how to rigorously prove that, via trigonometry and Bretschneider, as explained in (2). ☐

## 6c. Three angles

We have seen that some interesting mathematics appears in relation with the sines and cosines of sums of angles, $x+y$. This suggests, as a continuation, summing 3 or more angles, and we will explore this here. To start with, we have the following result:

THEOREM 6.13. *The sines of sums of* 3 *angles are given by the formula*

$$\begin{aligned}\sin(x+y+z) &= \sin x\cos y\cos z+\cos x\sin y\cos z\\ &\quad+\cos x\cos y\sin z-\sin x\sin y\sin z\end{aligned}$$

*the cosines of sums of* 3 *angles are given by the formula*

$$\begin{aligned}\cos(x+y+z) &= \cos x\cos y\cos z-\cos x\sin y\sin z\\ &\quad-\sin x\cos y\sin z-\sin x\sin y\cos z\end{aligned}$$

*and we have a formula for the tangent too, namely*

$$\tan(x+y+z)=\frac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan x\tan y-\tan x\tan z-\tan y\tan z}$$

*provided of course that the denominator is nonzero.*

PROOF. We use the addition formulae from Theorem 6.1, namely:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

In what regards the sine, the computation here is as follows:

$$\begin{aligned}
&\sin(x+y+z)\\
=\ &\sin x\cos(y+z) + \cos x\sin(y+z)\\
=\ &\sin x(\cos y\cos z - \sin y\sin z) + \cos x(\sin y\cos z + \cos y\sin z)\\
=\ &\sin x\cos y\cos z + \cos x\sin y\cos z + \cos x\cos y\sin z - \sin x\sin y\sin z
\end{aligned}$$

In what regards the cosine, the computation here is similar, as follows:

$$\begin{aligned}
&\cos(x+y+z)\\
=\ &\cos x\cos(y+z) - \sin x\sin(y+z)\\
=\ &\cos x(\cos y\cos z - \sin y\sin z) - \sin x(\sin y\cos z + \cos y\sin z)\\
=\ &\cos x\cos y\cos z - \cos x\sin y\sin z - \sin x\cos y\sin z - \sin x\sin y\cos z
\end{aligned}$$

Regarding now the tangent, this follows by taking the quotient, or by using:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$

Indeed, by using this formula twice, we obtain, for a sum of three angles:

$$\begin{aligned}
\tan(x+y+z) &= \frac{\tan x + \tan(y+z)}{1 - \tan x\tan(y+z)}\\
&= \frac{\tan x + \frac{\tan y + \tan z}{1 - \tan y\tan z}}{1 - \tan x\,\frac{\tan y + \tan z}{1 - \tan y\tan z}}\\
&= \frac{\tan x + \tan y + \tan z - \tan x\tan y\tan z}{1 - \tan x\tan y - \tan x\tan z - \tan y\tan z}
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a consequence of the above result, obtained with $x = y = z = t$, we have:

THEOREM 6.14. *The sines and cosines of sums of triple of angles are given by*

$$\sin(3t) = 3\sin t - 4\sin^3 t$$

$$\cos(3t) = 4\cos^3 t - 3\cos t$$

*and we have a formula for the tangent too, namely*

$$\tan(3t) = \frac{3\tan t - \tan^3 t}{1 - 3\tan^2 t}$$

*provided of course that the denominator is nonzero.*

PROOF. With $x = y = z = t$ in the sine formula from Theorem 6.13, we obtain:

$$\begin{aligned}\sin(3t) &= 3\sin t\cos^2 t - \sin^3 t\\ &= 3\sin t(1-\sin^2 t) - \sin^3 t\\ &= 3\sin t - 4\sin^3 t\end{aligned}$$

Similarly, with $x = y = z = t$ in the cosine formula from Theorem 6.13, we obtain:

$$\begin{aligned}\cos(3t) &= \cos^3 t - 3\cos t\sin^2 t\\ &= \cos^3 t - 3\cos t(1-\cos^2 t)\\ &= 4\cos^3 t - 3\cos t\end{aligned}$$

Finally, with $x = y = z = t$ in the tangent formula from Theorem 6.13 we obtain the formula for the tangent in the statement, without any further manipulation. □

Getting now to numeric applications, the above formulae raise the possibility of computing the trigonometric functions of 10° and its multiples, by solving the corresponding cubic equations. However, this will not work very well, because do we really know how to solve the cubic equations. So, let us record here something modest, as follows:

THEOREM 6.15. *The quantities $a = \sin 10°$, $b = \cos 10°$, $c = \tan 10°$ satisfy*

$$3a - 4a^3 = \frac{1}{2} \quad , \quad 4b^3 - 3b = \frac{\sqrt{3}}{2} \quad , \quad \frac{3c - c^3}{1 - 3c^2} = \frac{1}{\sqrt{3}}$$

*and we have similar equations, for the other multiples of* 10°.

PROOF. By taking $t = 10°$ in the formulae from Theorem 6.14, we obtain:

$$\sin(30°) = 3a - 4a^3$$

$$\cos(30°) = 4b^3 - 3b$$

$$\tan(30°) = \frac{3c - c^3}{1 - 3c^2}$$

Thus, we are led indeed to the formulae in the statement. □

In order to comment now on all this, we have to talk a bit about degree 3 equations. Here is a well-known result of Cardano, regarding them:

THEOREM 6.16. *For a normalized degree* 3 *equation, namely*

$$x^3 + 3px + 2q = 0$$

*the discriminant is $\Delta = -108(p^3 + q^2)$, and assuming $\Delta < 0$, the number*

$$x = \sqrt[3]{-q + \sqrt{p^3 + q^2}} + \sqrt[3]{-q - \sqrt{p^3 + q^2}}$$

*is a solution of our equation.*

PROOF. This is something quite tricky, the idea being as follows:

(1) Regarding the discriminant $\Delta$, the idea is that this is the degree 3 analogue of the well-known quantity $\Delta = b^2 - 4ac$, and with the precise formula being the above one, $\Delta = -108(p^3 + q^2)$. We will talk more in detail about this later in this book.

(2) Next, assuming $\Delta < 0$, which in practice means $p^3 + q^2 > 0$, the number $x$ in the statement is well-defined, and by using $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$, we have:

$$\begin{aligned} x^3 &= \left(\sqrt[3]{-q+\sqrt{p^3+q^2}} + \sqrt[3]{-q-\sqrt{p^3+q^2}}\right)^3 \\ &= -2q + 3\sqrt[3]{-q+\sqrt{p^3+q^2}} \cdot \sqrt[3]{-q-\sqrt{p^3+q^2}} \cdot x \\ &= -2q + 3\sqrt[3]{q^2 - p^3 - q^2} \cdot x \\ &= -2q - 3px \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Which sounds good, but getting back now to trigonometry, we have bad news:

PROPOSITION 6.17. *The degree* 3 *equation for* $y = 2\cos t$ *in terms of* $c = \cos(3t)$,

$$y^3 - 3y - 2c = 0$$

*has discriminant* $\Delta = 108(1 - c^2) > 0$, *so Theorem 6.16 does not apply to it.*

PROOF. This is something quite self-explanatory. Indeed, according to Theorem 6.14, the equation for $x = \cos t$ in terms of $c = \cos(3t)$ is as follows:

$$4x^3 - 3x = c$$

Now with $y = 2x$ as in the statement, this equation takes the following form:

$$y^3 - 3y = 2c$$

But this is a normalized 3 equation, as in Theorem 6.14, with parameters $p = -1$ and $q = -c$, so its discriminant is given by the following formula:

$$\Delta = -108(-1 + c^2) = 108(1 - c^2)$$

Thus, assuming of course $c \neq \pm 1$, we are led to the conclusion in the statement. □

The problem is now, what to do. Obviously, we would need a version of Theorem 6.16 dealing with the case $\Delta > 0$, which in practice requires talking about $\sqrt{p^3 + q^2}$ when $p^3 + q^2 < 0$. So, let us be crazy, and introduce a formal number $i$ satisfying:

$$i^2 = -1$$

And with this, we will certainly have our extension of Theorem 6.16. However, before doing that, let us first study the simplest cubic equation, $x^3 = 1$. We have:

$$x^3 = 1 \iff (x-1)(x^2 + x + 1) = 0$$

Now when looking at $x^2+x+1=0$, the discriminant is $\Delta=-3$, and so $\sqrt{\Delta}=\sqrt{3}\,i$. We conclude that $x^3=1$ has in fact 3 solutions, namely $x=1$, and:

$$w=\frac{-1+\sqrt{3}\,i}{2}\quad,\quad w^2=\frac{-1-\sqrt{3}\,i}{2}$$

Quite nice all this, and we can now extend Theorem 6.16, as follows:

THEOREM 6.18. *For a normalized degree 3 equation, namely*

$$x^3+3px+2q=0$$

*the discriminant is $\Delta=-108(p^3+q^2)$, and the formal numbers*

$$x=w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}$$

*with $w^3=1$ are the solutions of our equation.*

PROOF. As before, by using $(a+b)^3=a^3+b^3+3ab(a+b)$, we have:

$$\begin{aligned}
x^3 &= \left(w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}\right)^3\\
&= -2q+3\sqrt[3]{-q+\sqrt{p^3+q^2}}\cdot\sqrt[3]{-q-\sqrt{p^3+q^2}}\cdot x\\
&= -2q+3\sqrt[3]{q^2-p^3-q^2}\cdot x\\
&= -2q-3px
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Very good, and getting back now to trigonometry, along the lines of Proposition 6.17, the question is, do we have a win, with Theorem 6.18? And unfortunately, no way:

THEOREM 6.19. *The degree 3 equation for $y=2\cos t$ in terms of $c=\cos(3t)$,*

$$y^3-3y-2c=0$$

*having $\Delta=108(1-c^2)>0$, when approached via Theorem 6.18, gives nothing.*

PROOF. Strange statement that we have here, with such things being called "no-go results", and being the realm of pure mathematics. However, I have seen so many applied mathematicians, myself included a few times, saying "good work that we did, and we'll get that missing cosine via Cardano", that this is definitely worth some discussion:

(1) Theorem 6.18 applies to our equation, and gives, with $s=\sin(3t)$:

$$\begin{aligned}
x &= w\sqrt[3]{1+\sqrt{c^2-1}}+w^2\sqrt[3]{1-\sqrt{c^2-1}}\\
&= w\sqrt[3]{1+\sqrt{1-c^2}\cdot i}+w^2\sqrt[3]{1-\sqrt{1-c^2}\cdot i}\\
&= w\sqrt[3]{1+si}+w^2\sqrt[3]{1-si}
\end{aligned}$$

(2) The problem is now, how to extract that cubic roots, of the formal numbers $1\pm si$. And here, we get nothing, because when attempting to solve $(a\pm ib)^3=1\pm si$, we end up with some complicated equations, which are in fact more or less equivalent to the cubic equation $y^3-3y=2c$ that we started with. Thus, we get indeed nothing. □

We will be back to this, with more explanations, later in this book, when studying the formal numbers $a+ib$ as above, which are called complex numbers. More later.

## 6d. Higher formulae

Moving on now, let us see what happens for a sum of 4 angles. In view of Theorem 6.13, we do not really want to deal with the sine and the cosine, where the formulae will be most likely quite complicated, so we will focus on the tangent instead. We have:

THEOREM 6.20. *The tangents of sums of* 4 *angles are given by*

$$\tan(x+y+z+t)=\frac{\begin{pmatrix}\tan x+\tan y+\tan z+\tan t-\tan x\tan y\tan z\\ -\tan x\tan y\tan t-\tan x\tan z\tan t-\tan y\tan z\tan t\end{pmatrix}}{\begin{pmatrix}1-\tan x\tan y-\tan x\tan z-\tan x\tan t-\tan y\tan z\\ -\tan y\tan t-\tan z\tan t+\tan x\tan y\tan z\tan t\end{pmatrix}}$$

*provided of course that the denominator is nonzero.*

PROOF. We use the formula for the tangents of sums from Theorem 6.1, namely:

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

By using this formula twice we obtain, for a sum of four angles:

$$\begin{aligned}
&\tan(x+y+z+t)\\
=&\ \frac{\tan(x+y)+\tan(z+t)}{1-\tan(x+y)\tan(z+t)}\\
=&\ \frac{\frac{\tan x+\tan y}{1-\tan x\tan y}+\frac{\tan z+\tan t}{1-\tan z\tan t}}{1-\frac{\tan x+\tan y}{1-\tan x\tan y}\cdot\frac{\tan z+\tan t}{1-\tan z\tan t}}\\
=&\ \frac{\begin{pmatrix}\tan x+\tan y+\tan z+\tan t-\tan x\tan y\tan z\\ -\tan x\tan y\tan t-\tan x\tan z\tan t-\tan y\tan z\tan t\end{pmatrix}}{\begin{pmatrix}1-\tan x\tan y-\tan x\tan z-\tan x\tan t-\tan y\tan z\\ -\tan y\tan t-\tan z\tan t+\tan x\tan y\tan z\tan t\end{pmatrix}}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

And the problem is now, is what we found in Theorem 6.20 good news, or not? You would probably say, definitely no, that looks like the end of the world.

However, listen to the old man here, who has seen all sorts of complicated formulae, over his career, what we have in Theorem 6.20 is in fact not that bad. Indeed, we can now formulate the following result, which is something quite nice:

THEOREM 6.21. *The tangents of the sums of angles are given by*

$$\tan(x+y) = \frac{a+b}{1-ab}$$

$$\tan(x+y+z) = \frac{a+b+c-abc}{1-ab-ac-bc}$$

$$\tan(x+y+z+t) = \frac{a+b+c+d-abc-abd-acd-bcd}{1-ab-ac-ad-bc-bd-cd+abcd}$$

$$\vdots$$

*where $a = \tan x$, $b = \tan y$, $c = \tan z$, $d = \tan t$, $\ldots$, with on top odd symmetric functions of $a, b, c, d, \ldots$, and on the bottom even symmetric functions of $a, b, c, d, \ldots$*

PROOF. Here the formulae in the statement are those from Theorems 6.1, 6.13 and 6.20, and the conclusion at the end is something quite self-explanatory. We will leave some thinking here as an exercise, and we will be back to this, later in this book. □

Getting back now to the sine and cosine, we have seen in the above that for small $k \in \mathbb{N}$ we have formulae as follows, with $P_k, Q_k$ being certain polynomials:

$$\cos(kt) = P_k(\cos t) \quad , \quad \sin((k+1)t) = Q_k(\cos t)\sin t$$

To be more precise, in what regards the cosine, we have the following formulae:

$$\cos(2t) = 2\cos^2 t - 1$$

$$\cos(3t) = 4\cos^3 t - 3\cos t$$

$$\vdots$$

As for the sine, the formulae here, coming from what we know, are as follows:

$$\sin(2t) = 2\cos t \sin t$$

$$\sin(3t) = (4\cos^2 t - 1)\sin t$$

$$\vdots$$

To be more precise here, in what regards the formula of $\sin(3t)$, we have:

$$\begin{aligned} \sin(3t) &= 3\sin t - 4\sin^3 t \\ &= (3 - 4\sin^2 t)\sin t \\ &= (3 - 4 + 4\cos^2 t)\sin t \\ &= (4\cos^2 t - 1)\sin t \end{aligned}$$

In order to see now if our conjecture regarding $P_k, Q_k$ is true, let us compute as well the sine and cosine of $4t$. We have here the following result, confirming our conjecture:

PROPOSITION 6.22. *We have the following formulae,*

$$\cos(4t) = 8\cos^4 t - 8\cos^2 t + 1$$

$$\sin(4t) = (8\cos^3 t - 4\cos t)\sin t$$

*confirming our conjectures* $\cos(kt) = P_k(\cos t)$ *and* $\sin((k+1)t) = Q_k(\cos t)\sin t$.

PROOF. Regarding the cosine, we have the following computation:

$$\begin{aligned}\cos(4t) &= 2\cos^2(2t) - 1\\ &= 2(2\cos^2 t - 1)^2 - 1\\ &= 2(4\cos^4 t - 4\cos^2 t + 1) - 1\\ &= 8\cos^4 t - 8\cos^2 t + 1\end{aligned}$$

Regarding the sine, we have the following computation:

$$\begin{aligned}\sin(4t) &= 2\sin(2t)\cos(2t)\\ &= 4\sin t\cos t(2\cos^2 t - 1)\\ &= (8\cos^3 t - 4\cos t)\sin t\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

In general now, we can proceed by recurrence, and we obtain:

THEOREM 6.23. *The cosines and sines of multiplied angles are given by*

$$\cos(kt) = P_k(\cos t) \quad , \quad \sin((k+1)t) = Q_k(\cos t)\sin t$$

*with $P_k, Q_k$ being certain polynomials with integer coefficients, given by*

$$P_{k+1}(x) = P_k(x)x - Q_{k-1}(x)(1 - x^2)$$

$$Q_k(x) = Q_{k-1}(x)x + P_k(x)$$

*called Chebycheff polynomials of the first and second kind.*

PROOF. This is indeed something very standard, the idea being as follows:

(1) We use our basic formulae for the sums, which are as follows:

$$\cos(x + y) = \cos x\cos y - \sin x\sin y$$

$$\sin(x + y) = \sin x\cos y + \cos x\sin y$$

We conclude that we have the following formulae, valid for any $k \in \mathbb{N}$:

$$\cos((k+1)t) = \cos(kt)\cos t - \sin(kt)\sin t$$

$$\sin((k+1)t) = \sin(kt)\cos t + \cos(kt)\sin t$$

Now by recurrence, these formulae take the following form:

$$\cos((k+1)t) = P_k(\cos t)\cos t - Q_{k-1}(\cos t)\sin^2 t$$

$$\sin((k+1)t) = Q_{k-1}(\cos t)\sin t\cos t + P_k(\cos t)\sin t$$

We can write these latter formulae in a more convenient way, as follows:

$$\cos((k+1)t) = P_k(\cos t)\cos t - Q_{k-1}(\cos t)(1-\cos^2 t)$$

$$\sin((k+1)t) = (Q_{k-1}(\cos t)\cos t + P_k(\cos t))\sin t$$

Thus, we have the formulae in the statement, with $P_k, Q_k$ being as follows:

$$P_{k+1}(x) = P_k(x)x - Q_{k-1}(x)(1-x^2)$$

$$Q_k(x) = Q_{k-1}(x)x + P_k(x)$$

Observe in particular that both $P_k, Q_k$ much have integer coefficients.

(2) Let us do as well some numerics, as a matter of doublechecking what we found. As input for our computations, we have the following initial values:

$$P_0 = 1 \quad , \quad P_1 = x \quad , \quad Q_0 = 1$$

At the first step of our recurrence we obtain the following formulae:

$$P_2 = 2x^2 - (1-x^2) = 2x^2 - 1$$

$$Q_1 = x + x = 2x$$

At the second step of our recurrence we obtain the following formulae:

$$P_3 = (2x^3 - x) - (2x - 2x^3) = 4x^3 - 3x$$

$$Q_2 = 2x^2 + (2x^2 - 1) = 4x^2 - 1$$

At the third step of our recurrence we obtain the following formulae:

$$P_4 = (4x^4 - 3x^2) - (4x^2 - 1)(1 - x^2) = 8x^4 - 8x^2 + 1$$

$$Q_3 = (4x^3 - x) + (4x^3 - 3x) = 8x^3 - 4x$$

And, good news, this agrees with what we found in Proposition 6.22, and before. ∎

For future reference, let us record now the above numerics, along with some more:

Proposition 6.24. *The Chebycheff polynomials of the first kind are*

$$1\ ,\quad x\ ,\quad 2x^2-1\ ,\quad 4x^3-3x\ ,\quad 8x^4-8x^2+1\ ,\quad 16x^5-20x^3+5x\ ,\quad \ldots$$

*and the Chebycheff polynomials of the second kind are*

$$1\ ,\quad 2x\ ,\quad 4x^2-1\ ,\quad 8x^3-4x\ ,\quad 16x^4-12x^2+1\ ,\quad 32x^5-32x^3+6x\ ,\quad \ldots$$

*and this list can be indefinitely enlarged, by recurrence, when needed.*

Proof. Here the formulae of $P_0, P_1, P_2, P_3, P_4$ and $Q_0, Q_1, Q_2, Q_3$ are those found above, and those of $P_5$ and $Q_4, Q_5$ can be found similarly, by recurrence. ∎

As a continuation of this, and forgetting a bit about trigonometry, many useful and interesting things can be said about the Chebycheff polynomials, with the quantity of available information being truly remarkable, potentially covering dozens of pages.

So, exercise for you to learn more about this, when needed, and in what concerns us, for the end of the present chapter, we would like to record a main result regarding the Chebycheff polynomials, which is something quite advanced, as follows:

THEOREM 6.25. *The orthogonal polynomials for $L^2[-1,1]$, with measure*

$$d\mu(x)=(1-x)^a(1+x)^bdx$$

*called Jacobi polynomials, satisfy a degree 2 equation, namely*

$$(1-x^2)J_k''(x)+(b-a-(a+b+2)x)J_k'(x)+k(k+a+b+1)J_k(x)=0$$

*and are given by the following formula, featuring derivatives:*

$$J_k(x)=\frac{(-1)^k}{2^kk!}(1-x)^{-a}(1+x)^{-b}\frac{d^k}{dx^k}\left[(1-x)^a(1+x)^b(1-x^2)^k\right]$$

*At $a=b=0$ we recover the Legendre polynomials from physics, and at $a=b=\pm\frac{1}{2}$ we recover the Chebycheff polynomials of the first and second kind.*

PROOF. This is obviously something quite advanced, that we included here only with the aim of telling the truth regarding the Chebycheff polynomials, say for later when you will need such things, and with the idea of all this being as follows:

(1) Generally speaking, the statement appears as a generalization of the result for Legendre polynomials, which corresponds to the particular case $a=b=0$, and the proof is quite similar. We will leave learning more about all this as an exercise.

(2) Regarding now the main particular cases of the Jacobi polynomials, these are the Gegenbauer polynomials, appearing at $a=b$. However, there is not that much of a simplification when passing from general parameters $a,b$ to equal parameters, $a=b$, so in practice, the main particular cases are those indicated in the statement, namely:

– The Legendre polynomials, which naturally appear in questions from quantum mechanics, coming at the simplest values of the parameters, namely $a=b=0$.

– The Chebycheff polynomials of the first kind $P_k$, which are given by the formula $P_k(\cos t)=\cos(kt)$ from trigonometry, appearing at $a=b=-\frac{1}{2}$.

– The Chebycheff polynomials of the second kind $Q_k$, which are given by the formula $Q_k(\cos t)\sin t=\sin((k+1)t)$, appearing at $a=b=\frac{1}{2}$. $\square$

## 6e. Exercises

This was a standard trigonometry chapter, and as exercises on this, we have:

EXERCISE 6.26. *Compute the trigonometric functions of all multiples of* $22.5°$.

EXERCISE 6.27. *Compute the trigonometric functions of all multiples of* $7.5°$.

EXERCISE 6.28. *Compute the trigonometric functions of all multiples of* $3.75°$.

EXERCISE 6.29. *Learn about iterated square roots, and the case where they simplify.*

EXERCISE 6.30. *Learn more about degree* 3 *equations, and the Cardano formula.*

EXERCISE 6.31. *Learn also about degree* 4 *equations, and about Cardano there.*

EXERCISE 6.32. *Learn more, as much as you can, about Chebycheff polynomials.*

EXERCISE 6.33. *Learn also about the other types of orthogonal polynomials.*

As bonus exercise, work out more numerics, as many as you can. All good work.

CHAPTER 7

# Numeric angles

## 7a. Circles, arcs

With the trigonometry basics reasonably understood, and more on this of course in the remainder of this book, let us turn now into a philosophical question, regarding the angles themselves. We have been using, since chapter 2, a quite reasonable way of assigning numeric values to them, according to the following recipe:

(1) The square angle is worth $90^\circ$, and with this choice coming from astronomy, and more specifically, from the approximately 30 days that a lunar month has.

(2) Its half is $45^\circ$, its third is $30^\circ$, its quarter is $22.5^\circ$, its fifth is $18^\circ$, its sixth is $15^\circ$, and so on, so we can measure all angles of type $90^\circ/N$, with $N \in \mathbb{N}$.

(3) But then, by taking multiples of these latter angles, $90^\circ/N$ with $N \in \mathbb{N}$, we can measure all angles of type $90^\circ \cdot M/N$, with $M, N \in \mathbb{N}$.

(4) Thus, we can measure all angles of type $90^\circ \cdot r$, with $r \in \mathbb{Q}$, and since any real number can be approximated rationals, we can in fact measure all angles.

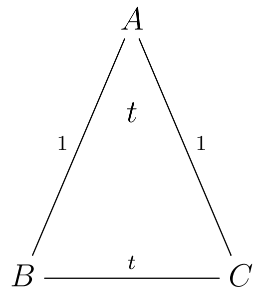
Which sounds very reasonable, and we will keep of course using this system, like everyone does, but as a philosophical question in relation with this, we have:

Question 7.1. *Is there a better way of measuring the angles, with the* $90^\circ$ *from astronomy being replaced by something that can help in our math computations?*

And modern question this is, because in ancient times, and in fact up to not that long ago, astronomy was the same thing as mathematics. But well, this is the situation, we cannot really argue with modernity, so we will have to answer our question.

In answer to this now, we would like to find a way of assigning to angles the lengths or areas that they "produce". And here, the situation is as follows:

(1) The first thought goes to triangles, say by assigning to each angle the side of an isosceles triangle having at the distinguished vertex sides $1, 1$, and angle $t$:

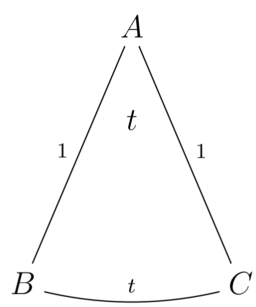



Which in practice means setting, by definition, $t = 2\sin(t/2)$. But this won't work well, because when doubling the angle the side obviously won't double.

(2) Along the same lines, as a second idea, we can use the same triangle, but look this time at its area, which would amount in setting $t = (\sin t)/2$. But this won't work either, because when doubling the angle the area obviously won't double.

In short, we are in trouble here, and we must invent something else. And the answer here comes from using disk slices instead of triangles, as follows:

Theorem 7.2. *We can measure angles by putting them in the middle of a circle of radius* 1*, and assigning to them the corresponding arc lengths:*




*Equivalently, we can use twice the area of the disk slice, which equals the arc length.*

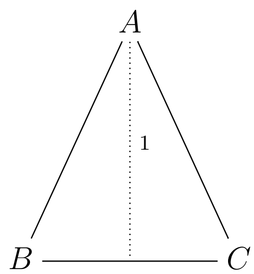
Proof. This is obviously something quite philosophical, and you might wonder, what exactly is to be proved here. In answer, there are two things to be proved, as follows:

(1) First is the fact that our measuring method is indeed good, in the sense that doubling the angles will double their values, tripling the angles will triple their values, and so on. But this is something which is plainly obvious, so done with this.

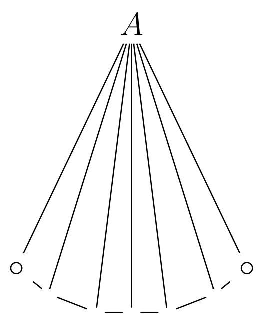
(2) And then, there is the last assertion, claiming that we have the following formula, with on the left the area of the disk slice $ABC$, and on the right the arc length $BC$:

$$2 \times area(ABC) = BC$$

But this is something which is clear for isosceles triangles having altitude 1:




Now back to our disk slice, this can be approximated by unions of such isosceles triangles, having altitude 1, in the obvious way, as follows:

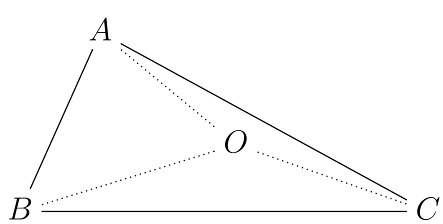



Thus, we conclude that our area formula holds indeed, as desired. □

Very nice all this, but in practice, will our discovery answer indeed Question 7.1, and help with our mathematics? At this stage of things, this is not clear at all, hope you agree with me with this, is it really worth it to bother with these arc lengths, or disk slices.

In short, patience, we have still a long way to go. Getting to work now, browsing through what we did so far in the present book, not very good news here, with everything or almost being not obviously related to arc lengths, or disk slices. With the exception, however, of a key result from the beginning of chapter 3, which was as follows:

Proposition 7.3. *Given a triangle $ABC$ lying on a circle,*




*the angle at $A$ does not depend on the position of $A$, and equals half the angle $BOC$.*

Proof. We already know this from chapter 3, but always good to talk about it again. Due to the various isosceles triangles on the picture, the angles of our triangle $ABC$ must be as follows, with $p, q, r$ being the smaller angles on the picture, from left to right:

$$a = p + q \quad , \quad b = p + r \quad , \quad c = r + q$$

Now let us look at the angle $BOC$. With the symbol $\heartsuit$ standing for new value of the flat angle, as per Theorem 7.2, this angle $BOC$ is given by the following formula:

$$\begin{aligned} BOC &= \heartsuit - 2r \\ &= (a + b + c) - 2r \\ &= 2p + 2q \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

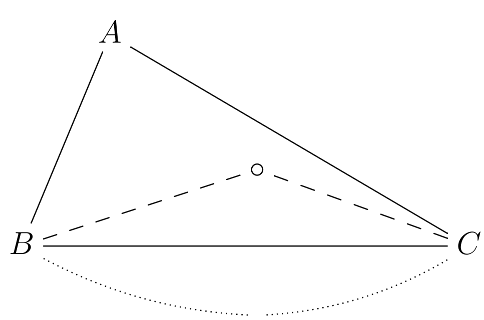
So, what does Proposition 7.3 teach us? As a first observation, its proof makes appear the elephant in the room, which is the new value $\heartsuit$ of the flat angle, measured by using the method in Theorem 7.2. So, let us record the following question:

$$\heartsuit = ?$$

In practice, this question asks for the computation of half of the length of the circle of radius 1. But more on this later, such elephants are usually to be ignored, right.

Next, and getting now to what Proposition 7.3 actually says, there is an obvious relation there with Theorem 7.2. But, thinking a bit at all this, when it comes to exploit this fact, we are rather led to a technical continuation of Theorem 7.2, as follows:

Theorem 7.4 (addendum). *We can equally measure the angles $A$ by putting them on a circle, as follows, and assigning to $A$ half of the length of the arc $BC$,*



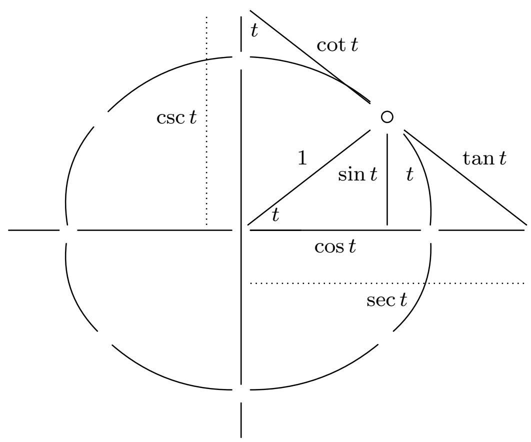

*and with this yielding the same quantity as the one defined before.*

Proof. This is indeed something self-explanatory, coming from Proposition 7.3. By the way, observe that nothing can be said in relation with the area of $ABC$, taken as triangle or disk slice, so we cannot add anything about areas, to our statement. □

With this discussed, what is next? Well, many things I guess, or rather everything, because we have nothing so far. Indeed, our Theorem 7.2, even complemented by Theorem 7.4 as above, looks more like a terrible complication, at the present stage of things.

Fortunately, trigonometry comes to the rescue. Remember indeed the catch-all picture for all 6 trigonometric functions, from chapter 5? Well, that picture features a circle, originally drawn there for understanding the secant and cosecant. And we can exploit that circle, in relation with Theorem 7.2, as to formulate the following finding:

THEOREM 7.5. *In the context of the catch-all picture for trigonometric functions*

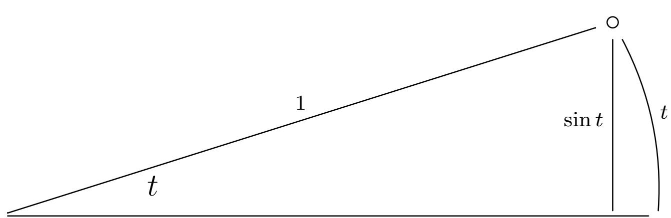



*the angle $t$ appears as an arc length, as indicated. In particular, we have*

$$\sin t \simeq t$$

*for small angles $t$, and with this being due to our new method of measuring angles.*

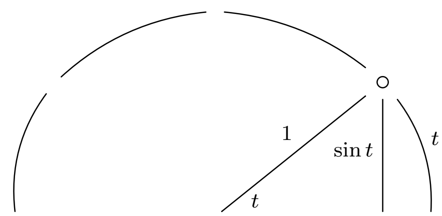
PROOF. This is again something self-explanatory, with the picture being something that we know well, from chapter 5, with just the arc length $t$ being added. As for the last assertion, assume that $t$ is small, and let us zoom on the area containing $\sin t$ and $t$:




We can see that we have indeed $\sin t \simeq t$, somehow by obvious reasons, and with a formal proof coming by comparing the various lengths and areas appearing there. Thus done, and we will be back to this later, with some further details and explanations. □

Still with me, I hope, after all this exploration and philosophy. You would probably say, after all, what's the point with Theorem 7.5, and in answer, my claim is that this is a win. To further make my case, let me reformulate the essentials of the main results that we learned so far in this chapter, Theorem 7.2 and Theorem 7.5, as follows:

THEOREM 7.6. *We can measure angles by putting them in the middle of a circle of radius* 1*, and assigning to them the corresponding arc lengths,*




*and in this way, we have the following key estimate, valid for small angles t,*

$$\sin t \simeq t$$

*which can be potentially useful, when doing more advanced trigonometry.*

PROOF. This is indeed a summary from what we know from Theorem 7.2 and Theorem 7.5, with some weeds removed, and with the last assertion being of course something subjective, which remains to be justified. But no worries, we will get to this, soon. □

Very nice all this, we seem to have some new theory going on, potentially answering the philosophical questions raised at the beginning of this chapter. Of course, still long way to go. In the meantime, as usual in such delicate situations, let us ask the cat, what he thinks about all this. And the answer comes quite encouraging, as follows:

CAT 7.7. *This world is made of small angles and forces adding up, and* $\sin t \simeq t$ *might be useful indeed.*

Okay, thanks cat, nice to hear that, sounds quite motivating, all this underlying physics, and we will have the remainder of this book, for exploring the subject.

## 7b. The number pi

Getting now seriously to work, with our new method for measuring the angles in hand, approved by cat, let us discuss the first obvious question that appears on the way, namely the new formula for the right angle 90°. In practice, this amounts in computing a quarter of the length of the unit circle, and we will study here this problem.

Before starting, however, a remark. While the 90° angle was certainly the "unit", when doing geometry in Part I, in relation with the various trigonometry considerations from chapters 5-6, or just with the unit circle itself, regarded as such, the full angle 360°, corresponding to the length of the unit circle, would be more appropriate as unit.

However, with the full angle $360°$ being something quite abstract, pictorially corresponding to a dumb point, or perhaps to a bizarre half-line, it is better to make a compromise here, and declare the flat angle $180°$, corresponding to half of the lenght of the unit circle, as being the unit. So, let us do so, and call our new angle unit $\pi$:

DEFINITION 7.8. *We call $\pi$ our new angle unit, according to the formula*

$$180° = \pi$$

*which in practice tells us that $\pi$ is half of the length of the unit circle.*

Before getting to the computation of $\pi$, now that we have our unit, let us do some angle conversions. For the basic angles, the conversion formulae are as follows:

$$0° = 0 \quad , \quad 90° = \frac{\pi}{2} \quad , \quad 180° = \pi \quad , \quad 270° = \frac{3\pi}{2}$$

Let us record as well the conversion formulae for the halves of these angles:

$$45° = \frac{\pi}{4} \quad , \quad 135° = \frac{3\pi}{4} \quad , \quad 225° = \frac{5\pi}{4} \quad , \quad 315° = \frac{7\pi}{4}$$

Finally, let us record as well the formulae for the thirds of the basic angles:

$$30° = \frac{\pi}{6} \quad , \quad 60° = \frac{\pi}{3} \quad , \quad 120° = \frac{2\pi}{3} \quad , \quad 150° = \frac{5\pi}{6}$$

$$210° = \frac{7\pi}{6} \quad , \quad 240° = \frac{4\pi}{3} \quad , \quad 300° = \frac{5\pi}{3} \quad , \quad 330° = \frac{11\pi}{6}$$

And so on, and good luck in memorizing all this. And by the way, with apologies in advance if I keep using sometimes the old $90°$ conventions, like everyone does, when it is not a matter of life and death, I mean, not a matter of advanced trigonometry.

Getting back now to what we wanted to do, as our most pressing issue, we have:

QUESTION 7.9. *What is the value of $\pi$?*

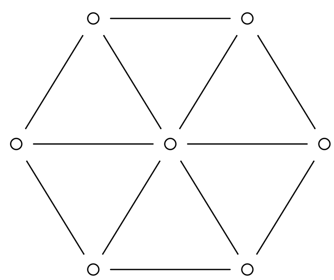
And good this question this is, because, as we will soon discover, although estimates of type $\pi \simeq 3.14$ are relatively easy to establish, $\pi$ is not a rational number, and even worse, there is no equation satisfied by $\pi$, or any kind of simple formula for $\pi$.

In short, patience, understanding what $\pi$ is will take us some time. To start with, we have the following fact, which can be regarded as being something quite axiomatic:

THEOREM 7.10. *The following two definitions of $\pi$ are equivalent:*

1. *The length of the unit circle is $L = 2\pi$.*
2. *The area of the unit disk is $A = \pi$.*

PROOF. This is something that we already know, coming from Theorem 7.2, but for mathematical pleasure, let us prove this again. We can cut the unit disk as a pizza, into $N$ slices, and forgetting about gastronomy, leave aside the rounded parts:



The area to be eaten can be then computed as follows, where $H$ is the height of the slices, $S$ is the length of their sides, and $P = NS$ is the total length of the sides:

$$\begin{aligned} A &= N \times \frac{HS}{2} \\ &= \frac{HP}{2} \\ &\simeq \frac{1 \times L}{2} \end{aligned}$$

Thus, with $N \to \infty$ we obtain that we have $A = L/2$, as desired. □

In what regards now the precise value of $\pi$, the above picture at $N = 6$ shows that we have $\pi > 3$, but not by much. More can be said by using some basic trigonometry, by replacing the hexagon used in the above with other polygons, and we have here:

THEOREM 7.11. *We have the following approximations of $\pi$,*

$$2.828 < \pi < 4$$

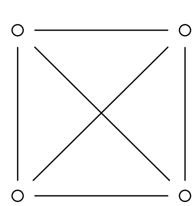
$$3 < \pi < 3.464$$

$$3.061 < \pi < 3.314$$

*obtained respectively by using squares, hexagons and octagons.*

PROOF. This is something which is quite long and routine, as follows:

(1) Let us first see what we can get by approximating the unit circle with squares:



The squares inscribed and circumscribed to the unit circle have edges as follows:

$$e=\sqrt{2}\quad,\quad E=2$$

Thus by looking at half perimeters, we obtain the following estimate for $\pi$:

$$2e<\pi<2E\implies 2\sqrt{2}<\pi<4$$

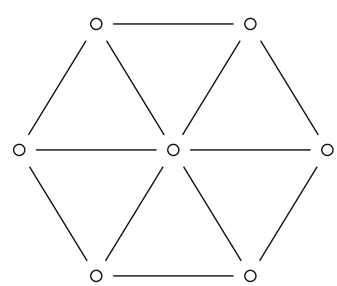
We have as well an estimate for $\pi$ coming by looking at areas, which is:

$$e^2<\pi<E^2\implies 2<\pi<4$$

Summarizing, the half perimeter method appears to be better than the area one, and gives the following numeric estimates, by using $\sqrt{2}=1.414\ldots$:

$$2.828<\pi<4$$

(2) Leaving the pentagons aside, let us see what happens by using hexagons:



The hexagons inscribed and circumscribed to the unit circle have edges as follows, with the formula for $E$ coming from an equilateral triangle having altitude 1:

$$e=1\quad,\quad E=\frac{2}{\sqrt{3}}$$

Thus by looking at half perimeters, we obtain the following estimate for $\pi$:

$$3e<\pi<3E\implies 3<\pi<2\sqrt{3}$$

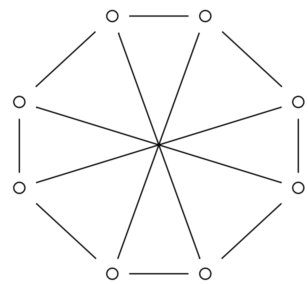
We have as well an estimate for $\pi$ coming by looking at areas, which is as follows, using the fact that the area of an equilateral triangle is $\sqrt{3}/4\times\text{edge}^2$:

$$\frac{3\sqrt{3}}{2}\,e^2<\pi<\frac{3\sqrt{3}}{2}\,E^2\implies\frac{3\sqrt{3}}{2}<\pi<2\sqrt{3}$$

Summarizing, the half perimeter method appears again to be better than the area one, and gives the following numeric estimates, by using $\sqrt{3}=1.732\ldots$:

$$3<\pi<3.464$$

(3) Leaving the heptagon aside, next we have the octagon, which is as follows:



The octagons inscribed and circumscribed to the unit circle have edges as follows, coming by looking at the respective "pizza slices", which are $\pi/4-3\pi/8-3\pi/8$ triangles, the inner ones having radial edge 1, and the outer ones having radial altitude 1:

$$e=2\sin\left(\frac{\pi}{8}\right)\quad,\quad E=2\tan\left(\frac{\pi}{8}\right)$$

Thus by looking at half perimeters, we obtain the following estimate for $\pi$:

$$4e<\pi<4E\quad\implies\quad 8\sin\left(\frac{\pi}{8}\right)<\pi<8\tan\left(\frac{\pi}{8}\right)$$

We have as well an estimate for $\pi$ coming by looking at areas, which is as follows, using the fact that the area of a $\pi/4-3\pi/8-3\pi/8$ triangle is $\cot(\pi/8)\times\text{edge}^2/4$:

$$\begin{aligned}
&2\cot(\pi/8)e^2<\pi<2\cot(\pi/8)E^2\\
\implies\ &8\sin(\pi/8)\cos(\pi/8)<\pi<8\tan(\pi/8)\\
\implies\ &4\sin(\pi/4)<\pi<8\tan(\pi/8)
\end{aligned}$$

(4) In order to reach now to some concrete numeric estimates, by using these octagon methods, we must ask Conor and Khabib for some help with trigonometry, and GSP for some assistance with the square roots. The half-perimeter estimate for $\pi$ reads:

$$\begin{aligned}
&8\sin(\pi/8)<\pi<8\tan(\pi/8)\\
\implies\ &4\sqrt{2-\sqrt{2}}<\pi<8(\sqrt{2}-1)\\
\implies\ &3.061<\pi<3.314
\end{aligned}$$

As for the area estimate for $\pi$, this gives something weaker, as follows:

$$\begin{aligned}
&4\sin(\pi/4)<\pi<8\tan(\pi/8)\\
\implies\ &4/\sqrt{2}<\pi<8(\sqrt{2}-1)\\
\implies\ &2.828<\pi<3.314
\end{aligned}$$

Thus, we are led to the conclusions in the statement. $\square$

As a conclusion to this, certainly good work that we did, but all this remains a bit frustrating, because any kid on the street knows that $\pi = 3.14$, and with our methods above we only have $\pi = 3.0$ or $\pi = 3.1$ or $\pi = 3.2$ or $\pi = 3.3$, which is quite lame.

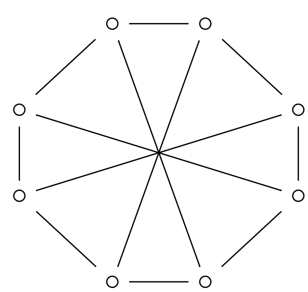
Nevermind. Life goes on, and passing now to the general case, that of the arbitrary regular polygons, we have the following result, further building on the above:

THEOREM 7.12. *We have the following estimates for $\pi$, obtained by computing the half perimeter of an inscribed and circumscribed regular $N$-gon,*

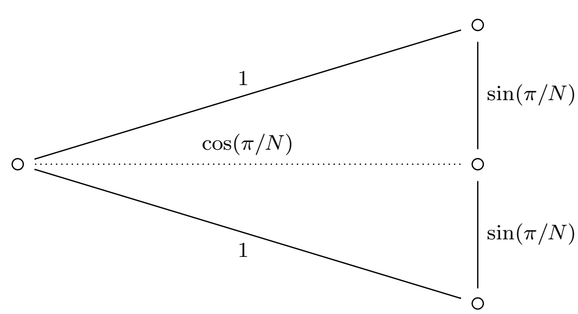
$$N\sin\left(\frac{\pi}{N}\right) < \pi < N\tan\left(\frac{\pi}{N}\right)$$

*and with a bit of patience, with $N = 2^n$ with $n >> 0$, this gives $\pi = 3.14159\ldots$*

PROOF. We use the same method as before. Consider indeed the unit circle, and an inscribed or circumscribed regular $N$-gon, as follows:



(1) In order to compute the edge of the inscribed $N$-gon, let us look at the corresponding "pizza slices". These are isosceles triangles with angle $2\pi/N$ and edge 1, so when drawing an altitude and computing the lengths, the picture is as follows:



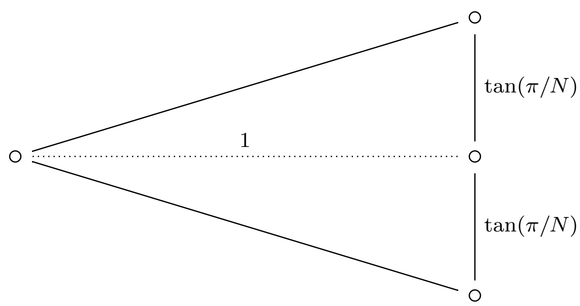

Thus, the edge and half-perimeter of the inscribed $N$-gon are as follows:

$$e = 2\sin\left(\frac{\pi}{N}\right) \quad , \quad p = N\sin\left(\frac{\pi}{N}\right)$$

We can compute as well the area of the inscribed $N$-gon, and we get:

$$\begin{aligned} a &= N \times area(slice) \\ &= N \sin\left(\frac{\pi}{N}\right)\cos\left(\frac{\pi}{N}\right) \\ &= \frac{N}{2}\sin\left(\frac{2\pi}{N}\right) \end{aligned}$$

(2) Getting now to the circumscribed $N$-gon, the "pizza slices" here are again isosceles triangles with angle $2\pi/N$, but this time with altitude 1. Thus when drawing these altitudes and computing the relevant lengths, the picture is as follows:




Thus, the edge and half-perimeter of the circumscribed $N$-gon are as follows:

$$E = 2\tan\left(\frac{\pi}{N}\right) \quad , \quad P = N\tan\left(\frac{\pi}{N}\right)$$

We can compute as well the area of the circumscribed $N$-gon, and we get:

$$A = N \times area(slice) = N\tan\left(\frac{\pi}{N}\right)$$

(3) Time now to derive some conclusions, from our study. With the perimeter method, we obtain the estimate from the statement, namely:

$$N\sin\left(\frac{\pi}{N}\right) < \pi < N\tan\left(\frac{\pi}{N}\right)$$

As for the area method, this gives the following estimate:

$$\frac{N}{2}\sin\left(\frac{2\pi}{N}\right) < \pi < N\tan\left(\frac{\pi}{N}\right)$$

But this latter estimate is to be ignored, because the upper bound is the same, and the lower bound is lower, and so unuseful, and this due to the following fact:

$$\sin(2t) = 2\sin t\cos t < 2\sin t$$

Thus, done with our study, and we have proved the estimate in the statement.

(4) Next, as a matter of doublechecking, let us see what we get at $N = 4, 6, 8$. And here, what we get is as follows, which is in tune with what we found before:

$$4\sin(\pi/4) < \pi < 4\tan(\pi/4) \implies 2\sqrt{2} < \pi < 4$$

$$6\sin(\pi/6) < \pi < 6\tan(\pi/6) \implies 3 < \pi < 2\sqrt{3}$$

$$8\sin(\pi/8) < \pi < 8\tan(\pi/8) \implies 4\sqrt{2-\sqrt{2}} < \pi < 8(\sqrt{2}-1)$$

(5) Finally, in what regards the last assertion of the theorem, by using our formulae for duplication of the angles it is pretty much clear that, with a bit of patience, we can succesively do the computations for the $N$-gons with $N$ being a power of 2:

$$N = 2, 4, 6, 8, 16, 32, 64, \ldots$$

And this will give us as many decimals of $\pi$ as we want, and in the end we will get $\pi = 3.14159\ldots$, as claimed. Thus, good work that we did, and theorem proved. □

Excuse me, but cat is here, meowing something. What is it, cat?

CAT 7.13. *I would be curious about the computations at $N = 16, 32, 64$, see if you humans can really do them, and how many decimals of $\pi$ you get.*

Humm, good point, it's true that Theorem 7.12 was quite nice, but at the level of numerics, there was in fact nothing new there, with respect to Theorem 7.11. So, here we go with some more computations. We have the following result, convincive, I hope:

THEOREM 7.14. *We have the the following estimates for $\pi$, obtained by computing the half perimeter of an inscribed and circumscribed regular hexadecagon,*

$$3.121 < \pi < 3.183$$

*and with a bit more work, with $N = 2^n$, $n >> 0$, the same method gives $\pi = 3.14159\ldots$*

PROOF. We recall from Theorem 7.12 that we have the following estimate, that we would like to use now at $N = 16$, and at higher powers of 2:

$$N\sin\left(\frac{\pi}{N}\right) < \pi < N\tan\left(\frac{\pi}{N}\right)$$

(1) Let us begin with some general trigonometry, regarding the sines and tangents of the halves of angles. For the sines, we have the following computation:

$$\begin{aligned}\cos(2t) = 1 - 2\sin^2 t \quad &\implies \quad 1 - 2\sin^2 t = \sqrt{1-\sin^2(2t)}\\ &\implies \quad 2\sin^2 t = 1 - \sqrt{1-\sin^2(2t)}\\ &\implies \quad \sin t = \sqrt{\frac{1-\sqrt{1-\sin^2(2t)}}{2}}\end{aligned}$$

As for the tangents, we have here the following computation, and with exercise here for you to figure out why we chose the sign plus, for the square root:

$$\begin{aligned}\tan(2t)=\frac{2\tan t}{1-\tan^2 t}\quad&\implies\quad \tan(2t)\tan^2 t+2\tan t-\tan(2t)=0\\&\implies\quad \tan t=\frac{-2+\sqrt{4+4\tan^2(2t)}}{2\tan(2t)}\\&\implies\quad \tan t=\frac{\sqrt{1+\tan^2(2t)}-1}{\tan(2t)}\end{aligned}$$

(2) Now let us see what we get at $N=16$. With $t=\pi/16$, and with trigonometric input concerning $2t=\pi/8$ from the proof of Theorem 7.11, we obtain:

$$\sin(\pi/16)=\sqrt{\frac{1-\sqrt{1-\frac{2-\sqrt{2}}{4}}}{2}}=\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

As for the tangent, again with trigonometric input from before, we get:

$$\tan(\pi/16)=\frac{\sqrt{1+(\sqrt{2}-1)^2}-1}{\sqrt{2}-1}=\frac{\sqrt{4-2\sqrt{2}}-1}{\sqrt{2}-1}$$

(3) At the level of the numerics, the formulae are as follows:

$$\sin(\pi/16)=0.195\ldots\quad,\quad\tan(\pi/16)=0.198\ldots$$

By multiplying by 16, as required by our method from Theorem 7.12, we have:

$$16\sin(\pi/16)=3.121\ldots\quad,\quad 16\tan(\pi/16)=3.182\ldots$$

Thus, we are led to the $N=16$ estimate in the statement. Moreover, with a bit more work, it is pretty much clear that we can do the same at $N=32,64,128,\ldots$, and we can only end up in this way with the known estimate for $\pi$, namely $\pi=3.14159\ldots$ □

Good work that we did, I bet the cat will be satisfied. However, cat declares:

Cat 7.15. *Boss, you are cheating, I saw you using your calculator for computing the square roots. But that calculator, if allowed, can give you* $\pi=3.14159\ldots$ *right away.*

Well, what can I say. Guess you win, dear cat, so as a grand conclusion to all the computations that we did so far, let us formulate:

Grand conclusion 7.16. *Computing the decimals of* $\pi$ *is no easy business, and*

$$\pi=3.14159\ldots$$

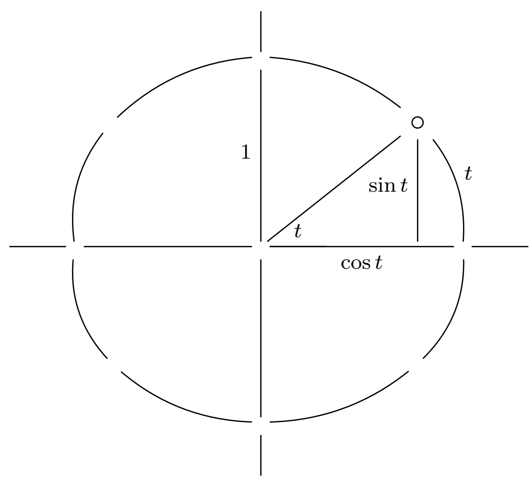
*is something that we will leave for later, when we will know more things.*

And with this, end of our study of $\pi$. More later, on several occasions.

## 7c. Basic estimates

Getting back now to what we wanted to do in this chapter, in relation with the angles, and their measuring, let us first recall that the following happens:

THEOREM 7.17. *We can measure angles by putting them in the middle of a circle of radius* 1*, and assigning to them the corresponding arc lengths,*



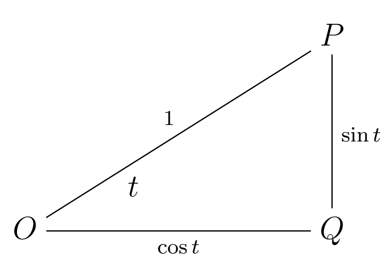

*and this gives as well a method for measuring sines and cosines, as indicated, and with this working for all angles $t \in \mathbb{R}$, with our usual conventions for signed segments.*

PROOF. This is something that we already know, the idea being as follows:

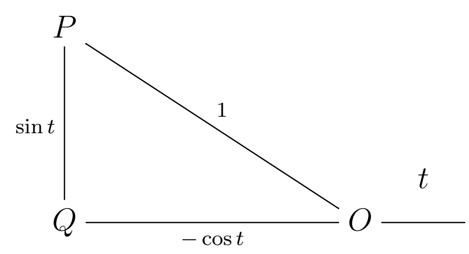
(1) To start with, the above picture is our usual one, from chapter 5, with the arc length $t$ added, and with the secondary trigonometric functions removed.

(2) As for the last assertion, regarding the arbitrary angles $t \in \mathbb{R}$, this comes from our general trigonometry discussion from chapter 5. However, for full clarity, let us review now that discussion, in the present setting. We have several cases, as follows:

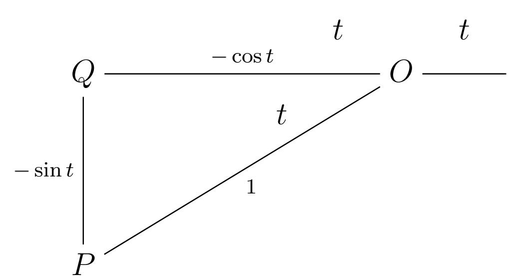
– In the simplest case, namely $t \in [0, \pi/2]$, the sine and cosine are indeed computed according to the following picture, which is the one in the statement:



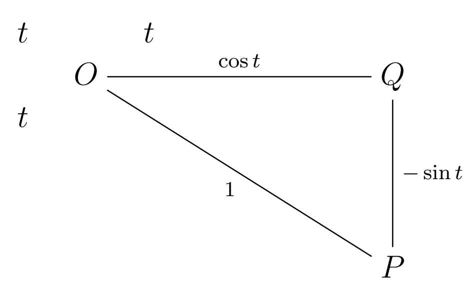
– In the case of obtuse angles, $t \in [\pi/2, \pi]$, the picture becomes as follows:




– In the next case, namely $t \in [\pi, 3\pi/2]$, the picture becomes as follows:



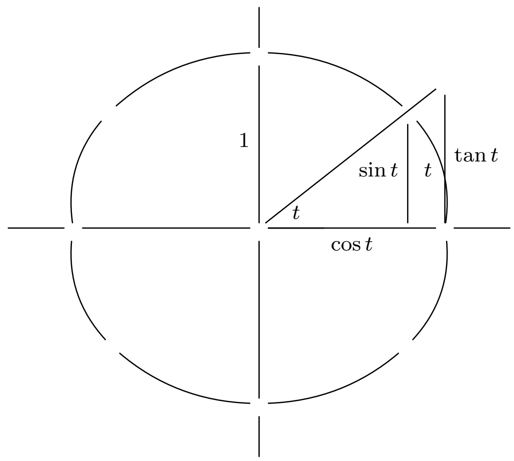

– As for the last case, namely $t \in [3\pi/2, 2\pi]$, here our picture is as follows:




And with this we are done, because with things fine for $t \in [0, 2\pi]$, they will be fine for any $t \in \mathbb{R}$, by using the $2\pi$ periodicity of both the sine and cosine, and of our geometric constructions. Thus, we are led to the conclusions in the statement. □

Regarding now the tangent, things are more complicated here, as follows:

THEOREM 7.18 (addendum). *The tangent can be added as well to the picture, as being a length of a vertical segment, as indicated,*




*with the convention however that this length is positive in the first and third quadrant, and negative in the second and fourth quadrant.*

PROOF. This is indeed something self-explanatory, and with the comment that there is no simple way of fixing things, so in a word, better not mess with the tangent. □

Let us get now into an interesting question, namely estimating $\sin, \cos, \tan$ and the other trigonometric functions. For this purpose, let us first recall the basic formulae for the sums of angles, that we established in chapter 6, which were as follows:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Let us recall as well, also from chapter 6, the formula for the tangent of the sum of two angles, which comes by dividing the two formulae above, namely:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Obviously, these formulae allow us in practice to transport our approximation questions around $t = 0$. Indeed, in what regards the sine, we have for instance the following estimate, with $t \simeq 0$, which technically proves that the sine is continuous, at any $x$:

$$\begin{aligned}
\sin(x+t) &= \sin x \cos t + \cos x \sin t \\
&\simeq \sin x \cos 0 + \cos x \sin 0 \\
&= \sin x \cdot 1 + \cos x \cdot 0 \\
&= \sin x
\end{aligned}$$

In what regards the cosine, the continuity computation here is similar, as follows:

$$\begin{aligned}\cos(x+t) &= \cos x\cos t - \sin x\sin t\\ &\simeq \cos x\cos 0 - \sin x\sin 0\\ &= \cos x\cdot 1 - \sin x\cdot 0\\ &= \cos x\end{aligned}$$

As for the tangent, again we have here a similar computation, as follows:

$$\begin{aligned}\tan(x+t) &= \frac{\tan x+\tan t}{1-\tan x\tan t}\\ &\simeq \frac{\tan x+\tan 0}{1-\tan x\tan 0}\\ &= \frac{\tan x+0}{1-\tan x\cdot 0}\\ &= \tan x\end{aligned}$$

Many other things can be said, along these lines, and we will be back to such things later in this book, on a more systematic basis, when doing calculus for the trigonometric functions. In any case, let us record these findings as an informal fact, as follows:

FACT 7.19. *We can use the standard formulae for the sums of angles in order to transport our various approximation questions around $t = 0$.*

With this understood, let us get now to what happens with trigonometric functions around 0. And here, to start with, we have the following basic estimates:

THEOREM 7.20. *We have the following estimates,*

$$\sin t \leq t \leq \tan t$$

*valid for small angles, coming from our convention for numeric angles.*

PROOF. Many things can be said here, the idea being as follows:

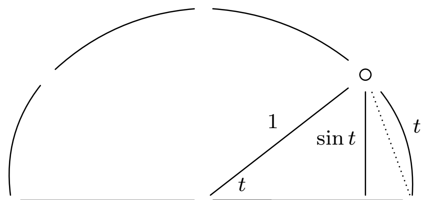
(1) As a first observation, we have already met such estimates before, in Theorem 7.12. To be more precise, we have seen estimates for $\pi$ as follows, obtained by evaluating the perimeters of the regular $N$-gons inscribed and circumscribed to the unit circle:

$$N\sin\left(\frac{\pi}{N}\right) < \pi < N\tan\left(\frac{\pi}{N}\right)$$

But these estimates are of the type of the one in the statement:

$$\sin\left(\frac{\pi}{N}\right) < \frac{\pi}{N} < \tan\left(\frac{\pi}{N}\right)$$

(2) In general now, the idea is that the estimates are both clear from our circle picture for the angles, and trigonometric functions. Indeed, the picture for the sine is:




Now by using the standard fact that the shortest distance between a point and a line is achieved by constructing the orthogonal projection on that line, we conclude that for any angle $t \in [0, \pi/2]$ we have indeed the following estimate, as claimed:

$$\sin t \leq t$$

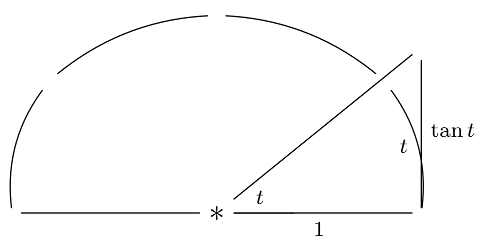
(3) Equivalently, and a bit more rigorously, we can draw the dotted segment above, having length $2\sin(t/2)$, and with Pythagoras on the left, followed by shortest distance between two points being achieved by that dotted segment on the right, we obtain:

$$\sin t \leq 2\sin(t/2) \leq t$$

(4) As yet another proof, we can compare the area of the above isosceles triangle with the area of the disk slice, which gives right away the following estimate, as desired:

$$\frac{\sin t}{2} \leq \frac{t}{2}$$

(5) Regarding now the tangent, again for $t \in [0, \pi/2]$, the picture is as follows:




But here we can argue that the arc $t$ and segment $\tan t$ are related by a projection from $*$, which lands orthogonally on the arc, and obliquely on the segment, and since orthogonal projections notoriously provide the best view, we obtain, as claimed:

$$t \leq \tan t$$

(6) Equivalently, and more rigorously, by comparing areas we get, as desired:

$$\frac{t}{2} \leq \frac{\tan t}{2}$$

(7) Thus, done. Finally, one remaining question concerns the exact range of the above estimates, and we will leave the discussion here as an interesting exercise. ☐

In fact, by using our circle technology, we are led to the following result:

THEOREM 7.21. *The following happen, for small angles, again coming from our convention for numeric angles, and best justifying this convention:*

(1) $\sin t \simeq t$.
(2) $\cos t \simeq 1 - t^2/2$.
(3) $\tan t \simeq t$.

PROOF. This can be indeed established as follows:

(1) This is clear indeed on the circle, by arguing like in the previous proof, and we will leave the various details here as an instructive exercise. Equivalently, this follows from $\sin t \leq t \leq \tan t$, by using $\tan t = \sin t / \cos t \simeq \sin t$, coming from $\cos t \simeq 1$.

(2) This comes from (1), and from Pythagoras. Indeed, knowing $\sin t \simeq t$, when looking for a quantity $\cos t$ making the Pythagoras formula $\sin^2 t + \cos^2 t = 1$ hold, we are led, via some quick thinking, to the formula $\cos t \simeq 1 - t^2/2$, as stated. Here is the verification, and with the result itself coming via some reverse engineering, from this:

$$\begin{aligned}\left(1-\frac{t^2}{2}\right)^2+t^2 &= \left(1-t^2+\frac{t^4}{4}\right)+t^2\\ &\simeq 1-t^2+t^2\\ &= 1\end{aligned}$$

(3) This is again clear on the circle, or simply follows from (1,2), by dividing. □

Many other things can be said, as a continuation of this. We will be back to this, on several occasions, with various improvements of the above results.

## 7d. More about pi

Getting back now to $\pi$ itself, we know that this jointly appears as half of the length of the unit circle, $\pi = L/2$, or as the area of the unit disk, $\pi = A$. In view of this, it is interesting to work out too what happens in our usual 3 dimensions, as a matter of deciding if $\pi$ rules there too, or if we will have some work to be done there, later.

And here, good news, things just fine, with $\pi$ being once again the king:

THEOREM 7.22. *The volume of the unit sphere is given by*

$$V = \frac{4\pi}{3}$$

*and the corresponding area is $A = 4\pi$.*

Proof. This is something very standard, by knowing a bit of integration theory, or even without knowing it, as explained below, the idea being as follows:

(1) To start with, we can talk about the integral of a continuous function $f : [a, b] \to \mathbb{R}$, as being the signed area below its graph. As a basic example here, for the function $f(x) = x$ we have to compute the area of a right trapezoid, and we obtain:

$$\begin{aligned}\int_a^b x dx &= \text{length} \times \text{average height} \\ &= (b-a) \times \frac{b+a}{2} \\ &= \frac{b^2-a^2}{2}\end{aligned}$$

(2) In general now, by drawing rectangles, we have the following formula for the integral, which can stand as a formal definition for this integral:

$$\int_a^b f(x)dx = \lim_{N\to\infty} \sum_{k=1}^N \frac{b-a}{N} \cdot f\left(a + \frac{b-a}{N} \cdot k\right)$$

To be more precise, when dividing the interval $[a, b]$ into $N$ equal parts, the common length of these equal parts is $(b - a)/N$, and on the vertical we can approximate the average height by the above values of $f$. Thus, we have indeed the above formula.

(3) As an illustration for this method, let us integrate $f(x) = x^2$. For this purpose, we will need the following two formulae, which are both well-known:

$$1 + 2 + \ldots + N = \frac{N(N+1)}{2}$$

$$1^2 + 2^2 + \ldots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

To be more precise, in what regards the first formula, this is best seen by arguing that the average of the numbers $1, 2, \ldots, N$ being the number in the middle, we have:

$$\frac{1 + 2 + \ldots + N}{N} = \frac{N+1}{2}$$

Thus, we obtain the following formula, which is the one given above:

$$1 + 2 + \ldots + N = \frac{N(N+1)}{2}$$

(4) Next, let us compute $1^2 + 2^2 + \ldots + N^2$. This is not obvious at all, so as a preliminary here, let us go back to the computation of $1 + \ldots + N$, and try to find a new

proof there, which might have some chances to extend to $1^2 + 2^2 + \ldots + N^2$. The trick is to consider the following picture, with stacks going from 1 to $N$:

$$\begin{array}{ccccc}
 & & & & \square \\
 & & & & \vdots \\
 & & \square & \ldots & \square \\
 & \square & \square & \ldots & \square \\
\square & \square & \square & \ldots & \square
\end{array}$$

Now if we take two copies of this, and put them one on the top of the other, with a twist, in the obvious way, we obtain a rectangle having size $N \times (N+1)$. Thus:

$$2(1 + 2 + \ldots + N) = N(N+1)$$

But this gives the same formula as the one found before, in (3), namely:

$$1 + 2 + \ldots + N = \frac{N(N+1)}{2}$$

(5) Armed with this new method, let us study now $1^2+2^2+\ldots+N^2$. Here we obviously need to do some 3D geometry, namely taking the picture $P$ formed by a succession of solid squares, having sizes $1 \times 1$, $2 \times 2$, $3 \times 3$, and so on up to $N \times N$. Some quick thinking suggests that stacking 3 copies of $P$, with some obvious twists, will lead us to a parallelepiped. But this is not exactly true, and some further thinking shows that what we have to do is to add 3 more copies of $P$, leading to the following formula:

$$1^2 + 2^2 + \ldots + N^2 = \frac{N(N+1)(2N+1)}{6}$$

Alternatively, this latter formula can be of course proved by recurrence, if you prefer doing so. And so, one way or another, both formulae in (3) are now proved.

(6) With this discussed, let us go back now to our problem raised in (3), namely integrating the function $f(x) = x^2$. We can do this by using the general formula in (2), with technical help from the formulae established in (3), as follows:

$$\begin{aligned}
\int_a^b x^2 dx &= \lim_{N\to\infty} \sum_{k=1}^N \frac{b-a}{N} \left( a + \frac{b-a}{N} \cdot k \right)^2 \\
&= \lim_{N\to\infty} a^2(b-a) + a(b-a)^2 \frac{N+1}{N} + (b-a)^3 \frac{(N+1)(2N+1)}{6N^2} \\
&= a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} \\
&= \frac{b^3 - a^3}{3}
\end{aligned}$$

(7) Getting now, eventually, to the unit sphere in 3D, its equation is as follows:

$$x^2 + y^2 + z^2 = 1$$

As a first observation, the range of the first coordinate $x$ is as follows:

$$x \in [-1, 1]$$

Now when this first coordinate $x$ is fixed, the other coordinates $y, z$ vary on a circle, given by the equation $y^2 + z^2 = 1 - x^2$, and so having radius as follows:

$$r(x) = \sqrt{1 - x^2}$$

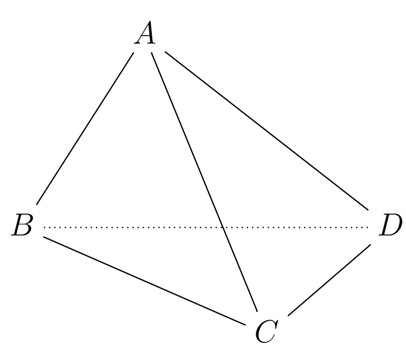
Thus, the vertical slice of our sphere at $x$ has area as follows:

$$a(x) = \pi r(x)^2 = \pi(1 - x^2)$$

(8) We conclude from this, using (6), that the volume of the sphere is given by:

$$\begin{aligned} V &= \pi \int_{-1}^{1} 1 - x^2 \, dx \\ &= \pi \left( \int_{-1}^{1} 1 \, dx - \int_{-1}^{1} x^2 dx \right) \\ &= \pi \left( 2 - \frac{2}{3} \right) \\ &= \frac{4\pi}{3} \end{aligned}$$

(9) Finally, the second assertion follows from the first one, by using the same "pizza" argument as in 2 dimensions, but this time with a 3 factor appearing, from the volume formula for tetrahedra. Indeed, consider a tetrahedron, in 3 dimensional space:




The volume of this tetrahedron is then given by the following formula, coming for instance by constructing a triangular prism, out of 3 copies of this tetrahedron:

$$\text{volume} = \frac{1}{3} \times \text{basis area} \times \text{height}$$

(10) Now recall the "pizza" argument from 2 dimensions, that we used several times, in this chapter. By using (9), the same will apply in 3D, giving the following formula:

$$\begin{aligned} A &= 3V \\ &= 3 \times \frac{4\pi}{3} \\ &= 4\pi \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Many other things can be said, as a continuation of the above, and we will come back to this later in this book, when systematically discussing analysis and integrals.

## 7e. Exercises

This was another standard trigonometry chapter, and as exercises, we have:

EXERCISE 7.23. *Further meditate on the two possible definitions for $\pi$, as length and area, and on the equivalence between them.*

EXERCISE 7.24. *Further build on our various computations above, for the decimals of $\pi$. The more decimals you compute here, the better that is.*

EXERCISE 7.25. *Learn a bit about some other properties of the number $\pi$, such as being irrational. Learn as well about common rational approximations of $\pi$.*

EXERCISE 7.26. *Learn also about more specialized properties of the number $\pi$, such as being transcendental. Although we will back to this, later.*

EXERCISE 7.27. *Meditate on how to recover the tangent, and the secondary trigonometric functions too, geometrically, as certain signed segments.*

EXERCISE 7.28. *Further work on the various proofs of $\sin t \leq t \leq \tan t$ given above, with the aim of making fully rigorous all of them.*

EXERCISE 7.29. *Try to improve the estimate $\sin t \simeq t$ for $t$ small, into something of type $\sin t \simeq t + at^2$, or even $\sin t \simeq t + at^2 + bt^3$, with $a, b \in \mathbb{R}$ to be found.*

EXERCISE 7.30. *Learn more about integrating functions, and then, following Einstein who told us to look at $\mathbb{R}^4$, compute the volume of the 4D sphere.*

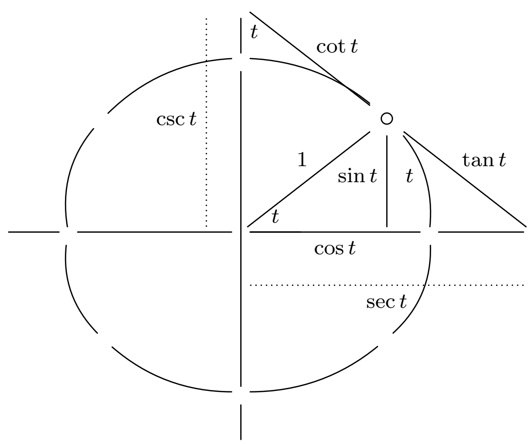
As bonus exercise, quite refreshing, learn about the Buffon needle too.

CHAPTER 8

# Inverse functions

## 8a. Functions, graphs

Time now for some calculus, in order to better understand the various trigonometric functions, namely sin, cos, tan, sec, csc, cot. We recall that these functions appear geometrically, according to the following catch-all picture, that we know well:




We have seen that we can do many things, using this picture. However, geometry is not everything, and a bit of algebraic abstraction, called "basic mathematical analysis", might help us. Let us start our study abstractly, with the following definition:

Definition 8.1. *A real function is a correspondence as follows:*

$$f:\mathbb{R}\to\mathbb{R}\quad,\quad x\to f(x)$$

*More generally, we can talk about functions* $f:X\to\mathbb{R}$*, with* $X\subset\mathbb{R}$*.*

Here the first notion is indeed something very intuitive, with this covering countless functions that we already know, as for instance the usual power functions:

$$f:\mathbb{R}\to\mathbb{R}\quad,\quad f(x)=x^n$$

In relation with trigonometry, the sine and cosine are functions of this type:

$$\sin:\mathbb{R}\to\mathbb{R}\quad,\quad\cos:\mathbb{R}\to\mathbb{R}$$

As for the second notion, this is something more general, which is useful too. As a basic example here, we have the inverse function, which cannot be defined at $x = 0$:

$$f : \mathbb{R} - \{0\} \to \mathbb{R} \quad , \quad f(x) = \frac{1}{x}$$

In relation with trigonometry, the tangent function $\tan x = \sin x / \cos x$ is obviously of this type, not defined at the points $x \in \mathbb{R}$ where the cosine vanishes, which are the odd multiples of $\pi/2$, and the same can be said about the secant $\sec x = 1/\cos x$:

$$\tan, \sec : \mathbb{R} - \left(\mathbb{Z}\pi + \frac{\pi}{2}\right) \to \mathbb{R}$$

As for the cosecant $\csc x = 1/\sin x$ and cotangent $\cot x = \cos x/\sin x$, these are not defined at the points $x \in \mathbb{R}$ where the sine vanishes, which are the multiples of $\pi$:

$$\csc, \cot : \mathbb{R} - \mathbb{Z}\pi \to \mathbb{R}$$

All this is quite interesting, bringing a bit of analytic order to what we have been doing so far, in relation with trigonometry, so let us record our findings, as follows:

PROPOSITION 8.2. *The trigonometric functions are as follows:*

(1) $\sin, \cos : \mathbb{R} \to \mathbb{R}$.
(2) $\tan, \sec : \mathbb{R} - (\mathbb{Z}\pi + \pi/2) \to \mathbb{R}$.
(3) $\csc, \cot : \mathbb{R} - \mathbb{Z}\pi \to \mathbb{R}$.

PROOF. This follows indeed from the above discussion, which itself basically comes from $\sin 0 = \cos(\pi/2) = 0$, and from our various rules for the trigonometric functions, when it comes to deal with angles $t \in \mathbb{R}$, outside the familiar domain $t \in [0, \pi/2]$. □

Now back to the general context of Definition 8.1, since we eventually allowed there the domain of the function to be an arbitrary set $X \subset \mathbb{R}$, why not doing the same for the image. We are led in this way into the following refinement of Definition 8.1:

DEFINITION 8.3 (update). *More generally, we call function any correspondence*

$$f : X \to Y \quad , \quad x \to f(x)$$

*with $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$.*

In practice, however, this will not change much to what we already had, from Definition 8.1. Indeed, any function $f : X \to Y$ with $Y \subset \mathbb{R}$ can be regarded as a function $f : X \to \mathbb{R}$ in the obvious way, by composing it with the inclusion $Y \subset \mathbb{R}$, as follows:

$$f : X \to Y \qquad \rightsquigarrow \qquad f : X \to Y \subset \mathbb{R}$$

However, Definition 8.3 can be something useful, in relation with the notions of injectivity, or surjectivity. Consider for instance the usual square function:

$$f : \mathbb{R} \to \mathbb{R} \quad , \quad f(x) = x^2$$

This function is certainly not injective, but we can make it injective, as follows:

$$f:[0,\infty)\to\mathbb{R}\quad,\quad f(x)=x^2$$

Which is good, but this latter function is still not surjective. However, we can make it surjective, by using the framework of Definition 8.3, as follows:

$$f:[0,\infty)\to[0,\infty)\quad,\quad f(x)=x^2$$

Obviously, this latter trick, in relation with surjectivity, can work for any function, in obvious way, by setting $Y=f(X)$. Let us record this finding, as follows:

PROPOSITION 8.4. *Any function $f:X\to\mathbb{R}$ can be made into a function*

$$f:X\to Y$$

*which is surjective, simply by setting $Y=f(X)$.*

PROOF. This is indeed something clear from definitions, as explained above. □

With this done, you might perhaps ask at this point, why not pulling now a similar trick for injectivity, a bit as we did before for $f(x)=x^2$, by restricting the domain. Well, the problem is that this is not really possible, in a general way, convenient for all functions, because depending on the exact function $f:\mathbb{R}\to\mathbb{R}$ that we have in mind, restricting the domain to this or that $X\subset\mathbb{R}$, as to have $f$ injective, remains something subjective.

Back now to trigonometry, all this is quite interesting, and Proposition 8.2 has the following refinement, in terms of our new function formalism, from Definition 8.3:

PROPOSITION 8.5. *The trigonometric functions are as follows, surjective:*

(1) $\sin:\mathbb{R}\to[-1,1]$.
(2) $\cos:\mathbb{R}\to[-1,1]$.
(3) $\tan:\mathbb{R}-(\mathbb{Z}\pi+\pi/2)\to\mathbb{R}$.
(4) $\sec:\mathbb{R}-(\mathbb{Z}\pi+\pi/2)\to\mathbb{R}-(-1,1)$.
(5) $\csc:\mathbb{R}-\mathbb{Z}\pi\to\mathbb{R}-(-1,1)$.
(6) $\cot:\mathbb{R}-\mathbb{Z}\pi\to\mathbb{R}$.

PROOF. This follows indeed Proposition 8.2, with the only change in the formulae coming by restricting the range of the sine, cosine, secant and cosecant. □

Getting now to injectivity issues, as explained above, restricting the domain of an arbitrary function $f:X\to Y$ is something quite subjective. However, for the trigonometric functions, such things can be done, with some common sense, by using:

THEOREM 8.6. *The trigonometric functions are as follows:*

(1) $\sin,\cos$ *are periodic, of period* $2\pi$.
(2) $\sec,\csc$ *are also periodic of period* $2\pi$.
(3) $\tan,\cot$ *are periodic too, of period* $\pi$.

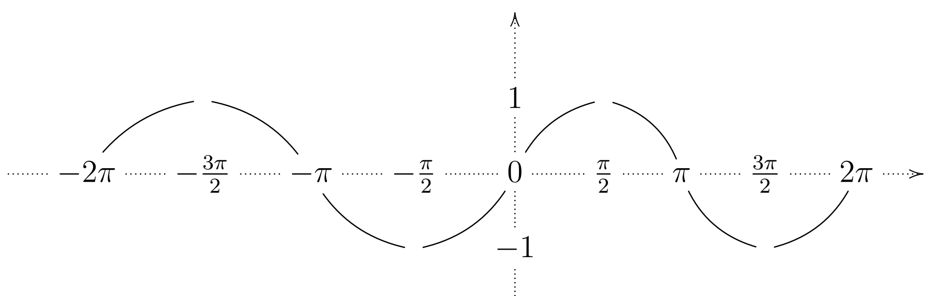
PROOF. Since the angles $t \in \mathbb{R}$, regarded geometrically, are periodic of period $2\pi$, so must be all their trigonometric functions, and this gives (1,2,3), with $2\pi$ everywhere. However, for the tangent something interesting happens, as follows:

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x} = \tan x$$

Thus tan is indeed periodic of period $\pi$, and the same happens for the cotangent. □

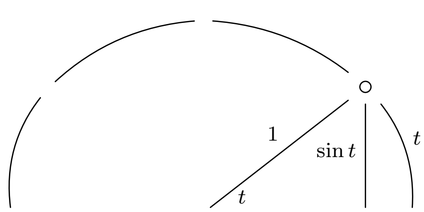
Before going further and exploiting this fact, in relation with injectivity, along the above lines, let us draw as well some pictures. Regarding the sine, we have:

PROPOSITION 8.7. *The graph of* $\sin : \mathbb{R} \to [-1,1]$ *is as follows,*




*with this pattern being repeated indefinitely, to the left and to the right.*

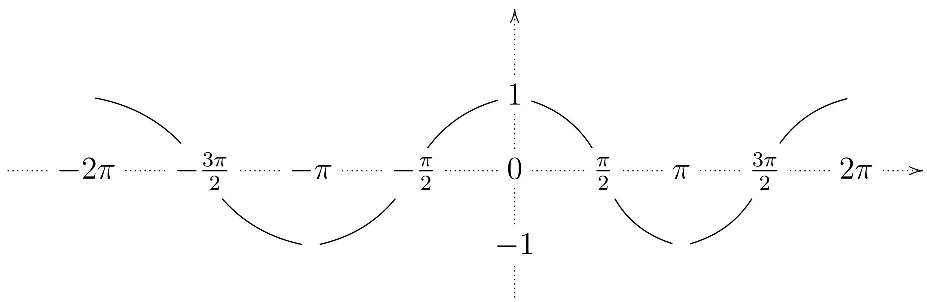
PROOF. We know that the sine appears as follows, and a bit of thinking, say with the angle $t$ turning counterclockwise, leads to the picture in the statement:




Observe also that the graph crosses the horizontal axis at 0 at a 45° angle, and with the other crossing angles being $45°, 135°$ too, by periodicity, due to $\sin t \simeq t$. □

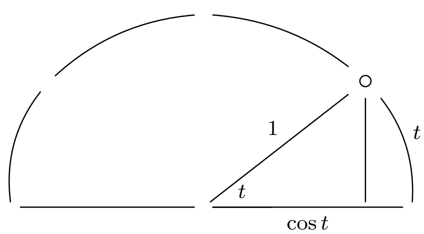
Regarding the cosine, we have a similar result here, as follows:

PROPOSITION 8.8. *The graph of* $\cos:\mathbb{R}\to[-1,1]$ *is as follows,*




*with this pattern being repeated indefinitely, to the left and to the right.*

PROOF. We know that the cosine appears as follows, and a bit of thinking, with the angle $t$ turning counterclockwise, leads to the picture in the statement:




Observe that what we get is the graph of the sine function, translated by $\pi/2$, and this due to the following formula, that we know well since chapter 5:

$$\cos x=\sin\left(x+\frac{\pi}{2}\right)$$

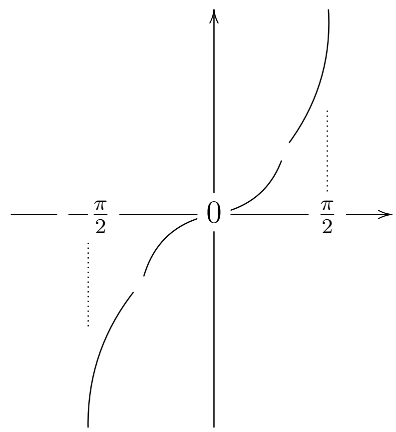
In particular, the crossings with the horizontal axis all happen at $45°,135°$ angles. Moreover, we can also say that the maximum at 0, and so by periodicity, all maxima and minima, happen with the graph being flattened there, with this coming from the following estimate, coming from $\sin t\simeq t$ and Pythagoras, that we know from chapter 7:

$$\cos t\simeq 1-\frac{t^2}{2}$$

To be more precise, with a bit of analysis know-how, or just thinking, at what slope of a curve should mean, the lack of degree 1 term, in $t$ itself, in the above formula, tells us precisely this, that the graph must be flattened around 0. Note also that, by translation, the same can be said about the various minima and maxima of the sine function. □

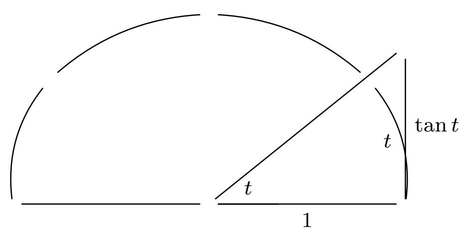
Regarding now the tangent, the result here is as follows:

PROPOSITION 8.9. *The graph of* $\tan:\mathbb{R}-(\mathbb{Z}\pi+\pi/2)\to\mathbb{R}$ *is as follows,*




*with this pattern being repeated indefinitely, to the left and to the right.*

PROOF. We know that the tangent appears as follows, and a bit of thinking, with the angle $t$ turning counterclockwise, leads to the picture in the statement:




As before with the sine and cosine, we can say more about this, in regards with the slopes at various particular points. Indeed, the graph must cross the horizontal axis at 0 at a 45° angle, due to the following estimate, that we know from chapter 7:

$$\tan t\simeq t$$

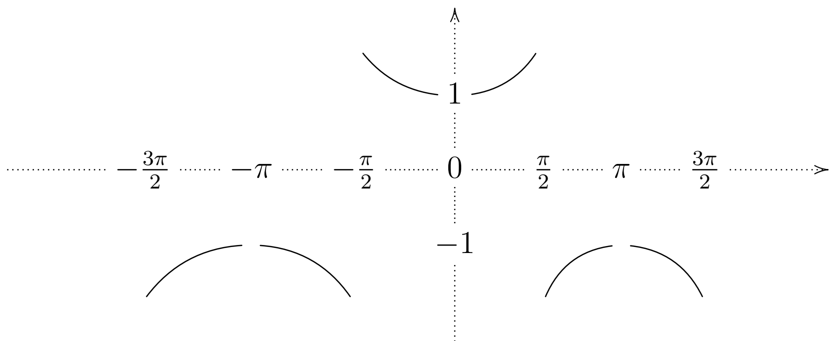
Also, the graph must be tangent to the vertical dotted lines at $\pm\pi/2$, as the above picture suggests, with this coming straight from the formula $\tan=\sin/\cos$. $\square$

With this discussed, let us turn now to the secondary trigonometric functions:

$$\sec x=\frac{1}{\cos x}\quad,\quad \csc x=\frac{1}{\sin x}\quad,\quad \cot x=\frac{1}{\tan x}$$

Regarding the secant, the result here is as follows, coming from the one for cos:

Proposition 8.10. *The graph of* $\sec:\mathbb{R}-(\mathbb{Z}\pi+\pi/2)\to\mathbb{R}$ *is as follows,*

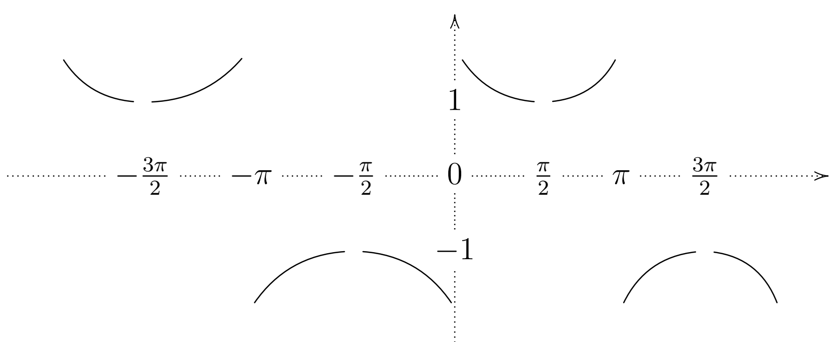



*with this pattern being repeated indefinitely, to the left and to the right.*

Proof. This comes indeed from $\sec=1/\cos$, by applying $x\to 1/x$ to the graph of cos, from Proposition 8.8. Thus, we obtain the graph in the statement, with flattened curves at the multiples of $\pi$, and with asymptotes at the multiples of $\pi$ plus $\pi/2$. □

Regarding now the cosecant, the result here is quite similar, as follows:

Proposition 8.11. *The graph of* $\csc:\mathbb{R}-\mathbb{Z}\pi\to\mathbb{R}$ *is as follows,*

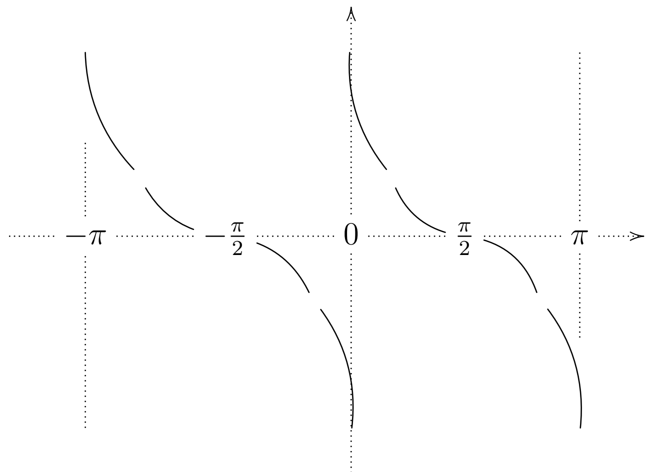



*with this pattern being repeated indefinitely, to the left and to the right.*

Proof. This comes indeed from $\csc=1/\sin$, by applying $x\to 1/x$ to the graph of sin, from Proposition 8.7. Equivalently, we can get this from Proposition 8.10 via $\csc x=\sec(x-\pi/2)$, which shows that the graph is the one of sec, translated by $\pi/2$. □

Finally, regarding cot, the result here is similar to Proposition 8.9, as follows:

Proposition 8.12. *The graph of* $\cot:\mathbb R-\mathbb Z\pi\to\mathbb R$ *is as follows,*




*with this pattern being repeated indefinitely, to the left and to the right.*

Proof. This comes indeed from $\cot=1/\tan$, by applying $x\to 1/x$ to the graph of tan, from Proposition 8.9. In practice, this amounts in symmetrizing the graph there, and then translating it by $\pi/2$, and we will leave some thinking here as an exercise. □

Summarizing, many interesting things going on, when it comes to draw the graphs of trigonometric functions, and with this being, quite obviously, just the tip of the iceberg. We will talk more about this in Part IV, when systematically doing calculus.

Getting back now to what we wanted to do, namely updating Proposition 8.5 by using the various periodicity properties from Theorem 8.6, this remains something a bit subjective, but armed with a bit of common sense, we can do this as follows:

Theorem 8.13. *The trigonometric functions are as follows, modulo periodicity, and with these functions being surjective:*

(1) $\sin:[0,2\pi]\to[-1,1]$.
(2) $\cos:[0,2\pi]\to[-1,1]$.
(3) $\tan:(-\pi/2,\pi/2)\to\mathbb R$.
(4) $\sec:(-\pi/2,3\pi/2)-\{\pi/2\}\to\mathbb R-(-1,1)$.
(5) $\csc:(0,2\pi)-\{\pi\}\to\mathbb R-(-1,1)$.
(6) $\cot:(0,\pi)\to\mathbb R$.

Proof. This follows from Proposition 8.5 and Theorem 8.6, but with the choice of the domain remaining something quite subjective, our comments being as follows:

(1) Here the choice $\sin:[0,2\pi]\to[-1,1]$ is reasonable, but if you prefer your mathematics to be centered at 0, you will rather want to go with $\sin:[-\pi,\pi]\to[-1,1]$.

(2) Some comment here, with $\cos : [0, 2\pi] \to [-1, 1]$ being reasonable, and with an alternative choice, which can be sometimes useful, being $\cos : [-\pi, \pi] \to [-1, 1]$.

(3) No comment here, $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ being obviously optimal.

(4) This is more tricky, another choice being $\sec : (0, 2\pi) - \{\pi/2, 3\pi/2\} \to \mathbb{R} - (-1, 1)$, and yet another choice being $\sec : (-\pi, \pi) - \{-\pi/2, \pi/2\} \to \mathbb{R} - (-1, 1)$.

(5) Our choice here, $\csc : (0, 2\pi) - \{\pi\} \to \mathbb{R} - (-1, 1)$, is something quite reasonable, although $\csc : (-\pi, \pi) - \{0\} \to \mathbb{R} - (-1, 1)$ can be sometimes useful too.

(6) No comment here, $\cot : (0, \pi) \to \mathbb{R}$ being obviously optimal. □

Moving on, let us recall that the trigonometric functions have as well a number of supplementary properties, coming in relation with $x \to -x$, which are not captured by the periodicity properties from Theorem 8.6, and can be summarized as follows:

PROPOSITION 8.14. *The trigonometric functions are as follows:*

(1) $\sin, \csc$ *are even,* $f(x) = f(-x)$.
(2) $\cos, \sec$ *are odd,* $f(x) = -f(x)$.
(3) $\tan, \cot$ *are odd too,* $f(x) = -f(x)$.

PROOF. This is indeed something that we know well, which for $\sin, \cos$ comes from definitions, and for the other trigonometric functions comes from this. □

But with this, we can further fine-tune Theorem 8.13, and again armed with a bit of common sense, in order to deal with uncertainty, we are led in this way to:

THEOREM 8.15. *The trigonometric functions are as follows, modulo periodicity and parity, and with these functions being bijective:*

(1) $\sin : [-\pi/2, \pi/2] \to [-1, 1]$.
(2) $\cos : [0, \pi] \to [-1, 1]$.
(3) $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$.
(4) $\sec : (0, \pi) - \{\pi/2\} \to \mathbb{R} - (-1, 1)$.
(5) $\csc : (-\pi/2, \pi/2) - \{0\} \to \mathbb{R} - (-1, 1)$.
(6) $\cot : (0, \pi) \to \mathbb{R}$.

*Moreover,* sin *is increasing,* cos *decreasing,* tan *increasing, and* cot *decreasing.*

PROOF. This follows indeed from Theorem 8.13 and Proposition 8.14. Observe also, as an interesting fact, that there is no reasonable way of making sec, csc monotone. □

And with this, end of our preliminary analytic study of the trigonometric functions. We have learned many interesting things about them, and looking retrospectively at what we did so far in this chapter, all this learning came from some trivialities regarding the general functions, namely those in Definition 8.1 and Definition 8.3. Nice.

## 8b. Continuity basics

Following the same strategy as before, namely trust in pure mathematics, let us temporarily say goodbye now to our beloved trigonometric functions, but we will be back to them soon, no worries, and develop some more theory for the general functions.

The idea will be that of focusing our study on the functions $f:\mathbb{R}\to\mathbb{R}$ which are suitably regular, with the hope of getting into interesting mathematics. And, in what regards these regularity properties, the most basic of them is continuity:

DEFINITION 8.16. *A function $f:\mathbb{R}\to\mathbb{R}$, or more generally $f:X\to\mathbb{R}$, with $X\subset\mathbb{R}$ being a subset, is called continuous when, for any $x_n,x\in X$:*

$$x_n\to x\implies f(x_n)\to f(x)$$

*Also, we say that $f:X\to\mathbb{R}$ is continuous at a given point $x\in X$ when the above condition is satisfied, for that point $x$.*

Regarding now the basic examples of continuous functions, there are many of them, and we will discuss them in a moment, once we will have some basic tools, in order to prove that this or that function is continuous or not, without much pain. As a matter, however, of having a first illustration for Definition 8.16, let us record here:

PROPOSITION 8.17. *The basic power functions, namely*

$$f(x)=x^k$$

*with $k\in\mathbb{N}$, are all continuous.*

PROOF. According to Definition 8.16, we want to prove that we have:

$$x_n\to x\implies x_n^k\to x^k$$

(1) A first method is by using the standard results regarding the sequences. To be more precise, we know from basic analysis that the following formula holds:

$$\lim_{n\to\infty}x_ny_n=\lim_{n\to\infty}x_n\lim_{n\to\infty}y_n$$

But with $x_n=y_n$, this leads to the following formula:

$$\lim_{n\to\infty}x_n^2=\left(\lim_{n\to\infty}x_n\right)^2$$

Obviously, we can iterate this method, and so for any $k\in\mathbb{N}$, we have:

$$\lim_{n\to\infty}x_n^k=\left(\lim_{n\to\infty}x_n\right)^k$$

But now, assuming $x_n\to x$ as above, this formula gives, as desired:

$$\lim_{n\to\infty}x_n^k=x^k$$

(2) As a second method, more direct, we must estimate quantities $(x+t)^k - x^k$, with $t$ small. But we can do this with the binomial formula, which gives, for $|t| \leq 1$:

$$\begin{aligned}
|(x+t)^k - x^k| &= \left|\sum_{s=0}^{k}\binom{k}{s}x^{k-s}t^s - x^k\right| \\
&= \left|\sum_{s=1}^{k}\binom{k}{s}x^{k-s}t^s\right| \\
&\leq \sum_{s=1}^{k}\binom{k}{s}|x|^{k-s}|t|^s \\
&\leq |t|\sum_{s=1}^{k}\binom{k}{s}|x|^{k-s} \\
&\leq |t|\sum_{s=0}^{k}\binom{k}{s}|x|^{k-s} \\
&= |t|(1+|x|)^k
\end{aligned}$$

Now assume $x_n \to x$. We can then write $x_n = x + t_n$, and by choosing our $n >> 0$ as to have $|t_n| \leq 1$, we can use the above estimate, which gives:

$$|x_n^k - x^k| \leq |t_n|(1+|x|^k)$$

Now since we have $t_n \to 0$, we obtain from this $x_n^k \to x^k$, as desired. □

Getting back now to general theory, and to Definition 8.16 as stated, many things can be said, about the continuous functions. We will discuss this, slowly, in what follows.

To start with, there are many other equivalent formulations of the notion of continuity, with a well-known, useful, and much feared one, being as follows:

THEOREM 8.18. *A function $f : X \to \mathbb{R}$ is continuous when*

$$\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0, |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

*holds.*

PROOF. Let us prove this, with no fear. According to Definition 8.16, in order for our function $f$ to be continuous, the following must happen, for any $x \in X$:

$$x_n \to x \implies f(x_n) \to f(x)$$

Now when reminding what convergence of a sequence exactly means, for both the convergences $x_n \to x$ and $f(x_n) \to f(x)$, we are led to the conclusion in the statement. □

In order to get now towards examples of continuous functions, let us start with the following theoretical result, regarding the various operations on functions:

THEOREM 8.19. *If $f,g$ are continuous, then so are:*
(1) $f+g$.
(2) $fg$.
(3) $f/g$.
(4) $f\circ g$.

PROOF. Before anything, we should mention that the claim is that (1-4) hold indeed, provided that at the level of domains and ranges, the statement makes sense. For instance in (1,2,3) we are talking about functions having the same domain, and with $g(x)\neq 0$ for the needs of (3), and there is a similar discussion regarding (4).

(1) The claim here is that if both $f,g$ are continuous at a point $x$, then so is the sum $f+g$. But this is clear from the similar result for sequences, namely:

$$\lim_{n\to\infty}(x_n+y_n)=\lim_{n\to\infty}x_n+\lim_{n\to\infty}y_n$$

(2) Again, the statement here is similar, and the result follows from:

$$\lim_{n\to\infty}x_ny_n=\lim_{n\to\infty}x_n\lim_{n\to\infty}y_n$$

(3) Here the claim is that if both $f,g$ are continuous at $x$, with $g(x)\neq 0$, then $f/g$ is continuous at $x$. In order to prove this, observe that by continuity, $g(x)\neq 0$ shows that $g(y)\neq 0$ for $|x-y|$ small enough. Thus we can assume $g\neq 0$, and with this assumption made, the result follows from the similar result for sequences, namely:

$$\lim_{n\to\infty}x_n/y_n=\lim_{n\to\infty}x_n/\lim_{n\to\infty}y_n$$

(4) Here the claim is that if $g$ is continuous at $x$, and $f$ is continuous at $g(x)$, then $f\circ g$ is continuous at $x$. But this is clear, coming from:

$$\begin{aligned}x_n\to x\quad&\implies\quad g(x_n)\to g(x)\\&\implies\quad f(g(x_n))\to f(g(x))\end{aligned}$$

Alternatively, using that scary $\varepsilon,\delta$ condition from Theorem 8.18, let us pick $\varepsilon>0$. Since $f$ is continuous at $g(x)$, we can find $\delta>0$ such that:

$$|g(x)-z|<\delta\implies|f(g(x))-f(z)|<\varepsilon$$

On the other hand, since $g$ is continuous at $x$, we can find $\gamma>0$ such that:

$$|x-y|<\gamma\implies|g(x)-g(y)|<\delta$$

Now by combining the above two inequalities, with $z=g(y)$, we obtain:

$$|x-y|<\gamma\implies|f(g(x))-f(g(y))|<\varepsilon$$

Thus, the composition $f\circ g$ is continuous at $x$, as desired. □

At the level of examples now, we first have the following result:

THEOREM 8.20. *The following functions are continuous:*
(1) $x^n$, *with* $n \in \mathbb{Z}$.
(2) $P/Q$, *with* $P, Q \in \mathbb{R}[X]$.
(3) $e^x = \sum_k x^k/k!$.

PROOF. This is a mixture of trivial and non-trivial results, as follows:

(1) Since $f(x) = x$ is continuous, by using Theorem 8.19 we obtain the result for exponents $n \in \mathbb{N}$, and then for general exponents $n \in \mathbb{Z}$ too. Observe that this generalizes Proposition 8.17, gone all that computations, by some kind of abstract miracle.

(2) The statement here, which generalizes (1), follows exactly as (1), by using the various findings from Theorem 8.19, and with the comment of course that, a bit like in Theorem 8.19 before, $P/Q$ is considered as function outside the zeroes of $Q$.

(3) This is something quite tricky, that will take us some time to understand, but since $e^x$ is crucially related to trigonometry, as we will discover later, all this effort will be worth it. To start with, we can define a number $e = 2.71828\ldots$ as follows:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

To be more precise, it is quite routine to show, by using the binomial formula and the arithmetic-geometric inequality, that the sequence on the right converges indeed, to a certain $e \in [2, 3]$. Next, again by using the binomial formula and various estimates, we have the following formula, which can stand as an alternative definition for $e$:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Observe that this series converges very fast, and in constrast with the $\pi$ mess from chapter 7, this can be used in order to reach to the known figure for $e$, namely:

$$e = 2.71828\ldots$$

As a main claim now, we have the following formula, valid for any $x \in \mathbb{R}$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

In order to prove this, consider the series on the right, $f(x)=\sum_k x^k/k!$, whose convergence is fast too, and easy to establish. By using the binomial formula, we have:

$$\begin{aligned}
f(x+y) &= \sum_{k=0}^\infty \frac{(x+y)^k}{k!}\\
&= \sum_{k=0}^\infty \sum_{s=0}^k \binom{k}{s}\cdot\frac{x^s y^{k-s}}{k!}\\
&= \sum_{k=0}^\infty \sum_{s=0}^k \frac{x^s y^{k-s}}{s!(k-s)!}\\
&= f(x)f(y)
\end{aligned}$$

Thus, we have good evidence that $f(x)$ should be a power function, and so we are on the way of proving $f(x)=e^x$, with $e=f(1)$. Now let us prove that $f$ is continuous. The continuity of $f$ is clear at $x=0$, and in general, this can be deduced as follows:

$$\begin{aligned}
\lim_{t\to0} f(x+t) &= \lim_{t\to0} f(x)f(t)\\
&= f(x)\lim_{t\to0} f(t)\\
&= f(x)\cdot 1\\
&= f(x)
\end{aligned}$$

Time now to put everything together, and prove our claim. We know that the series $f(x)=\sum_k x^k/k!$ is continuous, and satisfies the following conditions:

$$f(0)=1 \quad , \quad f(1)=e \quad , \quad f(x+y)=f(x)f(y)$$

But this gives $f(x)=e^x$, as desired, first for $x\in\mathbb{N}$, in the obvious way, then for $x\in\mathbb{Z}$, and even $x\in\mathbb{Q}$, again by simple algebra, and finally for $x\in\mathbb{R}$, by continuity. □

Good news, we can go back now to the trigonometric functions, and we have:

THEOREM 8.21. *The trigonometric functions, considered on their maximal domains,*

(1) $\sin,\cos:\mathbb{R}\to\mathbb{R}$.
(2) $\tan,\sec:\mathbb{R}-(\mathbb{Z}\pi+\pi/2)\to\mathbb{R}$.
(3) $\csc,\cot:\mathbb{R}-\mathbb{Z}\pi\to\mathbb{R}$.

*are continuous on these maximal domains.*

PROOF. This is something that we already talked about in chapter 7, in some detail, but always good to talk about this again. The idea with all this is as follows:

(1) We must first prove here that $x_n\to x$ implies $\sin x_n\to\sin x$, which in practice amounts in proving that $\sin(x+y)\simeq\sin x$ for $y$ small. But this follows from:

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$

Indeed, with this formula in hand, we can establish the continuity of $\sin x$, as follows, with the limits at 0 which are used being both clear on pictures:

$$\begin{aligned}\lim_{y\to 0}\sin(x+y) &= \lim_{y\to 0}(\sin x\cos y+\cos x\sin y)\\ &= \sin x\lim_{y\to 0}\cos y+\cos x\lim_{y\to 0}\sin y\\ &= \sin x\cdot 1+\cos x\cdot 0\\ &= \sin x\end{aligned}$$

Moving ahead now with $\cos x$, here the continuity follows from the continuity of $\sin x$, by using the following formula, which is obvious from definitions:

$$\cos x=\sin\left(\frac{\pi}{2}-x\right)$$

Alternatively, we can use the same method as for sin, and we get, as desired:

$$\begin{aligned}\lim_{y\to 0}\cos(x+y) &= \lim_{y\to 0}(\cos x\cos y-\sin x\sin y)\\ &= \cos x\lim_{y\to 0}\cos y-\sin x\lim_{y\to 0}\sin y\\ &= \cos x\cdot 1-\sin x\cdot 0\\ &= \cos x\end{aligned}$$

(2) The fact that the functions $\tan x$ and $\sec x$ are continuous too is clear from the fact that $\sin x$, $\cos x$ are continuous, by using Theorem 8.19 (3).

(3) As for the fact that the functions $\csc x$ and $\cot x$ are continuous too, this is again clear from the fact that $\sin x$, $\cos x$ are continuous, by using Theorem 8.19 (3). □

As a last piece of general theory, regarding the continuous functions, some functions are "more continuous than some other", as shown by the following result:

THEOREM 8.22. *Consider the following properties, regarding $f:X\to\mathbb{R}$ with $X\subset\mathbb{R}$:*

(1) *$f$ has the following property, for some $K>0$, called Lipschitz property:*

$$|f(x)-f(y)|\leq K|x-y|$$

(2) *$f$ is uniformly continuous, in the sense that the following happens:*

$$\forall\varepsilon>0,\exists\delta>0,|x-y|<\delta\implies|f(x)-f(y)|<\varepsilon$$

(3) *$f$ is continuous in the usual sense, namely:*

$$\forall x\in X,\forall\varepsilon>0,\exists\delta>0,|x-y|<\delta\implies|f(x)-f(y)|<\varepsilon$$

*We have then* (1) $\implies$ (2) $\implies$ (3). *Also, the converse implications do not hold.*

Proof. This is something quite self-explanatory, the idea being as follows:

(1) $\implies$ (2) This is clear, coming by talking $\delta = \varepsilon/K$.

(2) $\implies$ (3) This is something which is plainly trivial.

(3) $\not\Longrightarrow$ (2) Indeed, $x^2$ is continuous but not uniformly continuous.

(2) $\not\Longrightarrow$ (1) Indeed, $\sqrt{|x|}$ is uniformly continuous but not Lipschitz. □

In practice now, and for instance in relation with the trigonometric functions, restricted to various intervals $[a,b]$, all this seems to have something to do with the slope of the graph of $f$, computed at various points of $[a,b]$. We will be back to this later, in Part IV, when talking slopes of graphs, or derivatives, which can help with this.

## 8c. Inverse functions

Getting back now to questions raised in the beginning of this chapter, let us discuss now bijectivity and inversion problems. To start with, we have:

Theorem 8.23. *Given a bijective function $f : X \to Y$, its inverse function*

$$f^{-1} : Y \to X$$

*is obtained by flipping the graph over the $x = y$ diagonal of the plane.*

Proof. This is indeed something quite clear and intuitive, because by definition of the inverse function $f^{-1} : Y \to X$, this is given by the following formula:

$$y = f(x) \iff f^{-1}(y) = x$$

Thus, in practice, drawing the graph of $f^{-1} : Y \to X$ amounts in taking the graph of $f : X \to Y$ and interchanging the coordinates, $x \leftrightarrow y$, as indicated. □

As a basic application of this technology, dealing with the various trigonometric functions that we are interested in, from Theorem 8.15, we have the following result:

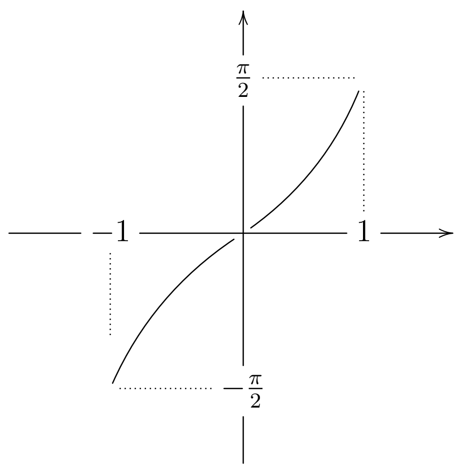
Theorem 8.24. *We can talk about the inverse trigonometric functions,*

(1) $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$.
(2) $\arccos : [-1,1] \to [0,\pi]$.
(3) $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$.
(4) $\text{arcsec} : \mathbb{R} - (-1,1) \to (0,\pi) - \{\pi/2\}$.
(5) $\text{arccsc} : \mathbb{R} - (-1,1) \to (-\pi/2,\pi/2) - \{0\}$.
(6) $\text{arccot} : \mathbb{R} \to (0,\pi)$.

*whose graphs can be obtained by flipping those of* $\sin, \cos, \tan, \sec, \csc, \cot$.

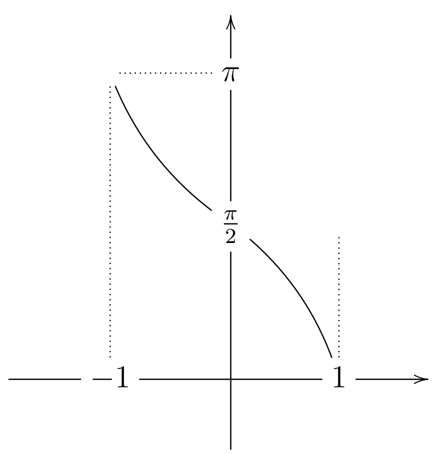
Proof. This is something self-explanatory, based on Theorem 8.15, as follows:

(1) Consider the function $\sin : [-\pi/2, \pi/2] \to [-1, 1]$, which is bijective. Its inverse function $\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$, obtained by flipping the graph, is as follows:



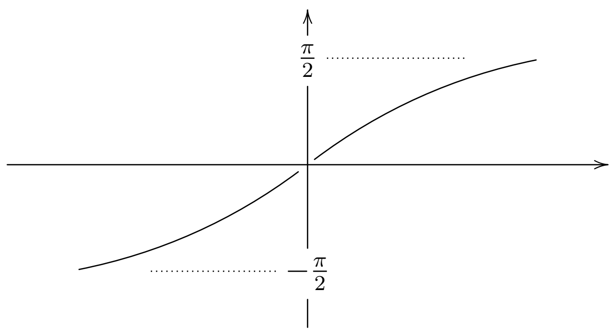

Observe that the graph crosses the horizontal axis at 45°, and also that we have vertical tangents at left and at right, with this coming from our knowledge of sin.

(2) Consider the function $\cos : [0, \pi] \to [-1, 1]$, which is bijective. Its inverse function $\arccos : [-1, 1] \to [0, \pi]$, obtained by flipping the graph, is then as follows:



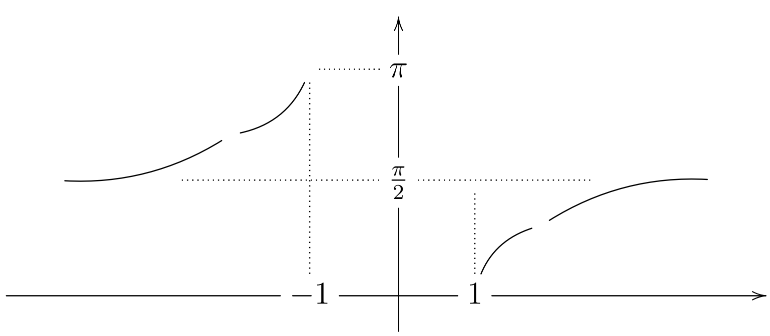

Observe that the graph crosses the vertical axis at 45°, and also that we have vertical tangents at left and at right, with this coming from our knowledge of cos.

(3) Consider the function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$, which is bijective. Its inverse function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$, obtained by flipping the graph, is as follows:



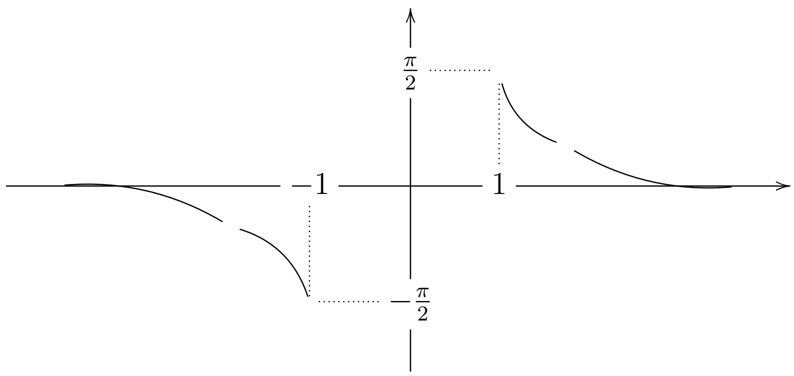

Observe that the graph crosses the coordinate axes at 45°, and that we have horizontal asymptotes on top and bottom, with this coming from our knowledge of tan.

(4) Consider the function $\sec : (0, \pi) - \{\pi/2\} \to \mathbb{R} - (-1, 1)$, which is bijective. Its inverse function $\text{arcsec} : \mathbb{R} - (-1, 1) \to (0, \pi) - \{\pi/2\}$ is then as follows:



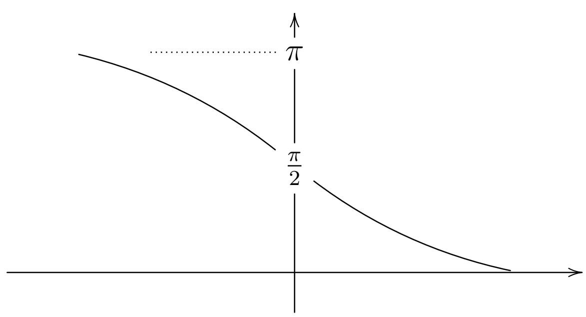

Observe that we have various asymptotes on the horizontal and vertical, as indicated by dotted lines in the above picture, with this coming from our knowledge of sec.

(5) Consider the function $\csc : (-\pi/2, \pi/2) - \{0\} \to \mathbb{R} - (-1, 1)$, which is bijective. Its inverse function $\operatorname{arccsc} : \mathbb{R} - (-1, 1) \to (-\pi/2, \pi/2) - \{0\}$ is then as follows:




Observe that we have various asymptotes on the horizontal and vertical, as indicated by dotted lines in the above picture, with this coming from our knowledge of csc.

(6) Consider the function $\cot : (0, \pi) \to \mathbb{R}$, which is bijective. Its inverse function $\operatorname{arccot} : \mathbb{R} \to (0, \pi)$, obtained by flipping the graph, is as follows:




Observe that the graph crosses the vertical axis at 45°, and that we have horizontal asymptotes on top and bottom, with this coming from our knowledge of cot. □

Many other things can be said about the inverse trigonometric functions, notably with all sorts of formulae for them, coming from the formulae that we know well for the usual trigonometric functions. We will leave some exploration here as an exercise.

## 8d. Approximation

Getting back now to the basics, are the limits of continuous functions continuous? And the answer here is no, as shown by the following result, featuring arctan:

THEOREM 8.25. *The basic discontinuous function, namely*

$$sgn(x)=\begin{cases}-1 & , \quad x<0\\ 0 & , \quad x=0\\ 1 & , \quad x>0\end{cases}$$

*can be approximated by suitable modifications of* $\arctan(x)$.

PROOF. We know that $\arctan(x)$ looks a bit like $sgn(x)$, but one problem comes from the fact that its image is $[-\pi/2,\pi/2]$, instead of the desired $[-1,1]$. Thus, we must first rescale $\arctan(x)$ by $\pi/2$, which amounts in considering the following function:

$$f(x)=\frac{2}{\pi}\arctan(x)$$

Now with this done, we must stretch the variable $x$, as to get our function closer and closer to $sgn(x)$. This can be done in several ways, a standard one being as follows:

$$g_n(x)=\frac{2}{\pi}\arctan(nx)$$

So, let us see if this works. First, we have the following computation, for $x>0$:

$$\begin{aligned}\lim_{n\to\infty}g_n(x) &= \frac{2}{\pi}\lim_{n\to\infty}\arctan(nx)\\ &= \frac{2}{\pi}\arctan(\infty)\\ &= \frac{2}{\pi}\cdot\frac{\pi}{2}\\ &= 1\end{aligned}$$

Similarly, we have the following computation, this time for $x<0$:

$$\begin{aligned}\lim_{n\to\infty}g_n(x) &= \frac{2}{\pi}\lim_{n\to\infty}\arctan(nx)\\ &= \frac{2}{\pi}\arctan(-\infty)\\ &= \frac{2}{\pi}\left(-\frac{\pi}{2}\right)\\ &= -1\end{aligned}$$

Finally, for $x=0$ the limit is that of the constant 0 sequence, as follows:

$$\lim_{n\to\infty}g_n(0)=\lim_{n\to\infty}0=0$$

We conclude from this that we have the following pointwise convergence:

$$\lim_{n\to\infty} g_n(x) = \begin{cases} -1 & , \quad x<0 \\ 0 & , \quad x=0 \\ 1 & , \quad x>0 \end{cases}$$

In other words, we have proved that we have the following approximation:

$$\lim_{n\to\infty} \frac{2}{\pi}\arctan(nx) = sgn(x)$$

Thus, we are led to the conclusion in the statement. □

Sumarizing, we are a bit in trouble, because we would like to have in our bag of theorems something saying that $f_n \to f$ with $f_n$ continuous implies $f$ continuous. Fortunately, this can be done, with a suitable refinement of the notion of convergence, as follows:

Definition 8.26. *We say that $f_n$ converges uniformly to $f$, and write $f_n \to_u f$, if:*

$$\forall\varepsilon>0, \exists N\in\mathbb{N}, \forall n\geq N, |f_n(x)-f(x)|<\varepsilon, \forall x$$

*That is, the same condition as for $f_n \to f$ must be satisfied, but with the $\forall x$ at the end.*

And it is this "$\forall x$ at the end" which makes the difference, and will make our theory work. Indeed, we have the following result, based on the above definition:

Theorem 8.27. *Assuming that $f_n$ are continuous, and that*

$$f_n \to_u f$$

*then $f$ is continuous. That is, uniform limit of continuous functions is continuous.*

Proof. Let us try indeed to prove that the limit $f$ is continuous at some point $x$. For this, we pick a number $\varepsilon>0$. Since $f_n \to_u f$, we can find $N\in\mathbb{N}$ such that:

$$|f_N(z)-f(z)| < \frac{\varepsilon}{3} \quad , \quad \forall z$$

On the other hand, since $f_N$ is continuous at $x$, we can find $\delta>0$ such that:

$$|x-y|<\delta \implies |f_N(x)-f_N(y)| < \frac{\varepsilon}{3}$$

But with this, we are done. Indeed, for $|x-y|<\delta$ we have:

$$\begin{aligned} |f(x)-f(y)| &\leq |f(x)-f_N(x)| + |f_N(x)-f_N(y)| + |f_N(y)-f(y)| \\ &\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Thus, the limit function $f$ is continuous at $x$, and we are done. □

Getting now to more concrete things, we have the following fundamental result, due to Weierstrass, regarding the approximation of functions by polynomials:

THEOREM 8.28 (Weierstrass). *Any continuous function on a closed interval*

$$f : [a, b] \to \mathbb{R}$$

*can be uniformly approximated by polynomials.*

PROOF. This is indeed something very classical, with a well-known, constructive proof, being by using an approximation by suitable Bernstein polynomials, namely:

$$f_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{kn}(x)$$

To be more precise, we assume here that $[a, b] = [0, 1]$, and we set:

$$b_{kn}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

As for the proof of this, this is something well-known, which goes as follows:

(1) Consider indeed the basic Bernstein polynomials $b_{kn}$, as constructed above. These remind the binomial laws, so it is with some probability that we will start. We have the following formulae, which are all elementary to establish, and which in probabilistic terms are dealing with the moments of order $0, 1, 2$ of the binomial laws:

$$\sum_k \binom{n}{k} x^k (1-x)^{n-k} = 1$$

$$\sum_k \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} = x$$

$$\sum_k \left(x - \frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k} = \frac{x(1-x)}{n}$$

(2) In terms of the basic Bernstein polynomials $b_{kn}$, the above formulae read:

$$\sum_k b_{kn}(x) = 1$$

$$\sum_k \frac{k}{n} \cdot b_{kn}(x) = x$$

$$\sum_k \left(x - \frac{k}{n}\right)^2 b_{kn}(x) = \frac{x(1-x)}{n}$$

(3) Now consider our arbitrary continuous function $f : [0, 1] \to \mathbb{R}$, and construct for any $n \in \mathbb{N}$ the approximation indicated above, namely:

$$f_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{kn}(x)$$

In order to estimate the error $|f_n - f|$, we will use a standard result, called Heine-Cantor theorem, stating that our continuous function $f : [a,b] \to \mathbb{R}$ is automatically uniformly continuous. So, pick $\varepsilon > 0$, and then $\delta > 0$ such that the following happens:

$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Now with this done, we have the following estimate, using the first formula in (2) at the first step, the uniform continuity at the last step, and with $M = \sup|f|$:

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\sum_k \left(f\left(\frac{k}{n}\right) - f(x)\right) b_{kn}(x)\right| \\ &\leq \sum_k \left|f\left(\frac{k}{n}\right) - f(x)\right| b_{kn}(x) \\ &= \sum_{\left|x-\frac{k}{n}<\delta\right|} \left|f\left(\frac{k}{n}\right) - f(x)\right| b_{kn}(x) + \sum_{\left|x-\frac{k}{n}\geq\delta\right|} \left|f\left(\frac{k}{n}\right) - f(x)\right| b_{kn}(x) \\ &\leq \varepsilon + M \sum_{\left|x-\frac{k}{n}\geq\delta\right|} b_{kn}(x) \end{aligned}$$

(4) The point now is that the last sum on the right can be estimated by using the Chebycheff inequality, based on the third formula from (2), and we obtain:

$$\begin{aligned} \sum_{\left|x-\frac{k}{n}\geq\delta\right|} b_{kn}(x) &\leq \sum_k \delta^{-2}\left(x - \frac{k}{n}\right)^2 b_{kn}(x) \\ &= \delta^{-2}\,\frac{x(1-x)}{n} \\ &\leq \frac{\delta^{-2}}{4n} \end{aligned}$$

(5) Now by putting everything together, we obtain the following estimate:

$$|f_n(x) - f(x)| \leq \varepsilon + \frac{\delta^{-2}M}{4n}$$

Thus we have indeed $|f_n - f| \to 0$, uniform convergence, as desired.

(6) Summarizing, present theorem proved, modulo some learning in relation with the Heine-Cantor theorem, which is something very standard, and with Chebycheff inequality too, which is something very standard too, that we will leave as an exercise. □

The above result is quite interesting, and as a question coming from this, we would like for instance to know how to explicitly approximate the basic trigonometric functions, defined on suitable intervals, by polynomials. However, this latter question is something non-trivial. We will be back to it later in this book, when discussing calculus.

## 8e. Exercises

This was a more advanced chapter, and as exercises on this, we have:

EXERCISE 8.29. *Learn more about the continuous and discontinuous functions, and notably about the notion of jump, and what can be done with it.*

EXERCISE 8.30. *Learn also about the alternative definition of continuity in terms of open and closed sets, and what can be done with it.*

EXERCISE 8.31. *Learn more about Lipschitz functions, and what can be done with them. Also, compute Lipschitz constants for all functions that you know.*

EXERCISE 8.32. *Learn more about uniform continuity, and what can be done with it. Also, investigate the uniform continuity of all functions that you know.*

EXERCISE 8.33. *Work out the basic formulae for inverse trigonometric functions, based on the basic formulae for usual trigonometric functions.*

EXERCISE 8.34. *Work out the basic estimates for inverse trigonometric functions, based on the basic estimates for usual trigonometric functions.*

EXERCISE 8.35. *Learn more about the general continuous functions, notably with their uniform continuity property on compact sets.*

EXERCISE 8.36. *Learn more about the uniform convergence of continuous functions, with its various properties, and what can be done with it.*

As bonus exercise, and no surprise here, start reading some calculus.

# Part III

# Affine coordinates

*There is a house in New Orleans*
*They call the Rising Sun*
*And it's been the ruin of many a poor boy*
*Dear God, I know I was one*

CHAPTER 9

# Affine coordinates

## 9a. The real plane

Welcome to geometry and trigonometry, take two. What we have been doing so far was certainly great work, needed for understanding what is going on, no question about this, but that material was a bit old, essentially going back to the old Greeks. Time now for some true modern things, from a few hundred centuries ago, no longer than that.

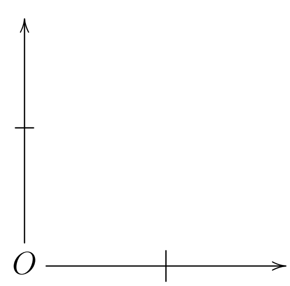
The general principle of modern geometry, coming from the work of Descartes and others, is something very simple and bright, as follows:

PRINCIPLE 9.1. *Everything that we know about plane geometry, including angles and trigonometry, can be better understood, and substantially generalized, by using vectors,*

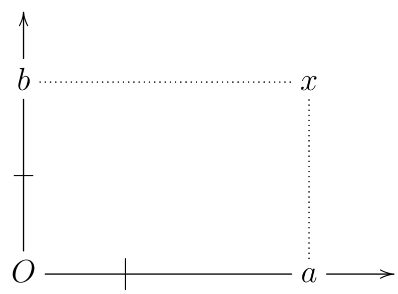
$$x=\begin{pmatrix}a\\ b\end{pmatrix}$$

*with $a,b\in\mathbb R$, which such a vector describing the position of a point $x$ in the plane with respect to a given system of coordinates, with $a,b\in\mathbb R$ being the coordinates of $x$.*

To be more precise here, let us fix a system of coordinates in the plane, with this meaning fixing a point $O$, called the origin, and then a pair of orthogonal lines passing through $O$. We will assume in addition that these two orthogonal lines are oriented, by marking arrows on them, and we will also specify the unit length on each of them, with the complete picture of our coordinate system being as follows:




Now given a point $x$ in the plane, we can project it onto the coordinate axes, and call the numbers $a,b\in\mathbb R$ describing the positions of these projections, with respect to the origin $O$, the coordinates of $x$, with the picture for this being as follows:




Observe now that, conversely, given two real numbers $a,b\in\mathbb R$, these will uniquely determine a certain point $x$ is the plane, constructed according to the above picture. That is, we draw $a$ on the horizontal axis, $b$ on the vertical axis, than we draw perpendiculars as above, and $x$ will be then the intersection of these two perpendiculars.

Summarizing, a point $x$ in the plane and a pair of real numbers $a,b\in\mathbb R$ is the same thing. In view of this, we agree to use the following notation, for this correspondence, and also make the convention that, with $x$ viewed in this way, it will be called vector:

$$x=\begin{pmatrix}a\\ b\end{pmatrix}$$

In practice now, with all this digested, it is actually convenient to forget about the plane, coordinates and projections, and summarize this discussion as follows:

Definition 9.2. *A vector is a pair of real numbers, written vertically:*

$$x=\begin{pmatrix}a\\ b\end{pmatrix}$$

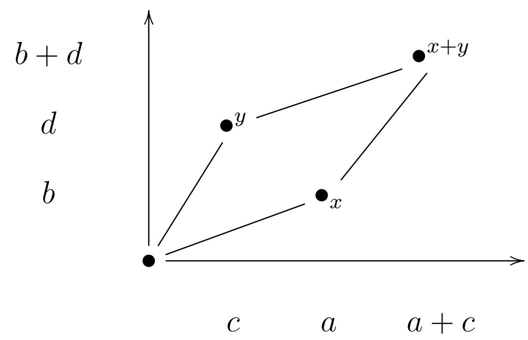
*We identify the vectors with the points in the plane, in the obvious way.*

Many interesting things can be done with vectors, and of particular interest is the summing operation for such vectors, given by the following formula:

$$x=\begin{pmatrix}a\\ b\end{pmatrix}\ ,\ y=\begin{pmatrix}c\\ d\end{pmatrix}\implies x+y=\begin{pmatrix}a+c\\ b+d\end{pmatrix}$$

Geometrically, and coming as a simple application of the Thales theorem, the idea with this operation is that the vectors add by forming a parallelogram, as shown by:

THEOREM 9.3. *The vector addition can be understood geometrically,*

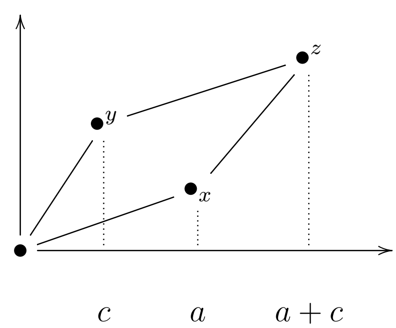



*with $x + y$ completing the parallelogram based at $O, x, y$.*

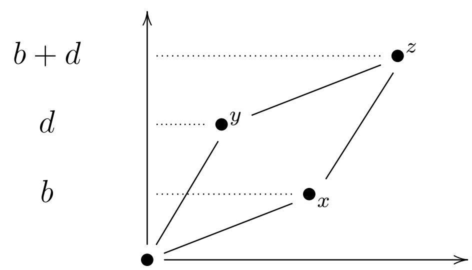
PROOF. This is something quite self-explanatory. Consider indeed a parallelogram in the plane, with three of its vertices being as follows:

$$O = \begin{pmatrix}0\\0\end{pmatrix} \quad , \quad x = \begin{pmatrix}a\\b\end{pmatrix} \quad , \quad y = \begin{pmatrix}c\\d\end{pmatrix}$$

Now let us draw verticals from $x, y$, and from the fourth vertex $z$ too. From Thales we obtain that the first coordinate of $z$ is $a + c$, according to the following picture:




Similarly, if we draw horizontals from $x, y$, and from $z$ too, from Thales we obtain that the second coordinate of $z$ is $b + d$, according to the following picture:



Thus we are led to the picture in the statement, and with the final conclusion being that the coordinates of the fourth vertex $z$ can be computed according to:

$$x=\begin{pmatrix}a\\b\end{pmatrix}\ ,\ y=\begin{pmatrix}c\\d\end{pmatrix}\implies z=\begin{pmatrix}a+c\\b+d\end{pmatrix}$$

But this is exactly the summing formula for the vectors, as desired. □

In practice, the summing operation is usefully complemented by the multiplication by scalars operation, which is given by the following very intuitive formula:

$$x=\begin{pmatrix}a\\b\end{pmatrix}\implies\lambda x=\begin{pmatrix}\lambda a\\\lambda b\end{pmatrix}$$

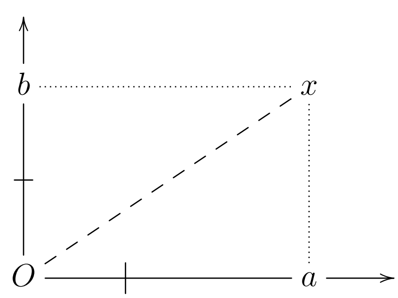
Next, of particular interest too, in relation with the computation of the lengths, is the following result, allowing us to compute the length of any vector:

Theorem 9.4. *The length of a vector is given by the following formula:*

$$x=\begin{pmatrix}a\\b\end{pmatrix}\implies||x||=\sqrt{a^2+b^2}$$

*Also, the vector lengths satisfy* $||\lambda x||=|\lambda|\cdot||x||$, *and* $||x+y||\leq||x||+||y||$.

Proof. In what regards the first assertion, which is the main one, this follows as a basic application of the theorem of Pythagoras, according to the following picture:



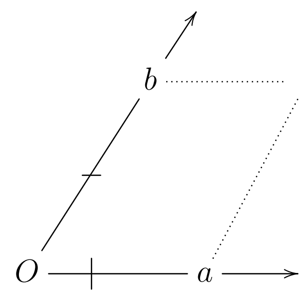

Regarding now the second assertion, with $x=\binom{a}{b}$, we have indeed:

$$\begin{aligned}||\lambda x|| &= \left|\left|\begin{pmatrix}\lambda a\\\lambda b\end{pmatrix}\right|\right|\\ &= \sqrt{(\lambda a)^2+(\lambda b)^2}\\ &= |\lambda|\sqrt{a^2+b^2}\\ &= |\lambda|\cdot||x||\end{aligned}$$

Finally, the last assertion is something clear, geometrically. Of course, this can be proved algebraically as well, by raising $||x+y||\leq||x||+||y||$ to the square, simplifying, and raising to the square again, and with this being a good exercise for you. □

And with this, good news, we have now all the needed vector calculus tools, in our bag, and we can start exploring what we can do, geometrically, with this new formalism. Before doing that, however, one more thing, which can be something useful:

FACT 9.5. *We can talk as well about oblique coordinates, according to*




*and in this setting, our knowledge about sums of vectors extends.*

To be more precise here, we can talk indeed about oblique coordinates, as above, by using a system of coordinates which is not necessarily orthogonal, and with the unit vectors on the coordinate axes being not necessarily of the same length. As mentioned, our basic knowledge about affine coordinates, including the parallelogram rule for the sums of vectors, extends to this setting, and this can be something useful. On the opposite, in this setting, we cannot use our formulae above for the lengths. More on this later.

## 9b. Points and lines

Getting back now to geometry, as a first good surprise, in what regards the axiomatics from chapter 1, that is literally nuked by coordinates. We have indeed, regarding the first axiom of geometry, that we started chapter 1 with, the following trivial theorem:

THEOREM 9.6. *Any two distinct points $P\neq Q$ determine a line, given by*

$$L=\left\{\lambda P+(1-\lambda)Q\Big|\lambda\in\mathbb{R}\right\}$$

*in affine coordinates.*

PROOF. We can say that the line $L$ determined by the points $P\neq Q$ consists by definition of the points $R$ such that we have, for a certain $\lambda\in\mathbb{R}$:

$$QR=\lambda QP$$

By using now the standard rules of vector calculus, this equation reads:

$$\begin{aligned}QR=\lambda QP&\iff R-Q=\lambda(P-Q)\\&\iff R=Q+\lambda(P-Q)\\&\iff R=\lambda P+(1-\lambda)Q\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Thus, very good news, axiom becoming theorem, what more can we wish for. Still talking lines, let us have some further look at them. We have the following result:

THEOREM 9.7. *The lines in the plane are the solutions of equations of type*

$$ax + by + c = 0$$

*with* $(a, b) \neq (0, 0)$*, and in addition, the following happen:*

(1) *Two such lines coincide when their triples* $(a, b, c)$ *are proportional.*
(2) *Two such lines are parallel or coincide when their pairs* $(a, b)$ *are proportional.*

PROOF. We have several things to be proved, the idea being as follows:

(1) As explained and Theorem 9.6 and its proof, with the convention that a line appears by uniting two points, the equations of these lines are as follows, with $P \neq Q$:

$$L = \left\{ \lambda P + (1 - \lambda)Q \middle| \lambda \in \mathbb{R} \right\}$$

Thus, in terms of coordinates, the lines are given by equations of the following type, with $(p, r) \neq (q, s)$, and with $\lambda \in \mathbb{R}$ being a parameter which varies:

$$\begin{cases} x = \lambda p + (1 - \lambda)q \\ y = \lambda r + (1 - \lambda)s \end{cases}$$

Equivalently, we can say that the lines are given by equations of the following type, with $(p, r) \neq (q, s)$, and with $\lambda \in \mathbb{R}$ being a parameter which varies:

$$\begin{cases} x = q + \lambda(p - q) \\ y = s + \lambda(r - s) \end{cases}$$

But now, by eliminating $\lambda$, in the obvious way, we are led to the conclusion that the lines are given by equations of the following type, with $(a, b) \neq (0, 0)$:

$$ax + by + c = 0$$

(2) In what regards now the second assertion, stating that two such lines coincide when their triples $(a, b, c)$ are proportional, this is something clear.

(3) As for the last assertion, stating that two such lines are parallel or coincide when their pairs $(a, b)$ are proportional, this is something clear too. □

In what follows we will often use the formula in Theorem 9.7, which is more convenient than the one in Theorem 9.6, for various algebraic computations. However, one problem with this sometimes comes from our lack of intuition regarding the parameters $a, b, c$:

QUESTION 9.8. *Given a line in the plane, written as above as*

$$ax + by + c = 0$$

*with* $(a, b) \neq (0, 0)$*, what is the geometric meaning of the parameters* $a, b, c$*?*

This question, which is something quite subtle, will be answered in due time. Moving on, and still following the material from the beginning of chapter 1, as a second piece of good news, our second geometry axiom becomes a theorem too, as follows:

THEOREM 9.9. *Given a point not lying on a line, $P \notin L$, we can draw through $P$ a unique parallel to $L$. That is, we can find a line $K$ satisfying $P \in K$, $K||L$.*

PROOF. According to Theorem 9.7, we can assume that our line $L$ is given by an equation of the following type, with $(a, b) \neq (0, 0)$:

$$ax + by + c = 0$$

As for the point $P$, with the notation $P = (x, y)$, the condition in the statement, namely $P \notin L$, tells us that the following must happen:

$$ax + by + c \neq 0$$

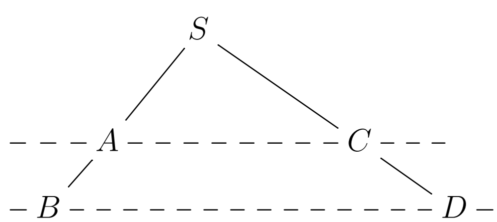
In view of this, let us pick now $\gamma \in \mathbb{R}$ such that the following equality holds:

$$ax + by + \gamma = 0$$

But this formula, with $x, y$ being now variables again, defines a certain line $K$, which certainly passes through $P$, and which is parallel to $L$ too, as desired. □

Getting now to the next thing that we did in chapter 1, namely the Thales theorem, and coming as further good news, that simplifies too with coordinates, as follows:

THEOREM 9.10 (Thales). *Proportions are kept, along parallel lines. That is, given a configuration as follows, consisting of two parallel lines, and of two extra lines,*




*the following equality holds:*

$$\frac{SA}{SB} = \frac{SC}{SD}$$

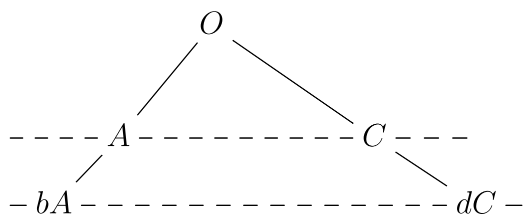
*Moreover, the converse of this holds too, in the sense that, in the context of a picture as above, if this equality is satisfied, then the lines $AC$ and $BD$ must be parallel.*

PROOF. Many things can be said here, the idea being as follows:

(1) In what regards the main assertion, we can assume if we want, by translation, that the point $S$ is the origin, $S = O$. Now with this assumption made, since $O, A, B$ are collinear, and since $O, C, D$ are collinear too, we must have, for certain $b, d \in \mathbb{R}$:

$$B = bA \quad , \quad D = dC$$

Thus, the picture of the Thales configuration becomes as follows, with $b,d\in\mathbb{R}$:




(2) Now let us prove the main assertion. We have the following equivalences:

$$\begin{aligned} AC||BD &\iff D-B=\lambda(C-A)\\ &\iff dC-bA=\lambda(C-A)\\ &\iff d=b \end{aligned}$$

But with this in hand, $d=b$, we obtain indeed the Thales formula, as follows:

$$\frac{OA}{OB}=\frac{1}{b}=\frac{1}{d}=\frac{OC}{OD}$$

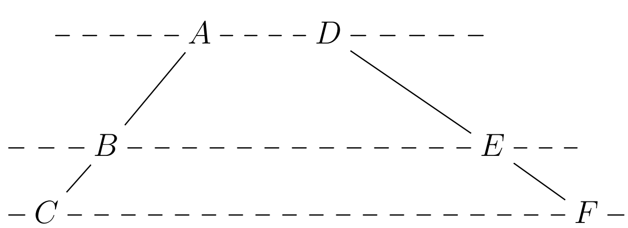
(3) Conversely now, we can still use the convention $S=O$ and the equalities $B=bA$ and $D=dC$ found in (1), and the picture there too, and we have, as claimed:

$$\begin{aligned} \frac{OA}{OB}=\frac{OC}{OD} &\implies \frac{1}{b}=\frac{1}{d}\\ &\implies b=d\\ &\implies AC||BD \end{aligned}$$

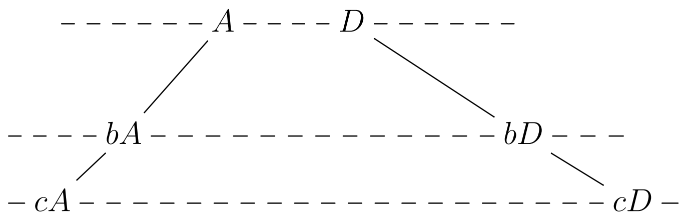
(4) Finally, let us mention that the other formulations of the Thales theorem, also from chapter 1, are also clear with coordinates. Indeed, for the above configuration, with the convention $S=O$, the improved conclusion, Thales 2, is as follows:

$$\frac{OA}{OB}=\frac{OC}{OD}=\frac{AC}{BD}$$

(5) Getting now to the Thales 3 configuration, also by following the material from chapter 1, this was as follows, with two lines meeting two parallel lines:




But here, save for a discussion of the case $AC||DF$, where the Thales 3 formula is clear, we can assume that $AC\cap DF$ is the origin $O$. And then, by proceeding as in (1), our picture becomes as follows, with $b, c \in \mathbb{R}$ being certain parameters:




We conclude that the Thales 3 formula holds indeed, as follows:

$$\frac{AB}{BC} = \frac{b-1}{c-1} = \frac{DE}{EF}$$

Thus, fully done with Thales, in all its formulations, using coordinates. $\square$

Getting now to the continuation of what we did in Part I, there are countless results there, and time perhaps to make a summary of what we have seen there:

(1) In chapter 1 we have seen the theorems of Desargues and Pappus, along with the duality of points and lines, and the cross ratio technology, used there for their proof.

(2) In chapter 2 we have seen the existence of the barycenter, then the Pythagoras theorem, and then the existence of the incenter, circumcenter and orthocenter.

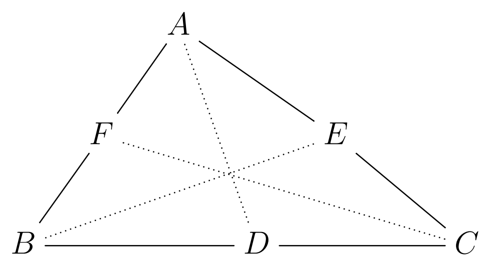
(3) In chapter 3 we have discussed the nine-point circle and Euler line, and then the Feuerbach points, with the promise that come back to these later, with details.

(4) In chapter 4 we have seen the theorems of Menelaus and Ceva, then the Gergonne and Nagel points, and then the theorems of Pascal and Brianchon.

Summarizing, many things to be done now, with coordinates, and contrary to what we did in Part I, where the material was organized in relation with the philosophical meaning of the theorems, rather than with the difficulty of the proofs, we will choose now to discuss what is the simplest first, and leave more complicated things for later.

So, getting now to the material from chapter 2, triangles and their centers, we first have the barycenter theorem, which drastically simplifies with coordinates, as follows:

Theorem 9.11 (Barycenter). *Given a triangle $ABC$, its medians cross,*




*at a point called barycenter, lying at $1/3 - 2/3$ on each median.*

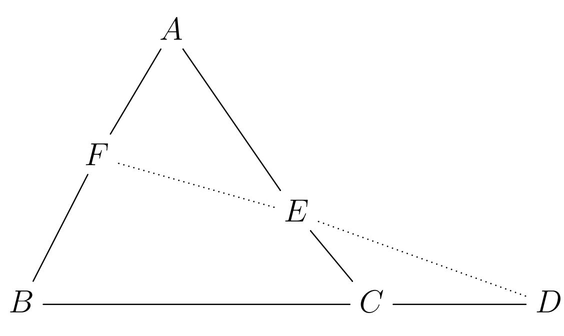
Proof. Let us call $A, B, C \in \mathbb{R}^2$ the coordinates of the vertices $A, B, C$, and consider the average $P = (A + B + C)/3$. We have then:

$$P = \frac{1}{3} \cdot A + \frac{2}{3} \cdot \frac{B + C}{2}$$

Thus $P$ lies on the median emanating from $A$, and a similar argument shows that $P$ lies as well on the medians emanating from $B, C$. Thus, we have our barycenter. $\square$

Very nice all this, hope you agree with me. In what regards now the rest of the material from chapter 2, the Pythagoras theorem is trivial with coordinates, while the rest, namely incenter, circumcenter and orthocenter, looks less trivial with coordinates, so we will keep this for later. Skipping as well chapter 3, obviously for later too, and getting now to the material from chapter 4, starting with Menelaus, we have:

Theorem 9.12 (Menelaus). *In a configuration of the following type, with a triangle $ABC$ cut by a line $DEF$,*




*we have the following formula, with all segments being taken oriented:*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$$

*Moreover, the converse holds, with this formula guaranteeing that $D, E, F$ are colinear.*

Proof. This is something fundamental, worth a detailed discussion, as follows:

(1) To start with, let us recall the proof that we already know, from chapter 4. The argument there was that by projecting the vertices $A,B,C$ on the line $DEF$, into points $A',B',C'$, we have the following computation, for unoriented segments:

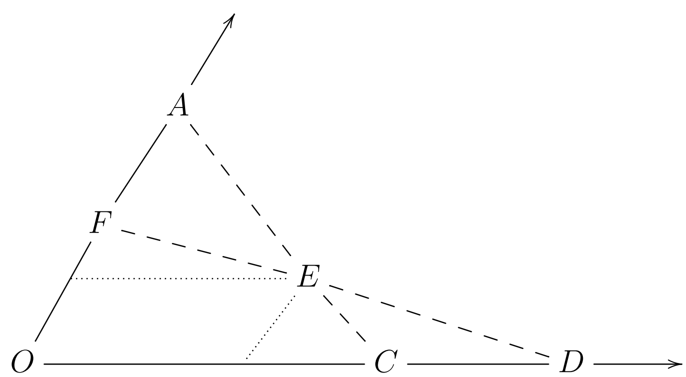
$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{AA'}{BB'}\cdot\frac{BB'}{CC'}\cdot\frac{CC'}{AA'}=1$$

But with this, we are basically done, because when adding orientations the study is elementary, as explained in chapter 4, and so is the proof of the converse.

(2) The point now is that, what we did in the above, with that projection trick, can be considered as being an affine coordinate proof. Indeed, by fixing the origin $O$ anywhere on the line $DEF$, and this line $DEF$ as being the $Ox$ axis, our computation reads:

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{A_2}{B_2}\cdot\frac{B_2}{C_2}\cdot\frac{C_2}{A_2}=1$$

(3) This being said, let us present now a second proof, which, although being a bit longer and less smart, will contain some interesting computations. When seeing the statement of the theorem, the first thought goes to oblique coordinates, according to:




To be more precise, consider a system of oblique coordinates as above, and points $A,F$ and $C,D$ on the axes. The problem, which is something quite fundamental, is that of computing the following intersection, that we called $E$ on the above picture:

$$AC\cap FD=?$$

(4) So, let us denote the coordinates of our various points as follows:

$$A=\begin{pmatrix}0\\a\end{pmatrix}\quad,\quad F=\begin{pmatrix}0\\f\end{pmatrix}\quad,\quad C=\begin{pmatrix}c\\0\end{pmatrix}\quad,\quad D=\begin{pmatrix}d\\0\end{pmatrix}\quad,\quad E=\begin{pmatrix}x\\y\end{pmatrix}$$

We have the following computation, giving the equation of the line $AC$:

$$\frac{AE}{AC}=\frac{x}{c}\quad,\quad\frac{CE}{CA}=\frac{y}{a}\quad\implies\quad\frac{x}{c}+\frac{y}{a}=1$$

Similarly, we have the following computation, giving the equation of $FD$:

$$\frac{FE}{FD}=\frac{x}{d}\quad,\quad\frac{DE}{DF}=\frac{y}{f}\quad\implies\quad\frac{x}{d}+\frac{y}{f}=1$$

Summarizing, the equations for the coordinates of $E=AC\cap FD$ are as follows:

$$\begin{cases}\frac{x}{c}+\frac{y}{a}=1\\ \frac{x}{d}+\frac{y}{f}=1\end{cases}$$

(5) Now let us solve this system. For finding $x$, we can write our system as:

$$\begin{cases}\frac{x}{cf}+\frac{y}{af}=\frac{1}{f}\\ \frac{x}{ad}+\frac{y}{af}=\frac{1}{a}\end{cases}$$

By making the difference, we obtain the following equation for $x$:

$$x\left(\frac{1}{cf}-\frac{1}{ad}\right)=\frac{1}{f}-\frac{1}{a}$$

Thus, the $x$ coordinate of $E=AC\cap FD$ is given by the following formula:

$$x=\frac{cd(a-f)}{ad-cf}$$

Similarly, the $y$ coordinate of $E=AC\cap FD$ is given by the following formula:

$$y=\frac{af(c-d)}{cf-ad}$$

(6) We can now prove Menelaus. We have indeed the following computation:

$$\begin{aligned}\frac{CE}{EA}&=\frac{CA}{EA}-1\\&=\frac{c}{x}-1\\&=\frac{ad-cf}{d(a-f)}-1\\&=\frac{f(d-c)}{d(a-f)}\end{aligned}$$

But with this, we have the following computation:

$$\frac{AF}{FO}\cdot\frac{OD}{DC}\cdot\frac{CE}{EA}=\frac{a-f}{f}\cdot\frac{d}{c-d}\cdot\frac{f(d-c)}{d(a-f)}=-1$$

But this is exactly what Menelaus says, modulo our relabeling of points. □

## 9c. Scalar products

Getting now to the Ceva theorem and various triangle centers, we have seen all across chapters 2-5 that computations can be quite complicated, basically leading us into trigonometry. In view of this, it is perhaps wiser to develop first some more vector mathematics, with some new tools, more powerful, that can help in our quest.

With this idea in mind, we first have the following result, providing us with a powerful vector calculus tool in order to deal with orthogonality, and related topics:

THEOREM 9.13. *If we define the scalar product of two vectors by*

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = ac + bd$$

*then the following happen:*

(1) $< A + B, C > = < A, C > + < B, C >$.
(2) $< A, B + C > = < A, B > + < A, C >$.
(3) $< \lambda A, B > = < A, \lambda B > = \lambda < A, B >$.
(4) $||A|| = \sqrt{< A, A >}$.
(5) $A \perp B \iff < A, B > = 0$.
(6) $< A, B > = ||A|| \cdot ||B|| \cdot \cos t$, *with $t$ being the angle between $A, B$.*
(7) $< A, B > = < A', B > = < A, B' >$, *prime being projection on the other vector.*

PROOF. Many things going on here, the idea being as follows:

(1-3) These formulae, very useful in practice, are all clear from definitions.

(4-7) To start with, the formula in (4) is clear, coming from:

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = a^2 + b^2 = \left|\left| \begin{pmatrix} a \\ b \end{pmatrix} \right|\right|^2$$

Observe that this formula agrees with what (6) says. In fact, more generally, the scalar product of two proportional vectors is as follows, again in agreement with (6):

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} \right\rangle = \lambda a^2 + \lambda b^2 = \pm \left|\left| \begin{pmatrix} a \\ b \end{pmatrix} \right|\right| \cdot \left|\left| \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} \right|\right|$$

In order to prove now (5), we can assume using (3) that we have $||A|| = ||B|| = 1$. But here, assuming $A \perp B$, if $s$ is the angle formed by $A$ with the $Ox$ axis, we have:

$$< A, B > = \left\langle \begin{pmatrix} \cos s \\ \sin s \end{pmatrix}, \pm \begin{pmatrix} -\sin s \\ \cos s \end{pmatrix} \right\rangle = 0$$

Getting now to (6), which will prove as well the converse of this, again we can assume $||A||=||B||=1$, and if $s$ is the angle formed by $A$ with the $Ox$ axis, we have:

$$\begin{aligned} < A,B > &= \left\langle \begin{pmatrix}\cos s\\ \sin s\end{pmatrix}, \begin{pmatrix}\cos(s+t)\\ \sin(s+t)\end{pmatrix}\right\rangle \\ &= \cos s\cos(s+t)+\sin s\sin(s+t) \\ &= \cos((s+t)-s) \\ &= \cos t \end{aligned}$$

As for (7), this is a reformulation of (6), using the above formula of $< A, \lambda A >$. □

As an application of our scalar product technology, we can solve now two questions that we had left. To start with, we can solve Question 9.8, majestically, as follows:

THEOREM 9.14. *The line equation $ax+by+c=0$ can be written as*

$$\left\langle \begin{pmatrix}a\\ b\end{pmatrix}, \begin{pmatrix}x\\ y\end{pmatrix}\right\rangle = -c$$

*with the vector $\binom{a}{b}$ being any nonzero vector orthogonal to the line.*

PROOF. This is indeed something self-explanatory, and very beautiful. □

As a second application, which is more philosophical, and solving a question that we had open since the beginning of chapter 5, some 100 pages ago, we have:

THEOREM 9.15. *The formula for scalar products, namely*

$$< A,B >= ||A|| \cdot ||B|| \cdot \cos t$$

*with $t$ being the angle between $A, B$, tells us what the cosine is good for.*

PROOF. This is again something self-explanatory, and very beautiful. □

Very nice all this, and we will make a good use of the scalar products, in what follows. Moving on now, shall we stop here, or develop some more vector calculus tools?

In answer, I don't know about you, but in what regards me, I'm still a bit scared by trigonometry, and would feel more confident with more vector weapons on me. So, with orthogonality understood, let us discuss now areas, that can only help us, later.

We are very used to the areas of triangles, but for our purposes here, it is better to deal with areas of parallelograms. And here, we have the following key result:

THEOREM 9.16. *The area of the parallelogram formed by $\binom{a}{c}$ and $\binom{b}{d}$ is*

$$A = \left|\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right|$$

*with the determinant on the right being given by* $\det = ad - bc$.

Proof. Many things can be said here, the idea with this being as follows:

(1) As a first observation, the vectors are proportional, $\binom{a}{c} \sim \binom{b}{d}$, or equivalently, their parallelogram area is 0, precisely when the following quantity vanishes:

$$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-bc$$

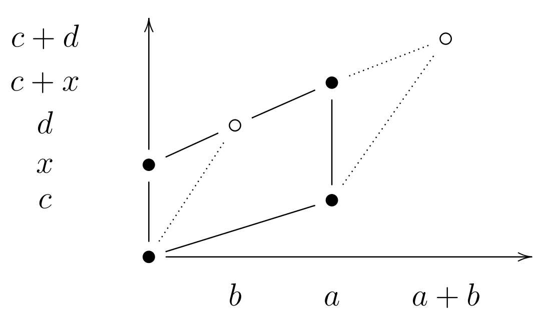
Thus, we know one thing, and with this justifying the introduction of det.

(2) Getting now to the general case, this does not look hard to prove, with the variety of techniques that we have, at our disposal. However, since what we will be doing here extends in fact to arbitrary $N$ dimensions, and we will need this later in this book, it is better to manufacture a proof starting from the basics, namely the Thales theorem.

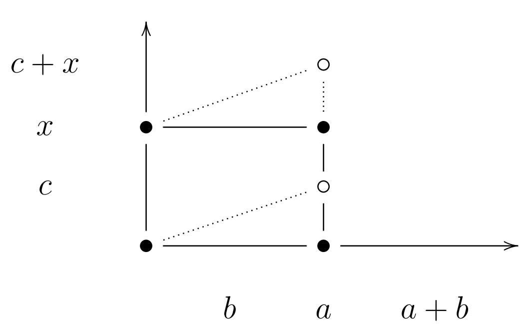
(3) So, getting started now, in general, we can assume for simplifying that we are in the case $a,b,c,d>0$, the proof in general being similar. Moreover, by interchanging if needed the vectors $\binom{a}{c},\binom{b}{d}$, we can assume that we are in the following situation:

$$\frac{a}{c}>\frac{b}{d}$$

(4) Now let us slide the upper side of the parallelogram downwards left, until we reach $Oy$. Our parallelogram, which has not changed its area in this process, becomes:



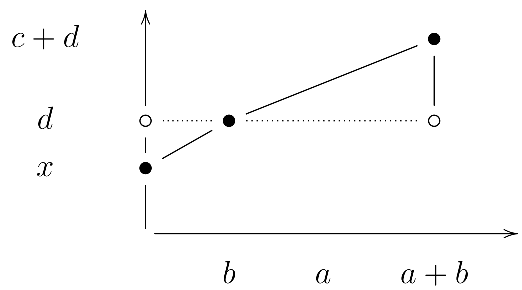

(5) Moreover, we can further modify this parallelogram, once again by not altering its area, by sliding the right side downwards, until we reach the $Ox$ axis:



(6) Let us compute now the area. Since our two sliding operations have not changed the area of the original parallelogram, this area is given by the following formula:

$$A = ax$$

In order to compute the quantity $x$, observe that in the context of the first move, we have two similar triangles, according to the following picture:




Thus, we are led to the following equation for the number $x$:

$$\frac{d-x}{b} = \frac{c}{a}$$

By solving this equation, we obtain the following value for $x$:

$$x = d - \frac{bc}{a}$$

(7) Thus the area of our parallelogram, or rather of the final rectangle obtained from it, which has the same area as the original parallelogram, is given by:

$$A = ax = ad - bc$$

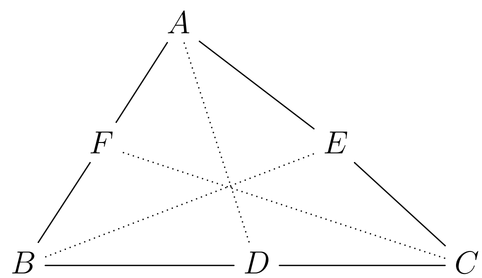
We are therefore led to the conclusion in the statement. □

Many other things can be said about determinants, and we will be back to this on several occasions. In the meantime, let us record the following formula, which shows that the determinants appear as particular cases of scalar products, or perhaps vice versa:

$$\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \left\langle \begin{pmatrix}a\\ c\end{pmatrix}, \begin{pmatrix}d\\ -b\end{pmatrix}\right\rangle$$

And with this, end of our general vector study, we have now tools for attacking all sorts of questions. Getting back to geometry, we can reprove the Ceva theorem:

Theorem 9.17 (Ceva). *In a configuration of the following type, with a triangle $ABC$ containing inner lines $AD, BE, CF$ which cross,*




*we have the following formula:*

$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1$$

*Moreover, the converse holds, with this formula guaranteeing that $AD, BE, CF$ cross.*

Proof. As explained in chapter 4, this follows either from Menelaus applied 3 times, or directly, by computing some areas, or via the duality between points and lines, from Menelaus. So, let us try now to manufacture a fourth proof, using coordinates:

(1) We can choose the origin $O$ to be the middle point on the above picture, so that the vectors $D, E, F$ are given by formulae as follows, for certain $d, e, f \in \mathbb{R}$:

$$D = dA \quad , \quad E = eB \quad , \quad F = fC$$

Now let us try to compute $E = eB$. This lies on $AC$, so we must have:

$$eB = \lambda A + (1-\lambda)C$$

We are therefore led to the following system of equations, for $E = eB$:

$$\begin{cases} eB_1 = \lambda A_1 + (1-\lambda)C_1 \\ eB_2 = \lambda A_2 + (1-\lambda)C_2 \end{cases}$$

(2) Equivalently, we have the following system of equations:

$$\begin{cases} \lambda(A_1 - C_1) = eB_1 - C_1 \\ \lambda(A_2 - C_2) = eB_2 - C_2 \end{cases}$$

By multiplying these equations by $B_2, B_1$ and substracting, we get:

$$\lambda = \frac{B_1C_2 - B_2C_1}{(A_1 - C_1)B_2 - (A_2 - C_2)B_1}$$

(3) Which looks quite complicated, but for Ceva, what we need is the following ratio:

$$\begin{aligned}\frac{CE}{EA} &= \frac{\lambda}{1-\lambda}\\ &= \frac{B_1C_2-B_2C_1}{(A_1-C_1)B_2-(A_2-C_2)B_1-B_1C_2+B_2C_1}\\ &= \frac{B_1C_2-B_2C_1}{A_1B_2-A_2B_1}\\ &= \frac{\det(BC)}{\det(AB)}\end{aligned}$$

(4) And with this, good news, we can reprove the Ceva theorem, as follows:

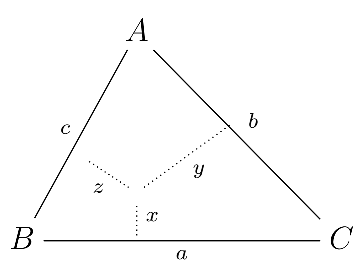
$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=\frac{\det(CA)}{\det(CB)}\cdot\frac{\det(AB)}{\det(AC)}\cdot\frac{\det(BC)}{\det(AB)}=1$$

(5) Finally, let us mention that this proof is in fact not very far from what we were doing in chapter 4 with areas, because the above $2\times 2$ determinants do compute areas, which are in fact exactly the same areas as those that we used in chapter 4. □

## 9d. Triangles, revised

As a continuation of the above, we still have many things to be done, in relation with triangles and their centers. Let us start with the following key result:

THEOREM 9.18. *We can talk about trilinear coordinates with respect to a triangle*




*with these being the distances $(x,y,z)$ to the sides, as indicated, up to a common scalar, and the trilinear coordinates of the basic triangle centers are as follows:*

(1) *Barycenter:* $(1/a,1/b,1/c)$.
(2) *Incenter:* $(1,1,1)$.
(3) *Excenters:* $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.
(4) *Circumcenter:* $(\cos A,\cos B,\cos C)$.
(5) *Orthocenter:* $(\sec A,\sec B,\sec C)$.
(6) *Nine-point circle center:* $(\cos(B-C),\cos(C-A),\cos(A-B))$.

PROOF. To start with, we can certainly talk about the distances $(x, y, z)$ to the sides of a given point, which uniquely determine the point. However, it is technically convenient to allow a scalar in all this, by making the following convention, for any $\lambda \neq 0$:

$$(x, y, z) = (\lambda x, \lambda y, \lambda z)$$

Observe that the value of the scalar, and so of the original distances $(x, y, z)$, called "absolute" trilinear coordinates, can be recaptured by using areas, as follows:

$$S = \frac{ax + by + cz}{2}$$

To be more precise, here $S$ is the area of the triangle, and if the point lies outside of the triangle, this formula must be fine-tuned with some $-$ signs, in the obvious way. Getting now to explicit computations, for various centers, these are as follows:

(1) The altitude lengths being $(2S/a, 2S/b, 2S/c)$, the absolute coordinates of the barycenter are $2S/3(1/a, 1/b, 1/c)$, which by simplifying leads to $(1/a, 1/b, 1/c)$.

(2) For the incenter things are trivial, the absolute coordinates being $(r, r, r)$, with $r$ being the radius of the incircle, which by simplifying leads to $(1, 1, 1)$.

(3) Same situation for the excenters, their absolute coordinates being respectively $(-r_a, r_a, r_a)$, $(r_b, -r_b, r_b)$, $(r_c, r_c, -r_c)$, with $r_a, r_b, r_c$ being the external circle radii.

(4) The absolute coordinates of the circumcenter are $(R\cos A, R\cos B, R\cos C)$, with $R$ being the circle radius, which by simplifying leads to $(\cos A, \cos B, \cos C)$.

(5) For the orthocenter, we have seen in chapter 5 that we have $x = 2R\cos B\cos C$, and with this being proportional to $\sec A$, we obtain $(\sec A, \sec B, \sec C)$.

(6) As for the center of the nine-point circle, we have seen in chapter 5 that we have $x = R\cos(B - C)$, so we obtain $(\cos(B - C), \cos(C - A), \cos(A - B))$. □

Getting now to the other triangle centers, these can be computed as well by using our previous formulae in this book, with the Gergonne point appearing as follows:

$$\left(\frac{a}{b+c-a}\,,\,\frac{b}{a+c-b}\,,\,\frac{c}{a+b-c}\right)$$

As for the Nagel point, this appears similarly, with all fractions inverted:

$$\left(\frac{b+c-a}{a}\,,\,\frac{a+c-b}{b}\,,\,\frac{a+b-c}{c}\right)$$

Next, regarding the Feuerbach points, the result here is as follows:

THEOREM 9.19. *The Feuerbach point of a triangle $ABC$ has trilinear coordinates*

$$(1 - \cos(B - C), 1 - \cos(C - A), 1 - \cos(A - B))$$

*and a similar result holds for the secondary Feuerbach points, with $-$ signs added.*

Proof. This is something a bit more technical, the idea being as follows:

(1) Consider indeed the point $Z$ in the statement. There is a clear relation here with the incenter $I$ and the nine-point circle center $N$, coming from the following formulae:

$$I = (1, 1, 1)$$
$$N = (\cos(B - C), \cos(C - A), \cos(A - B))$$
$$Z = (1 - \cos(B - C), 1 - \cos(C - A), 1 - \cos(A - B))$$

To be more precise, we can see right away from this that $I, N, Z$ are collinear.

(2) Next, we must prove the Feuerbach theorem, stating that the incircle, centered at $I$ and having radius $r$, and the nine-point circle, centered at $N$ and having radius $\rho = R/2$, are indeed tangent at $Z$. This amounts in proving the following formulae:

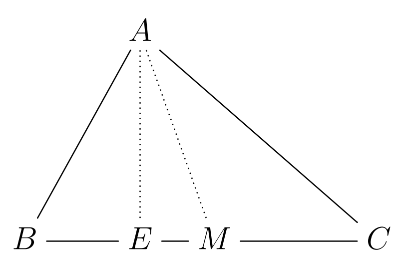
$$IZ = r \quad , \quad NZ = \rho \quad , \quad IN = \rho - r$$

(3) To be more precise, thanks to our observation in (1) that $I, N, Z$ are collinear, we just have to prove two of these formulae, with the third one being automatic. But this can be done indeed, with some trigonometric pain, by computing first the absolute coordinates of $Z$, and then computing the above distances, using Pythagoras.

(4) Finally, the same arguments apply to the secondary Feuerbach points $Z_a, Z_b, Z_c$, which can be constructed similarly, by formally removing $N$ from the excenters $I_a, I_b, I_c$. And we will leave this as an exercise for you too, all good mathematics, enjoy. □

Moving on, we can construct as well new triangle centers by using our trilinear coordinate technology, notably with the symmedian point, which appears as follows:

Theorem 9.20 (Lemoine). *The symmedians of a triangle, appearing by symmetrizing each median with respect to the corresponding angle bisector*




*meet at a point having coordinates $(a, b, c)$, called symmedian or Lemoine point.*

Proof. There are many known proofs here, but with the trilinear coordinates of the intesection being given by such a simple formula, this is an invitation to coordinates:

(1) So, consider the point $K = (a, b, c)$, and let us project it on $AB, AC$, into points $X, Y$. We have then the following computation, involving the angles on top:

$$\frac{\sin(KAB)}{\sin(KAC)} = \frac{KX/KA}{KY/KA} = \frac{KX}{KY} = \frac{c}{b}$$

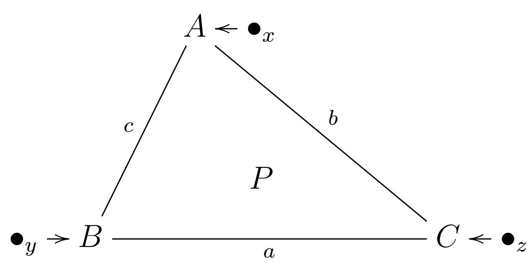
(2) On the other hand, regarding the median, if we project $B, C$ on it, to points $Z, T$, we have the following computation, again involving the angles on top:

$$\frac{\sin(MAB)}{\sin(MAC)} = \frac{BZ/AB}{CT/AC} = \frac{AC}{AB} = \frac{b}{c}$$

(3) Now since the ratios that we found in (1,2) are inverted, we conclude that $K$ lies indeed on the symmedian emanating from $A$, and this gives the result. $\square$

As a last piece of general theory, as a useful technical version of the trilinear coordinates, we can talk as well about barycentric coordinates, as follows:

THEOREM 9.21. *We can talk about barycentric coordinates with respect to a triangle*




*corresponding to the weights $(x, y, z)$ that must be installed at vertices, as for $P$ to be the physical barycenter. These are again taken up to a scalar, defined according to*

$$P = \frac{xA + yB + zC}{x + y + z}$$

*and called absolute when $x + y + z = 1$. The conversion formulae are*

$$\text{trilinear } (x, y, z) \longrightarrow \text{ barycentric } (ax, by, cz)$$

$$\text{barycentric } (x, y, z) \longrightarrow \text{ trilinear } (x/a, y/b, z/c)$$

*with $a, b, c$ being the sides. In practice, the formulae for known triangle centers are similar to those before, with some simplifying, and some becoming more complicated.*

PROOF. Many things going on here, the idea being as follows:

(1) It is best to start algebrically, by defining the absolute barycentric coordinates of a point $P$ as being the unique numbers $x, y, z$ such that the following happens:

$$P = xA + yB + xC \quad , \quad x + y + z = 1$$

Then, as before with the trilinear coordinates, it is technically convenient to allow a scalar in all this, by making the following convention, for any $\lambda \neq 0$:

$$(x, y, z) = (\lambda x, \lambda y, \lambda z)$$

(2) Next, we have the physical interpretation in the statement, say coming via the considerations from chapter 2, and with the remark that the weights to be installed at vertices are unique up to a scalar, that is, are subject to $(x,y,z)=(\lambda x,\lambda y,\lambda z)$.

(3) Getting now to the point, explicit computation of the barycentric coordinates, our claim is that we have the following key formula, with all areas being signed:

$$(x,y,z)=(area(PBC),area(PAC),area(PAB))$$

(4) But this follows from the following computation, for the corresponding absolute barycentric coordinates $(x,y,z)$, based on the area formula from Theorem 9.16:

$$\begin{aligned}
2\cdot area(PBC) &= \det(P-B,C-B)\\
&= \det(xA+(y-1)B+zC,C-B)\\
&= \det(xA-(x+z)B+zC,C-B)\\
&= \det(x(A-B)+z(C-B),C-B)\\
&= \det(x(A-B),C-B)\\
&= x\det(A-B,C-B)\\
&= x\times 2\cdot area(ABC)
\end{aligned}$$

(5) Next, and coming as a consequence of the above formula, we have the conversion formulae between trilinear and barycentric coordinates in the statement.

(6) Finally, regarding the last assertion, many things can be said here, starting with the fact that the barycenter becomes simpler, $(1,1,1)$, while the incenter becomes more complicated, $(a,b,c)$. Exercise of course for you, to learn more about all this. □

## 9e. Exercises

This was a key chapter, featuring many computations with coordinates, in relation with everything that we knew. Our exercises are about completing these computations:

EXERCISE 9.22. *Prove the Desargues theorem, using coordinates.*

EXERCISE 9.23. *Duality between points and lines, using coordinates.*

EXERCISE 9.24. *Prove the Pappus theorem, using coordinates.*

EXERCISE 9.25. *Work out the basics of the cross ratio, using coordinates.*

EXERCISE 9.26. *Construct all basic triangle centers, using coordinates.*

EXERCISE 9.27. *Nine-point circle and Euler line, using coordinates.*

EXERCISE 9.28. *Feuerbach points fully done, using coordinates.*

EXERCISE 9.29. *Also Pascal and Brianchon, using coordinates.*

As bonus exercise, which might actually help with this, learn a bit more linear algebra, in 3D, which can be very useful, in order to manipulate the trilinear coordinates.

CHAPTER 10

# Ellipses, conics

## 10a. Matrices, rotations

As another application of the vector calculus, that we will need among others in order to study the conics, let us discuss now the transformations of the plane. The transformations of the plane $\mathbb{R}^2$ that we are interested in are as follows:

DEFINITION 10.1. *A map $f:\mathbb{R}^2\to\mathbb{R}^2$ is called affine when it maps lines to lines,*

$$f(tx+(1-t)y)=tf(x)+(1-t)f(y)$$

*for any $x,y\in\mathbb{R}^2$ and any $t\in\mathbb{R}$. If in addition $f(0)=0$, we call $f$ linear.*

As a first observation, our "maps lines to lines" interpretation of the equation in the statement assumes that the points are degenerate lines, and this in order for our interpretation to work when $x=y$, or when $f(x)=f(y)$. Also, what we call line is not exactly a set, but rather a dynamic object, think trajectory of a point on that line. We will be back to this later, once we will know more about such maps.

Here are some basic examples of symmetries, all being linear in the above sense:

PROPOSITION 10.2. *The symmetries with respect to $Ox$ and $Oy$ are:*

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\-y\end{pmatrix}\quad,\quad\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-x\\y\end{pmatrix}$$

*The symmetries with respect to the $x=y$ and $x=-y$ diagonals are:*

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}y\\x\end{pmatrix}\quad,\quad\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-y\\-x\end{pmatrix}$$

*All these maps are linear, in the above sense.*

PROOF. The fact that all these maps are linear is clear, because they map lines to lines, in our sense, and they also map 0 to 0. As for the explicit formulae in the statement, these are clear as well, by drawing pictures for each of the maps involved. □

Here are now some basic examples of rotations, once again all being linear:

PROPOSITION 10.3. *The rotations of angle* $0°$ *and of angle* $90°$ *are:*

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\y\end{pmatrix}\quad,\quad\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-y\\x\end{pmatrix}$$

*The rotations of angle* $180°$ *and of angle* $270°$ *are:*

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}-x\\-y\end{pmatrix}\quad,\quad\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}y\\-x\end{pmatrix}$$

*All these maps are linear, in the above sense.*

PROOF. As before, these rotations are all linear, for obvious reasons. As for the formulae in the statement, these are clear as well, by drawing pictures. □

Here are some basic examples of projections, once again all being linear:

PROPOSITION 10.4. *The projections on* $Ox$ *and* $Oy$ *are:*

$$\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\0\end{pmatrix}\quad,\quad\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}0\\y\end{pmatrix}$$

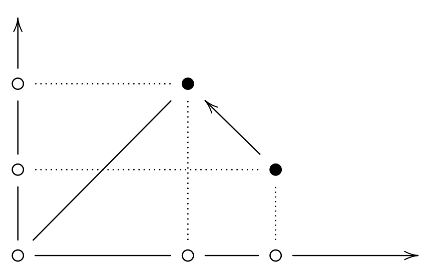
*The projections on the* $x=y$ *and* $x=-y$ *diagonals are:*

$$\begin{pmatrix}x\\y\end{pmatrix}\to\frac{1}{2}\begin{pmatrix}x+y\\x+y\end{pmatrix}\quad,\quad\begin{pmatrix}x\\y\end{pmatrix}\to\frac{1}{2}\begin{pmatrix}x-y\\y-x\end{pmatrix}$$

*All these maps are linear, in the above sense.*

PROOF. Again, these projections are all linear, and the formulae are clear as well, by drawing pictures, with only the last 2 formulae needing some explanations:

(1) In what regards the projection on the $x=y$ diagonal, the picture here is as follows:



But this gives the result, since the $45°$ triangle shows that this projection leaves invariant $x+y$, so we can only end up with the average $(x+y)/2$, as double coordinate.

(2) As for the projection on the $x=-y$ diagonal, the proof here is similar. □

Finally, we have the translations, which are as follows:

PROPOSITION 10.5. *The translations are exactly the maps of the form*

$$\begin{pmatrix}x\\ y\end{pmatrix}\to\begin{pmatrix}x+p\\ y+q\end{pmatrix}$$

*with $p,q\in\mathbb{R}$, and these maps are all affine, in the above sense.*

PROOF. A translation $f:\mathbb{R}^2\to\mathbb{R}^2$ is clearly affine, because it maps lines to lines. Also, such a translation is uniquely determined by the following vector:

$$f\begin{pmatrix}0\\ 0\end{pmatrix}=\begin{pmatrix}p\\ q\end{pmatrix}$$

To be more precise, $f$ must be the map which takes a vector $\binom{x}{y}$, and adds this vector $\binom{p}{q}$ to it. But this gives the formula in the statement. □

Summarizing, we have many interesting examples of linear and affine maps. Let us develop now some general theory, for such maps. As a first result, we have:

THEOREM 10.6. *For a map $f:\mathbb{R}^2\to\mathbb{R}^2$, the following are equivalent:*

(1) *$f$ is linear in our sense, mapping lines to lines, and $0$ to $0$.*
(2) *$f$ maps sums to sums, $f(x+y)=f(x)+f(y)$, and satisfies $f(\lambda x)=\lambda f(x)$.*

PROOF. This is something which comes from definitions, as follows:

(1) $\implies$ (2) We know that $f$ satisfies the following equation, and $f(0)=0$:

$$f(tx+(1-t)y)=tf(x)+(1-t)f(y)$$

By setting $y=0$, and by using our assumption $f(0)=0$, we obtain, as desired:

$$f(tx)=tf(x)$$

As for the first condition, regarding sums, this can be established as follows:

$$\begin{aligned}f(x+y)&=f\left(2\cdot\frac{x+y}{2}\right)\\ &=2f\left(\frac{x+y}{2}\right)\\ &=2\cdot\frac{f(x)+f(y)}{2}\\ &=f(x)+f(y)\end{aligned}$$

(2) $\implies$ (1) Conversely now, assuming that $f$ satisfies $f(x+y)=f(x)+f(y)$ and $f(\lambda x)=\lambda f(x)$, then $f$ must map lines to lines, as shown by:

$$\begin{aligned}f(tx+(1-t)y)&=f(tx)+f((1-t)y)\\ &=tf(x)+(1-t)f(y)\end{aligned}$$

Also, we have $f(0)=f(2\cdot 0)=2f(0)$, which gives $f(0)=0$, as desired. □

The above result is very useful, and in practice, we will often use the condition (2) there, somewhat as a new definition for the linear maps. Let us record this as follows:

Definition 10.7 (upgrade). *A map $f:\mathbb{R}^2\to\mathbb{R}^2$ is called:*

(1) *Linear, when it satisfies $f(x+y)=f(x)+f(y)$ and $f(\lambda x)=\lambda f(x)$.*
(2) *Affine, when it is of the form $f=g+x$, with $g$ linear, and $x\in\mathbb{R}^2$.*

Done with the axiomatics of plane transformations, you would say? You must be kidding, because coming on top of this definition, we have the following powerful result, which can stand as yet another axiomatization of the linear and affine maps:

Theorem 10.8. *The linear maps $f:\mathbb{R}^2\to\mathbb{R}^2$ are precisely the maps of type*

$$f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}$$

*and the affine maps $f:\mathbb{R}^2\to\mathbb{R}^2$ are precisely the maps of type*

$$f\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}+\begin{pmatrix}p\\q\end{pmatrix}$$

*depending respectively on 4, and on 6 real parameters.*

Proof. Assuming that $f$ is linear in the sense of Definition 10.7, we have:

$$\begin{aligned}f\begin{pmatrix}x\\y\end{pmatrix}&=f\left(x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}\right)\\&=xf\begin{pmatrix}1\\0\end{pmatrix}+yf\begin{pmatrix}0\\1\end{pmatrix}\end{aligned}$$

Thus, we obtain the formula in the statement, with $a,b,c,d\in\mathbb{R}$ being given by:

$$f\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}\quad,\quad f\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}b\\d\end{pmatrix}$$

In the affine case now, we have as extra piece of data a vector, as follows:

$$f\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}p\\q\end{pmatrix}$$

Indeed, if $f:\mathbb{R}^2\to\mathbb{R}^2$ is affine, then the following map is linear:

$$f-\begin{pmatrix}p\\q\end{pmatrix}:\mathbb{R}^2\to\mathbb{R}^2$$

Thus, by using the formula for linear maps, we obtain the result. □

As a further twist to the story, in what regards the linear maps, appearing as above, we can put our parameters $a,b,c,d$ into a matrix, in the following way:

Definition 10.9. *A matrix $A\in M_2(\mathbb R)$ is an array as follows:*

$$A=\begin{pmatrix}a&b\\ c&d\end{pmatrix}$$

*These matrices act on the vectors in the following way,*

$$\begin{pmatrix}a&b\\ c&d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}ax+by\\ cx+dy\end{pmatrix}$$

*the rule being "multiply the rows of the matrix by the vector".*

The above multiplication formula might seem a bit complicated, at a first glance, but it is not. Here is an example for it, quickly worked out:

$$\begin{pmatrix}1&2\\ 5&6\end{pmatrix}\begin{pmatrix}3\\ 1\end{pmatrix}=\begin{pmatrix}1\cdot 3+2\cdot 1\\ 5\cdot 3+6\cdot 1\end{pmatrix}=\begin{pmatrix}5\\ 21\end{pmatrix}$$

Now with the above multiplication convention for matrices and vectors, we can turn Theorem 10.8 into something even better, and more powerful, as follows:

Theorem 10.10. *The linear maps $f:\mathbb R^2\to\mathbb R^2$ are precisely the maps of type*

$$f(v)=Av$$

*and the affine maps $f:\mathbb R^2\to\mathbb R^2$ are precisely the maps of type*

$$f(v)=Av+w$$

*with $A$ being a $2\times 2$ matrix, and with $v,w\in\mathbb R^2$ being vectors, written vertically.*

Proof. This comes indeed from Theorem 10.8, via Definition 10.9, with:

$$A=\begin{pmatrix}a&b\\ c&d\end{pmatrix}\quad,\quad v=\begin{pmatrix}x\\ y\end{pmatrix}\quad,\quad w=\begin{pmatrix}p\\ q\end{pmatrix}$$

Thus, we are led to the conclusions in the statement. □

At the level of basic examples, the symmetries from Proposition 10.2, with respect to $Ox$, $Oy$ and to the diagonals $x=y$, $x=-y$, come from the following matrices:

$$\begin{pmatrix}1&0\\ 0&-1\end{pmatrix}\quad,\quad\begin{pmatrix}-1&0\\ 0&1\end{pmatrix}\quad,\quad\begin{pmatrix}0&1\\ 1&0\end{pmatrix}\quad,\quad\begin{pmatrix}0&-1\\ -1&0\end{pmatrix}$$

Indeed, this is clear from the formulae in Proposition 10.2. Next, the rotations from Proposition 10.3, of angles 0°, 90°, 180°, 270°, come from the following matrices:

$$\begin{pmatrix}1&0\\ 0&1\end{pmatrix}\quad,\quad\begin{pmatrix}0&-1\\ 1&0\end{pmatrix}\quad,\quad\begin{pmatrix}-1&0\\ 0&-1\end{pmatrix}\quad,\quad\begin{pmatrix}0&1\\ -1&0\end{pmatrix}$$

And finally, the various projections from Proposition 10.4, on $Ox$, $Oy$ and on the diagonals $x=y$, $x=-y$, come from the following matrices:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad,\quad\begin{pmatrix}0&0\\0&1\end{pmatrix}\quad,\quad\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\quad,\quad\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

In view of this, let us discuss now the computation of the arbitrary symmetries, rotations and projections. We begin with the rotations, whose formula is a must-know:

PROPOSITION 10.11. *The rotation of angle $t\in\mathbb{R}$ is given by the matrix*

$$R_t=\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}$$

*depending on $t\in\mathbb{R}$ taken modulo $2\pi$.*

PROOF. The rotation being linear, it must correspond to a certain matrix $\binom{a\ b}{c\ d}$. But we can guess this matrix via its action on $\binom{1}{0}$ and $\binom{0}{1}$. Indeed, pictures show that:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\quad,\quad\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$$

Guessing now the matrix is not complicated, because the first equation gives us the first column, and the second equation gives us the second column:

$$\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\quad,\quad\begin{pmatrix}b\\d\end{pmatrix}=\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$$

Thus, we can just put together these two vectors, and we obtain our matrix. □

Regarding now the symmetries, the formula here is as follows:

PROPOSITION 10.12. *The symmetry with respect to $Ox$ rotated by $t/2\in\mathbb{R}$ is*

$$S_t=\begin{pmatrix}\cos t&\sin t\\ \sin t&-\cos t\end{pmatrix}$$

*depending on $t\in\mathbb{R}$ taken modulo $2\pi$.*

PROOF. As before, we can guess the matrix via its action on the basic coordinate vectors $\binom{1}{0}$ and $\binom{0}{1}$. Indeed, some quick pictures show that we must have:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}\quad,\quad\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\sin t\\ -\cos t\end{pmatrix}$$

Thus, we can just put together these two vectors, and we obtain our matrix. □

Finally, regarding the projections, the formula here is as follows:

PROPOSITION 10.13. *The projection on $Ox$ rotated by $t/2 \in \mathbb{R}$ is*

$$P_t = \frac{1}{2}\begin{pmatrix}1+\cos t&\sin t\\ \sin t&1-\cos t\end{pmatrix}$$

*depending on $t \in \mathbb{R}$ taken modulo $2\pi$.*

PROOF. Indeed, some quick pictures, using similarity of triangles, show that:

$$P_t\begin{pmatrix}1\\0\end{pmatrix} = \cos\frac{t}{2}\begin{pmatrix}\cos\frac{t}{2}\\ \sin\frac{t}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1+\cos t\\ \sin t\end{pmatrix}$$

$$P_t\begin{pmatrix}0\\1\end{pmatrix} = \sin\frac{t}{2}\begin{pmatrix}\cos\frac{t}{2}\\ \sin\frac{t}{2}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\sin t\\ 1-\cos t\end{pmatrix}$$

Now by putting together these two vectors, and we obtain our matrix. □

In order to formulate now a second theorem, dealing with compositions of maps, let us make the following multiplication convention, between matrices and matrices:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix} = \begin{pmatrix}ap+br&aq+bs\\cp+dr&cq+ds\end{pmatrix}$$

This might look a bit complicated, but as before, in what was concerning multiplying matrices and vectors, the idea is very simple, namely "multiply the rows of the first matrix by the columns of the second matrix". With this convention, we have:

THEOREM 10.14. *If we denote by $f_A : \mathbb{R}^2 \to \mathbb{R}^2$ the linear map associated to $A$,*

$$f_A(v) = Av$$

*then we have the following multiplication formula for such maps:*

$$f_A f_B = f_{AB}$$

*That is, the composition of linear maps corresponds to the multiplication of matrices.*

PROOF. We want to prove that we have the following formula, valid for any two matrices $A, B \in M_2(\mathbb{R})$, and any vector $v \in \mathbb{R}^2$:

$$A(Bv) = (AB)v$$

For this purpose, let us write our matrices and vector as follows:

$$A = \begin{pmatrix}a&b\\c&d\end{pmatrix} \quad , \quad B = \begin{pmatrix}p&q\\r&s\end{pmatrix} \quad , \quad v = \begin{pmatrix}x\\y\end{pmatrix}$$

The formula that we want to prove becomes:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\left[\begin{pmatrix}p&q\\r&s\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right] = \left[\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}p&q\\r&s\end{pmatrix}\right]\begin{pmatrix}x\\y\end{pmatrix}$$

But this is the same as saying that we have:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}px+qy\\rx+sy\end{pmatrix}=\begin{pmatrix}ap+br&aq+bs\\cp+dr&cq+ds\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

And this latter formula does hold indeed, because on both sides we get:

$$\begin{pmatrix}apx+aqy+brx+bsy\\cpx+cqy+drx+dsy\end{pmatrix}$$

Thus, we have proved the result. □

As a verification for the above result, let us compose two rotations. The computation here is as follows, yieding a rotation, as it should, and of the correct angle:

$$\begin{aligned}R_sR_t&=\begin{pmatrix}\cos s&-\sin s\\\sin s&\cos s\end{pmatrix}\begin{pmatrix}\cos t&-\sin t\\\sin t&\cos t\end{pmatrix}\\&=\begin{pmatrix}\cos s\cos t-\sin s\sin t&-\cos s\sin t-\sin t\cos s\\\sin s\cos t+\cos s\sin t&-\sin s\sin t+\cos s\cos t\end{pmatrix}\\&=\begin{pmatrix}\cos(s+t)&-\sin(s+t)\\\sin(s+t)&\cos(s+t)\end{pmatrix}\\&=R_{s+t}\end{aligned}$$

Finally, in relation with the considerations from chapter 9, we have:

THEOREM 10.15. *A matrix is invertible precisely when its determinant*

$$\det\begin{pmatrix}a&b\\c&d\end{pmatrix}=ad-bc$$

*is nonzero, and in this case, we have the following inversion formula:*

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$$

*Moreover, this result can be used for inverting the linear maps.*

PROOF. Many things can be said here, the idea being as follows:

(1) To start with, $ad-bc=0$ means $\binom{a}{c}\sim\binom{b}{d}$, so the associated linear map is not surjective, and by Theorem 10.14 it follows that our matrix is not invertible.

(2) Next, assuming $ad-bc\neq 0$, let us try to solve the following problem:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}*&*\\ *&*\end{pmatrix}=\begin{pmatrix}ad-bc&0\\0&ad-bc\end{pmatrix}$$

But the solution here is a no-brainer, and we are led to the inversion formula in the statement. As for the last assertion, this follows from Theorem 10.14.

(3) Finally, you might wonder what the determinant really is. In answer, we can say that we have by definition the following formula, for any $x,y\in\mathbb{R}^2$, with the sign being $+$ when $x,y$ come in this order, counterclockwise, and being $-$ otherwise:

$$\det(x\ y)=\pm area(O,x,y,x+y)$$

Indeed, this makes the link with the considerations from chapter 9, and the results there tell us that this is indeed the determinant, as appearing in the statement. □

## 10b. Ellipses, conics

Getting back now to geometry, let us talk about conics. We have already met them in chapter 4, and time to have a systematic look at all this. We first have:

THEOREM 10.16. *The ellipses, taken centered at the origin* $0$*, and squarely oriented with respect to* $Oxy$*, can be defined in* $4$ *possible ways, as follows:*

(1) *As the curves given by an equation as follows, with* $a,b>0$*:*

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

(2) *Or given by an equation as follows, with* $q>0$*,* $p=-q$*, and* $l\in(0,2q)$*:*

$$d(z,p)+d(z,q)=l$$

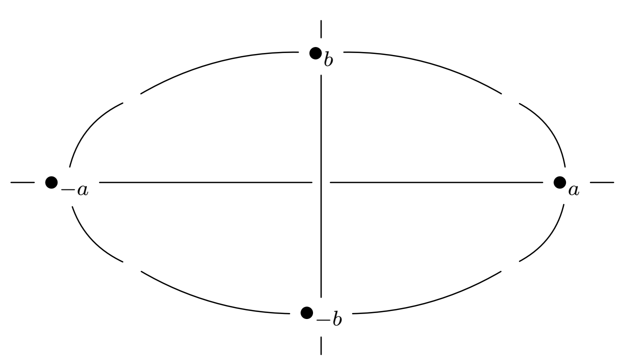
(3) *As the curves appearing when drawing a circle, from various perspectives:*

$$\bigcirc\quad\to\quad ?$$

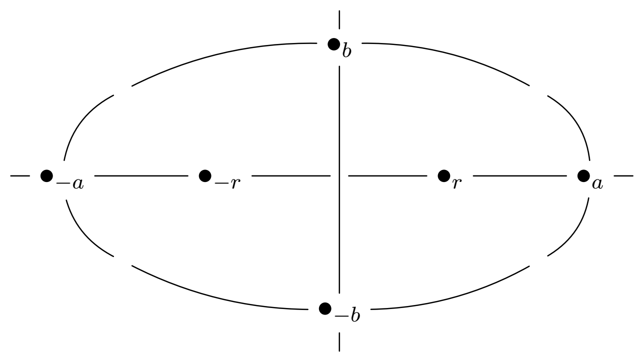
(4) *As the closed non-degenerate curves appearing by cutting a cone with a plane.*

PROOF. This might look a bit confusing, and you might say, what exactly is to be proved here. Good point, and in answer, what is to be proved is that the above constructions (1-4) give rise to the same class of curves. And this can be done as follows:

(1) To start with, let us draw a picture from what comes out of (1), which will be our main definition for the ellipses, in what follows. Here that is, making it clear what the parameters $a,b>0$ stand for, with $2a\times 2b$ being the gift box size for our ellipse:



(2) Let us prove now that such an ellipse has two focal points, as stated in (2). We must look for a number $r > 0$, and a number $l > 0$, such that our ellipse appears as $d(z,p) + d(z,q) = l$, with $p = (0,-r)$ and $q = (0,r)$, according to the following picture:




(3) Let us first compute these numbers $r, l > 0$. Assuming that our result holds indeed as stated, by taking $z = (0,a)$, we see that the length $l$ is:

$$l = (a-r) + (a+r) = 2a$$

As for the parameter $r$, by taking $z = (b,0)$, we conclude that we must have:

$$2\sqrt{b^2+r^2} = 2a \implies r = \sqrt{a^2-b^2}$$

(4) With these observations made, let us prove now the result. Given $l, r > 0$, and setting $p = (0,-r)$ and $q = (0,r)$, we have the following computation, with $z = (x,y)$:

$$\begin{aligned}
& d(z,p) + d(z,q) = l \\
\iff\quad & \sqrt{(x+r)^2+y^2} + \sqrt{(x-r)^2+y^2} = l \\
\iff\quad & \sqrt{(x+r)^2+y^2} = l - \sqrt{(x-r)^2+y^2} \\
\iff\quad & (x+r)^2 + y^2 = (x-r)^2 + y^2 + l^2 - 2l\sqrt{(x-r)^2+y^2} \\
\iff\quad & 2l\sqrt{(x-r)^2+y^2} = l^2 - 4xr \\
\iff\quad & 4l^2(x^2+r^2-2xr+y^2) = l^4 + 16x^2r^2 - 8l^2xr \\
\iff\quad & 4l^2x^2 + 4l^2r^2 + 4l^2y^2 = l^4 + 16x^2r^2 \\
\iff\quad & (4x^2-l^2)(4r^2-l^2) = 4l^2y^2
\end{aligned}$$

(5) Now observe that we can further process the equation that we found as follows:

$$\begin{aligned}(4x^2-l^2)(4r^2-l^2)=4l^2y^2 &\iff \frac{4x^2-l^2}{l^2}=\frac{4y^2}{4r^2-l^2}\\ &\iff \frac{4x^2-l^2}{l^2}=\frac{y^2}{r^2-l^2/4}\\ &\iff \left(\frac{x}{2l}\right)^2-1=\left(\frac{y}{\sqrt{r^2-l^2/4}}\right)^2\\ &\iff \left(\frac{x}{2l}\right)^2+\left(\frac{y}{\sqrt{r^2-l^2/4}}\right)^2=1\end{aligned}$$

(6) Thus, our result holds indeed, and with the numbers $l,r>0$ appearing, and no surprise here, via the formulae $l=2a$ and $r=\sqrt{a^2-b^2}$, found in (3) above.

(7) Getting back to our theorem, we have two other assertions there at the end, (3,4). But, thinking a bit, these assertions are equivalent, and (4) can be established by doing some 3D computations, that we will leave here as an instructive exercise, for you. $\square$

Along the same lines, at a more advanced level, we have the following result:

THEOREM 10.17. *The conics, which are the algebraic curves of degree* 2 *in the plane,*

$$C=\left\{(x,y)\in\mathbb{R}^2\Big|P(x,y)=0\right\}$$

*with* $\deg P\leq 2$*, appear modulo degeneration by cutting a* 2*-sided cone with a plane, and can be classified into ellipses, parabolas and hyperbolas.*

PROOF. This follows by further building on Theorem 10.16, as follows:

(1) Let us first classify the conics up to non-degenerate linear transformations of the plane, which are by definition transformations as follows, with $\det A\neq 0$:

$$\begin{pmatrix}x\\ y\end{pmatrix}\to A\begin{pmatrix}x\\ y\end{pmatrix}$$

Our claim is that as solutions we have the circles, parabolas, hyperbolas, along with some degenerate solutions, namely $\emptyset$, points, lines, pairs of lines, $\mathbb{R}^2$.

(2) As a first remark, it looks like we forgot precisely the ellipses, but via linear transformations these become circles, so things fine. As a second remark, all our claimed solutions can appear. Indeed, the circles, parabolas, hyperbolas can appear as follows:

$$x^2+y^2=1\quad,\quad x^2=y\quad,\quad xy=1$$

As for $\emptyset$, points, lines, pairs of lines, $\mathbb{R}^2$, these can appear too, as follows, and with our polynomial $P$ chosen, whenever possible, to be of degree exactly 2:

$$x^2 = -1 \quad , \quad x^2 + y^2 = 0 \quad , \quad x^2 = 0 \quad , \quad xy = 0 \quad , \quad 0 = 0$$

Observe here that, when dealing with these degenerate cases, assuming $\deg P = 2$ instead of $\deg P \leq 2$ would only rule out $\mathbb{R}^2$ itself, which is not worth it.

(3) Getting now to the proof of our claim in (1), classification up to linear transformations, consider an arbitrary conic, written as follows, with $a, b, c, d, e, f \in \mathbb{R}$:

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

Assume first $a \neq 0$. By making a square out of $ax^2$, up to a linear transformation in $(x, y)$, we can get rid of the term $cxy$, and we are left with:

$$ax^2 + by^2 + dx + ey + f = 0$$

In the case $b \neq 0$ we can make two obvious squares, and again up to a linear transformation in $(x, y)$, we are left with an equation as follows:

$$x^2 \pm y^2 = k$$

In the case of positive sign, $x^2 + y^2 = k$, the solutions are the circle, when $k \geq 0$, the point, when $k = 0$, and $\emptyset$, when $k < 0$. As for the case of negative sign, $x^2 - y^2 = k$, which reads $(x - y)(x + y) = k$, here once again by linearity our equation becomes $xy = l$, which is a hyperbola when $l \neq 0$, and two lines when $l = 0$.

(4) In the case $b \neq 0$ the study is similar, with the same solutions, so we are left with the case $a = b = 0$. Here our conic is as follows, with $c, d, e, f \in \mathbb{R}$:

$$cxy + dx + ey + f = 0$$

If $c \neq 0$, by linearity our equation becomes $xy = l$, which produces a hyperbola or two lines, as explained before. As for the remaining case, $c = 0$, here our equation is:

$$dx + ey + f = 0$$

But this is generically the equation of a line, unless we are in the case $d = e = 0$, where our equation is $f = 0$, having as solutions $\emptyset$ when $f \neq 0$, and $\mathbb{R}^2$ when $f = 0$.

(5) Thus, done with the classification, up to linear transformations as in (1). But this classification leads to the classification in general too, by applying now linear transformations to the solutions that we found. So, done with this, and very good.

(6) It remains to discuss the cone cutting. By suitably choosing our coordinate axes $(x, y, z)$, we can assume that our cone is given by an equation as follows, with $k > 0$:

$$x^2 + y^2 = kz^2$$

In order to prove the result, we must in principle intersect this cone with an arbitrary plane, which has an equation as follows, with $(a,b,c)\neq(0,0,0)$:

$$ax+by+cz=d$$

(7) However, before getting into computations, observe that what we want to find is a certain degree 2 equation in the above plane, for the intersection. Thus, it is convenient to change the coordinates, as for our plane to be given by the following equation:

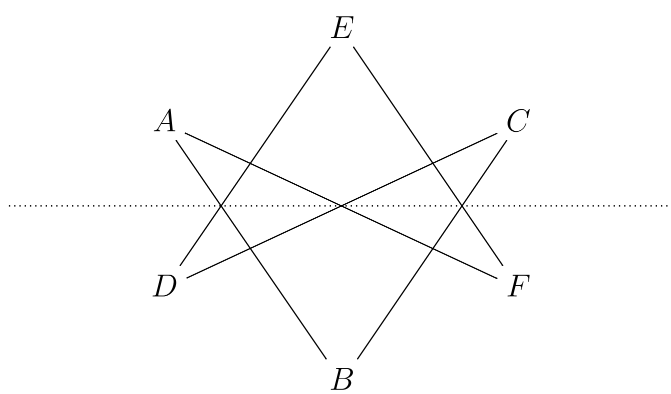
$$z=0$$

(8) But with this done, what we have to do is to see how the cone equation $x^2+y^2=kz^2$ changes, under this change of coordinates, and then set $z=0$, as to get the $(x,y)$ equation of the intersection. But this leads, via some thinking or computations, to the conclusion that the cone equation $x^2+y^2=kz^2$ becomes in this way a degree 2 equation in $(x,y)$, which can be arbitrary, and so to the final conclusion in the statement. □

In practice now, we already know many things about ellipses, from the beginning of this section. Similar things can be said about parabolas and hyperbolas.

Getting now to more advanced plane geometry, we have the following result:

THEOREM 10.18 (Pascal). *Given a hexagon lying on a conic*




*the pairs of opposite sides intersect in points which are collinear.*

PROOF. This is something quite standard, the idea being as follows:

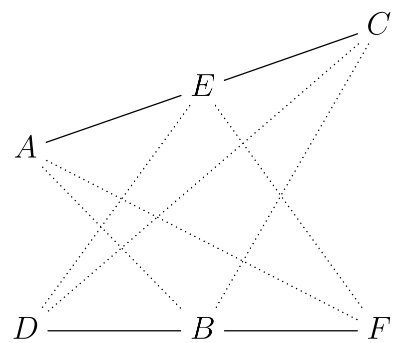
(1) We already know this from chapter 4 for the circles, and by using the cone cutting from Theorem 10.17, and projecting, we have this, more generally, for all ellipses.

(2) But then, based on the fact that the ellipses are generic conics, we can conclude, via a standard abstract algebra argument, that the result must hold for all conics.

(3) Alternatively, all this can be proved of course directly, say by using affine coordinates, and with the computations here being an excellent exercise. □

As an interesting consequence of the above result, worth recording, we have:

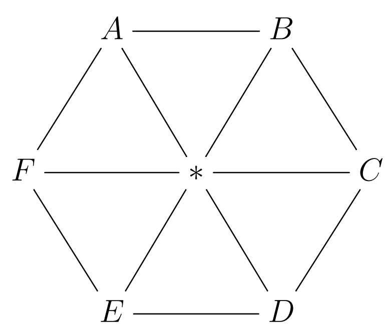
FACT 10.19. *The Pascal theorem for conics generalizes the Pappus theorem*




*which corresponds to the case where the conic in question consists of* 2 *lines.*

In short, good news, if you had troubles in understanding the proof of Pappus, back when struggling with chapter 1, no worries, because you can have it now via Pascal. Still following the material in chapter 4, we have as well the following result:

THEOREM 10.20 (Brianchon). *Given a hexagon circumscribed around on a conic*




*the main diagonals intersect.*

PROOF. The story here is quite similar to that of the Pascal theorem, the idea being that we know this from chapter 4 for the circles, and with this actually coming from Pascal itself, via duality, then the extension to the ellipses is standard, by using Theorem 10.17 and projecting, and finally the extension to all conics is standard too, by using a standard abstract algebra argument, involving genericity. Alternatively, this can be proved as well by using coordinates, and with the computations here being an excellent exercise. □

And with this, good news, looking back restrospectively at what we wanted to do in the present Part III, namely reviewing the geometry from Part I by using coordinates, all that we knew from Part I, or almost, has been successfully reviewed. Very nice.

## 10c. Polar coordinates

Moving on with our mathematics, and with the sky being the limit, we can in fact do better than the affine coordinates, by introducing polar coordinates, as follows:

THEOREM 10.21. *The points of the plane $x \in \mathbb{R}^2$, written as vectors*

$$x = \begin{pmatrix} a \\ b \end{pmatrix}$$

*can be written in polar coordinates, in the following way,*

$$x = \begin{pmatrix} r\cos t \\ r\sin t \end{pmatrix}$$

*with the connecting formulae being as follows,*

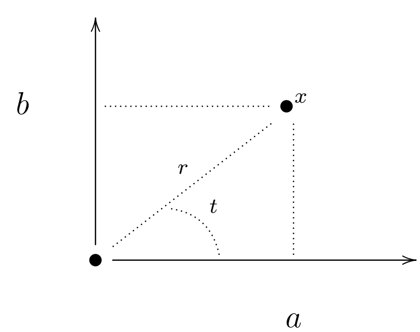
$$a = r\cos t \quad , \quad b = r\sin t$$

*and in the other sense being as follows,*

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

*and with the numbers $r, t$ being called modulus, and argument.*

PROOF. This is something self-explanatory and intuitive, with $r = \sqrt{a^2 + b^2}$ being as usual the length of the vector, and with $t$ being the angle made by the vector with the $Ox$ axis. That is, with the picture for what is going on in the above being as follows:




Thus, we are led to the conclusions in the statement. □

As a complement to the above result, in relation with notations, let us record:

COMMENT 10.22. *In the presence of physics, and time $t$, it is better to write:*

$$x = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

*However, with no physics present, it is better to use Theorem 10.21 as such.*

And I am saying this because I am usually faster than my students to do computations with polar coordinates, thanks to my dumb $t$, which is quicker to write than their $\theta$.

As an application now of this technology, and coming as a continuation of our work on conics, let us investigate more general curves. Let us start our discussion with:

DEFINITION 10.23. *An algebraic curve in $\mathbb{R}^2$ is the vanishing set*

$$C=\left\{(x,y)\in\mathbb{R}^2\Big|P(x,y)=0\right\}$$

*of a polynomial $P\in\mathbb{R}[X,Y]$ of arbitrary degree.*

We already know well the algebraic curves in degree 2, which are the conics, and a first problem is, what results from what we learned about conics have a chance to be relevant to the arbitrary algebraic curves. And normally none, because the ellipses, parabolas and hyperbolas are obviously very particular curves, having very particular properties.

Let us record however a useful statement here, as follows:

PROPOSITION 10.24. *The conics can be written in cartesian, polar, or parametric coordinates, with the equations for the unit circle being*

$$x^2+y^2=1 \qquad , \qquad r=1 \qquad , \qquad x=\cos t\,,\ y=\sin t$$

*and with the equations for ellipses, parabolas and hyperbolas being similar.*

PROOF. The equations for the circle are clear, those for ellipses were found in the above, and we will leave as an exercise those for parabolas and hyperbolas. □

As a true answer to our question now, coming this time from a very modest conic, namely $xy=0$, that we dismissed in the above as being "degenerate", we have:

THEOREM 10.25. *The following happen, for curves $C$ defined by polynomials $P$:*

1. *In degree $d=2$, curves can have singularities, such as $xy=0$ at $(0,0)$.*
2. *In general, assuming $P=P_1\ldots P_k$, we have $C=C_1\cup\ldots\ldots\cup C_k$.*
3. *A union of curves $C_i\cup C_j$ is generically non-smooth, unless disjoint.*
4. *Due to this, we say that $C$ is non-degenerate when $P$ is irreducible.*

PROOF. All this is self-explanatory, the details being as follows:

(1) This is something obvious, just the story of two lines crossing.

(2) This comes from the following trivial fact, with the notation $z=(x,y)$:

$$P_1\ldots P_k(z)=0\iff P_1(z)=0,\text{ or }P_2(z)=0,\ \ldots\ ,\text{ or }P_k(z)=0$$

(3) This is something very intuitive, and it actually takes a bit of time to imagine a situation where $C_1\cap C_2\neq\emptyset$, $C_1\not\subset C_2$, $C_2\not\subset C_1$, but $C_1\cup C_2$ is smooth. In practice now, "generically" has of course a mathematical meaning, in relation with probability,

and our assertion does say something mathematical, that we are supposed to prove. But, we will not insist on this, and leave this as an instructive exercise, precise formulation of the claim, and its proof, in the case you are familiar with probability theory.

(4) This is just a definition, based on the above, that we will use in what follows. □

With degree 1 and 2 investigated, and our conclusions recorded, let us get now to degree 3, see what new phenomena appear here. And here, to start with, we have the following remarkable curve, well-known from calculus, because 0 is not a maximum or minimum of the function $x \to y$, despite the derivative vanishing there:

$$x^3 = y$$

Also, in relation with set theory and logic, and with the foundations of mathematics in general, we have the following curve, which looks like the empyset $\emptyset$:

$$(x - y)(x^2 + y^2 - 1) = 0$$

But, it is not about counterexamples to calculus, or about logic, that we want to talk about here. As a first truly remarkable degree 3 curve, or cubic, we have the cusp:

PROPOSITION 10.26. *The standard cusp, which is the cubic given by*

$$x^3 = y^2$$

*has a singularity at* $(0, 0)$*, with only* 1 *tangent line at that singularity.*

PROOF. The two branches of the cusp are indeed both tangent to $Ox$, because:

$$y' = \pm\frac{3}{2}\sqrt{x} \implies y'(0) = 0$$

Observe also that what happens for the cusp is different from what happens for $xy = 0$, precisely because we have 1 line tangent at the singularity, instead of 2. □

As a second remarkable cubic, which gets the crown, and the right to have a Theorem about it, we have the Tschirnhausen curve, which is as follows:

THEOREM 10.27. *The Tschirnhausen cubic, given by the following equation,*

$$x^3 = x^2 - 3y^2$$

*makes the dream of* $xy = 0$ *come true, by self-intersecting, and being non-degenerate.*

PROOF. This is something self-explanatory, by drawing a picture, but there are several other interesting things that can be said about this curve, as follows:

(1) Let us start with the curve written in polar coordinates, as follows:

$$r \cos^3\left(\frac{t}{3}\right) = a$$

With $k=\tan(\theta/3)$, the equations of the coordinates are as follows:

$$x=a(1-3k^2)\quad,\quad y=ak(3-k^2)$$

Now by eliminating $k$, we reach to the following equation:

$$(a-x)(8a+x)^2=27ay^2$$

(2) By translating horizontally by $8a$, and changing signs of variables, we have:

$$x=3a(3-k^2)\quad,\quad y=ak(3-k^2)$$

Now by eliminating $k$, we reach to the following equation:

$$x^3=9a(x^2-3y^2)$$

But with $a=1/9$ this is precisely the equation in the statement. □

In degree 4 now, quartics, we have enough dimensions for "improving" the cusp and the Tschirnhausen curve. First we have the cardioid, which is as follows:

Proposition 10.28. *The cardioid, which is a quartic, given in polar coordinates by*

$$2r=a(1-\cos t)$$

*makes the dream of $x^3=y^2$ come true, by being a closed curve, with a cusp.*

Proof. As before with the Tschirnhausen curve, this is something self-explanatory, by drawing a picture, but there are several things that must be said, as follows:

(1) The cardioid appears by definition by rolling a circle of radius $c>0$ around another circle of same radius $c>0$. With $t$ being the rolling angle, we have:

$$x=2c(1-\cos t)\cos t$$

$$y=2c(1-\cos t)\sin t$$

(2) Thus, in polar coordinates we get the equation in the statement, with $a=4c$:

$$r=2c(1-\cos t)$$

(3) Finally, in cartesian coordinates, the equation is as follows:

$$(x^2+y^2)^2+4cx(x^2+y^2)=4c^2y^2$$

Thus, what we have is indeed a degree 4 curve, as claimed. □

Still in degree 4, the crown gets to the Bernoulli lemniscate, which is as follows:

Theorem 10.29. *The Bernoulli lemniscate, a quartic, which is given by*

$$r^2=a^2\cos 2t$$

*makes the dream of $x^3=x^2-3y^2$ come true, by being closed, and self-intersecting.*

PROOF. As usual, this is something self-explanatory, by drawing a picture, which looks like an $\infty$ sign, but there are several other things that must be said. For instance in cartesian coordinates, the equation of our curve is as follows, with $a^2 = 2c^2$:

$$(x^2 + y^2)^2 = c^2(x^2 - y^2)$$

Thus, we are led to the conclusions in in the statement. $\square$

We will be back to curves, with a discussion at $N = 5$ and higher, in chapter 12.

## 10d. The Solar system

Getting back now to the basics, conics, we have seen in the above that all basic mathematics relies on them. Importantly, the same is true for physics, with the planets and comets in our Solar system moving aroung the Sun on ellipses, and with certain asteroids, which are here in our system just for visiting, moving on parabolas and hyperbolas.

Excited about this? So am I, and here is the result, due to Kepler and Newton:

THEOREM 10.30. *Planets and other celestial bodies move around the Sun on conics,*

$$C = \left\{(x, y) \in \mathbb{R}^2 \Big| P(x, y) = 0\right\}$$

*with $P \in \mathbb{R}[x, y]$ being of degree 2, which can be ellipses, parabolas or hyperbolas.*

PROOF. This is something quite long, requiring some knowledge of calculus and equations, and more specifically, of multivariable calculus, and with this knowledge being scheduled for later in this book, at the very end, in chapter 16, we are a bit in trouble. This being said, no way back, so here is the proof, technically based on that material that we will learn later, and please relax, take this as a physics class, and enjoy:

(1) According to observations and calculations performed over the centuries, since the ancient times, and first formalized by Newton, following some groundbreaking work of Kepler, the force of attraction between two bodies of masses $M, m$ is given by:

$$||F|| = G \cdot \frac{Mm}{d^2}$$

Here $d$ is the distance between the two bodies, and $G \simeq 6.674 \times 10^{-11}$ is a constant. Now assuming that $M$ is fixed at $0 \in \mathbb{R}^3$, the force exterted on $m$ positioned at $x \in \mathbb{R}^3$, regarded as a vector $F \in \mathbb{R}^3$, is given by the following formula:

$$F = -||F|| \cdot \frac{x}{||x||} = -\frac{GMm}{||x||^2} \cdot \frac{x}{||x||} = -\frac{GMmx}{||x||^3}$$

But $F = ma = m\ddot{x}$, with $a = \ddot{x}$ being the acceleration, second derivative of the position, so the equation of motion of $m$, assuming that $M$ is fixed at 0, is:

$$\ddot{x} = -\frac{GMx}{||x||^3}$$

(2) Obviously, the problem happens in 2 dimensions, and you can even find, as an exercise, a formal proof of that, based on the above equation. Now here the most convenient is to use standard $x, y$ coordinates, and denote our point as $z = (x, y)$. With this change made, and by setting $K = GM$, the equation of motion becomes:

$$\ddot{z} = -\frac{Kz}{||z||^3}$$

In other words, in terms of the coordinates $x, y$, the equations are:

$$\ddot{x} = -\frac{Kx}{(x^2+y^2)^{3/2}} \quad , \quad \ddot{y} = -\frac{Ky}{(x^2+y^2)^{3/2}}$$

(3) Let us begin with a simple particular case, that of the circular solutions. To be more precise, we are interested in solutions of the following type:

$$x = r\cos\alpha t \quad , \quad y = r\sin\alpha t$$

In this case we have $||z|| = r$, so our equation of motion becomes:

$$\ddot{z} = -\frac{Kz}{r^3}$$

On the other hand, differentiating $x, y$ leads to the following formula:

$$\ddot{z} = (\ddot{x}, \ddot{y}) = -\alpha^2(x, y) = -\alpha^2 z$$

Thus, we have a circular solution when the parameters $r, \alpha$ satisfy:

$$r^3\alpha^2 = K$$

(4) In the general case now, the problem can be solved via some calculus. Let us write indeed our vector $z = (x, y)$ in polar coordinates, as in Comment 10.22, as follows:

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

We have then $||z|| = r$, and our equation of motion becomes, as in (3):

$$\ddot{z} = -\frac{Kz}{r^3}$$

Let us differentiate now $x, y$. By using standard calculus rules, we have:

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta}$$

Differentiating one more time gives the following formulae:

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\sin\theta\cdot\dot{\theta} - r\cos\theta\cdot\dot{\theta}^2 - r\sin\theta\cdot\ddot{\theta}$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\cos\theta\cdot\dot{\theta} - r\sin\theta\cdot\dot{\theta}^2 + r\cos\theta\cdot\ddot{\theta}$$

Consider now the following two quantities, appearing as coefficients in the above:

$$a = \ddot{r} - r\dot{\theta}^2 \quad , \quad b = 2\dot{r}\dot{\theta} + r\ddot{\theta}$$

In terms of these quantities, our second derivative formulae read:

$$\ddot{x} = a\cos\theta - b\sin\theta$$

$$\ddot{y} = a\sin\theta + b\cos\theta$$

(5) We can now solve the equation of motion from (4). Indeed, with the formulae that we found for $\ddot{x}, \ddot{y}$, our equation of motion takes the following form:

$$a\cos\theta - b\sin\theta = -\frac{K}{r^2}\cos\theta$$

$$a\sin\theta + b\cos\theta = -\frac{K}{r^2}\sin\theta$$

But these two formulae can be written in the following way:

$$\left(a + \frac{K}{r^2}\right)\cos\theta = b\sin\theta$$

$$\left(a + \frac{K}{r^2}\right)\sin\theta = -b\cos\theta$$

By making now the product, and assuming that we are in a non-degenerate case, where the angle $\theta$ varies indeed, we obtain by positivity that we must have:

$$a + \frac{K}{r^2} = b = 0$$

(6) We are almost there. Let us first examine the second equation, $b = 0$. Remembering who $b$ is, from (4), this equation can be solved as follows:

$$\begin{aligned}
b = 0 \quad &\Longleftrightarrow \quad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0\\
&\Longleftrightarrow \quad \frac{\ddot{\theta}}{\dot{\theta}} = -2\frac{\dot{r}}{r}\\
&\Longleftrightarrow \quad (\log\dot{\theta})' = (-2\log r)'\\
&\Longleftrightarrow \quad \log\dot{\theta} = -2\log r + c\\
&\Longleftrightarrow \quad \dot{\theta} = \frac{\lambda}{r^2}
\end{aligned}$$

As for the first equation the we found, namely $a + K/r^2 = 0$, remembering from (4) that $a$ was by definition given by $a = \ddot{r} - r\dot{\theta}^2$, this equation now becomes:

$$\ddot{r} - \frac{\lambda^2}{r^3} + \frac{K}{r^2} = 0$$

(7) As a conclusion to all this, in polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, our equations of motion are as follows, with $\lambda$ being a constant, not depending on $t$:

$$\ddot{r} = \frac{\lambda^2}{r^3} - \frac{K}{r^2} \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

Even better now, by writing $K = \lambda^2/c$, these equations read:

$$\ddot{r} = \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right) \quad , \quad \dot{\theta} = \frac{\lambda}{r^2}$$

(8) As an illustration, let us quickly work out the case of a circular motion, where $r$ is constant. Here $\ddot{r} = 0$, so the first equation gives $c = r$. Also we have $\dot{\theta} = \alpha$, with:

$$\alpha = \frac{\lambda}{r^2}$$

Assuming $\theta = 0$ at $t = 0$, from $\dot{\theta} = \alpha$ we obtain $\theta = \alpha t$, and so, as in (3) above:

$$x = r\cos\alpha t \quad , \quad y = r\sin\alpha t$$

Observe also that the condition found in (3) is indeed satisfied:

$$r^3\alpha^2 = \frac{\lambda^2}{r} = \frac{\lambda^2}{c} = K$$

(9) Back to the general case now, our claim is that we have the following formula, for the distance $r = r(t)$ as function of the angle $\theta = \theta(t)$, for some $\varepsilon, \delta \in \mathbb{R}$:

$$r = \frac{c}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

Let us first check that this formula works indeed. With $r$ being as above, and by using our second equation found before, $\dot{\theta} = \lambda/r^2$, we have the following computation:

$$\begin{aligned}\dot{r} &= \frac{c(\varepsilon\sin\theta - \delta\cos\theta)\dot{\theta}}{(1 + \varepsilon\cos\theta + \delta\sin\theta)^2}\\ &= \frac{\lambda c(\varepsilon\sin\theta - \delta\cos\theta)}{r^2(1 + \varepsilon\cos\theta + \delta\sin\theta)^2}\\ &= \frac{\lambda(\varepsilon\sin\theta - \delta\cos\theta)}{c}\end{aligned}$$

Thus, the second derivative of the above function $r$ is given, as desired, by:

$$\begin{aligned}\ddot{r} &= \frac{\lambda(\varepsilon\cos\theta + \delta\sin\theta)\dot{\theta}}{c}\\ &= \frac{\lambda^2(\varepsilon\cos\theta + \delta\sin\theta)}{r^2 c}\\ &= \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right)\end{aligned}$$

(10) The above check was something quite informal, and now we must prove that our formula is indeed the correct one. For this purpose, we use a trick. Let us write:

$$r(t) = \frac{1}{f(\theta(t))}$$

Abbreviated, and by always reminding that $f$ takes $\theta = \theta(t)$ as variable, this reads:

$$r = \frac{1}{f}$$

With the convention that dots mean as usual derivatives with respect to $t$, and that the primes will denote derivatives with respect to $\theta = \theta(t)$, we have:

$$\dot{r} = -\frac{f'\dot{\theta}}{f^2} = -\frac{f'}{f^2} \cdot \frac{\lambda}{r^2} = -\lambda f'$$

By differentiating one more time with respect to $t$, we obtain:

$$\ddot{r} = -\lambda f''\dot{\theta} = -\lambda f'' \cdot \frac{\lambda}{r^2} = -\frac{\lambda^2}{r^2} f''$$

On the other hand, our equation for $\ddot{r}$ found in (7) reads:

$$\ddot{r} = \frac{\lambda^2}{r^2}\left(\frac{1}{r} - \frac{1}{c}\right) = \frac{\lambda^2}{r^2}\left(f - \frac{1}{c}\right)$$

Thus, in terms of $f = 1/r$ as above, our equation for $\ddot{r}$ simply reads:

$$f'' + f = \frac{1}{c}$$

But this latter equation is elementary to solve. Indeed, both functions $\cos t, \sin t$ satisfy $g" + g = 0$, so any linear combination of them satisfies as well this equation. But the solutions of $f'' + f = 1/c$ being those of $g'' + g = 0$ shifted by $1/c$, we obtain:

$$f = \frac{1 + \varepsilon\cos\theta + \delta\sin\theta}{c}$$

Now by inverting, we obtain the formula announced in (9), namely:

$$r = \frac{c}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

(11) But this leads to the conclusion that the trajectory is a conic. Indeed, in terms of the parameter $\theta$, the formulae of the coordinates are:

$$x = \frac{c\cos\theta}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

$$y = \frac{c\sin\theta}{1 + \varepsilon\cos\theta + \delta\sin\theta}$$

But these are precisely the equations of conics in polar coordinates.

(12) To be more precise, in order to find the precise equation of the conic, observe that the two functions $x, y$ that we found above satisfy the following formula:

$$\begin{aligned} x^2+y^2 &= \frac{c^2(\cos^2\theta+\sin^2\theta)}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \\ &= \frac{c^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \end{aligned}$$

On the other hand, these two functions satisfy as well the following formula:

$$\begin{aligned} (\varepsilon x+\delta y-c)^2 &= \frac{c^2\big(\varepsilon\cos\theta+\delta\sin\theta-(1+\varepsilon\cos\theta+\delta\sin\theta)\big)^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \\ &= \frac{c^2}{(1+\varepsilon\cos\theta+\delta\sin\theta)^2} \end{aligned}$$

We conclude that our coordinates $x, y$ satisfy the following equation:

$$x^2+y^2=(\varepsilon x+\delta y-c)^2$$

But what we have here is an equation of a conic, as claimed. □

## 10e. Exercises

This was a truly advanced geometry chapter, and as exercises here, we have:

Exercise 10.31. *Find the matrices of all plane transformations that you know.*

Exercise 10.32. *Find the transformations coming from all matrices $A \in M_2(0,1)$.*

Exercise 10.33. *Then, do the same for all matrices $A \in M_2(-1,0,1)$.*

Exercise 10.34. *Learn more linear algebra, in particular with diagonalization.*

Exercise 10.35. *Learn more about the focal points of ellipses, and other conics.*

Exercise 10.36. *Prove the Pascal and Brianchon theorems directly, using coordinates.*

Exercise 10.37. *Work out the precise formulae for conics, in polar coordinates.*

Exercise 10.38. *Learn more about Kepler and Newton, and their findings.*

As bonus exercise, find and start reading an old-style algebraic geometry book.

CHAPTER 11

# Complex numbers

## 11a. Complex numbers

Let us discuss now the complex numbers, bringing our geometry and trigonometry discussion in this book to yet another level, hopefully final. There is a lot of magic here, and we will carefully explain this material. The starting definition is as follows:

DEFINITION 11.1. *The complex numbers are variables of the form*

$$x = a + ib$$

*with $a, b \in \mathbb{R}$, which add in the obvious way, and multiply according to the following rule:*

$$i^2 = -1$$

*Each real number can be regarded as a complex number, $a = a + i \cdot 0$.*

In other words, we consider variables as above, without bothering for the moment with their precise meaning. Now consider two such complex numbers:

$$x = a + ib \quad , \quad y = c + id$$

The formula for the sum is then the obvious one, as follows:

$$x + y = (a + c) + i(b + d)$$

As for the formula of the product, by using the rule $i^2 = -1$, we obtain:

$$\begin{aligned} xy &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2bd \\ &= ac + iad + ibc - bd \\ &= (ac - bd) + i(ad + bc) \end{aligned}$$

Thus, the complex numbers as introduced above are well-defined. The multiplication formula is of course quite tricky, and hard to memorize, but we will see later some alternative ways, which are more conceptual, for performing the multiplication.

The advantage of using the complex numbers comes from the fact that the equation $x^2 = 1$ has now a solution, $x = i$. In fact, this equation has two solutions, namely:

$$x = \pm i$$

This is of course very good news. More generally, we have the following result, regarding the arbitrary degree 2 equations, with real coefficients:

THEOREM 11.2. *The complex solutions of $ax^2+bx+c=0$ with $a,b,c\in\mathbb{R}$ are*

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

*with the square root of negative real numbers being defined as*

$$\sqrt{-m}=\pm i\sqrt{m}$$

*and with the square root of positive real numbers being the usual one.*

PROOF. We can write our equation in the following way:

$$\begin{aligned}ax^2+bx+c=0 &\iff x^2+\frac{b}{a}x+\frac{c}{a}=0\\ &\iff \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0\\ &\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\\ &\iff x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We will see later that any degree 2 complex equation has solutions as well, and that more generally, any polynomial equation, real or complex, has solutions. Moving ahead now, we can represent the complex numbers in the plane, in the following way:

PROPOSITION 11.3. *The complex numbers, written as usual*

$$x=a+ib$$

*can be represented in the plane, according to the following identification:*

$$x=\begin{pmatrix}a\\ b\end{pmatrix}$$

*With this convention, the sum of complex numbers is the usual sum of vectors.*

PROOF. Consider indeed two arbitrary complex numbers:

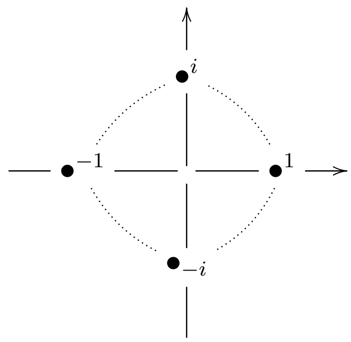
$$x=a+ib\quad,\quad y=c+id$$

Their sum is then by definition the following complex number:

$$x+y=(a+c)+i(b+d)$$

Thus, we are led to the conclusion in the statement. □

As an illustration for this, let us record the following basic picture, with some key complex numbers, namely $1, i, -1, -i$, represented according to our conventions:




We have so far a quite good understanding of their complex numbers, and their addition. In order to understand now the multiplication operation, we must do something more complicated, namely using polar coordinates. Let us start with:

DEFINITION 11.4. *The complex numbers $x = a + ib$ can be written in polar coordinates,*

$$x = r(\cos t + i \sin t)$$

*with the connecting formulae being as follows,*

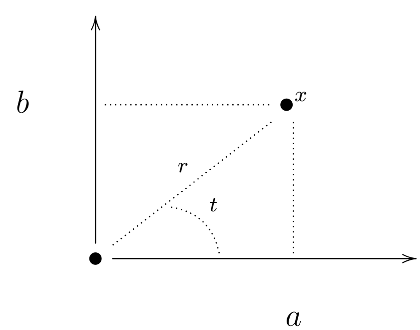
$$a = r \cos t \quad , \quad b = r \sin t$$

*and in the other sense being as follows,*

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

*and with $r, t$ being called modulus, and argument.*

There is a clear relation here with the vector notation from Proposition 11.3, because $r$ is the length of the vector, and $t$ is the angle made by the vector with the $Ox$ axis. To be more precise, the picture for what is going on in Definition 11.4 is as follows:



As a basic example here, the number $i$ takes the following form:

$$i=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$$

The point now is that in polar coordinates, the multiplication formula for the complex numbers, which was so far something quite opaque, takes a very simple form:

THEOREM 11.5. *Two complex numbers written in polar coordinates,*

$$x=r(\cos s+i\sin s)\quad,\quad y=p(\cos t+i\sin t)$$

*multiply according to the following formula:*

$$xy=rp(\cos(s+t)+i\sin(s+t))$$

*In other words, the moduli multiply, and the arguments sum up.*

PROOF. We can assume that we have $r=p=1$, by dividing everything by these numbers. Now with this assumption made, we have the following computation:

$$\begin{aligned}xy &= (\cos s+i\sin s)(\cos t+i\sin t)\\ &= (\cos s\cos t-\sin s\sin t)+i(\cos s\sin t+\sin s\cos t)\\ &= \cos(s+t)+i\sin(s+t)\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result, which is based on some non-trivial trigonometry, is quite powerful. As a basic application of it, we can now compute powers, as follows:

THEOREM 11.6. *The powers of a complex number, written in polar form,*

$$x=r(\cos t+i\sin t)$$

*are given by the following formula, valid for any exponent $k\in\mathbb{N}$:*

$$x^k=r^k(\cos kt+i\sin kt)$$

*Moreover, this formula holds in fact for any $k\in\mathbb{Z}$, and even for any $k\in\mathbb{Q}$.*

PROOF. We have the following computation, with $k$ terms everywhere:

$$\begin{aligned}x^k &= x\ldots x\\ &= r(\cos t+i\sin t)\ldots r(\cos t+i\sin t)\\ &= r^k([\cos(t+\ldots+t)+i\sin(t+\ldots+t))\\ &= r^k(\cos kt+i\sin kt)\end{aligned}$$

Thus, we are done with the case $k\in\mathbb{N}$. Regarding now the generalization to the case $k\in\mathbb{Z}$, it is enough here to do the verification for $k=-1$, where the formula is:

$$x^{-1}=r^{-1}(\cos(-t)+i\sin(-t))$$

But this number $x^{-1}$ is indeed the inverse of $x$, as shown by:

$$\begin{aligned} xx^{-1} &= r(\cos t + i\sin t)\cdot r^{-1}(\cos(-t)+i\sin(-t)) \\ &= \cos(t-t)+i\sin(t-t) \\ &= \cos 0 + i\sin 0 \\ &= 1 \end{aligned}$$

Finally, regarding the generalization to the case $k \in \mathbb{Q}$, it is enough to do the verification for exponents of type $k = 1/n$, with $n \in \mathbb{N}$. The claim here is that:

$$x^{1/n} = r^{1/n}\left[\cos\left(\frac{t}{n}\right) + i\sin\left(\frac{t}{n}\right)\right]$$

In order to prove this, let us compute the $n$-th power of this number. We can use the power formula for the exponent $n \in \mathbb{N}$, that we already established, and we obtain:

$$\begin{aligned} (x^{1/n})^n &= (r^{1/n})^n\left[\cos\left(n\cdot\frac{t}{n}\right) + i\sin\left(n\cdot\frac{t}{n}\right)\right] \\ &= r(\cos t + i\sin t) \\ &= x \end{aligned}$$

Thus, we have indeed a $n$-th root of $x$, and our proof is now complete. □

We should mention that there is a bit of ambiguity in the above, in the case of the exponents $k \in \mathbb{Q}$, due to the fact that the square roots, and the higher roots as well, can take multiple values, in the complex number setting. We will be back to this.

As a basic application of Theorem 11.6, we have the following result:

PROPOSITION 11.7. *Each complex number, written in polar form,*

$$x = r(\cos t + i\sin t)$$

*has two square roots, given by the following formula:*

$$\sqrt{x} = \pm\sqrt{r}\left[\cos\left(\frac{t}{2}\right) + i\sin\left(\frac{t}{2}\right)\right]$$

*When $x > 0$, these roots are $\pm\sqrt{x}$. When $x < 0$, these roots are $\pm i\sqrt{-x}$.*

PROOF. The first assertion is clear indeed from the general formula in Theorem 11.6, at $k = 1/2$. As for its particular cases with $x \in \mathbb{R}$, these are clear from it. □

As a comment here, for $x > 0$ we are very used to call the usual $\sqrt{x}$ square root of $x$. However, for $x < 0$, or more generally for $x \in \mathbb{C} - \mathbb{R}_+$, there is less interest in choosing one of the possible $\sqrt{x}$ and calling it "the" square root of $x$, because all this is based on our convention that $i$ comes up, instead of down, which is something rather arbitrary.

We can go back now to the degree 2 equations, and we have:

THEOREM 11.8. *The complex solutions of $ax^2+bx+c=0$ with $a,b,c\in\mathbb{C}$ are*

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

*with the square root of complex numbers being defined as above.*

PROOF. This is clear, the computations being the same as in the real case. To be more precise, our degree 2 equation can be written as follows:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Thus, we are led to the conclusion in the statement. □

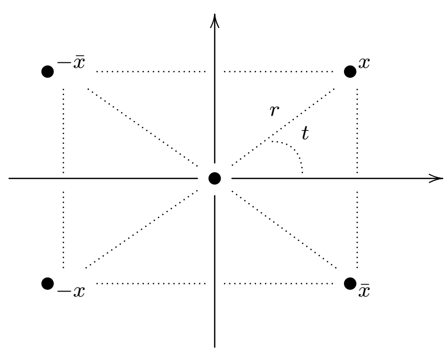
As a last general topic regarding the complex numbers, let us discuss conjugation. This is something quite tricky, complex number specific, as follows:

DEFINITION 11.9. *The complex conjugate of $x=a+ib$ is the following number,*

$$\bar{x}=a-ib$$

*obtained by making a reflection with respect to the $Ox$ axis.*

As before with other such operations on the complex numbers, a quick picture says it all. Here is the picture, with the numbers $x,\bar{x},-x,-\bar{x}$ being all represented:




There are many things that can be said about conjugation, summarized as follows:

THEOREM 11.10. *The conjugation operation $x\to\bar{x}$ has the following properties:*

(1) *$x=\bar{x}$ precisely when $x$ is real.*
(2) *$x=-\bar{x}$ precisely when $x$ is purely imaginary.*
(3) *$x\bar{x}=|x|^2$, with $|x|=r$ being as usual the modulus.*
(4) *With $x=r(\cos t+i\sin t)$, we have $\bar{x}=r(\cos t-i\sin t)$.*
(5) *We have the formula $\overline{xy}=\bar{x}\bar{y}$, for any $x,y\in\mathbb{C}$.*
(6) *The solutions of $ax^2+bx+c=0$ with $a,b,c\in\mathbb{R}$ are conjugate.*

PROOF. These results are all elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something clear too, because with $x = a + ib$ our equation $x = -\bar{x}$ reads $a + ib = -a + ib$, and so $a = 0$, which amounts in saying that $x$ is purely imaginary.

(3) This is a key formula, which can be proved as follows, with $x = a + ib$:

$$\begin{aligned} x\bar{x} &= (a+ib)(a-ib) \\ &= a^2 + b^2 \\ &= |x|^2 \end{aligned}$$

(4) This is clear indeed from the picture following Definition 11.9.

(5) This is something quite magic, which can be proved as follows:

$$\begin{aligned} \overline{(a+ib)(c+id)} &= \overline{(ac-bd)+i(ad+bc)} \\ &= (ac-bd)-i(ad+bc) \\ &= (a-ib)(c-id) \end{aligned}$$

(6) This comes from the formula of the solutions, that we know from Theorem 11.2, but we can deduce this as well directly, without computations. Indeed, by using our assumption that the coefficients are real, $a, b, c \in \mathbb{R}$, we have:

$$\begin{aligned} ax^2 + bx + c = 0 &\implies \overline{ax^2 + bx + c} = 0 \\ &\implies \bar{a}\bar{x}^2 + \bar{b}\bar{x} + \bar{c} = 0 \\ &\implies a\bar{x}^2 + b\bar{x} + c = 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

## 11b. Euler formula

We would like to discuss now the final and most convenient writing of the complex numbers, which is a variation on the polar writing, as follows:

$$x = re^{it}$$

For this purpose, let us start with the following basic result:

THEOREM 11.11. *We can exponentiate the complex numbers, according to the formula*

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

*and the function* $x \to e^x$ *satisfies* $e^{x+y} = e^x e^y$.

PROOF. We must first prove that the series converges. But this follows from:

$$\begin{aligned} |e^x| &= \left|\sum_{k=0}^\infty \frac{x^k}{k!}\right| \\ &\leq \sum_{k=0}^\infty \left|\frac{x^k}{k!}\right| \\ &= \sum_{k=0}^\infty \frac{|x|^k}{k!} \\ &= e^{|x|} < \infty \end{aligned}$$

Regarding the formula $e^{x+y} = e^x e^y$, this follows too as in the real case, as follows:

$$\begin{aligned} e^{x+y} &= \sum_{k=0}^\infty \frac{(x+y)^k}{k!} \\ &= \sum_{k=0}^\infty \sum_{s=0}^k \binom{k}{s} \cdot \frac{x^s y^{k-s}}{k!} \\ &= \sum_{k=0}^\infty \sum_{s=0}^k \frac{x^s y^{k-s}}{s!(k-s)!} \\ &= e^x e^y \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

As a consequence of the above formula $e^{x+y} = e^x e^y$, we have the following result:

PROPOSITION 11.12. *The exponential of complex numbers is given by*

$$e^{s+it} = e^s e^{it}$$

*with $e^s$ being a usual real exponential, and with $e^{it}$, in need to be computed.*

PROOF. This is indeed something self-explanatory, coming from $e^{x+y} = e^x e^y$, and with the somewhat non-standard notation $x = s + it$ being something needed later. □

Now let us get to the remaining problem, computation of $e^{it}$ with $t \in \mathbb{R}$. Here are a few elementary observations, regarding the operation $t \to e^{it}$:

PROPOSITION 11.13. *For $t \in \mathbb{R}$ the number $e^{it}$ belongs to the unit circle,*

$$e^{it} \in \mathbb{T}$$

*and the operation $t \to e^{it}$ is subject to the following formulae,*

$$e^{i(s+t)} = e^{is} e^{it} \quad , \quad e^{i0} = 1 \quad , \quad (e^{it})^{-1} = e^{it}$$

*telling us $t \to e^{it}$ is a group morphism $\mathbb{R} \to \mathbb{T}$.*

PROOF. There are several things going on here, the idea being as follows:

(1) To start with, we have the following formula, valid for any $x \in \mathbb{C}$:

$$e^{\bar{x}} = \sum_{k=0}^\infty \frac{\bar{x}^k}{k!} = \overline{\sum_{k=0}^\infty \frac{x^k}{k!}} = \overline{e^x}$$

We have as well the following computation, again valid for any $x \in \mathbb{C}$:

$$e^x e^{-x} = e^{x-x} = e^0 = 1 \implies (e^x)^{-1} = e^{-x}$$

(2) But with these two formulae in hand, we can prove the first assertion. Indeed, the first formula, applied with $x = it$, with $t \in \mathbb{R}$, gives the following equality:

$$e^{-it} = \overline{e^{it}}$$

As for the second formula above, again applied with $x = it$, this gives:

$$(e^{it})^{-1} = e^{it}$$

We conclude that the complex number $z = e^{it}$ has the following property:

$$z^{-1} = \bar{z}$$

But this is exactly the equation of the unit circle $\mathbb{T}$, as desired.

(3) Regarding now the various formulae in the statement, for the operation $t \to e^{it}$, these are all trivial, coming from the multiplicativity formula $e^{x+y} = e^x e^y$.

(4) As for the final conclusion, this is something quite intuitive, telling us that $t \to e^{it}$ transforms the additive structure of $\mathbb{R}$ into the multiplicative structure of $\mathbb{T}$. □

What is next? Well, we will have to improvise a bit, and we are led in this way to the following fundamental result of Euler, regarding the complex exponential:

THEOREM 11.14. *We have the following formula,*

$$e^{it} = \cos t + i \sin t$$

*valid for any $t \in \mathbb{R}$.*

PROOF. There are several possible proofs of this, the idea being as follows:

(1) Intuitive proof. We know from Proposition 11.13 that $t \to e^{it}$ is a group morphism $\mathbb{R} \to \mathbb{T}$. But in view of this, barring any pathologies, this operation can only appear by "wrapping". That is, we must have a formula as follows, for a certain $\alpha \in \mathbb{R}$:

$$e^{it} = \cos(\alpha t) + i \sin(\alpha t)$$

In order now to find the parameter $\alpha \in \mathbb{R}$, let us look at what happens around $t = 0$. As a first observation, at $t = 0$ precisely, our formula is as follows, true:

$$e^0 = \cos 0 + i \sin 0$$

The point now is that, around $t = 0$, we have the following elementary estimate, simply obtained by truncating the series defining the exponential:

$$e^{it} \simeq 1 + it$$

On the other hand, we know from chapter 7 that we have $\sin t \simeq t$ and $\cos t \simeq 1 - t^2/2$, for $t \simeq 0$. We conclude that we have the following estimate, for $t \simeq 0$:

$$\cos(\alpha t) + i\sin(\alpha t) \simeq 1 + i\alpha t$$

Thus we must have $\alpha = 1$, and we are led to the Euler formula in the statement.

(2) Calculus proof. This is something more solid, obtained by differentiating the following function, using the various available calculus rules, and getting 0:

$$f(t) = e^{-it}(\cos t + i\sin t)$$

Indeed, this shows that our function $f$ must be constant, equal to $f(0) = 1$, as desired. We will discuss this in detail in Part IV, when doing calculus. □

Now back to our $x = re^{it}$ objectives, with the above theory in hand we can indeed use from now on this notation, the complete statement being as follows:

THEOREM 11.15. *The complex numbers $x = a + ib$ can be written in polar coordinates,*

$$x = re^{it}$$

*with the connecting formulae being*

$$a = r\cos t \quad , \quad b = r\sin t$$

*and in the other sense being*

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = \frac{b}{a}$$

*and with $r, t$ being called modulus, and argument.*

PROOF. This is a reformulation of our previous Definition 11.4, by using the formula $e^{it} = \cos t + i\sin t$ from Theorem 11.14, and multiplying everything by $r$. □

With this in hand, we can now go back to the basics, namely the addition and multiplication of the complex numbers. We have the following result:

THEOREM 11.16. *In polar coordinates, the complex numbers multiply as*

$$re^{is} \cdot pe^{it} = rp\, e^{i(s+t)}$$

*with the arguments $s, t$ being taken modulo $2\pi$.*

PROOF. This is something that we already know, from Theorem 11.5, reformulated by using the notations from Theorem 11.15. Observe that this follows as well directly, from the fact that we have $e^{x+y} = e^x e^y$, that we know from Theorem 11.11. □

The above formula is obviously very powerful. However, in polar coordinates we do not have a simple formula for the sum. Thus, this formalism has its limitations.

We can investigate as well more complicated operations, as follows:

THEOREM 11.17. *We have the following operations on the complex numbers, written in polar form, as above:*
(1) *Inversion:* $(re^{it})^{-1} = r^{-1}e^{-it}$.
(2) *Square roots:* $\sqrt{re^{it}} = \pm\sqrt{r}e^{it/2}$.
(3) *Powers:* $(re^{it})^a = r^a e^{ita}$.
(4) *Conjugation:* $\overline{re^{it}} = re^{-it}$.

PROOF. This is something that we already know, from Theorem 11.6, but we can now discuss all this, from a more conceptual viewpoint, the idea being as follows:

(1) We have indeed the following computation, using Theorem 11.16:

$$\begin{aligned}(re^{it})(r^{-1}e^{-it}) &= rr^{-1} \cdot e^{i(t-t)} \\ &= 1 \cdot 1 \\ &= 1\end{aligned}$$

(2) Once again by using Theorem 11.16, we have:

$$(\pm\sqrt{r}e^{it/2})^2 = (\sqrt{r})^2 e^{i(t/2+t/2)} = re^{it}$$

(3) Given an arbitrary number $a \in \mathbb{R}$, we can define, as stated:

$$(re^{it})^a = r^a e^{ita}$$

Due to Theorem 11.16, this operation $x \to x^a$ is indeed the correct one.

(4) This comes from the fact, that we know from Theorem 11.10, that the conjugation operation $x \to \bar{x}$ keeps the modulus, and switches the sign of the argument. □

## 11c. Polynomials, roots

Getting now to the real thing, recall from Theorem 11.8 that any degree 2 equation has 2 complex roots. We can in fact prove that any polynomial equation, of arbitrary degree $N \in \mathbb{N}$, has exactly $N$ complex solutions, counted with multiplicities:

THEOREM 11.18. *Any polynomial* $P \in \mathbb{C}[X]$ *decomposes as*

$$P = c(X - a_1) \ldots (X - a_N)$$

*with* $c \in \mathbb{C}$ *and with* $a_1, \ldots, a_N \in \mathbb{C}$.

Proof. The problem is that of proving that our polynomial has at least one root, because afterwards we can proceed by recurrence. We prove this by contradiction. So, assume that $P$ has no roots, and pick a number $x\in\mathbb{C}$ where $|P|$ attains its minimum:

$$|P(x)|=\min_{y\in\mathbb{C}}|P(y)|>0$$

Since $Q(t)=P(x+t)-P(x)$ is a polynomial which vanishes at $t=0$, this polynomial must be of the form $ct^k$ + higher terms, with $c\neq 0$, and with $k\geq 1$ being an integer. We obtain from this that, with $t\in\mathbb{C}$ small, we have the following estimate:

$$P(x+t)\simeq P(x)+ct^k$$

Now let us write $t=rw$, with $r>0$ small, and with $|w|=1$. Our estimate becomes:

$$P(x+rw)\simeq P(x)+cr^kw^k$$

Now recall that we assumed $P(x)\neq 0$. We can therefore choose $w\in\mathbb{T}$ such that $cw^k$ points in the opposite direction to that of $P(x)$, and we obtain in this way:

$$\begin{aligned}|P(x+rw)| &\simeq |P(x)+cr^kw^k|\\ &= |P(x)|(1-|c|r^k)\end{aligned}$$

Now by choosing $r>0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(x+rw)|<|P(x)|$$

But this contradicts our definition of $x\in\mathbb{C}$, as a point where $|P|$ attains its minimum. Thus $P$ has a root, and by recurrence it has $N$ roots, as stated. □

In practice now, the above proof being by contradiction, and so being not very useful, when it comes to explicitly compute the roots, the following question remains open:

Question 11.19. *Given $P\in\mathbb{C}[X]$, how to compute its roots?*

And good question this is, which will bring us into many interesting things. In degree 2, to start with, we already know how to do this, from Theorem 11.8, but the formula there relies on some trigonometry for extracting the square roots, according to:

$$\sqrt{r(\cos t+i\sin t)}=\pm\sqrt{r}\left[\cos\left(\frac{t}{2}\right)+i\sin\left(\frac{t}{2}\right)\right]$$

Now regarding this trigonometry part, we certainly know how to deal with it, by using our formulae from chapter 6 for the halving of angles, which were as follows:

$$\cos\left(\frac{t}{2}\right)=\sqrt{\frac{1+\cos t}{2}}\quad,\quad\sin\left(\frac{t}{2}\right)=\sqrt{\frac{1-\cos t}{2}}$$

However, this makes it for too many operations, when solving our degree 2 equations, and in practice, it is often better to use the following result, for the square roots:

THEOREM 11.20. *Any complex number $x = a + ib$ has two square roots, given by*

$$\sqrt{x} = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} \pm i\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}$$

*with the signs being identical when $b > 0$, and opposite when $b < 0$.*

PROOF. This is something quite routine, the idea being as follows:

(1) With $x = a + ib$ as in the statement, and $\sqrt{x} = c + id$, our equation is:

$$(c + id)^2 = a + ib$$

In terms of the real and imaginary parts, we have two equations, as follows:

$$c^2 - d^2 = a \quad , \quad 2cd = b$$

(2) Let us first compute the number $u = c^2$. The equation for it is as follows:

$$u - \frac{b^2}{4u} = a$$

Thus, the number $u = c^2$ satisfies the following degree 2 equation:

$$u^2 - au - \frac{b^2}{4} = 0$$

But this latter equation has a unique positive solution, given by:

$$u = \frac{a+\sqrt{a^2+b^2}}{2}$$

Thus, we are led to the formula of $c = \pm\sqrt{u}$ in the statement.

(3) Similarly, let us compute now $v = d^2$. The equation for it is as follows:

$$\frac{b^2}{4v} - v = a$$

Thus, the number $v = d^2$ satisfies the following degree 2 equation:

$$v^2 + av - \frac{b^2}{4} = 0$$

But this latter equation has a unique positive solution, given by:

$$v = \frac{-a+\sqrt{a^2+b^2}}{2}$$

Thus, we are led to the formula of $d = \pm\sqrt{v}$ in the statement, and this gives the result, with the last assertion regarding signs being clear, coming from $2cd = b$. □

With this being said, I don't know about you, but personally, for better sleeping at night, I would rather prefer to have this doublechecked. So, given two numbers $a, b \in \mathbb{R}$, consider the following numbers $c, d \in \mathbb{R}$, with the sign on the right being that of $b$:

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \quad , \quad d = \pm\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}$$

We have then $(c + id)^2 = (c^2 - d^2) + 2icd$, whose real part is given by:

$$\begin{aligned} c^2 - d^2 &= \frac{a + \sqrt{a^2 + b^2}}{2} - \frac{-a + \sqrt{a^2 + b^2}}{2} \\ &= \frac{a}{2} + \frac{a}{2} \\ &= a \end{aligned}$$

As for the imaginary part, this can be computed as follows:

$$\begin{aligned} 2cd &= \pm 2\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2} \cdot \frac{-a + \sqrt{a^2 + b^2}}{2}} \\ &= \pm 2\sqrt{\frac{-a^2 + a^2 + b^2}{4}} \\ &= \pm |b| \\ &= b \end{aligned}$$

Thus we have indeed $(c + id)^2 = a + ib$, as desired. Now by getting back to the degree 2 equations, we can formulate a new result regarding them, as follows:

THEOREM 11.21. *The complex solutions of $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{C}$ are*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

*with the square root of $b^2 - 4ac = p + iq$ being extracted as above, namely*

$$\sqrt{p + iq} = \pm\sqrt{\frac{p + \sqrt{p^2 + q^2}}{2}} \pm i\sqrt{\frac{-p + \sqrt{p^2 + q^2}}{2}}$$

*with the signs being identical when $q > 0$, and opposite when $q < 0$.*

PROOF. This follows indeed from our old degree 2 computation, from the proof of Theorem 11.8, with the square roots being extracted as in Theorem 11.20. □

Getting now to degree 3 equations, let us try to solve $ax^3 + bx^2 + cx + d = 0$. By linear transformations we can assume $a = 1, b = 0$, and then it is convenient to write $c = 3p, d = 2q$. Thus, our equation becomes $x^3 + 3px + 2q = 0$, and regarding such equations, we have the following famous result, that we already met in chapter 6:

THEOREM 11.22 (Cardano). *For a normalized degree 3 equation, namely*

$$x^3+3px+2q=0$$

*the following three complex numbers,*

$$x_{1,2,3}=w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}$$

*with $w=1,e^{2\pi i/3},e^{4\pi i/3}$ are the solutions of our equation.*

PROOF. With $x$ as above, by using $(a+b)^3=a^3+b^3+3ab(a+b)$, we have:

$$\begin{aligned}
x^3 &= \left(w\sqrt[3]{-q+\sqrt{p^3+q^2}}+w^2\sqrt[3]{-q-\sqrt{p^3+q^2}}\right)^3\\
&= -2q+3\sqrt[3]{-q+\sqrt{p^3+q^2}}\cdot\sqrt[3]{-q-\sqrt{p^3+q^2}}\cdot x\\
&= -2q+3\sqrt[3]{q^2-p^3-q^2}\cdot x\\
&= -2q-3px
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We refer to chapter 6 for the continuation of the story, the idea being as follows:

WARNING 11.23. *Do not use Cardano when $p^3+q^2<0$, because this will lead you into extracting third roots of numbers $y\in\mathbb{C}-\mathbb{R}$, which cannot be explicitly done.*

Getting now to degree 4, as before it is possible to write the equations in a more convenient form, namely $x^4+6px^2+4qx+3r=0$, and quite remarkably, we have:

THEOREM 11.24. *The roots of a normalized degree 4 equation, written as*

$$x^4+6px^2+4qx+3r=0$$

*are as follows, with $y$ satisfying the equation $(y^2-3r)(y-3p)=2q^2$,*

$$x_1=\frac{1}{\sqrt{2}}\left(-\sqrt{y-3p}+\sqrt{-y-3p+\frac{4q}{\sqrt{2y-6p}}}\right)$$

$$x_2=\frac{1}{\sqrt{2}}\left(-\sqrt{y-3p}-\sqrt{-y-3p+\frac{4q}{\sqrt{2y-6p}}}\right)$$

$$x_3=\frac{1}{\sqrt{2}}\left(\sqrt{y-3p}+\sqrt{-y-3p-\frac{4q}{\sqrt{2y-6p}}}\right)$$

$$x_4=\frac{1}{\sqrt{2}}\left(\sqrt{y-3p}-\sqrt{-y-3p-\frac{4q}{\sqrt{2y-6p}}}\right)$$

*and with $y$ being computable via the Cardano formula.*

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, let us write our equation in the following form:

$$x^4 = -6px^2 - 4qx - 3r$$

Here comes the trick. Assume that we have found a number $y$ satisfying the following equation, and we will see in a moment why we are doing this:

$$(y^2 - 3r)(y - 3p) = 2q^2$$

The point indeed is that with this magic number $y$ in hand, our degree 4 equation takes a particularly simple form, as follows:

$$\begin{aligned}
(x^2+y)^2 &= x^4 + 2x^2y + y^2 \\
&= -6px^2 - 4qx - 3r + 2x^2y + y^2 \\
&= (2y - 6p)x^2 - 4qx + y^2 - 3r \\
&= (2y - 6p)x^2 - 4qx + \frac{2q^2}{y - 3p} \\
&= \left(\sqrt{2y - 6p}\cdot x - \frac{2q}{\sqrt{2y - 6p}}\right)^2
\end{aligned}$$

(2) Which looks very good, leading us to the following degree 2 equations:

$$x^2 + y + \sqrt{2y - 6p}\cdot x - \frac{2q}{\sqrt{2y - 6p}} = 0$$

$$x^2 + y - \sqrt{2y - 6p}\cdot x + \frac{2q}{\sqrt{2y - 6p}} = 0$$

Now let us write these two degree 2 equations in standard form, as follows:

$$x^2 + \sqrt{2y - 6p}\cdot x + \left(y - \frac{2q}{\sqrt{2y - 6p}}\right) = 0$$

$$x^2 - \sqrt{2y - 6p}\cdot x + \left(y + \frac{2q}{\sqrt{2y - 6p}}\right) = 0$$

(3) Regarding the first equation, the solutions there are as follows:

$$x_1 = \frac{1}{2}\left(-\sqrt{2y - 6p} + \sqrt{-2y - 6p + \frac{8q}{\sqrt{2y - 6p}}}\right)$$

$$x_2 = \frac{1}{2}\left(-\sqrt{2y - 6p} - \sqrt{-2y - 6p + \frac{8q}{\sqrt{2y - 6p}}}\right)$$

As for the second equation, the solutions there are as follows:

$$x_3=\frac{1}{2}\left(\sqrt{2y-6p}+\sqrt{-2y-6p-\frac{8q}{\sqrt{2y-6p}}}\right)$$

$$x_4=\frac{1}{2}\left(\sqrt{2y-6p}-\sqrt{-2y-6p-\frac{8q}{\sqrt{2y-6p}}}\right)$$

Thus, we are led to the formulae in the statement. □

We still have to compute the number $y$ appearing in the above via Cardano, and the result here, adding to what we already have in Theorem 11.24, is as follows:

THEOREM 11.25 (continuation). *The value of $y$ in the previous theorem is*

$$y=t+p+\frac{a}{t}$$

*where the number $t$ is given by the formula*

$$t=\sqrt[3]{b+\sqrt{b^2-a^3}}$$

*with $a=p^2+r$ and $b=2p^2-3pr+q^2$.*

PROOF. The legend has it that this is what comes from Cardano, but depressing and normalizing and solving $(y^2-3r)(y-3p)=2q^2$ makes it for too many operations, so the most pragmatic way is to simply check this equation. With $y$ as above, we have:

$$\begin{aligned}
y^2-3r &= t^2+2pt+(p^2+2a)+\frac{2pa}{t}+\frac{a^2}{t^2}-3r\\
&= t^2+2pt+(3p^2-r)+\frac{2pa}{t}+\frac{a^2}{t^2}
\end{aligned}$$

With this in hand, we have the following computation:

$$\begin{aligned}
(y^2-3r)(y-3p) &= \left(t^2+2pt+(3p^2-r)+\frac{2pa}{t}+\frac{a^2}{t^2}\right)\left(t-2p+\frac{a}{t}\right)\\
&= t^3+(a-4p^2+3p^2-r)t+(2pa-6p^3+2pr+2pa)\\
&\quad +(3p^2a-ra-4p^2a+a^2)\frac{1}{t}+\frac{a^3}{t^3}\\
&= t^3+(a-p^2-r)t+2p(2a-3p^2+r)+a(a-p^2-r)\frac{1}{t}+\frac{a^3}{t^3}\\
&= t^3+2p(-p^2+3r)+\frac{a^3}{t^3}
\end{aligned}$$

Now by using the formula of $t$ in the statement, this gives:

$$\begin{aligned}(y^2-3r)(y-3p) &= b+\sqrt{b^2-a^3}-4p^2+6pr+\frac{a^3}{b+\sqrt{b^2-a^3}} \\ &= b+\sqrt{b^2-a^3}-4p^2+6pr+b-\sqrt{b^2-a^3} \\ &= 2b-4p^2+6pr \\ &= 2(2p^2-3pr+q^2)-4p^2+6pr \\ &= 2q^2\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In degree 5 and more, things become fairly complicated, and we have:

THEOREM 11.26. *There is no general formula for the roots of polynomials of degree $N=5$ and higher, with the reason for this, coming from Galois theory, being that the group $S_5$ is not solvable. The simplest numeric example is $P=X^5-X-1$.*

PROOF. This is something quite tricky, the idea being as follows:

(1) The first assertion, for generic polynomials, is due to Abel-Ruffini, but Galois theory helps in better understanding this, and comes with a number of bonus points too, namely the possibility of formulating a finer result, with Abel-Ruffini's original "generic", which was something algebraic, being now replaced by an analytic "generic", and also with the possibility of dealing with concrete polynomials, such as:

$$P=X^5-X-1$$

(2) Regarding now the details of the Galois proof of the Abel-Ruffini theorem, assume that the roots of a polynomial $P\in F[X]$ can be computed by using iterated roots, a bit as for the degree 2 equation, or for the degree 3 and 4 equations, via Cardano. Then, algebrically speaking, this gives rise to a tower of fields as folows, with $F_0=F$, and each $F_{i+1}$ being obtained from $F_i$ by adding a root, $F_{i+1}=F_i(x_i)$, with $x_i^{n_i}\in F_i$:

$$F_0\subset F_1\subset\ldots\subset F_k$$

(3) In order for Galois theory to apply well to this situation, we must make all the extensions normal, which amounts in replacing each $F_{i+1}=F_i(x_i)$ by its extension $K_i(x_i)$, with $K_i$ extending $F_i$ by adding a $n_i$-th root of unity. Thus, with this replacement, we can assume that the tower in (2) in normal, meaning that all Galois groups are cyclic.

(4) Now by Galois theory, at the level of the corresponding Galois groups we obtain a tower of groups as follows as follows, which is a resolution of the last group $G_k$, the Galois group of $P$, in the sense of group theory, in the sense that all quotients are cyclic:

$$G_1\subset G_2\subset\ldots\subset G_k$$

As a conclusion, Galois theory tells us that if the roots of a polynomial $P\in F[X]$ can be computed by using iterated roots, then its Galois group $G=G_k$ must be solvable.

(5) In the generic case, the conclusion is that Galois theory tells us that, in order for all polynomials of degree 5 to be solvable, via square roots, the group $S_5$, which appears there as Galois group, must be solvable, in the sense of group theory. But this is wrong, because the alternating subgroup $A_5 \subset S_5$ is simple, and therefore not solvable.

(6) Finally, regarding the polynomial $P = X^5 - X - 1$, some elementary computations here, based on arithmetic over $\mathbb{F}_2, \mathbb{F}_3$, and involving various cycles of length $2, 3, 5$, show that its Galois group is $S_5$. Thus, we have our counterexample.

(7) To be more precise, our polynomial factorizes over $\mathbb{F}_2$ as follows:

$$X^5 - X - 1 = (X^2 + X + 1)(X^3 + X^2 + 1)$$

We deduce from this the existence of an element $\tau\sigma \in G \subset S_5$, with $\tau \in S_5$ being a transposition, and with $\sigma \in S_5$ being a 3-cycle, disjoint from it. Thus, we have:

$$\tau = (\tau\sigma)^3 \in G$$

(8) On the other hand since $P = X^5 - X - 1$ is irreducible over $\mathbb{F}_5$, we have as well available a certain 5-cycle $\rho \in G$. Now since $< \tau, \rho >= S_5$, we conclude that the Galois group of $P$ is full, $G = S_5$, and by (4) and (5) we have our counterexample.

(9) Finally, as mentioned in (1), all this shows as well that a random polynomial of degree 5 or higher is not solvable by square roots, and with this being an elementary consequence of the main result from (5), via some standard analysis arguments. □

There is a lot of further interesting theory that can be developed here, following Galois and others. For more on all this, we recommend any solid algebra book.

## 11d. Napoleon, Fermat

We have kept the best for the end. As a last topic regarding the complex numbers, which is something really beautiful, we have the roots of unity. Let us start with:

THEOREM 11.27. *The equation $x^N = 1$ has $N$ complex solutions, namely*

$$\left\{ w^k \Big| k = 0, 1, \ldots, N-1 \right\} \quad , \quad w = e^{2\pi i/N}$$

*which are called roots of unity of order $N$.*

PROOF. This follows from the general multiplication formula for the complex numbers in polar form. Indeed, with the notation $x = re^{it}$, our equation reads:

$$r^N e^{itN} = 1$$

Thus $r = 1$, and $t \in [0, 2\pi)$ must be a multiple of $2\pi/N$, as stated. □

As an illustration here, the roots of unity of small order, along with some of their basic properties, which are very useful for computations, are as follows:

$\underline{N=1}$. Here the unique root of unity is 1.

$\underline{N=2}$. Here we have two roots of unity, namely 1 and $-1$.

$\underline{N=3}$. Here we have 1, then $w=e^{2\pi i/3}$, and then $w^2=\bar{w}=e^{4\pi i/3}$.

$\underline{N=4}$. Here the roots of unity, read as usual counterclockwise, are $1,i,-1,-i$.

$\underline{N=5}$. Here, with $w=e^{2\pi i/5}$, the roots of unity are $1,w,w^2,w^3,w^4$.

$\underline{N=6}$. Here a useful alternative writing is $\{\pm1,\pm w,\pm w^2\}$, with $w=e^{2\pi i/3}$.

$\underline{N=7}$. Here, with $w=e^{2\pi i/7}$, the roots of unity are $1,w,w^2,w^3,w^4,w^5,w^6$.

$\underline{N=8}$. Here the roots of unity, read as usual counterclockwise, are the numbers $1,w,i,iw,-1,-w,-i,-iw$, with $w=e^{\pi i/4}$, which is also given by $w=(1+i)/\sqrt{2}$.

The roots of unity are very useful variables, and have many interesting properties. As a first application, we can now solve the ambiguity questions related to the extraction of $N$-th roots, that we met in the above, the statement here being as follows:

THEOREM 11.28. *Any nonzero $x=re^{it}$ has exactly $N$ roots of order $N$, namely*

$$y=r^{1/N}e^{it/N}$$

*multiplied by the $N$ roots of unity of order $N$.*

PROOF. We must solve the equation $z^N=x$, over the complex numbers. Since the number $y$ in the statement clearly satisfies $y^N=x$, our equation is equivalent to:

$$z^N=y^N$$

Now observe that we can write this equation in the following way:

$$\left(\frac{z}{y}\right)^N=1$$

We conclude from this that the solutions $z$ of our equation appear by multiplying $y$ by the solutions of $t^N=1$, which are the $N$-th roots of unity, as claimed. □

The roots of unity appear in connection with many other interesting questions, and there are many useful formulae relating them, which are good to know. Here is a basic such formula, very beautiful, which has many applications, all across mathematics:

THEOREM 11.29. *The roots of unity, $\{w^k\}$ with $w=e^{2\pi i/N}$, have the property*

$$\sum_{k=0}^{N-1}(w^k)^s=N\delta_{N|s}$$

*for any exponent $s\in\mathbb{N}$, where on the right we have a Kronecker symbol.*

PROOF. The numbers in the statement, when written more conveniently as $(w^s)^k$ with $k = 0, \ldots, N-1$, form a certain regular polygon in the plane $P_s$. Thus, if we denote by $C_s$ the barycenter of this polygon, we have the following formula:

$$\frac{1}{N}\sum_{k=0}^{N-1} w^{ks} = C_s$$

Now observe that in the case $N \not| s$ our polygon $P_s$ is non-degenerate, circling around the unit circle, and having center $C_s = 0$. As for the case $N|s$, here the polygon is degenerate, lying at 1, and having center $C_s = 1$. Thus, we have the following formula:

$$C_s = \delta_{N|s}$$

Thus, we obtain the formula in the statement. □

And with this, end of our theoretical discussion regarding the complex numbers, we have certainly amassed here more nuclear-grade weapons than all the Cold War main powers combined. And the question is, getting now to geometry, can all this help?

In answer, depends on the question, with the philosophy being as follows:

PRINCIPLE 11.30. *The complex numbers tell a certain story of the real plane, and with this story being usually the accurate one, when it comes to modern mathematics or physics. In what regards classical plane geometry and triangles, however, which are old disciplines, some things can be done with complex numbers, while some other, not.*

This principle is something quite deep, and to be more precise, the dichotomy comes somehow from classical vs quantum mathematics and physics, as follows:

(1) In what regards classical mathematics and physics, that is basically built around things like gravity and conics, which naturally live over $\mathbb{R}$.

(2) As for the more modern, quantum mathematics and physics, that is built around electromagnetism and quantum mechanics, which live over $\mathbb{C}$.

But all this is perhaps too abstract. In order to sense whether your plane geometry or triangle problem is worth an approach via complex numbers, here is a method:

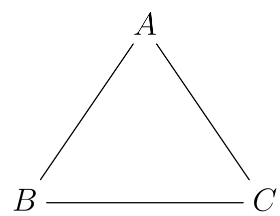
METHOD 11.31. *If your plane geometry or triangle problem features*

1. *Equilateral triangles, which can be dealt with using $w^3 = 1$,*
2. *Or squares, which can be dealt with using the numbers $1, i, -1, -i$,*
3. *Or other regular polygons, which can be dealt with using $w^N = 1$,*
4. *Or circles, which simply read $|x - c| = r$ in complex notation,*

*then the complex numbers might be the way, for solving your problem.*

So, this was the general idea, hope this was understandable, and in what follows we will discuss, as an illustration for (1), certain geometry questions featuring equilateral triangles, following Napoleon, Fermat, Torricelli and others. Let us start with:

Theorem 11.32. *A triangle $ABC$, with $A, B, C$ appearing counterclockwise*

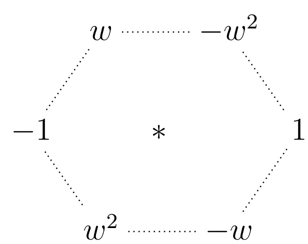



*is equilateral precisely when its vertices, regarded as complex numbers, satisfy*

$$A + wB + w^2C = 0$$

*with $w = e^{2\pi i/3}$. When $A, B, C$ appear clockwise, the same happens, with $w \to w^2$.*

Proof. The roots of unity of order 3, and their opposites, are as follows:



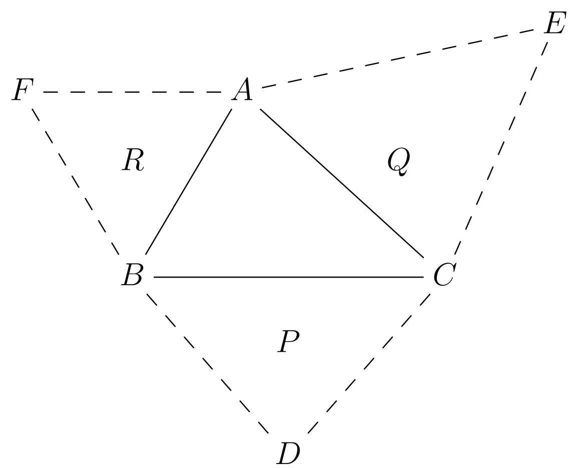

Thus the clockwise rotation by 60° is $P \to -wP$, and by using this, along with $1 + w + w^2$, coming from $w^3 = 1$, the condition for $ABC$ to be equilateral reads:

$$\begin{aligned} A - C = -w(B - C) &\iff A + wB - (1 + w)C = 0 \\ &\iff A + wB + w^2C = 0 \end{aligned}$$

As for the last assertion, this follows from this, by interchanging $B \leftrightarrow C$. □

Getting now to arbitrary triangles $ABC$, we saw earlier in this book that these passioned many people, including al-Mutaman, king of Zaragoza, and discoverer or the Ceva theorem. More recently Napoleon, emperor of the French, spent some time in studying the configuration involving equilateral triangles erected on the sides of $ABC$. His main findings, along with those of Fermat and Torricelli, can be summarized as follows:

THEOREM 11.33. *In the context of the Napoleon configuration, namely*




*with equilateral triangles, and their barycenters drawn, the following happen,*

(1) *Napoleon theorem: the triangle $PQR$ is equilateral.*
(2) *Torricelli circles: the circles $ABF$, $BCD$, $ACE$ are concurrent.*
(3) *Torricelli point: $AD, BE, CF$ cross, on this circle concurrence point.*
(4) *More Torricelli: these lines $AD, BE, CF$ cross at $60° - 120°$ angles.*
(5) *Fermat point: the Torricelli point minimizes $AX + BX + CX$.*
(6) *Napoleon point: $AP, BQ, CR$ cross too, at the Napoleon point.*

*with the assumption that all angles of $ABC$ are $\leq 120°$ being needed for* (5).

PROOF. Many things going on here, the idea being as follows:

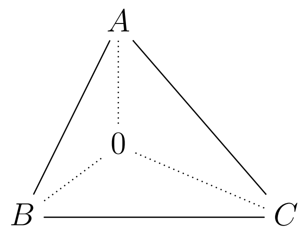
(1) The Napoleon theorem follows, majestically, using Theorem 11.32, as follows:

$$\begin{aligned} P + wQ + w^2R &= \frac{B+C+D}{3} + w \cdot \frac{A+C+E}{3} + w^2 \cdot \frac{A+B+F}{3} \\ &= \frac{B+wA+w^2F}{3} + \frac{C+wE+w^2A}{3} + \frac{D+wC+w^2B}{3} \\ &= 0+0+0 \\ &= 0 \end{aligned}$$

(2,3,4) These assertions, which are all related, are all elementary, and follow from some angle hunting, without any major difficulty. We will leave them as exercises, for you.

(5) Let us define the Fermat point of a triangle $ABC$ as being the point which minimizes $AX + BX + CX$, with the existence being clear, but with the uniqueness, not. Our claim is that when the triangle $ABC$ has all angles $\leq 120°$, this Fermat point is the

Torricelli point from (2,3,4), appearing as follows, with all angles around it being $120°$:




In order to prove this, we use vector calculus. By fixing the origin 0 at the Torricelli point, as indicated above, we have the following estimate, for any point $X$ in the plane, with $i, j, k$ denoting the unit vectors along $A, B, C$, which satisfy $i + j + k = 0$:

$$\begin{aligned} ||A|| + ||B|| + ||C|| &= < A, i > + < B, j > + < C, k > \\ &= < A - X, i > + < B - X, j > + < C - X, k > \\ &\leq ||A - X|| + ||B - X|| + ||C - X|| \end{aligned}$$

Thus, claim proved. As for the case where one of the angles of $ABC$ is $\geq 120°$, here the Fermat point must be that vertex, and we will leave this as an exercise.

(6) Well, I must admit that I tried to prove this with my favorite plane geometry method, complex numbers, and failed, the computations being quite complicated. Moral of the story, not everyone is Napoleon, and I will leave this to you, as an exercise.

(7) Finally, let us mention that most of the above results hold as well when drawing the equilateral triangles in an inward way, notably leading to the second Napoleon point of $ABC$. And exercise of course for you, to learn more about all this. □

## 11e. Exercises

This was a quite exciting chapter, certainly mathematical, but of philosophical and physics flavor too, and as exercises about this, we have:

Exercise 11.34. *What becomes $\mathbb{C}$ when the complex plane is drawn upside-down?*

Exercise 11.35. *What about drawing the reals from right to left? Or doing both?*

Exercise 11.36. *Further meditate on the need for $\pm$ when extracting square roots.*

Exercise 11.37. *Work out all the details for the existence of roots of polynomials.*

Exercise 11.38. *Learn a bit about the resultant of two polynomials.*

Exercise 11.39. *Learn also about the discriminant, and its various properties.*

Exercise 11.40. *Practice a bit with Cardano in degree 3, and in degree 4 too.*

Exercise 11.41. *Learn some Galois theory, including various algebraic preliminaries.*

As bonus exercise, which is long, but very instructive, have a look at everything that we did since the beginning of this book, by using complex number technology.

CHAPTER 12

# Curves, surfaces

## 12a. Plane curves

Time to end the present Part III of this book, on geometry and trigonometry, and in a beautiful way, and this because the remaining Part IV will be dedicated to calculus methods and their applications to technical trigonometry, not exactly sweet things.

In our plan, we would like talk more about plane curves, as a continuation of our discussion about conics and other basic curves from chapter 10, and then explore curves and surfaces, and geometry in general, in 3D space, and in higher dimensions too.

Getting started, let us recall from chapter 10 the following key definition:

Definition 12.1. *An algebraic curve in $\mathbb{R}^2$ is the vanishing set*

$$C=\left\{(x,y)\in\mathbb{R}^2\Big|P(x,y)=0\right\}$$

*of a polynomial $P\in\mathbb{R}[X,Y]$ of arbitrary degree.*

As explained in chapter 10, this definition is something very general. A bit of basic theory can be developed at this level, the conclusions being as follows:

Theorem 12.2. *The following happen, for curves $C$ defined by polynomials $P$:*

1. *In degree $d=2$, curves can have singularities, such as $xy=0$ at $(0,0)$.*
2. *In general, assuming $P=P_1\ldots P_k$, we have $C=C_1\cup\ldots\ldots\cup C_k$.*
3. *A union of curves $C_i\cup C_j$ is generically non-smooth, unless disjoint.*
4. *Due to this, we say that $C$ is non-degenerate when $P$ is irreducible.*

Proof. We know all this from chapter 10, the idea being as follows:

(1) This is something obvious, just the story of two lines crossing.

(2) This comes from the following trivial fact, with the notation $z=(x,y)$:

$$P_1\ldots P_k(z)=0\iff P_1(z)=0,\ \text{or } P_2(z)=0,\ \ldots\ ,\ \text{or } P_k(z)=0$$

(3) This is something very intuitive, and it actually takes a bit of time to imagine a situation where $C_1\cap C_2\neq\emptyset$, $C_1\not\subset C_2$, $C_2\not\subset C_1$, but $C_1\cup C_2$ is smooth.

(4) This is just a definition, based on the above, that we will use in what follows. □

Still following the material from chapter 10, at the level of examples, we have:

THEOREM 12.3. *The low degree plane curves are as follows:*

(1) *Degree* 1*: the points, needing no presentation.*
(2) *Degree* 2*: the conics, namely ellipses and hyperbolas, then circles and parabolas, and the two lines too, needing no presentation either.*
(3) *Degree* 3*: the cubics, notably featuring the standard cusp* $x^3 = y^2$*, and the Tschirnhausen curve* $x^3 = x^2 - 3y^2$*.*
(4) *Degree* 4*: The quartics, notably featuring the cardioid* $2r = a(1 - \cos t)$*, and the Bernoulli lemniscate* $r^2 = a^2 \cos 2t$*.*

PROOF. Again, this is something that we know well from chapter 10, and we refer to the material there for a full discussion of this, including the special curves in (3,4). □

What is next? Higher degree of course, and in the lack of anything nice in degree 5, quintics, let us discuss now the degree 6, sextics. We first have here:

PROPOSITION 12.4. *The trefoil sextic, or Kiepert curve, which is given by*

$$r^3 = a^3 \cos 3t$$

*looks like a trefoil, closed curve, with a triple self-intersection.*

PROOF. As before with other such curves, drawing a picture, which reveals the trefoil in question, is mandatory. Next, with $z = x + iy = re^{it}$, we have:

$$\begin{aligned}
r^3 = a^3 \cos 3t \quad &\iff \quad r^3 \cos 3t = \left(\frac{r^2}{a}\right)^3 \\
&\iff \quad z^3 + \bar{z}^3 = 2\left(\frac{z\bar{z}}{a}\right)^3 \\
&\iff \quad (x+iy)^3 + (x-iy)^3 = 2\left(\frac{x^2+y^2}{a}\right)^3 \\
&\iff \quad x^3 - 3xy^2 = \left(\frac{x^2+y^2}{a}\right)^3 \\
&\iff \quad (x^2+y^2)^3 = a^3(x^3 - 3xy^2)
\end{aligned}$$

We conclude that we have indeed a sextic, as claimed. □

We also have in degree 6 the most beautiful of curves them all, the Cayley sextic:

THEOREM 12.5. *The Cayley sextic, given in polar coordinates by*

$$r = a \cos^3\left(\frac{t}{3}\right)$$

*makes the dream of previous curves come true, by looking like a self-intersecting heart.*

PROOF. As before, picture mandatory. With $z = re^{it}$ and $u = z^{1/3}$ we have:

$$\begin{aligned}
r = a\cos^3\left(\frac{t}{3}\right) &\iff ar\cos^3\left(\frac{t}{3}\right) = r^2 \\
&\iff a\left(\frac{u+\bar{u}}{2}\right)^3 = r^2 \\
&\iff a(u^3 + \bar{u}^3 + 3u\bar{u}(u+\bar{u})) = 8r^2 \\
&\iff 3au\bar{u}\cdot\frac{u+\bar{u}}{2} = 4r^2 - ax \\
&\iff 27a^3r^6\cdot\frac{r^2}{a} = (4r^2 - ax)^3 \\
&\iff 27a^2(x^2+y^2)^2 = (4x^2 + 4y^2 - ax)^3
\end{aligned}$$

We conclude that have indeed a sextic, as claimed. □

And we will stop here with examples, what else can we wish more for, than the Cayley sextic. Quite remarkably now, most of the above curves are sinusoidal spirals, in the following sense, and with actually the term "sinusoidal spiral" being a bit unfortunate:

THEOREM 12.6. *The sinusoidal spirals, which are as follows,*

$$r^n = a^n \cos nt$$

*with $a \neq 0$ and $n \in \mathbb{Q} - \{0\}$, include the following curves:*

1. *$n = -1$ line.*
2. *$n = 1$ circle, $n = -1/2$ parabola, $n = -2$ hyperbola.*
3. *$n = -3$ Humbert cubic, $n = -1/3$ Tschirnhausen curve.*
4. *$n = 1/2$ cardioid, $n = 2$ Bernoulli lemniscate.*
5. *$n = 3$ Kiepert trefoil, $n = 1/3$ Cayley sextic.*

PROOF. We first have to prove that the sinusoidal spirals are indeed algebraic curves. But this is best done by using the complex coordinate $z = re^{it}$, as follows:

$$\begin{aligned}
r^n = a^n\cos nt &\iff r^n\cos nt = \left(\frac{r^2}{a}\right)^n \\
&\iff z^n + \bar{z}^n = 2\left(\frac{z\bar{z}}{a}\right)^n \\
&\iff (x+iy)^n + (x-iy)^n = 2\left(\frac{x^2+y^2}{a}\right)^n
\end{aligned}$$

As a first observation now, in the case $n\in\mathbb N$ we can simply use the binomial formula, and we get an algebraic equation of degree $2n$, as follows:

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k}y^{2k}=\left(\frac{x^2+y^2}{a}\right)^n$$

In general, things are a bit more complicated, as shown for instance by our computation for the Cayley sextic. However, the same idea as there applies, and we are led in this way to the equation of an algebraic curve, as claimed. Regarding now the examples:

(1) At $n=-1$ the equation is as follows, producing a line:

$$r\cos t=a\iff x=a$$

(2) At $n=1$ the equation is as follows, producing a circle:

$$r=a\cos t\iff r^2=ax\iff x^2+y^2=ax$$

(3) At $n=-1/2$ the equation is as follows, producing a parabola:

$$a=r\cos^2(t/2)\iff r+x=2a\iff y^2=4a(a-x)$$

(4) At $n=-2$ the equation is as follows, producing a hyperbola:

$$a^2=r\cos^2 2t\iff a^2=2x^2-r^2\iff (x+y)(x-y)=a^2$$

(5) At $n=-3$ the equation is as follows, producing a curve with 3 components, which looks like some sort of "trivalent hyperbola", called Humbert cubic:

$$r^3\cos 3t=a^3\iff z^3+\bar z^3=2a^3\iff x^3-3xy^2=a^3$$

(6) As for the other curves, this follows from our various formulae above. □

Let us study now more in detail the sinusoidal spirals. We first have:

PROPOSITION 12.7. *The sinusoidal spirals, which with $z=x+iy$ are*

$$z^n+\bar z^n=2\left(\frac{z\bar z}{a}\right)^n$$

*with $a\neq 0$ and $n\in\mathbb Q-\{0\}$, are as follows:*

1. *With $n=-m$, $m\in\mathbb N$, the equation is $z^m+\bar z^m=2a^m$, degree $m$.*
2. *With $n=m$, $m\in\mathbb N$, the equation is $z^m+\bar z^m=2(z\bar z/a)^m$, degree $2m$.*
3. *With $n=-1/m$, $m\in\mathbb N$, the equation is $(z^{1/m}+\bar z^{1/m})^m=2^ma$.*
4. *With $n=1/m$, $m\in\mathbb N$, the equation is $(z^{1/m}+\bar z^{1/m})^m=2^mz\bar z/a$.*

PROOF. This is something self-explanatory, the details being as follows:

(1) With $n=-m$ and $m\in\mathbb N$ as in the statement, the equation is, as claimed:

$$z^{-m}+\bar z^{-m}=2\left(\frac{z\bar z}{a}\right)^{-m}\iff z^m+\bar z^m=2a^m$$

(2) This is an empty statement, just a matter of using the new variable $m=n$.

(3) With $n=-1/m$ and $m\in\mathbb N$ as in the statement, the equation is, as claimed:

$$z^{-1/m}+\bar{z}^{-1/m}=2\left(\frac{z\bar{z}}{a}\right)^{-1/m}\iff z^{1/m}+\bar{z}^{1/m}=2a^{1/m}$$
$$\iff (z^{1/m}+\bar{z}^{1/m})^m=2^m a$$

(4) With $n=1/m$ and $m\in\mathbb N$ as in the statement, the equation is, as claimed:

$$z^{1/m}+\bar{z}^{1/m}=2\left(\frac{z\bar{z}}{a}\right)^{1/m}\iff (z^{1/m}+\bar{z}^{1/m})^m=2^m\cdot\frac{z\bar{z}}{a}$$

Thus, we are led to the conclusions in the statement. □

Observe that in the fractionary cases, $n=\pm1/m$, the equations in the above statement are not polynomial in $x,y$, unless at very small values of $m$. To be more precise:

(1) In the case $n=-1/m$, we certainly have at $m=1,2,3$ the $d=1$ line, $d=2$ parabola, and $d=3$ Tschirnhausen curve, but at $m=4$ things change, with the equation $(z^{1/4}+\bar{z}^{1/4})^4=16a$ being no longer polynomial in $x,y$, and requiring a further square operation to make it polynomial, and therefore leading to a curve of degree $d=8$.

(2) As for the case $n=1/m$, this is more complicated, with the data that we have at $m=1,2,3$, namely the $d=2$ circle, $d=3$ cardioid, and $d=6$ Cayley sextic, being not very good, and with things getting even more complicated at $m=4$ and higher.

In short, things quite complicated, and the general case, $n=\pm p/q$ with $p,q\in\mathbb N$, is certainly even more complicated. Instead of insisting on this, let us focus now on the simplest sinusoidal spirals that we have, namely those with $n=\pm m$, with $m\in\mathbb N$.

The point indeed is that the sinusoidal spirals with $n\in\mathbb N$ are also part of another remarkable family of plane algebraic curves, going back to Cassini, as follows:

Theorem 12.8. *The polynomial lemniscates, which are as follows,*

$$|P(z)|=b^n$$

*with $P\in\mathbb C[X]$ having $n$ distinct roots, and $b>0$, include the following curves:*

(1) *The sinusoidal spirals with $n\in\mathbb N$, including the $n=1$ circle, $n=2$ Bernoulli lemniscate, and $n=3$ Kiepert trefoil.*
(2) *The Cassini ovals, which are the quartics given by $|z+c|\cdot|z-c|=b^2$, covering too the Bernoulli lemniscate, appearing at $b=c$.*

Proof. This is something quite self-explanatory, the details being as follows:

(1) Regarding the sinusoidal spirals with $n\in\mathbb N$, their equation is, with $a^n=2c^n$:

$$\begin{aligned}
z^n+\bar{z}^n=2\left(\frac{z\bar{z}}{a}\right)^n
&\iff c^n(z^n+\bar{z}^n)=(z\bar{z})^n\\
&\iff (z^n-c^n)(\bar{z}^n-c^n)=c^{2n}\\
&\iff |z^n-c^n|=c^n
\end{aligned}$$

(2) Regarding the Cassini ovals, these correspond to the case where the polynomial $P\in\mathbb C[X]$ has degree 2, and we already know from the above that these cover the Bernoulli lemniscate. In general, the equation for the Cassini ovals is:

$$\begin{aligned}
|z+c|\cdot|z-c|=b^2
&\iff |z^2-c^2|=b^2\\
&\iff (z^2-c^2)(\bar{z}^2-c^2)=b^4\\
&\iff (z\bar{z})^2-c^2(z^2+\bar{z}^2)+c^4=b^4\\
&\iff (x^2+y^2)^2-c^2(x^2-y^2)+c^4=b^4\\
&\iff (x^2+y^2)^2=c^2(x^2-y^2)+b^4-c^4
\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The polynomial lemniscates can be geometrically understood as follows:

THEOREM 12.9. *The equation $|P(z)|=b$ defining the polynomial lemniscates can be written as follows, in terms of the roots $c_1,\ldots,c_n$ of the polynomial $P$,*

$$\sqrt[n]{\prod_{k=1}^n|z-c_i|}=b$$

*telling us that the geometric mean of the distances from $z$ to the vertices of the polygon formed by $c_1,\ldots,c_n$ must be the constant $b>0$.*

PROOF. This is something self-explanatory, and as an illustration, let us work out the case of sinusoidal spirals with $n\in\mathbb N$. Here with $w=e^{2\pi i/n}$ we have:

$$z^n-c^n=\prod_{k=1}^n(z-cw^k)$$

Thus, the sinusoidal spiral equation reformulates as follows:

$$|z^n-c^n|=c^n\iff\prod_{k=1}^n|z-cw^k|=c^n\iff\sqrt[n]{\prod_{k=1}^n|z-cw^k|}=c$$

Thus, for a sinusoidal spiral with positive integer parameter, the geometric mean of the distances to the vertices of a regular polygon must equal the radius of the polygon. □

Regarding now the sinusoidal spirals with $n\in-\mathbb N$, these are too part of another remarkable family of plane algebraic curves, constructed as follows:

THEOREM 12.10. *Given points in the plane $c_1,\ldots,c_n\in\mathbb C$ and a number $d\in\mathbb R$, construct the associated stelloid as being the set of points $z\in\mathbb C$ verifying*

$$\frac{1}{n}\sum_{k=1}^n\alpha_v(z-c_i)=d$$

*with $\alpha_v$ denoting the angle with respect to a direction $v$. Then the stelloid is an algebraic curve, not depending on $v$, and at the level of examples we have the sinusoidal spirals with $n\in-\mathbb N$, including the $n=-1$ line, $n=-2$ hyperbola, and $n=-3$ Humbert cubic.*

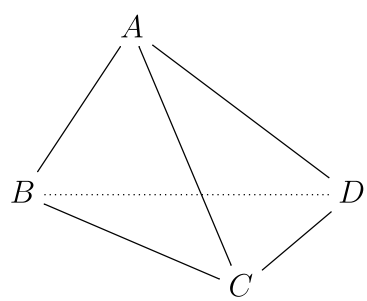
PROOF. All this is quite self-explanatory, and we will leave the verification of the various generalities regarding the stelloids, as well as the verification of the relation with the sinusoidal spirals with $n\in-\mathbb N$, as an instructive exercise. □

So long for plane algebraic curves. Needless to say, all the above is old-style, first class mathematics, having countless applications. For instance when doing classical mechanics or electrodynamics, you will certainly meet polynomial lemniscates and stelloids, when looking at the field lines. Also, the image of any circle passing though 0 by $z\to z^2$ is a cardioid, and the famous Mandelbrot set is organized around such a cardioid.

## 12b. Space geometry

Getting now the usual 3 dimensions that we live in, to start with, many interesting things can be said, in analogy with what we know about triangles. We first have:

THEOREM 12.11. *Any tetrahedron in three-dimensional space*




*has a barycenter, lying $1/4-3/4$ on the medians, uniting vertices to opposite barycenters.*

PROOF. The barycenter of our tetrahedron can only be given by:

$$P=\frac{A+B+C+D}{4}$$

Now observe that this formula can be written in the following way:

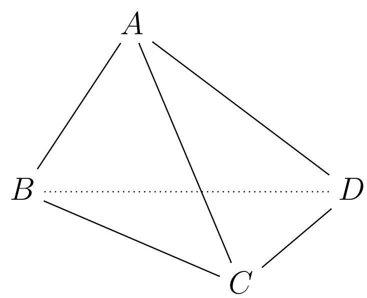
$$P=\frac{1}{4}\cdot A+\frac{3}{4}\cdot\frac{B+C+D}{3}$$

Thus, we are led to the conclusion in the statement. □

As in the case of the triangles, there is some further discussion here in relation with physical barycenters, when considering that the vertices, or edges, or faces, or the whole solid body itself, have mass. We will leave the study here as an interesting exercise.

Moving on, as a second basic result, again as for the triangles, we have:

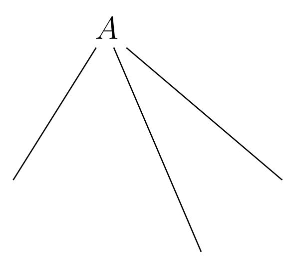
THEOREM 12.12. *Any tetrahedron in three-dimensional space*




*has an incenter, where the solid angle bisectors cross.*

PROOF. Again, this is something quite self-explanatory, and as in the case of the triangles, there are several ways of precisely stating and proving this, as follows:

(1) As a first approach, which is straightforward, we can base our study on the notion of solid angle bisector, as stated. Consider indeed a solid angle, as follows:



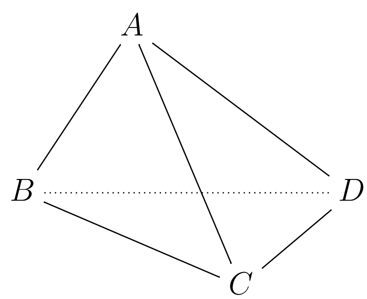

This solid angle has then a bisector, and with this best seen by fitting a sphere into our angle. Indeed, if $O$ is the center of the sphere, $AO$ is the angle bisector.

(2) Now the point is that the 4 angle bisectors cross indeed, and this can be seen for instance by interpreting each angle bisector as being an intersection of 3 planes, in the obvious way. Indeed, the total of 12 planes that we have must intersect.

(3) But the simplest is to argue that the incenter appears by fitting, or rather by inflating, a sphere inside our tetrahedron. Indeed, once our sphere is duly inflated, as to touch the faces, its center will be the incenter of our tetrahedron. □

Along the same lines, we have as well the following result:

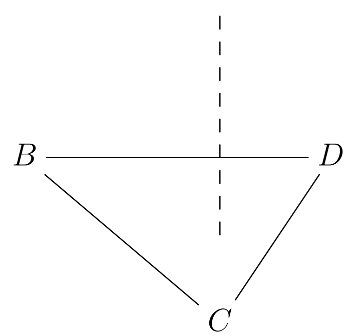
THEOREM 12.13. *Any tetrahedron in three-dimensional space*




*has a circumcenter, where the perpendicular bisectors cross.*

PROOF. Again, this is something quite self-explanatory, and as in the case of the triangles, there are several ways of precisely stating and proving this, as follows:

(1) As a first approach, which is straightforward, we can base our study on the notion of perpendicular bisector, as stated. Consider indeed a triangle in space:



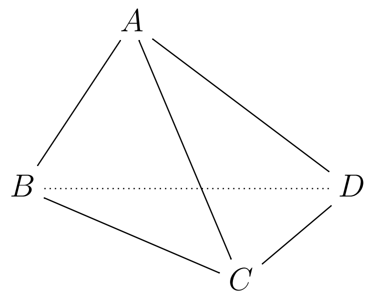

This triangle has then a perpendicular bisector, emanating from the circumcenter, as shown above, and with this best seen by fitting our triangle into a sphere. Indeed, if $O$ is the center of the sphere, $O$ projects on the triangle via the perpendicular bisector.

(2) Now the point is that the 4 perpendicular bisectors cross indeed, and this can be seen for instance by interpreting each perpendicular bisector as being an intersection of 3 planes, in the obvious way. Indeed, the total of 12 planes that we have must intersect.

(3) But the simplest is to argue that the circumcenter appears by fitting, or rather by deflating, a sphere outside our tetrahedron. Indeed, once our sphere is duly deflated, as to touch the vertices, its center will be the circumcenter of our tetrahedron. □

Regarding now the orthocenter, things here are quite complicated, as follows:

THEOREM 12.14. *Under suitable assumptions, the tetrahedra in 3D space*

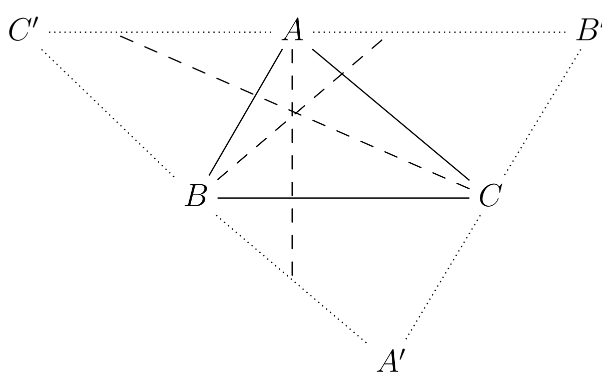



*have an orthocenter, where the altitudes cross.*

PROOF. This is something quite subtle, the idea being as follows:

(1) To start with, the altitudes of a tetrahedron do not cross, in general. We will leave some thinking here, and the construction of counterexamples, as an exercise.

(2) Along the same lines, but a bit more philosophically, let us look at the 2-dimensional proof, of the existence of the orthocenter. The trick and picture were as follows:




But such things won't work in three dimensions, somehow for obvious reasons, and again, we will leave some thinking here as an instructive exercise.

(3) Getting now to what can be done, as to have some theory and results going on, following Monge and others, the idea is that we can talk about orthocentric tetrahedra. Consider indeed a tetrahedron whose opposite edges are orthogonal:

$$AB \perp CD \quad , \quad AC \perp BD \quad , \quad AD \perp BC$$

In this situation the altitudes cross, and their intersection, the orthocenter, coincides with the Monge point, appearing as the intersection of the 6 midplanes, which pass through the middle of each of the 6 edges, and are orthogonal to the opposite edge. □

## 12c. Linear algebra

Getting now to vector calculus and linear algebra, we are already a bit experts in that, in 2 dimensions. So, we will be quite brief, the extension of most of the results being quite straightforward. Also, for various reasons that will become clear in a moment, it is convenient to discuss directly the $N$-dimensional case. As a starting point, we have:

DEFINITION 12.15. *The points $x \in \mathbb{R}^N$ can be represented as vectors*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

*and are subject to the addition and multiplication by scalars operations*

$$x + y = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_N + y_N \end{pmatrix} \quad , \quad \lambda x = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_N \end{pmatrix}$$

*geometrically corresponding to forming a parallelogram, and dilating by $\lambda$.*

Along the same lines, at a more advanced level, we can talk about scalar products and lengths of vectors, with the basic theory here being summarized as follows:

THEOREM 12.16. *We can talk about scalar products and lengths, according to*

$$< x, y >= \sum_i x_i y_i \quad , \quad ||x|| = \sqrt{\sum_i x_i^2}$$

*which are related by the following conversion formulae,*

$$||x|| = \sqrt{< x, x >} \quad , \quad < x, y >= \frac{||x + y||^2 - ||x - y||^2}{4}$$

*and the following happen:*

(1) $< \lambda x, y >=< x, \lambda y >= \lambda < x, y >$.
(2) $< x + y, z >=< x, z > + < y, z >$.
(3) $< x, y + z >=< x, z > + < y, z >$.
(4) $||\lambda x|| = |\lambda| \cdot ||x||$.
(5) $| < x, y > | \leq ||x|| \cdot ||y||$.
(6) $||x + y|| \leq ||x|| + ||y||$.
(7) $x \perp y \iff < x, y >= 0$, *by definition.*
(8) $< x, y >= ||x|| \cdot ||y|| \cdot \cos t$, *with $t$ being the angle between $x, y$, by definition.*
(9) $< x, y >=< x', y >=< x, y' >$, *prime being the projection on the other vector.*

PROOF. We can certainly talk about scalar products and lengths, as above, and with the second conversion formula, called polarization identity, coming from:

$$\begin{aligned}||x+y||^2-||x-y||^2 &= <x+y,x+y>-<x-y,x-y>\\ &= ||x||^2+||y||^2+2<x,y>-||x||^2-||y||^2+2<x,y>\\ &= 4<x,y>\end{aligned}$$

By the way, talking useful identities, we have as well a parallelogram rule, as in 2D, that I forgot to mention in the above statement, which is as follows:

$$\begin{aligned}||x+y||^2+||x-y||^2 &= <x+y,x+y>+<x-y,x-y>\\ &= ||x||^2+||y||^2+2<x,y>+||x||^2+||y||^2-2<x,y>\\ &= 2(||x||^2+||y||^2)\end{aligned}$$

As for the various claims in the statement, these basically follow as in 2D:

(1-4) All the verifications here are trivial, as before in 2D.

(5-6) Given two vectors $x,y\in\mathbb{R}^N$, consider the following function $f:\mathbb{R}\to\mathbb{R}$:

$$\begin{aligned}f(t) &= ||x+ty||^2\\ &= <x+ty,x+ty>\\ &= ||x||^2+2t<x,y>+t^2||y||^2\end{aligned}$$

Thus $f$ is a degree 2 polynomial, and since this polynomial is positive, its discriminant must be negative, $\Delta\leq 0$. But the discriminant is given by the following formula:

$$\Delta=4<x,y>^2-4||x||^2||y||^2$$

Thus, we obtain $|<x,y>|\leq||x||\cdot||y||$, as claimed. As for $||x+y||\leq||x||+||y||$, this follows from this, by raising to the square and simplifying.

(7-9) This is something more subtle, because do we really know what orthogonality and angles really are, in $\mathbb{R}^N$, so the best is to proceed as indicated, with orthogonality and angles being defined as above, and with the last formula being an easy exercise. □

Good work that we did, we are now experts in vector calculus, but as a matter of making sure that we have not forgotten anything, let us ask the cat. And cat says:

CAT 12.17. *Normally vector calculus in 3D is about vector products,*

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\times\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}x_2y_3-x_3y_2\\x_3y_1-x_1y_3\\x_1y_2-x_2y_1\end{pmatrix}$$

*but you can probably reach to some interesting things using $<x,y>$ only.*

Thanks cat, and I am afraid indeed that we won't have time here to talk about $x \times y$. Thus being said, shall you ever get into advanced mechanics, angular momentum, rotating frames, fluid dynamics, special and general relativity, or electromagnetism and quantum mechanics, ask your cat about vector products $x \times y$, which can be useful.

Moving on, with some linear algebra, let us start with the following definition:

Definition 12.18. *A map $f : \mathbb{R}^N \to \mathbb{R}^M$ is called linear when it satisfies:*

$$f(x+y) = f(x) + f(y) \quad , \quad f(\lambda x) = \lambda f(x)$$

*That is, $f$ must behave well with respect to the basic operations on vectors.*

As a first question that you might have, why calling linear such beasts? In answer, observe that the above linearity conditions can be merged into one, as follows:

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

But this latter condition tells us that our map $f$ must map lines into lines, or rather points moving on lines to points moving on lines, as follows:

$$f : \ [x - y] \ \rightsquigarrow \ [f(x) - f(y)]$$

Thus, the terminology is justified. In what regards now the mathematics of the linear maps, again by following the 2D material from the previous chapters, we have:

Theorem 12.19. *The linear maps $f : \mathbb{R}^N \to \mathbb{R}^M$ are in correspondence with the matrices $A \in M_{M\times N}(\mathbb{R})$, with the linear map associated to such a matrix being*

$$f(x) = Ax$$

*and with the matrix associated to a linear map being given by the formula*

$$A_{ij} = < f(e_j), e_i >$$

*with $\{e_i\}$ being the standard bases, and $< x, y > = \sum_i x_i y_i$ being the scalar product.*

Proof. There are several things going on here, the idea being as follows:

(1) According to Definition 12.18, a linear map $f : \mathbb{R}^N \to \mathbb{R}^M$ must send a vector $x \in \mathbb{R}^N$ to a certain vector $f(x) \in \mathbb{R}^M$, all whose components are linear combinations of the components of $x$. Thus, we can write, for certain numbers $A_{ij} \in \mathbb{R}$:

$$f\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix} = \begin{pmatrix}A_{11}x_1 + \ldots + A_{1N}x_N\\ \vdots\\ A_{M1}x_1 + \ldots + A_{MN}x_N\end{pmatrix}$$

Now observe that the parameters $A_{ij} \in \mathbb{R}$ can be regarded as being the entries of a rectangular matrix $A \in M_{M\times N}(\mathbb{R})$. Thus, we have a correspondence, as follows:

$$f \ \leftrightarrow \ A$$

(2) In order to understand now how this correspondence works, let us make the following convention, for the multiplication of the rectangular matrices:

$$(AB)_{ij}=\sum_k A_{ik}B_{kj}$$

To be more precise, we assume here $A\in M_{M\times N}(\mathbb{R})$ and $B\in M_{N\times K}(\mathbb{R})$, and we obtain in this way a certain matrix $AB\in M_{M\times K}(\mathbb{R})$. Now observe that in the case $K=1$, and by omitting the corresponding trivial index $j$, our multiplication formula reads:

$$(AB)_i=\sum_k A_{ik}B_k$$

But this is quite similar to what we have in (1), with the formula there taking the following form, which is the one in the statement, with our present conventions:

$$f(x)=Ax$$

(3) Regarding now the second assertion, with $f(x)=Ax$ as above, if we denote by $e_1,\ldots,e_N$ the standard basis of $\mathbb{R}^N$, then we have the following formula:

$$f(e_j)=\begin{pmatrix}A_{1j}\\ \vdots\\ A_{Mj}\end{pmatrix}$$

But this gives the formula $<f(e_j),e_i>=A_{ij}$ in the statement, as desired. □

As a first consequence of the above result, of great practical interest, we have:

THEOREM 12.20. *Regarding the linear maps, written as $f_A(x)=Ax$:*

1. *These compose according to $f_Af_B=f_{AB}$.*
2. *$f_A$ is invertible when $A$ is invertible, and $f_A^{-1}=f_{A^{-1}}$.*
3. *When $A$ is invertible, $f_A(x)=y$ is solved by $x=f_{A^{-1}}(y)$.*

PROOF. This is something self-explanatory, with (1) being clear from definitions, (2) coming from (1), and (3) coming from (2). As a comment, however, in order to understand the meaning of this, let us see what (3) tells us. The equation $f_A(x)=y$ reads:

$$\begin{cases}A_{11}x_1+\ldots+A_{1N}x_N=y_1\\ \qquad\qquad\vdots\\ A_{N1}x_1+\ldots+A_{NN}x_N=y_N\end{cases}$$

We recognize here an arbitrary linear system, which is something that is certainly not easy to solve, with bare hands. But with our linear algebra technology, assuming that

$A = (A_{ij})$ is invertible, say with inverse $B = (B_{ij})$, the solution is given by:

$$\begin{cases} x_1 = B_{11}y_1 + \ldots + B_{1N}y_N \\ \vdots \\ x_N = B_{N1}y_1 + \ldots + B_{NN}y_N \end{cases}$$

Which sounds quite amazing, hope you agree with me. In practice, however, inverting matrices is something non-trivial. We will be back to this later, with a solution. □

As a continuation of this, inspired by the $2 \times 2$ determinants, let us formulate:

THEOREM 12.21. *We can talk about the determinants of $N \times N$ matrices, defined according to the following formula, valid for any vectors $v_1, \ldots, v_N \in \mathbb{R}^N$,*

$$\det(v_1 \ \ldots \ v_N) = \pm vol < v_1, \ldots, v_N >$$

*with $< v_1, \ldots, v_N > \subset \mathbb{R}^N$ being the parallelepiped spanned by these vectors, and with the sign being $+$ if we can continuously pass $e_i \to v_i$, and being $-$ otherwise. And:*

1. *In 2 dimensions, we have* $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.
2. *In general,* det *enjoys the same properties as in 2D.*
3. *In particular, $A$ is invertible when $\det A \neq 0$, and $A^{-1}$ can be computed.*

PROOF. There is a long story here, the idea being as follows:

(1) We can certainly define the determinant as in the statement, and in 2D we obtain indeed $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, according to our computations in chapter 9.

(2) Things here are more complicated, depending on your exact knowledge of 2D determinants, but believe me, all formulae that you presumably know in 2D, except of course for $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} =< \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} d \\ -b \end{pmatrix} >$, which is something quite special, extend indeed to $N$ dimensions, and with the proofs being quite elementary, using the Thales theorem, a bit as we did in chapter 9. Among others, we obtain the following formula, at $N = 3$:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

(3) Here the first assertion comes from Theorem 12.20, and the second assertion is more technical, with for instance the inversion formula at $N = 3$ being as follows:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix}$$

Summarizing, many things to be learned here, and have a look at [11] for more. □

Finally, let us talk about diagonalization. The basic theory here is as follows:

THEOREM 12.22. *A vector $v\in\mathbb R^N$ is called eigenvector of $A\in M_N(\mathbb R)$, with corresponding eigenvalue $\lambda$, when $A$ multiplies by $\lambda$ in the direction of $v$:*

$$Av=\lambda v$$

*In the case where $\mathbb R^N$ has a basis $v_1,\ldots,v_N$ formed by eigenvectors of $A$, with corresponding eigenvalues $\lambda_1,\ldots,\lambda_N$, in this new basis $A$ becomes diagonal, as follows:*

$$A\sim\begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_N\end{pmatrix}$$

*Equivalently, if we denote by $D=diag(\lambda_1,\ldots,\lambda_N)$ the above diagonal matrix, and by $P=[v_1\ldots v_N]$ the square matrix formed by the eigenvectors of $A$, we have:*

$$A=PDP^{-1}$$

*In this case we say that the matrix $A$ is diagonalizable.*

PROOF. The first assertion is clear, and the second one follows from it, by changing the basis. Alternatively, we can prove this by a direct computation too, as follows:

$$PDP^{-1}v_i=PDe_i=P\lambda_ie_i=\lambda_iPe_i=\lambda_iv_i$$

Thus, the matrices $A$ and $PDP^{-1}$ coincide, as stated. □

In practice now, in order to diagonalize the real matrices $A\in M_N(\mathbb R)$, it is better to regards them as complex matrices, $A\in M_N(\mathbb C)$, because there are more chances to find eigenvectors in $\mathbb C^N$, than in its subspace $\mathbb R^N$. And, with this trick adopted, we have:

THEOREM 12.23. *Given a matrix $A\in M_N(\mathbb C)$, consider its characteristic polynomial*

$$P(X)=\det(A-X1_N)$$

*then factorize this polynomial, by computing the complex roots, with multiplicities,*

$$P(X)=(-1)^N(X-\lambda_1)^{n_1}\ldots(X-\lambda_k)^{n_k}$$

*and finally compute the corresponding eigenspaces, for each eigenvalue found:*

$$E_i=\left\{v\in\mathbb C^N\Big|Av=\lambda_iv\right\}$$

*The dimensions of these eigenspaces satisfy then the following inequalities,*

$$\dim(E_i)\leq n_i$$

*and $A$ is diagonalizable precisely when we have equality for any $i$.*

PROOF. This is something more technical, based on some routine algebra, and with the occurrence of the characteristic polynomial being not surprising, due to:

$$Av=\lambda v\iff\det(A-\lambda)=0$$

For details, you can have a look at any linear algebra book, including mine [11]. □

Let us end this discussion with something very concrete and useful, as follows:

THEOREM 12.24. *Any matrix $A \in M_N(\mathbb{R})$ which is symmetric, $A = A^t$, is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^t$$

*with $U \in M_N(\mathbb{R})$ orthogonal, and $D \in M_N(\mathbb{R})$ diagonal. The converse holds too.*

PROOF. Many things going on here, the idea being as follows:

(1) To start with, the operation $A \to A^t$ is the transposition, $(A^t)_{ij} = A_{ji}$. Also, $U \in M_N(\mathbb{R})$ is called orthogonal when it satisfies the following conditions, with the equivalence coming from the polarization identity from Theorem 12.16:

$$\begin{aligned} ||Ux|| = ||x|| &\iff < Ux, Uy >=< x, y > \\ &\iff < U^tUx, y >=< x, y > \\ &\iff U^t = U^{-1} \end{aligned}$$

(2) Getting now to the proof, the last assertion trivially holds, because if we take a matrix of the form $A = UDU^t$, with $U$ orthogonal and $D$ diagonal, we have:

$$A^t = (UDU^t)^t = UDU^t = A$$

(3) In the other sense now, assume that $A$ is symmetric, $A = A^t$. Our first claim is that the eigenvalues are real. Indeed, assuming $Av = \lambda v$, we have, as desired:

$$\begin{aligned} \lambda < v, v > &= < Av, v > \\ &= < v, Av > \\ &= \bar{\lambda} < v, v > \end{aligned}$$

(4) Our next claim is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Indeed, assuming $Av = \lambda v$, $Aw = \mu w$ we have, using $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned} \lambda < v, w > &= < Av, w > \\ &= < v, Aw > \\ &= \mu < v, w > \end{aligned}$$

We conclude that $\lambda \neq \mu$ implies $v \perp w$, which proves our claim.

(5) In order to finish the proof, it remains to prove that the eigenspaces of $A$ span the whole $\mathbb{R}^N$. For this purpose, observe that assuming $Av = \lambda v$ and $v \perp w$, we have:

$$\begin{aligned} < Aw, v > &= < w, Av > \\ &= < w, \lambda v > \\ &= \lambda < w, v > \\ &= 0 \end{aligned}$$

Thus $v^\perp$ is invariant under $A$, so we can do the recurrence, and we get the result. □

## 12d. Surfaces, manifolds

We would like to end the present chapter and Part III with a discussion on what happens to curves in higher dimensions, with an introduction to modern algebraic geometry. As before with other things in this chapter, we will be quite quick, and advanced.

Let us first get to $\mathbb{R}^3$. Here we are right away into a dillema, because the plane curves have two possible generalizations. First we have the algebraic curves in $\mathbb{R}^3$:

Definition 12.25. *An algebraic curve in $\mathbb{R}^3$ is a curve as follows,*

$$C=\left\{(x,y,z)\in\mathbb{R}^3\Big|P(x,y,z)=0,\,Q(x,y,z)=0\right\}$$

*appearing as the joint zeroes of two polynomials $P,Q$.*

These curves look of course like the usual plane curves, and at the level of the phenomena that can appear, these are similar to those in the plane, involving singularities and so on, but also knotting, which is a new phenomenon. However, it is hard to say something with bare hands about knots, and we will not get into this, in this book.

On the other hand, as another natural generalization of the plane curves, and this might sound a bit surprising, we have the surfaces in $\mathbb{R}^3$, constructed as follows:

Definition 12.26. *An algebraic surface in $\mathbb{R}^3$ is a surface as follows,*

$$S=\left\{(x,y,z)\in\mathbb{R}^3\Big|P(x,y,z)=0\right\}$$

*appearing as the zeroes of a polynomial $P$.*

The point indeed is that, as it was the case with the plane curves, what we have here is something defined by a single equation. And with respect to many questions, having a single equation matters a lot, and this is why surfaces in $\mathbb{R}^3$ are "simpler" than curves in $\mathbb{R}^3$. In fact, believe me, they are even the correct generalization of the curves in $\mathbb{R}^2$.

As an example of what can be done with surfaces, which is very similar to what we did with the conics $C\subset\mathbb{R}^2$ before, we have the following result:

Theorem 12.27. *The degree $2$ surfaces $S\subset\mathbb{R}^3$, called quadrics, are the ellipsoid*

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$$

*which is the only compact one, plus $16$ more, which can be explicitly listed.*

Proof. We will be quite brief here, because we intend to rediscuss all this in a moment, with more details, in arbitrary $N$ dimensions, the idea being as follows:

(1) The equations for a quadric $S\subset\mathbb{R}^2$ are best written as follows, with $A\in M_3(\mathbb{R})$ being a matrix, $B\in M_{1\times 3}(\mathbb{R})$ being a row vector, and $C\in\mathbb{R}$ being a constant:

$$< Au,u > +Bu+C=0$$

(2) Since $A\in M_3(\mathbb{R})$ can be chosen symmetric, by using Theorem 12.24 we are left, modulo degeneracy and linear transformations, with signed sums of squares:

$$\pm x^2\pm y^2\pm z^2=0,1$$

(3) Thus the sphere is the only compact quadric, up to linear transformations, and by applying now linear transformations to it, we are led to the ellipsoids in the statement.

(4) As for the other quadrics, there are many of them, a bit similar to the parabolas and hyperbolas in 2 dimensions, and some work here leads to a 16 item list. □

With this discussed, instead of further insisting on the surfaces $S\subset\mathbb{R}^3$, or getting into their rivals, the curves $C\subset\mathbb{R}^3$, which appear as intersections of such surfaces, $C=S\cap S'$, let us get instead to arbitrary $N$ dimensions, see what the axiomatics looks like there, with the hope that this will clarify our dimensionality dillema, curves vs surfaces.

So, moving to $N$ dimensions, we have here the following definition, to start with:

Definition 12.28. *An algebraic hypersurface in $\mathbb{R}^N$ is a space of the form*

$$S=\left\{(x_1,\ldots,x_N)\in\mathbb{R}^N\Big|P(x_1,\ldots,x_N)=0,\forall i\right\}$$

*appearing as the zeroes of a polynomial $P\in\mathbb{R}[x_1,\ldots,x_N]$.*

Again, this is a quite general definition, covering both the plane curves $C\subset\mathbb{R}$ and the surfaces $S\subset\mathbb{R}^2$, which is certainly worth a systematic exploration. But, no hurry with this, for the moment we are here for talking definitions and axiomatics.

In order to have now a full collection of beasts, in all possible dimensions $N\in\mathbb{N}$, and of all possible dimensions $k\in\mathbb{N}$, we must intersect such algebraic hypersurfaces. We are led in this way to the zeroes of families of polynomials, as follows:

Definition 12.29. *An algebraic manifold in $\mathbb{R}^N$ is a space of the form*

$$X=\left\{(x_1,\ldots,x_N)\in\mathbb{R}^N\Big|P_i(x_1,\ldots,x_N)=0,\forall i\right\}$$

*with $P_i\in\mathbb{R}[x_1,\ldots,x_N]$ being a family of polynomials.*

As a first observation, as already mentioned, such a manifold appears as an intersection of hypersurfaces $S_i$, those associated to the various polynomials $P_i$:

$$X=S_1\cap\ldots\cap S_k$$

There is actually a bit of a discussion needed here, regarding the parameter $k\in\mathbb{N}$, shall we allow this parameter to be $k=\infty$ too, or not. We will discuss this later.

Let us first look more in detail at the hypersurfaces. We have here:

THEOREM 12.30. *The degree 2 hypersurfaces $S \subset \mathbb{R}^N$, called quadrics, are up to degeneracy and to linear transformations the hypersurfaces of the following form,*

$$\pm x_1^2 \pm \ldots \pm x_N^2 = 0, 1$$

*and with the sphere being the only compact one.*

PROOF. We have two statements here, the idea being as follows:

(1) The equations for a quadric $S \subset \mathbb{R}^N$ are best written as follows, with $A \in M_N(\mathbb{R})$ being a matrix, $B \in M_{1\times N}(\mathbb{R})$ being a row vector, and $C \in \mathbb{R}$ being a constant:

$$< Ax, x > +Bx + C = 0$$

(2) Since $A \in M_N(\mathbb{R})$ can be chosen symmetric, by using Theorem 12.24 we are left, modulo degeneracy and linear transformations, with signed sums of squares:

$$\pm x_1^2 \pm \ldots \pm x_N^2 = 0, 1$$

(3) To be more precise, by changing the basis of $\mathbb{R}^N$, as to have $A \in M_N(\mathbb{R})$ diagonal, our equation becomes as follows, with $D \in M_N(\mathbb{R})$ being now diagonal:

$$< Dx, x > +Ex + F = 0$$

(4) But now, by making squares in the obvious way, which amounts in applying yet another linear transformation to our quadric, the equation takes the following form, with $G \in M_N(-1, 0, 1)$ being diagonal, and with $H \in \{0, 1\}$ being a constant:

$$< Gx, x >= H$$

(5) Now barring the degenerate cases, we can further assume $G \in M_N(-1, 1)$, and we are led in this way to the equation claimed in (2) above, namely:

$$\pm x_1^2 \pm \ldots \pm x_N^2 = 0, 1$$

(6) In particular we see that, up to some degenerate cases, namely emptyset and point, the only compact quadric, up to linear transformations, is the one given by:

$$x_1^2 + \ldots + x_N^2 = 1$$

(7) But this is the unit sphere, so are led to the conclusions in the statement. □

Getting now to the arbitrary general manifolds, we have the following question:

QUESTION 12.31. *Given an algebraic manifold in $\mathbb{R}^N$, appearing as*

$$X = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \middle| P_i(x_1, \ldots, x_N) = 0, \forall i \right\}$$

*what are the polynomials $P \in \mathbb{R}[x_1, \ldots, x_N]$ vanishing on $X$? Conversely, given a set*

$$I \subset \mathbb{R}[x_1, \ldots, x_N]$$

*what is the manifold $X$ where all the polynomials $P \in I$ vanish?*

Obviously, this is something important, because assuming that we managed to find an answer, we would have a useful "algebraic geometry" correspondence, as follows:

$$\left(X\subset\mathbb R^N\right)\quad\longleftrightarrow\quad\left(I\subset\mathbb R[x_1,\ldots,x_N]\right)$$

In practice now, we already know a bit about the beasts on the left $X$, so let us study the beasts on the right $I$. Here are a few basic observations, about them:

– To start with, assuming that $X\subset\mathbb R^N$ comes from polynomials $\{P_i\}$, the set $I\subset\mathbb R[x_1,\ldots,x_N]$ of polynomials vanishing on $X$ obviously contains $\{P_i\}$.

– However, much more is true. Indeed, if we come with any family of polynomials $\{Q_i\}\subset\mathbb R[x_1,\ldots,x_N]$, it is then clear that we must have $\sum_iP_iQ_i\in I$.

– Getting now a bit abstract, we can see that, more generally, $I\subset\mathbb R[x_1,\ldots,x_N]$ must be stable under sums, and must satisfy $P\in I\implies PQ\in I$, $\forall Q$.

And so, question now, in view of all this, what are the beasts $I\subset\mathbb R[x_1,\ldots,x_N]$ that we are looking for? In answer, these must be ideals, in the following sense:

Definition 12.32. *We have notions of rings and ideals, as follows:*

1. *A ring $R$ is a set with operations $+$ and $\times$, satisfying the usual conditions for such operations, except for $ab=ba$, and for $a\neq0\implies\exists a^{-1}$.*
2. *An ideal $I\subset R$ is a subgroup with the left ideal property $i\in I,r\in R\implies ir\in I$, or the right ideal property $i\in I,r\in R\implies ri\in I$, or both.*

In what follows we will be mainly interested in the ring $R=\mathbb R[x_1,\ldots,x_N]$, which is commutative, $ab=ba$. For such rings, the 3 notions of ideals in (2) coincide.

In relation now with our algebraic geometry questions, we can reformulate our notion of algebraic manifold, in commutative algebra terms, as follows:

Theorem 12.33. *The algebraic manifolds are precisely the sets of the form*

$$X=\left\{x\in\mathbb R^N\Big|P(x)=0,\forall P\in I\right\}$$

*with $I\subset\mathbb R[x_1,\ldots,x_N]$ being a certain ideal.*

Proof. In one sense, this comes from the discussion after Question 12.31, and in the other sense this is trivial, because we can write $I=\{P_i|i\in I\}$, with $P_i=i$. □

In order to further discuss now the correspondence $X\leftrightarrow I$, we need to know more algebra. Let us start with the following basic fact, in the context of Definition 12.32:

Theorem 12.34. *Let $R$ be a ring, and $I\subset R$ be an additive subgroup.*

1. *$I$ is a two-sided ideal precisely when $F=R/I$ is a ring.*
2. *If $R$ is commutative, $I\subset R$ is a maximal ideal precisely when $F$ is a field.*

Proof. This is something very standard, the idea being as follows:

(1) Since the additive group $(R,+)$ is abelian, given an additive subgroup $I\subset R$ we can form the quotient group $F=R/I$, which is abelian too, with addition as follows:

$$(a+I)+(b+I)=(a+b+I)$$

The question is now, can we turn this abelian group $F$ into a ring? Normally the multiplication can only be as follows, and with this clarifying our statement:

$$(a+I)(b+I)=(ab+I)$$

But, will this work. In practice, the following condition must be satisfied:

$$(a+I)=(a'+I)\ ,\ (b+I)=(b'+I)\quad\implies\quad(ab+I)=(a'b'+I)$$

But this shows that $I\subset R$ must be a two-sided ideal, as claimed.

(2) Assume first that $F=R/I$ is a field. This means that any nonzero element of $F$ is invertible, and with our usual conventions for $F$, this reads:

$$\forall a\notin I\ ,\ \exists b\in R\ ,\ (ab+I)=(1+I)$$

Now assume by contradiction that $I\subset R$ is not maximal, so that we have a bigger ideal $I\subset J\subset R$. If we pick $a\in J-I$, we obtain, by the above, the following:

$$a\in J-I\ ,\ b\in R\ ,\ ab=1+i\ ,\ i\in I$$

But this is contradictory, because since $J$ is an ideal, containing $I$, we must have $ab,i\in J$, and so we conclude that we have $1\in J$, which in turn gives:

$$J=R$$

As for the converse, this follows via some similar arguments, exercise for you. □

Getting back now to algebraic geometry, we first have the following result:

Theorem 12.35 (Hilbert basis theorem). *Any ideal of polynomials*

$$I\subset\mathbb{R}[x_1,\ldots,x_N]$$

*is finitely generated,* $I=(P_1,\ldots,P_k)$*, for some* $P_i\in\mathbb{R}[x_1,\ldots,x_N]$.

Proof. This is something quite tricky, the idea being as follows:

(1) Following Emmy Noether, let us call a ring $R$ Noetherian when any ideal $I\subset R$ is finitely generated. Equivalently, any increasing sequence of ideals $I_1\subset I_2\subset\ldots$ must stabilize, in the sense that we must have $I_n=I_{n+1}=\ldots$, for some $n\in\mathbb{N}$.

(2) We want to prove that $\mathbb{R}[x_1,\ldots,x_N]$ is Noetherian, and we will do this by recurrence on $N$. Since $R=\mathbb{R}$ is clearly Noetherian, as being a field, we are left with proving the recurrence step. And, for this purpose, we will prove something which is a bit more general, namely that if a ring $R$ is Noetherian, then so is the ring $R[X]$.

(3) We do this by contradiction. So, assume that $R$ is Noetherian, and that $R[X]$ is not Noetherian, so that we have an ideal $I \subset R[X]$ which is not finitely generated.

(4) In order to find a contradiction, let us pick $P_1 \in I$ of minimial degree $d_1 \in \mathbb{N}$, then $P_2 \in I/(P_1)$ of minimal degree $d_2 \in \mathbb{N}$, then $P_3 \in I/(P_1, P_2)$ of minimal degree $d_3 \in \mathbb{N}$, and so on. Since our ideal $I \subset R[X]$ was assumed to be not finitely generated, this procedure will not stop, and we obtain an increasing sequence, as follows:

$$d_1 \leq d_2 \leq d_3 \leq \ldots$$

(5) Now let $a_i \in R$ be the leading coefficient of each $P_i$, and set $J = (a_1, a_2, \ldots) \subset R$. Since $R$ was assumed to be Noetherian, we can find $n \in \mathbb{N}$ such that $J = (a_1, \ldots, a_n)$. Thus, we have a formula as follows, for certain scalars $\lambda_i \in R$:

$$a_{n+1} = \sum_{i=1}^{n} \lambda_i a_i$$

(6) With this done, consider the following polynomial, with $\lambda_i \in R$ as above:

$$Q = \sum_{i=1}^{n} \lambda_i X^{d_{n+1} - d_i} P_i$$

This polyomial satisfies then $Q \in (P_1, \ldots, P_n)$, and has the same leading coefficient as $P_{n+1} \notin (P_1, \ldots, P_n)$. Thus, the following polynomial has degree $< d_{n+1}$:

$$P_{n+1} - Q \in I/(P_1, \ldots, P_n)$$

But this is a contradiction, as desired, and this finishes the proof. $\square$

In practice, Theorem 12.35 is best remembered geometrically, as follows:

THEOREM 12.36. *The algebraic manifolds $X \subset \mathbb{R}^N$ are precisely the intersections*

$$X = S_1 \cap \ldots \cap S_k$$

*with $S_i \subset \mathbb{R}^N$ being hypersurfaces.*

PROOF. Indeed, given an algebraic manifold $X \subset \mathbb{R}^N$, we can consider the ideal $I \subset \mathbb{R}[x_1, \ldots, x_N]$ of polynomials vanishing on $X$, then write $I = (P_1, \ldots, P_k)$ with $k < \infty$, as in Theorem 12.35, and then set $S_i \subset \mathbb{R}^N$ to be the set of zeroes of $P_i$. $\square$

Moving ahead now, let us further investigate the correspondence $X \leftrightarrow I$. We would like this to be bijective, but there are at least 2 obstructions to this, as follows:

– To start with, assuming $P^k = 0$ on $X$, we have $P = 0$ on $X$. In view of this, we must restrict the attention to the ideals $I$ which are "radical", $P^k \in I \implies P \in I$.

– Also, at $N = 1$, the ideal $I = (x^2 + 1) \subset \mathbb{R}[x]$ produces the manifold $X = \emptyset$. In view of this, we must trade $\mathbb{R}$ for $\mathbb{C}$, where arbitrary polynomials have roots.

So, these are two obvious obstructions, with respective solutions to them, and coming now as good news, there is no third obstruction, as shown by the following result:

Theorem 12.37 (Nullstellensatz). *We have a correspondence*

$$\Big(X\subset\mathbb C^N\Big)\quad\longleftrightarrow\quad\Big(I\subset\mathbb C[x_1,\ldots,x_N]\Big)$$

*between algebraic manifolds in $\mathbb C^N$, and radical ideals of $\mathbb C[x_1,\ldots,x_N]$.*

Proof. This is something quite tricky, due to Hilbert, as follows:

(1) We know that at $N=1$ polynomials have roots, so here $I=(P)\implies X_I\neq\emptyset$. The point now is that, by doing some algebra, in the spirit of what we did in the proof of Theorem 12.35, something similar happens in arbitrary $N$ dimensions, in the sense that any proper ideal $I\subset\mathbb C[x_1,\ldots,x_N]$ produces a non-empty manifold, $X_I\neq\emptyset$.

(2) Next, what we want to prove is that given an ideal $I\subset\mathbb C[x_1,\ldots,x_N]$, any polynomial $P\in\mathbb C[x_1,\ldots,x_N]$ vanishing on $X_I$ has the property $P^k\in I$, for some $k\in\mathbb N$. For this purpose, we can add 1 dimension, and consider the following ideal:

$$J=<I,x_{N+1}P(x_1,\ldots,x_N)-1>$$

(3) Now since we have $X_J=\emptyset$, by (1) we conclude that $J$ is trivial. In order now to best interpret this finding, consider the following algebra:

$$\mathbb C[x_1,\ldots,x_N][P^{-1}]=\mathbb C[x_1,\ldots,x_{N+1}]/(x_{N+1}P-1)$$

The triviality of $J$ gives then a formula of the following type, with $f_i\in I$:

$$1=f_0+f_1x_{N+1}+\ldots+f_kx_{N+1}^k$$

Now by multiplying by $P^k$, we obtain from this $P^k\in I$, as desired. □

## 12e. Exercises

Welcome to space geometry, and as exercises on this, we have:

Exercise 12.38. *Learn more about sinusoidal spirals, and their properties.*

Exercise 12.39. *Learn also more about polynomial lemniscates, and stelloids.*

Exercise 12.40. *Read also some electrostatics, featuring various algebraic curves.*

Exercise 12.41. *Clarify what we said, in relation with orthocentric tetrahedra.*

Exercise 12.42. *Mediate on the notions of orthogonality and angles, in $\mathbb R^N$.*

Exercise 12.43. *Learn also about vector products in $\mathbb R^3$, and their applications.*

Exercise 12.44. *Learn the full theory of the determinant, defined as a volume.*

Exercise 12.45. *Read, as much as you can, about matrix diagonalization.*

As bonus exercise, reiterated, start reading a nice algebraic geometry book.

# Part IV

# Calculus methods

*If you're going to San Francisco*
*Be sure to wear some flowers in your hair*
*If you're going to San Francisco*
*You're gonna meet some gentle people there*

CHAPTER 13

# Functions, derivatives

## 13a. Functions, derivatives

With algebra and geometry reasonably understood, time to get, with no fear, into analysis and related topics. Our main motivation is very simple, as follows:

Motivation 13.1. *We have been advised by felines to trade our beloved* $90^\circ$ *from astronomy for the quite abstract* $\pi/2$ *from mathematics, based on the fact that*

$$\sin t \simeq t$$

*can be of help, in relation with the fact that this world is made of small angles and forces, adding up. But is this really a good idea? We want to see applications of this.*

But all this is perhaps a bit too philosophical. At a more concrete level, still in relation with our rescaling $90^\circ \to \pi/2$, and with the resulting estimate $\sin t \simeq t$, we have:

Motivations 13.2. *In with relation with the above, and more concretely:*

(1) *Can we further improve* $\sin t \simeq t$*,* $\cos t \simeq 1 - t^2/2$*,* $\tan t \simeq t$*?*
(2) *What about estimating* $\sin x, \cos x, \tan x$*, at an arbitrary* $x \in \mathbb{R}$*?*
(3) *In fact, how to best approximate* $\sin, \cos, \tan$ *by polynomials?*
(4) *What are the averages of* $\sin, \cos, \tan$*, over suitable intervals?*
(5) *What about more complicated averages, say of* $\sin^p x \cos^q x$*?*
(6) *Also, what about* $\pi$ *itself, can we reach to* $\pi = 3.1415\ldots$*?*

All this was trigonometry, which is certainly useful for engineering and related topics. On top of this, we have accumulated as well a number of motivations coming from pure mathematics and physics, more philosophical, which can be summarized as follows:

Motivations 13.3. *In relation with pure mathematics and physics:*

(1) *How to rigorously prove* $e^{it} = \cos t + i \sin t$*?*
(2) *What are the volumes of spheres, in arbitrary dimensions?*
(3) *Is a matrix* $A \in M_N(\mathbb{C})$*, picked at random, diagonalizable?*
(4) *Can we approximate the functions* $f : \mathbb{R}^N \to \mathbb{R}^M$ *by linear maps?*
(5) *How to define tangent spaces to our various curves and surfaces?*
(6) *Can we understand the computations of Newton, for gravity?*

Getting to work now, there is an answer to all these questions, namely "calculus". The idea of calculus is very simple. To start with, we will be interested in the functions $f : \mathbb{R} \to \mathbb{R}$. We know that when $f$ is continuous at $x$, we can write an approximation formula as follows, for the values of our function $f$ around that point $x$:

$$f(x+t) \simeq f(x)$$

The problem is now, how to improve this? And a bit of thinking at all this suggests to look at the slope of $f$ at the point $x$. Which leads us into the following notion:

Definition 13.4. *A function $f : \mathbb{R} \to \mathbb{R}$ is called differentiable at $x$ when*

$$f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$$

*called derivative of $f$ at that point $x$, exists.*

As a first remark, in order for $f$ to be differentiable at $x$, that is to say, in order for the above limit to converge, the numerator must go to 0, as the denominator $t$ does:

$$\lim_{t \to 0} [f(x+t) - f(x)] = 0$$

Thus, $f$ must be continuous at $x$. However, the converse is not true, a basic counterexample being $f(x) = |x|$ at $x = 0$. Let us summarize these findings as follows:

Proposition 13.5. *If $f$ is differentiable at $x$, then $f$ must be continuous at $x$. However, the converse is not true, a basic counterexample being $f(x) = |x|$, at $x = 0$.*

Proof. The first assertion is something that we already know, from the above. As for the second assertion, regarding $f(x) = |x|$, this is something quite clear on the picture of $f$, but let us prove this mathematically, based on Definition 13.4. We have:

$$\lim_{t \searrow 0} \frac{|0+t| - |0|}{t} = \lim_{t \searrow 0} \frac{t - 0}{t} = 1$$

On the other hand, we have as well the following computation:

$$\lim_{t \nearrow 0} \frac{|0+t| - |0|}{t} = \lim_{t \nearrow 0} \frac{-t - 0}{t} = -1$$

Thus, the limit in Definition 13.4 does not converge, as desired. $\square$

Generally speaking, the last assertion in Proposition 13.5 should not bother us much, because most of the basic continuous functions are differentiable, and we will see examples in a moment. Before that, however, let us recall why we are here, namely improving the basic estimate $f(x+t) \simeq f(x)$. We can now do this, using the derivative, as follows:

Theorem 13.6. *Assuming that $f$ is differentiable at $x$, we have:*

$$f(x+t) \simeq f(x) + f'(x)t$$

*In other words, $f$ is, approximately, locally affine at $x$.*

PROOF. Assume indeed that $f$ is differentiable at $x$, and let us set, as before:

$$f'(x)=\lim_{t\to0}\frac{f(x+t)-f(x)}{t}$$

By multiplying by $t$, we obtain that we have, once again in the $t\to0$ limit:

$$f(x+t)-f(x)\simeq f'(x)t$$

Thus, we are led to the conclusion in the statement. □

All this is very nice, and before developing more theory, let us work out some examples. As a first illustration, the derivatives of the power functions are as follows:

THEOREM 13.7. *We have the differentiation formula*

$$(x^p)'=px^{p-1}$$

*valid for any exponent $p\in\mathbb{R}$.*

PROOF. We can do this in three steps, as follows:

(1) In the case $p\in\mathbb{N}$ we can use the binomial formula, which gives, as desired:

$$\begin{aligned}(x+t)^p&=\sum_{k=0}^n\binom{p}{k}x^{p-k}t^k\\&=x^p+px^{p-1}t+\ldots+t^p\\&\simeq x^p+px^{p-1}t\end{aligned}$$

(2) Let us discuss now the general case $p\in\mathbb{Q}$. We write $p=m/n$, with $m\in\mathbb{Z}$ and $n\in\mathbb{N}$. In order to do the computation, we use the following formula:

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\ldots+b^{n-1})$$

We set in this formula $a=(x+t)^{m/n}$ and $b=x^{m/n}$. We obtain, as desired:

$$\begin{aligned}(x+t)^{m/n}-x^{m/n}&=\frac{(x+t)^m-x^m}{(x+t)^{m(n-1)/n}+\ldots+x^{m(n-1)/n}}\\&\simeq\frac{(x+t)^m-x^m}{nx^{m(n-1)/n}}\\&\simeq\frac{mx^{m-1}t}{nx^{m(n-1)/n}}\\&=\frac{m}{n}\cdot x^{m-1-m+m/n}\cdot t\\&=\frac{m}{n}\cdot x^{m/n-1}\cdot t\end{aligned}$$

(3) In the general case now, where $p \in \mathbb{R}$ is real, we can use a similar argument. Indeed, given any integer $n \in \mathbb{N}$, we have the following computation:

$$\begin{aligned}(x+t)^p - x^p &= \frac{(x+t)^{pn} - x^{pn}}{(x+t)^{p(n-1)} + \ldots + x^{p(n-1)}} \\ &\simeq \frac{(x+t)^{pn} - x^{pn}}{nx^{p(n-1)}}\end{aligned}$$

Now observe that we have the following estimate, with [.] being the integer part:

$$(x+t)^{[pn]} \leq (x+t)^{pn} \leq (x+t)^{[pn]+1}$$

By using the binomial formula on both sides, for the integer exponents $[pn]$ and $[pn]+1$ there, we deduce that with $n >> 0$ we have the following estimate:

$$(x+t)^{pn} \simeq x^{pn} + pnx^{pn-1}t$$

Thus, we can finish our computation started above as follows:

$$(x+t)^p - x^p \simeq \frac{pnx^{pn-1}t}{nx^{pn-p}} = px^{p-1}t$$

But this gives $(x^p)' = px^{p-1}$, which finishes the proof. □

Here are some further computations, for other basic functions that we know:

Theorem 13.8. *We have the following results:*

(1) $(\sin x)' = \cos x$.
(2) $(\cos x)' = -\sin x$.
(3) $(e^x)' = e^x$.
(4) $(\log x)' = x^{-1}$.

Proof. This is quite tricky, as always when computing derivatives, as follows:

(1) Regarding sin, the computation here goes as follows:

$$\begin{aligned}(\sin x)' &= \lim_{t\to 0} \frac{\sin(x+t) - \sin x}{t} \\ &= \lim_{t\to 0} \frac{\sin x \cos t + \cos x \sin t - \sin x}{t} \\ &= \lim_{t\to 0} \sin x \cdot \frac{\cos t - 1}{t} + \cos x \cdot \frac{\sin t}{t} \\ &= \cos x\end{aligned}$$

Here we have used the fact, that we know well from chapter 7, obtained by drawing the trigonometric circle, that we have $\sin t \simeq t$ for $t \simeq 0$, plus the fact, which follows from this and Pythagoras, $\sin^2 + \cos^2 = 1$, that we have as well $\cos t \simeq 1 - t^2/2$, for $t \simeq 0$.

(2) The computation for cos is similar, based on the same ingredients, as follows:

$$\begin{aligned}(\cos x)' &= \lim_{t\to 0}\frac{\cos(x+t)-\cos x}{t}\\ &= \lim_{t\to 0}\frac{\cos x\cos t-\sin x\sin t-\cos x}{t}\\ &= \lim_{t\to 0}\cos x\cdot\frac{\cos t-1}{t}-\sin x\cdot\frac{\sin t}{t}\\ &= -\sin x\end{aligned}$$

(3) For the exponential, the derivative can be computed as follows:

$$\begin{aligned}(e^x)' &= \left(\sum_{k=0}^\infty\frac{x^k}{k!}\right)'\\ &= \sum_{k=0}^\infty\frac{kx^{k-1}}{k!}\\ &= e^x\end{aligned}$$

(4) As for the logarithm, the computation here is as follows, using $\log(1+y)\simeq y$ for $y\simeq 0$, which follows from $e^y\simeq 1+y$ that we found in (3), by taking the logarithm:

$$\begin{aligned}(\log x)' &= \lim_{t\to 0}\frac{\log(x+t)-\log x}{t}\\ &= \lim_{t\to 0}\frac{\log(1+t/x)}{t}\\ &= \frac{1}{x}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

Speaking exponentials, we can now formulate a nice result about them:

THEOREM 13.9. *The exponential function, namely*

$$e^x=\sum_{k=0}^\infty\frac{x^k}{k!}$$

*is the unique power series satisfying $f'=f$ and $f(0)=1$.*

PROOF. Consider indeed a power series satisfying $f'=f$ and $f(0)=1$. Due to $f(0)=1$, the first term must be 1, and so our function must look as follows:

$$f(x)=1+\sum_{k=1}^\infty c_kx^k$$

According to our differentiation rules, the derivative of this series is given by:

$$f(x)=\sum_{k=1}^\infty kc_kx^{k-1}$$

Thus, the equation $f'=f$ is equivalent to the following equalities:

$$c_1=1\quad,\quad 2c_2=c_1\quad,\quad 3c_3=c_2\quad,\quad 4c_4=c_3\quad,\quad\ldots$$

But this system of equations can be solved by recurrence, as follows:

$$c_1=1\quad,\quad c_2=\frac{1}{2}\quad,\quad c_3=\frac{1}{2\times 3}\quad,\quad c_4=\frac{1}{2\times 3\times 4}\quad,\quad\ldots$$

Thus we have $c_k=1/k!$, leading to the conclusion in the statement. □

Observe that the above result leads to a more conceptual explanation for the number $e$ itself. To be more precise, $e\in\mathbb R$ is the unique number satisfying:

$$(e^x)'=e^x$$

Let us work out now some general results. We have here the following statement:

THEOREM 13.10. *We have the following formulae:*
(1) $(f+g)'=f'+g'$.
(2) $(fg)'=f'g+fg'$.
(3) $(f\circ g)'=(f'\circ g)\cdot g'$.

PROOF. All these formulae are elementary, the idea being as follows:

(1) This follows indeed from definitions, the computation being as follows:

$$\begin{aligned}(f+g)'(x)&=\lim_{t\to0}\frac{(f+g)(x+t)-(f+g)(x)}{t}\\&=\lim_{t\to0}\left(\frac{f(x+t)-f(x)}{t}+\frac{g(x+t)-g(x)}{t}\right)\\&=\lim_{t\to0}\frac{f(x+t)-f(x)}{t}+\lim_{t\to0}\frac{g(x+t)-g(x)}{t}\\&=f'(x)+g'(x)\end{aligned}$$

(2) This follows from definitions too, the computation, by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative, being as follows:

$$\begin{aligned}(fg)(x+t)&=f(x+t)g(x+t)\\&\simeq(f(x)+f'(x)t)(g(x)+g'(x)t)\\&\simeq f(x)g(x)+(f'(x)g(x)+f(x)g'(x))t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of $t$, namely:

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x)$$

(3) Regarding compositions, the computation here is as follows, again by using the more convenient formula $f(x+t)\simeq f(x)+f'(x)t$ as a definition for the derivative:

$$\begin{aligned}(f\circ g)(x+t) &= f(g(x+t))\\ &\simeq f(g(x)+g'(x)t)\\ &\simeq f(g(x))+f'(g(x))g'(x)t\end{aligned}$$

Indeed, we obtain from this that the derivative is the coefficient of $t$, namely:

$$(f\circ g)'(x)=f'(g(x))g'(x)$$

Thus, we are led to the conclusions in the statement. □

We can of course combine the above formulae, and we obtain for instance:

THEOREM 13.11. *The derivatives of fractions are given by:*

$$\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$$

*In particular, we have the following formula, for the derivative of inverses:*

$$\left(\frac{1}{f}\right)'=-\frac{f'}{f^2}$$

*In fact, we have $(f^p)'=pf^{p-1}$, for any exponent $p\in\mathbb{R}$.*

PROOF. This statement is written a bit upside down, and for the proof it is better to proceed backwards. To be more precise, by using $(x^p)'=px^{p-1}$ and Theorem 13.10 (3), we obtain the third formula. Then, with $p=-1$, we obtain from this the second formula. And finally, by using this second formula and Theorem 13.10 (2), we obtain:

$$\begin{aligned}\left(\frac{f}{g}\right)' &= f'\cdot\frac{1}{g}+f\left(\frac{1}{g}\right)'\\ &= \frac{f'}{g}-\frac{fg'}{g^2}\\ &= \frac{f'g-fg'}{g^2}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

With the above formulae in hand, we can now do all sorts of computations for the other basic functions that we know, including $\tan x$, or $\arctan x$:

THEOREM 13.12. *We have the following formulae,*

$$(\tan x)'=\frac{1}{\cos^2 x}\quad,\quad(\arctan x)'=\frac{1}{1+x^2}$$

*and the derivatives of the remaining trigonometric functions can be computed as well.*

PROOF. For the tangent, we have the following computation:

$$\begin{aligned}(\tan x)' &= \frac{\sin' x\cos x - \sin x\cos' x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\end{aligned}$$

As for arctan, we can use here the following computation:

$$\begin{aligned}(\tan\circ\arctan)'(x) &= \tan'(\arctan x)\arctan'(x)\\ &= \frac{1}{\cos^2(\arctan x)}\arctan'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x' = 1$, we obtain from this:

$$\arctan'(x) = \cos^2(\arctan x)$$

On the other hand, with $t = \arctan x$ we know that we have $\tan t = x$, and so:

$$\cos^2(\arctan x) = \cos^2 t = \frac{1}{1+\tan^2 t} = \frac{1}{1+x^2}$$

Thus, we are led to the formula in the statement. As for the last assertion, we will leave this as an exercise for now, and come back later to it, in chapter 14. □

At the theoretical level now, further building on Theorem 13.6, we have:

THEOREM 13.13. *The local minima and maxima of a differentiable function $f:\mathbb{R}\to\mathbb{R}$ appear at the points $x\in\mathbb{R}$ where:*

$$f'(x) = 0$$

*However, the converse of this fact is not true in general.*

PROOF. The first assertion follows from the formula in Theorem 13.6, namely:

$$f(x+t) \simeq f(x) + f'(x)t$$

Indeed, saying that our function $f$ has a local maximum at $x\in\mathbb{R}$ means that there exists a number $\varepsilon > 0$ such that the following happens:

$$f(x+t) \geq f(x) \quad , \quad \forall t\in[-\varepsilon,\varepsilon]$$

We conclude that we must have $f'(x)t \geq 0$ for sufficiently small $t$, and since this small $t$ can be both positive or negative, this gives, as desired:

$$f'(x) = 0$$

As for the study of the local minima, this is similar. Finally, in what regards the converse, the simplest counterexample here is $f(x) = x^3$, taken at $x = 0$. □

As an important consequence now of Theorem 13.13, we have:

THEOREM 13.14. *Assuming that $f:[a,b]\to\mathbb{R}$ is differentiable, we have*

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

*for some $c\in(a,b)$, called mean value property of $f$.*

PROOF. In the case $f(a)=f(b)$, the result, called Rolle theorem, states that we have $f'(c)=0$ for some $c\in(a,b)$, and follows from Theorem 13.13. Now in what regards our statement, due to Lagrange, this follows from Rolle, applied to the following function:

$$g(x)=f(x)-\frac{f(b)-f(a)}{b-a}\cdot x$$

Indeed, we have $g(a)=g(b)$, due to our choice of the constant on the right, so we get $g'(c)=0$ for some $c\in(a,b)$, which translates into the formula in the statement. □

Back now to Theorem 13.13, this can be used in order to find the minimum and maximum of any differentiable function, and this method is best recalled as follows:

ALGORITHM 13.15. *In order to find the minimum and maximum of $f:[a,b]\to\mathbb{R}$:*
1. *Compute the derivative $f'$.*
2. *Solve the equation $f'(x)=0$.*
3. *Add $a,b$ to your set of solutions.*
4. *Compute $f(x)$, for all your solutions.*
5. *Compute the min/max of all these $f(x)$ values.*
6. *Then this is the min/max of your function.*

Needless to say, all this is very interesting, and powerful. The general problem in any type of applied mathematics is that of finding the minimum or maximum of some function, and we have now an algorithm for dealing with such questions. Very nice.

## 13b. Second derivatives

The derivative theory that we have is already quite powerful, and can be used in order to solve all sorts of interesting questions, but with a bit more effort, we can do better. Indeed, at a more advanced level, we can come up with the following notion:

DEFINITION 13.16. *We say that $f:\mathbb{R}\to\mathbb{R}$ is twice differentiable if it is differentiable, and its derivative $f':\mathbb{R}\to\mathbb{R}$ is differentiable too. The derivative of $f'$ is denoted*

$$f'':\mathbb{R}\to\mathbb{R}$$

*and is called second derivative of $f$.*

You might probably wonder why coming with this definition, which looks a bit abstract and complicated, instead of further developing the theory of the first derivative, which looks like something very reasonable and useful. Good point, and answer to this coming in a moment. But before that, let us get a bit familiar with $f''$. We first have:

INTERPRETATION 13.17. *The second derivative $f''(x) \in \mathbb{R}$ is the number which:*

(1) *Expresses the growth rate of the slope $f'(z)$ at the point $x$.*
(2) *Gives us the acceleration of the function $f$ at the point $x$.*
(3) *Computes how much different is $f(x)$, compared to $f(z)$ with $z \simeq x$.*
(4) *Tells us how much convex or concave is $f$, around the point $x$.*

So, this is the truth about the second derivative, making it clear that what we have here is a very interesting notion. In practice now, (1) follows from the usual interpretation of the derivative, as both a growth rate, and a slope. Regarding (2), this is some sort of reformulation of (1), using the intuitive meaning of the word "acceleration", with the relevant physics equations, due to Newton, being as follows:

$$v = \dot{x} \quad , \quad a = \dot{v}$$

Regarding now (3) in the above, this is something more subtle, of statistical nature, that we will clarify with some mathematics, in a moment. As for (4), this is something quite subtle too, that we will again clarify with some mathematics, in a moment.

In practice now, let us first compute the second derivatives of the functions that we are familiar with, see what we get. The result here, which is perhaps not very enlightening at this stage of things, but which certainly looks technically useful, is as follows:

PROPOSITION 13.18. *The second derivatives of the basic functions are as follows:*

(1) $(x^p)'' = p(p-1)x^{p-2}$.
(2) $\sin'' = -\sin$.
(3) $\cos'' = -\cos$.
(4) $\exp' = \exp$.
(5) $\log'(x) = -1/x^2$.

*Also, there are functions which are differentiable, but not twice differentiable.*

PROOF. We have several assertions here, the idea being as follows:

(1) Regarding the various formulae in the statement, these all follow from the various formulae for the derivatives established before, as follows:

$$(x^p)'' = (px^{p-1})' = p(p-1)x^{p-2}$$

$$(\sin x)'' = (\cos x)' = -\sin x$$

$$(\cos x)'' = (-\sin x)' = -\cos x$$

$$(e^x)'' = (e^x)' = e^x$$

$$(\log x)'' = (-1/x)' = -1/x^2$$

Of course, this is not the end of the story, because these formulae remain quite opaque, and must be examined in view of Interpretation 13.17, in order to see what exactly is going on. Also, we have tan and the inverse trigonometric functions too. In short, plenty of good exercises here, for you, and the more you solve, the better your calculus will be.

(2) Regarding now the counterexample, recall first that the simplest example of a function which is continuous, but not differentiable, was $f(x) = |x|$, the idea behind this being to use a "piecewise linear function whose branches do not fit well". In connection now with our question, piecewise linear will not do, but we can use a similar idea, namely "piecewise quadratic function whose branches do not fit well". So, let us set:

$$f(x) = \begin{cases} -x^2 & (x \leq 0) \\ x^2 & (x \geq 0) \end{cases}$$

The derivative is then $f'(x) = 2|x|$, which is not differentiable, as desired. □

Getting now to general theory, we first have the following key result:

THEOREM 13.19. *Any twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ is locally quadratic,*

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

*with $f''(x)$ being as usual the derivative of the function $f' : \mathbb{R} \to \mathbb{R}$ at the point $x$.*

PROOF. Assume indeed that $f$ is twice differentiable at $x$, and let us try to construct an approximation of $f$ around $x$ by a quadratic function, as follows:

$$f(x+t) \simeq a + bt + ct^2$$

We must have $a = f(x)$, and we also know from Theorem 13.6 that $b = f'(x)$ is the correct choice for the coefficient of $t$. Thus, our approximation must be as follows:

$$f(x+t) \simeq f(x) + f'(x)t + ct^2$$

In order to find the correct choice for $c \in \mathbb{R}$, observe that the function $t \to f(x+t)$ matches with $t \to f(x) + f'(x)t + ct^2$ in what regards the value at $t = 0$, and also in what regards the value of the derivative at $t = 0$. Thus, the correct choice of $c \in \mathbb{R}$ should be the one making match the second derivatives at $t = 0$, and this gives:

$$f''(x) = 2c$$

We are therefore led to the formula in the statement, namely:

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

In order to prove now that this formula holds indeed, we will use L'Hôpital's rule, which states that the $0/0$ type limits can be computed as follows:

$$\frac{f(x)}{g(x)} \simeq \frac{f'(x)}{g'(x)}$$

Observe that this formula holds indeed, as an application of Theorem 13.6. Now by using this, if we denote by $\varphi(t) \simeq P(t)$ the formula to be proved, we have:

$$\begin{aligned}\frac{\varphi(t)-P(t)}{t^2} &\simeq \frac{\varphi'(t)-P'(t)}{2t}\\ &\simeq \frac{\varphi''(t)-P''(t)}{2}\\ &= \frac{f''(x)-f''(x)}{2}\\ &= 0\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result substantially improves Theorem 13.6, and there are many applications of it. As a first such application, justifying Interpretation 13.17 (3), we have the following statement, which is a bit heuristic, but we will call it however Proposition:

PROPOSITION 13.20. *Intuitively speaking, the second derivative $f''(x) \in \mathbb{R}$ computes how much different is $f(x)$, compared to the average of $f(z)$, with $z \simeq x$.*

PROOF. As already mentioned, this is something a bit heuristic, but which is good to know. Let us write the formula in Theorem 13.19 as such, and with $t \to -t$ too:

$$f(x+t) \simeq f(x) + f'(x)t + \frac{f''(x)}{2}\,t^2$$

$$f(x-t) \simeq f(x) - f'(x)t + \frac{f''(x)}{2}\,t^2$$

By making the average, we obtain the following formula:

$$\frac{f(x+t)+f(x-t)}{2} \simeq f(x) + \frac{f''(x)}{2}\,t^2$$

But this is what our statement says, save for some uncertainties regarding the averaging method, and for the precise value of $I(t^2/2)$. We will leave this for later. □

Back to rigorous mathematics, we can improve as well Theorem 13.13, as follows:

THEOREM 13.21. *The local minima and local maxima of a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ appear at the points $x \in \mathbb{R}$ where*

$$f'(x) = 0$$

*with the local minima corresponding to the case $f'(x) \geq 0$, and with the local maxima corresponding to the case $f''(x) \leq 0$.*

PROOF. The first assertion is something that we already know. As for the second assertion, we can use the formula in Theorem 13.19, which in the case $f'(x) = 0$ reads:

$$f(x+t) \simeq f(x) + \frac{f''(x)}{2}\,t^2$$

Indeed, assuming $f''(x) \neq 0$, it is clear that the condition $f''(x) > 0$ will produce a local minimum, and that the condition $f''(x) < 0$ will produce a local maximum. □

As before with Theorem 13.13, the above result is not the end of the story with the mathematics of the local minima and maxima, because things are undetermined when:

$$f'(x) = f''(x) = 0$$

In answer, in such cases, the third derivative must be used. More on this later.

## 13c. Convex functions

As a main concrete application of the second derivative, which is something very useful in practice, and related to Interpretation 13.17 (4), we have the following result:

THEOREM 13.22 (Jensen). *Given a convex function $f : \mathbb{R} \to \mathbb{R}$, we have the following inequality, for any $x_1, \ldots, x_N \in \mathbb{R}$, and any $\lambda_1, \ldots, \lambda_N > 0$ summing up to 1,*

$$f(\lambda_1 x_1 + \ldots + \lambda_N x_N) \leq \lambda_1 f(x_1) + \ldots + \lambda_N x_N$$

*with equality when $x_1 = \ldots = x_N$. In particular, by taking the weights $\lambda_i$ to be all equal, we obtain the following inequality, valid for any $x_1, \ldots, x_N \in \mathbb{R}$,*

$$f\left(\frac{x_1 + \ldots + x_N}{N}\right) \leq \frac{f(x_1) + \ldots + f(x_N)}{N}$$

*and once again with equality when $x_1 = \ldots = x_N$. A similar statement holds for the concave functions, with all the inequalities being reversed.*

PROOF. This is indeed something quite routine, the idea being as follows:

(1) First, we can talk about convex functions in a usual, intuitive way, with this meaning by definition that the following inequality must be satisfied:

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

(2) But this means, via a simple argument, by approximating numbers $t \in [0,1]$ by sums of powers $2^{-k}$, that for any $t \in [0,1]$ we must have:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Alternatively, via yet another simple argument, this time by doing some geometry with triangles, this means that we must have:

$$f\left(\frac{x_1+\ldots+x_N}{N}\right)\leq\frac{f(x_1)+\ldots+f(x_N)}{N}$$

But then, again alternatively, by combining the above two simple arguments, the following must happen, for any $\lambda_1,\ldots,\lambda_N>0$ summing up to 1:

$$f(\lambda_1x_1+\ldots+\lambda_Nx_N)\leq\lambda_1f(x_1)+\ldots+\lambda_Nx_N$$

(3) Summarizing, all our Jensen inequalities, at $N=2$ and at $N\in\mathbb N$ arbitrary, are equivalent. The point now is that, if we look at what the first Jensen inequality, that we took as definition for the convexity, exactly means, this is simply equivalent to:

$$f''(x)\geq0$$

(4) Thus, we are led to the conclusions in the statement, regarding the convex functions. As for the concave functions, the proof here is similar. Alternatively, we can say that $f$ is concave precisely when $-f$ is convex, and get the results from what we have. □

As a basic application of the Jensen inequality, widely useful in practice, we have:

THEOREM 13.23 (Young). *We have the following inequality,*

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}$$

*valid for any $a,b\geq0$, and any exponents $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$.*

PROOF. We use the logarithm function, which is concave on $(0,\infty)$, due to:

$$(\log x)''=\left(-\frac{1}{x}\right)'=-\frac{1}{x^2}$$

Thus we can apply the Jensen inequality, and we obtain in this way:

$$\begin{aligned}\log\left(\frac{a^p}{p}+\frac{b^q}{q}\right)&\geq\frac{\log(a^p)}{p}+\frac{\log(b^q)}{q}\\&=\log(a)+\log(b)\\&=\log(ab)\end{aligned}$$

Now by exponentiating, we obtain the Young inequality. □

Observe that for the simplest exponents, namely $p=q=2$, the Young inequality gives something which is trivial, but is very useful and basic, namely:

$$ab\leq\frac{a^2+b^2}{2}$$

Things over with this, you would say? You must be kidding, because as a key application now of the Young inequality, also widely useful in practice, we have:

THEOREM 13.24 (Hölder). *Assuming that $p,q\geq 1$ are conjugate, in the sense that*

$$\frac{1}{p}+\frac{1}{q}=1$$

*we have the following inequality, valid for any two vectors $x,y\in\mathbb C^N$,*

$$\sum_i|x_iy_i|\leq\left(\sum_i|x_i|^p\right)^{1/p}\left(\sum_i|y_i|^q\right)^{1/q}$$

*with the convention that an $\infty$ exponent produces a $\max|x_i|$ quantity.*

PROOF. This is something very standard, the idea being as follows:

(1) Assume first that we are dealing with finite exponents, $p,q\in(1,\infty)$. By linearity we can assume that $x,y$ are normalized, in the following way:

$$\sum_i|x_i|^p=\sum_i|y_i|^q=1$$

But in this case, by applying Young and summing we obtain, as desired:

$$\begin{aligned}\sum_i|x_iy_i|&\leq\sum_i\frac{|x_i|^p}{p}+\sum_i\frac{|y_i|^q}{q}\\&=\frac{1}{p}+\frac{1}{q}\\&=1\end{aligned}$$

(2) In the case $p=1$ and $q=\infty$, or vice versa, the inequality holds too, trivially, with the convention that an $\infty$ exponent produces a max quantity, according to:

$$\lim_{p\to\infty}\left(\sum_i|x_i|^p\right)^{1/p}=\max|x_i|$$

Thus, we are led to the conclusion in the statement. □

As a consequence now of the Hölder inequality, we have:

THEOREM 13.25 (Minkowski). *Assuming $p\in[1,\infty]$, we have the inequality*

$$\left(\sum_i|x_i+y_i|^p\right)^{1/p}\leq\left(\sum_i|x_i|^p\right)^{1/p}+\left(\sum_i|y_i|^p\right)^{1/p}$$

*for any two vectors $x,y\in\mathbb C^N$, with our usual conventions at $p=\infty$.*

PROOF. We have indeed the following estimate, using the Hölder inequality, and the conjugate exponent $q\in[1,\infty]$, given by $1/p+1/q=1$:

$$\begin{aligned}\sum_i|x_i+y_i|^p&=\sum_i|x_i+y_i|\cdot|x_i+y_i|^{p-1}\\&\leq\sum_i|x_i|\cdot|x_i+y_i|^{p-1}+\sum_i|y_i|\cdot|x_i+y_i|^{p-1}\\&\leq\left(\sum_i|x_i|^p\right)^{1/p}\left(\sum_i|x_i+y_i|^{(p-1)q}\right)^{1/q}\\&+\left(\sum_i|y_i|^p\right)^{1/p}\left(\sum_i|x_i+y_i|^{(p-1)q}\right)^{1/q}\\&=\left[\left(\sum_i|x_i|^p\right)^{1/p}+\left(\sum_i|y_i|^p\right)^{1/p}\right]\left(\sum_i|x_i+y_i|^p\right)^{1-1/p}\end{aligned}$$

Here we have used the following fact, at the end:

$$\frac{1}{p}+\frac{1}{q}=1\implies\frac{1}{q}=\frac{p-1}{p}\implies(p-1)q=p$$

Thus, we are led to the inequality in the statement. □

Good news, done with inequalities, and as a consequence of our results, and more specifically of the Minkowski inequality obtained above, we can formulate:

THEOREM 13.26. *Given an exponent $p\in[1,\infty]$, the formula*

$$||x||_p=\left(\sum_i|x_i|^p\right)^{1/p}$$

*with our usual conventions at $p=\infty$, defines a norm on $\mathbb{C}^N$.*

PROOF. This follows indeed from Minkowski, and with the norm axioms being by definition something intuitive, inspired from the properties of the length of vectors $x\in\mathbb{R}^N$, namely $||x||>0$ for $x\neq0$, $||\lambda x||=|\lambda|\cdot||x||$, and $||x+y||\leq||x||+||y||$. □

And with this, we are now experts in functional analysis. If you ever fail solving your problem by using the usual distance $||.||_2$, switch to $||.||_p$, with a suitably chosen $p$.

## 13d. Taylor formula

Back now to the general theory of the derivatives, and their theoretical applications, we can further develop our basic approximation method, at order 3, at order 4, and so on, the ultimate result on the subject, called Taylor formula, being as follows:

THEOREM 13.27. *Any function $f:\mathbb{R}\to\mathbb{R}$ can be locally approximated as*

$$f(x+t)=\sum_{k=0}^\infty\frac{f^{(k)}(x)}{k!}\,t^k$$

*where $f^{(k)}(x)$ are the higher derivatives of $f$ at the point $x$.*

PROOF. Consider the function to be approximated, namely:

$$\varphi(t)=f(x+t)$$

Let us try to best approximate this function at a given order $n\in\mathbb{N}$. We are therefore looking for a certain polynomial in $t$, of the following type:

$$P(t)=a_0+a_1t+\ldots+a_nt^n$$

The natural conditions to be imposed are those stating that $P$ and $\varphi$ should match at $t=0$, at the level of the actual value, of the derivative, second derivative, and so on up the $n$-th derivative. Thus, we are led to the approximation in the statement:

$$f(x+t)\simeq\sum_{k=0}^n\frac{f^{(k)}(x)}{k!}\,t^k$$

In order to prove now that this approximation holds indeed, we can use L'Hôpital's rule, applied several times, as in the proof of Theorem 13.19. To be more precise, if we denote by $\varphi(t)\simeq P(t)$ the approximation to be proved, we have:

$$\begin{aligned}\frac{\varphi(t)-P(t)}{t^n}&\simeq\frac{\varphi'(t)-P'(t)}{nt^{n-1}}\\&\simeq\frac{\varphi''(t)-P''(t)}{n(n-1)t^{n-2}}\\&\ \vdots\\&\simeq\frac{\varphi^{(n)}(t)-P^{(n)}(t)}{n!}\\&=\frac{f^{(n)}(x)-f^{(n)}(x)}{n!}\\&=0\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Here is a related interesting statement, inspired from the above proof:

PROPOSITION 13.28. *For a polynomial of degree $n$, the Taylor approximation*

$$f(x+t)\simeq\sum_{k=0}^n\frac{f^{(k)}(x)}{k!}\,t^k$$

*is an equality. The converse of this statement holds too.*

Proof. By linearity, it is enough to check the equality in question for the monomials $f(x)=x^p$, with $p\leq n$. But here, the formula to be proved is as follows:

$$(x+t)^p\simeq\sum_{k=0}^p\frac{p(p-1)\ldots(p-k+1)}{k!}\,x^{p-k}t^k$$

We recognize the binomial formula, so our result holds indeed. As for the converse, this is clear, because the Taylor approximation is a polynomial of degree $n$. $\square$

There are many other things that can be said about the Taylor formula, at the theoretical level, notably with a study of the remainder, when truncating this formula at a given order $n\in\mathbb{N}$. We will be back to this, later in this book, towards the end.

In relation now with the local extrema, we have the following result:

Theorem 13.29. *Given a differentiable function $f:\mathbb{R}\to\mathbb{R}$, we can always write*

$$f(x+t)\simeq f(x)+\frac{f^{(n)}(x)}{n!}\,t^n$$

*with $f^{(n)}(x)\neq 0$, and this tells us if $x$ is a local minimum, or maximum of $f$.*

Proof. This is indeed something self-explanatory, coming from Theorem 13.27, with the number $n\in\mathbb{N}$ in question being the smallest one such that $f^{(n)}(x)\neq 0$. $\square$

As a concrete application now of the Taylor formula, we have:

Theorem 13.30. *We have the following formulae,*

$$e^x=\sum_{k=0}^\infty\frac{x^k}{k!}\quad,\quad \log(1+x)=\sum_{k=0}^\infty(-1)^{k+1}\frac{x^k}{k}$$

*as well as the following formulae,*

$$\sin x=\sum_{l=0}^\infty(-1)^l\frac{x^{2l+1}}{(2l+1)!}\quad,\quad \cos x=\sum_{l=0}^\infty(-1)^l\frac{x^{2l}}{(2l)!}$$

*as Taylor series, and in general as well, with $x\in(-1,1]$ needed for* log.

Proof. There are several assertions here, the proofs being as follows:

(1) Regarding the Taylor series statements, we can use here the following formulae:

$$(e^x)'=e^x\quad,\quad(\log x)'=x^{-1}$$

$$(\sin x)'=\cos x\quad,\quad(\cos x)'=-\sin x$$

Thus we can differentiate exp, log, sin, cos, as many times as we want to, and compute the corresponding Taylor series, and we obtain the formulae in the statement.

(2) Regarding now the convergence away from 0, we already know that this happens for $e^x$. In order to discuss sin and cos, we will need the Euler formula, namely:

$$e^{ix} = \cos x + i \sin x$$

To be more precise, this is a formula that we more or less established in chapter 11, with the promise to come back later to it, after learning calculus. So, let us set:

$$f(x) = \frac{\cos x + i \sin x}{e^{ix}}$$

The point is that we can compute the derivative of $f$, and we obtain:

$$\begin{aligned} f'(x) &= (e^{-ix}(\cos x + i \sin x))' \\ &= -ie^{-ix}(\cos x + i \sin x) + e^{-ix}(-\sin x + i \cos x) \\ &= e^{-ix}(-i \cos x + \sin x) + e^{-ix}(-\sin x + i \cos x) \\ &= 0 \end{aligned}$$

We conclude from this that $f$ is constant, equal to $f(0) = 1$, as desired.

(3) Getting back now to our questions regarding sin and cos, we have the following computation, valid for any $x \in \mathbb{R}$, based on the usual formula of the exponential:

$$\begin{aligned} e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \\ &= \sum_{l=0}^{\infty} \frac{(ix)^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{(ix)^{2l+1}}{(2l+1)!} \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l}}{(2l)!} + i \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+1}}{(2l+1)!} \end{aligned}$$

Now by comparing this with $e^{ix} = \cos x + i \sin x$, we obtain, for any $x \in \mathbb{R}$:

$$\cos x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l}}{(2l)!} \quad , \quad \sin x = \sum_{l=0}^{\infty} (-1)^l \frac{x^{2l+1}}{(2l+1)!}$$

(4) Finally, in what regards the logarithm, we know that we have, as Taylor series:

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

By using now the general theory of series, from basic analysis, we can see that this series does obviously not converge for $|x| > 1$, nor at $x = -1$. Thus, we are left with the question whether the above formula holds or not, at $x \in (-1, 1]$.

(5) And in order to prove that it is so, we must check one of the following formulae, with $\log(1+x)$ standing here by definition for the above Taylor series of $\log(1+x)$:

$$\exp(\log(1+x)) = 1+x \quad , \quad \log(\exp(x)) = x$$

But this can be done indeed, with some patience, and we will leave the computations here, based on the binomial formula, as an instructive exercise. □

As another application of our Taylor formula technology, we have:

THEOREM 13.31. *We have the following generalized binomial formula, with $p \in \mathbb{R}$,*

$$(x+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^{p-k} t^k$$

*with the generalized binomial coefficients being given by the formula*

$$\binom{p}{k} = \frac{p(p-1)\ldots(p-k+1)}{k!}$$

*valid for any $|t| < |x|$. With $p \in \mathbb{N}$, we recover the usual binomial formula.*

PROOF. It is customary to divide everything by $x$, which is the same as assuming $x = 1$. The formula to be proved is then as follows, under the assumption $|t| < 1$:

$$(1+t)^p = \sum_{k=0}^{\infty} \binom{p}{k} t^k$$

Let us discuss now the validity of this formula, depending on $p \in \mathbb{R}$:

(1) Case $p \in \mathbb{N}$. According to our definition of the generalized binomial coefficients, we have $\binom{p}{k} = 0$ for $k > p$, so the series is stationary, and the formula to be proved is:

$$(1+t)^p = \sum_{k=0}^{p} \binom{p}{k} t^k$$

But this is the usual binomial formula, which holds for any $t \in \mathbb{R}$.

(2) Case $p = -1$. Here we can use the following formula, valid for $|t| < 1$:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots$$

But this is exactly our generalized binomial formula at $p = -1$, because:

$$\binom{-1}{k} = \frac{(-1)(-2)\ldots(-k)}{k!} = (-1)^k$$

(3) Case $p \in -\mathbb{N}$. With $p = -m$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{-m}{k} &= \frac{(-m)(-m-1)\ldots(-m-k+1)}{k!} \\ &= (-1)^k\frac{(m+k-1)!}{(m-1)!k!} \\ &= (-1)^k\binom{m+k-1}{m-1}\end{aligned}$$

Thus, our generalized binomial formula at $p = -m$ reads:

$$\frac{1}{(1+t)^m} = \sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{m-1}t^k$$

But this is something standard, and we will leave the proof as an exercise.

(4) General case, $p \in \mathbb{R}$. As we can see, things escalate quickly, so we will skip the next step, $p \in \mathbb{Q}$, and discuss directly the case $p \in \mathbb{R}$. Consider the following function:

$$f(x) = x^p$$

The derivatives at $x = 1$ are then given by the following formula:

$$f^{(k)}(1) = p(p-1)\ldots(p-k+1)$$

Thus, the Taylor approximation at $x = 1$ is as follows:

$$f(1+t) = \sum_{k=0}^{\infty}\frac{p(p-1)\ldots(p-k+1)}{k!}\,t^k$$

But this is exactly our generalized binomial formula, so we are done with the case where $t$ is small. As for the general case, which reads $|t| < 1$ with our normalization $x = 1$ above, this follows from $(1+t)^p = \exp(p\log(1+t))$, using Theorem 13.30. □

As a main application now of our generalized binomial formula, we have:

THEOREM 13.32. *We have the following formula,*

$$\sqrt{1+t} = 1 - 2\sum_{k=1}^{\infty}C_{k-1}\left(\frac{-t}{4}\right)^k$$

*with $C_k = \frac{1}{k+1}\binom{2k}{k}$ being the Catalan numbers. Also, we have*

$$\frac{1}{\sqrt{1+t}} = \sum_{k=0}^{\infty}D_k\left(\frac{-t}{4}\right)^k$$

*with $D_k = \binom{2k}{k}$ being the central binomial coefficients.*

PROOF. At $p = 1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{1/2}{k} &= \frac{1/2(-1/2)\ldots(3/2-k)}{k!}\\ &= (-1)^{k-1}\frac{(2k-2)!}{2^{k-1}(k-1)!2^k k!}\\ &= -2\left(\frac{-1}{4}\right)^k C_{k-1}\end{aligned}$$

Also, at $p = -1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{-1/2}{k} &= \frac{-1/2(-3/2)\ldots(1/2-k)}{k!}\\ &= (-1)^k\frac{(2k)!}{2^k k! 2^k k!}\\ &= \left(\frac{-1}{4}\right)^k D_k\end{aligned}$$

Thus, Theorem 13.31 at $p = \pm 1/2$ gives the formulae in the statement. □

Quite nice all this. Eventually, we learned how to extract square roots. We will see many other concrete applications of calculus, in what follows.

## 13e. Exercises

Good to have you here, at this calculus chapter, and as exercises, we have:

EXERCISE 13.33. *Clarify all the details in the proof of $(x^p)' = px^{p-1}$.*

EXERCISE 13.34. *Compute the derivatives of remaining trigonometric functions.*

EXERCISE 13.35. *Compute the second derivatives of all trigonometric functions.*

EXERCISE 13.36. *Further meditate on the interpretations of the second derivative.*

EXERCISE 13.37. *Clarify everything that we said, in relation with convex functions.*

EXERCISE 13.38. *Compute the third and fourth derivatives of all basic functions.*

EXERCISE 13.39. *Learn more about the general Taylor formula, and its remainder.*

EXERCISE 13.40. *See what happens to the binomial formula, at exponents $p \in \mathbb{Z}/3$.*

As bonus exercise, recommended, find and solve 100 basic calculus exercises.

CHAPTER 14

# Trigonometric functions

## 14a. First derivatives

Time to see how our calculus technology works for the trigonometric functions. Let us start with the following key result, that we already know from chapter 13:

THEOREM 14.1. *We have the following formulae,*

$$(\sin x)' = \cos x \quad , \quad (\cos x)' = -\sin x \quad , \quad (\tan x)' = \frac{1}{\cos^2 x}$$

*provided that the denominator at right does not vanish.*

PROOF. This is something that we know from chapter 13, but as a matter of having the present chapter rather self-contained, let us briefly recall the proofs:

(1) Regarding sin, the computation here goes as follows, using the basic estimate $\sin t \simeq t$ for $t \simeq 0$, along with $\cos t \simeq 1 - t^2/2$, coming from it via Pythagoras:

$$\begin{aligned}
(\sin x)' &= \lim_{t\to 0} \frac{\sin(x+t) - \sin x}{t} \\
&= \lim_{t\to 0} \frac{\sin x \cos t + \cos x \sin t - \sin x}{t} \\
&= \lim_{t\to 0} \sin x \cdot \frac{\cos t - 1}{t} + \cos x \cdot \frac{\sin t}{t} \\
&= \cos x
\end{aligned}$$

(2) The computation for cos is similar, based on the same ingredients, as follows:

$$\begin{aligned}
(\cos x)' &= \lim_{t\to 0} \frac{\cos(x+t) - \cos x}{t} \\
&= \lim_{t\to 0} \frac{\cos x \cos t - \sin x \sin t - \cos x}{t} \\
&= \lim_{t\to 0} \cos x \cdot \frac{\cos t - 1}{t} - \sin x \cdot \frac{\sin t}{t} \\
&= -\sin x
\end{aligned}$$

(3) For the tangent, by using the rules in chapter 13, we have indeed:

$$\begin{aligned}(\tan x)' &= \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\end{aligned}$$

(4) Alternatively, we can get this by using $\tan t \simeq t$ for $t \simeq 0$, as follows:

$$\begin{aligned}(\tan x)' &= \lim_{t\to 0}\frac{\tan(x+t)-\tan x}{t}\\ &= \lim_{t\to 0}\frac{\frac{\tan x+\tan t}{1-\tan x\tan t}-\tan x}{t}\\ &= \lim_{t\to 0}\frac{\tan t+\tan^2 x\tan t}{t(1-\tan x\tan t)}\\ &= \lim_{t\to 0}\frac{\tan t+\tan^2 x\tan t}{t}\\ &= 1+\tan^2 x\\ &= \frac{1}{\cos^2 x}\end{aligned}$$

Thus, we are led to the formulae in the statement. □

As a comment now, observe that the formula for the tangent can be written as follows, in terms of the secant function, and with this looking like an improvement:

$$(\tan x)' = \sec^2 x$$

However, it is better not to do so, and this for a quite subtle reason, as follows:

FACT 14.2. *We will learn later that the operation inverse to $f \to f'$, called integration, is something interesting too, and in view of this, it is better to express our $f'$ functions in terms of* $\sin, \cos$ *only, for subsequent quick identification and integration, when needed.*

Well, hope you get my point, while the various secondary trigonometric functions are certainly very interesting objects, worth the study, and valuable as input for our present derivative computations, they are not recommended as output, for the above reasons.

Talking now secondary trigonometric functions, we first have:

THEOREM 14.3. *We have the following formulae,*

$$(\sec x)' = \frac{\sin x}{\cos^2 x} \quad , \quad (\csc x)' = -\frac{\cos x}{\sin^2 x} \quad , \quad (\cot x)' = -\frac{1}{\sin^2 x}$$

*provided that the denominators do not vanish.*

Proof. For the secant, we have the following computation:

$$\begin{aligned}(\sec x)' &= \left(\frac{1}{\cos x}\right)'\\ &= -\frac{\cos' x}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

For the cosecant, we have a similar computation, as follows:

$$\begin{aligned}(\csc x)' &= \left(\frac{1}{\sin x}\right)'\\ &= -\frac{\sin' x}{\sin^2 x}\\ &= -\frac{\cos x}{\sin^2 x}\end{aligned}$$

For the cotangent, we have the following computation, as for the tangent:

$$\begin{aligned}(\cot x)' &= \left(\frac{\cos x}{\sin x}\right)'\\ &= \frac{\cos' x\sin x - \cos x\sin' x}{\sin^2 x}\\ &= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\\ &= -\frac{1}{\sin^2 x}\end{aligned}$$

Alternatively, we can use our previous formula for the tangent, and we obtain:

$$\begin{aligned}(\cot x)' &= \left(\frac{1}{\tan x}\right)'\\ &= -\frac{\tan' x}{\tan^2 x}\\ &= -\frac{1/\cos^2 x}{\sin^2 x/\cos^2 x}\\ &= -\frac{1}{\sin^2 x}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Time now for the inverse trigonometric functions. We have here:

THEOREM 14.4. *The derivatives of basic inverse trigonometric functions are*

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad , \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \quad , \quad (\arctan x)' = \frac{1}{1+x^2}$$

*and the derivatives of secondary inverse trigonometric functions are*

$$(\mathrm{arcsec}\, x)' = \frac{1}{|x|\sqrt{x^2-1}} \ , \ (\mathrm{arccsc}\, x)' = -\frac{1}{|x|\sqrt{x^2-1}} \ , \ (\mathrm{arccot}\, x)' = -\frac{1}{1+x^2}$$

*provided that the denominators do not vanish.*

PROOF. This is something routine, by using what we already have, along with the formula $(f \circ g)' = (f' \circ g) \cdot g'$ from chapter 13, as follows:

(1) For the arcsine, we can use the following computation:

$$\begin{aligned}(\sin \circ \arcsin)'(x) &= \sin'(\arcsin x) \arcsin'(x) \\ &= \cos(\arcsin x)\ \arcsin'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x' = 1$, we obtain from this:

$$\arcsin'(x) = \frac{1}{\cos(\arcsin x)}$$

On the other hand, with $t = \arcsin x$ we know that we have $\sin t = x$, and so:

$$\begin{aligned}\cos(\arcsin x) &= \cos t \\ &= \sqrt{1-\sin^2 t} \\ &= \sqrt{1-x^2}\end{aligned}$$

Thus, we are led to the formula in the statement, namely:

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

(2) For the arcosine, we have a similar computation, as follows:

$$\begin{aligned}(\cos \circ \arccos)'(x) &= \cos'(\arccos x) \arccos'(x) \\ &= -\sin(\arccos x)\ \arccos'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x' = 1$, we obtain from this:

$$\arccos'(x) = -\frac{1}{\sin(\arccos x)}$$

On the other hand, with $t = \arccos x$ we know that we have $\cos t = x$, and so:

$$\begin{aligned}\sin(\arccos x) &= \sin t \\ &= \sqrt{1-\cos^2 t} \\ &= \sqrt{1-x^2}\end{aligned}$$

Thus, we are led to the formula in the statement, namely:

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

(3) For the arctangent, we can use the following computation:

$$\begin{aligned}(\tan\circ\arctan)'(x) &= \tan'(\arctan x)\arctan'(x)\\ &= \frac{1}{\cos^2(\arctan x)}\arctan'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x'=1$, we obtain from this:

$$\arctan'(x) = \cos^2(\arctan x)$$

On the other hand, with $t=\arctan x$ we know that we have $\tan t = x$, and so:

$$\begin{aligned}\cos^2(\arctan x) &= \cos^2 t\\ &= \frac{1}{1+\tan^2 t}\\ &= \frac{1}{1+x^2}\end{aligned}$$

Thus, we are led to the formula in the statement, namely:

$$(\arctan x)' = \frac{1}{1+x^2}$$

(4) For the arcsecant, we can use the following computation:

$$\begin{aligned}(\sec\circ\operatorname{arcsec})'(x) &= \sec'(\operatorname{arcsec} x)\operatorname{arcsec}'(x)\\ &= \frac{\sin(\operatorname{arcsec} x)}{\cos^2(\operatorname{arcsec} x)}\operatorname{arcsec}'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x'=1$, we obtain from this:

$$\operatorname{arcsec}'(x) = \frac{\cos^2(\operatorname{arcsec} x)}{\sin(\operatorname{arcsec} x)}$$

On the other hand, with $t=\operatorname{arcsec} x$ we know that we have $\sec t = x$, and so:

$$\cos(\operatorname{arcsec} x) = \cos t = \frac{1}{x}$$

As for the sine of the arcsecant, we can compute it as well, as follows:

$$\begin{aligned}\sin(\operatorname{arcsec} x) &= \sin t\\ &= \sqrt{1-\cos^2 t}\\ &= \sqrt{1-\frac{1}{x^2}}\\ &= \frac{\sqrt{x^2-1}}{|x|}\end{aligned}$$

Thus, we are led to the formula in the statement, namely:

$$\begin{aligned}(\operatorname{arcsec} x)' &= \frac{\cos^2(\operatorname{arcsec} x)}{\sin(\operatorname{arcsec} x)}\\ &= \frac{1}{x^2}\cdot\frac{|x|}{\sqrt{x^2-1}}\\ &= \frac{1}{|x|\sqrt{x^2-1}}\end{aligned}$$

(5) For the arcosecant, we can use the following computation:

$$\begin{aligned}(\csc\circ\operatorname{arccsc})'(x) &= \csc'(\operatorname{arccsc} x)\operatorname{arccsc}'(x)\\ &= -\frac{\cos(\operatorname{arcsec} x)}{\sin^2(\operatorname{arccsc} x)}\operatorname{arccsc}'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x'=1$, we obtain from this:

$$\operatorname{arccsc}'(x) = -\frac{\sin^2(\operatorname{arccsc} x)}{\cos(\operatorname{arccsc} x)}$$

On the other hand, with $t=\operatorname{arccsc} x$ we know that we have $\csc t = x$, and so:

$$\sin(\operatorname{arccsc} x) = \sin t = \frac{1}{x}$$

As for the cosine of the arcosecant, we can compute it as well, as follows:

$$\begin{aligned}\cos(\operatorname{arccsc} x) &= \cos t\\ &= \sqrt{1-\sin^2 t}\\ &= \sqrt{1-\frac{1}{x^2}}\\ &= \frac{\sqrt{x^2-1}}{|x|}\end{aligned}$$

Thus, we are led to the formula in the statement, namely:

$$\begin{aligned}(\operatorname{arccsc} x)' &= -\frac{\sin^2(\operatorname{arccsc} x)}{\cos(\operatorname{arccsc} x)}\\ &= -\frac{1}{x^2}\cdot\frac{|x|}{\sqrt{x^2-1}}\\ &= -\frac{1}{|x|\sqrt{x^2-1}}\end{aligned}$$

(6) For the arcotangent, we can use the following computation:

$$\begin{aligned}(\cot\circ\operatorname{arccot})'(x) &= \cot'(\operatorname{arccot} x)\operatorname{arccot}'(x)\\ &= -\frac{1}{\sin^2(\operatorname{arccot} x)}\operatorname{arccot}'(x)\end{aligned}$$

Indeed, since the term on the left is simply $x'=1$, we obtain from this:

$$\operatorname{arccot}'(x) = -\sin^2(\operatorname{arccot} x)$$

On the other hand, with $t=\operatorname{arccot} x$ we know that we have $\cot t = x$, and so:

$$\begin{aligned}\sin^2(\operatorname{arccot} x) &= \sin^2 t\\ &= \frac{1}{1+\cot^2 t}\\ &= \frac{1}{1+x^2}\end{aligned}$$

Thus, we are led to the formula in the statement, namely:

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

And so, theorem proved, we are now experts in computing derivatives. □

## 14b. Higher derivatives

In what regards now the second derivatives, we first have the following result:

THEOREM 14.5. *We have the following formulae,*

$$(\sin x)'' = -\sin x \quad , \quad (\cos x)'' = -\cos x \quad , \quad (\tan x)'' = \frac{2\sin x}{\cos^3 x}$$

*as well as the following formulae,*

$$(\sec x)'' = \frac{1+\sin^2 x}{\cos^3 x} \quad , \quad (\csc x)'' = \frac{1+\cos^2 x}{\sin^3 x} \quad , \quad (\cot x)'' = \frac{2\cos x}{\sin^3 x}$$

*provided that the denominators do not vanish.*

PROOF. This comes from the formulae in Theorem 14.1 and Theorem 14.3, with help from the general derivation formulae from chapter 13 when needed, as follows:

(1) The formula for the sine is clear, coming as follows:

$$(\sin x)'' = (\cos x)' = -\sin x$$

(2) The formula for the cosine is clear too, coming as follows:

$$(\cos x)'' = (-\sin x)' = -\cos x$$

(3) For the tangent, we have the following computation:

$$\begin{aligned}(\tan x)'' &= \left(\frac{1}{\cos^2 x}\right)' \\ &= -\frac{(\cos^2 x)'}{\cos^4 x} \\ &= \frac{2\cos x \sin x}{\cos^4 x} \\ &= \frac{2\sin x}{\cos^3 x}\end{aligned}$$

(4) For the secant, we have the following computation:

$$\begin{aligned}(\sec x)'' &= \left(\frac{\sin x}{\cos^2 x}\right)' \\ &= \frac{\sin' x\cos^2 x - \sin x(\cos^2 x)'}{\cos^4 x} \\ &= \frac{\cos x\cdot\cos^2 x + \sin x\cdot 2\cos x\sin x}{\cos^4 x} \\ &= \frac{\cos^2 x + 2\sin^2 x}{\cos^3 x} \\ &= \frac{1+\sin^2 x}{\cos^3 x}\end{aligned}$$

(5) For the cosecant, we have a similar computation, as follows:

$$\begin{aligned}(\csc x)'' &= \left(-\frac{\cos x}{\sin^2 x}\right)' \\ &= -\frac{\cos' x \sin^2 x - \cos x(\sin^2 x)'}{\sin^4 x} \\ &= \frac{\sin x \cdot \sin^2 x + \cos x \cdot 2 \sin x \cos x}{\sin^4 x} \\ &= \frac{\sin^2 x + 2\cos^2 x}{\sin^3 x} \\ &= \frac{1 + \cos^2 x}{\sin^3 x}\end{aligned}$$

(6) For the cotangent, we have the following computation, as for the tangent:

$$\begin{aligned}(\cot x)'' &= \left(-\frac{1}{\sin^2 x}\right)' \\ &= \frac{(\sin^2 x)'}{\sin^4 x} \\ &= \frac{2\sin x \cos x}{\sin^4 x} \\ &= \frac{2\cos x}{\sin^3 x}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Regarding now the inverse trigonometric functions, we have here:

THEOREM 14.6. *The second derivatives of basic inverse trigonometric functions are*

$$(\arcsin x)'' = \frac{x}{(1-x^2)^{3/2}} \ , \ (\arccos x)'' = -\frac{x}{(1-x^2)^{3/2}} \ , \ (\arctan x)'' = -\frac{2x}{(1+x^2)^2}$$

*and the second derivatives of secondary inverse trigonometric functions are*

$$(\text{arcsec}\, x)'' = \frac{|x|(1-2x^2)}{x^3(x^2-1)^{3/2}} \ , \ (\text{arccsc}\, x)'' = -\frac{|x|(1-2x^2)}{x^3(x^2-1)^{3/2}} \ , \ (\text{arccot}\, x)'' = \frac{2x}{(1+x^2)^2}$$

*provided that the denominators do not vanish.*

PROOF. This is routine, by using the formulae from Theorem 14.4, as follows:

(1) For the arcsine, the computation is as follows:

$$\begin{aligned}(\arcsin x)'' &= \left(\frac{1}{\sqrt{1-x^2}}\right)' \\ &= -\frac{\sqrt{1-x^2}'}{1-x^2} \\ &= \frac{x/\sqrt{1-x^2}}{1-x^2} \\ &= \frac{x}{(1-x^2)^{3/2}}\end{aligned}$$

(2) For the arcosine the computation, using the one above, is as follows:

$$(\arccos x)'' = \left(-\frac{1}{\sqrt{1-x^2}}\right)' = -\frac{x}{(1-x^2)^{3/2}}$$

(3) For the arctangent, the computation is as follows:

$$(\arctan x)'' = \left(\frac{1}{1+x^2}\right)' = -\frac{2x}{(1+x^2)^2}$$

(4) For the arcsecant, the computation is as follows:

$$\begin{aligned}(\mathrm{arcsec}\, x)'' &= \left(\frac{1}{|x|\sqrt{x^2-1}}\right)' \\ &= -\frac{(|x|\sqrt{x^2-1})'}{x^2(x^2-1)} \\ &= -\frac{|x|'\sqrt{x^2-1}+|x|\sqrt{x^2-1}'}{x^2(x^2-1)} \\ &= -\frac{sgn(x)\sqrt{x^2-1}+|x|x/\sqrt{x^2-1}}{x^2(x^2-1)} \\ &= -\frac{sgn(x)(x^2-1)+|x|x}{x^2(x^2-1)^{3/2}} \\ &= -\frac{|x|(x^2-1)+|x|x^2}{x^3(x^2-1)^{3/2}} \\ &= \frac{|x|(1-2x^2)}{x^3(x^2-1)^{3/2}}\end{aligned}$$

(5) For the arcosecant the computation, using the one above, is as follows:

$$(\mathrm{arccsc}\, x)'' = \left(-\frac{1}{|x|\sqrt{x^2-1}}\right)' = -\frac{|x|(1-2x^2)}{x^3(x^2-1)^{3/2}}$$

(6) For the arcotangent the computation, using the one for the tangent, is:

$$(\text{arccot}\, x)'' = \left(-\frac{1}{1+x^2}\right)' = \frac{2x}{(1+x^2)^2}$$

Thus, we are led to the formulae in the statement. □

Regarding now the third derivatives, for the sine and cosine we have:

$$(\sin x)''' = (-\sin x)' = -\cos x$$

$$(\cos x)''' = (-\cos x)' = \sin x$$

As for the fourth derivatives of the sine and cosine, these are as follows:

$$(\sin x)'''' = (-\cos x)' = \sin x$$

$$(\cos x)'''' = (\sin x)' = \cos x$$

In view of this, which looks interesting, let us see as well what happens for the tangent. However, the result here is as follows, making it clear that we have no periodicity:

THEOREM 14.7. *The first two derivatives of the tangent function are*

$$(\tan x)' = \frac{1}{\cos^2 x} \quad , \quad (\tan x)'' = \frac{2\sin x}{\cos^3 x}$$

*and the third and fourth derivatives are*

$$(\tan x)''' = \frac{2+4\sin^2 x}{\cos^4 x} \quad , \quad (\tan x)'''' = \frac{16\sin x + 8\sin^3 x}{\cos^5 x}$$

*provided that the denominators do not vanish.*

PROOF. We already know the first two formulae, from Theorem 14.1 and Theorem 14.5. Regarding now the third formula, the computation here is as follows:

$$\begin{aligned}
(\tan x)''' &= \left(\frac{2\sin x}{\cos^3 x}\right)' \\
&= \frac{2\cos x \cdot \cos^3 x + 2\sin x \cdot 3\cos^2 x \sin x}{\cos^6 x} \\
&= \frac{2\cos^2 x + 6\sin^2 x}{\cos^4 x} \\
&= \frac{2+4\sin^2 x}{\cos^4 x}
\end{aligned}$$

As for the fourth formula, the computation here is as follows:

$$\begin{aligned}(\tan x)'''' &= \left(\frac{2+4\sin^2 x}{\cos^4 x}\right)'\\ &= \frac{8\sin x\cos x\cdot\cos^4 x+(2+4\sin^2 x)\cdot 4\cos^3 x\sin x}{\cos^8 x}\\ &= \frac{8\sin x\cos^2 x+(2+4\sin^2 x)\cdot 4\sin x}{\cos^5 x}\\ &= \frac{8\sin x(1-\sin^2 x)+(2+4\sin^2 x)\cdot 4\sin x}{\cos^5 x}\\ &= \frac{16\sin x+8\sin^3 x}{\cos^5 x}\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

Regarding now the higher derivatives of sec, csc, cot, and of the inverse functions arcsin, arccos, arctan and arcsec, arccsc, arccot, these can be certainly computed too, up to the needed order, but the complexity grows with the order, a bit like for tan.

Summarizing, we are here back to the basics, with sin, cos being the nice trigonometric functions, and with everything else being quite complicated. Good to know.

## 14c. Taylor series

Getting now to the Taylor series of the various trigonometric functions, that we would like to compute at $x=0$, some simplifications appear here, due to the fact that we will not need for this all the derivatives $f^{(k)}(x)$, but just their values $f^{(k)}(0)$. And, as we will soon discover, this will allow the computation for all 12 basic functions, namely:

| | | |
|---|---|---|
| sin | cos | tan |
| sec | csc | cot |
| arcsin | arccos | arctan |
| arcsec | arccsc | arccot |

Getting started now, with what is most likely the simplest, namely the sine and cosine, we have here the following result, that we already know from chapter 13:

Theorem 14.8. *We have the following formulae for* sin *and* cos*,*

$$\sin x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n+1}}{(2n+1)!}\quad,\quad \cos x=\sum_{n=0}^\infty(-1)^n\frac{x^{2n}}{(2n)!}$$

*as Taylor series, around* 0*, and also in general.*

PROOF. This is something that we saw in chapter 13, and as a matter of having the present section rather complete and self-contained, here is the detailed proof:

(1) Regarding the sine, we can use here the following formulae:

$$(\sin x)' = \cos x \quad , \quad (\cos x)' = -\sin x$$

Indeed, we can differentiate sin as many times as we want to, and we get:

$$\sin^{(k)}(x) = \begin{cases} \sin x & k = 0(4) \\ \cos x & k = 1(4) \\ -\sin x & k = 2(4) \\ -\cos x & k = 3(4) \end{cases}$$

In particular, we obtain the following formula for the derivatives, at $x = 0$:

$$\sin^{(k)}(0) = \begin{cases} 0 & k = 0(4) \\ 1 & k = 1(4) \\ 0 & k = 2(4) \\ -1 & k = 3(4) \end{cases}$$

Thus, when constructing the Taylor series, the even powers of $x$ dissapear, and for the odd powers, the derivative formulae that we need to know are as follows:

$$\sin^{(2n+1)}(0) = (-1)^n$$

But this gives the formula in the statement for the Taylor series, namely:

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(2) For the cosine, the story is similar, again based on the following formulae:

$$(\sin x)' = \cos x \quad , \quad (\cos x)' = -\sin x$$

Indeed, we can differentiate cos as many times as we want to, and we get:

$$\cos^{(k)}(x) = \begin{cases} \cos x & k = 0(4) \\ -\sin x & k = 1(4) \\ -\cos x & k = 2(4) \\ \sin x & k = 3(4) \end{cases}$$

In particular, we obtain the following formula for the derivatives, at $x = 0$:

$$\cos^{(k)}(0) = \begin{cases} 1 & k = 0(4) \\ 0 & k = 1(4) \\ -1 & k = 2(4) \\ 0 & k = 3(4) \end{cases}$$

Thus, when constructing the Taylor series, the odd powers of $x$ dissapear, and for the even powers, the derivative formulae that we need to know are as follows:

$$\cos^{(2n)}(0) = (-1)^n$$

But this gives the formula in the statement for the Taylor series, namely:

$$\cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!}$$

Thus, we are led to the formulae in the statement.

(3) Regarding now the convergence of the Taylor series away from 0, coming as a bonus, and which actually reproves the above results, via a different method, we have the following computation, for any $x \in \mathbb{R}$, based on the usual formula of the exponential:

$$\begin{aligned} e^{ix} &= \sum_{k=0}^\infty \frac{(ix)^k}{k!} \\ &= \sum_{n=0}^\infty \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^\infty \frac{(ix)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

Now by comparing this with the Euler formula $e^{ix} = \cos x + i \sin x$, which itself comes from $\sin' = \cos$, $\cos' = -\sin$, by differentiating the function $f(x) = e^{-ix}(\cos x + i \sin x)$, and getting $f'(x) = 0$, and so $f(x) = f(0) = 1$, we obtain, for any $x \in \mathbb{R}$:

$$\cos x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} \quad , \quad \sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Thus, we are led to the conclusions in the statement. □

The problem is now, in practice, how to memorize the above formulae? Here is my personal method, which works every single time, in a matter of few seconds:

Method 14.9. *In order to recover the series of* $\sin, \cos$*, all you need to know is*

$$e^z = \sum_{k=0}^\infty \frac{z^k}{k!}$$

*along with the Euler formula,* $e^{ix} = \cos x + i \sin x$*.*

Getting now to more specialized results, let us discuss the computation of the Taylor series of the other basic trigonometric functions. We will be interested, as usual in this chapter, in the fundamental 12 trigonometric functions, which are as follows:

$$\begin{matrix} \sin & \cos & \tan \\ \sec & \csc & \cot \\ \arcsin & \arccos & \arctan \\ \mathrm{arcsec} & \mathrm{arccsc} & \mathrm{arccot} \end{matrix}$$

We already have 2 Taylor series, that of sin, cos, but in what regards the other 10 functions, things can be quite tricky. Our plan will be as follows:

(1) In what regards arcsin, arccos, arctan, arccot, the first derivatives, computed before, look quite good, suggesting that the Taylor series can be computed with the generalized binomial formula, for square roots. We can expect here to have formulae making appear the Catalan numbers $C_k$, and the central binomial coefficients $D_k$.

(2) In what regards arcsec, arccsc, these do not converge at 0, but we can multiply them by $x$, and use the above results for the other inverse trigonometric functions. Thus, modulo some quantities of type $1/x$, we can expect again to have formulae making appear the Catalan numbers $C_k$, and the central binomial coefficients $D_k$.

(3) Next come tan, csc, cot, whose study is more tricky, leading to Taylor series featuring some numbers which are not explicitly computable, namely the tangent numbers $T_k$, and the Bernoulli numbers $B_k$, and with these latter numbers being related to each other, and useful for many other purposes. Expect some tricky mathematics here.

(4) Finally, we have sec, whose study is again quite tricky, leading again to Taylor series featuring some numbers which are not explicitly computable, namely the Euler numbers $E_k$. And with these latter numbers being related to tangent numbers $T_k$, and the Bernoulli numbers $B_k$. Again, expect some tricky mathematics here.

Getting started now, we first need to talk about arcsin, arccos, arctan, arccot:

THEOREM 14.10. *The Taylor series of* arcsin*,* arccos *are given by*

$$\arcsin x=\sum_{n=0}^\infty\frac{D_n}{4^n(2n+1)}\,x^{2n+1}\quad,\quad\arccos x=\frac{\pi}{2}-\sum_{n=0}^\infty\frac{D_n}{4^n(2n+1)}\,x^{2n+1}$$

*and the Taylor series of* arctan*,* arccot *are given by*

$$\arctan x=\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\,x^{2n+1}\quad,\quad\mathrm{arccot}\,x=\frac{\pi}{2}-\sum_{n=0}^\infty\frac{(-1)^n}{2n+1}\,x^{2n+1}$$

*with* $D_n=\binom{2n}{n}$ *being the central binomial coefficients.*

PROOF. This is something routine, by using the formulae of the first derivatives that we computed before, in Theorem 14.4, which were as follows:

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad , \quad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2} \quad , \quad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

(1) Indeed, let us recall from the end of chapter 13 that we can extract the inverse square roots as follows, with $D_n = \binom{2n}{n}$ being the central binomial coefficients:

$$\frac{1}{\sqrt{1+t}} = \sum_{n=0}^{\infty} D_n \left(\frac{-t}{4}\right)^n$$

With the change of variables $t = -x^2$, this formula becomes:

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} D_n \left(\frac{x^2}{4}\right)^n$$

The question is now, what is the function having this as derivative? Since the arcsine must vanish at $x = 0$, we are led to the formula in the statement, namely:

$$\arcsin x = \sum_{n=0}^{\infty} \frac{D_n}{4^n(2n+1)}\, x^{2n+1}$$

(2) A similar study applies to the arcosine, and we obtain here, again as claimed:

$$\arccos x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{D_n}{4^n(2n+1)}\, x^{2n+1}$$

Alternatively, we can simply say that this formula follows from the one of arcsin.

(3) Regarding now the arctangent, we can use here the following formula:

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

By arguing like before for the arcsine, we obtained here, as claimed:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1}$$

(4) Finally, a similar study applies to the arcotangent, and we obtain, as claimed:

$$\operatorname{arccot} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1}$$

Alternatively, we can simply say that this formula follows from the one of arctan. □

Next, we need to talk about arcsec, arccsc, the result here being as follows:

THEOREM 14.11. *The Taylor series of* arcsec *is given by*

$$\operatorname{arcsec} x = \frac{\pi}{2} - \sum_{n=0}^\infty \frac{D_n}{4^n(2n+1)}\, x^{-(2n+1)}$$

*and the Taylor series of* arccsc *is given by*

$$\operatorname{arccsc} x = \sum_{n=0}^\infty \frac{D_n}{4^n(2n+1)}\, x^{-(2n+1)}$$

*with* $D_n = \binom{2n}{n}$ *being the central binomial coefficients.*

PROOF. This is again routine, by using the formulae of the first derivatives that we computed before, in Theorem 14.4, which were as follows:

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}} \quad , \quad (\operatorname{arccsc} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$$

Thus, by using the binomial formula, we obtain the formulae in the statement. Alternatively, these formulae follow from those for arccos, arcsin, from Theorem 14.10. □

Next, we need to talk about tan, csc, cot. The result here is more complicated, and a bit theoretical, involving the tangent and Bernoulli numbers, as follows:

THEOREM 14.12. *The Taylor series of* tan *is given by*

$$\tan x = \sum_{n=0}^\infty (-1)^n \frac{T_{2n+1}}{(2n+1)!}\, x^{2n+1}$$

*the Taylor series of* csc *is given by*

$$x \csc x = 1 + \sum_{n=1}^\infty (-1)^{n-1} \frac{(4^n-2)B_{2n}}{(2n)!}\, x^{2n}$$

*and the Taylor series of* cot *is given by*

$$x \cot x = 1 - \sum_{n=1}^\infty (-1)^{n-1} \frac{4^n B_{2n}}{(2n)!}\, x^{2n}$$

*with* $T_k$ *being the tangent numbers, and* $B_k$ *being the Bernoulli numbers.*

PROOF. This is something more tricky, and many things can be said here. Let us mention that the Bernoulli numbers are defined recursively, according to:

$$\sum_{n=1}^k \binom{2k}{2n-1} \frac{B_{2n}}{2n} = \frac{2k-1}{2(2k+1)}$$

As for the tangent numbers, these are modifications of the Bernoulli numbers:

$$T_{2n+1}=4^{n+1}(4^{n+1}-1)\frac{B_{2n+2}}{2n+2}$$

As for the proof, this is something quite technical. We will be back to this in chapter 15, when discussing more in detail the Bernoulli numbers, and their properties. □

Finally, we need to talk about sec, the result here being as follows:

THEOREM 14.13. *The Taylor series of* sec *is given by*

$$\sec x=\sum_{n=0}^\infty(-1)^n\frac{E_{2n}}{(2n)!}\,x^{2n}$$

*with* $E_k$ *being Euler numbers.*

PROOF. This is again something more tricky, and many things can be said here. Let us mention that the Euler numbers are defined recursively, according to:

$$\sum_{n=1}^k\binom{2k}{2n}E_{2n}=-1$$

As for the proof, this is again something more technical. We will be back to this in chapter 15, when discussing more in detail this type of formulae, and related topics. □

And with this, end of our discussion regarding the 12 basic trigonometric functions. Finally, let us mention that, in analogy with what we know from Theorem 14.8 regarding sin and cos, the above Taylor series formulae hold not only around $x=0$, but in general too, provided of course that the series converges. Which is of course, good to know.

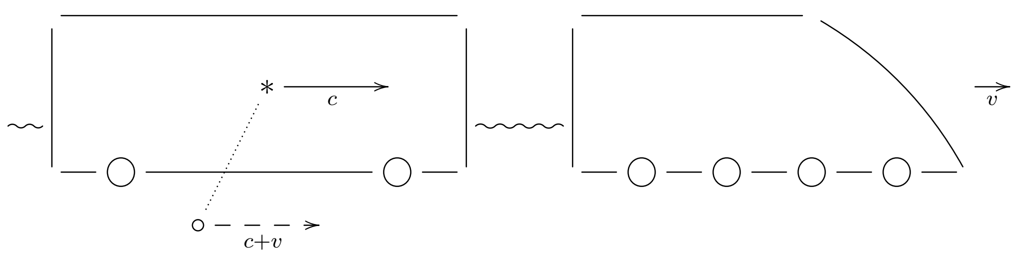
## 14d. Hyperbolic functions

Ready for some physics? We would like to talk now about the hyperbolic functions, which appear for instance in Einstein's relativity theory. Let us start with:

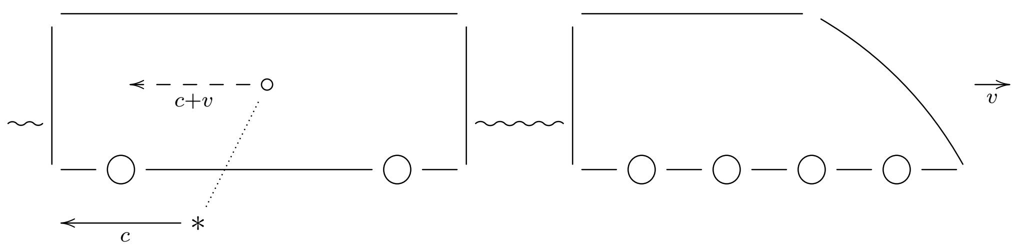
FACT 14.14 (Einstein principles). *The following happen:*

(1) *Light travels in vacuum at a finite speed,* $c<\infty$.
(2) *This speed* $c$ *is the same for all inertial observers.*
(3) *In non-vacuum, the light speed is lower,* $v<c$.
(4) *Nothing can travel faster than light,* $v\not>c$.

The point now is that, obviously, something is wrong here. Indeed, assuming for instance that we have a train, running in vacuum at speed $v>0$, and someone on board lights a flashlight $*$ towards the locomotive, then an observer $\circ$ on the ground will see the light traveling at speed $c + v > c$, which is a contradiction:




Equivalently, with the same train running, in vacuum at speed $v > 0$, if the observer on the ground lights a flashlight $*$ towards the back of the train, then viewed from the train, that light will travel at speed $c + v > c$, which is a contradiction again:




Summarizing, Fact 14.14 implies $c + v = c$, so contradicts classical mechanics, which therefore needs a fix. By dividing all speeds by $c$, as to have $c = 1$, and by restricting the attention to the 1D case, to start with, we are led to the following puzzle:

Puzzle 14.15. *How to define speed addition on the space of* 1D *speeds, which is*

$$I = [-1, 1]$$

*with our $c = 1$ convention, as to have $1 + c = 1$, as required by physics?*

In view of our geometric knowledge so far, a natural idea here would be that of wrapping $[-1, 1]$ into a circle, and then stereographically projecting on $\mathbb{R}$. Indeed, we can then "import" to $[-, 1, 1]$ the usual addition on $\mathbb{R}$, via the inverse of this map. So, let us see where all this leads us. First, the formula of our map is as follows:

Theorem 14.16. *The map wrapping $[-1, 1]$ into the unit circle, and then stereographically projecting on $\mathbb{R}$ is given by the formula*

$$\varphi(u) = \tan\left(\frac{\pi u}{2}\right)$$

*with the convention that our wrapping is the most straightforward one, making correspond $\pm 1 \to i$, with negatives on the left, and positives on the right.*

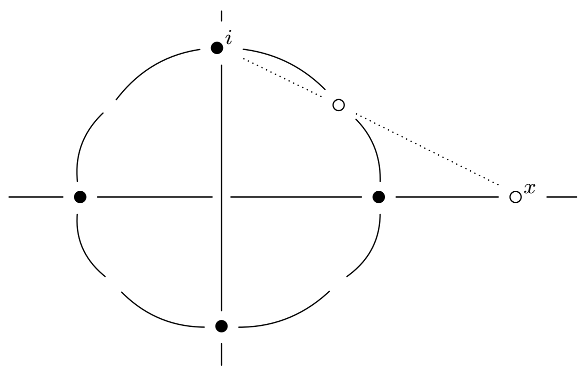
PROOF. Regarding the wrapping, as indicated, this is given by:

$$u\to e^{it}\quad,\quad t=\pi u-\frac{\pi}{2}$$

Indeed, this correspondence wraps $[-1,1]$ as above, the basic instances of our correspondence being as follows, and with everything being fine modulo $2\pi$:

$$-1\to\frac{\pi}{2}\quad,\quad -\frac{1}{2}\to-\pi\quad,\quad 0\to-\frac{\pi}{2}\quad,\quad \frac{1}{2}\to 0\quad,\quad 1\to\frac{\pi}{2}$$

Regarding now the stereographic projection, the picture here is as follows:




Thus, by Thales, the formula of the stereographic projection is as follows:

$$\frac{\cos t}{x}=\frac{1-\sin t}{1}\implies x=\frac{\cos t}{1-\sin t}$$

Now if we compose our wrapping operation above with the stereographic projection, what we get is, via the above Thales formula, and some trigonometry:

$$\begin{aligned}
x &= \frac{\cos t}{1-\sin t}\\
&= \frac{\cos\left(\pi u-\frac{\pi}{2}\right)}{1-\sin\left(\pi u-\frac{\pi}{2}\right)}\\
&= \frac{\cos\left(\frac{\pi}{2}-\pi u\right)}{1+\sin\left(\frac{\pi}{2}-\pi u\right)}\\
&= \frac{\sin(\pi u)}{1+\cos(\pi u)}\\
&= \frac{2\sin\left(\frac{\pi u}{2}\right)\cos\left(\frac{\pi u}{2}\right)}{2\cos^2\left(\frac{\pi u}{2}\right)}\\
&= \tan\left(\frac{\pi u}{2}\right)
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result is very nice, but when it comes to physics, things do not work, for instance because of the wrong slope of the function $\varphi(u) = \tan\left(\frac{\pi u}{2}\right)$ at the origin, which makes our summing on $[-1, 1]$ not compatible with the Galileo addition, at low speeds.

So, what to do? Obviously, trash Theorem 14.16, and start all over again. Getting back now to Puzzle 14.15, this has in fact a simpler solution, based this time on algebra, and which in addition is the good, physically correct solution, as follows:

THEOREM 14.17. *If we sum the speeds according to the Einstein formula*

$$u +_e v = \frac{u+v}{1+uv}$$

*then the Galileo formula still holds, approximately, at low speeds*

$$u +_e v \simeq u + v$$

*and if we have $u = 1$ or $v = 1$, the resulting sum is $u +_e v = 1$.*

PROOF. All this is self-explanatory, and clear from definitions, and with the Einstein formula of $u +_e v$ itself being just an obvious solution to Puzzle 14.15, provided that, importantly, we know 0 geometry, and rely on very basic algebra only. $\square$

So, very nice, problem solved, at least in 1D. But, shall we give up with geometry, and the stereographic projection? Certainly not, let us try to recycle that material. In order to do this, let us recall that the usual trigonometric functions are given by:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad , \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad , \quad \tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$

The point now is that, mathematically speaking, the above functions have some natural "hyperbolic" or "imaginary" analogues, constructed as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad , \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

But the function on the right, tanh, starts reminding the formula of Einstein addition, from Theorem 14.17. So, we have our idea, and we are led to the following result:

THEOREM 14.18. *The Einstein speed summation in 1D is given by*

$$\tanh x +_e \tanh y = \tanh(x+y)$$

*with* $\tanh : [-\infty, \infty] \to [-1, 1]$ *being the hyperbolic tangent function.*

PROOF. This follows by putting together our various formulae above, but it is perhaps better, for clarity, to prove this directly. Our claim is that we have:

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

But this can be checked via direct computation, from the definitions, as follows:

$$\begin{aligned}\frac{\tanh x+\tanh y}{1+\tanh x\tanh y} &= \left(\frac{e^x-e^{-x}}{e^x+e^{-x}}+\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)\Big/\left(1+\frac{e^x-e^{-x}}{e^x+e^{-x}}\cdot\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)\\ &= \frac{(e^x-e^{-x})(e^y+e^{-y})+(e^x+e^{-x})(e^y-e^{-y})}{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y+e^{-y})}\\ &= \frac{2(e^{x+y}-e^{-x-y})}{2(e^{x+y}+e^{-x-y})}\\ &= \tanh(x+y)\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Very nice all this, hope you agree. As a conclusion, passing from the Riemann stereographic projection sum to the Einstein summation basically amounts in replacing:

$$\tan\to\tanh$$

Let us formulate as well this finding more philosophically, as follows:

CONCLUSION 14.19. *The Einstein speed summation in 1D is the imaginary analogue of the summation on* $[-1,1]$ *obtained via the Riemann stereographic projection.*

As a continuation of this, many other things can be said about relativity, with the next obvious challenge being that of understanding what happens to the Einstein summation formula when passing to 3D. And here, the summation formula is as follows, making appear the vector products $\times$ that our cat advisor was talking about, in chapter 12:

$$u+_e v=\frac{1}{1+<u,v>}\left(u+v+\frac{u\times(u\times v)}{1+\sqrt{1-||u||^2}}\right)$$

Well, quite interesting all this, hope you agree with me, and the temptation is high to keep talking about this, in the remainder of this chapter, and even of this book.

This being said, let us be reasonable. As mathematicians, we definitely have good reasons for adopting sinh, cosh, tanh, as trigonometric functions. But then we can talk about secondary and inverse functions too, in the obvious way, which leads us to:

CONCLUSION 14.20 (mathematics). *There are in fact* 24 *trigonometric functions,*

| | | | | | |
|---|---|---|---|---|---|
| sin | cos | tan | sec | csc | cot |
| arcsin | arccos | arctan | arcsec | arccsc | arccot |
| sinh | cosh | tanh | sech | csch | coth |
| arcsinh | arccosh | arctanh | arcsech | arccsch | arccoth |

*with the hyperbolic ones being useful in relativity, and perhaps in other physics too.*

Getting back now to what we know how to do, with precision and speed, namely computing derivatives, and then Taylor series, we first have the following result:

THEOREM 14.21. *The derivatives of basic hyperbolic trigonometric functions are*

$$(\sinh x)' = \cosh x \quad , \quad (\cosh x)' = \sinh x \quad , \quad (\tanh x)' = \frac{1}{\cosh^2 x}$$

*and the derivatives of secondary hyperbolic trigonometric functions are*

$$(\mathrm{sech}\, x)' = -\frac{\sinh x}{\cosh^2 x} \quad , \quad (\mathrm{csch}\, x)' = -\frac{\cosh x}{\sinh^2 x} \quad , \quad (\coth x)' = -\frac{1}{\sinh^2 x}$$

*provided that the denominators do not vanish.*

PROOF. This is indeed something very standard, say exercise for you. □

Regarding the inverse hyperbolic trigonometric functions, we have:

THEOREM 14.22. *The derivatives of basic inverse hyperbolic functions are given by*

$$(\mathrm{arcsinh}\, x)' = \frac{1}{\sqrt{1+x^2}} \quad , \quad (\mathrm{arccosh}\, x)' = \frac{1}{\sqrt{x^2-1}} \quad , \quad (\mathrm{arctanh}\, x)' = \frac{1}{1-x^2}$$

*and the derivatives of secondary inverse hyperbolic functions are given by*

$$(\mathrm{arcsech}\, x)' = -\frac{1}{|x|\sqrt{1+x^2}} \ , \ (\mathrm{arccsch}\, x)' = -\frac{1}{|x|\sqrt{1-x^2}} \ , \ (\mathrm{arccoth}\, x)' = \frac{1}{1-x^2}$$

*provided that the denominators do not vanish.*

PROOF. This is again something very standard, again exercise for you. □

Regarding now the Taylor series, we first have the following result:

THEOREM 14.23. *The Taylor series of* sinh, cosh *are given by*

$$\sinh x = \sum_{n=0}^\infty \frac{x^{2n+1}}{(2n+1)!} \quad , \quad \cosh x = \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!}$$

*the Taylor series of* tanh, sech *are given by*

$$\tanh x = \sum_{n=0}^\infty \frac{T_{2n+1}}{(2n+1)!}\, x^{2n+1} \quad , \quad \mathrm{sech}\, x = \sum_{n=0}^\infty \frac{E_{2n}}{(2n)!}\, x^{2n}$$

*and the Taylor series of* csch, coth *are given by*

$$x\, \mathrm{csch}\, x = 1 - \sum_{n=1}^\infty \frac{(4^n-2)B_{2n}}{(2n)!}\, x^{2n} \quad , \quad x \coth x = 1 + \sum_{n=1}^\infty \frac{4^n B_{2n}}{(2n)!}\, x^{2n}$$

*with $B_k, E_k, T_k$ being the Bernoulli, Euler and tangent numbers.*

PROOF. This is again something very standard, again exercise for you. □

Finally, regarding the Taylor series of the inverse functions, we have:

THEOREM 14.24. *The Taylor series of* arcsinh, arccosh *are given by*

$$\operatorname{arcsinh} x = \sum_{n=0}^\infty (-1)^n \frac{D_n}{4^n(2n+1)}\, x^{2n+1} \quad , \quad \operatorname{arccosh} x = \log(2x) - \frac{1}{2}\sum_{n=1}^\infty \frac{D_n}{4^n n}\, x^{-2n}$$

*the Taylor series of* arctanh, arcsech *are given by*

$$\operatorname{arctanh} x = \sum_{n=0}^\infty \frac{x^{2n+1}}{2n+1} \quad , \quad \operatorname{arcsech} x = \log\left(\frac{2}{x}\right) - \frac{1}{2}\sum_{n=1}^\infty \frac{D_n}{4^n n}\, x^{2n}$$

*and the Taylor series of* arccsch *and* arccoth *are given by*

$$\operatorname{arccsch} x = \sum_{n=0}^\infty (-1)^n \frac{D_n}{4^n(2n+1)}\, x^{-(2n+1)} \quad , \quad \operatorname{arccoth} x = \sum_{n=0}^\infty \frac{x^{-(2n+1)}}{2n+1}$$

*with* $D_n = \binom{2n}{n}$ *being the central binomial coefficients.*

PROOF. This is again something very standard, again exercise for you. □

And with this, good news, done with all 24 trigonometric functions. Memorize all this, which is useful, and pass the word to your children, and grandchildren.

## 14e. Exercises

This was a quite tough calculus chapter, no wonder we had only Theorems here, with not a single Proposition present, and as exercises on all this, we have:

EXERCISE 14.25. *Finish the computations for* tan.

EXERCISE 14.26. *Finish the computations for* sec.

EXERCISE 14.27. *Finish the computations for* csc.

EXERCISE 14.28. *Finish the computations for* cot.

EXERCISE 14.29. *Finish the computations for* tanh.

EXERCISE 14.30. *Finish the computations for* sech.

EXERCISE 14.31. *Finish the computations for* csch.

EXERCISE 14.32. *Finish the computations for* coth.

As bonus exercise, and no surprise here, read some relativity theory, in 1D, 2D, 3D, and special and general, say from the book of Einstein [31], which is a must-read.

CHAPTER 15

# Sums, estimates

## 15a. Integration theory

With the trigonometric functions reasonably understood, time to focus now on $e, \pi$, which are at the origins of trigonometry. Here are some questions, regarding $e$:

Questions 15.1. *Regarding $e$, which produces trigonometry via $e^{it} = \cos t + i \sin t$:*

(1) *$e$ is most likely irrational, but why?*
(2) *In fact, $e$ is most likely transcendental, but why?*
(3) *How to reach to the known figure $e = 2.71828\ldots$?*
(4) *What else can be said, of fundamental nature, about $e$?*

And in the hope that you agree with me, all these questions look important, worth some study. As for the number $\pi$, our questions regarding it are identical, as follows:

Questions 15.2. *Regarding $\pi$, which needs no presentation either:*

(1) *$\pi$ is most likely irrational, but why?*
(2) *In fact, $\pi$ is most likely transcendental, but why?*
(3) *How to reach to the known figure $\pi = 3.14159\ldots$?*
(4) *What else can be said, of fundamental nature, about $\pi$?*

In answe to all this, more calculus, I guess. So, let us start with some calculus basics, coming as a continuation of the material from chapter 13. What we discussed there, derivatives, was in fact only half of the story. The other half of the story, that we will discuss now, involves the integrals, which are constructed as follows:

Definition 15.3. *The integral of a continuous function $f : [a, b] \to \mathbb{R}$, denoted*

$$\int_a^b f(x)dx$$

*is the area below the graph of $f$, signed $+$ where $f \geq 0$, and signed $-$ where $f \leq 0$.*

We have already met in fact this notion, in chapter 7, and we refer to the material there for more explanations, and for some computations too, notably for $f(x) = x^2$.

Generally speaking, in order to compute integrals, we can use our geometric knowledge. Here are some basic results, coming from various areas that we know how to compute:

Proposition 15.4. *We have the following results:*

(1) *When $f$ is linear, we have the following formula:*

$$\int_a^b f(x)dx = (b-a)\cdot\frac{f(a)+f(b)}{2}$$

(2) *In fact, when $f$ is piecewise linear on $[a=a_1,a_2,\ldots,a_n=b]$, we have:*

$$\int_a^b f(x)dx = \sum_{i=1}^{n-1}(a_{i+1}-a_i)\cdot\frac{f(a_i)+f(a_{i+1})}{2}$$

(3) *We have as well the formula $\int_{-1}^1\sqrt{1-x^2}\,dx=\pi/2$.*

Proof. These results all follow from basic geometry, as follows:

(1) Assuming $f\geq 0$, we must compute the area of a trapezoid having sides $f(a),f(b)$, and height $b-a$. But this is the same as the area of a rectangle having side $(f(a)+f(b))/2$ and height $b-a$, and we obtain $(b-a)(f(a)+f(b))/2$, as claimed.

(2) This is clear indeed from the formula found in (1), by additivity.

(3) The integral in the statement is by definition the area of the upper unit half-disc. But since the area of the whole unit disc is $\pi$, this half-disc area is $\pi/2$. □

As an interesting observation, (2) in the above result makes it quite clear that $f$ does not necessarily need to be continuous, in order to talk about its integral. Indeed, assuming that $f$ is piecewise linear on $[a=a_1,a_2,\ldots,a_n=b]$, but not necessarily continuous, we can still talk about its integral, in the obvious way, exactly as in Definition 15.3, and we have an explicit formula for this integral, generalizing the one found in (2), namely:

$$\int_a^b f(x)dx = \sum_{i=1}^{n-1}(a_{i+1}-a_i)\cdot\frac{f(a_i^+)+f(a_{i+1}^-)}{2}$$

Based on this observation, let us upgrade our formalism, as follows:

Definition 15.5. *We say that a function $f:[a,b]\to\mathbb{R}$ is integrable when the area below its graph is computable. In this case we denote by*

$$\int_a^b f(x)dx$$

*this area, signed $+$ where $f\geq 0$, and signed $-$ where $f\leq 0$.*

As basic examples of integrable functions, we have the continuous ones, somewhat for obvious reasons, and we will take this as granted. As further examples, we have the functions which are piecewise linear, or piecewise continuous. As another class of examples, the monotone, or piecewise monotone functions, can be shown to be integrable as well. But all this is rather philosophy, let us not bother much with this.

Getting to work now, here are some general results regarding the integrals:

PROPOSITION 15.6. *We have the following formulae,*

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b \lambda f(x) = \lambda \int_a^b f(x)$$

*valid for any functions $f, g$ and any scalar $\lambda \in \mathbb{R}$.*

PROOF. Both these formulae are indeed clear from definitions. □

Moving ahead, passed the above elementary results, we must do some analysis, in order to compute integrals. This is something quite tricky, and we have here:

THEOREM 15.7. *We have the Riemann integration formula,*

$$\int_a^b f(x)dx = (b-a) \times \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N f\left(a + \frac{b-a}{N} \cdot k\right)$$

*which can serve as a definition for the integral.*

PROOF. This is standard, by drawing rectangles. We have indeed the following formula, which can stand as a definition for the signed area below the graph of $f$:

$$\int_a^b f(x)dx = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N \frac{b-a}{N} \cdot f\left(a + \frac{b-a}{N} \cdot k\right)$$

Thus, we are led to the formula in the statement. □

Observe that the above formula suggests that $\int_a^b f(x)dx$ is the length of the interval $[a, b]$, namely $b - a$, times the average of $f$ on the interval $[a, b]$. Thinking a bit, this is indeed something true, with no need for Riemann sums, coming directly from Definition 15.3, because area means side times average height. Thus, we can formulate:

THEOREM 15.8. *The integral of a function $f : [a, b] \to \mathbb{R}$ is given by*

$$\int_a^b f(x)dx = (b-a) \times A(f)$$

*where $A(f)$ is the average of $f$ over the interval $[a, b]$.*

PROOF. As explained above, this is clear from Definition 15.3, via some geometric thinking. Alternatively, this is something which certainly comes from Theorem 15.7. □

The point of view in Theorem 15.8 is something quite useful, and as an illustration for this, let us review the results that we already have, by using this interpretation. First, we have the formula for linear functions from Proposition 15.4, namely:

$$\int_a^b f(x)dx = (b-a)\cdot\frac{f(a)+f(b)}{2}$$

But this formula is totally obvious with our new viewpoint, from Theorem 15.8. The same goes for the results in Proposition 15.6, which become even more obvious with the viewpoint from Theorem 15.8. However, not everything trivializes in this way, and the result which is left, namely the formula $\int_{-1}^1\sqrt{1-x^2}\,dx=\pi/2$ from Proposition 15.4 (3), not only does not trivialize, but becomes quite opaque with our new philosophy.

Going ahead with more interpretations of the integral, we have:

THEOREM 15.9. *We have the Monte Carlo integration formula,*

$$\int_a^b f(x)dx = (b-a)\times\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f(x_i)$$

*with $x_1,\ldots,x_N\in[a,b]$ being random.*

PROOF. We recall from Theorem 15.7 that the idea is that we have a formula as follows, with the points $x_1,\ldots,x_N\in[a,b]$ being uniformly distributed:

$$\int_a^b f(x)dx = (b-a)\times\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f(x_i)$$

But this works as well when the points $x_1,\ldots,x_N\in[a,b]$ are randomly distributed, for somewhat obvious reasons, and this gives the result. □

Observe that Monte Carlo integration works better than Riemann, when it comes to computing as usual, by estimating, and refining the estimate. Also, Monte Carlo is smarter than Riemann, because the symmetries of the function can fool Riemann, but not Monte Carlo. All this is good to know, say when integrating by using a computer.

Finally, here is one more useful interpretation of the integral:

THEOREM 15.10. *The integral of a function $f:[a,b]\to\mathbb{R}$ is given by*

$$\int_a^b f(x)dx = (b-a)\times E(f)$$

*where $E(f)$ is the expectation of $f$, regarded as random variable.*

PROOF. This is just some sort of fancy reformulation of Theorem 15.9, the idea being that what we can "expect" from a random variable is of course its average. □

Going ahead with more theory, here is one key result regarding the integrals:

THEOREM 15.11. *Given a continuous function $f : [a,b] \to \mathbb{R}$, we have*

$$\exists c \in [a,b] \quad , \quad \int_a^b f(x)dx = (b-a)f(c)$$

*with this being called mean value property.*

PROOF. We have indeed the following trivial estimate:

$$\min(f) \leq \frac{\int_a^b f(x)dx}{b-a} \leq \max(f)$$

Since $f$ must takes all values on $[\min(f), \max(f)]$, we get a $c \in [a,b]$ such that:

$$\frac{\int_a^b f(x)dx}{b-a} = f(c)$$

Thus, we are led to the conclusion in the statement. □

Next, we have the following key result, called fundamental theorem of calculus:

THEOREM 15.12. *Given a continuous function $f : [a,b] \to \mathbb{R}$, if we set*

$$F(x) = \int_a^x f(s)ds$$

*then $F' = f$. That is, the derivative of the integral is the function itself.*

PROOF. This follows from the Riemann integration picture, and more specifically, from the mean value property from Theorem 15.11. Indeed, we have:

$$\frac{F(x+t) - F(x)}{t} = \frac{1}{t}\int_x^{x+t} f(x)dx$$

On the other hand, our function $f$ being continuous, by using the mean value property from Theorem 15.11, we can find a number $c \in [x, x+t]$ such that:

$$\frac{1}{t}\int_x^{x+t} f(x)dx = f(x)$$

Thus, putting our formulae together, we conclude that we have:

$$\frac{F(x+t) - F(x)}{t} = f(c)$$

Now with $t \to 0$, no matter how the number $c \in [x, x+t]$ varies, one thing that we can be sure about is that we have $c \to x$. Thus, by continuity of $f$, we obtain:

$$\lim_{t \to 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

But this means exactly that we have $F' = f$, and we are done. □

We have as well the following result, which is something equivalent, and a hair more beautiful, also called fundamental theorem of calculus:

THEOREM 15.13. *Given a function $F:\mathbb{R}\to\mathbb{R}$, we have*

$$\int_a^b F'(x)dx = F(b)-F(a)$$

*for any interval $[a,b]$.*

PROOF. As already mentioned, this is something which follows from Theorem 15.12, and is in fact equivalent to it. Indeed, consider the following function:

$$G(s)=\int_a^s F'(x)dx$$

By using Theorem 15.12 we have $G'=F'$, and so our functions $F,G$ differ by a constant. But with $s=a$ we have $G(a)=0$, and so the constant is $F(a)$, and we get:

$$F(s)=G(s)+F(a)$$

Now with $s=b$ this gives $F(b)=G(b)+F(a)$, which reads:

$$F(b)=\int_a^b F'(x)dx+F(a)$$

Thus, we are led to the conclusion in the statement. □

The fundamental theorem of calculus is something quite powerful, and as an illustration for this, destroying our previous ad-hoc computations from chapter 7, we have:

THEOREM 15.14. *We have the following integration formulae,*

$$\int_a^b x^p dx = \frac{b^{p+1}-a^{p+1}}{p+1} \quad , \quad \int_a^b \frac{1}{x}\,dx = \log\left(\frac{b}{a}\right)$$

$$\int_a^b \sin x\,dx = \cos a - \cos b \quad , \quad \int_a^b \cos x\,dx = \sin b - \sin a$$

$$\int_a^b e^x dx = e^b - e^a \quad , \quad \int_a^b \log x\,dx = b\log b - a\log a - b + a$$

*all obtained, in case you ever forget them, via the fundamental theorem of calculus.*

PROOF. This is indeed something self-explanatory, with only the last formula being in need of some explanations. So, we are looking for a function $F$ satisfying:

$$F'(x)=\log x$$

In order to solve this, speaking logarithm and derivatives, what we know is:

$$(\log x)' = \frac{1}{x}$$

But then, in order to make appear log on the right, the idea is quite clear, namely multiplying on the left by $x$. We obtain in this way the following formula:

$$(x\log x)' = 1\cdot\log x + x\cdot\frac{1}{x} = \log x + 1$$

We are almost there, all we have to do now is to substract $x$ from the left, as to get:

$$(x\log x - x)' = \log x$$

But this this formula in hand, we can go back to our problem, and we get the result. □

Getting back now to theory, inspired by the above, let us formulate:

DEFINITION 15.15. *Given $f$, we call primitive of $f$ any function $F$ satisfying:*

$$F' = f$$

*We denote such primitives by $\int f$, and also call them indefinite integrals.*

Observe that the primitives are unique up to an additive constant, in the sense that if $F$ is a primitive, then so is $F+c$, for any $c\in\mathbb{R}$, and conversely, if $F,G$ are two primitives, then we must have $G=F+c$, for some $c\in\mathbb{R}$, with this latter fact coming from a result from chapter 13, saying that the derivative vanishes when the function is constant.

As for the convention at the end, $F=\int f$, this comes from the fundamental theorem of calculus, which can be written as follows, by using this convention:

$$\int_a^b f(x)dx = \left(\int f\right)(b) - \left(\int f\right)(a)$$

By the way, observe that there is no contradiction here, coming from the indeterminacy of $\int f$. Indeed, when adding a constant $c\in\mathbb{R}$ to the chosen primitive $\int f$, when conputing the above difference the $c$ quantities will cancel, and we will obtain the same result.

We can now reformulate Theorem 15.14 in a more digest form, as follows:

THEOREM 15.16. *We have the following formulae for primitives,*

$$\int x^p = \frac{x^{p+1}}{p+1} \quad , \quad \int\frac{1}{x} = \log x$$

$$\int \sin x = -\cos x \quad , \quad \int\cos x = \sin x$$

$$\int e^x = e^x \quad , \quad \int\log x = x\log x - x$$

*allowing us to compute the corresponding definite integrals too.*

PROOF. Here the various formulae in the statement follow from Theorem 15.14, and the last assertion comes from the integration formula given after Definition 15.15. □

Getting back now to theory, we have the following very useful result:

THEOREM 15.17. *We have the formula*

$$\int f'g + \int fg' = fg$$

*called integration by parts.*

PROOF. This follows by integrating the Leibnitz formula, namely:

$$(fg)' = f'g + fg'$$

Indeed, with our convention for primitives, this gives the above formula. □

It is then possible to pass to usual integrals, and we obtain a formula here as well, as follows, also called integration by parts, with the convention $[\varphi]_a^b = \varphi(b) - \varphi(a)$:

$$\int_a^b f'g + \int_a^b fg' = \Big[fg\Big]_a^b$$

In practice, the most interesting case is that when $fg$ vanishes on the boundary $\{a, b\}$ of our interval, leading to the following formula:

$$\int_a^b f'g = -\int_a^b fg'$$

Examples of this usually come with $[a, b] = [-\infty, \infty]$, and more on this later. Now still at the theoretical level, we have as well the following result:

THEOREM 15.18. *We have the change of variable formula*

$$\int_a^b f(x)dx = \int_c^d f(\varphi(t))\varphi'(t)dt$$

*where* $c = \varphi^{-1}(a)$ *and* $d = \varphi^{-1}(b)$.

PROOF. This follows with $f = F'$, from the following differentiation rule, that we know from chapter 13, and whose proof is something elementary:

$$(F\varphi)'(t) = F'(\varphi(t))\varphi'(t)$$

Indeed, by integrating between $c$ and $d$, we obtain the result. □

And with this, we have now in our pocket the full collection of basic calculus tools.

## 15b. More on e, pi

Time now for some tough computations, in order to answer our various questions, from the beginning of this chapter. We first have the following result, about $e$:

THEOREM 15.19. *The number $e$ from analysis, given by*

$$e=\sum_{k=0}^\infty\frac{1}{k!}$$

*which numerically means $e=2.7182818284\ldots$, is irrational.*

PROOF. Many things can be said here, as follows:

(1) To start with, the series of $e$ converges very fast, as shown by:

$$\begin{aligned}
e &= \sum_{k=0}^{N-1}\frac{1}{k!}+\frac{1}{N!}\left(1+\frac{1}{N+1}+\frac{1}{(N+1)(N+2)}+\ldots\right)\\
&< \sum_{k=0}^{N-1}\frac{1}{k!}+\frac{1}{N!}\left(1+\frac{1}{N+1}+\frac{1}{(N+1)^2}+\ldots\right)\\
&= \sum_{k=0}^{N-1}\frac{1}{k!}+\frac{1}{N!}\left(1+\frac{1}{N}\right)\\
&= \sum_{k=0}^{N}\frac{1}{k!}+\frac{1}{N\cdot N!}
\end{aligned}$$

Indeed, the error term in the approximation is really tiny, the estimate being:

$$\sum_{k=0}^N\frac{1}{k!}<e<\sum_{k=0}^N\frac{1}{k!}+\frac{1}{N\cdot N!}$$

(2) Now by using this, you can easily compute the decimals of $e$. Actually, you can't call yourself mathematician, or scientist, if you haven't done this by hand, just for the fun, but just in case, here is how the approximation goes, for small values of $N$:

$$N=2\implies 2.5<e<2.75$$

$$N=3\implies 2.666\ldots<e<2.722\ldots$$

$$N=4\implies 2.70833\ldots<e<2.71875\ldots$$

$$N=5\implies 2.71666\ldots<e<2.71833\ldots$$

$$N=6\implies 2.71805\ldots<e<2.71828\ldots$$

$$N=7\implies 2.71825\ldots<e<2.71828\ldots$$

Thus, first 4 decimals computed, and I would leave the continuation to you.

(3) Getting now to irrationality, a look at $e = 2.7182818284\ldots$ might suggest that the $81, 82, 84\ldots$ values might eventually, after some internal fight, decide for a winner, and so that $e$ might be rational. However, this is wrong, and $e$ is in fact irrational.

(4) So, let us prove now this, that $e$ is irrational. Following Fourier, we will do this by contradiction. So, assume $e = m/n$, and let us look at the following number:

$$x = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$$

As a first observation, $x$ is an integer, as shown by the following computation:

$$\begin{aligned} x &= n!\left(\frac{m}{n} - \sum_{k=0}^{n} \frac{1}{k!}\right) \\ &= m(n-1)! - \sum_{k=0}^{n} n(n-1)\ldots(n-k+1) \end{aligned}$$

(5) On the other hand $x > 0$, and we have as well the following estimate:

$$\begin{aligned} x &= n! \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots \\ &= \frac{1}{n} \end{aligned}$$

Thus $x \in (0,1)$, which contradicts our previous finding $x \in \mathbb{Z}$, as desired. □

As a continuation, we have the following result, which is substantially harder:

THEOREM 15.20. *The number $e$ is transcendental.*

PROOF. Assume by contradiction that $e$ is algebraic, with this meaning that it is a root of a polynomial with integer coefficients, $c_i \in \mathbb{Z}$, as follows:

$$c_0 + c_1 e + \ldots + c_n e^n = 0$$

(1) Following Hermite, consider the following polynomials, and we will see later why:

$$f_k(x) = x^k \left[(x-1)\ldots(x-n)\right]^{k+1}$$

Consider also the following quantities, that we will study more in detail later:

$$A_k = \int_0^\infty f_k(x) e^{-x} dx$$

By multiplying our equation for $e$ by this quantity $A_k$, we obtain:

$$c_0\int_0^\infty f_k(x)e^{-x}dx+c_1\int_0^\infty f_k(x)e^{1-x}dx+\ldots c_n\int_0^\infty f_k(x)e^{n-x}dx=0$$

(2) Here comes the trick. Consider the following two quantities:

$$P=c_0\int_0^\infty f_k(x)e^{-x}dx+c_1\int_1^\infty f_k(x)e^{1-x}dx+\ldots+c_n\int_n^\infty f_k(x)e^{n-x}dx$$

$$Q=c_1\int_0^1 f_k(x)e^{-x}dx+\ldots+c_n\int_0^n f_k(x)e^{n-x}dx$$

In terms of these quantities, the formula that we found in (1) reads:

$$P+Q=0$$

(3) Now let us look at $P$. Our claim is that this is an integer, $P\in\mathbb{Z}$, and that there are arbitrarily large numbers $k>>0$ for which the following holds:

$$\frac{P}{k!}\in\mathbb{Z}-\{0\}$$

Indeed, according to our formula above defining $P$, we have:

$$\begin{aligned}P&=\sum_{r=0}^n c_r\int_r^\infty f_k(x)e^{r-x}dx\\&=\sum_{r=0}^n c_r\int_0^\infty f_k(x+r)e^{-x}dx\\&=\int_0^\infty\left(\sum_{r=0}^n c_rf_k(x+r)\right)e^{-x}dx\end{aligned}$$

On the other hand, integrating such functions is easy, according to:

$$\begin{aligned}\int_0^\infty x^se^{-x}dx&=\int_0^\infty\left(\frac{x^{s+1}}{s+1}\right)'e^{-x}dx\\&=\int_0^\infty\frac{x^{s+1}}{s+1}\,e^{-x}dx\\&=\frac{1}{s+1}\int_0^\infty x^{s+1}e^{-x}dx\end{aligned}$$

Thus, we are led by recurrence on $s\in\mathbb{N}$ to the following formula:

$$\int_0^\infty x^se^{-x}dx=s!$$

For a linear combination now, we are led to the following formula:

$$g(x)=\sum_s a_sx^s\implies\int_0^\infty g(x)e^{-x}dx=\sum_s a_ss!$$

But this shows that we have indeed $P\in\mathbb{Z}$, and also, via a bit of study based on the exact formula of $f_k$, from the beginning of (1), that we have in fact:

$$\frac{P}{k!}\in\mathbb{Z}$$

Finally, we still have to prove that we have $P\neq 0$, for arbitrarily large numbers $k>>0$. But the point here is that for $k+1>n,|c_0|$, chosen prime, a detailed study of our integral shows that we have $(k+1)\not|P$, and so $P\neq 0$ indeed, as desired.

(4) With this done, let us look now at $Q$. Our claim is that for $k>>0$ we have:

$$\left|\frac{Q}{k!}\right|<1$$

Indeed, by using the exact formula of $f_k$, from the beginning of (1), we have:

$$\begin{aligned}f_k(x)e^{-x}&=x^k\left[(x-1)\ldots(x-n)\right]^{k+1}e^{-x}\\&=\left[x(x-1)\ldots(x-n)\right]^k\times(x-1)\ldots(x-n)e^{-x}\end{aligned}$$

We conclude that for $x\in[0,n]$ we have an estimate as follows, with $G,H>0$ being certain constants, appearing as maxima of the two components appearing above:

$$|f_k(x)e^{-x}|<G^kH$$

Now by integrating, we obtain from this the following estimate for $Q$ itself:

$$\begin{aligned}|Q|&=\left|c_1\int_0^1 f_k(x)e^{-x}dx+\ldots+c_ne^n\int_0^n f_k(x)e^{-x}dx\right|\\&\leq|c_1|\int_0^1|f_k(x)e^{-x}|dx+\ldots+|c_n|e^n\int_0^n|f_k(x)e^{-x}|dx\\&\leq|c_1|\cdot G^kH+\ldots+|c_n|e^n\cdot nG^kH\\&=(|c_1|e+\ldots+|c_n|e^n)\frac{n(n+1)}{2}G^kH\end{aligned}$$

But in this estimate the only term depending on $k$ is the power $G^k$, and since since $k!$ grows much faster than this power $G^k$, this proves our claim:

$$\left|\frac{Q}{k!}\right|\approx\frac{G^k}{k!}\to 0$$

(5) And with this, done, because what we found in (3,4) contradicts the formula $P+Q=0$ from (2). Thus $e$ is indeed transcendental, as claimed. $\square$

As a continuation of the above material, let us prove now, a bit as for $e$ before, that $\pi$ is irrational, and even transcendental. The result here is as follows:

THEOREM 15.21. *The number $\pi = 3.14159\ldots$ has the following properties:*

(1) *It is irrational.*
(2) *It is transcendental.*

PROOF. This is indeed something quite routine, by using the same ideas as before for $e$, but with everything being now a bit more technical, the idea being as follows:

(1) In what regards the irrationality of $\pi$, no simple argument as for $e$ is available, so we rather have to take our inspiration from the Hermite proof of the transcendence of $e$, given above. With this idea in mind, consider the following quantities:

$$I_n(t) = \int_{-1}^1 (1-x^2)^n \cos(xt)dx$$

By double partial integration we obtain the following formula:

$$\begin{aligned}
I_n(t) &= \int_{-1}^1 (1-x^2)^n \cos(xt)dx \\
&= \int_{-1}^1 2nx(1-x^2)^{n-1}\frac{\sin(xt)}{t}\,dx \\
&= \frac{2n}{t}\int_{-1}^1 x(1-x^2)^{n-1}\sin(xt)dx \\
&= \frac{2n}{t}\int_{-1}^1 \left[(1-x^2)^{n-1} - 2(n-1)x^2(1-x^2)^{n-2}\right]\frac{\cos(xt)}{t}\,dx \\
&= \frac{2n}{t^2}\int_{-1}^1 (1-x^2)^{n-2}\left[1-x^2-2(n-1)x^2\right]\cos(xt)dx \\
&= \frac{2n}{t^2}\int_{-1}^1 (1-x^2)^{n-2}\left[1-(2n-1)x^2\right]\cos(xt)dx \\
&= \frac{2n}{t^2}\int_{-1}^1 (1-x^2)^{n-2}\left[(2n-1)(1-x^2)-(2n-2)\right]\cos(xt)dx \\
&= \frac{2n}{t^2}\left[(2n-1)I_{n-1}(t)-(2n-2)I_{n-2}(t)\right]
\end{aligned}$$

Thus, we have the following recurrence relation for our quantities:

$$t^2 I_n(t) = 2n(2n-1)I_{n-1}(t) - 4n(n-1)I_{n-2}(t)$$

In terms of $J_n(t) = t^{2n+1}I_n(t)$, this recurrence formula becomes:

$$J_n(t) = 2n(2n-1)J_{n-1}(t) - 4n(n-1)t^2 J_{n-2}(t)$$

Regarding now the initial data, for this latter recurrence, this is as follows:

$$J_0(t) = 2\sin t \quad , \quad J_1(t) = -4t\cos t + 4\sin t$$

We conclude from this that we must have a formula as follows, with $P_n, Q_n$ being certain polynomials of degree $\leq n$, with integer coefficients:

$$J_n(t) = n!(P_n(t)\sin t + Q_n(t)\cos t)$$

Now observe that with $t = \pi/2$, we obtain from this the following formula:

$$\left(\frac{\pi}{2}\right)^{2n+1} I_n\left(\frac{\pi}{2}\right) = n!P_n\left(\frac{\pi}{2}\right)$$

Assume now by contradiction that $\pi$ is rational, so that $\pi/2 = a/b$ with $a, b \in \mathbb{N}$. We can rewrite the formula found above in the following more convenient way:

$$\frac{a^{2n+1}}{n!} I_n\left(\frac{a}{b}\right) = b^{2n+1}P_n\left(\frac{a}{b}\right)$$

But, by definition of the integrals $I_n$, we have $I_n(a/b) = I_n(\pi/2) \in (0, 2)$. Thus with $n >> 0$ the number on the left belongs to $(0, 1)$, which is contradictory, because the number on the right is an integer. We conclude that $\pi$ is irrational, as claimed.

(2) Regarding now the transcendence of $\pi$, again it is possible to adapt the ideas of Hermite for $e$, from the proof of Theorem 15.20, but this remains something quite technical. Instead, it is better to have an algebraic look at this, by using the Lindemann-Weierstrass theorem, which states that if $a_1, \ldots, a_n \in \mathbb{C}$ are algebrically independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebrically independent too over $\mathbb{Q}$. To be more precise:

– To start with, observe that the Lindemann-Weierstrass theorem shows with $n = 1$ that $e = e^1$ is transcendental. Thus, this is definitely something non-trivial.

– However, this is something that can be proved, with some knowledge of algebra and field theory. For more on this, you can consult any advanced number theory book.

– And, in relation with $\pi$, we can use again $n = 1$, but this time in conjunction with the Euler formula $e^{\pi i} = -1$, and we obtain that $\pi$ is transcendental. □

So, this was for the story with $\pi$, quickly told, and in practice, in order to fully understand all this, there are of course many things to be learned. So, find a good old book on number theory, such as Hardy and Wright [50], and start reading.

## 15c. Playing games

Still talking about $e, \pi$, I don't know about you, but personally I would like to have more interpretations of them. And, regarding $e$, here is something quite interesting:

THEOREM 15.22. *The probability for a random permutation $\sigma \in S_N$ to be a derangement, that is, to have no fixed points, is given by the following formula:*

$$P = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^N \frac{1}{N!}$$

*Thus we have the following asymptotic formula, in the $N \to \infty$ limit,*

$$P \simeq \frac{1}{e}$$

*with $e = 2.71828\ldots$ being the usual constant from analysis.*

PROOF. This is something very classical, and beautiful, which is best viewed by using the inclusion-exclusion principle. Consider indeed the following sets:

$$S_N^i = \left\{ \sigma \in S_N \middle| \sigma(i) = i \right\}$$

By inclusion-exclusion, the probability that we are interested in is given by:

$$\begin{aligned} P &= \frac{1}{N!}\left( |S_N| - \sum_i |S_N^i| + \sum_{i<j} |S_N^i \cap S_N^j| - \ldots + (-1)^N \sum_{i_1<\ldots<i_N} |S_N^{i_1} \cap \ldots \cap S_N^{i_N}| \right) \\ &= \frac{1}{N!} \sum_{k=0}^N (-1)^k \sum_{i_1<\ldots<i_k} (N-k)! \\ &= \frac{1}{N!} \sum_{k=0}^N (-1)^k \binom{N}{k} (N-k)! \\ &= \sum_{k=0}^N \frac{(-1)^k}{k!} \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

In order to further build on this, let us formulate the following key definition:

DEFINITION 15.23. *The Poisson law of parameter 1 is the following measure,*

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

*and the Poisson law of parameter $t > 0$ is the following measure,*

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

*with the letter "p" standing for Poisson.*

Observe that our laws have indeed mass 1, as they should, and this due to:

$$e^t=\sum_k\frac{t^k}{k!}$$

Getting back now to permutations, we have the following result:

THEOREM 15.24. *The main character of $S_N$, which counts the fixed points,*

$$\chi(\sigma)=\#\left\{i\in\{1,\ldots,N\}\Big|\sigma(i)=i\right\}$$

*follows the Poisson law $p_1$, in the $N\to\infty$ limit. More generally, the variable*

$$\chi_t(\sigma)=\#\left\{i\in\{1,\ldots,[tN]\}\Big|\sigma(i)=i\right\}$$

*with $t\in(0,1]$ follows the Poisson law $p_t$, in the $N\to\infty$ limit.*

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In order to establish the first result in the statement, regarding the main character, we must prove the following formula, for any $r\in\mathbb N$, in the $N\to\infty$ limit:

$$P(\chi=r)\simeq\frac{1}{r!e}$$

We already know, from Theorem 15.22, that this formula holds at $r=0$:

$$P(\chi=0)\simeq\frac{1}{e}$$

In the general case, we have to count the permutations $\sigma\in S_N$ having exactly $r$ points. Now since having such a permutation amounts in choosing $r$ points among $1,\ldots,N$, and then permuting the $N-r$ points left, without fixed points allowed, we have:

$$\begin{aligned}\#\left\{\sigma\in S_N\Big|\chi(\sigma)=r\right\}&=\binom{N}{r}\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}\\&=\frac{N!}{r!(N-r)!}\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}\\&=N!\times\frac{1}{r!}\times\frac{\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}}{(N-r)!}\end{aligned}$$

By dividing everything by $N!$, we obtain from this the following formula:

$$\frac{\#\left\{\sigma\in S_N\Big|\chi(\sigma)=r\right\}}{N!}=\frac{1}{r!}\times\frac{\#\left\{\sigma\in S_{N-r}\Big|\chi(\sigma)=0\right\}}{(N-r)!}$$

Now by using the computation at $r=0$, that we already have, from Theorem 15.22, it follows that with $N\to\infty$ we have the following estimate:

$$P(\chi=r)\simeq\frac{1}{r!}\cdot P(\chi=0)\simeq\frac{1}{r!}\cdot\frac{1}{e}$$

Thus, we obtain as limiting measure the Poisson law of parameter 1, as stated.

(2) Regarding now the second assertion, involving an arbitrary parameter $t\in(0,1]$, the proof here is similar. To be more precise, by using the inclusion-exclusion principle, as in the proof of Theorem 15.22, we first have the following formula:

$$P(\chi_t=0)\simeq\frac{1}{e^t}$$

But then, we can generalize this formula, by proceeding as in (1) above, into:

$$P(\chi_t=r)\simeq\frac{t^r}{r!e^t}$$

Thus, we obtain as limiting measure the Poisson law of parameter $t$, as stated. □

Quite nice, all the above. In relation with $\pi$ now, and along the same lines, connections with probability and games, we have the following remarkable result, due to Buffon:

THEOREM 15.25. *The probability for a needle of length* 1*, when trown on a grid of parallel* 1*-spaced lines, to intersect one line, is:*

$$P=\frac{2}{\pi}$$

*Moreover, we have generalizations of this result, with needles of arbitrary length, thrown over a grid of parallel lines, with arbitrary spacing.*

PROOF. This is something quite tricky, and mandatory for properly learning probability theory, and science in general, because there are several possible modelings of the problem, leading, quite surprisingly, to different values of $P$. And, obviously, only one such modeling can be the correct one. We will leave this as an exercise, and enjoy. □

## 15d. Basel formula

We are not done with our mathematics, because when looking at the list of questions from the beginning of this chapter, all solved, save for the most important one:

QUESTION 15.26. *Is there any magic formula for $\pi$, allowing us to reach to*

$$\pi=3.1415926535\ldots$$

*and why not, to do some other things, too?*

In answer to this, following Euler, we have the following remarkable result:

THEOREM 15.27. *We have the following formula, due to Euler,*

$$\sum_{n=1}^\infty\frac{1}{n^2}=\frac{\pi^2}{6}$$

*answering the Basel problem, asking for the computation of this sum.*

PROOF. The original proof of Euler is as follows, based on the fact that the zeroes of $\sin x/x$ appear precisely at the points $x=k\pi$, with $k\in\mathbb Z$:

$$\begin{aligned}\frac{\sin x}{x}&=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\\&=\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\\&=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots\\&=1-\frac{1}{\pi^2}\sum_{n=1}^\infty\frac{1}{n^2}\,x^2+\ldots\end{aligned}$$

However, in practice, all this needs a bit more justification, of course. □

Summarizing, Question 15.26 solved, save for fully understanding the proof of the Basel formula. In order to discuss this, let us formulate the following key definition:

DEFINITION 15.28. *The Riemann zeta function is given by*

$$\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}$$

*with the exponent being an integer $s\geq 2$.*

According to Euler we have the following formula, that we would like to fully understand, and why not generalize too, into formulae for $\zeta(s)$ at higher values of $s\geq 2$:

$$\zeta(s)=\frac{\pi^2}{6}$$

In order to discuss this, we will need the following well-known fact:

THEOREM 15.29. *We can talk about the gamma function*

$$\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}\,dx$$

*extending the usual factorial of integers, $\Gamma(s)=(s-1)!$.*

PROOF. The integral converges indeed, and by partial integration we have:

$$\begin{aligned}\Gamma(s+1) &= \int_0^\infty x^s e^{-x}\,dx\\ &= \int_0^\infty s x^{s-1} e^{-x}\,dx\\ &= s\Gamma(s)\end{aligned}$$

Regarding now the case $s \in \mathbb{N}$, for the initial value $s = 1$ we have:

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

Thus, for $s \in \mathbb{N}$ we have indeed $\Gamma(s) = (s-1)!$, as claimed. □

Getting now to zeta, we can formulate a key result about it, as follows:

THEOREM 15.30. *We have the following formula,*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx$$

*valid for any $s \in \mathbb{C}$ with $Re(s) > 1$.*

PROOF. We have indeed the following computation:

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx &= \int_0^\infty \frac{x^{s-1}}{e^x} \cdot \frac{1}{1-e^{-x}}\,dx\\ &= \int_0^\infty x^{s-1}(e^{-x} + e^{-2x} + e^{-3x} + \ldots)\\ &= \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx}\,dx\\ &= \sum_{n=1}^\infty \int_0^\infty \left(\frac{y}{n}\right)^{s-1} e^{-y}\,\frac{dy}{n}\\ &= \sum_{n=1}^\infty \frac{1}{n^s} \int_0^\infty y^{s-1} e^{-y}\,dy\\ &= \zeta(s)\Gamma(s)\end{aligned}$$

Thus, we are led to the formula in the statement. □

At a more advanced level now, we can try to compute particular values of $\zeta$. Things are quite tricky here, and we have the following result, which is of interest to us:

THEOREM 15.31. *We have the following formula, for the even integers $s = 2k$,*

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}$$

*with $B_n$ being the Bernoulli numbers, which in practice gives the formulae*

$$\zeta(2) = \frac{\pi^2}{6} \quad , \quad \zeta(4) = \frac{\pi^4}{90} \quad , \quad \zeta(6) = \frac{\pi^6}{945} \quad , \quad \zeta(8) = \frac{\pi^8}{9450} \quad , \quad \ldots$$

*generalizing the formula $\zeta(2) = \pi^2/6$ of Euler, solving the Basel problem.*

PROOF. This is something quite tricky, the idea being as follows:

(1) We have the following computation, based on the formula in Theorem 15.30:

$$\begin{aligned}
\zeta(2k) &= \frac{1}{\Gamma(2k)}\int_0^\infty \frac{x^{2k-1}}{e^x-1}\,dx \\
&= \frac{1}{(2k-1)!}\int_0^\infty \frac{x^{2k-1}}{e^x-1}\,dx \\
&= \frac{1}{(2k-1)!}\int_0^\infty \frac{(2\pi t)^{2k-1}}{e^{2\pi t}-1}\,2\pi dt \\
&= \frac{(2\pi)^{2k}}{(2k-1)!}\int_0^\infty \frac{t^{2k-1}}{e^{2\pi t}-1}\,dt
\end{aligned}$$

(2) But, we recognize on the right the integral giving rise to the even Bernoulli numbers, with one of the many definitions of these numbers being as follows:

$$B_{2k} = 4k(-1)^{k+1}\int_0^\infty \frac{t^{2k-1}}{e^{2\pi t}-1}\,dt$$

Thus, we can finish our computation of the values $\zeta(2k)$ as follows:

$$\begin{aligned}
\zeta(2k) &= \frac{(2\pi)^{2k}}{(2k-1)!}\cdot(-1)^{k+1}\frac{B_{2k}}{4k} \\
&= (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2\cdot(2k)!}
\end{aligned}$$

(3) Regarding now the Bernoulli numbers, there is a long story here. At the beginning, we have the following formula of Bernoulli, standing as a definition for them:

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\sum_{k=0}^{m} B_k n^{m+1-k}$$

This leads to the following recurrence relation, which computes them:

$$B_m = -\frac{1}{m+1}\sum_{k=0}^{m-1}\binom{m+1}{k}B_k$$

In practice, we can see that the odd Bernoulli numbers all vanish, except for the first one, $B_1 = -1/2$, and that the even Bernoulli numbers are as follows:

$$\frac{1}{6}\quad,\quad -\frac{1}{30}\quad,\quad \frac{1}{42}\quad,\quad -\frac{1}{30}\quad,\quad \frac{5}{66}\quad,\quad -\frac{691}{2730}\quad,\quad \frac{7}{6}\quad,\quad \ldots$$

(4) For analytic purposes, the Bernoulli numbers are best viewed as follows, with this coming from the fact that the coefficients satisfy the above recurrence relation:

$$\begin{aligned}\frac{x}{e^x-1} &= \sum_{n=0}^\infty B_n\frac{x^n}{n!}\\ &= 1-\frac{1}{2}x+\frac{1}{6}\cdot\frac{x^2}{2!}-\frac{1}{30}\cdot\frac{x^4}{4!}+\frac{1}{42}\cdot\frac{x^6}{6!}-\frac{1}{30}\cdot\frac{x^8}{8!}+\ldots\end{aligned}$$

Observe that all this is related as well to the hyperbolic functions, via:

$$\frac{x}{2}\left(\coth\frac{x}{2}-1\right)=\frac{x}{e^x-1}=\sum_{n=0}^\infty B_n\frac{x^n}{n!}$$

The point now is that, in relation with our zeta business, the above analytic formulae give, after some calculus, the formula that we used in (3), namely:

$$B_{2k}=4k(-1)^{k+1}\int_0^\infty\frac{t^{2k-1}}{e^{2\pi t}-1}\,dt$$

(5) Finally, no discussion about the Bernoulli numbers would be complete without mentioning the Euler-Maclaurin formula, involving them, which is as follows:

$$\begin{aligned}\sum_{k=0}^{n-1}f(x) &\simeq \int_0^n f(x)dx-\frac{1}{2}(f(n)-f(0))\\ &+\frac{1}{6}\cdot\frac{f'(n)-f'(0)}{2!}-\frac{1}{30}\cdot\frac{f^{(3)}(n)-f^{(3)}(0)}{4!}\\ &+\frac{1}{42}\cdot\frac{f^{(5)}(n)-f^{(5)}(0)}{6!}-\frac{1}{30}\cdot\frac{f^{(7)}(n)-f^{(7)}(0)}{8!}+\ldots\end{aligned}$$

(6) And there is more coming from the complex extension of the zeta function, by analytic continuation, which is something quite standard. Indeed, the values of zeta at the negative integers $0,-1,-2,-3,\ldots$ are not $\infty$, but are rather given by:

$$\zeta(-n)=(-1)^n\frac{B_{n+1}}{n+1}$$

Alternatively, we have the following formula for the Bernoulli numbers:

$$B_n = (-1)^{n-1} n\zeta(1-n)$$

(7) In any case, we are led to the various conclusions in the statement, both theoretical and numeric. And exercise for you of course to learn more about the Bernoulli numbers, and beware of the freakish notations used by mathematicians there. □

As a more digest form of Theorem 15.31, let us record as well:

THEOREM 15.32. *The generating function of the numbers $\zeta(2k)$ with $k \in \mathbb{N}$ is*

$$\sum_{k=0}^{\infty} \zeta(2k)x^{2k} = -\frac{\pi x}{2}\cot(\pi x)$$

*and with this generalizing the formula involving Bernoulli numbers.*

PROOF. This is something tricky, again, the idea being as follows:

(1) A version of the recurrence formula for Bernoulli numbers is as follows:

$$B_{2n} = -\frac{1}{n+1/2}\sum_{k=1}^{n-1}\binom{2n}{2k} B_{2k}B_{2n-2k}$$

Now observe that this formula can be written in the following way:

$$\frac{B_{2n}}{(2n)!} = -\frac{1}{n+1/2}\sum_{k=1}^{n-1}\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2n-2k}}{(2n-2k)!}$$

In view of Theorem 15.31, we obtain the following formula, valid at any $n > 1$:

$$\zeta(2n) = \frac{1}{n+1/2}\sum_{k=1}^{n-1}\zeta(2k)\zeta(2n-2k)$$

(2) But this allows the computation of the series in the statement, by squaring that series. Indeed, consider the following modified version of that series:

$$f(x) = 2\sum_{k=0}^{\infty}\zeta(2k)\left(\frac{x}{\pi}\right)^{2k}$$

By squaring, and using the recurrence formula for the numbers $\zeta(2n)$ found in (1), with some care at the values $n = 0, 1$, not covered by that formula, we obtain:

$$f^2 + f + x^2 = xf'$$

(3) But this is precisely the functional equation satisfied by $g(x) = -x\cot x$. Indeed, by using the well-known formula $\cot' = -\cot^2 - 1$, we have:

$$\begin{aligned} xg' &= x(-\cot x - x\cot' x) \\ &= x(-\cot x + x\cot^2 x + x) \\ &= g + g^2 + x^2 \end{aligned}$$

(4) We conclude that we have $f = g$, which reads:

$$2\sum_{k=0}^\infty \zeta(2k)\left(\frac{x}{\pi}\right)^{2k} = -x\cot x$$

Now by replacing $x \to \pi x$, we obtain the formula in the statement. □

Regarding now the values $\zeta(2k+1)$ with $k \in \mathbb{N}$, the story here is more complicated, with the first such number being the Apéry constant, given by:

$$\zeta(3) = \sum_{n=1}^\infty \frac{1}{n^3}$$

There has been a lot of work on this number, by Apéry and others, and on the higher $\xi(2k+1)$ values as well. Let us record here the following result, a bit of physics flavor:

Theorem 15.33. *We have the following formula,*

$$\zeta(s) = \int_0^1 \ldots \int_0^1 \frac{dx_1 \ldots dx_s}{1 - x_1 \ldots x_s}$$

*valid for any $s \in \mathbb{N}$, $s \geq 2$.*

Proof. This follows as usual from some calculus, the idea being as follows:

(1) At $s = 2$ we have the following computation, using Theorem 15.30:

$$\begin{aligned} \int_0^1\int_0^1 \frac{1}{1-xy}\,dxdy &= \int_0^1 \left[-\frac{\log(1-xy)}{y}\right]_0^1 dy \\ &= -\int_0^1 \frac{\log(1-y)}{y}\,dy \\ &= -\int_0^\infty \frac{\log(e^{-t})}{1-e^{-t}}\,e^{-t}dt \\ &= \int_0^\infty \frac{t}{e^t - 1}\,dt \\ &= \zeta(2)\Gamma(2) \\ &= \zeta(2) \end{aligned}$$

(2) In the general case, $s \in \mathbb{N}$, the best is to start with the following formula:

$$\frac{1}{1-x_1\ldots x_s}=\sum_{n=0}^\infty (x_1\ldots x_s)^n$$

Thus, the integral in the statement is given by the following formula:

$$\int_0^1\ldots\int_0^1\frac{dx_1\ldots dx_s}{1-x_1\ldots x_s}=\sum_{n=0}^\infty\int_0^1\ldots\int_0^1(x_1\ldots x_s)^n dx_1\ldots dx_s$$

But this leads to the formula in the statement, after some computations. □

Many other things can be said about the Riemann zeta function and its special values, as a continuation of the above. Check here any advanced number theory book.

## 15e. Exercises

This was yet another key calculus chapter, and as exercises, we have:

EXERCISE 15.34. *Meditate on the random numbers needed for Monte Carlo.*

EXERCISE 15.35. *Find, then sell, a good algorithm for generating random numbers.*

EXERCISE 15.36. *Understand when exactly the integrals commute with limits, or sums.*

EXERCISE 15.37. *Find exercises about computing primitives, and solve them.*

EXERCISE 15.38. *Clarify all the details in relation with the transcedentality of $e$.*

EXERCISE 15.39. *Clarify all the details in relation with the irrationality of $\pi$.*

EXERCISE 15.40. *Clarify all the details in relation with the transcedentality of $\pi$.*

EXERCISE 15.41. *Experiment, abstractly and concretely, with the Buffon needle.*

As bonus exercise, and no surprise here, read more about the zeta function.

CHAPTER 16

# Spherical integrals

## 16a. Advanced calculus

Time to end this book, and looking back at the list of questions that we formulated at the beginning of the present Part IV, there are still many things to be discussed. These concern for the most analysis in arbitrary $N$ dimensions, and the role of $\pi$ and trigonometry there. We will be here, in this final chapter, for discussing this.

As a bonus, we will discuss as well the remaining 1-variable question that we have left, which concerns the study of the integrals of the following type, which will turn to appear quite naturally in relation with the $N$ variable questions to be investigated here:

$$I=\int_a^b\sin^p t\cos^p t\,dt$$

Getting started now, multivariable calculus is a matter of merging the linear algebra material from chapter 12, concerning the linear maps $f:\mathbb{R}^N\to\mathbb{R}^M$, and the basic calculus from chapter 13, concerning the arbitary maps $f:\mathbb{R}\to\mathbb{R}$. We have indeed the following result, to start with, at order 1, which creates a wide bridge between these topics:

THEOREM 16.1. *The derivative of a function $f:\mathbb{R}^N\to\mathbb{R}^M$, making the formula*

$$f(x+t)\simeq f(x)+f'(x)t$$

*work, must be the matrix of partial derivatives at $x$, namely*

$$f'(x)=\left(\frac{df_i}{dx_j}(x)\right)_{ij}\in M_{M\times N}(\mathbb{R})$$

*acting on the vectors $t\in\mathbb{R}^N$ by usual multiplication.*

PROOF. As a first observation, the formula in the statement makes sense indeed, as an equality, or rather approximation, of vectors in $\mathbb{R}^M$, as follows:

$$f\begin{pmatrix}x_1+t_1\\ \vdots\\ x_N+t_N\end{pmatrix}\simeq f\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}+\begin{pmatrix}\frac{df_1}{dx_1}(x)&\ldots&\frac{df_1}{dx_N}(x)\\ \vdots&&\vdots\\ \frac{df_M}{dx_1}(x)&\ldots&\frac{df_M}{dx_N}(x)\end{pmatrix}\begin{pmatrix}t_1\\ \vdots\\ t_N\end{pmatrix}$$

In order to prove now this formula, we can proceed by recurrence, as follows:

(1) First of all, at $N=M=1$ what we have is a usual 1-variable function $f:\mathbb{R}\to\mathbb{R}$, and the formula in the statement is something that we know well, namely:

$$f(x+t)\simeq f(x)+f'(x)t$$

(2) Let us discuss now the case $N=2,M=1$. Here what we have is a function $f:\mathbb{R}^2\to\mathbb{R}$, and by using twice the basic approximation result from (1), we obtain:

$$\begin{aligned}
f\begin{pmatrix}x_1+t_1\\x_2+t_2\end{pmatrix} &\simeq f\begin{pmatrix}x_1+t_1\\x_2\end{pmatrix}+\frac{df}{dx_2}(x)t_2\\
&\simeq f\begin{pmatrix}x_1\\x_2\end{pmatrix}+\frac{df}{dx_1}(x)t_1+\frac{df}{dx_2}(x)t_2\\
&= f\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}\frac{df}{dx_1}(x) & \frac{df}{dx_2}(x)\end{pmatrix}\begin{pmatrix}t_1\\t_2\end{pmatrix}
\end{aligned}$$

(3) More generally, we can deal in this way with the case $N\in\mathbb{N},M=1$, by recurrence. But this gives the result in the general case $N,M\in\mathbb{N}$ too. Indeed, let us write:

$$f=\begin{pmatrix}f_1\\ \vdots\\ f_M\end{pmatrix}$$

We can apply our result to each of the components $f_i:\mathbb{R}^N\to\mathbb{R}$, and we get:

$$f_i\begin{pmatrix}x_1+t_1\\ \vdots\\ x_N+t_N\end{pmatrix}\simeq f_i\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}+\begin{pmatrix}\frac{df_i}{dx_1}(x) & \ldots & \frac{df_i}{dx_N}(x)\end{pmatrix}\begin{pmatrix}t_1\\ \vdots\\ t_N\end{pmatrix}$$

But this is precisely what we want, at the level of the global map $f:\mathbb{R}^N\to\mathbb{R}^M$. □

As a technical complement to the above result, further clarifying things, we have:

THEOREM 16.2. *For a function $f:X\to\mathbb{R}^M$, with $X\subset\mathbb{R}^N$, the following conditions are equivalent, and in this case we say that $f$ is continuously differentiable:*

1. *$f$ is differentiable, and the map $x\to f'(x)$ is continuous.*
2. *$f$ has partial derivatives, which are continuous with respect to $x\in X$.*

*If these conditions are satisfied, $f'(x)$ is the matrix fomed by the partial derivatives at $x$.*

PROOF. We already know, from Theorem 16.1, that the last assertion holds. Regarding now the proof of the equivalence, this goes as follows:

(1) $\implies$ (2) Assuming that $f$ is differentiable, we know from Theorem 16.1 that $f'(x)$ must be the matrix fomed by the partial derivatives at $x$. Thus, for any $x,y\in X$:

$$\frac{df_i}{dx_j}(x)-\frac{df_i}{dx_j}(y)=f'(x)_{ij}-f'(y)_{ij}$$

By applying now the absolute value, we obtain from this the following estimate:

$$\begin{aligned}\left|\frac{df_i}{dx_j}(x)-\frac{df_i}{dx_j}(y)\right| &= |f'(x)_{ij}-f'(y)_{ij}|\\ &= |(f'(x)-f'(y))_{ij}|\\ &\leq ||f'(x)-f'(y)||\end{aligned}$$

But this gives the result, because if the map $x \to f'(x)$ is assumed to be continuous, then the partial derivatives follow to be continuous with respect to $x \in X$.

(2) $\implies$ (1) This is something more technical. For simplicity, let us assume $M = 1$, the proof in general being similar. Given $x \in X$ and $\varepsilon > 0$, let us pick $r > 0$ such that the ball $B = B_x(r)$ belongs to $X$, and such that the following happens, over $B$:

$$\left|\frac{df}{dx_j}(x)-\frac{df}{dx_j}(y)\right| < \frac{\varepsilon}{N}$$

Our claim is that, with this choice made, we have the following estimate, for any $t \in \mathbb{R}^N$ satisfying $||t|| < r$, with $A$ being the vector of partial derivatives at $x$:

$$|f(x+t)-f(x)-At| \leq \varepsilon||t||$$

In order to prove this claim, the idea will be that of suitably applying the mean value theorem, over the $N$ directions of $\mathbb{R}^N$. Indeed, consider the following vectors:

$$t^{(k)} = \begin{pmatrix} t_1\\ \vdots\\ t_k\\ 0\\ \vdots\\ 0\end{pmatrix}$$

In terms of these vectors, we have the following formula:

$$f(x+t)-f(x) = \sum_{j=1}^N f(x+t^{(j)})-f(x+t^{(j-1)})$$

Also, the mean value theorem gives a formula as follows, with $s_j \in [0,1]$:

$$f(x+t^{(j)})-f(x+t^{(j-1)}) = \frac{df}{dx_j}(x+s_jt^{(j)}+(1-s_j)t^{(j-1)})\cdot t_j$$

But, according to our assumption on $r > 0$ from the beginning, the derivative on the right differs from $\frac{df}{dx_j}(x)$ by something which is smaller than $\varepsilon/N$:

$$\left|\frac{df}{dx_j}(x+s_jt^{(j)}+(1-s_j)t^{(j-1)})-\frac{df}{dx_j}(x)\right| < \frac{\varepsilon}{N}$$

Now by putting everything together, we obtain the following estimate:

$$\begin{aligned}|f(x+t)-f(x)-At| &\leq \sum_{j=1}^N\left|f(x+t^{(j)})-f(x+t^{(j-1)})-\frac{df}{dx_j}(x)\cdot t_j\right|\\ &= \sum_{j=1}^N\left|\frac{df}{dx_j}(x+s_jt^{(j)}+(1-s_j)t^{(j-1)})\cdot t_j-\frac{df}{dx_j}(x)\cdot t_j\right|\\ &\leq \sum_{j=1}^N\frac{\varepsilon}{N}\cdot|t_j|\\ &\leq \varepsilon||t||\end{aligned}$$

Thus we have proved our claim, and this gives the result. □

Moving on, with this done, our next task will be that of extending to several variables our basic results from one-variable calculus. As a standard result here, we have:

Theorem 16.3. *We have the chain derivative formula*

$$(f\circ g)'(x)=f'(g(x))\cdot g'(x)$$

*as an equality of matrices.*

Proof. This is something standard in one variable, and in several variables the proof is similar, by using the abstract notion of derivative coming from Theorem 16.1. To be more precise, consider a composition of functions, as follows:

$$f:\mathbb R^N\to\mathbb R^M\quad,\quad g:\mathbb R^K\to\mathbb R^N\quad,\quad f\circ g:\mathbb R^K\to\mathbb R^M$$

According to Theorem 16.1, the derivatives of these functions are certain linear maps, corresponding to certain rectangular matrices, as follows:

$$f'(g(x))\in M_{M\times N}(\mathbb R)\quad,\quad g'(x)\in M_{N\times K}(\mathbb R)\qquad (f\circ g)'(x)\in M_{M\times K}(\mathbb R)$$

Thus, our formula makes sense indeed. As for proof, this comes from:

$$\begin{aligned}(f\circ g)(x+t) &= f(g(x+t))\\ &\simeq f(g(x)+g'(x)t)\\ &\simeq f(g(x))+f'(g(x))g'(x)t\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Also, again in analogy with what we know well from chapter 13, we have:

Theorem 16.4. *The Taylor formula at order* 1 *for a function* $f:\mathbb R^N\to\mathbb R$ *is*

$$f(x+t)\simeq f(x)+f'(x)t$$

*and in particular, in order for* $x$ *to be a local extremum, we must have* $f'(x)=0$*.*

PROOF. Here the first assertion is something that we know, and the second assertion follows from it. Indeed, let us look at the order 1 term, given by:

$$f'(x)t = \sum_{i=1}^N \frac{df}{dx_i}\, t_i$$

Now since this linear combination of the entries of $t \in \mathbb{R}^N$ can range among positives and negatives, unless all the coefficients are zero, which means $f'(x) = 0$, we are led to the conclusion that local extremum needs $f'(x) = 0$ to hold, as stated. □

Moving on, we can talk as well about higher derivatives, simply by performing the operation of taking derivatives recursively. As a first result here, we have:

THEOREM 16.5. *The double derivatives of a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy*

$$\frac{d^2f}{dxdy} = \frac{d^2f}{dydx}$$

*called Clairaut formula.*

PROOF. Given $z = (a, b)$, consider the following functions, with $h, k \in \mathbb{R}$ small:

$$u(h, k) = f(a + h, b + k) - f(a + h, b)$$
$$v(h, k) = f(a + h, b + k) - f(a, b + k)$$
$$w(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

By the mean value theorem, for $h, k \neq 0$ we can find $\alpha, \beta \in \mathbb{R}$ such that:

$$\begin{aligned} w(h,k) &= u(h,k) - u(0,k) \\ &= h \cdot \frac{d}{dx}\, u(\alpha h, k) \\ &= h\left(\frac{d}{dx}\, f(a + \alpha h, b + k) - \frac{d}{dx}\, f(a + \alpha h, b)\right) \\ &= hk \cdot \frac{d}{dy} \cdot \frac{d}{dx}\, f(a + \alpha h, b + \beta k) \end{aligned}$$

Similarly, again for $h, k \neq 0$, we can find $\gamma, \delta \in \mathbb{R}$ such that:

$$\begin{aligned} w(h,k) &= v(h,k) - v(h,0) \\ &= k \cdot \frac{d}{dy}\, v(h, \delta k) \\ &= k\left(\frac{d}{dy}\, f(a + h, b + \delta k) - \frac{d}{dy}\, f(a, b + \delta k)\right) \\ &= hk \cdot \frac{d}{dx} \cdot \frac{d}{dy}\, f(a + \gamma h, b + \delta k) \end{aligned}$$

Now by dividing everything by $hk\neq 0$, we conclude from this that the following equality holds, with the numbers $\alpha,\beta,\gamma,\delta\in\mathbb R$ being found as above:

$$\frac{d}{dy}\cdot\frac{d}{dx}\,f(a+\alpha h,b+\beta k)=\frac{d}{dx}\cdot\frac{d}{dy}\,f(a+\gamma h,b+\delta k)$$

But with $h,k\to 0$ we get from this the Clairaut formula at $z=(a,b)$, as desired. □

In arbitrary dimensions now, we have the following result:

THEOREM 16.6. *Given $f:\mathbb R^N\to\mathbb R$, we can talk about its higher derivatives,*

$$\frac{d^kf}{dx_{i_1}\ldots dx_{i_k}}=\frac{d}{dx_{i_1}}\cdots\frac{d}{dx_{i_k}}(f)$$

*provided that these derivatives exist indeed. Moreover, due to the Clairaut formula,*

$$\frac{d^2f}{dx_idx_j}=\frac{d^2f}{dx_jdx_i}$$

*the order in which these higher derivatives are computed is irrelevant.*

PROOF. This is indeed something self-explanatory, based on the Clairaut formula from Theorem 16.5, applied to the various 2-variable restrictions of $f:\mathbb R^N\to\mathbb R$. □

All this is very nice, and as an illustration, let us work out in detail the case $k=2$. Here things are quite special, and we can formulate the following definition:

DEFINITION 16.7. *Given a twice differentiable function $f:\mathbb R^N\to\mathbb R$, we set*

$$f''(x)=\left(\frac{d^2f}{dx_idx_j}\right)_{ij}$$

*which is a symmetric matrix, called Hessian matrix of $f$ at the point $x\in\mathbb R^N$.*

To be more precise, we know that when $f:\mathbb R^N\to\mathbb R$ is twice differentiable, its order 2 partial derivatives are the numbers in the statement. Now since these numbers naturally form a $N\times N$ matrix, the temptation is high to call this matrix $f''(x)$, and so we will do. And finally, we know from Clairaut that this matrix is symmetric:

$$f''(x)_{ij}=f''(x)_{ji}$$

Observe that at $N=1$ this is compatible with the usual definition of the second derivative $f''$, because in this case, the $1\times 1$ matrix from Definition 16.7 is:

$$f''(x)=(f''(x))\in M_{1\times 1}(\mathbb R)$$

As a word of warning, however, never use Definition 16.7 for functions $f:\mathbb R^N\to\mathbb R^M$, where the second derivative can only be something more complicated. Also, never attempt either to do something similar at $k=3$ or higher, for functions $f:\mathbb R^N\to\mathbb R$ with $N>1$, because again, that beast has too many indices, for being a true, honest matrix.

Back now to our usual business, approximation, we have the following result:

THEOREM 16.8. *Given a twice differentiable function $f:\mathbb{R}^N\to\mathbb{R}$, we have*

$$f(x+t)\simeq f(x)+f'(x)t+\frac{<f''(x)t,t>}{2}$$

*where $f''(x)\in M_N(\mathbb{R})$ stands as usual for the Hessian matrix.*

PROOF. This is something very standard, the idea being as follows:

(1) As a first observation, at $N=1$ the Hessian matrix is the usual $f''(x)$, and the formula in the statement is something that we know well from basic calculus, namely:

$$f(x+t)\simeq f(x)+f'(x)t+\frac{f''(x)t^2}{2}$$

(2) In general now, this is in fact something which does not need a new proof, because it follows from the one-variable formula above, applied to the restriction of $f$ to the following segment in $\mathbb{R}^N$, which can be regarded as being a one-variable interval:

$$I=[x,x+t]$$

To be more precise, let $y\in\mathbb{R}^N$, and consider the following function, with $r\in\mathbb{R}$:

$$g(r)=f(x+ry)$$

We know from (1) that the Taylor formula for $g$, at the point $r=0$, reads:

$$g(r)\simeq g(0)+g'(0)r+\frac{g''(0)r^2}{2}$$

And our claim is that, with $t=ry$, this is precisely the formula in the statement.

(3) So, let us see if our claim is correct. By using the chain rule, we have the following formula, with on the right, as usual, a row vector multiplied by a column vector:

$$g'(r)=f'(x+ry)\cdot y$$

By using again the chain rule, we can compute the second derivative as well:

$$\begin{aligned}
g''(r)&=(f'(x+ry)\cdot y)'\\
&=\left(\sum_i\frac{df}{dx_i}(x+ry)\cdot y_i\right)'\\
&=\sum_i\sum_j\frac{d^2f}{dx_idx_j}(x+ry)\cdot\frac{d(x+ry)_j}{dr}\cdot y_i\\
&=\sum_i\sum_j\frac{d^2f}{dx_idx_j}(x+ry)\cdot y_iy_j\\
&=<f''(x+ry)y,y>
\end{aligned}$$

(4) Time now to conclude. We know that we have $g(r) = f(x + ry)$, and according to our various computations above, we have the following formulae:

$$g(0) = f(x) \quad , \quad g'(0) = f'(x) \quad , \quad g''(0) =< f''(x)y, y >$$

Buit with this data in hand, the usual Taylor formula for our one variable function $g$, at order 2, at the point $r = 0$, takes the following form, with $t = ry$:

$$\begin{aligned} f(x + ry) &\simeq f(x) + f'(x)ry + \frac{< f''(x)y, y > r^2}{2} \\ &= f(x) + f'(x)t + \frac{< f''(x)t, t >}{2} \end{aligned}$$

Thus, we have obtained the formula in the statement. □

We can go back now to local extrema, and we have, improving Theorem 16.4:

Theorem 16.9. *In order for a twice differentiable function $f : \mathbb{R}^N \to \mathbb{R}$ to have a local minimum or maximum at $x \in \mathbb{R}^N$, the first derivative must vanish there,*

$$f'(x) = 0$$

*and the Hessian must be positive or negative, in the sense that the quantities*

$$< f''(x)t, t >\in \mathbb{R}$$

*must keep a constant sign, positive or negative, when $t \in \mathbb{R}^N$ varies.*

Proof. This comes from Theorem 16.8. Consider indeed the formula there, namely:

$$f(x + t) \simeq f(x) + f'(x)t + \frac{< f''(x)t, t >}{2}$$

We know from Theorem 16.4 that, in order for our function to have a local minimum or maximum at $x \in \mathbb{R}^N$, the first derivative must vanish there, $f'(x) = 0$. Moreover, with this assumption made, the approximation that we have around $x$ becomes:

$$f(x + t) \simeq f(x) + \frac{< f''(x)t, t >}{2}$$

Thus, we are led to the conclusion in the statement. □

At higher order now, things become more complicated, as follows:

Theorem 16.10. *Given an order $k$ differentiable function $f : \mathbb{R}^N \to \mathbb{R}$, we have*

$$f(x + t) \simeq f(x) + f'(x)t + \frac{< f''(x)t, t >}{2} + \ldots$$

*and this can help in identifying the local extrema, a bit as in the one-variable case.*

PROOF. The study here is very similar to that at $k = 2$, from the proof of Theorem 16.8, with everything coming from the usual Taylor formula, applied on:

$$I = [x, x + t]$$

Thus, it is pretty much clear that we are led to the conclusion in the statement. We will leave some study here as an instructive exercise. □

Now back to order 2, where most problems usually take place, the story is not over with Theorem 16.9. Indeed, the Hessian matrix being symmetric, the linear algebra theory from chapter 12 applies to it, and shows that it is diagonalizable. Thus, we can formulate the following result, complementing what was said in Theorem 16.9:

THEOREM 16.11. *Given a symmetric matrix $A \in M_N(\mathbb{R})$, as for instance a Hessian matrix $A = f''(x)$, with eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$, the following happen,*

1. *$< At, t > \geq 0$ for any $t \in \mathbb{R}^N$ precisely when $\lambda_1, \ldots, \lambda_N \geq 0$.*
2. *$< At, t >> 0$ for any $t \neq 0$ precisely when $\lambda_1, \ldots, \lambda_N > 0$.*
3. *$< At, t > \leq 0$ for any $t \in \mathbb{R}^N$ precisely when $\lambda_1, \ldots, \lambda_N \leq 0$.*
4. *$< At, t >< 0$ for any $t \neq 0$ precisely when $\lambda_1, \ldots, \lambda_N < 0$.*

*and with this helping identifying the minima and maxima of functions $f : \mathbb{R}^N \to \mathbb{R}$.*

PROOF. This is something self-explanatory, coming from our results from chapter 12, and with the last assertion being something that we know, from Theorem 16.9. □

As a comment, the above result is of course not the end of the story with the extrema of functions $f : \mathbb{R}^N \to \mathbb{R}$, because depending on how the Hessian $A = f''(x)$ looks like, we might been in need of a study at higher order, as suggested in Theorem 16.10. We will leave some exploration here, examples and conclusions, as an instructive exercise.

## 16b. Spherical coordinates

With the derivatives of the functions $f : \mathbb{R}^N \to \mathbb{R}^M$ understood, time now to discuss the integrals. We are led in this way to the following question:

QUESTION 16.12. *How to integrate the functions $f : \mathbb{R}^N \to \mathbb{R}$,*

$$f \to \int_{\mathbb{R}^N} f(z)dz$$

*in analogy with what we know about integrating functions $f : \mathbb{R} \to \mathbb{R}$?*

In answer, and taking $N = 2$ for simplifying, I bet that your answer would be that we can define the multivariable integral simply by iterating, as follows:

$$\int_{\mathbb{R}^2} f(z)dz = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)dxdy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)dydx$$

However, there is a major bug with all this, coming from the following result:

THEOREM 16.13. *The Fubini formula, namely*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dxdy = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)dydx$$

*can fail, for certain suitably chosen functions.*

PROOF. We have indeed the following computation, no question about this:

$$\begin{aligned}\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dxdy &= \int_0^1\left[\frac{x}{x^2+y^2}\right]_0^1 dy\\ &= \int_0^1 \frac{1}{1+y^2}\,dy\\ &= \frac{\pi}{4}\end{aligned}$$

On the other hand, by using this, and symmetry, we have as well:

$$\begin{aligned}\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dydx &= \int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,dxdy\\ &= -\int_0^1\int_0^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dxdy\\ &= -\frac{\pi}{4}\end{aligned}$$

Thus Fubini can fail for certain functions, as said in the statement. Damn. □

What do do? Well, there is a mathematical answer to this, which is however something quite complicated, whose essentials can be summarized as follows:

THEOREM 16.14 (Measure theory). *We can rigorously integrate the functions*

$$f:\mathbb{R}^N\to\mathbb{R}$$

*and assuming that $f$ is measurable and integrable, in the sense that we have*

$$\int_{\mathbb{R}^N}|f(z)|dz<\infty$$

*we have the following equalities, for any decomposition $N=N_1+N_2$:*

$$\int_{\mathbb{R}^{N_1}}\int_{\mathbb{R}^{N_2}} f(x,y)dydx = \int_{\mathbb{R}^{N_2}}\int_{\mathbb{R}^{N_1}} f(x,y)dxdy = \int_{\mathbb{R}^N} f(z)dz$$

*Moreover, the same holds when $f:\mathbb{R}^N\to\mathbb{R}$ is assumed positive, and measurable.*

PROOF. This is something quite long and complicated, due to Lebesgue, Riesz, Borel, Fubini, Tonelli and others, traditionally learned in measure theory class. Alternatively, have a look at the first few dozen pages of Rudin [79], which explain all this. □

Summarizing, we can talk about multiple integrals. Getting now to general theory and rules, for computing such integrals, the key result here is the change of variable formula. In order to discuss this, let us start with something that we know well, in 1D:

Proposition 16.15. *We have the change of variable formula*

$$\int_a^b f(x)dx = \int_c^d f(\varphi(t))\varphi'(t)dt$$

*where* $c = \varphi^{-1}(a)$ *and* $d = \varphi^{-1}(b)$.

Proof. This follows with $f = F'$, via the following differentiation rule:

$$(F\varphi)'(t) = F'(\varphi(t))\varphi'(t)$$

Indeed, by integrating between $c$ and $d$, we obtain the result. □

In several variables now, we can only expect the above $\varphi'(t)$ factor to get replaced by something similar, a sort of "derivative of $\varphi$, arising as a real number". But this can only be the determinant $\det(\varphi'(t))$, right, and with this in mind, we are led to:

Theorem 16.16. *Given a transformation* $\varphi = (\varphi_1, \ldots, \varphi_N)$, *we have*

$$\int_E f(x)dx = \int_{\varphi^{-1}(E)} f(\varphi(t))|J_\varphi(t)|dt$$

*with the* $J_\varphi$ *quantity, called Jacobian, being given by*

$$J_\varphi(t) = \det\left[\left(\frac{d\varphi_i}{dx_j}(x)\right)_{ij}\right]$$

*and with this generalizing the one-variable formula from Proposition 16.15.*

Proof. This is something quite tricky, the idea being as follows:

(1) Observe first that this generalizes indeed the change of variable formula in 1 dimension, from Proposition 16.15, the point here being that the absolute value on the derivative appears as to compensate for the lack of explicit bounds for the integral.

(2) As a second observation, we can assume if we want, by linearity, that we are dealing with the constant function $f = 1$. For this function, our formula reads:

$$vol(E) = \int_{\varphi^{-1}(E)} |J_\varphi(t)|dt$$

In terms of $D = \varphi^{-1}(E)$, this amounts in proving that we have:

$$vol(\varphi(D)) = \int_D |J_\varphi(t)|dt$$

Now since this latter formula is additive with respect to $D$, it is enough to prove it for small cubes $D$. And here, as a first remark, our formula is clear for the linear maps $\varphi$, by using the definition of the determinant of real matrices, as a signed volume.

(3) However, the extension of this to the case of non-linear maps $\varphi$ is something which looks non-trivial, so we will not follow this path, in what follows. So, while the above $f=1$ discussion is certainly something nice, our theorem is still in need of a proof.

(4) In order to prove the theorem, as stated, let us rather focus on the transformations used $\varphi$, instead of the functions to be integrated $f$. Our first claim is that the validity of the theorem is stable under taking compositions of such transformations $\varphi$.

(5) In order to prove this claim, consider a composition, as follows:

$$\varphi:E\to F\quad,\quad\psi:D\to E\quad,\quad\varphi\circ\psi:D\to F$$

Assuming that the theorem holds for $\varphi,\psi$, we have the following computation:

$$\begin{aligned}\int_F f(x)dx &= \int_E f(\varphi(s))|J_\varphi(s)|ds\\ &= \int_D f(\varphi\circ\psi(t))|J_\varphi(\psi(t))|\cdot|J_\psi(t)|dt\\ &= \int_D f(\varphi\circ\psi(t))|J_{\varphi\circ\psi}(t)|dt\end{aligned}$$

Thus, our theorem holds as well for $\varphi\circ\psi$, and we have proved our claim.

(6) Next, as a key ingredient, let us examine the case where we are in $N=2$ dimensions, and our transformation $\varphi$ has one of the following special forms:

$$\varphi(x,y)=(\psi(x,y),y)\quad,\quad\varphi(x,y)=(x,\psi(x,y))$$

By symmetry, it is enough to deal with the first case. Here the Jacobian is $d\psi/dx$, and by replacing if needed $\psi\to-\psi$, we can assume that this Jacobian is positive, $d\psi/dx>0$. Now by assuming as before that $D=\varphi^{-1}(E)$ is a rectangle, $D=[a,b]\times[c,d]$, we can prove our formula by using the change of variables in 1 dimension, as follows:

$$\begin{aligned}\int_E f(s)ds &= \int_{\varphi(D)} f(x,y)dxdy\\ &= \int_c^d\int_{\psi(a,y)}^{\psi(b,y)} f(x,y)dxdy\\ &= \int_c^d\int_a^b f(\psi(x,y),y)\frac{d\psi}{dx}\,dxdy\\ &= \int_D f(\varphi(t))J_\varphi(t)dt\end{aligned}$$

(7) But with this, we can now prove the theorem, in $N = 2$ dimensions. Indeed, given a transformation $\varphi = (\varphi_1, \varphi_2)$, consider the following two transformations:

$$\phi(x,y) = (\varphi_1(x,y), y) \quad , \quad \psi(x,y) = (x, \varphi_2 \circ \phi^{-1}(x,y))$$

We have then $\varphi = \psi \circ \phi$, and by using (6) for $\psi, \phi$, which are of the special form there, and then (3) for composing, we conclude that the theorem holds for $\varphi$, as desired.

(8) Thus, theorem proved in $N = 2$ dimensions, and the extension of the above proof to arbitrary $N$ dimensions is straightforward, that we will leave this as an exercise. □

Time now do some exciting computations, with the technology that we have. In what regards the applications of Theorem 16.16, these often come via:

PROPOSITION 16.17. *We have polar coordinates in 2 dimensions,*

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$$

*the corresponding Jacobian being $J = r$.*

PROOF. This is elementary, the Jacobian being:

$$\begin{aligned} J &= \begin{vmatrix} \frac{d(r\cos t)}{dr} & \frac{d(r\cos t)}{dt} \\ \\ \frac{d(r\sin t)}{dr} & \frac{d(r\sin t)}{dt} \end{vmatrix} \\ &= \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} \\ &= r\cos^2 t + r\sin^2 t \\ &= r \end{aligned}$$

Thus, we have indeed the formula in the statement. □

We can now compute the Gauss integral, which is the best calculus formula ever:

THEOREM 16.18. *We have the following formula,*

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

*called Gauss integral formula.*

PROOF. Let $I$ be the above integral. By using polar coordinates, we obtain:

$$\begin{aligned}
I^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2}dxdy\\
&= \int_0^{2\pi}\int_0^\infty e^{-r^2}rdrdt\\
&= 2\pi\int_0^\infty\left(-\frac{e^{-r^2}}{2}\right)'dr\\
&= 2\pi\left[0-\left(-\frac{1}{2}\right)\right]\\
&= \pi
\end{aligned}$$

Thus, we are led to the formula in the statement. □

Moving now to 3 dimensions, we have here the following result:

PROPOSITION 16.19. *We have spherical coordinates in* 3 *dimensions,*

$$\begin{cases}x = r\cos s\\ y = r\sin s\cos t\\ z = r\sin s\sin t\end{cases}$$

*the corresponding Jacobian being* $J(r,s,t) = r^2\sin s$.

PROOF. The fact that we have indeed spherical coordinates is clear. Regarding now the Jacobian, this is given by the following formula:

$$\begin{aligned}
&J(r,s,t)\\
&= \begin{vmatrix}\cos s & -r\sin s & 0\\ \sin s\cos t & r\cos s\cos t & -r\sin s\sin t\\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t\end{vmatrix}\\
&= r^2\sin s\sin t\begin{vmatrix}\cos s & -r\sin s\\ \sin s\sin t & r\cos s\sin t\end{vmatrix} + r\sin s\cos t\begin{vmatrix}\cos s & -r\sin s\\ \sin s\cos t & r\cos s\cos t\end{vmatrix}\\
&= r\sin s\sin^2 t\begin{vmatrix}\cos s & -r\sin s\\ \sin s & r\cos s\end{vmatrix} + r\sin s\cos^2 t\begin{vmatrix}\cos s & -r\sin s\\ \sin s & r\cos s\end{vmatrix}\\
&= r\sin s(\sin^2 t+\cos^2 t)\begin{vmatrix}\cos s & -r\sin s\\ \sin s & r\cos s\end{vmatrix}\\
&= r\sin s\times 1\times r\\
&= r^2\sin s
\end{aligned}$$

Thus, we have indeed the formula in the statement. □

Let us work out now the general spherical coordinate formula, in arbitrary $N$ dimensions. The formula here, which generalizes those at $N=2,3$, is as follows:

THEOREM 16.20. *We have spherical coordinates in $N$ dimensions,*

$$\begin{cases}x_1 &= r\cos t_1\\ x_2 &= r\sin t_1\cos t_2\\ \vdots\\ x_{N-1} &= r\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1}\\ x_N &= r\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1}\end{cases}$$

*the corresponding Jacobian being given by the following formula,*

$$J(r,t)=r^{N-1}\sin^{N-2}t_1\sin^{N-3}t_2\,\ldots\,\sin^2 t_{N-3}\sin t_{N-2}$$

*and with this generalizing the known formulae at $N=2,3$.*

PROOF. As before, the fact that we have spherical coordinates is clear. Regarding now the Jacobian, also as before, by developing over the last column, we have:

$$\begin{aligned}J_N &= r\sin t_1\ldots\sin t_{N-2}\sin t_{N-1}\times\sin t_{N-1}J_{N-1}\\ &+ r\sin t_1\ldots\sin t_{N-2}\cos t_{N-1}\times\cos t_{N-1}J_{N-1}\\ &= r\sin t_1\ldots\sin t_{N-2}(\sin^2 t_{N-1}+\cos^2 t_{N-1})J_{N-1}\\ &= r\sin t_1\ldots\sin t_{N-2}J_{N-1}\end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence. □

As an application, let us compute the volumes of spheres. For this purpose, we must understand how the products of coordinates integrate over spheres. Let us start with the case $N=2$. Here the sphere is the unit circle $\mathbb{T}$, and with $z=e^{it}$ the coordinates are $\cos t,\sin t$. We can first integrate arbitrary powers of these coordinates, as follows:

THEOREM 16.21 (Wallis). *We have the following formulae,*

$$\int_0^{\pi/2}\cos^p t\,dt=\int_0^{\pi/2}\sin^p t\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)}\frac{p!!}{(p+1)!!}$$

*where $\varepsilon(p)=1$ if $p$ is even, and $\varepsilon(p)=0$ if $p$ is odd, and where*

$$m!!=(m-1)(m-3)(m-5)\ldots$$

*with the product ending at 2 if $m$ is odd, and ending at 1 if $m$ is even.*

PROOF. Let us first compute the integral on the left in the statement:

$$I_p=\int_0^{\pi/2}\cos^p t\,dt$$

We do this by partial integration. We have the following formula:

$$\begin{aligned}(\cos^p t\sin t)' &= p\cos^{p-1}t(-\sin t)\sin t+\cos^p t\cos t\\ &= p\cos^{p+1}t-p\cos^{p-1}t+\cos^{p+1}t\\ &= (p+1)\cos^{p+1}t-p\cos^{p-1}t\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain the following formula:

$$(p+1)I_{p+1}=pI_{p-1}$$

Thus we can compute $I_p$ by recurrence, and we obtain:

$$\begin{aligned}I_p &= \frac{p-1}{p}\,I_{p-2}\\ &= \frac{p-1}{p}\cdot\frac{p-3}{p-2}\,I_{p-4}\\ &= \frac{p-1}{p}\cdot\frac{p-3}{p-2}\cdot\frac{p-5}{p-4}\,I_{p-6}\\ &\ \vdots\\ &= \frac{p!!}{(p+1)!!}\,I_{1-\varepsilon(p)}\end{aligned}$$

But $I_0=\frac{\pi}{2}$ and $I_1=1$, so we get the result. As for the second formula, this follows from the first one, with $t=\frac{\pi}{2}-s$. Thus, we have proved both formulae in the statement. □

We can now compute the volume of the sphere, as follows:

THEOREM 16.22. *The volume of the unit sphere in $\mathbb{R}^N$ is given by*

$$V=\left(\frac{\pi}{2}\right)^{[N/2]}\frac{2^N}{(N+1)!!}$$

*with our usual convention $N!!=(N-1)(N-3)(N-5)\ldots$*

PROOF. Let us denote by $B^+$ the positive part of the unit sphere, or rather unit ball $B$, obtained by cutting this unit ball in $2^N$ parts. At the level of volumes, we have:

$$V=2^NV^+$$

We have the following computation, using spherical coordinates:

$$\begin{aligned}
V^+ &= \int_{B^+} 1 \\
&= \int_0^1 \int_0^{\pi/2} \ldots \int_0^{\pi/2} r^{N-1} \sin^{N-2} t_1 \ldots \sin t_{N-2}\, dr dt_1 \ldots dt_{N-1} \\
&= \int_0^1 r^{N-1}\, dr \int_0^{\pi/2} \sin^{N-2} t_1\, dt_1 \ldots \int_0^{\pi/2} \sin t_{N-2} dt_{N-2} \int_0^{\pi/2} 1 dt_{N-1} \\
&= \frac{1}{N} \times \left(\frac{\pi}{2}\right)^{[N/2]} \times \frac{(N-2)!!}{(N-1)!!} \cdot \frac{(N-3)!!}{(N-2)!!} \ldots \frac{2!!}{3!!} \cdot \frac{1!!}{2!!} \cdot 1 \\
&= \frac{1}{N} \times \left(\frac{\pi}{2}\right)^{[N/2]} \times \frac{1}{(N-1)!!} \\
&= \left(\frac{\pi}{2}\right)^{[N/2]} \frac{1}{(N+1)!!}
\end{aligned}$$

Thus, we obtain the formula in the statement. □

## 16c. Normal variables

We have kept the best for the end. By using the Gauss formula $\int_{\mathbb{R}} e^{-x^2} = \sqrt{\pi}$ from Theorem 16.18, we can now introduce the normal laws, as follows:

Definition 16.23. *The normal law of parameter* $1$ *is the following measure:*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

*More generally, the normal law of parameter* $t > 0$ *is the following measure:*

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

*These are also called Gaussian distributions, with "g" standing for Gauss.*

Observe that the above laws have indeed mass 1, as they should. This follows indeed from the Gauss formula, which gives, with $x = \sqrt{2t}\, y$:

$$\begin{aligned}
\int_{\mathbb{R}} e^{-x^2/2t} dx &= \int_{\mathbb{R}} e^{-y^2} \sqrt{2t}\, dy \\
&= \sqrt{2t} \int_{\mathbb{R}} e^{-y^2} dy \\
&= \sqrt{2\pi t}
\end{aligned}$$

Generally speaking, the normal laws appear as bit everywhere, in real life. The reasons behind this phenomenon come from the Central Limit Theorem (CLT), that we will explain in a moment, after developing some general theory. As a first result, we have:

PROPOSITION 16.24. *We have the variance formula*

$$V(g_t) = t$$

*valid for any* $t > 0$.

PROOF. The first moment is 0, because our normal law $g_t$ is centered. As for the second moment, this can be computed as follows:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^2 e^{-x^2/2t}dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} (tx)\left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} te^{-x^2/2t}dx \\ &= t \end{aligned}$$

We conclude from this that the variance is $V = M_2 = t$. □

More generally now, the moments of the normal law are as follows:

THEOREM 16.25. *The even moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2} \times k!!$$

*where* $k!! = (k-1)(k-3)(k-5)\ldots$, *and the odd moments vanish.*

PROOF. We have the following computation, valid for any integer $k \in \mathbb{N}$:

$$\begin{aligned} M_k &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} y^k e^{-y^2/2t}dy \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} (ty^{k-1})\left(-e^{-y^2/2t}\right)' dy \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} t(k-1)y^{k-2}e^{-y^2/2t}dy \\ &= t(k-1)\times\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} y^{k-2}e^{-y^2/2t}dy \\ &= t(k-1)M_{k-2} \end{aligned}$$

Now recall from the proof of Proposition 16.24 that we have $M_1 = 0$, $M_2 = t$. Thus by recurrence, we are led to the formula in the statement. □

Here is another general result, which is the key one for the study of the normal laws, regarding the computation of their Fourier transform $F_f(x) = E(e^{ixf})$ :

Theorem 16.26. *We have the following formula, valid for any $t>0$:*

$$F_{g_t}(x)=e^{-tx^2/2}$$

*In particular, the normal laws satisfy $g_s*g_t=g_{s+t}$, for any $s,t>0$.*

Proof. The Fourier transform formula can be established as follows:

$$\begin{aligned}
F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-y^2/2t+ixy}dy\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2}dy\\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-z^2-tx^2/2}\sqrt{2t}dz\\
&= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\int_{\mathbb{R}}e^{-z^2}dz\\
&= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\cdot\sqrt{\pi}\\
&= e^{-tx^2/2}
\end{aligned}$$

As for the last assertion, this follows from the fact that $\log F_{g_t}$ is linear in $t$, via the well-known fact that the logarithm of the Fourier transform linearizes the convolution. □

We are now ready to state and prove the Central Limit Theorem, as follows:

Theorem 16.27 (CLT). *Given random variables $f_1,f_2,f_3,\ldots\in L^\infty(X)$ which are i.i.d., centered, and with variance $t>0$, we have, with $n\to\infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i\sim g_t$$

*with $g_t$ being the Gaussian law of parameter $t$.*

Proof. The Fourier transform $F_f(x)=E(e^{ixf})$ is given by the following formula:

$$\begin{aligned}
F_f(x) &= E\left(\sum_{k=0}^\infty\frac{(ixf)^k}{k!}\right)\\
&= \sum_{k=0}^\infty\frac{(ix)^kE(f^k)}{k!}\\
&= \sum_{k=0}^\infty\frac{i^kM_k(f)}{k!}\,x^k
\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is given by:

$$\begin{aligned}F(x) &= \left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n\\ &= \left[1-\frac{tx^2}{2n}+O(n^{-2})\right]^n\\ &\simeq \left[1-\frac{tx^2}{2n}\right]^n\\ &\simeq e^{-tx^2/2}\end{aligned}$$

But this latter function being the Fourier transform of $g_t$, we obtain the result. □

So long for the basics of probability theory, quickly discussed. In fact, we have already met probability in chapter 15, with a mysterious occurrence of the Poisson laws there. For more on all this, and related topics, you can have a look at my book [11].

## 16d. Hyperspherical laws

Let us discuss now the computation of arbitrary integrals over the sphere. We will need a technical result extending the Wallis formula from Theorem 16.21, as follows:

THEOREM 16.28 (Wallis 2). *We have the following formula,*

$$\int_0^{\pi/2}\cos^p t\sin^q t\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}$$

*where $\varepsilon(p)=1$ if $p$ is even, and $\varepsilon(p)=0$ if $p$ is odd, and where*

$$m!!=(m-1)(m-3)(m-5)\ldots$$

*with the product ending at 2 if $m$ is odd, and ending at 1 if $m$ is even.*

PROOF. We use the same idea as for Theorem 16.21. Let $I_{pq}$ be the integral in the statement. In order to do the partial integration, observe that we have:

$$\begin{aligned}(\cos^p t\sin^q t)' &= p\cos^{p-1}t(-\sin t)\sin^q t\\ &+ \cos^p t\cdot q\sin^{q-1}t\cos t\\ &= -p\cos^{p-1}t\sin^{q+1}t+q\cos^{p+1}t\sin^{q-1}t\end{aligned}$$

By integrating between 0 and $\pi/2$, we obtain, for $p,q>0$:

$$pI_{p-1,q+1}=qI_{p+1,q-1}$$

Thus, we can compute $I_{pq}$ by recurrence. When $q$ is even we have:

$$\begin{aligned}
I_{pq} &= \frac{q-1}{p+1}\,I_{p+2,q-2}\\
&= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\,I_{p+4,q-4}\\
&= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\cdot\frac{q-5}{p+5}\,I_{p+6,q-6}\\
&= \vdots\\
&= \frac{p!!q!!}{(p+q)!!}\,I_{p+q}
\end{aligned}$$

But the last term comes from the formula in Theorem 16.21, and we obtain the result:

$$\begin{aligned}
I_{pq} &= \frac{p!!q!!}{(p+q)!!}\,I_{p+q}\\
&= \frac{p!!q!!}{(p+q)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(p+q)}\frac{(p+q)!!}{(p+q+1)!!}\\
&= \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}
\end{aligned}$$

Observe that this gives the result for $p$ even as well, by symmetry. Indeed, we have $I_{pq}=I_{qp}$, by using the following change of variables:

$$t=\frac{\pi}{2}-s$$

In the remaining case now, where both $p,q$ are odd, we can use once again the formula $pI_{p-1,q+1}=qI_{p+1,q-1}$ established above, and the recurrence goes as follows:

$$\begin{aligned}
I_{pq} &= \frac{q-1}{p+1}\,I_{p+2,q-2}\\
&= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\,I_{p+4,q-4}\\
&= \frac{q-1}{p+1}\cdot\frac{q-3}{p+3}\cdot\frac{q-5}{p+5}\,I_{p+6,q-6}\\
&= \vdots\\
&= \frac{p!!q!!}{(p+q-1)!!}\,I_{p+q-1,1}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

Goood news, we can now integrate over the spheres, as follows:

THEOREM 16.29 (Wallis 3). *The polynomial integrals over the unit sphere $S^{N-1}_\mathbb{R}\subset\mathbb{R}^N$, with respect to the normalized, mass 1 measure, are given by the following formula,*

$$\int_{S^{N-1}_\mathbb{R}}x_1^{k_1}\ldots x_N^{k_N}\,dx=\frac{(N-1)!!k_1!!\ldots k_N!!}{(N+\Sigma k_i-1)!!}$$

*valid when all exponents $k_i$ are even. If an exponent $k_i$ is odd, the integral vanishes.*

PROOF. Assume first that one of the exponents $k_i$ is odd. We can make then the following change of variables, which shows that the integral in the statement vanishes:

$$x_i\to -x_i$$

Assume now that all exponents $k_i$ are even. As a first observation, the result holds indeed at $N=2$, due to the formula from Theorem 16.28, which reads:

$$\int_0^{\pi/2}\cos^pt\sin^qt\,dt=\left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}=\frac{p!!q!!}{(p+q+1)!!}$$

In the general case now, where the dimension $N\in\mathbb{N}$ is arbitrary, the integral in the statement can be written in spherical coordinates, as follows:

$$I=\frac{2^N}{A}\int_0^{\pi/2}\ldots\int_0^{\pi/2}x_1^{k_1}\ldots x_N^{k_N}\,J\,dt_1\ldots dt_{N-1}$$

Here $A$ is the area of the sphere, $J$ is the Jacobian, and the $2^N$ factor comes from the restriction to the $1/2^N$ part of the sphere where all the coordinates are positive. According to Theorem 16.22, coupled with a standard "pizza" argument, for passing from volumes to areas, the normalization constant in front of the integral is as follows:

$$\frac{2^N}{A}=\left(\frac{2}{\pi}\right)^{[N/2]}(N-1)!!$$

As for the unnormalized integral, this is given by:

$$\begin{aligned}I'=\int_0^{\pi/2}\ldots\int_0^{\pi/2}\quad&(\cos t_1)^{k_1}(\sin t_1\cos t_2)^{k_2}\\&\vdots\\&(\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1})^{k_{N-1}}\\&(\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1})^{k_N}\\&\sin^{N-2}t_1\sin^{N-3}t_2\ldots\sin^2t_{N-3}\sin t_{N-2}\\&dt_1\ldots dt_{N-1}\end{aligned}$$

By rearranging the terms, we obtain:

$$\begin{aligned} I' &= \int_0^{\pi/2} \cos^{k_1} t_1 \sin^{k_2+\ldots+k_N+N-2} t_1 \, dt_1 \\ & \quad \int_0^{\pi/2} \cos^{k_2} t_2 \sin^{k_3+\ldots+k_N+N-3} t_2 \, dt_2 \\ & \quad \vdots \\ & \quad \int_0^{\pi/2} \cos^{k_{N-2}} t_{N-2} \sin^{k_{N-1}+k_N+1} t_{N-2} \, dt_{N-2} \\ & \quad \int_0^{\pi/2} \cos^{k_{N-1}} t_{N-1} \sin^{k_N} t_{N-1} \, dt_{N-1} \end{aligned}$$

Now by using the above-mentioned formula at $N = 2$, this gives:

$$\begin{aligned} I' &= \frac{k_1!!(k_2+\ldots+k_N+N-2)!!}{(k_1+\ldots+k_N+N-1)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(N-2)} \\ & \quad \frac{k_2!!(k_3+\ldots+k_N+N-3)!!}{(k_2+\ldots+k_N+N-2)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(N-3)} \\ & \quad \vdots \\ & \quad \frac{k_{N-2}!!(k_{N-1}+k_N+1)!!}{(k_{N-2}+k_{N-1}+l_N+2)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(1)} \\ & \quad \frac{k_{N-1}!!k_N!!}{(k_{N-1}+k_N+1)!!}\left(\frac{\pi}{2}\right)^{\varepsilon(0)} \end{aligned}$$

Now let $F$ be the part involving the double factorials, and $P$ be the part involving the powers of $\pi/2$, so that $I' = F \cdot P$. Regarding $F$, by cancelling terms we have:

$$F = \frac{k_1!!\ldots k_N!!}{(\Sigma k_i + N - 1)!!}$$

As in what regards $P$, by summing the exponents, we obtain $P = \left(\frac{\pi}{2}\right)^{[N/2]}$. We can now put everything together, and we obtain:

$$\begin{aligned} I &= \frac{2^N}{A} \times F \times P \\ &= \left(\frac{2}{\pi}\right)^{[N/2]} (N-1)!! \times \frac{k_1!!\ldots k_N!!}{(\Sigma k_i + N - 1)!!} \times \left(\frac{\pi}{2}\right)^{[N/2]} \\ &= \frac{(N-1)!!k_1!!\ldots k_N!!}{(\Sigma k_i + N - 1)!!} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Now back to probability, as an application of the above formula, we have:

THEOREM 16.30. *The moments of the hyperspherical variables are*

$$\int_{S^{N-1}_\mathbb{R}} x_i^p dx = \frac{(N-1)!!p!!}{(N+p-1)!!}$$

*and the rescaled variables $y_i = \sqrt{N}x_i$ become normal and independent with $N \to \infty$.*

PROOF. We have two assertions here, the idea being as follows:

(1) The moment formula in the statement follows from the general formula from Theorem 16.29. As a consequence, with $N \to \infty$ we have the following estimate:

$$\begin{aligned}\int_{S^{N-1}_\mathbb{R}} x_i^p dx &\simeq N^{-p/2} \times p!! \\ &= N^{-p/2} M_p(g_1)\end{aligned}$$

By comparing now with the moment formula in Theorem 16.25, we conclude that the rescaled variables $\sqrt{N}x_i$ become normal with $N \to \infty$, as claimed.

(2) As for the proof of the asymptotic independence, this is standard too, once again by using Theorem 16.29. Indeed, the joint moments of $x_1, \ldots, x_N$ are given by:

$$\begin{aligned}\int_{S^{N-1}_\mathbb{R}} x_1^{k_1} \ldots x_N^{k_N} \, dx &= \frac{(N-1)!!k_1!! \ldots k_N!!}{(N+\Sigma k_i - 1)!!} \\ &\simeq N^{-\Sigma k_i} \times k_1!! \ldots k_N!!\end{aligned}$$

By rescaling, the joint moments of the variables $y_i = \sqrt{N}x_i$ are given by:

$$\int_{S^{N-1}_\mathbb{R}} y_1^{k_1} \ldots y_N^{k_N} \, dx \simeq k_1!! \ldots k_N!!$$

Thus, we have multiplicativity, and so independence with $N \to \infty$, as claimed. □

## 16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter. However, if interested in learning more geometry and trigonometry, both theory and applications, there are plenty of good choices here, with basically any advanced mathematics or physics book having concrete formulae inside doing the job. So, have a look at the various books referenced below, normally only good material there, and choose one.

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