

# Geometry and probability

Teo Banica

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CERGY-PONTOISE, F-95000  
CERGY-PONTOISE, FRANCE. [teo.banica@gmail.com](mailto:teo.banica@gmail.com)

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ABSTRACT. This is an introduction to geometric methods in probability, and vice versa, with all needed preliminaries included. We first discuss the basic equations of physics, from a geometric viewpoint, and with focus on the case of random input and output. Then we go on a detailed discussion regarding heat diffusion, in various dimensions, and the related notion of heat kernel. We then discuss the use of spherical coordinates, again in various dimensions, in relation with various questions from physics and probability. Finally, we discuss rotationally invariant problems, and other invariance questions.

## Preface

This is an introduction to geometric methods in probability, and vice versa, with all needed preliminaries included. The book is organized in 4 parts, as follows:

I. We first discuss the basic equations of physics, from a geometric viewpoint, and with focus on the case of random input and output.

II. Then we go on a more detailed discussion regarding heat diffusion, in various dimensions, and the related notion of heat kernel.

III. We then discuss the use of spherical coordinates, again in various dimensions, in relation with various questions from physics and probability.

IV. Finally, we discuss various rotationally invariant problems, both algebraic and analytic, and other invariance questions.

Many thanks to my cats, for some help with infinite dimensions.

*Cergy, January 2026*

*Teo Banica*



## Contents

|                                       |          |
|---------------------------------------|----------|
| Preface                               | 3        |
| <b>Part I. Probability, equations</b> | <b>9</b> |
| Chapter 1.                            | 11       |
| 1a.                                   | 11       |
| 1b.                                   | 11       |
| 1c.                                   | 11       |
| 1d.                                   | 11       |
| 1e. Exercises                         | 11       |
| Chapter 2.                            | 13       |
| 2a.                                   | 13       |
| 2b.                                   | 13       |
| 2c.                                   | 13       |
| 2d.                                   | 13       |
| 2e. Exercises                         | 13       |
| Chapter 3.                            | 15       |
| 3a.                                   | 15       |
| 3b.                                   | 15       |
| 3c.                                   | 15       |
| 3d.                                   | 15       |
| 3e. Exercises                         | 15       |
| Chapter 4.                            | 17       |
| 4a.                                   | 17       |
| 4b.                                   | 17       |
| 4c.                                   | 17       |
| 4d.                                   | 17       |
| 4e. Exercises                         | 17       |

|  |    |
|--|----|
| <b>Part II. The heat kernel</b>        | 19 |
| Chapter 5.                             | 21 |
| 5a.                                    | 21 |
| 5b.                                    | 21 |
| 5c.                                    | 21 |
| 5d.                                    | 21 |
| 5e. Exercises                          | 21 |
| Chapter 6.                             | 23 |
| 6a.                                    | 23 |
| 6b.                                    | 23 |
| 6c.                                    | 23 |
| 6d.                                    | 23 |
| 6e. Exercises                          | 23 |
| Chapter 7.                             | 25 |
| 7a.                                    | 25 |
| 7b.                                    | 25 |
| 7c.                                    | 25 |
| 7d.                                    | 25 |
| 7e. Exercises                          | 25 |
| Chapter 8.                             | 27 |
| 8a.                                    | 27 |
| 8b.                                    | 27 |
| 8c.                                    | 27 |
| 8d.                                    | 27 |
| 8e. Exercises                          | 27 |
| <b>Part III. Spherical coordinates</b> | 29 |
| Chapter 9.                             | 31 |
| 9a.                                    | 31 |
| 9b.                                    | 31 |
| 9c.                                    | 31 |
| 9d.                                    | 31 |
| 9e. Exercises                          | 31 |

## CONTENTS

7

|                                      |           |
|--------------------------------------|-----------|
| Chapter 10.                          | 33        |
| 10a.                                 | 33        |
| 10b.                                 | 33        |
| 10c.                                 | 33        |
| 10d.                                 | 33        |
| 10e. Exercises                       | 33        |
| Chapter 11.                          | 35        |
| 11a.                                 | 35        |
| 11b.                                 | 35        |
| 11c.                                 | 35        |
| 11d.                                 | 35        |
| 11e. Exercises                       | 35        |
| Chapter 12.                          | 37        |
| 12a.                                 | 37        |
| 12b.                                 | 37        |
| 12c.                                 | 37        |
| 12d.                                 | 37        |
| 12e. Exercises                       | 37        |
| <b>Part IV. Invariance questions</b> | <b>39</b> |
| Chapter 13.                          | 41        |
| 13a.                                 | 41        |
| 13b.                                 | 41        |
| 13c.                                 | 41        |
| 13d.                                 | 41        |
| 13e. Exercises                       | 41        |
| Chapter 14.                          | 43        |
| 14a.                                 | 43        |
| 14b.                                 | 43        |
| 14c.                                 | 43        |
| 14d.                                 | 43        |
| 14e. Exercises                       | 43        |
| Chapter 15.                          | 45        |

|                |    |
|----------------|----|
| 15a.           | 45 |
| 15b.           | 45 |
| 15c.           | 45 |
| 15d.           | 45 |
| 15e. Exercises | 45 |
| Chapter 16.    | 47 |
| 16a.           | 47 |
| 16b.           | 47 |
| 16c.           | 47 |
| 16d.           | 47 |
| 16e. Exercises | 47 |
| Bibliography   | 49 |

Part I

Probability, equations



## CHAPTER 1

1a.

1b.

1c.

1d.

### 1e. Exercises

Exercises:

EXERCISE 1.1.

EXERCISE 1.2.

EXERCISE 1.3.

EXERCISE 1.4.

EXERCISE 1.5.

EXERCISE 1.6.

EXERCISE 1.7.

EXERCISE 1.8.

Bonus exercise.



## CHAPTER 2

**2a.**

**2b.**

**2c.**

**2d.**

### **2e. Exercises**

Exercises:

EXERCISE 2.1.

EXERCISE 2.2.

EXERCISE 2.3.

EXERCISE 2.4.

EXERCISE 2.5.

EXERCISE 2.6.

EXERCISE 2.7.

EXERCISE 2.8.

Bonus exercise.



## CHAPTER 3

**3a.**

**3b.**

**3c.**

**3d.**

### **3e. Exercises**

Exercises:

EXERCISE 3.1.

EXERCISE 3.2.

EXERCISE 3.3.

EXERCISE 3.4.

EXERCISE 3.5.

EXERCISE 3.6.

EXERCISE 3.7.

EXERCISE 3.8.

Bonus exercise.



## CHAPTER 4

4a.

4b.

4c.

4d.

### 4e. Exercises

Exercises:

EXERCISE 4.1.

EXERCISE 4.2.

EXERCISE 4.3.

EXERCISE 4.4.

EXERCISE 4.5.

EXERCISE 4.6.

EXERCISE 4.7.

EXERCISE 4.8.

Bonus exercise.



## Part II

# The heat kernel



## CHAPTER 5

5a.

5b.

5c.

5d.

### 5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

EXERCISE 5.7.

EXERCISE 5.8.

Bonus exercise.



## CHAPTER 6

**6a.**

**6b.**

**6c.**

**6d.**

### **6e. Exercises**

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

EXERCISE 6.6.

EXERCISE 6.7.

EXERCISE 6.8.

Bonus exercise.



## CHAPTER 7

7a.

7b.

7c.

7d.

### 7e. Exercises

Exercises:

EXERCISE 7.1.

EXERCISE 7.2.

EXERCISE 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

EXERCISE 7.7.

EXERCISE 7.8.

Bonus exercise.



## CHAPTER 8

8a.

8b.

8c.

8d.

### 8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.



## Part III

# Spherical coordinates



## CHAPTER 9

**9a.**

**9b.**

**9c.**

**9d.**

### **9e. Exercises**

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.



## CHAPTER 10

**10a.**

**10b.**

**10c.**

**10d.**

**10e. Exercises**

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

EXERCISE 10.8.

Bonus exercise.



## CHAPTER 11

**11a.**

**11b.**

**11c.**

**11d.**

**11e. Exercises**

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.



## CHAPTER 12

**12a.**

**12b.**

**12c.**

**12d.**

**12e. Exercises**

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.



## Part IV

# Invariance questions



## CHAPTER 13

**13a.**

**13b.**

**13c.**

**13d.**

**13e. Exercises**

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Bonus exercise.



## CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.



## CHAPTER 15

**15a.**

**15b.**

**15c.**

**15d.**

**15e. Exercises**

Exercises:

EXERCISE 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

EXERCISE 15.7.

EXERCISE 15.8.

Bonus exercise.



## CHAPTER 16

**16a.**

**16b.**

**16c.**

**16d.**

**16e. Exercises**

Congratulations for having read this book, and no exercises for this final chapter.



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