Deformed Hadamard matrices

Teo Banica

"Introduction to Hadamard matrices", 3/6

07/20

Hadamard matrices

The complex Hadamard matrix manifold appears by definition as an intersection of smooth real algebraic manifolds:

$$X_N = M_N(\mathbb{T}) \cap \sqrt{N} U_N$$

This intersection is far from being smooth. Given a point $H \in X_N$, the problem is that of understanding the singularity at H.

Affine deformations

Notation. We denote by X_p an unspecified neighborhood of a point in a manifold, $p \in X$.

Theorem. For $H \in X_N, A \in M_N(\mathbb{R})$, the following are equivalent:

(1) $H_{ij}^q = H_{ij} q^{A_{ij}}$ is an Hadamard matrix, for any $q \in \mathbb{T}_1$.

(2) $\sum_k H_{ik} \bar{H}_{jk} q^{A_{ik} - A_{jk}} = 0$, for any $i \neq j$ and any $q \in \mathbb{T}_1$.

(3) $\sum_k H_{ik} \bar{H}_{jk} \varphi(A_{ik} - A_{jk}) = 0$, for any $i \neq j$ and any $\varphi : \mathbb{R} \to \mathbb{C}$.

(4) $\sum_{k \in E_{ij}^r} H_{ik} \bar{H}_{jk} = 0$, $\forall i \neq j, r \in \mathbb{R}$, with $E_{ij}^r = \{k | A_{ik} - A_{jk} = r\}$.

Proof. (1) $\iff$ (2) $\implies$ (4) $\implies$ (3) $\implies$ (2).

General deformations

Notation. We consider functions of type $f : X_p \to Y_q$, which by definition satisfy $f(p) = q$.

Definition. Let $H \in M_N(\mathbb{C})$ be a complex Hadamard matrix.

(1) A deformation of H is a smooth function $f : \mathbb{T}_1 \to (X_N)_H$.

(2) Called "affine" if $f_{ij}(q) = H_{ij} q^{A_{ij}}$, with $A \in M_N(\mathbb{R})$.

(3) Called "trivial" when $f_{ij}(q) = H_{ij} q^{a_i + b_j}$, with $a, b \in \mathbb{R}^N$.

Examples. The Diţă deformations, with $Q_{ij} = q^{A_{ij}}$, are affine.

Tangent cones

Definition. Associated to a point $H \in X_N$ are:

(1) The enveloping tangent space: obtained as intersection of tangent spaces, $\widetilde{T}_H X_N = T_H M_N(\mathbb{T}) \cap T_H \sqrt{N} U_N$.

(2) The tangent cone $T_H X_N$: the set of tangent vectors to the deformations of H.

(3) The affine tangent cone $T_H^\circ X_N$: same as above, using affine deformations only.

(4) The trivial tangent cone $T_H^\times X_N$: same as above, using trivial deformations only.

$\Longrightarrow$ We have $T_H^\times X_N \subset T_H^\circ X_N \subset T_H X_N \subset \widetilde{T}_H X_N$.

Basic geometry

Theorem. The cones $T_H^\times X_N \subset T_H^\circ X_N \subset T_H X_N \subset \widetilde{T}_H X_N$ can be computed as follows:

(1) $\widetilde{T}_H X_N$ is the linear space formed by the matrices $A \in M_N(\mathbb{R})$ satisfying $\sum_k H_{ik}\bar{H}_{jk}(A_{ik} - A_{jk}) = 0$, for any i, j.

(2) $T_H X_N$ consists of matrices of the form $A_{ij} = g_{ij}'(0)$, with $g : M_N(\mathbb{R})_0 \to M_N(\mathbb{R})_0$, $\sum_k H_{ik}\bar{H}_{jk} e^{i(g_{ik}(t) - g_{jk}(t))} = 0$.

(3) $T_H^\circ X_N$ consists of the matrices $A \in M_N(\mathbb{R})$ satisfying $\sum_k H_{ik}\bar{H}_{jk} q^{A_{ik} - A_{jk}} = 0$, for any $i \neq j$ and any $q \in \mathbb{T}$.

(4) $T_H^\times X_N$ consists of the matrices $A \in M_N(\mathbb{R})$ which are of the form $A_{ij} = a_i + b_j$, for certain vectors $a, b \in \mathbb{R}^N$.

Summary, defect

Definition. The defect of an Hadamard matrix $H\in X_N$ is the real dimension $d(H)$ of the associated enveloping tangent space,

$$\widetilde{T}_HX_N=T_HM_N(\mathbb{T})\cap T_H\sqrt{N}U_N$$

which can be computed according to the formula

$$\widetilde{T}_HX_N=\left\{A\in M_N(\mathbb{R})\Big|\sum_kH_{ik}\bar{H}_{jk}(A_{ik}-A_{jk})=0,\forall i,j\right\}$$

and whose elements are those making the matrix

$$H_{ij}^q=H_{ij}q^{A_{ij}}$$

"complex Hadamard at order 1", with respect to $q\in\mathbb{T}$.

Basic properties

Theorem. Let $H \in X_N$ be a complex Hadamard matrix.

(1) If $H \simeq \widetilde{H}$ then $d(H) = d(\widetilde{H})$.

(2) We have $2N - 1 \leq d(H) \leq N^2$.

(3) If $d(H) = 2N - 1$, the dephased image of H is isolated.

Proof. (1) The equations at $K_{ij} = a_i b_j H_{ij}$, $|a_i| = |b_j| = 1$ are:

$$\sum_k a_i b_k H_{ik} \bar{a}_j \bar{b}_k \bar{H}_{jk} (A_{ik} - A_{jk}) = 0$$

By simplifying we obtain the equations for H, as desired.

(2) This follows from $T_H^\times X_N \subset T_H X_N \subset \widetilde{T}_H X_N$.

(3) If $d(H) = 2N - 1$ then $T_H X_N = T_H^\times X_N$, as needed.

Real matrices 1/2

Theorem. We have a linear space isomorphism

$$\widetilde{T}_H X_N \simeq \left\{ E \in M_N(\mathbb{C}) \Big| E = E^*, (EH)_{ij}\bar{H}_{ij} \in \mathbb{R}, \forall i,j \right\}$$

with $A \to E$ and $E \to A$ being constructed as follows:

$$E_{ij} = \sum_k H_{ik}\bar{H}_{jk}A_{ik} \quad , \quad A_{ij} = (EH)_{ij}\bar{H}_{ij}$$

Proof. Given $A \in M_N(\mathbb{C})$, if we set $R_{ij} = A_{ij}H_{ij}$ and $E = RH^*$, then $A \to R \to E$ is bijective onto $M_N(\mathbb{C})$, and we have:

$$E_{ij} = \sum_k H_{ik}\bar{H}_{jk}A_{ik}$$

The equations become $E_{ij} = \bar{E}_{ji}$, and we are left with $A_{ij} \in \mathbb{R}$.

Real matrices 2/2

Theorem. For any real Hadamard matrix $H \in M_N(\pm 1)$ we have

$$\widetilde{T}_H X_N \simeq M_N(\mathbb{R})^{symm}$$

and so the corresponding defect is $d(H) = N(N+1)/2$.

Proof. We use the previous result. Since H is real the condition

$$(EH)_{ij}\bar{H}_{ij} \in \mathbb{R}$$

simply tells us that E must be real, and this gives the result.

Fourier 1/4

<u>Theorem</u>. For $F = F_G$, the matrices $A \in \widetilde{T}_F X_N$, with $N = |G|$, are those of the form $A = PF^*$, with $P \in M_N(\mathbb{C})$ satisfying

$$P_{ij} = P_{i+j,j} = \bar{P}_{i,-j}$$

where the indices i, j are by definition taken in the group G.

<u>Proof</u>. By decomposing G, with $w_k = e^{2\pi i/k}$ we have:

$$F_{i_1 \ldots i_r, j_1 \ldots j_r} = (w_{N_1})^{i_1 j_1} \ldots (w_{N_r})^{i_r j_r}$$

The equations $\sum_k F_{ik} \bar{F}_{jk} (A_{ik} - A_{jk}) = 0$ become:

$$(AF)_{i_1 \ldots i_r, i_1 - j_1 \ldots i_r - j_r} - (AF)_{j_1 \ldots j_r, i_1 - j_1 \ldots i_r - j_r} = 0$$

Thus with $P = AF$ our system is simply $P_{i,i-j} = P_{j,i-j}$.

Fourier 2/4

Theorem. The defect of a Fourier matrix F_G is given by

$$d(F_G) = \sum_{g \in G} \frac{|G|}{ord(g)}$$

and equals as well the number of 1 entries of the matrix F_G.

Proof. The first assertion follows by counting the solutions of $P_{ij} = P_{i+j,j} = \bar{P}_{i,-j}$. Also, we have

$$\begin{aligned} \#(1 \in F_G) &= \# \left\{ (g, \chi) \in G \times \widehat{G} \middle| \chi(g) = 1 \right\} \\ &= \sum_{g \in G} \frac{|G|}{ord(g)} \end{aligned}$$

so the second assertion follows from the first one.

Fourier 3/4

Theorem. The defect of a usual Fourier matrix F_N is given by

$$d(F_N) = N \prod_{i=1}^{s} \left(1 + a_i - \frac{a_i}{p_i}\right)$$

where $N = p_1^{a_1} \dots p_s^{a_s}$ is the decomposition of N into prime factors.

Proof. This follows by counting, either from

$$d(F_G) = \sum_{g \in G} \frac{|G|}{ord(g)}$$

or from $d(F_G) = \#(1 \in F_G)$.

Fourier 4/4

Theorem. Given a finite abelian group G, the quantity

$$\delta(G) = \sum_{g \in G} \frac{1}{ord(g)}$$

decomposes as follows, over the isotypic components:

$$\delta(G) = \prod_{p} \delta(G_p)$$

For p-groups we have $\delta(G) = 1 + \sum_{k \geq 1} \frac{c_k - c_{k-1}}{p^k}$ with

$$c_k = \# \left\{ g \in G \middle| ord(g) \leq p^k \right\}$$

and these quantities satisfy $c_k(G \times H) = c_k(G) c_k(H)$.

$\Longrightarrow$ Thus, we can compute $d(F_G)$ for any G.

Exponential writing

<u>Theorem</u>. Assume that $H \in M_N(\mathbb{C})$ is Hadamard, let $A \in M_N(\mathbb{C})$ be antihermitian, and consider the matrix

$$H^{(t)} = e^{tA}H$$

with $t \in \mathbb{R}$. We have then:

(1) $H^{(t)}$ is Hadamard when $|\sum_{rs} H_{rq}\bar{H}_{sq}(e^{tA})_{pr}(e^{-tA})_{sp}| = 1$.

(2) $H^{(t)}$ is Hadamard at order 0 when $|(AH)_{pq}| = 1$.

<u>Proof</u>. Here (1) follows by computing the quantities $|H_{pq}^{(t)}|^2$, and (2) follows from (1) by differentiating at 0.

Nicoara-White

Theorem. Let G be a finite abelian group, and for $g,h\in G$, set:

$$B_{pq}=\begin{cases}1 & \text{if } \exists k\in\mathbb{N}, p=h^kg, q=h^{k+1}g\\ 0 & \text{otherwise}\end{cases}$$

When $(g,h)\in G^2$ range in suitable cosets, the unitary matrices

$$e^{it(B+B^t)}F_G\quad,\quad e^{t(B-B^t)}F_G$$

are both Hadamard, and make the defect of F_G be attained.

Proof. The previous equations simplify, in the case of a Fourier matrix, and we are led into linear algebra and group theory.

Master Hadamard

Definition. The MHM are the Hadamard matrices of type

$$H_{ij} = \lambda_i^{n_j}$$

with $\lambda_i \in \mathbb{T}, n_j \in \mathbb{R}$. $f(z) = \sum_j z^{n_j}$ is called "master function".

Theorem. $F_M \otimes_Q F_N$ is master Hadamard, in the case

$$Q_{ib} = q^{i(Np_b + b)}$$

with $q = e^{2\pi i/MNk}$ and $k \in \mathbb{N}$, and $p_0, \ldots, p_{N-1} \in \mathbb{R}$.

Conjecture. The MHM appear as Diță deformations of F_N.

Defect

Theorem. The defect of a master Hadamard matrix is given by

$$d(H)=\dim_{\mathbb R}\left\{B\in M_N(\mathbb C)\Big|\bar{B}=\frac{1}{N}BL,(BR)_{i,ij}=(BR)_{j,ij}\ \forall i,j\right\}$$

where the matrices L,R on the right are given by

$$L_{ij}=f\left(\frac{1}{\lambda_i\lambda_j}\right)\quad,\quad R_{i,jk}=f\left(\frac{\lambda_j}{\lambda_i\lambda_k}\right)$$

with f being the associated master function.

Proof. This is a standard computation, with $B=AH^t$.

Conjecture. The only isolated master Hadamard matrices are the Fourier matrices F_p, with p prime.

McNulty-Weigert

Theorem 1. Assuming that $K \in M_N(\mathbb{C})$ is Hadamard, so is

$$H_{ia,jb} = \frac{1}{\sqrt{Q}} K_{ij} (L_i^* R_j)_{ab}$$

when $\{L_1, \ldots, L_N\} \subset \sqrt{Q} U_Q$ and $\{R_1, \ldots, R_N\} \subset \sqrt{Q} U_Q$ are such that each of the matrices $\frac{1}{\sqrt{Q}} L_i^* R_j \in \sqrt{Q} U_Q$ is Hadamard.

Theorem 2. For $q \geq 3$ prime, $\{F_q, DF_q, \ldots, D^{q-1} F_q\}$, where

$$D = diag\left(1, 1, w, w^3, w^6, w^{10}, \ldots, w^{\frac{q^2-1}{8}}, \ldots, w^{10}, w^6, w^3, w\right)$$

with $w = e^{2\pi i/q}$, are such that $\frac{1}{\sqrt{q}} E_i^* E_j$ is Hadamard, for any $i \neq j$.

Isolation

By combining the above results, we are led to complex Hadamard matrices which are often isolated, such as the Tao matrix:

$$T_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & w & w & w^2 & w^2 \\ 1 & w & 1 & w^2 & w^2 & w \\ 1 & w & w^2 & 1 & w & w^2 \\ 1 & w^2 & w^2 & w & 1 & w \\ 1 & w^2 & w & w^2 & w & 1 \end{pmatrix}$$

As an interesting consequence, T_6 is not "exceptional". Also, most of the known isolated matrices are of McNulty-Weigert type.