Almost Hadamard matrices

Teo Banica

"Introduction to Hadamard matrices", 5/6

07/20

Hadamard matrices

The set formed by the $N \times N$ real Hadamard matrices is:

$$X_N = M_N(\pm 1) \cap \sqrt{N} O_N$$

(1) How to locate analytically these matrices?

(2) What to do when $N \notin 4\mathbb{N}$?

(3) What about the complex case?

Hadamard bound

Theorem. Given a matrix $H \in M_N(\pm 1)$, we have

$$|\det(H)| \leq N^{N/2}$$

with equality precisely when H is Hadamard.

Proof. The determinant of a system of N vectors in $\mathbb{R}^N$ is:

$$\det(H_1, \ldots, H_N) = \pm vol < H_1, \ldots, H_N >$$

In our case, ± 1 entries, we have the following inequality,

$$|\det(H_1, \ldots, H_N)| \leq ||H_1|| \times \ldots \times ||H_N|| = (\sqrt{N})^N$$

with equality when our vectors are pairwise orthogonal.

$\implies$ "quasi-Hadamard matrices", at $N \notin 4\mathbb{N}$

Norm estimates

Theorem. Given a matrix $U \in O_N$, we have

$$||U||_1 \leq N\sqrt{N}$$

with equality precisely when $H = U/\sqrt{N}$ is Hadamard.

Proof. We have the following Cauchy-Schwarz estimate:

$$||U||_1 = \sum_{ij} |U_{ij}| \leq N \left(\sum_{ij} |U_{ij}|^2 \right)^{1/2} = N\sqrt{N}$$

The equality case holds when $|U_{ij}| = \sqrt{N}$ for any i, j, and so when the rescaled matrix $H = U/\sqrt{N}$ satisfies $H \in M_N(\pm 1)$.

$\implies$ "almost Hadamard matrices", at $N \notin 4\mathbb{N}$

Jensen, Hölder

Theorem. Given $\psi : [0, \infty) \to \mathbb{R}$, define $F : U_N \to \mathbb{R}$ by:

$$F(U) = \sum_{ij} \psi(|U_{ij}|^2)$$

(1) ψ concave $\implies$ F maximized when $H = \sqrt{N}U$ Hadamard.

(2) ψ convex $\implies$ F minimized when $H = \sqrt{N}U$ Hadamard.

Theorem. Let $U \in U_N$, and set $H = \sqrt{N}U$.

(1) At $p < 2$, $||U||_p \leq N^{2/p-1/2}$, equality when H Hadamard.

(2) At $p > 2$, $||U||_p \geq N^{2/p-1/2}$, equality when H Hadamard.

Almost Hadamard

Definition. Given $U \in U_N$, the matrix $H = \sqrt{N}U$ is called:

(1) Almost Hadamard, if U locally maximizes the 1-norm on U_N.

(2) p-almost Hadamard, with $p < 2$, if U locally maximizes the p-norm on U_N.

(3) p-almost Hadamard, with $p > 2$, if U locally minimizes the p-norm on U_N.

(4) Absolute almost Hadamard, if it is p-almost Hadamard at any $p \neq 2$.

Also: real versions of these notions, with U_N replaced by O_N.

Rotation trick

Theorem. If $U \in O_N$ locally maximizes the 1-norm, then

$$U_{ij} \neq 0$$

for any i, j. We write $U \in O_N^*$.

Proof. This uses a rotation trick (BCS), as follows:

$$\begin{pmatrix} \cos t & \sin t & & & \\ -\sin t & \cos t & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \ldots \\ U_N \end{pmatrix} = \begin{pmatrix} \cos t \cdot U_1 + \sin t \cdot U_2 \\ -\sin t \cdot U_1 + \cos t \cdot U_2 \\ U_3 \\ \ldots \\ U_N \end{pmatrix}$$

By differentiating $||.||_1$ with respect to t, we obtain the result.

Complex case

Theorem. If $U \in U_N$ locally maximizes the 1-norm, then

$$U_{ij} \neq 0$$

for any i, j. We write $U \in U_N^*$.

Proof. As in the real case, this follows from a rotation trick. The computations however are more complicated (BN).

Problem. Find such results at any $p \in [1, \infty] - \{2\}$.

Critical points 1/4

Theorem. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a differentiable function. A matrix $U \in U_N^*$ is a critical point of the quantity

$$F(U) = \sum_{ij} \varphi(|U_{ij}|)$$

when WU^* is self-adjoint, where $W_{ij} = \mathrm{sgn}(U_{ij})\varphi'(|U_{ij}|)$.

Proof. Lagrange multipliers. We have the following formula:

$$dF = \sum_{ij} \varphi'(|U_{ij}|)d|U_{ij}| = \frac{1}{2}\sum_{ij} W_{ij}d\bar{U}_{ij} + \bar{W}_{ij}dU_{ij}$$

We are led to $W = 2M^tU$, $\bar{W} = 2M\bar{U}$, and so to $WU^* = UW^*$.

Critical points 2/4

Definition. Given $U \in U_N$, we consider its "color decomposition"

$$U = \sum_{r>0} rU_r$$

with $U_r \in M_N(\mathbb{T} \cup \{0\})$ containing the phases, and we call U:

(1) Semi-balanced, if $U_r U^*$, $U^* U_r$ are all self-adjoint.

(2) Balanced, if $U_r U_s^*$, $U_r^* U_s$ are all self-adjoint.

Theorem. For a matrix $U \in U_N^*$, the following are equivalent:

(1) U is a critical point of $F(U) = \sum_{ij} \varphi(|U_{ij}|)$, for any φ.

(2) U is a critical point of all p-norms, with $p \in [1, \infty)$.

(3) U is semi-balanced, in the above sense.

Critical points 3/4

Theorem. The class of balanced matrices is as follows:

(1) It contains $U = H/\sqrt{N}$, with $H \in M_N(\mathbb{C})$ Hadamard.

(2) It is stable under transposition.

(3) It is stable under complex conjugation.

(4) It is stable under and taking adjoints.

(5) It is stable under taking tensor products.

(6) It is stable under the Hadamard equivalence relation.

(7) It contains $V_N = \frac{1}{N}(2\mathbb{I}_N - N1_N)$, where $\mathbb{I}_N$ is the all-1 matrix.

Critical points 4/4

We call (a,b,c) pattern any $M \in M_N(0,1)$, with $N = a + 2b + c$, such that any two rows look as follows, up to a permutation:

$$\begin{matrix} 0\ldots0 & 0\ldots0 & 1\ldots1 & 1\ldots1 \\ \underbrace{0\ldots0}_{a} & \underbrace{1\ldots1}_{b} & \underbrace{0\ldots0}_{b} & \underbrace{1\ldots1}_{c} \end{matrix}$$

Examples from BIBD. These produce two-entry unitary matrices, by replacing the $0,1$ entries with suitable numbers x,y.

Theorem. The following matrices are balanced:

(1) The orthogonal matrices coming from (a,b,c) patterns.

(2) The unitary matrices which are circulant and self-adjoint.

Hessians 1/2

<u>Theorem</u>. Given $U \in U_N$, set $S_{ij} = \mathrm{sgn}(U_{ij})$, and $X = S^* U$.

(1) U locally maximizes the 1-norm on U_N when $X \geq 0$, and

$$\Phi(U, B) = Tr(XB^2) - \sum_{ij} \frac{Re\left[(UB)_{ij}\overline{S}_{ij}\right]^2}{|U_{ij}|}$$

is positive, for any hermitian matrix $B \in M_N(\mathbb{C})$.

(2) In the real case, $U \in O_N$, this matrix locally maximizes the 1-norm on O_N when the matrix $X = S^* U$ is self-adjoint, and the sum of its two smallest eigenvalues is positive.

Hessians 2/2

The difference between the real and complex cases comes from:

Theorem. For $X \in M_N(\mathbb{C})$ self-adjoint, the following are equivalent:
(1) $Tr(XA^2) \leq 0$, for any anti-hermitian matrix $A \in M_N(\mathbb{C})$.
(2) $Tr(XB^2) \geq 0$, for any hermitian matrix $B \in M_N(\mathbb{C})$.
(3) $Tr(XC) \geq 0$, for any positive matrix $C \in M_N(\mathbb{C})$.
(4) $X \geq 0$.

Theorem. For $X \in M_N(\mathbb{R})$ symmetric, the following are equivalent:
(1) $Tr(XA^2) \leq 0$, for any antisymmetric matrix A.
(2) The sum of the two smallest eigenvalues of X is positive.

Real AHM, 1/4

Theorem. If $U = U(x, y)$ is orthogonal, coming from an (a, b, c) pattern, with

$$(N(a-b)+2b)|x|+(N(c-b)+2b)|y| \geq 0$$

the matrix $H = \sqrt{N}U$ is almost Hadamard, in the real sense.

Proof. Since any row of U consists of $a+b$ copies of x and $b+c$ copies of y, we have:

$$(SU^t)_{ij} = \begin{cases} (a+b)|x|+(b+c)|y| & (i=j) \\ (a-b)|x|+(c-b)|y| & (i \neq j) \end{cases}$$

After come computations, this gives the result.

Real AHM, 2/4

As a basic example for the above construction, we have the following matrix:

$$K_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 2-N & 2 & \ldots & 2 \\ 2 & 2-N & \ldots & 2 \\ \ldots & \ldots & \ldots & \ldots \\ 2 & 2 & \ldots & 2-N \end{pmatrix}$$

This is absolute almost Hadamard, in the real sense (Mohan).

Other interesting examples, coming from block designs (BNZ).

Real AHM, 3/4

Theorem. Consider a circulant matrix $H \in M_N(\mathbb{R}^*)$, $H_{ij} = \gamma_{j-i}$. If the following conditions are satisfied, H is almost Hadamard:

(1) The vector $q = F^*\gamma$ satisfies $q \in \mathbb{T}^N$.

(2) With $\varepsilon = \mathrm{sgn}(\gamma)$, $\rho_i = \sum_r \varepsilon_r \gamma_{i+r}$, $\nu = F^*\rho$, we have $\nu > 0$.

Proof. We have the following computation:

$$(S^t H)_{ij} = \sum_k S_{ki} H_{kj} = \sum_k \varepsilon_{i-k} \gamma_{j-k} = \sum_r \varepsilon_r \gamma_{j-i+r} = \rho_{j-i}$$

Thus $S^t U$ is circulant, with $\rho/\sqrt{N}$ as first row.

Real AHM, 4/4

Consider the following vector, having length $N = 2n + 1$:

$$q = (-1)^n(1, -1, 1, \ldots, -1, 1, 1, -1, \ldots, 1, -1)$$

This vector produces the following circulant $N \times N$ real AHM:

$$L_N = \frac{1}{N}\begin{pmatrix} 1 & -\cos^{-1}\frac{\pi}{N} & \cos^{-1}\frac{2\pi}{N} & \ldots\ldots & \cos^{-1}\frac{(N-1)\pi}{N} \\ \cos^{-1}\frac{(N-1)\pi}{N} & 1 & -\cos^{-1}\frac{\pi}{N} & \ldots\ldots & -\cos^{-1}\frac{(N-2)\pi}{N} \\ -\cos^{-1}\frac{(N-2)\pi}{N} & \cos^{-1}\frac{(N-1)\pi}{N} & 1 & \ldots\ldots & \cos^{-1}\frac{(N-3)\pi}{N} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ -\cos^{-1}\frac{\pi}{N} & \cos^{-1}\frac{2\pi}{N} & -\cos^{-1}\frac{3\pi}{N} & \ldots\ldots & 1 \end{pmatrix}$$

$\Longrightarrow$ many other interesting examples (BNZ)

$\Longrightarrow$ applications to submatrices of Hadamard matrices (BNS)

Complex AHM

The complex AHM are conjectured to be all Hadamard (!)

Conjecture. Any local maximizer of the 1-norm on U_N must be a global maximizer, i.e. must be a rescaled Hadamard matrix.

$\Longrightarrow$ Verified for real AHM and their complex versions (BN).

$\Longrightarrow$ Potential powerful analytic characterization of the CHM.

Random derivatives

Let $OSC_N \subset USC_N$ be the orthogonal symmetric circulant matrices, and unitary self-adjoint circulant matrices, and USB_N be the unitary bistochastic self-adjoint matrices. We have:

Conjecture. Given $U \in USB_N$ satisfying $S^*U \geq 0$, there exists a simple function $B \to B^U$ (passage to another coset?), such that

$$\int_{OSC_N} \Phi(U, B^U) dB \leq 0$$

with equality when $H = \sqrt{N}U$ is Hadamard.