Hadamard matrix models

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"Introduction to Hadamard matrices", 6/6

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Fourier 1/4

Theorem. The Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$, which in standard matrix form, with indices $i, j = 0, 1, \ldots, N-1$, is

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{N-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{(2N-1)} & \ldots & w^{(N-1)^2} \end{pmatrix}$$

is a complex Hadamard matrix, in dephased form.

Proof. The scalar products between distinct rows, rescaled by $1/N$, are barycenters of regular polygons, and so they vanish.

Fourier 2/4

Theorem. Given G finite abelian, with dual $\widehat{G} = \{\chi : G \to \mathbb{T}\}$, consider the Fourier coupling $\mathcal{F}_G : G \times \widehat{G} \to \mathbb{T}$:

$$(i, \chi) \to \chi(i)$$

(1) Via the standard isomorphism $G \simeq \widehat{G}$, this Fourier coupling is a square matrix, $F_G \in M_G(\mathbb{T})$, which is complex Hadamard.

(2) For a cyclic group $G = \mathbb{Z}_N$ we obtain in this way, via the standard identification $\mathbb{Z}_N = \{1, \ldots, N\}$, the Fourier matrix F_N.

(3) In general, when using a decomposition $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$, the corresponding Fourier matrix is $F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$.

Proof. All this is elementary group theory.

Fourier 3/4

Results about the Fourier matrices:

(1) They exist at any $N \in \mathbb{N}$! (no complex HC)

(2) Circulants: Björck F_N' circulant (no CHC), Backelin..

(3) Geometry/Defect: K,N,TZ,BB,T,B + Nicoara-White..

(4) Isolation, versions: McNulty-Weigert, Master Hadamard..

(5) Analysis: $F_N \otimes' F_N$ bistochastic, glow up to order 4

Fourier 4/4

Question. Given a complex Hadamard matrix $H \in M_N(\mathbb{T})$, is it the "Fourier matrix" of something?

Answer. YES, in a certain sense, the relevant group-type object being a quantum permutation group $G \subset S_N^+$.

Bonus. The construction $H \to G$ makes the link with von Neumann algebras, subfactors, planar algebras, spin models.

Quantum groups 1/4

<u>C^*-algebra</u>: complex algebra A, with norm $||.||$ making it a Banach algebra, and involution $*$ satisfying $||aa^*|| = ||a||^2$.

<u>Basic examples</u>: $A \subset B(H)$ closed $*$-algebra ("generic"), also $A = C(X)$ with X compact space, $||.|| =$ sup norm.

<u>Gelfand theorem</u>: the commutative C^*-algebras are those of the form $C(X)$, with X compact space [proof: $X = Spec(A)$].

$\implies$ in general, write $A = C(X)$, with $X =$ "<u>NC space</u>".

Quantum groups 2/4

Let G be a compact Lie group. Then $G \subset U_N$. Multiplication:

$$(UV)_{ij} = \sum_k U_{ik} V_{kj}$$

By Stone-Weierstrass we have $C(G) = < u_{ij} >$, where:

$$u_{ij}(U) = U_{ij}$$

The multiplication $G \times G \to G$ transposes as:

$$u_{ij} \to \sum_k u_{ik} \otimes u_{kj}$$

Thus G is well described by $C(G)$, together with $u = (u_{ij})$.

Quantum groups 3/4

Definition. Let A be a C^*-algebra, with $u \in M_N(A)$ biunitary (u, u^t unitaries), whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

We write $A = C(G)$, and call G a compact quantum group.

[axioms due to Woronowicz, 1987, slightly modified]

Quantum groups 4/4

Theorem 1. Haar integration functional:

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

Theorem 2. Peter-Weyl theory, and in particular:

$$C^\infty(G) \simeq \bigoplus_{r\in Irr(G)} B(H_r)$$

Theorem 3. Tannaka-Krein duality, between G and:

$$C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

[refs: Woronowicz's CMP and TKD papers, late 80s]

Quantum permutations

Permutation matrices $S_N \subset O_N$. Coordinates of S_N are:

$$u_{ij} = \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

Magic (entries are projections, sum 1 on each row/column)

<u>Definition</u>. The quantum permutation group S_N^+ is defined via:

$$C(S_N^+) = C^*\left((u_{ij}) \middle| u = N \times N \text{ magic}\right)$$

[it's compact (!) verification of the axioms is routine: Wang 98]

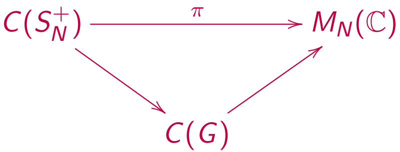
The construction

Given an Hadamard matrix $H \in M_N(\mathbb{T})$, the rank 1 projections

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

<u>Definition</u>. We associate to H the quantum permutation group $G \subset S_N^+$ given by the following "Hopf image" factorization,



where $\pi(u_{ij}) = Proj(H_i/H_j)$ are the above rank 1 projections.

Results 1/4

Theorem. For a Fourier matrix F_G we obtain in this way the finite abelian group G itself, acting on itself.

Proof. Assume first $H = F_N$. Here the rows of H are given by $H_i = \rho^i$, where $\rho = (1, w, w^2, \ldots, w^{N-1})$. Thus, we have:

$$\frac{H_i}{H_j} = \rho^{i-j}$$

Thus the rank 1 projections $P_{ij} = Proj(H_i/H_j)$ form a circulant matrix, all whose entries commute, and we obtain $G = \mathbb{Z}_N$.

In the general case, $H = F_G$ with G arbitrary, the proof is similar. Alternatively, we can use $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$.

Results 2/4

Theorem. For a tensor product of Hadamard matrices $H = H' \otimes H''$ we obtain a product of quantum groups, $G = G' \times G''$.

Proof. We have a diagram as follows:

$$\begin{array}{ccccc}
C(S_{N'}^+) \otimes C(S_{N''}^+) & \longrightarrow & C(G') \otimes C(G'') & \longrightarrow & M_{N'}(\mathbb{C}) \otimes M_{N''}(\mathbb{C}) \\
\uparrow & & & & \downarrow \\
C(S_N^+) & \longrightarrow & C(G) & \longrightarrow & M_N(\mathbb{C})
\end{array}$$

Thus the representation factorizes through $C(G') \otimes C(G'')$.

Results 3/4

The idea is that the inner faithful models $\pi : C(G) \to M_K(C(T))$ "remind" the quantum group. We have indeed:

Theorem. The Tannakian category of G is given by

$$C_{kl} = Hom(U^{\otimes k}, U^{\otimes l})$$

where $U_{ij} = \pi(u_{ij})$, and with the formal Hom-spaces at right.

Theorem. The integration over G is given by

$$\int_G = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$, with $\varphi = tr \otimes \int_T$.

Results 4/4

<u>Theorem</u>. Given two finite abelian groups G, H, with $|G| = M$, $|H| = N$, consider the main character χ of the quantum group associated to the Diță deformation $\mathcal{F}_{G\times H}$. We have then

$$law\left(\frac{\chi}{N}\right) = \left(1 - \frac{1}{M}\right)\delta_0 + \frac{1}{M}\,\pi_t$$

in moments, with $M = tN \to \infty$, where π_t is the free Poisson law of parameter $t > 0$. In addition, this holds for any generic fiber.

<u>Proof</u>. Long story here (B, BB, B, B).

Versions 1/2

Definition. An Hadamard matrix over a unital C^*-algebra A is a square matrix $H \in M_N(A)$ satisfying the following conditions:

(1) All the entries of H are unitaries, $H_{ij} \in U_A$.

(2) These entries commute on all rows and all columns of H.

Theorem. If $H \in M_N(A)$ is Hadamard, the following matrices $P_{ij} \in M_N(A)$ form a magic matrix $P = (P_{ij})$, over $M_N(A)$:

$$(P_{ij})_{ab} = \frac{1}{N} H_{ia} H_{ja}^* H_{jb} H_{ib}^*$$

Thus, we have a representation $\pi : C(S_N^+) \to M_N(A)$, that we can factorize $\pi : C(S_N^+) \to C(G) \to M_N(A)$, with $G \subset S_N^+$ minimal.

Versions 2/2

Definition. The quantum partial permutation semigroup $\widetilde{S}_N^+$ is defined via the formula

$$C(S_N^+) = C^*\left((u_{ij})\Big| u = N \times N \text{ submagic}\right)$$

where "submagic" means that the entries are projections, pairwise orthogonal on rows and columns.

Theorem. Assuming that $H \in M_{M\times N}(\mathbb{T})$ is partial Hadamard, with rows $H_1, \ldots, H_M \in \mathbb{T}^N$, the following matrix is submagic:

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

Thus H produces a representation $\pi_H : C(\widetilde{S}_M^+) \to M_N(\mathbb{C})$, that we can factorize through $C(G)$, with $G \subset \widetilde{S}_M^+$ minimal.

Von Neumann

Von Neumann algebras: $A \subset B(H)$, involution $*$, weakly closed.

Theorem 1. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Theorem 2. When writing the center as $Z(A) = L^\infty(X)$, the whole algebra decomposes as $A = \int_X A_x dx$.

Theorem 3. The theory of factors, $Z(A) = \mathbb{C}$, reduces to that of the II_1 factors ($\dim A = \infty$, trace $tr : A \to \mathbb{C}$).

[this is heavy: Murrray-von Neumann, Tomita-Takeaski, Connes..]

Popa

A pair of orthogonal MASA is a pair of maximal abelian subalgebras

$$B, C \subset A$$

which are orthogonal: $tr(bc) = tr(b)tr(c)$, for any $b \in B, c \in C$.

Theorem. Up to a unitary, the pairs of orthogonal MASA in the simplest von Neumann factor, namely $M_N(\mathbb{C})$, are

$$A = \Delta \quad , \quad B = H\Delta H^*$$

with $\Delta =$ diagonal matrices, and $H \in M_N(\mathbb{T})$ Hadamard.

Proof. Write the orthogonality condition, then conclude.

Jones

(1) Given $H \in M_N(\mathbb{T})$ Hadamard, the associated pair of MASA fit into a "commuting square" in the sense of subfactor theory:

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

(2) By "basic construction" we obtain a subfactor $Q \subset R$, whose invariants can be computed using "Ocneanu compactness".

(3) The factor Q appears as fixed point algebra under the action of the corresponding quantum permutation group $G \subset S_N^+$.

$\implies$ spin models?