

# QUANTUM GROUPS AND HADAMARD MATRICES

TEODOR BANICA AND REMUS NICOARA

ABSTRACT. To any complex Hadamard matrix we associate a quantum permutation group. The correspondence is not one-to-one, but the quantum group encapsulates a number of subtle properties of the matrix. We investigate various aspects of the construction: compatibility to product operations, characterization of matrices which give usual groups, explicit computations for small matrices.

## INTRODUCTION

A complex Hadamard matrix is a matrix  $h \in M_n(\mathbb{C})$ , having the following property: entries are on the unit circle, and rows are mutually orthogonal. Equivalently,  $n^{-1/2}h$  is a unitary matrix with all entries having the same absolute value.

The basic example is the Fourier  $n \times n$  matrix, given by  $F_{ij} = w^{ij}$ , where  $w = e^{2\pi i/n}$ . There are many other examples, but there is no other known family of complex Hadamard matrices that exist for every  $n$ .

These matrices are related to several questions:

- (1) The story begins with work of Sylvester, who studied real Hadamard matrices [S]. These have  $\pm 1$  entries, and can be represented as black and white pavements of a  $n \times n$  square. Such a matrix must have order  $n = 2$  or  $n = 4k$ , and the main problem here is whether there are such matrices for any  $k$ . At the time of writing, the verification goes up to  $k = 166$ .
- (2) One can try to replace the  $\pm 1$  entries by numbers of the form  $\pm 1, \pm i$ , or by  $n$ -th roots of unity (the Fourier matrix appears), or by roots of unity of arbitrary order. These are natural generalizations of real Hadamard matrices, first investigated by Butson in the sixties [Bu].
- (3) The matrices with arbitrary complex entries appeared in the eighties. Popa discovered that such a matrix gives rise to the operator algebra condition  $\Delta \perp h\Delta h^*$ , where  $\Delta \subset M_n(\mathbb{C})$  is the algebra of diagonal matrices [Po]. Then work of Jones led to the conclusion that associated to  $h$  are the following objects: a statistical mechanical model, a knot invariant, a subfactor, and a planar algebra. The computation of algebraic invariants for these objects appears to be remarkably subtle and difficult. See [J].
- (4) In the meantime, motivated by a problem of Enflo, Björck discovered that circulant Hadamard matrices correspond to cyclic  $n$ -roots, and started to

---

2000 *Mathematics Subject Classification.* 46L65 (05B20, 46L37).

*Key words and phrases.* Quantum permutation, Hadamard matrix.

R.N. was supported by NSF under Grant No. DMS-0500933.

construct many examples [Bj]. In a key paper, Haagerup proved that for  $n = 5$  the Fourier matrix is the only complex Hadamard matrix, up to permutations and multiplication by diagonal unitaries [H]. Self-adjoint complex Hadamard matrices of order 6 have been recently classified [BN].

- (5) Another important result is that of Petrescu, who discovered a one-parameter family at  $n = 7$ , providing a counterexample to a conjecture of Popa regarding the finiteness of the number of complex Hadamard matrices of prime dimension [Pe]. The notion of deformation is further investigated in [N].
- (6) Several applications of complex Hadamard matrices were discovered in the late nineties. These appear in a remarkable number of areas of mathematics and physics, and the various existence and classification problems are fast evolving. See [TZ] for a catalogue of most known complex Hadamard matrices and a summary of their applications.

The purpose of this paper is to present a definition for what might be called symmetry of a Hadamard matrix. This is done by associating to  $h$  a certain Hopf algebra  $A$ . This algebra is of a very special type: it is a quotient of Wang's quantum permutation algebra [Wa2]. In other words, we have the heuristic formula  $A = \mathbb{C}(G)$ , where  $G$  is a quantum group which permutes the set  $\{1, \dots, n\}$ .

As an example, the Fourier matrix gives  $G = \mathbb{Z}_n$ . In the general case  $G$  doesn't really exist as a concrete object, but some partial understanding of the quantum permutation phenomenon is available, via methods from finite, compact and discrete groups, subfactors, planar algebras, low-dimensional topology, statistical mechanical models, classical and free probability, geometry, random matrices etc. See [BB], [BBC], [BC1], [BC2] and the references there. The hope is that the series of papers on the subject will grow quickly, and evolve towards the Hadamard matrix problematics. A precise comment in this sense is made in the last section.

Finally, let us mention that the fact that Hadamard matrices produce quantum groups is known since [Ba1], [Wa2]. The relation between coinvolutive compact quantum groups, abstract statistical mechanical models and associated commuting squares, subfactors and standard  $\lambda$ -lattices is worked out in [Ba2]. Another result is the one in [Ba3], where a Tannaka-Galois type correspondence between spin planar algebras and quantum permutation groups is obtained. In both cases the situations discussed are more general than those involving Hadamard matrices, where some simplifications are expected to appear. However, the whole subject is quite technical, and it is beyond our purposes to discuss the global picture. This paper should be regarded as an introduction to the subject.

The paper is organized as follows. 1 is a quick introduction to quantum permutation groups. 2 contains a few basic facts regarding complex Hadamard matrices, the construction of the correspondence, and the computation for the Fourier matrix. In 3-6 we investigate several aspects of the correspondence: compatibility between tensor products of quantum groups and of Hadamard matrices, the relation with magic squares, characterization of matrices which give usual groups, explicit computations for small matrices, and some comments about subfactors and deformation.

**Acknowledgements.** This work was started in March 2006 at Vanderbilt University, and we would like to thank Dietmar Bisch for the kind hospitality and help. The paper also benefited from several discussions with Stefaan Vaes.

## 1. QUANTUM PERMUTATION GROUPS

This section is an introduction to  $A(S_n)$ , the algebra of free coordinates on the symmetric group  $S_n$ . This algebra was discovered by Wang in [Wa2].

The idea of noncommuting coordinates goes back to Heisenberg, the specific idea of using algebras of free coordinates on algebraic groups should be attributed to Brown, and a detailed study of these algebras, from a K-theoretic point of view, is due to McClanahan. Brown's algebras are in fact too big, for instance they have no antipode, and the continuation of the story makes use of Woronowicz's work on the axiomatization of compact quantum groups. The first free quantum groups, corresponding to  $U_n$  and  $O_n$ , appeared in Wang's thesis. The specific question about free analogues of  $S_n$  was asked by Connes. See [Br], [M], [Wa1], [Wa2].

We can see from this brief presentation that the algebra  $A(S_n)$  we are interested in comes somehow straight from Heisenberg, after a certain period of time. In fact, as we will see now, the background required in order to define  $A(S_n)$  basically reduces to the definition of  $C^*$ -algebras, and to some early work on the subject:

Let  $A$  be a  $C^*$ -algebra. That is, we have a complex algebra with a norm and an involution, such that Cauchy sequences converge, and  $\|aa^*\| = \|a\|^2$ .

The basic example is  $B(H)$ , the algebra of bounded operators on a Hilbert space  $H$ . In fact, any  $C^*$ -algebra appears as subalgebra of some  $B(H)$ .

The key example is  $C(X)$ , the algebra of continuous functions on a compact space  $X$ . This algebra is commutative, and any commutative  $C^*$ -algebra is of this form.

The above two statements are results of Gelfand-Naimark-Segal and Gelfand. The proofs make use of standard results in commutative algebra, complex analysis, functional analysis, and measure theory. The relation between the two statements is quite subtle, and comes from the spectral theorem for self-adjoint operators. The whole material can be found in any book on operator algebras.

Wang's idea makes a fundamental use of the notion of projection:

**Definition 1.1.** *Let  $A$  be a  $C^*$ -algebra.*

- (1) *A projection is an element  $p \in A$  satisfying  $p^2 = p = p^*$ .*
- (2) *Two projections  $p, q \in A$  are called orthogonal when  $pq = 0$ .*
- (3) *A partition of unity is a set of orthogonal projections, which sum up to 1.*

A projection in  $B(H)$  is an orthogonal projection  $P(K)$ , where  $K \subset H$  is a closed subspace. Orthogonality of projections corresponds to orthogonality of subspaces, and partitions of unity correspond to decompositions of  $H$ .

A projection in  $C(X)$  is a characteristic function  $\chi(Y)$ , where  $Y \subset X$  is an open and closed subset. Orthogonality of projections corresponds to disjointness of subsets, and partitions of unity correspond to partitions of  $X$ .

**Definition 1.2.** *A magic unitary is a square matrix  $u \in M_n(A)$ , all whose rows and columns are partitions of unity in  $A$ .*

The terminology comes from a vague similarity with magic squares, to be investigated later on. For the moment we are rather interested in the continuing the above classical/quantum analogy, for projections and partitions of unity:

A magic unitary over  $B(H)$  is of the form  $P(K_{ij})$ , with  $K$  magic decomposition of  $H$ , in the sense that all rows and columns of  $K$  are decompositions of  $H$ . The basic examples here are of the form  $K_{ij} = \mathbb{C}\xi_{ij}$ , where  $\xi$  is a magic basis of  $H$ , in the sense that all rows and columns of  $\xi$  are bases of  $H$ .

A magic unitary over  $C(X)$  is of the form  $\chi(Y_{ij})$ , with  $Y$  magic partition of  $X$ , in the sense that all rows and columns of  $Y$  are partitions of  $X$ . The key example here comes from a finite group  $G$  acting on a finite set  $X$ : the characteristic functions  $\chi_{ij} = \{\sigma \in G \mid \sigma(j) = i\}$  form a magic unitary over  $C(G)$ .

In the particular case of the symmetric group  $S_n$  acting on  $\{1, \dots, n\}$ , we have the following presentation result, which follows from the Gelfand theorem:

**Theorem 1.1.**  *$C(S_n)$  is the universal commutative  $C^*$ -algebra generated by  $n^2$  elements  $\chi_{ij}$ , with relations making  $(\chi_{ij})$  a magic unitary matrix. Moreover, the maps*

$$\begin{aligned} \Delta(\chi_{ij}) &= \sum \chi_{ik} \otimes \chi_{kj} \\ \varepsilon(\chi_{ij}) &= \delta_{ij} \\ S(\chi_{ij}) &= \chi_{ji} \end{aligned}$$

*are the comultiplication, counit and antipode of  $C(S_n)$ .*

In other words, when regarding  $S_n$  as an algebraic group, the relations satisfied by the  $n^2$  coordinates are those expressing magic unitarity. Indeed, the characteristic functions  $\chi_{ij}$  are nothing but the  $n^2$  coordinates on the group  $S_n \subset O_n$ , where the embedding is the one given by permutation matrices.

See the preliminary section in [BBC] for a proof of the above result, and the introduction of [BC2] for more comments on the subject.

We are interested in the algebra of free coordinates on  $S_n$ . This is obtained by removing commutativity in the above presentation result:

**Definition 1.3.**  *$A(S_n)$  is the universal  $C^*$ -algebra generated by  $n^2$  elements  $x_{ij}$ , with relations making  $(x_{ij})$  a magic unitary matrix. The maps*

$$\begin{aligned} \Delta(x_{ij}) &= \sum x_{ik} \otimes x_{kj} \\ \varepsilon(x_{ij}) &= \delta_{ij} \\ S(x_{ij}) &= x_{ji} \end{aligned}$$

*are called the comultiplication, counit and antipode of  $A(S_n)$ .*

This algebra, discovered by in [Wa2], fits into the quantum group formalism developed by Woronowicz in [Wo1], [Wo2]. In fact, the quantum group  $G_n$  defined by the formula  $A(S_n) = C(G_n)$  is a free analogue of the symmetric group  $S_n$ . This quantum group doesn't exist of course: the idea is just that various properties of  $A(S_n)$

can be expressed in terms of it. As an example, the canonical map  $A(S_n) \rightarrow C(S_n)$  should be thought of as coming from an embedding  $S_n \subset G_n$ .

**Proposition 1.1.** *For  $n = 1, 2, 3$  we have  $A(S_n) = C(S_n)$ .*

This follows from the fact that for  $n \leq 3$ , the entries of a  $n \times n$  magic unitary matrix have to commute. For instance at  $n = 2$  the matrix must be of the form

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

where  $p$  is a projection, and entries of this matrix obviously commute.

The result is no longer true for  $n = 4$ , where more complicated examples of magic unitary matrices are available, for instance via diagonal concatenation:

$$u = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

In fact,  $A(S_n)$  with  $n \geq 4$  is not commutative, and infinite dimensional.

Consider now an arbitrary magic unitary matrix  $u \in M_n(A)$ . We say that  $u$  is a corepresentation of  $A$  if its coefficients generate  $A$ , and if the formulae

$$\begin{aligned} \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji} \end{aligned}$$

define morphisms of  $C^*$ -algebras. These morphisms have by definition values in the algebras  $A \otimes A, \mathbb{C}$  and  $A^{op}$ , and they are uniquely determined if they exist. In case they exist, these morphisms make  $A$  into a Hopf algebra in the sense of [Wo1].

This notion provides an axiomatization for quotients of  $A(S_n)$ :

**Definition 1.4.** *A quantum permutation algebra is a  $C^*$ -algebra  $A$ , given with a magic unitary corepresentation  $u \in M_n(A)$ .*

We have the following purely combinatorial approach to these algebras:

Recall first that the unitary representations of a discrete group  $L$  are in correspondence with representations of the group algebra  $C^*(L)$ . The same is known to hold for discrete quantum groups, so we have a correspondence between representations

$$\begin{aligned} L_n &\rightarrow U(H) \\ A(S_n) &\rightarrow B(H) \end{aligned}$$

where  $L_n$  is the discrete quantum group whose group algebra is  $A(S_n)$ . We can therefore consider the quantum permutation algebra  $A = C^*(Im(L_n))$ .

All this is quite heuristic, but the construction of  $A$  is definitely possible:

**Definition 1.5.** *Associated to a representation  $\pi : A(S_n) \rightarrow B(H)$  is the smallest quantum permutation algebra  $A_\pi$  realizing a factorization of  $\pi$ .*

In this definition we assume of course that the morphism making  $\pi$  factorize is the canonical one, given by  $x_{ij} \rightarrow u_{ij}$ . The construction of  $A_\pi$  is standard, for instance by taking the quotient of  $A(S_n)$  by an appropriate Hopf algebra ideal. The uniqueness up to isomorphism is also clear. See [Ba1].

We have the following examples:

- (1) For the counit representation  $\pi : A(S_n) \rightarrow \mathbb{C}$  we get  $A_\pi = \mathbb{C}$ .
- (2) For a faithful representation  $\pi : A(S_n) \subset B(H)$  we get  $A_\pi = A(S_n)$ .
- (3) If  $A$  is a quantum permutation algebra,  $A \subset B(H)$ , and  $\pi : A(S_n) \rightarrow A$  is a surjective representation of  $A(S_n)$ , then  $A_\pi = A$ .

These examples are all trivial. Note however that the third one has the following interesting consequence:

**Proposition 1.2.** *Any quantum permutation algebra is of the form  $A_\pi$ .*

*Proof.* Let  $A$  be a quantum permutation algebra. We can compose the canonical quotient map  $A(S_n) \rightarrow A$  with an embedding  $A \subset B(H)$ , say given by the GNS theorem, and we get a representation  $\pi$  as in the statement.  $\square$

## 2. HADAMARD MATRICES

Let  $h \in M_n(\mathbb{C})$  be an Hadamard matrix. This means that all entries of  $h$  have modulus 1, and that rows of  $h$  are mutually orthogonal. In other words, we have

$$h = \begin{pmatrix} h_1 \\ h_2 \\ \dots \\ h_n \end{pmatrix}$$

where the vectors  $h_i$  are formed by complex numbers of modulus 1, and are orthogonal with respect to the usual scalar product of  $\mathbb{C}^n$ , given by:

$$\langle x, y \rangle = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$$

It follows from definitions that the columns of  $h$  are orthogonal as well.

We have the following characterization of such matrices:

**Proposition 2.1.** *For a matrix  $h \in M_n(\mathbb{C})$ , the following are equivalent:*

- (1)  $h$  is an Hadamard matrix.
- (2)  $n^{-1/2}h$  is a unitary matrix, all whose entries have same modulus.

We should mention that in the operator algebra literature, it is rather  $n^{-1/2}h$  that is called Hadamard. The other warning is that in the combinatorics literature, the Hadamard matrices are just the real ones, meaning those having  $\pm 1$  entries. We hope that the hybrid terminology used in this paper won't cause any trouble.

We use capital letters to denote explicit Hadamard matrices. The first example, which is in fact the only basic example, is the Fourier matrix:

**Definition 2.1.** *The Fourier matrix of order  $n$  is given by  $F_{ij} = w^{ij}$ , where  $w = e^{2\pi i/n}$ .*

We have a natural equivalence relation for Hadamard matrices, given by  $h \sim k$  if one can pass from  $h$  to  $k$  by permutations of rows and columns, and by multiplications of rows and columns by complex numbers of modulus 1. See [TZ].

For  $n = 1, 2, 3, 5$  any Hadamard matrix is equivalent to the Fourier one, see [H]. At  $n = 4$  we have the following example, depending on  $q$  on the unit circle:

$$M_q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & q & -1 & -q \\ 1 & -1 & 1 & -1 \\ 1 & -q & -1 & q \end{pmatrix}$$

These are, up to equivalence, all  $4 \times 4$  Hadamard matrices. As an example, the Fourier matrix corresponds to the value  $q = \pm i$ . See Haagerup [H].

At  $n = 6$  we have several examples, as for instance:

$$H = \begin{pmatrix} i & 1 & 1 & 1 & 1 & 1 \\ 1 & i & 1 & -1 & -1 & 1 \\ 1 & 1 & i & 1 & -1 & -1 \\ 1 & -1 & 1 & i & 1 & -1 \\ 1 & -1 & -1 & 1 & i & 1 \\ 1 & 1 & -1 & -1 & 1 & i \end{pmatrix}$$

This matrix appears in the paper of Haagerup [H]. Another example appears in the paper of Tao [T]. This is given by the following formula, with  $w = e^{2\pi i/3}$ :

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & w & w & w^2 & w^2 \\ 1 & w & 1 & w^2 & w^2 & w \\ 1 & w & w^2 & 1 & w & w^2 \\ 1 & w^2 & w^2 & w & 1 & w \\ 1 & w^2 & w & w^2 & w & 1 \end{pmatrix}$$

Observe that the Fourier, Haagerup and Tao matrices are based on certain roots of unity. We have here the following notion:

**Definition 2.2.** *The Butson class  $H_l(n)$  consists of Hadamard matrices  $h \in M_n(\mathbb{C})$  having the property  $h_{ij}^l = 1$  for any  $i, j$ .*

In this definition  $l$  is a positive integer. It is convenient to use as well the notation  $H_\infty(n)$ , for the class of all  $n \times n$  Hadamard matrices. Here are a few examples:

- (1) The Fourier matrix is in  $H_n(n)$ .
- (2) The Haagerup matrix is in  $H_4(6)$ .
- (3) The Tao matrix is in  $H_3(6)$ .
- (4) The real Hadamard matrices are in  $H_2(n)$ .
- (5) The matrix  $M(q)$  is in  $H_l(4)$ , where  $l$  is the order of  $q^2$ .

We call  $n$  size, and  $l$  level. In lack of some better idea, the complexity of an Hadamard matrix will be measured by its size and level, the size coming first.

At infinite level the self-adjoint matrices of order 6 are classified in [BN]. The complete list of Hadamard matrices at  $n = 6$  is not known.

We end this discussion with the following well-known fact:

**Proposition 2.2.** *Each Hadamard matrix is equivalent to a Hadamard matrix which is dephased, meaning that the first row and column and the diagonal consist of 1's.*

Consider now an arbitrary Hadamard matrix  $h \in M_n(\mathbb{C})$ . The rows of  $h$ , denoted as usual  $h_1, \dots, h_n$ , can be regarded as elements of the algebra  $\mathbb{C}^n$ .

Since each  $h_i$  is formed by complex numbers of modulus 1, this element is invertible. We can therefore consider the following matrix of elements of  $\mathbb{C}^n$ :

$$\xi_{ij} = \frac{h_j}{h_i}$$

The scalar products on rows of  $\xi$  are computed as follows:

$$\begin{aligned} \langle \xi_{ij}, \xi_{ik} \rangle &= \langle h_j/h_i, h_k/h_i \rangle \\ &= n \langle h_j, h_k \rangle \\ &= n \cdot \delta_{jk} \end{aligned}$$

In other words, each row of  $\xi$  is an orthogonal basis of  $\mathbb{C}^n$ . A similar computation works for columns, so  $\xi$  is a magic basis of  $\mathbb{C}^n$ . Thus we can apply the procedure from previous section, and we get a magic unitary matrix, a representation, and a quantum permutation algebra:

**Definition 2.3.** *Let  $h \in M_n(\mathbb{C})$  be an Hadamard matrix.*

- (1)  $\xi(h)$  is the magic basis given by  $\xi_{ij} = h_j/h_i$ .
- (2)  $P(h)$  is the magic unitary given by  $P_{ij} = P(\xi_{ij})$ .
- (3)  $\pi_h$  is the representation given by  $\pi_h(u_{ij}) = P_{ij}$ .
- (4)  $A(h)$  is the quantum permutation algebra associated to  $\pi$ .

In other words, associated to  $h$  are the rank one projections  $P(h_j/h_i)$ , which form a magic unitary over  $M_n(\mathbb{C})$ . We consider the corresponding representation

$$\pi : A(S_n) \rightarrow M_n(\mathbb{C})$$

we say that this comes from a representation  $L_n \rightarrow U(n)$  of the dual of the  $n$ -th quantum permutation group, and we consider the algebra  $A = C^*(\text{Im}(L_n))$ .

It is routine to check that equivalent Hadamard matrices  $h_1, h_2$  give the same algebra,  $A(h_1) = A(h_2)$ . This is because the standard equivalence operations, namely permutation of rows and columns, and multiplication of rows and columns by scalars of modulus 1, give conjugate magic unitaries, hence conjugate representations of  $A(S_n)$ . As for the converse, this is expected not to hold, see the conclusion.

As a first example, consider the Fourier matrix:

**Theorem 2.1.** *For the Fourier matrix  $F$  of order  $n$  we have  $A(F) = C(\mathbb{Z}_n)$ .*

*Proof.* We have the following computation, where  $\rho = (w, w^2, \dots, w^n)$ :

$$F_{ij} = w^{ij} \Rightarrow F_i = \rho^i$$



$$\begin{aligned} \Rightarrow \xi_{ij} &= \rho^{j-i} \\ \Rightarrow P_{ij} &= P(\rho^{j-i}) \end{aligned}$$

Consider the cycle  $c(i) = i - 1$ . We regard  $\mathbb{Z}_n = \{0, 1, \dots, n - 1\}$  as a subgroup of  $S_n$ , via the embedding  $k \rightarrow c^k$ . Consider the following diagram:

$$\begin{array}{ccc} A(S_n) & \rightarrow & M_n(\mathbb{C}) \\ \downarrow & & \uparrow \\ C(S_n) & \longrightarrow & C(\mathbb{Z}_n) \end{array}$$

We define the connecting maps in the following way:

- (1) The map  $A(S_n) \rightarrow M_n(\mathbb{C})$  is the representation  $\pi$ .
- (2) The map  $A(S_n) \rightarrow C(S_n)$  is the canonical one, given by  $x_{ij} \rightarrow \chi_{ij}$ .
- (3) The map  $C(S_n) \rightarrow C(\mathbb{Z}_n)$  is the transpose of  $\mathbb{Z}_n \subset S_n$ .
- (4) The map  $C(\mathbb{Z}_n) \rightarrow M_n(\mathbb{C})$  is given by  $\delta_k \rightarrow P(\rho^k)$ .

We compute the image of  $\chi_{ij}$  by the third map:

$$\begin{aligned} \text{Im}(\chi_{ij}) &= \text{Im}(\chi\{\sigma \mid \sigma(j) = i\}) \\ &= \chi\{k \mid c^k(j) = i\} \\ &= \chi\{k \mid j - k = i\} \\ &= \delta_{j-i} \end{aligned}$$

Thus at level of generators, we have the following diagram:

$$\begin{array}{ccc} x_{ij} & \rightarrow & P(\rho^{j-i}) \\ \downarrow & & \uparrow \\ \chi_{ij} & \longrightarrow & \delta_{j-i} \end{array}$$

This diagram commutes, so the above diagram of algebras commutes as well. Thus we have a factorization of  $\pi$  through the quantum permutation algebra  $C(\mathbb{Z}_n)$ . Moreover, this algebra is the minimal one making  $\pi$  factorize, for instance because it is isomorphic to the image of  $\pi$ . This gives the result.  $\square$

### 3. TENSOR PRODUCTS

Given two Hadamard matrices  $h \in M_n(\mathbb{C})$  and  $k \in M_m(\mathbb{C})$ , we can consider their tensor product. This is an element as follows:

$$h \otimes k \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$$

In order to view  $h \otimes k$  as a usual matrix, we have to identify the algebra on the right with the matrix algebra  $M_{nm}(\mathbb{C})$ . This isomorphism is not canonical, so we proceed as follows: given  $h, k$ , we define  $h \otimes k \in M_{nm}(\mathbb{C})$  by the formula

$$(h \otimes k)_{ia,jb} = h_{ij}k_{ab}$$

where double indices  $ia, jb \in \{1, \dots, nm\}$  come from indices  $i, j \in \{1, \dots, n\}$  and  $a, b \in \{1, \dots, m\}$ , via some fixed decomposition of the index set. Most convenient here is to use the lexicographic decomposition:  $\{1, 2, \dots, nm\} = \{11, 12, \dots, nm\}$ .

It is well-known that the above two definitions of  $h \otimes k$  coincide.

**Proposition 3.1.**  *$h \otimes k$  is an Hadamard matrix.*

*Proof.* We can use for instance Proposition 2.1: since  $h, k$  are Hadamard matrices,  $n^{-1/2}h, m^{-1/2}k$  are unitaries. Now the tensor product of two unitaries being a unitary, we get that  $(nm)^{-1/2}h \otimes k$  is unitary, so  $h \otimes k$  is Hadamard.  $\square$

It is known from [BB] that various operations at level of combinatorial objects lead to similar operations at level of quantum permutation algebras. In this section we prove such a result for tensor products of Hadamard matrices:

**Theorem 3.1.** *We have  $A(h \otimes k) = A(h) \otimes A(k)$ .*

*Proof.* It is convenient here to use tensor product notations, at the same time with bases and indices. We use the lexicographic identification of Hilbert spaces

$$\begin{aligned} \mathbb{C}^n \otimes \mathbb{C}^m &= \mathbb{C}^{nm} \\ e_i \otimes e_a &= e_{ia} \end{aligned}$$

along with the corresponding identification of operator algebras:

$$\begin{aligned} M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) &= M_{nm}(\mathbb{C}) \\ e_{ij} \otimes e_{ab} &= e_{ia, jb} \end{aligned}$$

We denote by  $\xi, P, \pi$  the objects associated in Definition 2.3 to  $h, k, h \otimes k$ , with the choice of indices telling which is which. We have the following computation:

- (1) The  $ia$ -th row of  $h \otimes k$  is given by  $(h \otimes k)_{ia} = h_i \otimes k_a$ .
- (2) The corresponding magic basis is given by  $\xi_{ia, jb} = \xi_{ij} \otimes \xi_{ab}$ .
- (3) The corresponding magic unitary is given by  $P_{ia, jb} = P_{ij} \otimes P_{ab}$ .

Consider now the factorizations associated to  $h, k$ :

$$\begin{aligned} A(S_n) &\rightarrow A(h) \rightarrow M_n(\mathbb{C}) \\ A(S_m) &\rightarrow A(k) \rightarrow M_m(\mathbb{C}) \end{aligned}$$

These are given by the following formulae:

$$\begin{aligned} x_{ij} &\rightarrow u_{ij} \rightarrow P_{ij} \\ x_{ab} &\rightarrow u_{ab} \rightarrow P_{ab} \end{aligned}$$

We can form the tensor product of these maps, then we add to the picture the factorization associated to  $h \otimes k$ , along with two vertical arrows:

$$\begin{array}{ccccc} A(S_n) \otimes A(S_m) & \rightarrow & A(h) \otimes A(k) & \rightarrow & M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \\ \uparrow & & & & \downarrow \\ A(S_{nm}) & \rightarrow & A(h \otimes k) & \rightarrow & M_{nm}(\mathbb{C}) \end{array}$$

Here the arrow on the right is the identification mentioned in the beginning of the proof. As for the arrow on the left, this is constructed as follows. Recall first

that  $A(S_{nm})$  is the universal algebra generated by the entries of a  $nm \times nm$  magic unitary matrix  $x_{ia,jb}$ . The matrix  $x_{ij} \otimes x_{ab}$  being a magic unitary as well, we get a morphism of algebras mapping  $x_{ia,jb} \rightarrow x_{ij} \otimes x_{ab}$ , that we put at left.

At level of generators, we have the following diagram:

$$\begin{array}{ccccc} x_{ij} \otimes x_{ab} & \rightarrow & u_{ij} \otimes u_{ab} & \rightarrow & P_{ij} \otimes P_{ab} \\ \uparrow & & & & \downarrow \\ x_{ia,jb} & \rightarrow & u_{ia,jb} & \rightarrow & P_{ia,jb} \end{array}$$

This diagram commutes, so the above diagram of algebras commutes as well. Thus the factorization associated to  $h \otimes k$  factorizes through the algebra  $A(h) \otimes A(k)$ . Now since  $A(h \otimes k)$  is the minimal algebra producing such a factorization, this gives a morphism  $A(h) \otimes A(k) \rightarrow A(h \otimes k)$ , mapping  $u_{ij} \otimes u_{ab} \rightarrow u_{ia,jb}$ .

This morphism is surjective, because the elements  $u_{ia,jb}$  generate the algebra  $A(h \otimes k)$ . Now since  $A(h \otimes k)$  makes a factorization of the representation associated to  $h \otimes k$ , this algebra produces as well factorizations of the representations associated to  $h, k$ . Thus our morphism is injective, and this gives the result.  $\square$

As a first application, we can start classification work for small Hadamard matrices. Recall from previous section that each matrix has a size  $n$  and a level  $l$ , according to the formula  $h \in H_l(n)$  with  $l$  minimal, and that we decided to list matrices in terms of  $n, l$ , with  $n$  coming first. The situation is as follows:

- (1) For  $n = 1, 2, 3$  there is only one matrix, namely the Fourier one. As shown by Theorem 2.1, this matrix produces the algebra  $C(\mathbb{Z}_n)$ .
- (2) For  $n = 4$  the smallest possible level is  $l = 2$ . We have here the matrices  $M_q$ , with parameter  $q = \pm 1$ .
- (3) The next possible level is  $l = 4$ . We have here the matrices  $M_q$  with  $q = \pm i$ , both equivalent to the Fourier matrix, which gives  $C(\mathbb{Z}_4)$ .

Thus at  $nl \leq 44$ , we have just two matrices left. But these can be investigated by using Theorem 3.1:

**Corollary 3.1.** *For  $q = \pm 1$ , the Hadamard matrix*

$$M_q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & q & -1 & -q \\ 1 & -1 & 1 & -1 \\ 1 & -q & -1 & q \end{pmatrix}$$

*produces the algebra  $C(\mathbb{Z}_2 \times \mathbb{Z}_2)$ .*

*Proof.* The  $2 \times 2$  Fourier matrix is given by  $F_{ij} = (-1)^{ij}$ , so we have:

$$F = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Consider the tensor square of  $F$ , given by  $(F \otimes F)_{ia,jb} = F_{ia}F_{jb}$ . As for  $F$ , this is a  $\pm 1$  matrix. Now since the  $-1$  entry of  $F$  appears on the 11 position, the  $-1$

entries of  $F \otimes F$  appear on positions  $(ia, jb)$  satisfying  $ij = 11$  or  $ab = 11$ , with the position  $(11, 11)$  excluded. There are six such positions:

$$(11, 12), (11, 21), (12, 11), (12, 12), (21, 11), (21, 21)$$

Now by using the lexicographic identification  $\{1, 2, 3, 4\} = \{11, 12, 21, 22\}$ , these six positions are 12,13,21,22,31,33. This gives the following formula:

$$F \otimes F = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

By permuting rows and columns of this matrix we can get both matrices in the statement. The result follows from Theorem 2.1 and Theorem 3.1, by using the canonical identification  $C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2) = C(\mathbb{Z}_2 \times \mathbb{Z}_2)$ .  $\square$

#### 4. MAGIC SQUARES

In this section and in next one we investigate the case when  $A$  is commutative. In this situation, we must have  $A = C(G)$ , for a certain subgroup  $G \subset S_n$ .

We already know that  $A$  is commutative for the Fourier matrix, where we have  $G = \mathbb{Z}_n$ , and for the matrix  $M_q$  with  $q = \pm 1$ , where we have  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ . In fact, these two examples are the only ones that have been investigated so far. This makes our general problem quite unclear, because we have no counterexample.

Let us go back however to proof of Theorem 2.1. A careful examination shows that the proof relies on the fact that  $P$  is a circulant matrix. The idea will be to generalize this proof, with ‘‘circulant’’ replaced by ‘‘magic’’.

Let us first recall the following well-known definition:

**Definition 4.1.** *A magic square is a square matrix  $\sigma$ , having as entries the numbers  $1, \dots, n$ , such that each row and each column of  $\sigma$  is a permutation of  $1, \dots, n$ .*

*We assume that all magic squares are normalized, in the sense that the first row is  $(1, \dots, n)$ , and the diagonal is  $(1, \dots, 1)$ .*

In this definition, we use the fact that it is always possible to normalize a magic square: a permutation of the columns makes the first row  $(1, \dots, n)$ , then a permutation of the rows makes the diagonal  $(1, \dots, 1)$ .

As a first example, we have the  $n \times n$  circulant matrix  $\sigma(i, j) = j - i$ , with  $j - i \in \{1, \dots, n\}$  taken modulo  $n$ . Here is another example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

The magic squares produce magic unitaries, in the following way:

**Proposition 4.1.** *If  $E = (E_1, \dots, E_n)$  is a partition of unity with rank one projections and  $\sigma = \sigma(i, j)$  is a magic square, then  $(E_\sigma)_{ij} = E_{\sigma(i, j)}$  is a magic unitary.*

In other words,  $E_\sigma$  is obtained by putting  $E$  superscripts to elements of  $\sigma$ . For instance the above  $4 \times 4$  matrix gives:

$$E_\sigma = \begin{pmatrix} E_1 & E_2 & E_3 & E_4 \\ E_2 & E_1 & E_4 & E_3 \\ E_3 & E_4 & E_1 & E_2 \\ E_4 & E_3 & E_2 & E_1 \end{pmatrix}$$

In the above statement, the condition that  $\sigma$  is normalized is not necessary; nor the fact that the projections  $E_i$  are of rank one. We make these assumptions for some technical reasons, to become clear later on.

We denote by  $\sigma_1, \dots, \sigma_n$  the rows of a magic square  $\sigma$ , and we regard them as permutations of  $\{1, \dots, n\}$ . For instance, for the above  $4 \times 4$  matrix we get:

$$\begin{aligned} \sigma_1 &= (1 \ 2 \ 3 \ 4) \\ \sigma_2 &= (2 \ 1 \ 4 \ 3) \\ \sigma_3 &= (3 \ 4 \ 1 \ 2) \\ \sigma_4 &= (4 \ 3 \ 2 \ 1) \end{aligned}$$

With these notations, we have the following result about commutativity, which deals with a slightly more general situation than that of Hadamard matrices:

**Theorem 4.1.** *For a representation  $\pi : A(S_n) \rightarrow M_n(\mathbb{C})$  having the property that  $P_{ij} = \pi(x_{ij})$  are rank one projections, the following are equivalent:*

- (1)  $A_\pi$  is commutative.
- (2)  $A_\pi = C(G)$ , for a certain subgroup  $G \subset S_n$ .
- (3)  $P = E_\sigma$ , for a certain partition of unity  $E$  and magic matrix  $\sigma$ .

Moreover, in this situation  $G$  is the group generated by the rows of  $\sigma$ .

*Proof.* Assume that  $A = A_\pi$  is commutative. The algebra  $\text{Im}(\pi)$  being a quotient of  $A$ , it is commutative as well. Thus the projections  $P_{ij}$  mutually commute.

On the other hand, two rank one projections commute if and only if their images are equal, or orthogonal. Together with the magic unitarity of  $P$ , this shows that each row of  $P$  is a permutation of the first row.

So, consider the first row of  $P$ , regarded as a partition of unity:

$$E = (E_1, \dots, E_n)$$

By the above remark, the  $i$ -th row of  $P$  must be of the following form:

$$E_{\sigma_i} = (E_{\sigma_i(1)}, \dots, E_{\sigma_i(n)})$$

Here  $\sigma_i \in S_n$  is a certain permutation. Now the formula  $\sigma(i, j) = \sigma_i(j)$  defines a matrix  $\sigma$ , which is magic. With the above  $E$  and this matrix  $\sigma$ , we have  $P = E_\sigma$ .

In other words, we have (1)  $\Rightarrow$  (3). Also, the last assertion implies (2), which in turn implies (1). So, it remains to prove that (3) implies the last assertion.

Consider the following diagram, where  $X = \{\sigma_1, \dots, \sigma_n\}$  is the subset of  $S_n$  formed by rows of  $\sigma$ :

$$\begin{array}{ccc} A(S_n) & \rightarrow & M_n(\mathbb{C}) \\ \downarrow & & \uparrow \\ C(S_n) & \longrightarrow & C(X) \end{array}$$

We define the connecting maps in the following way:

- (1) The map  $A(S_n) \rightarrow M_n(\mathbb{C})$  is the representation  $\pi$ .
- (2) The map  $A(S_n) \rightarrow C(S_n)$  is the canonical one, given by  $x_{ij} \rightarrow \chi_{ij}$ .
- (3) The map  $C(S_n) \rightarrow C(X)$  is the transpose of  $X \subset S_n$ .
- (4) The map  $C(X) \rightarrow M_n(\mathbb{C})$  is given by  $\delta_{\sigma_k} \rightarrow E_k$ .

We compute the image of  $\chi_{ij}$  by the third map:

$$\begin{aligned} \text{Im}(\chi_{ij}) &= \text{Im}(\chi\{\sigma \mid \sigma(j) = i\}) \\ &= \chi\{\sigma_k \mid \sigma_k(j) = i\} \\ &= \delta_{\sigma_{\sigma(i,j)}} \end{aligned}$$

Thus at level of generators, we have the following diagram:

$$\begin{array}{ccc} x_{ij} & \rightarrow & E_{\sigma(i,j)} \\ \downarrow & & \uparrow \\ \chi_{ij} & \longrightarrow & \delta_{\sigma_{\sigma(i,j)}} \end{array}$$

This diagram commutes, so the above diagram of algebras commutes as well. Now since  $C(S_n)$  is a Hopf algebra, the Hopf algebra  $A_\pi$  we are looking for must be a quotient of it. In other words, we must have  $A = C(G)$ , where  $G$  is a subgroup of  $S_n$ . On the other hand, from minimality of  $A_\pi$  we get that this algebra must be the minimal one containing  $C(X)$ . Thus  $G$  is the group generated by  $X$ .  $\square$

As an illustration, we get new proofs for Theorem 2.1 and Corollary 3.1:

- (1) For the Fourier matrix we have  $P = E_\sigma$ , where  $\sigma$  is the magic square given by  $\sigma(i, j) = j - i$ , and  $E$  is the first row of  $P$ . We have  $\sigma_i = c^i$ , where  $c$  is the cycle  $c(i) = i - 1$ , so we get the group  $\{1, c, \dots, c^{n-1}\} = \mathbb{Z}_n$ .
- (2) For the matrix  $M_q$  with  $q = \pm 1$  we have  $P = E_\sigma$ , where  $\sigma$  is the  $4 \times 4$  matrix in the beginning of this section, and  $E$  is the first row of  $P$ . We get the group formed by rows of  $\sigma$ , namely  $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ .

In the general case, the construction  $\sigma \rightarrow \pi \rightarrow A_\pi$  is not understood. Some comments in this sense are presented in the end of next section.

## 5. THE COMMUTATIVE CASE

We have now all ingredients for answering the question raised in the beginning of previous section. The result is as follows:

**Theorem 5.1.** *For an Hadamard matrix  $h$ , the following are equivalent:*

- (1)  $A$  is commutative.
- (2)  $h$  is a tensor product of Fourier matrices.

*Proof.* The statement is of course up to equivalence of Hadamard matrices. We use this fact several times in the proof, without special mention: that is, we agree to allow suitable permutations of rows and columns of  $h$ , as well as multiplication of rows and columns by scalars of modulus one.

(2)  $\Rightarrow$  (1): this follows from Theorem 2.1 and Theorem 3.1.

(1)  $\Rightarrow$  (2): assume that  $A$  is commutative. Theorem 4.1 tells us that we have  $P = E_\sigma$ , for a certain magic square  $\sigma$ . Here  $E$  is a certain partition of unity, but since  $\sigma$  is normalized,  $E$  must be the first row of  $P$ .

We have  $P_{ij} = P(h_j/h_i)$ , so the condition  $P = E_\sigma$  becomes:

$$P \left( \frac{h_j}{h_i} \right) = P \left( \frac{h_{\sigma(i,j)}}{h_1} \right)$$

By using Proposition 2.2. we can assume  $h_1 = 1$ . This gives:

$$P \left( \frac{h_j}{h_i} \right) = P (h_{\sigma(i,j)})$$

Now two rank one projections being equal if and only if the corresponding vectors are proportional, we must have complex scalars  $\lambda_{ij}$  such that:

$$\frac{h_j}{h_i} = \lambda_{ij} h_{\sigma(i,j)}$$

Now remember that the row vectors  $h_i$  have entries of modulus one, so they are elements of the group  $\mathbb{T}^n$ , where  $\mathbb{T}$  is the unit circle. We denote by  $H_i \in \mathbb{T}^n/\mathbb{T}$  their images modulo constant vectors in  $\mathbb{T}^n$ . The above condition becomes:

$$\frac{H_j}{H_i} = H_{\sigma(i,j)}$$

This relation shows that  $X = \{H_1, \dots, H_n\}$  is stable by quotients. Since  $X$  contains as well the neutral element  $H_1$ , it must be a subgroup of  $\mathbb{T}^n/\mathbb{T}$ .

Now since  $X$  is an abelian group, it must be a product of cyclic groups, and we can proceed as follows:

(1) Assume first that we have  $X \simeq \mathbb{Z}_n$ . We can permute rows of  $h$ , as to have  $H_i = H^{i-1}$  for any  $i$ , for a certain element  $H \in \mathbb{T}^n/\mathbb{T}$ . This gives  $h_i = \lambda_i \rho^{i-1}$  for any  $i$ , for a certain element  $\rho \in \mathbb{T}^n$ , and certain scalars  $\lambda_i$ . Now since the vectors  $h_i$  are orthogonal, their multiples  $\rho^{i-1}$  must be orthogonal as well.

Consider now the vector  $\rho$ . Its image in  $\mathbb{T}^n/\mathbb{T}$  has order  $n$ , so by multiplying  $h$  by a suitable scalar, we may assume that we have  $\rho^n = 1$ . In other words, we have  $\rho = (w_i)$ , where each  $w_i$  is a  $n$ -th root of unity.

Now from the relation  $\rho^{i-1} \perp \rho^{j-1}$  for  $1 \leq i < j \leq n$  we get that the sum of  $k$ -th powers of the numbers  $w_i$  vanishes, for  $1 \leq k < n$ . This shows that the product of degree one polynomials  $X - w_i$  is the polynomial  $X^n - 1$ , so the set of numbers  $w_i$

is the set of  $n$ -th roots of unity. Thus by permuting rows of  $h$  we can assume that we have  $w_i = w^i$ , with  $w = e^{2\pi i/n}$ .

Now the relation  $h_i = \lambda_i \rho^{i-1}$  tells us that  $h$  is obtained from the Fourier matrix by multiplying rows by scalars. Thus  $h$  is equivalent to the Fourier matrix.

(2) Assume now that we have  $X \simeq Y \times Z$ , for certain groups  $Y, Z$ . By replacing  $n$  by  $nm$  and single indices by double indices, we can assume that we have  $H_{ia} = K_i L_a$ , where  $Y = \{K_1, \dots, K_n\}$  and  $Z = \{L_1, \dots, L_m\}$  are subgroups of  $\mathbb{T}^{nm}/\mathbb{T}$ .

We take now arbitrary lifts  $k_1, \dots, k_n$  and  $l_1, \dots, l_m$  for the elements of  $Y, Z$ . The formula  $H_{ia} = K_i L_a$  becomes  $h_{ia} = \lambda_{ia} k_i l_a$ , for certain scalars  $\lambda_{ia}$ . Now these vectors being orthogonal, by keeping  $a$  fixed we get that the vectors  $k_i$  are orthogonal, so  $k$  is an Hadamard matrix. Also, with  $i$  fixed we get that  $l$  is an Hadamard matrix.

On the other hand, we can get rid of the scalars  $\lambda_{ij}$  by multiplying rows of  $h$  by their inverses. Thus we have  $h_{ia} = k_i l_a$ , where  $k, l$  are Hadamard matrices.

With notations from previous section, this gives  $h = k \otimes l$ .

(3) We can conclude now by induction. Assume that the statement is proved for  $n < N$ , and let  $h$  be a Hadamard matrix of order  $N$ . In the case  $X \simeq \mathbb{Z}_N$  we can apply (1) and we are done. If not, we have  $X \simeq Y \times Z$  as in (2), so we get  $h = k \otimes l$ . Now Theorem 3.1 gives  $A(h) = A(k) \otimes A(l)$ , so both  $A(k), A(l)$  are commutative. Thus we can apply the induction assumption, and we are done.  $\square$

The permutation groups coming from arbitrary magic squares can be non-abelian. Consider for instance the following matrix:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \\ 4 & 5 & 1 & 3 & 2 \\ 2 & 4 & 5 & 1 & 3 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

The rows of this matrix generate the group  $S_5$ , which cannot be obtained by using  $5 \times 5$  Hadamard matrices, because of [H]. In general, we don't know what are the groups which can appear from the magic square construction.

The other question is about what happens to Theorem 4.1 when the rank one assumption is removed. Once again, we don't know the answer.

## 6. SMALL MATRICES

We recall from [H] that for  $n = 1, 2, 3, 5$  the only Hadamard matrix is the Fourier one, and that at  $n = 4$  we have only the matrices  $M_q$ , with  $|q| = 1$ :

$$M_q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & q & -1 & -q \\ 1 & -1 & 1 & -1 \\ 1 & -q & -1 & q \end{pmatrix}$$

By results in previous sections, for parameters  $q$  satisfying  $q^4 = 1$ , this matrix  $M_q$  produces a commutative algebra:



- (1) For  $q = \pm 1$  we get  $C(G)$ , with  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ .
- (2) For  $q = \pm i$  we get  $C(G)$ , with  $G = \mathbb{Z}_4$ .

Now remember that for a finite abelian group  $G$ , the algebra of complex functions  $C(G)$  is canonically isomorphic to the group algebra  $C^*(G)$ . The isomorphism is given by the discrete Fourier transform in the case  $G = \mathbb{Z}_n$ , and by a product of such transforms, or just by Pontrjagin duality, in the general case.

We can therefore reformulate the above result in the following way:

- (1) For  $q = \pm 1$  we get  $C^*(G)$ , with  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ .
- (2) For  $q = \pm i$  we get  $C^*(G)$ , with  $G = \mathbb{Z}_4$ .

In order to be generalized, this result has to be reformulated one more time. We have the following equivalent statement, in terms of crossed products:

- (1) For  $q = \pm 1$  we get  $C^*(G)$ , with  $G = \mathbb{Z}_2 \rtimes \mathbb{Z}_2$ .
- (2) For  $q = \pm i$  we get  $C^*(G)$ , with  $G = \mathbb{Z}_1 \rtimes \mathbb{Z}_4$ .

Recall now that for a discrete group  $G$ , meaning a possibly infinite group, without topology on it, the group algebra  $C^*(G)$  is a  $C^*$ -algebra, obtained from the usual group algebra  $\mathbb{C}[G]$  by using a standard completion procedure.

With these preliminaries and notations, we have the following result:

**Theorem 6.1.** *Let  $n$  be the order of  $q^2$ , and for  $n < \infty$  write  $n = 2^s m$ , with  $m$  odd. The matrix  $M_q$  produces the algebra  $C^*(G)$ , where  $G$  is as follows:*

- (1) For  $s = 0$  we have  $G = \mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$ .
- (2) For  $s = 1$  we have  $G = \mathbb{Z}_{n/2} \rtimes \mathbb{Z}_4$ .
- (3) For  $s \geq 2$  we have  $G = \mathbb{Z}_n \rtimes \mathbb{Z}_4$ .
- (4) For  $n = \infty$  we have  $G = \mathbb{Z} \rtimes \mathbb{Z}_2$ .

*Proof.* Consider the second and the third row of  $M_q$ :

$$\begin{aligned} \rho &= (1 \quad q \quad -1 \quad -q) \\ \varepsilon &= (1 \quad -1 \quad 1 \quad -1) \end{aligned}$$

We regard these vectors as elements of the algebra  $\mathbb{C}^4$ . The product of these vectors being the last row of  $M_q$ , we have the following formula:

$$M_q = \begin{pmatrix} 1 \\ \rho \\ \varepsilon \\ \varepsilon\rho \end{pmatrix}$$

By taking quotients of these vectors we get the corresponding magic basis:

$$\xi = \begin{pmatrix} 1 & \rho & \varepsilon & \varepsilon\rho \\ \rho^{-1} & 1 & \varepsilon\rho^{-1} & \varepsilon \\ \varepsilon & \varepsilon\rho & 1 & \rho \\ \varepsilon\rho^{-1} & \varepsilon & \rho^{-1} & 1 \end{pmatrix}$$

Each entry of this matrix is of the form  $\rho^k$  or  $\varepsilon\rho^k$ , with  $k \in \mathbb{Z}$ . So, consider the vectors  $\rho_+^k = \rho^k$  and  $\rho_-^k = \varepsilon\rho^k$ . These are given by the following global formula:

$$\rho_{\pm}^k = (1 \quad \pm q^k \quad (-1)^k \quad \pm(-q)^k)$$

Now the orthogonal projection onto a vector  $(x_i)$  being the matrix  $(\bar{x}_i x_j)$ , we have the following formula for the corresponding projection:

$$P(\rho_{\pm}^k) = \begin{pmatrix} 1 & \pm q^{-k} & (-1)^k & \pm(-q)^{-k} \\ \pm q^k & 1 & \pm(-q)^k & (-1)^k \\ (-1)^k & \pm(-q)^{-k} & 1 & \pm q^{-k} \\ \pm(-q)^k & (-1)^k & \pm q^k & 1 \end{pmatrix}$$

The idea is to express this matrix as a suitable linear combination of elements of  $U(4)$ . We write  $n = 2^s m$  as in the statement, and we proceed as follows:

Case  $s = 0$ . Here  $n$  is odd, and we have  $q^{2n} = 1$ , so  $q^n = \pm 1$ . We can replace if necessary  $q$  by  $-q$ , as to have  $q^n = 1$ .

Consider the canonical subgroup  $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset U(4)$ , consisting of the identity matrix 1, and of the following three matrices:

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Consider also the following matrix, which is in  $U(4)$  as well:

$$\sigma = \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^{-1} \\ q & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \end{pmatrix}$$

We have the following formulae, valid for  $k$  odd:

$$\sigma^k = \begin{pmatrix} 0 & 0 & q^k & 0 \\ 0 & 0 & 0 & q^{-k} \\ q^k & 0 & 0 & 0 \\ 0 & q^{-k} & 0 & 0 \end{pmatrix} \quad \alpha\sigma^k = \begin{pmatrix} 0 & 0 & 0 & q^{-k} \\ 0 & 0 & q^k & 0 \\ 0 & q^{-k} & 0 & 0 \\ q^k & 0 & 0 & 0 \end{pmatrix}$$

We also have the following formulae, valid for  $k$  even:

$$\sigma^k = \begin{pmatrix} q^k & 0 & 0 & 0 \\ 0 & q^{-k} & 0 & 0 \\ 0 & 0 & q^k & 0 \\ 0 & 0 & 0 & q^{-k} \end{pmatrix} \quad \alpha\sigma^k = \begin{pmatrix} 0 & q^{-k} & 0 & 0 \\ q^k & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-k} \\ 0 & 0 & q^k & 0 \end{pmatrix}$$

In particular we have  $\sigma^n = \beta$  and  $\sigma^{2n} = 1$ .

For  $k$  odd the number  $n + k$  is even, and we have:

$$\begin{aligned} P(\rho_{\pm}^k) &= 1 \pm \alpha\sigma^{n+k} - \sigma^n \mp \alpha\sigma^k \\ &= (1 - \sigma^n)(1 \mp \alpha\sigma^k) \end{aligned}$$

For  $k$  even the number  $n + k$  is odd, and we have:

$$\begin{aligned} P(\rho_{\pm}^k) &= 1 \pm \alpha\sigma^k + \sigma^n \mp \alpha\sigma^{n+k} \\ &= (1 + \sigma^n)(1 \pm \alpha\sigma^k) \end{aligned}$$

This gives the following global formula:

$$P(\rho_{\pm}^k) = (1 + (-1)^k \sigma^n)(1 \pm (-1)^k \alpha\sigma^k)$$

It is routine to check that  $\alpha, \sigma \in U(4)$  generate the dihedral group  $G = \mathbb{Z}_{2n} \rtimes \mathbb{Z}_2$ . Consider now the following element of the abstract group algebra  $C^*(G)$ :

$$P_{\pm}^k = (1 + (-1)^k \sigma^n)(1 \pm (-1)^k \alpha\sigma^k)$$

From  $\sigma^{2n} = 1$  we get that  $\sigma^n$  is self-adjoint, and of square 1. From  $\alpha^2 = 1$  and  $(\alpha\sigma)^2 = 1$  we get  $\sigma\alpha\sigma = \alpha$ , hence  $\sigma^k\alpha\sigma^k = \alpha$  for any  $k$ . It follows that  $\alpha\sigma^k$  is also self-adjoint, and of square 1. This shows that in the above formula, both elements in brackets are projections. Moreover, these two projections commute. It follows that each element  $P_{\pm}^k$  is a projection.

Now remember that the magic unitary associated to  $\pi$  has as entries elements of the form  $P(\rho_{\pm}^k)$ . By making the replacement  $P(\rho_{\pm}^k) \rightarrow P_{\pm}^k$  we get a square matrix over  $C^*(G)$ , all whose entries are projections. The sums on rows and columns being 1, this is a magic unitary, and we get a factorization of  $\pi$  through  $C^*(G)$ .

Case  $n = \infty$ . What changes in the above proof is that  $\alpha, \sigma \in U(4)$  generate now the infinite dihedral group  $G = \mathbb{Z} \rtimes \mathbb{Z}_2$ . The second part of the proof applies as well, with this modification, and gives the result.

Case  $s = 1$ . Here  $n/2$  is odd. We have  $q^n = -1$ , so  $q^{n/2} = \pm i$ , which gives  $(-iq)^{n/2} = \pm 1$ . We replace if necessary  $q$  by  $-q$ , as to have  $(-iq)^{n/2} = 1$ .

Consider the canonical subgroup  $\mathbb{Z}_4 \subset U(4)$ , consisting of the identity matrix 1, and of the following three matrices:

$$\delta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \delta^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \delta^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Consider also the following matrix, which is in  $U(4)$  as well:

$$\tau = \begin{pmatrix} -q & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & -q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

Since  $n/2$  is odd and  $(-iq)^{n/2} = 1$ , we have  $(i\tau)^{n/2} = 1$ .

We have the following formulae:

$$\delta\tau^k = \begin{pmatrix} 0 & q^{-k} & 0 & 0 \\ 0 & 0 & (-q)^k & 0 \\ 0 & 0 & 0 & q^{-k} \\ (-q)^k & 0 & 0 & 0 \end{pmatrix}$$

$$\delta^3 \tau^k = \begin{pmatrix} 0 & 0 & 0 & q^{-k} \\ (-q)^k & 0 & 0 & 0 \\ 0 & q^{-k} & 0 & 0 \\ 0 & 0 & (-q)^k & 0 \end{pmatrix}$$

This gives the following formula for the above projection:

$$\begin{aligned} P(\rho_{\pm}^k) &= 1 \pm \delta \tau^k + (-1)^k \delta^2 \pm (-1)^k \delta^3 \tau^k \\ &= (1 + (-1)^k \delta^2)(1 \pm \delta \tau^k) \end{aligned}$$

Consider now the matrix  $\nu = i\tau$ . We have  $\nu^{n/2} = 1$ , and it is routine to check that  $\delta, \nu \in U(4)$  generate the group  $G = \mathbb{Z}_{n/2} \rtimes \mathbb{Z}_4$ . Consider now the following element of the abstract group algebra  $C^*(G)$ , where  $\tau = -i\nu$ :

$$P_{\pm}^k = (1 + (-1)^k \delta^2)(1 \pm \delta \tau^k)$$

From  $\delta^4 = 1$  and from  $\tau \delta \tau = -\delta$  we get that each  $P_{\pm}^k$  is a projection.

Now remember that the magic unitary associated to  $\pi$  has as entries elements of the form  $P(\rho_{\pm}^k)$ . By making the replacement  $P(\rho_{\pm}^k) \rightarrow P_{\pm}^k$  we get a square matrix over  $C^*(G)$ , all whose entries are projections. The sums on rows and columns being 1, this is a magic unitary, and we get a factorization of  $\pi$  through  $C^*(G)$ .

Case  $s \geq 2$ . Here  $n/2$  is even, and we have  $q^n = -1$ , so  $q^{n/2} = \pm i$ .

We use the matrices  $\delta, \tau$ . The above formulae for  $\delta \tau^k$  and  $\delta^3 \tau^k$  hold again, and lead to the above formula for  $P(\rho_{\pm}^k)$ . What changes is the  $n/2$ -th power of  $\tau$ :

$$\tau^{n/2} = \begin{pmatrix} \pm i & 0 & 0 & 0 \\ 0 & \mp i & 0 & 0 \\ 0 & 0 & \pm i & 0 \\ 0 & 0 & 0 & \mp i \end{pmatrix}$$

Observe also that we have  $\tau^n = -1$ , so  $(w\tau)^n = 1$ , where  $w = e^{\pi i/n}$ .

This shows that we don't have a factorization like in the case where  $n/2$  is odd, so we must use the group generated by  $\delta, w\tau \in U(4)$ , which is  $G = \mathbb{Z}_n \rtimes \mathbb{Z}_4$ .  $\square$

The above result has several interpretations. First, it provides examples of subgroups of the 4-th quantum permutation group, also called Pauli quantum group [BC2]. In other words, the whole thing should be regarded as being part of a general McKay correspondence for the Pauli quantum group, not available yet.

Another interpretation is in terms of subfactors: the above result can be regarded as being a particular case of the *ADE* classification of index 4 subfactors. Unlike in the quantum group case, the classification is available here in full generality. See [EK]. The principal graphs corresponding to the matrices  $M_q$  are those of type  $A_4^{(1)}$ ,  $D_n^{(1)}$ ,  $D_{\infty}$ . This follows for instance by combining Theorem 6.1 with general results in [Ba2], and with the well-known fact that cyclic groups correspond to type *A* graphs, and dihedral groups correspond to type *D* graphs.

As a last remark, the above result provides the first example of a deformation situation for quantum permutation groups. This is of course just an example, but the general fact that it suggests would be use of the unit circle as parameter space,

for more general deformation situations. We should mention that this idea, while being fundamental in most theories emerging from Drinfeld's original work [D], is quite new in the compact quantum group area, where the deformation parameter traditionally belongs to the real line. It is our hope that further developments of the subject will be of use in connection with several problems, regarding both Hadamard matrices and compact quantum groups.

## REFERENCES

- [Ba1] T. Banica, Hopf algebras and subfactors associated to vertex models, *J. Funct. Anal.* **159** (1998), 243–266.
- [Ba2] T. Banica, Compact Kac algebras and commuting squares, *J. Funct. Anal.* **176** (2000), 80–99.
- [Ba3] T. Banica, Quantum automorphism groups of homogeneous graphs, *J. Funct. Anal.* **224** (2005), 243–280.
- [BB] T. Banica and J. Bichon, Free product formulae for quantum permutation groups, *J. Math. Inst. Jussieu*, to appear.
- [BBC] T. Banica, J. Bichon and G. Chenevier, Graphs having no quantum symmetry, *Ann. Inst. Fourier*, to appear.
- [BC1] T. Banica and B. Collins, Integration over quantum permutation groups, *J. Funct. Anal.*, to appear.
- [BC2] T. Banica and B. Collins, Integration over the Pauli quantum group, [math.QA/0610041](#).
- [BN] K. Beauchamp and R. Nicoara, Orthogonal maximal abelian  $*$ -subalgebras of the  $6 \times 6$  matrices, [math.OA/0609076](#).
- [Bj] G. Björck, Functions of modulus 1 on  $Z_n$  whose Fourier transforms have constant modulus, and cyclic  $n$ -roots, *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* **315** (1990), 131–140.
- [Br] L. Brown, Ext of certain free product  $C^*$ -algebras, *J. Operator Theory* **6** (1981), 135–141.
- [Bu] A.T. Butson, Generalized Hadamard matrices, *Proc. Amer. Math. Soc.* **13** (1962), 894–898.
- [D] V. Drinfeld, Quantum groups, Proc. ICM Berkeley (1986), 798–820.
- [EK] D. Evans and Y. Kawahigashi, Quantum symmetries on operator algebras, Oxford University Press (1998).
- [H] U. Haagerup, Orthogonal maximal abelian  $*$ -subalgebras of the  $n \times n$  matrices, in “Operator algebras and quantum field theory”, International Press (1997), 296–323.
- [J] V.F.R. Jones, Planar algebras I, [math.QA/9909027](#).
- [M] K. McClanahan,  $C^*$ -algebras generated by elements of a unitary matrix, *J. Funct. Anal.* **107** (1992), 439–457.
- [N] R. Nicoara, A finiteness result for commuting squares of matrix algebras, *J. Operator Theory* **55** (2006), 295–310.
- [Pe] M. Petrescu, Existence of continuous families of complex Hadamard matrices of certain prime dimensions and related results, Ph.D. Thesis, UCLA (1997).
- [Po] S. Popa, Orthogonal pairs of  $*$ -subalgebras in finite von Neumann algebras, *J. Operator Theory* **9** (1983), 253–268.
- [S] J.J. Sylvester, Thoughts on inverse orthogonal matrices, simultaneous sign-successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers, *Phil. Mag.* **34** (1867), 461–475.
- [TZ] W. Tadej and K. Życzkowski, A concise guide to complex Hadamard matrices, *Open Syst. Inf. Dyn.* **13** (2006), 133–177.
- [T] T. Tao, Fuglede's conjecture is false in 5 and higher dimensions, *Math. Res. Lett.* **11** (2004), 251–258.
- [Wa1] S. Wang, Free products of compact quantum groups, *Comm. Math. Phys.* **167** (1995), 671–692.

- [Wa2] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.
- [Wo1] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.
- [Wo2] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted  $SU(N)$  groups, *Invent. Math.* **93** (1988), 35–76.

T.B.: DEPARTMENT OF MATHEMATICS, PAUL SABATIER UNIVERSITY, 118 ROUTE DE NARBONNE, 31062 TOULOUSE, FRANCE  
*E-mail address:* `banica@picard.ups-tlse.fr`

R.N.: DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, 1326 STEVENSON CENTER, NASHVILLE, TN 37240, USA  
*E-mail address:* `remus.nicoara@vanderbilt.edu`