TRUNCATION AND DUALITY RESULTS FOR HOPF IMAGE ALGEBRAS

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Abstract. Associated to an Hadamard matrix $H \in M_N(\mathbb{C})$ is the spectral measure $\mu \in \mathcal{P}[0, N]$ of the corresponding Hopf image algebra, $A = C(G)$ with $G \subset S_N^+$. We study here a certain family of discrete measures $\mu^r \in \mathcal{P}[0, N]$, coming from the idempotent state theory of G, which converge in Cesàro limit to μ. Our main result is a duality formula of type $\int_0^N (x/N)^p d\mu^r(x) = \int_0^N (x/N)^r d\nu^p(x)$, where μ^r, ν^r are the truncations of the spectral measures μ, ν associated to H, H^t. We prove as well, using these truncations μ^r, ν^r, that for any deformed Fourier matrix $H = F_M \otimes_Q F_N$ we have $\mu = \nu$.

Introduction

A complex Hadamard matrix is a square matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal. The basic example of such a matrix is the Fourier one, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$:

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{N-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & w^{N-1} & w^{2(N-1)} & \ldots & w^{(N-1)^2} \end{pmatrix}$$

In general, the theory of complex Hadamard matrices can be regarded as a "non-standard" branch of discrete Fourier analysis. For a number of potential applications to quantum physics and quantum information theory questions, see [4], [8], [10].

Each Hadamard matrix $H \in M_N(\mathbb{C})$ is known to produce a subfactor $M \subset R$ of the Murray-von Neumann hyperfinite factor R, having index $[R : M] = N$. The associated planar algebra $P = (P_k)$ has a direct description in terms of H, worked out in [7], and a key problem is that of computing the corresponding Poincaré series, given by:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

An alternative approach to this question is via quantum groups [11], [12]. The idea is that associated to $H \in M_N(\mathbb{C})$ is a quantum subgroup $G \subset S_N^+$ of Wang's quantum

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permutation group [9], constructed by using to the Hopf image method, developed in [2]. More precisely, $G \subset S_N^+$ appears via a factorization diagram, as follows:

$$\begin{tikzcd} C(S_N^+) \arrow[rr, "\pi"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G) \arrow[ur, "\rho"'] & \end{tikzcd}$$

Here the upper arrow is defined by $\pi : u_{ij} \to P_{ij} = Proj(H_i/H_j)$, where u_{ij} are the standard generators of $C(S_N^+)$, and where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H. The lower left arrow is by definition transpose to the embedding $G \subset S_N^+$, and the quantum group $G \subset S_N^+$ itself is by definition the minimal one producing such a factorization.

With this notion in hand, the problem is that of computing the spectral measure μ of the main character $\chi : G \to \mathbb{C}$. This is indeed the same problem as above, because by Woronowicz's Tannakian duality [12], f is the Stieltjes transform of μ:

$$f(z) = \int_G \frac{1}{1 - z\chi}$$

Here, and in what follows, we use the integration theory developed in [11].

As a basic example, for a Fourier matrix F_N the associated quantum group $G \subset S_N^+$ is the cyclic group $\mathbb{Z}_N$, and we therefore have $\mu = (1 - \frac{1}{N})\delta_0 + \frac{1}{N}\delta_N$ in this case. In general, however, the computation of μ is a quite difficult question. See [3].

In this paper we discuss a certain truncation procedure for the main spectral measure, coming from the idempotent state theory of the associated quantum group [3], [6]. Consider the following functionals, where $*$ is the convolution, $\psi * \phi = (\psi \otimes \phi)\Delta$:

$$\int_G^r = (tr \circ \rho)^{*r}$$

The point with these functionals is that, as explained in [3], we have the following Cesàro limiting result, coming from the general results of Woronowicz in [11]:

$$\int_G \varphi = \lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^k \int_G^r \varphi$$

This formula can be of course used in order to estimate or exactly compute the various integrals over G, and doing so will be the main idea in the present paper.

At the level of the main character, we have the following result:

Theorem A. *The law χ with respect to $\int_G^r$ equals the law of the Gram matrix*

$$X_{i_1\ldots i_r, j_1\ldots j_r} = < \xi_{i_1\ldots i_r}, \xi_{j_1\ldots j_r} >$$

of the norm one vectors $\xi_{i_1\ldots i_r} = \frac{1}{\sqrt{N}} \cdot \frac{H_{i_1}}{H_{i_2}} \otimes \ldots \otimes \frac{1}{\sqrt{N}} \cdot \frac{H_{i_r}}{H_{i_1}}$.

Here the law of X is by definition its spectral measure, with respect to the trace.

Observe that with $r \to \infty$, via the above-mentioned Cesàro limiting procedure, we obtain from the laws in Theorem A the spectral measure μ that we are interested in.

Our second, and main theoretical result, is as follows:

Theorem B. *We have the moment/truncation duality formula*

$$\int_{G_H}^r \left(\frac{\chi}{N}\right)^p = \int_{G_{H^t}}^p \left(\frac{\chi}{N}\right)^r$$

where G_H, G_{H^t} are the quantum groups associated to H, H^t.

This formula, which is quite non-trivial, is probably quite interesting, in connection with the duality between the quantum groups $G_H, G_{\overline{H}}, G_{H^t}, G_{H^*}$ studied in [1].

As an illustration for the above methods, we will work out the case of the deformed Fourier matrices, $H = F_N \otimes_Q F_M$, with the following result:

Theorem C. *For $H = F_N \otimes_Q F_M$ we have the self-duality formula*

$$\int_{G_H} \varphi(\chi) = \int_{G_{H^t}} \varphi(\chi)$$

valid for any parameter matrix $Q \in M_{M\times N}(\mathbb{T})$.

The paper is organized as follows: 1-2 are preliminary sections, and in 3-4-5 we present the truncation procedure, and we prove Theorems A-B-C above.

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1. Hadamard matrices

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, and whose rows are pairwise orthogonal. The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. More generally, we have as example the Fourier matrix $F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$ of any finite abelian group $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$. See [8].

The complex Hadamard matrices are usually regarded modulo equivalence:

Definition 1.1. *Two complex Hadamard matrices $H, K \in M_N(\mathbb{C})$ are called equivalent, and we write $H \sim K$, if one can pass from one to the other by permuting the rows and columns, or by multiplying these rows and columns by numbers in $\mathbb{T}$.*

As explained in the introduction, each complex Hadamard matrix produces a subfactor $M \subset R$ of the Murray-von Neumann hyperfinite factor R, having index $[R : M] = N$, which can be understood in terms of quantum groups. Indeed, let us call "magic" any square matrix $u = (u_{ij})$ whose entries are projections ($p = p^2 = p^*$), summing up to 1 on each row and column. We have then the following key definition, due to Wang [9]:

Definition 1.2. *$C(S_N^+)$ is the universal C^*-algebra generated by the entries of a $N\times N$ magic matrix $u=(u_{ij})$, with comultiplication, counit and antipode maps defined on the standard generators by $\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$, $\varepsilon(u_{ij})=\delta_{ij}$ and $S(u_{ij})=u_{ji}$.*

As explained in [9], this algebra satisfies Woronowicz's axioms in [11], and so S_N^+ is a compact quantum group, called quantum permutation group. Since the functions $v_{ij}:S_N\to\mathbb C$ given by $v_{ij}(\sigma)=\delta_{i\sigma(j)}$ form a magic matrix, we have a quotient map $C(S_N^+)\to C(S_N)$, which corresponds to an embedding $S_N\subset S_N^+$. This embedding is an isomorphism at $N=1,2,3$, but not at $N\geq 4$, where S_N^+ is not finite. See [9].

The link with the Hadamard matrices comes from:

Definition 1.3. *Associated to an Hadamard matrix $H\in M_N(\mathbb T)$ is the minimal quantum group $G\subset S_N^+$ producing a factorization of type*

$$\begin{tikzcd} C(S_N^+) \arrow[rr, "\pi"] \arrow[dr] & & M_N(\mathbb C) \\ & C(G) \arrow[ur, "\rho"'] & \end{tikzcd}$$

where $\pi:u_{ij}\to P_{ij}=Proj(H_i/H_j)$, where $H_1,\ldots,H_N\in\mathbb T^N$ are the rows of H.

Here the fact that π is indeed well-defined follows from the fact that $P=(P_{ij})$ is magic, which comes from the fact that the rows of H are pairwise orthogonal. As for the existence and uniqueness of the quantum group $G\subset S_N^+$ as in the statement, this comes from Hopf algebra theory, by dividing $C(S_N^+)$ by a suitable ideal. See [2].

At the level of examples, it is known that the Fourier matrix F_G produces the group G itself. In general, the computation of G is a quite difficult question. See [3].

At a theoretical level, it is known that the above-mentioned subfactor $M\subset R$ associated to H appears as a fixed point subfactor associated to G. See [1].

In what follows we will rather use a representation-theoretic formulation of this latter result. Let $u=(u_{ij})$ be the fundamental representation of G.

Definition 1.4. *We let $\mu\in\mathcal P[0,N]$ be the law of variable $\chi=\sum_i u_{ii}$, with respect to the Haar integration functional of $C(G)$.*

Note that the main character $\chi=\sum_i u_{ii}$ being a sum of N projections, we have the operator-theoretic formula $0\leq\chi\leq N$, and so $supp(\mu)\subset[0,N]$, as stated above.

Observe also that the moments of μ are integers, because we have the following computation, based on Woronowicz's general Peter-Weyl type results in [11]:

$$\int_0^N x^k d\mu(x)=\int_G Tr(u)^k=\int_G Tr(u^{\otimes k})=\dim(Fix(u^{\otimes k}))$$

The above moments, or rather the fixed point spaces appearing on the right, can be computed by using the following fundamental result, from [2]:

Theorem 1.5. *We have an equality of complex vector spaces*

$$Fix(u^{\otimes k}) = Fix(P^{\otimes k})$$

where for $X \in M_N(A)$ we set $X^{\otimes k} = (X_{i_1 j_1} \ldots X_{i_k j_k})_{i_1 \ldots i_k, j_1 \ldots j_k}$.

Now back to the subfactor problematics, it is known from [7] that the planar algebra associated to H is given by $P_k = Fix(P^{\otimes k})$. Thus, Theorem 1.5 tells us that the Poincaré series $f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$ is nothing but the Stieltjes transform of μ:

$$f(z) = \int_G \frac{1}{1 - z\chi}$$

Summarizing, modulo some standard correspondences, the main subfactor problem regarding H consists in computing the spectral measure μ in Definition 1.4.

2. Finiteness, duality

We discuss in this section a key issue, namely the formulation of the duality between the quantum permutation groups associated to the matrices $H, \overline{H}, H^t, H^*$. Our claim is that the general scheme for this duality is, roughly speaking, as follows:

$$\begin{array}{ccc} H & \text{———} & H^t \\ | & & | \\ \overline{H} & \text{———} & H^* \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} G & \text{———} & \widehat{G} \\ | & & | \\ G^\sigma & \text{———} & \widehat{G}^\sigma \end{array}$$

More precisely, this scheme fully works when the quantum groups are finite. In the general case the situation is more complicated, as explained in [1].

The results in [1], written some time ago, in the general context of vertex models, and without using the Hopf image formalism in [2], are in fact not very enlightening in the Hadamard matrix case. We will present below an updated approach. First, we have:

Proposition 2.1. *The matrices $P = (P_{ij})$ for $H, \overline{H}, H^t, H^*$ are related by:*

$$\begin{array}{ccc} H & \text{———} & H^t \\ | & & | \\ \overline{H} & \text{———} & H^* \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} (P_{ij})_{kl} & \text{———} & (P_{kl})_{ij} \\ | & & | \\ (P_{ji})_{kl} & \text{———} & (P_{kl})_{ji} \end{array}$$

In addition, we have the formula $(P_{ij})_{kl} = (P_{ji})_{lk}$.

Proof. The magic matrix associated to H is given by $P_{ij} = Proj(H_i/H_j)$. Now since $H \to \overline{H}$ transforms $H_i/H_j \to H_j/H_i$, we conclude that the magic matrices $P^H, P^{\overline{H}}$ associated to $H, \overline{H}$ are related by the formula $P^H_{ij} = P^{\overline{H}}_{ji}$, as stated above.

In matrix notation, the formula for the matrix P^H is as follows:

$$(P^H_{ij})_{kl} = \frac{1}{N} \cdot \frac{H_{ik} H_{jl}}{H_{il} H_{jk}}$$

Now by replacing $H\to H^t$, we obtain the following formula:

$$(P_{ij}^{H^t})_{kl}=\frac{1}{N}\cdot\frac{H_{ki}H_{lj}}{H_{li}H_{kj}}=(P_{kl}^H)_{ij}$$

Finally, the last assertion is clear from the above formula of P^H. □

Let us compute now Hopf images. First, regarding the operation $H\to\overline{H}$, we have:

Proposition 2.2. *The quantum groups associated to $H,\overline{H}$ are related by*

$$G_{\overline{H}}=G_H^\sigma$$

where the Hopf algebra $C(G^\sigma)$ is $C(G)$ with comultiplication $\Sigma\Delta$, where Σ is the flip.

Proof. Our claim is that, starting from a factorization for H as in Definition 1.3 above, we can construct a factorization for $\overline{H}$, as follows:

$$\begin{tikzcd} u_{ij} \arrow[rr] \arrow[dr] & & P_{ij} \\ & v_{ij}\in C(G) \arrow[ur] & \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} u_{ij} \arrow[rr] \arrow[dr] & & P_{ji} \\ & v_{ji}\in C(G^\sigma) \arrow[ur] & \end{tikzcd}$$

Indeed, observe first that since $v_{ij}\in C(G)$ are the coefficients of a corepresentation, then so are the elements $v_{ji}\in C(G^\sigma)$. Thus, in order to produce the factorization on the right, it is enough to take the diagram on the left, and compose at top left with the canonical map $C(S_N^+)\to C(S_N^{+\sigma})$ given by $u_{ij}\to u_{ji}$, and we are done. □

Let us investigate now the operation $H\to H^t$. We use the notion of dual of a finite quantum group, see e.g. [11]. The result here is as follows:

Theorem 2.3. *The quantum groups associated to H,H^t are related by usual duality,*

$$G_{H^t}=\widehat{G}_H$$

provided that the quantum group G_H is finite.

Proof. Our claim is that, starting from a factorization for H as in Definition 1.3 above, we can construct a factorization for H^t, as follows:

$$\begin{tikzcd} C(S_N^+) \arrow[rr, "\pi_H"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G) \arrow[ur, "\rho"'] & \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} C(S_N^+) \arrow[rr, "\pi_{H^t}"] \arrow[dr] & & M_N(\mathbb{C}) \\ & C(G)^* \arrow[ur, "\eta"'] & \end{tikzcd}$$

More precisely, having a factorization as the one on the left, let us set:

$$\begin{aligned}\eta(\varphi)&=(\varphi(v_{kl}))_{kl}\\ w_{kl}(x)&=(\rho(x))_{kl}\end{aligned}$$

Our claim is that η is a representation, w is a corepresentation, and the factorization on the right holds indeed. Let us first check that η is a representation:

$$\begin{aligned}
\eta(\varphi\psi) &= (\phi\psi(v_{kl}))_{kl} = ((\varphi\otimes\psi)\Delta(v_{kl}))_{kl} = (\sum_a \varphi(v_{ka})\psi(v_{al}))_{kl} = \eta(\varphi)\eta(\psi)\\
\eta(\varepsilon) &= (\varepsilon(v_{kl}))_{kl} = (\delta_{kl})_{kl} = 1\\
\eta(\varphi^*) &= (\varphi^*(v_{kl}))_{kl} = (\overline{\varphi(S(v_{kl}^*))})_{kl} = (\overline{\varphi(v_{lk})})_{kl} = \eta(\varphi)^*
\end{aligned}$$

Let us check now the fact that w is a corepresentation:

$$\begin{aligned}
(\Delta w_{kl})(x\otimes y) &= w_{kl}(xy) = \rho(xy)_{kl} = \sum_i \rho(x)_{ki}\rho(y)_{il}\\
&= \sum_i w_{ki}(x)w_{il}(y) = (\sum_i w_{ki}\otimes w_{il})(x\otimes y)\\
\varepsilon(w_{kl}) &= w_{kl}(1) = 1_{kl} = \delta_{kl}
\end{aligned}$$

We check now the fact that the above diagram commutes on the generators u_{ij}:

$$\eta(w_{ab}) = (w_{ab}(v_{kl}))_{kl} = (\rho(v_{kl})_{ab})_{kl} = ((P^H_{kl})_{ab})_{kl} = ((P^{H^t}_{ab})_{kl})_{kl} = P^{H^t}_{ab}$$

It remains to prove that w is magic. We have the following formula:

$$\begin{aligned}
w_{a_0a_p}(v_{i_1j_1}\ldots v_{i_pj_p}) &= (\Delta^{(p-1)}w_{a_0a_p})(v_{i_1j_1}\otimes\ldots\otimes v_{i_pj_p})\\
&= \sum_{a_1\ldots a_{p-1}} w_{a_0a_1}(v_{i_1j_1})\ldots w_{a_{p-1}a_p}(v_{i_pj_p})\\
&= \frac{1}{N^p}\sum_{a_1\ldots a_{p-1}} \frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\cdots\cdots\frac{H_{i_pa_{p-1}}H_{j_pa_p}}{H_{i_pa_p}H_{j_pa_{p-1}}}
\end{aligned}$$

In order to check that each w_{ab} is an idempotent, observe that we have:

$$\begin{aligned}
w^2_{a_0a_p}(v_{i_1j_1}\ldots v_{i_pj_p}) &= (w_{a_0a_p}\otimes w_{a_0a_p})\sum_{k_1\ldots k_p} v_{i_1k_1}\ldots v_{i_pk_p}\otimes v_{k_1j_1}\ldots v_{k_pj_p}\\
&= \frac{1}{N^{2p}}\sum_{k_1\ldots k_p}\sum_{a_1\ldots a_{p-1}}\sum_{\alpha_1\ldots\alpha_{p-1}} \frac{H_{i_1a_0}H_{k_1a_1}}{H_{i_1a_1}H_{k_1a_0}}\cdots\cdots\frac{H_{i_pa_{p-1}}H_{k_pa_p}}{H_{i_pa_p}H_{k_pa_{p-1}}}\\
&\quad \frac{H_{k_1a_0}H_{j_1\alpha_1}}{H_{k_1\alpha_1}H_{j_1a_0}}\cdots\cdots\frac{H_{k_p\alpha_{p-1}}H_{j_pa_p}}{H_{k_pa_p}H_{j_p\alpha_{p-1}}}
\end{aligned}$$

The point now is that when summing over k_1 we obtain $N\delta_{a_1\alpha_1}$, then when summing over k_2 we obtain $N\delta_{a_2\alpha_2}$, and so on up to summing over k_{p-1}, where we obtain $N\delta_{a_{p-1}\alpha_{p-1}}$.

Thus, after performing all these summations, what we are left with is:

$$\begin{aligned}
w_{a_0a_p}^2(v_{i_1j_1}\ldots v_{i_pj_p}) &= \frac{1}{N^{p+1}}\sum_{k_p}\sum_{a_1\ldots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\cdot\ldots\ldots\cdot\frac{H_{i_pa_{p-1}}H_{k_pa_p}}{H_{i_pa_p}H_{k_pa_{p-1}}}\cdot\frac{H_{k_pa_{p-1}}H_{j_pa_p}}{H_{k_pa_p}H_{j_pa_{p-1}}}\\
&= \frac{1}{N^{p+1}}\sum_{k_p}\sum_{a_1\ldots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\cdot\ldots\ldots\cdot\frac{H_{i_pa_{p-1}}H_{j_pa_p}}{H_{i_pa_p}H_{j_pa_{p-1}}}\\
&= \frac{1}{N^p}\sum_{a_1\ldots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\cdot\ldots\ldots\cdot\frac{H_{i_pa_{p-1}}H_{j_pa_p}}{H_{i_pa_p}H_{j_pa_{p-1}}}\\
&= w_{a_0a_p}(v_{i_1j_1}\ldots v_{i_pj_p})
\end{aligned}$$

Regarding now the involutivity, the check here is simply:

$$\begin{aligned}
w_{a_0a_p}^*(v_{i_1j_1}\ldots v_{i_pj_p}) &= \overline{w_{a_0a_p}(S(v_{i_pj_p}\ldots v_{i_1j_1}))}\\
&= \overline{w_{a_0a_p}(v_{j_1i_1}\ldots v_{j_pi_p})}\\
&= w_{a_0a_p}^*(v_{i_1j_1}\ldots v_{i_pj_p})
\end{aligned}$$

Finally, for checking the first "sum 1" condition, observe that we have:

$$\sum_{a_0} w_{a_0a_p}(v_{i_1j_1}\ldots v_{i_pj_p}) = \frac{1}{N^p}\sum_{a_0\ldots a_{p-1}}\frac{H_{i_1a_0}H_{j_1a_1}}{H_{i_1a_1}H_{j_1a_0}}\cdot\ldots\ldots\cdot\frac{H_{i_pa_{p-1}}H_{j_pa_p}}{H_{i_pa_p}H_{j_pa_{p-1}}}$$

The point now is that when summing over a_0 we obtain $N\delta_{i_1j_1}$, then when summing over a_1 we obtain $N\delta_{i_2j_2}$, and so on up to summing over a_{p-1}, where we obtain $N\delta_{i_pj_p}$. Thus, after performing all these summations, what we are left with is:

$$\sum_{a_0} w_{a_0a_p}(v_{i_1j_1}\ldots v_{i_pj_p}) = \delta_{i_1j_1}\ldots\delta_{i_pj_p} = \varepsilon(v_{i_1j_1}\ldots v_{i_pj_p})$$

The proof of the other "sum 1" condition is similar, and this finishes the proof. □

3. The truncation procedure

Let us go back now to the factorization in Definition 1.3. Regarding the Haar functional of the quantum group G, we have the following key result, from [3]:

Proposition 3.1. *We have the Cesàro limiting formula*

$$\int_G = \lim_{k\to\infty}\frac{1}{k}\sum_{r=1}^k\int_G^r$$

where the functionals at right are by definition given by $\int_G^r = (tr\circ\rho)^{*r}$.

Regarding the functionals $\int_G^r$, their evaluation is a linear algebra problem. Several formulations of the problem were proposed in [3], and we will use here the following formula, which appears there, but in a somewhat technical form:

Proposition 3.2. *The functionals* $\int_G^r = (tr \circ \rho)^{*r}$ *are given by*

$$\int_G^r u_{a_1b_1}\ldots u_{a_pb_p} = (T_p^r)_{a_1\ldots a_p,b_1\ldots b_p}$$

where $(T_p)_{i_1\ldots i_p,j_1\ldots j_p} = tr(P_{i_1j_1}\ldots P_{i_pj_p})$, *with* $P_{ij} = Proj(H_i/H_j)$.

Proof. With $a_s = i_s^0$ and $b_s = i_s^{r+1}$, we have the following computation:

$$\begin{aligned}\int_G^r u_{a_1b_1}\ldots u_{a_pb_p} &= (tr\circ\rho)^{\otimes r}\Delta^{(r)}(u_{i_1^0i_1^{r+1}}\ldots u_{i_p^0i_p^{r+1}})\\ &= (tr\circ\rho)^{\otimes r}\sum_{i_1^1\ldots i_p^r} u_{i_1^0i_1^1}\ldots u_{i_p^0i_p^1}\otimes\ldots\ldots\otimes u_{i_1^ru_1^{r+1}}\ldots u_{i_p^ri_p^{r+1}}\\ &= tr^{\otimes r}\sum_{i_1^1\ldots i_p^r} P_{i_1^0i_1^1}\ldots P_{i_p^0i_p^1}\otimes\ldots\ldots\otimes P_{i_1^ri_1^{r+1}}\ldots P_{i_p^ri_p^{r+1}}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}(T_p^r)_{a_1\ldots a_p,b_1\ldots b_p} &= \sum_{i_1^1\ldots i_p^r}(T_p)_{i_1^0\ldots i_p^0,i_1^1\ldots i_p^1}\ldots\ldots(T_p)_{i_1^r\ldots i_p^r,i_1^{r+1}\ldots i_p^{r+1}}\\ &= \sum_{i_1^1\ldots i_p^r} tr(P_{i_1^0i_1^1}\ldots P_{i_p^0i_p^1})\ldots\ldots tr(P_{i_1^ri_1^{r+1}}\ldots P_{i_p^ri_p^{r+1}})\\ &= tr^{\otimes r}\sum_{i_1^1\ldots i_p^r} P_{i_1^0i_1^1}\ldots P_{i_p^0i_p^1}\otimes\ldots\ldots\otimes P_{i_1^ri_1^{r+1}}\ldots P_{i_p^ri_p^{r+1}}\end{aligned}$$

Thus we have obtained the formula in the statement, and we are done. □

We can now define the truncations of μ, as follows:

Proposition 3.3. *Let* μ^r *be the law of* χ *with respect to* $\int_G^r = (tr \circ \rho)^{*r}$.

1. μ^r *is a probability measure on* $[0, N]$.
2. *We have the formula* $\mu = \lim_{k\to\infty}\frac{1}{k}\sum_{r=1}^k \mu^r$.
3. *The moments of* μ^r *are the numbers* $c_p^r = Tr(T_p^r)$.

Proof. (1) The fact that μ^r is indeed a probability measure follows from the fact that the linear form $(tr \circ \rho)^{*r} : C(G) \to \mathbb{C}$ is a positive unital trace, and the assertion on the support comes from the fact that the main character χ is a sum of N projections.

(2) This follows from Proposition 3.1, i.e. from the main result in [3].

(3) This follows from Proposition 3.2 above, by summing over $a_i = b_i$. □

Let us recall now that associated to a complex Hadamard matrix $H \in M_N(\mathbb{C})$ is its profile matrix, given by:

$$Q_{ab,cd} = \frac{1}{N}\left\langle\frac{H_a}{H_b},\frac{H_c}{H_d}\right\rangle = \frac{1}{N}\sum_i\frac{H_{ia}H_{id}}{H_{ib}H_{ic}}$$

With this notation, we have the following result:

Proposition 3.4. *The measures μ^r have the following properties:*

(1) $\mu^0=\delta_N$.
(2) $\mu^1=(1-\frac{1}{N})\delta_0+\frac{1}{N}\delta_N$.
(3) $\mu^2=law(S)$, *where* $S_{ab,cd}=|Q_{ab,cd}|^2$.
(4) *For a Fourier matrix* F_G *we have* $\mu^1=\mu^2=\ldots=\mu$.

Proof. We use the formula $c_p^r=Tr(T_p^r)$ from Proposition 3.3 (3) above.

(1) At $r=0$ we have $c_p^0=Tr(T_p^0)=Tr(Id_{N^p})=N^p$, so $\mu^0=\delta_N$.

(2) At $r=1$, if we denote by J the flat matrix $(1/N)_{ij}$, we have indeed:

$$c_p^1=Tr(T_p)=\sum_{i_1\ldots i_p}tr(P_{i_1i_1}\ldots P_{i_pi_p})=\sum_{i_1\ldots i_p}tr(J^p)=\sum_{i_1\ldots i_p}tr(J)=N^{p-1}$$

(3) This can be checked directly, and is also a consequence of Theorem 3.5 below.

(4) For a Fourier matrix the representation ρ producing the factorization in Definition 1.3 is well-known to be faithful, and this gives the result. □

In the general case now, we have the following result:

Theorem 3.5. *We have* $\mu^r=law(X)$, *where*

$$X_{a_1\ldots a_r,b_1\ldots b_r}=Q_{a_1b_1,a_2b_2}\ldots Q_{a_rb_r,a_1b_1}$$

where Q denotes as usual the profile matrix.

Proof. We compute the moments of μ^r. We first have:

$$\begin{aligned}
c_p^r &= Tr(T_p^r)=\sum_{i^1\ldots i^r}(T_p)_{i^1i^2}\ldots(T_p)_{i^ri^1}\\
&= \sum_{i_1^1\ldots i_p^r}(T_p)_{i_1^1\ldots i_p^1,i_1^2\ldots i_p^2}\ldots\ldots(T_p)_{i_1^r\ldots i_p^r,i_1^1\ldots i_p^1}\\
&= \sum_{i_1^1\ldots i_p^r}tr(P_{i_1^1i_1^2}\ldots P_{i_p^1i_p^2})\ldots\ldots tr(P_{i_1^ri_1^1}\ldots P_{i_p^ri_p^1})
\end{aligned}$$

In terms of H, we obtain the following formula:

$$\begin{aligned}
c_p^r &= \frac{1}{N^r}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}(P_{i_1^1i_1^2})_{a_1^1a_2^1}\ldots(P_{i_p^1i_p^2})_{a_p^1a_1^1}\ldots\ldots(P_{i_1^ri_1^1})_{a_1^ra_2^r}\ldots(P_{i_p^ri_p^1})_{a_p^ra_1^r}\\
&= \frac{1}{N^{(p+1)r}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_1^2a_2^1}}{H_{i_1^1a_2^1}H_{i_1^2a_1^1}}\ldots\frac{H_{i_p^1a_p^1}H_{i_p^2a_1^1}}{H_{i_p^1a_1^1}H_{i_p^2a_p^1}}\ldots\ldots\frac{H_{i_1^ra_1^r}H_{i_1^1a_2^r}}{H_{i_1^ra_2^r}H_{i_1^1a_1^r}}\ldots\frac{H_{i_p^ra_p^r}H_{i_p^1a_1^r}}{H_{i_p^ra_1^r}H_{i_p^1a_p^r}}
\end{aligned}$$

Now by changing the order of the summation, we obtain:

$$\begin{aligned} c_p^r = \frac{1}{N^{(p+1)r}} \sum_{a_1^1\ldots a_p^r} \quad & \sum_{i_1^1} \frac{H_{i_1^1a_1^1}H_{i_1^1a_2^r}}{H_{i_1^1a_2^1}H_{i_1^1a_1^r}} \cdots\cdots \sum_{i_1^r} \frac{H_{i_1^ra_2^{r-1}}H_{i_1^ra_1^r}}{H_{i_1^ra_1^{r-1}}H_{i_1^ra_2^r}} \\ & \cdots \\ & \sum_{i_p^1} \frac{H_{i_p^1a_p^1}H_{i_p^1a_1^r}}{H_{i_p^1a_1^1}H_{i_p^1a_p^r}} \cdots\cdots \sum_{i_p^r} \frac{H_{i_p^ra_1^{r-1}}H_{i_p^ra_p^r}}{H_{i_p^ra_p^{r-1}}H_{i_p^ra_1^r}} \end{aligned}$$

In terms of Q, and then of the matrix X in the statement, we get:

$$\begin{aligned} c_p^r &= \frac{1}{N^r} \sum_{a_1^1\ldots a_p^r} (Q_{a_1^1a_2^1,a_1^ra_2^r} \ldots Q_{a_1^ra_2^r,a_1^{r-1}a_2^{r-1}}) \cdots\cdots (Q_{a_p^1a_1^1,a_p^ra_1^r} \ldots Q_{a_p^ra_1^r,a_p^{r-1}a_1^{r-1}}) \\ &= \frac{1}{N^r} \sum_{a_1^1\ldots a_p^r} X_{a_1^1\ldots a_1^r,a_2^1\ldots a_2^r} \cdots\cdots X_{a_p^1\ldots a_p^r,a_1^1\ldots a_1^r} \\ &= \frac{1}{N^r} Tr(X^p) = tr(X^p) \end{aligned}$$

But this gives the formula in the statement, and we are done. □

Observe that the above result covers the previous computations of μ^0, μ^1, μ^2, and in particular the formula of μ^2 in Proposition 3.4 (3). Indeed, at $r = 2$ we have:

$$X_{ab,cd} = Q_{ac,bd}Q_{bd,ac} = Q_{ab,cd}\overline{Q_{ab,cd}} = |Q_{ab,cd}|^2$$

We will discuss in the next section some further interpretations of μ^r.

4. Basic properties, examples

Let us first take a closer look at the matrices X appearing in Theorem 3.5. These are in fact Gram matrices, of certain norm one vectors:

Proposition 4.1. *We have $\mu^r = law(X)$, with $X_{a_1\ldots a_r,b_1\ldots b_r} =< \xi_{a_1\ldots a_r}, \xi_{b_1\ldots b_r} >$, where:*

$$\xi_{a_1\ldots a_r} = \frac{1}{\sqrt{N}} \cdot \frac{H_{a_1}}{H_{a_2}} \otimes \ldots \otimes \frac{1}{\sqrt{N}} \cdot \frac{H_{a_r}}{H_{a_1}}$$

In addition, these vectors $\xi_{a_1\ldots a_r}$ are all of norm one.

Proof. The first assertion follows from the following computation:

$$\begin{aligned} X_{a_1\ldots a_r,b_1\ldots b_r} &= \frac{1}{N^r} \left\langle \frac{H_{a_1}}{H_{b_1}}, \frac{H_{a_2}}{H_{b_2}} \right\rangle \ldots \left\langle \frac{H_{a_r}}{H_{b_r}}, \frac{H_{a_1}}{H_{b_1}} \right\rangle \\ &= \frac{1}{N^r} \left\langle \frac{H_{a_1}}{H_{a_2}}, \frac{H_{b_1}}{H_{b_2}} \right\rangle \ldots \left\langle \frac{H_{a_r}}{H_{a_1}}, \frac{H_{b_r}}{H_{b_1}} \right\rangle \\ &= \frac{1}{N^r} \left\langle \frac{H_{a_1}}{H_{a_2}} \otimes \ldots \otimes \frac{H_{a_r}}{H_{a_1}}, \frac{H_{b_1}}{H_{b_2}} \otimes \ldots \otimes \frac{H_{b_r}}{H_{b_1}} \right\rangle \end{aligned}$$

As for the second assertion, this is clear from the formula of $\xi_{a_1\ldots a_r}$. $\square$

At the level of concrete examples now, we first have:

Proposition 4.2. *For a Fourier matrix $H=F_G$ we have:*

(1) $Q_{ab,cd}=\delta_{a+d,b+c}$.
(2) $X_{a_1\ldots a_r,b_1\ldots b_r}=\delta_{a_1-b_1,\ldots,a_r-b_r}$.
(3) $X^2=NX$, *so X/N is a projection.*

Proof. We use the formulae $H_{ij}H_{ik}=H_{i,j+k}$, $\overline{H}_{ij}=H_{i,-j}$ and $\sum_iH_{ij}=N\delta_{j0}$.
(1) We have indeed the following computation:

$$Q_{ab,cd}=\frac{1}{N}\sum_iH_{i,a+d-b-c}=\delta_{a+d,b+c}$$

(2) This follows from the following computation:

$$\begin{aligned}X_{a_1\ldots a_r,b_1\ldots b_r}&=\delta_{a_1+b_2,b_1+a_2}\ldots\delta_{a_r+b_1,b_r+a_1}\\&=\delta_{a_1-b_1,a_2-b_2}\ldots\delta_{a_r-b-r,a_1-b_1}\\&=\delta_{a_1-b_1,\ldots,a_r-b_r}\end{aligned}$$

(3) By using the formula in (2) above, we obtain:

$$\begin{aligned}(X^2)_{a_1\ldots a_r,b_1\ldots b_r}&=\sum_{c_1\ldots c_r}X_{a_1\ldots a_r,c_1\ldots c_r}X_{c_1\ldots c_r,b_1\ldots b_r}\\&=\sum_{c_1\ldots c_r}\delta_{a_1-c_1,\ldots,a_r-c_r}\delta_{c_1-b_1,\ldots,c_r-b_r}\\&=N\delta_{a_1-b_1,\ldots,a_r-b_r}=NX_{a_1\ldots a_r,b_1\ldots b_r}\end{aligned}$$

Thus $(X/N)^2=X/N$, and since X/N is as well self-adjoint, it is a projection. $\square$

Another situation which is elementary is the tensor product one:

Proposition 4.3. *Let $L=H\otimes K$.*

(1) $Q^L_{iajb,kcld}=Q^H_{ij,kl}Q^K_{ab,cd}$.
(2) $X^L_{i_1a_1\ldots i_ra_r,j_1b_1\ldots j_rb_r}=X^H_{i_1\ldots i_r,j_1\ldots j_r}X^K_{a_1\ldots a_r,b_1\ldots b_r}$.
(3) $\mu_L^r=\mu_H^r*\mu_K^r$, *for any $r\geq0$.*

Proof. (1) This follows from the following computation:

$$\begin{aligned}Q^L_{iajb,kcld}&=\frac{1}{NM}\sum_{me}\frac{L_{me,ia}L_{me,ld}}{L_{me,kc}L_{me,jb}}=\frac{1}{NM}\sum_{me}\frac{H_{mi}K_{ea}H_{ml}K_{ld}}{H_{mk}K_{ec}H_{mj}K_{eb}}\\&=\frac{1}{N}\sum_m\frac{H_{mi}H_{ml}}{H_{mk}H_{mj}}\cdot\frac{1}{M}\sum_e\frac{K_{ea}K_{ed}}{K_{ec}K_{eb}}=Q^H_{ij,kl}Q^K_{ab,cd}\end{aligned}$$

(2) This follows from (2) above, because we have:

$$\begin{aligned} X^L_{i_1a_1\ldots i_ra_r,j_1b_1\ldots j_rb_r} &= Q^L_{i_1a_1j_1b_1,1_2a_2j_2b_2}\ldots Q^L_{i_ra_rj_rb_r,i_1a_1j_1b_1}\\ &= Q^H_{i_1j_1,i_2j_2}Q^K_{a_1b_1,a_2b_2}\ldots Q^H_{i_rj_r,i_1j_1}Q^K_{a_rb_r,a_1b_1}\\ &= X^H_{i_1\ldots i_r,j_1\ldots j_r}X^K_{a_1\ldots a_r,b_1\ldots b_r} \end{aligned}$$

(3) This follows from (3) above, which tells us that, modulo certain standard indentifications, we have $X^L=X^H\otimes X^K$. $\square$

We will be back in section 5 below to the study of concrete examples. Now let us discuss some general duality issues. We have here:

Theorem 4.4. *We have the moment/truncation duality formula*

$$\int_{G_H}^r\left(\frac{\chi}{N}\right)^p=\int_{G_{H^t}}^p\left(\frac{\chi}{N}\right)^r$$

where G_H,G_{H^t} are the quantum groups associated to H,H^t.

Proof. We use the following formula, from the proof of Theorem 3.5:

$$c_p^r=\frac{1}{N^{(p+1)r}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_1^2a_2^1}}{H_{i_1^1a_2^1}H_{i_1^2a_1^1}}\cdots\frac{H_{i_p^1a_p^1}H_{i_p^2a_1^1}}{H_{i_p^1a_1^1}H_{i_p^2a_p^1}}\cdots\cdots\frac{H_{i_1^ra_1^r}H_{i_1^1a_2^r}}{H_{i_1^ra_2^r}H_{i_1^1a_1^r}}\cdots\frac{H_{i_p^ra_p^r}H_{i_p^1a_1^r}}{H_{i_p^ra_1^r}H_{i_p^1a_p^r}}$$

By interchanging $p\leftrightarrow r$, and by transposing as well all the summation indices, according to the rules $i_x^y\to i_y^x$ and $a_x^y\to a_y^x$, we obtain the following formula:

$$c_r^p=\frac{1}{N^{(r+1)p}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_2^1a_1^2}}{H_{i_1^1a_1^2}H_{i_2^1a_1^1}}\cdots\frac{H_{i_1^ra_1^r}H_{i_2^ra_1^1}}{H_{i_1^ra_1^1}H_{i_2^ra_1^r}}\cdots\cdots\frac{H_{i_p^1a_p^1}H_{i_1^1a_p^2}}{H_{i_p^1a_p^2}H_{i_1^1a_p^1}}\cdots\frac{H_{i_p^ra_p^r}H_{i_1^ra_p^1}}{H_{i_p^ra_p^1}H_{i_1^ra_p^r}}$$

Now by interchaging all the summation indices, $i_x^y\leftrightarrow a_x^y$, we obtain:

$$c_r^p=\frac{1}{N^{(r+1)p}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}\frac{H_{a_1^1i_1^1}H_{a_2^1i_1^2}}{H_{a_1^1i_1^2}H_{a_2^1i_1^1}}\cdots\frac{H_{a_1^ri_1^r}H_{a_2^ri_1^1}}{H_{a_1^ri_1^1}H_{a_2^ri_1^r}}\cdots\cdots\frac{H_{a_p^1i_p^1}H_{a_1^1i_p^2}}{H_{a_p^1i_p^2}H_{a_1^1i_p^1}}\cdots\frac{H_{a_p^ri_p^r}H_{a_1^ri_p^1}}{H_{a_p^ri_p^1}H_{a_1^ri_p^r}}$$

With $H\to H^t$, we obtain the following formula, this time for H^t:

$$c_r^p=\frac{1}{N^{(r+1)p}}\sum_{i_1^1\ldots i_p^r}\sum_{a_1^1\ldots a_p^r}\frac{H_{i_1^1a_1^1}H_{i_1^2a_2^1}}{H_{i_1^2a_1^1}H_{i_1^1a_2^1}}\cdots\frac{H_{i_1^ra_1^r}H_{i_1^1a_2^r}}{H_{i_1^1a_1^r}H_{i_1^ra_2^r}}\cdots\cdots\frac{H_{i_p^1a_p^1}H_{i_p^2a_1^1}}{H_{i_p^2a_p^1}H_{i_p^1a_1^1}}\cdots\frac{H_{i_p^ra_p^r}H_{i_p^1a_1^r}}{H_{i_p^1a_p^r}H_{i_p^ra_1^r}}$$

The point now is that, modulo a permutation of terms, the quantity on the right is exactly the one as in the above formula of c_p^r. Thus, if we denote by α this quantity:

$$c_p^r(H)=\frac{\alpha}{N^{(p+1)r}},\qquad c_r^p(H^t)=\frac{\alpha}{N^{(r+1)p}}$$

Thus we have $N^r c_p^r(H) = N^p c_r^p(H^t)$, and by dividing by N^{p+r}, we obtain:

$$\frac{c_p^r(H)}{N^p} = \frac{c_r^p(H^t)}{N^r}$$

But this gives the formula in the statement, and we are done. $\square$

The above result shows that the normalized moments $\gamma_p^r = \frac{c_p^r}{N^p}$ are subject to the condition $\gamma_p^r(H) = \gamma_r^p(H^t)$. We have the following table of γ_p^r numbers for H:

$$\begin{bmatrix} p\backslash r & 1 & 2 & r & \infty \\ 1 & 1/N & 1/N & 1/N & 1/N \\ 2 & 1/N & tr(S/N)^2 & tr(S/N)^r & c_2 \\ p & 1/N & tr(S/N)^p & ? & c_p \\ \infty & 1/N & c_2 & \mu^r(1) & \mu(1) \end{bmatrix}$$

Here we have used the well-known fact that for $supp(\mu) \subset [0,1]$ we have $c_p \to \mu(1)$, something which is clear for discrete measures, and for continuous measures too.

Since the table for H^t is transpose to the table of H, we obtain:

Proposition 4.5. $\mu_H(1) = \mu_{H^t}(1)$.

Proof. This follows indeed from Theorem 4.4, by letting $p, r \to \infty$. $\square$

Observe that this result recovers a bit of Theorem 2.3, because we have:

Proposition 4.6. *For $G \subset S_N^+$ finite we have $\mu(1) = \frac{1}{|G|}$.*

Proof. The idea is to use the principal graph. So, let first Γ be an arbitrary finite graph, with a distinguished vertex denoted 1, let $A \in M_M(0,1)$ with $M = |\Gamma|$ be its adjacency matrix, set $N = ||\Gamma||$, and let $\xi \in \mathbb{R}^M$ be a Perron-Frobenius eigenvector for A, known to be unique up to multiplication by a scalar. Our claim is that we have:

$$\lim_{p\to\infty} \frac{(A^p)_{11}}{N^p} = \frac{\xi_1^2}{||\xi||^2}$$

Indeed, if we choose an orthonormal basis of eigenvectors (ξ^i), with $\xi^1 = \xi/||\xi||$, and write $A = UDU^t$, with $U = [\xi^1 \ldots \xi^M]$ and D diagonal, then we have, as claimed:

$$(A^p)_{11} = (UD^pU^t)_{11} = \sum_k U_{1k}^2 D_{kk}^p \simeq U_{11}^2 N^p = \frac{\xi_1^2}{||\xi||^2} N^p$$

Now back to our quantum group $G \subset S_N^+$, let Γ be its principal graph, having as vertices the elements $r \in Irr(G)$. The moments of μ being the numbers $c_p = (A^p)_{11}$, we have:

$$\mu(1) = \lim_{p\to\infty} \frac{c_p}{N^p} = \lim_{p\to\infty} \frac{(A^p)_{11}}{N^p} = \frac{\xi_1^2}{||\xi||^2}$$

On the other hand, it is known that with the normalization $\xi_1 = 1$, the entries of the Perron-Frobenius eigenvector are simply $\xi_r = \dim(r)$. Thus we have:

$$\frac{\xi_1^2}{||\xi||^2} = \frac{1}{\sum_r \dim(r)^2} = \frac{1}{|G|}$$

Together with the above formula of $\mu(1)$, this finishes the proof. □

5. Deformed Fourier matrices

In this section we study the deformed Fourier matrices, $L = F_M \otimes_Q F_N$, constructed by Diţă in [5]. These matrices are defined by $L_{ia,jb} = Q_{ib}(F_M)_{ij}(F_N)_{ab}$.

We first have the following technical result:

Proposition 5.1. *Let $H = F_M \otimes_Q F_N$, and set $R^x_{ab,cd} = \frac{1}{M}\sum_m w^{mx}\frac{Q_{ma}Q_{md}}{Q_{mc}Q_{mb}}$.*

(1) $Q_{iajb,kcld} = \delta_{a-b,c-d} R^{i+l-k-j}_{ab,cd}$.
(2) $X_{i_1a_1\ldots i_ra_r,j_1b_1\ldots j_rb_r} = \delta_{a_1-b_1,\ldots,a_r-b_r} R^{i_1+j_2-j_1-i_2}_{a_1b_1,a_2b_2} \ldots R^{i_r+j_1-j_r-i_1}_{a_rb_r,a_1b_1}$.

Proof. First, for a general deformation $H = K \otimes_Q L$, we have:

$$\begin{aligned} Q_{iajb,kcld} &= \frac{1}{MN}\sum_{me}\frac{H_{me,ia}H_{me,ld}}{H_{me,kc}H_{me,jb}} = \frac{1}{MN}\sum_{me}\frac{Q_{ma}K_{mi}L_{ea}Q_{md}K_{ml}L_{ld}}{Q_{mc}K_{mk}L_{ec}Q_{mb}K_{mj}L_{eb}} \\ &= \frac{1}{M}\sum_m \frac{Q_{ma}Q_{md}}{Q_{mc}Q_{mb}}\cdot\frac{K_{mi}K_{ml}}{K_{mk}K_{mj}}\cdot\frac{1}{N}\sum_e\frac{L_{ea}L_{ed}}{L_{ec}L_{eb}} \end{aligned}$$

Thus for a deformed Fourier matrix $H = F_M \otimes_Q F_N$ we have:

$$Q_{iajb,kcld} = \delta_{a+d,b+c}\frac{1}{M}\sum_m \frac{Q_{ma}Q_{md}}{Q_{mc}Q_{mb}} w^{m(i+l-k-j)}$$

But this gives (1), and then (2), and we are done. □

With the above formulae in hand, we can now state and prove:

Theorem 5.2. *For the matrix $H = F_M \otimes_Q F_N$ we have*

$$\mu_H = \mu_{H^t}$$

for any value of the parameter matrix $Q \in M_{M\times N}(\mathbb{T})$.

Proof. We use the matrices X,R constructed in Proposition 5.1 above. According to the result in Proposition 5.1 (2), we have the following formula:

$$\begin{aligned}
c_p^r &= \frac{1}{N^r}\sum_{a_1^1\ldots a_p^r} X_{a_1^1\ldots a_1^r,a_2^1\ldots a_2^r}\ldots\ldots X_{a_p^1\ldots a_p^r,a_1^1\ldots a_1^r}\\
&= \frac{1}{N^r}\sum_{a_1^1\ldots a_p^r}\sum_{i_1^1\ldots i_p^r}\delta_{a_1^1-a_2^1,\ldots,a_1^r-a_2^r}R^{i_1^1+i_2^2-i_1^2-i_2^1}_{a_1^1a_2^1,a_1^2a_2^2}\ldots\ldots R^{i_1^r+i_2^1-i_1^1-i_2^r}_{a_1^ra_2^r,a_1^1a_2^1}\\
&\ \ \ \ldots\\
&\ \ \ \delta_{a_p^1-a_1^1,\ldots,a_p^r-a_1^r}R^{i_p^1+i_1^2-i_1^1-i_p^2}_{a_p^1a_1^1,a_p^2a_1^2}\ldots\ldots R^{i_p^r+i_1^1-i_p^1-i_1^r}_{a_p^ra_1^r,a_p^1a_1^1}
\end{aligned}$$

Observe that the conditions on the a indices, coming from the Kronecker symbols, state that the columns of $a=(a_i^j)$ must differ by vertical vectors of type $(s,\ldots,s)$.

Now let us compute the sum over i indices, obtained by neglecting the Kronecker symbols. According to the formula of $R^x_{ab,cd}$ in Proposition 5.1, this is:

$$\begin{aligned}
S &= \frac{1}{N^{pr}}\sum_{i_1^1\ldots i_p^r}\sum_{m_1^1\ldots m_p^r} w^{E(i,m)}\frac{Q_{m_1^1a_1^1}Q_{m_1^1a_2^2}}{Q_{m_1^1a_2^1}Q_{m_1^1a_1^2}}\ldots\ldots\frac{Q_{m_1^ra_1^r}Q_{m_1^ra_2^1}}{Q_{m_1^ra_2^r}Q_{m_1^ra_1^1}}\\
&\ \ \ \ldots\\
&\ \ \ \frac{Q_{m_p^1a_p^1}Q_{m_p^1a_1^2}}{Q_{m_p^1a_1^1}Q_{m_p^1a_p^2}}\ldots\ldots\frac{Q_{m_p^ra_p^r}Q_{m_p^ra_1^1}}{Q_{m_p^ra_1^r}Q_{m_p^ra_p^1}}
\end{aligned}$$

Here the exponent appearing at right is given by:

$$\begin{aligned}
E(i,m) &= m_1^1(i_1^1+i_2^2-i_1^2-i_2^1)+\ldots+m_1^r(i_1^r+i_2^1-i_1^1-i_2^r)\\
&\ \ \ \ldots\\
&\ \ \ +m_p^1(i_p^1+i_1^2-i_p^2-i_1^1)+\ldots+m_p^r(i_p^r+i_1^1-i_p^1-i_1^r)
\end{aligned}$$

Now observe that this exponent can be written as:

$$\begin{aligned}
E(i,m) &= i_1^1(m_1^1-m_1^r-m_p^1+m_p^r)+\ldots+i_1^r(m_1^r-m_1^{r-1}-m_p^r+m_p^{r-1})\\
&\ \ \ \ldots\\
&\ \ \ +i_p^1(m_p^1-m_p^r-m_{p-1}^1+m_{p-1}^r)+\ldots+i_p^r(m_p^r-m_p^{r-1}-m_{p-1}^r+m_{p-1}^{r-1})
\end{aligned}$$

With this formula in hand, we can perform the sum over the i indices, and the point if that the resulting condition on the m indices will be exactly the same as the above-mentioned condition on the a indices. Thus, we obtain a formula as follows, where $\Delta(.)$

is a certain product of Kronecker symbols:

$$\begin{aligned} c_p^r &= \frac{1}{N^r}\sum_{a_1^1\ldots a_p^r}\sum_{m_1^1\ldots m_p^r}\Delta(a)\Delta(m)\frac{Q_{m_1^1a_1^1}Q_{m_1^1a_2^2}}{Q_{m_1^1a_2^1}Q_{m_1^1a_1^2}}\cdots\cdots\frac{Q_{m_1^ra_1^r}Q_{m_1^ra_2^1}}{Q_{m_1^ra_2^r}Q_{m_1^ra_1^1}} \\ &\quad \ldots \\ &\quad \frac{Q_{m_p^1a_p^1}Q_{m_p^1a_1^2}}{Q_{m_p^1a_1^1}Q_{m_p^1a_p^2}}\cdots\cdots\frac{Q_{m_p^ra_p^r}Q_{m_p^ra_1^1}}{Q_{m_p^ra_1^r}Q_{m_p^ra_p^1}} \end{aligned}$$

The point now is that when replacing $H = F_M \otimes_Q F_N$ with its transpose matrix, $H^t = F_N \otimes_{Q^t} F_M$, we will obtain exactly the same formula, with Q replaced by Q^t. But, with $a_x^y \leftrightarrow m_x^y$, this latter formula will be exactly the one above, and we are done. $\square$

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