Bistochastic Hadamard matrices

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"Introduction to Hadamard matrices", 4/6

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Bistochastic matrices

A complex Hadamard matrix $H \in M_N(\mathbb{T})$ is called bistochastic when the sums on all rows and all columns are equal.

It is known that any complex Hadamard matrix can be put in bistochastic form, up to the equivalence relation.

As a motivating remark, F_2 looks better in bistochastic form:

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} = F_2'$$

This suggests studying the Hadamard matrices $H \in M_N(\pm 1)$ by putting them in complex bistochastic form, $H' \in M_N(\mathbb{T})$.

Basic examples

Theorem. The class of bistochastic complex Hadamard matrices is stable under permuting rows and columns, and under taking tensor products. As basic examples, we have:

(1) The circulant and symmetric form F_N' of F_N.
(2) The bistochastic and symmetric form F_G' of F_G.
(3) The circulant Backelin matrices, having size MN with $M|N$.

Proof. Assuming that H, K are bistochastic, with sums λ, μ:

$$\sum_{ia}(H \otimes K)_{ia,jb} = \sum_{ia} H_{ij} K_{ab} = \sum_i H_{ij} \sum_a K_{ab} = \lambda\mu$$

$$\sum_{jb}(H \otimes K)_{ia,jb} = \sum_{jb} H_{ij} K_{ab} = \sum_j H_{ij} \sum_b K_{ab} = \lambda\mu$$

As for the other assertions, we already know all this.

Theory 1/4

Theorem. For $H \in M_N(\mathbb{T})$ Hadamard, the following are equivalent:

(1) H is bistochastic, with sums λ.

(2) H is row-stochastic, with sums λ, and $|\lambda|^2 = N$.

Proof. (1) $\implies$ (2) With $H_1, \ldots, H_N \in \mathbb{T}^N$ being the rows:

$$N = \sum_i < H_1, H_i >= \sum_j H_{1j} \cdot \bar{\lambda} = |\lambda|^2$$

(2) $\implies$ (1) With ξ being the all-one vector, we have:

$$\begin{aligned} H\xi = \lambda\xi \quad & \implies \quad H^*H\xi = \lambda H^*\xi \implies N^2\xi = \lambda H^*\xi \\ & \implies \quad N^2\xi = \bar{\lambda} H^t\xi \implies H^t\xi = \lambda\xi \end{aligned}$$

Thus row-stochastic with $|\lambda|^2 = N$ implies column-stochastic.

Theory 2/4

Theorem. For an Hadamard matrix $H \in M_N(\mathbb{T})$, the excess,

$$E(H) = \sum_{ij} H_{ij}$$

satisfies $|E(H)| \leq N\sqrt{N}$, with equality when H is bistochastic.

Proof. In terms of the all-one vector ξ, we have:

$$E(H) = \sum_{ij} H_{ij} = \sum_{ij} H_{ij}\xi_j\bar{\xi}_i =< H\xi, \xi >$$

By Cauchy-Schwarz we obtain, using $H/\sqrt{N} \in U_N$:

$$|E(H)| \leq ||H\xi|| \cdot ||\xi|| \leq ||H|| \cdot ||\xi||^2 = N\sqrt{N}$$

For equality we must have $H\xi \sim \xi$. But with $H\xi = \lambda\xi$, the above computation gives $|\lambda|^2 = N$, so the previous result applies.

Theory 3/4

Notations. The complex projective space appears as follows:

$$P^{N-1}_{\mathbb{C}} = (\mathbb{C}^N - \{0\}) / < x = \lambda y >$$

Inside this projective space, we have the Clifford torus:

$$\mathbb{T}^{N-1} = \left\{ (z_1, \ldots, z_N) \in P^{N-1}_{\mathbb{C}} \Big| |z_1| = \ldots = |z_N| \right\}$$

Theorem. For $U \in U_N$, the following are equivalent:

(1) $U' = LUR$ is bistochastic, with $L, R \in U_N$ diagonal.

(2) The standard torus $\mathbb{T}^N \subset \mathbb{C}^N$ satisfies $\mathbb{T}^N \cap U\mathbb{T}^N \neq \emptyset$.

(3) The Clifford torus $\mathbb{T}^{N-1} \subset P^{N-1}_{\mathbb{C}}$ satisfies $\mathbb{T}^{N-1} \cap U\mathbb{T}^{N-1} \neq \emptyset$.

Theory 4/4

Theorem. Any $U \in U_N$ can be put in bistochastic form,

$$U' = LUR$$

with $L, R \in U_N$ diagonal, via a certain non-explicit method.

Proof. It is known that $\mathbb{T}^{N-1} \subset P^{N-1}_{\mathbb{C}}$ is a Lagrangian submanifold, that $\mathbb{T}^{N-1} \to U\mathbb{T}^{N-1}$ is a Hamiltonian isotopy, and that $\mathbb{T}^{N-1}$ cannot be displaced from itself via a Hamiltonian isotopy (..)

Theorem. Any complex Hadamard matrix $H \in M_N(\mathbb{T})$ can be put in bistochastic form, modulo the equivalence relation.

Proof. Follows from Idel-Wolf, $U = H/\sqrt{N}$ being unitary.

Butson 1/2

Theorem. Assuming that $H_N(l)$ contains a bistochastic matrix,

$$\begin{aligned} a_0 + a_1 + \ldots + a_{l-1} &= N \\ |a_0 + a_1 w + \ldots + a_{l-1} w^{l-1}|^2 &= N \end{aligned}$$

with $w = e^{2\pi i/l}$ must have solutions, over the positive integers.

Proof. This comes from the formula $|\lambda|^2 = N$, established before. Indeed, if we denote by $a_i \in \mathbb{N}$ the number of w^i entries appearing in the first row of our matrix, the row sum is:

$$\lambda = a_0 + a_1 w + \ldots + a_{l-1} w^{l-1}$$

Thus, we obtain the system of equations in the statement.

Butson 2/2

Theorem. Assuming that $H_N(l)$ contains a bistochastic matrix, the following equations must have solutions, over the integers:

(1) $l = 2$: $4n^2 = N$.

(2) $l = 3$: $x^2 + y^2 + z^2 = 2N$, with $x + y + z = 0$.

(3) $l = 4$: $a^2 + b^2 = N$.

Proof. (1) This follows from the previous result.

(2) This follows by using the following identity:

$$|a + bw + cw^2|^2 = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

(3) This follows by using $|a + ib|^2 = a^2 + b^2$.

Fourier 1/2

Definition. We say that $H \in M_N(\mathbb{T})$ Hadamard is in "almost bistochastic form" when all row sums belong to $\sqrt{N} \cdot \mathbb{T}$.

Theorem. The matrix $F_N \otimes'_Q F_N$, with $Q \in M_N(\mathbb{T})$, given by

$$(F_N \otimes'_Q F_N)_{ia,jb} = \frac{w^{ij+ab}}{w^{bj+j}} \cdot \frac{Q_{ib}}{Q_{b+1,b}}$$

where $w = e^{2\pi i/N}$ is almost bistochastic, and $\sim F_N \otimes_Q F_N$.

Proof. Direct computation, using roots of unity.

Fourier 2/2

As an illustration, at $N = 2$ we have $w = -1$, and with $Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $u = \frac{p}{r}, v = \frac{s}{q}$, we obtain the following matrix:

$$F_2 \otimes_Q F_2 = \begin{pmatrix} \frac{p}{r} & \frac{q}{q} & -\frac{p}{r} & \frac{q}{q} \\ \frac{p}{r} & -\frac{q}{q} & -\frac{p}{r} & -\frac{q}{q} \\ \frac{r}{r} & \frac{s}{q} & \frac{r}{r} & -\frac{s}{q} \\ \frac{r}{r} & -\frac{s}{q} & \frac{r}{r} & \frac{s}{q} \end{pmatrix} = \begin{pmatrix} u & 1 & -u & 1 \\ u & -1 & -u & -1 \\ 1 & v & 1 & -v \\ 1 & -v & 1 & v \end{pmatrix}$$

Observe that this matrix is indeed almost bistochastic, with row sums $2, -2, 2, 2$.

Questions

We know from Idel-Wolf that any Hadamard matrix $H \in M_N(\mathbb{T})$ can be put in bistochastic form, at least in theory.

The problem is that of doing this explicitely, for instance for:

(1) $F_N \otimes_Q F_M$, with $N \neq M$.

(2) The Paley matrices.

(3) The Williamson matrices.

(4) Other known real Hadamard matrices.

These questions are interesting, because all this might lead to a "complex bistochastic formulation" of the HC, and the CHC.

The glow, 1/4

Definition. The glow of $H \in M_N(\pm 1)$ is the probability measure $\mu \in \mathcal{P}(\mathbb{Z})$ describing the distribution of the excess,

$$E = \sum_{ij} H_{ij}$$

over the real Hadamard equivalence class of H.

In other words, if we define $\varphi : \mathbb{Z}_2^N \times \mathbb{Z}_2^N \to \mathbb{Z}$ by

$$\varphi(a, b) = \sum_{ij} a_i b_j H_{ij}$$

then μ is the measure on $\mathbb{Z}$ given by $\mu(\{k\}) = P(\varphi = k)$.

The glow, 2/4

Square city, with N horizontal streets and N vertical streets, and with street lights at each crossroads. When evening comes the lights are switched on at the positions (i,j) where $H_{ij}=1$, and then, all night long, they are randomly switched on and off, with the help of $2N$ master switches, one at the end of each street:

$$\begin{matrix}
\to & & \diamondsuit & \diamondsuit & \diamondsuit & \diamondsuit \\
\to & & \diamondsuit & \times & \diamondsuit & \times \\
\to & & \diamondsuit & \diamondsuit & \times & \times \\
\to & & \diamondsuit & \times & \times & \diamondsuit \\
 & & & & & \\
 & & \uparrow & \uparrow & \uparrow & \uparrow
\end{matrix}$$

With this picture in mind, μ describes indeed the glow of the city. All this is related to the Gale-Berlekamp game.

The glow, 3/4

Theorem. Let $H \in M_N(\pm 1)$ be an Hadamard matrix of order $N \geq 4$, and denote by

$$\mu^{even}, \mu^{odd}$$

the mass one-rescaled restrictions of $\mu \in \mathcal{P}(4\mathbb{Z})$ to $8\mathbb{Z}, 8\mathbb{Z}+4$.

(1) At $N = 0(8)$ we have $\mu = \frac{3}{4}\mu^{even} + \frac{1}{4}\mu^{odd}$.

(2) At $N = 4(8)$ we have $\mu = \frac{1}{4}\mu^{even} + \frac{3}{4}\mu^{odd}$.

Proof. This follows by using the formula

$$\mu = \frac{1}{2^N} \sum_{b \in \mathbb{Z}_2^N} \beta_1(c_1) * \ldots * \beta_N(c_N)$$

with $\beta_r(c) = \left(\frac{\delta_r + \delta_{-r}}{2}\right)^{*c}$, $c_r = \#\{r \in |S_1|, \ldots, |S_N|\}$, $S = Hb$.

The glow, 4/4

Theorem. The glow moments of $H\in M_N(\pm 1)$ are given by:

$$\int_{\mathbb{Z}_2^N\times\mathbb{Z}_2^N}\left(\frac{E}{N}\right)^{2p}=(2p)!!+O(N^{-1})$$

In particular the variable E/N becomes Gaussian with $N\to\infty$.

Proof. The moments of $E=\sum_{ij}a_ib_jH_{ij}$ are given by

$$\begin{aligned}\int_{\mathbb{Z}_2^N\times\mathbb{Z}_2^N}E^r &= \sum_{ix}H_{i_1x_1}\ldots H_{i_rx_r}\int_{\mathbb{Z}_2^N}a_{i_1}\ldots a_{i_r}\int_{\mathbb{Z}_2^N}b_{x_1}\ldots b_{x_r}\\ &= \sum_{\pi,\sigma\in P_{even}(r)}\ \sum_{\ker i=\pi,\ker x=\sigma}H_{i_1x_1}\ldots H_{i_rx_r}\end{aligned}$$

and after some computations, this gives the result.

Complex glow, 1/4

<u>Definition</u>. The glow of a complex matrix $H \in M_N(\mathbb{C})$ is the probability measure $\mu \in \mathcal{P}(\mathbb{C})$ given by:

$$\int_{\mathbb{C}} \varphi(x) d\mu(x) = \int_{\mathbb{T}^N \times \mathbb{T}^N} \varphi\left(\sum_{ij} a_i b_j H_{ij}\right) d(a,b)$$

In other words, the glow is the law of the excess

$$E = \sum_{ij} H_{ij}$$

over the complex Hadamard equivalence class of H.

Complex glow, 2/4

Theorem. The glow has the following properties:

(1) $\mu = \varepsilon \times \mu^+$, where $\mu^+ = law(|E|)$.

(2) μ is invariant under rotations.

(3) $H \in \sqrt{N} U_N$ implies $supp(\mu) \subset N\sqrt{N}\,\mathbb{D}$.

(4) $H \in \sqrt{N} U_N$ implies as well $N\sqrt{N}\,\mathbb{T} \subset supp(\mu)$.

Proof. (1) Follows by using $H \to zH$ with $|z| = 1$.

(2) Follows from (1), the convolution with ε bringing the invariance.

(3) This folllows from Cauchy-Schwarz, cf. excess theory.

(4) This is something highly non-trivial, coming from Idel-Wolf.

Complex glow, 3/4

Theorem. The glow of a complex Hadamard matrix $H \in M_N(\mathbb{T})$ is given by:

$$\frac{1}{p!}\int_{\mathbb{T}^N\times\mathbb{T}^N}\left(\frac{|E|}{N}\right)^{2p} = 1 - \binom{p}{2}N^{-1} + O(N^{-2})$$

In particular, E/N becomes complex Gaussian in the $N \to \infty$ limit.

Proof. This uses the moment method, and combinatorics.

Complex glow, 4/4

Theorem. The glow of F_G, with $|G| = N$, is given by

$$\frac{1}{p!}\int_{\mathbb{T}^N\times\mathbb{T}^N}\left(\frac{|E|}{N}\right)^{2p} = 1 - K_1N^{-1} + K_2N^{-2} - K_3N^{-3} + O(N^{-4})$$

with $K_1 = \binom{p}{2}$, $K_2 = \binom{p}{2}\frac{3p^2+p-8}{12}$, $K_3 = \binom{p}{3}\frac{p^3+4p^2+p-18}{8}$.

Proof. Once again moment method, and combinatorics.

Remark. The next term depends on $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$,
$K_4 = \frac{8}{3}\binom{p}{3} + \frac{3}{4}\left(121 + \frac{2^e}{N}\right)\binom{p}{4} + 416\binom{p}{5} + \frac{2915}{2}\binom{p}{6} + 40\binom{p}{7} + 105\binom{p}{8}$
$e \in \mathbb{N}$ being the number of even numbers among $N_1, \ldots, N_k$.