Introduction to C*-algebras

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Abstract. This is an introduction to the theory of C^*-algebras. These are the norm closed $*$-algebras of bounded linear operators on a complex Hilbert space, $A \subset B(H)$, but can be axiomatized as well abstractly, as being the Banach $*$-algebras $(A, ||.||)$ whose norm is subject to the condition $||aa^*|| = ||a||^2$. Moreover, they can be interpreted as well as being the algebras of continuous functions on compact quantum spaces, $A = C(X)$. We first discuss the C^*-algebra basics, with the aim of understanding well the equivalence between these 3 points of view: operator theoretic, functional analytic, and geometric. Then we go into a more detailed study of the main classes of examples, and what can be done with them, with various geometric and analytic motivations in mind.

Preface

What is a quantum space? This is a good and appropriate question, in the present nuclear age, and this regardless of whether you are a mathematics or physics student, or even professor, or just some random guy in the street, with a vague high-school knowledge of modern science. Not that we can really control all these nuclear beasts, and what they can do by themselves, shall they ever start being animated by some form of intelligence, but at least, for having some theoretical understanding of them.

In answer, a quantum space is the dual of a C^*-algebra. To be more precise, the C^*-algebras are something precise and mathematical, that we can see, and manipulate, so to say, as humans, all routine work here, certainly no problem with that. As for the quantum spaces themselves, these appear as, well, so-called duals of these C^*-algebras, and in the lack of appropriate senses in order to see, smell or taste them, we can still work and work a lot on the C^*-algebras, in order to get familiar with them.

This was for the general idea, with two main points of view on the question, but in practice now, things further ramify, with four points of view, which are as follows:

(1) Concrete C^*-algebras. These are by definition the norm closed $*$-algebras of bounded linear operators on a complex Hilbert space, $A \subset B(H)$.

(2) Abstract C^*-algebras. These are the same thing, but axiomatized as being the Banach $*$-algebras $(A, ||.||)$ whose norm is subject to the condition $||aa^*|| = ||a||^2$.

(3) Abstract quantum spaces. These are the beasts X obtained by formally writing the C^*-algebras as being the algebras of continuous functions on them, $A = C(X)$.

(4) Concrete quantum spaces. Same beasts X, but appearing this time in relation with quantum physics, by zooming down and enjoying, with a good microscope.

Excited by this? So am I, despite having spent 30 years in this business, and having not understood much, but never too late, for remaining forever young. So, this will be what we will be talking about, in this book, introduction to (1-4).

In practice, the book is organized in four parts, with the first half, Parts I-II, discussing the C^*-algebra basics, with the aim of understanding well the equivalence between (1-2), and with a constant look into (3) too. Then, in the second half, Parts III-IV, we will get into a more detailed study of the main classes of examples, and what can be done with them, with various geometric and analytic motivations, in relation with (3-4), in mind.

In the hope that you will like this book, and do not hesitate of course to have on your desk at least 3-4 other books on the same topic, for some sort of simultaneous reading, mixing various viewpoints, learning C^*-algebras being no easy business, for us humans.

Speaking learning, many thanks to the many books on the subject, old or more recent, that I have been struggling with as a student, then as a young researcher, and then as, well, confirmed researcher. Many thanks as well to my cats, they say that (4) is trivial and that (1-3) come as corollaries, hope one day I'll reach to their level of wisdom.

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Contents

Part I

C*-algebras

I see a boat on the river
It’s sailing away
Down to the ocean
Where to I can’t say

CHAPTER 1

Operator theory

1a. Linear operators

We would like to first discuss the theory of linear operators $T : H \to H$ over a complex Hilbert space H, usually taken separable. Let us start with a basic result, as follows:

THEOREM 1.1. *Given a Hilbert space H, consider the linear operators $T : H \to H$, and for each such operator define its norm by the following formula:*

$$||T|| = \sup_{||x||=1} ||Tx||$$

The operators which are bounded, $||T|| < \infty$, form then a complex algebra $B(H)$, which is complete with respect to $||.||$. When H comes with a basis $\{e_i\}_{i \in I}$, we have

$$B(H) \subset \mathcal{L}(H) \subset M_I(\mathbb{C})$$

where $\mathcal{L}(H)$ is the algebra of all linear operators $T : H \to H$, and $\mathcal{L}(H) \subset M_I(\mathbb{C})$ is the correspondence $T \to M$ obtained via the usual linear algebra formulae, namely:

$$T(x) = Mx \quad , \quad M_{ij} =< Te_j, e_i >$$

In infinite dimensions, none of the above two inclusions is an equality.

PROOF. This is something straightforward, the idea being as follows:

(1) The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S + T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(2) Regarding now the completness assertion, if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x)+T_n(y) \\ &= \lim_{n\to\infty} T_n(x)+\lim_{n\to\infty} T_n(y) \\ &= T(x)+T(y) \end{aligned}$$

Similarly, we have $T(\lambda x) = \lambda T(x)$, and we conclude that $T \in \mathcal{L}(H)$.

(3) With this done, it remains to prove now that we have $T \in B(H)$, and that $T_n \to T$ in norm. For this purpose, observe that we have:

$$\begin{aligned} ||T_n - T_m|| \leq \varepsilon\ ,\ \forall n,m \geq N &\implies ||T_n x - T_m x|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n,m \geq N \\ &\implies ||T_n x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n \geq N \\ &\implies ||T_N x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1 \\ &\implies ||T_N - T|| \leq \varepsilon \end{aligned}$$

But this gives both $T \in B(H)$, and $T_N \to T$ in norm, and we are done.

(4) Regarding the embeddings, the correspondence $T \to M$ in the statement is indeed linear, and its kernel is $\{0\}$, so we have indeed an embedding as follows, as claimed:

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

In finite dimensions we have an isomorphism, because any $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \to \mathbb{C}^N$, given by $< Te_j, e_i >= M_{ij}$. However, in infinite dimensions, we have matrices not producing operators, as for instance the all-one matrix.

(5) As for the examples of linear operators which are not bounded, these are more complicated, coming from logic, and we will not really need them in what follows. □

As a second basic result regarding the operators, we will need:

Theorem 1.2. *Each operator $T \in B(H)$ has an adjoint $T^* \in B(H)$, given by:*

$$< Tx, y >=< x, T^*y >$$

The operation $T \to T^*$ is antilinear, antimultiplicative, involutive, and satisfies:

$$||T|| = ||T^*|| \quad , \quad ||TT^*|| = ||T||^2$$

When H comes with a basis $\{e_i\}_{i\in I}$, the operation $T \to T^*$ corresponds to

$$(M^*)_{ij} = \overline{M}_{ji}$$

at the level of the associated matrices $M \in M_I(\mathbb{C})$.

PROOF. This is standard too, and can be proved in 3 steps, as follows:

(1) The existence of the adjoint operator T^*, given by the formula in the statement, comes from the fact that the function $\varphi(x) =< Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\varphi(x) =< x, T^*y >$$

Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S+T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^* \quad , \quad (ST)^* = T^*S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y > \\ &= ||T^*|| \end{aligned}$$

(2) Regarding now $||TT^*|| = ||T||^2$, which is a key formula, observe that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, as desired.

(3) Finally, when H comes with a basis, the formula $< Tx, y >=< x, T^*y >$ applied with $x = e_i$, $y = e_j$ translates into the formula $(M^*)_{ij} = \overline{M}_{ji}$, as desired. ☐

Let us discuss now the diagonalization problem for the operators $T \in B(H)$, in analogy with the diagonalization problem for the usual matrices $A \in M_N(\mathbb{C})$. As a first observation, we can talk about eigenvalues and eigenvectors, as follows:

DEFINITION 1.3. *Given an operator $T \in B(H)$, assuming that we have*

$$Tx = \lambda x$$

we say that $x \in H$ is an eigenvector of T, with eigenvalue $\lambda \in \mathbb{C}$.

We know many things about eigenvalues and eigenvectors, in the finite dimensional case. However, most of these will not extend to the infinite dimensional case, or at least not extend in a straightforward way, due to a number of reasons:

(1) Most of basic linear algebra is based on the fact that $Tx = \lambda x$ is equivalent to $(T-\lambda)x = 0$, so that λ is an eigenvalue when $T-\lambda$ is not invertible. In the infinite dimensional setting $T-\lambda$ might be injective and not surjective, or vice versa, or invertible with $(T-\lambda)^{-1}$ not bounded, and so on.

(2) Also, in linear algebra $T-\lambda$ is not invertible when $\det(T-\lambda)=0$, and with this leading to most of the advanced results about eigenvalues and eigenvectors. In infinite dimensions, however, it is impossible to construct a determinant function $\det: B(H)\to\mathbb{C}$, and this even for the diagonal operators on $l^2(\mathbb{N})$.

Summarizing, we are in trouble. Forgetting about (2), which obviously leads nowhere, let us focus on the difficulties in (1). In order to cut short the discussion there, regarding the various properties of $T-\lambda$, we can just say that $T-\lambda$ is either invertible with bounded inverse, the "good case", or not. We are led in this way to the following definition:

Definition 1.4. *The spectrum of an operator $T\in B(H)$ is the set*

$$\sigma(T)=\left\{\lambda\in\mathbb{C}\Big|T-\lambda\notin B(H)^{-1}\right\}$$

where $B(H)^{-1}\subset B(H)$ is the set of invertible operators.

As a basic example, in the finite dimensional case, $H=\mathbb{C}^N$, the spectrum of a usual matrix $A\in M_N(\mathbb{C})$ is the collection of its eigenvalues, taken without multiplicities. We will see many other examples. In general, the spectrum has the following properties:

Proposition 1.5. *The spectrum of $T\in B(H)$ contains the eigenvalue set*

$$\varepsilon(T)=\left\{\lambda\in\mathbb{C}\Big|\ker(T-\lambda)\neq\{0\}\right\}$$

and $\varepsilon(T)\subset\sigma(T)$ is an equality in finite dimensions, but not in infinite dimensions.

Proof. We have several assertions here, the idea being as follows:

(1) First of all, the eigenvalue set is indeed the one in the statement, because $Tx=\lambda x$ tells us precisely that $T-\lambda$ must be not injective. The fact that we have $\varepsilon(T)\subset\sigma(T)$ is clear as well, because if $T-\lambda$ is not injective, it is not bijective.

(2) In finite dimensions we have $\varepsilon(T)=\sigma(T)$, because $T-\lambda$ is injective if and only if it is bijective, with the boundedness of the inverse being automatic.

(3) In infinite dimensions we can assume $H=l^2(\mathbb{N})$, and the shift operator $S(e_i)=e_{i+1}$ is injective but not surjective. Thus $0\in\sigma(T)-\varepsilon(T)$. □

Philosophically, the best way of thinking at this is as follows: the numbers $\lambda\notin\sigma(T)$ are good, because we can invert $T-\lambda$, the numbers $\lambda\in\sigma(T)-\varepsilon(T)$ are bad, because so they are, and the eigenvalues $\lambda\in\varepsilon(T)$ are evil. Welcome to operator theory.

Let us develop now some general theory. As a first goal, we would like to prove that the spectra are non-empty. This is something quite tricky, the result being as follows:

THEOREM 1.6. *The spectrum of a bounded operator $T \in B(H)$ is:*

(1) *Compact.*
(2) *Contained in the disc $D_0(||T||)$.*
(3) *Non-empty.*

PROOF. This can be proved by using some complex analysis, as follows:

(1) In view of (2) below, it is enough to prove that $\sigma(T)$ is closed. But this follows from the following computation, with $|\varepsilon|$ being small:

$$\begin{aligned}\lambda \notin \sigma(T) &\implies T - \lambda \in B(H)^{-1}\\ &\implies T - \lambda - \varepsilon \in B(H)^{-1}\\ &\implies \lambda + \varepsilon \notin \sigma(T)\end{aligned}$$

(2) This follows indeed from the following computation:

$$\begin{aligned}\lambda > ||T|| &\implies \left|\left|\frac{T}{\lambda}\right|\right| < 1\\ &\implies 1 - \frac{T}{\lambda} \in B(H)^{-1}\\ &\implies \lambda - T \in B(H)^{-1}\\ &\implies \lambda \notin \sigma(T)\end{aligned}$$

(3) Assume by contradiction $\sigma(T) = \emptyset$. Given a linear form $f \in B(H)^*$, consider the following map, which is well-defined, due to our assumption $\sigma(T) = \emptyset$:

$$\varphi : \mathbb{C} \to \mathbb{C} \quad , \quad \lambda \to f((T - \lambda)^{-1})$$

By using the fact that $T \to T^{-1}$ is differentiable, which is something elementary, we conclude that this map is differentiable, and so holomorphic. Also, we have:

$$\begin{aligned}\lambda \to \infty &\implies T - \lambda \to \infty\\ &\implies (T - \lambda)^{-1} \to 0\\ &\implies f((T - \lambda))^{-1} \to 0\end{aligned}$$

Thus by the Liouville theorem we obtain $\varphi = 0$. But, in view of the definition of φ, this gives $(T - \lambda)^{-1} = 0$, which is a contradiction, as desired. □

Here is now a second basic result regarding the spectra, inspired from what happens in finite dimensions, for the usual complex matrices, and which shows that things do not necessarily extend without troubles to the infinite dimensional setting:

THEOREM 1.7. *We have the following formula, valid for any operators S, T:*

$$\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$$

In finite dimensions we have $\sigma(ST) = \sigma(TS)$, but this fails in infinite dimensions.

PROOF. There are several assertions here, the idea being as follows:

(1) This is something that we know in finite dimensions, coming from the fact that the characteristic polynomials of the associated matrices A, B coincide:

$$P_{AB} = P_{BA}$$

Thus we obtain $\sigma(ST) = \sigma(TS)$ in this case, as claimed. Observe that this improves twice the general formula in the statement, first because we have no issues at 0, and second because what we obtain is actually an equality of sets with mutiplicities.

(2) In general now, let us first prove the main assertion, stating that $\sigma(ST), \sigma(TS)$ coincide outside 0. We first prove that we have the following implication:

$$1 \notin \sigma(ST) \implies 1 \notin \sigma(TS)$$

Assume indeed that $1 - ST$ is invertible, with inverse denoted R:

$$R = (1 - ST)^{-1}$$

We have then the following formulae, relating our variables R, S, T:

$$RST = STR = R - 1$$

By using $RST = R - 1$, we have the following computation:

$$\begin{aligned}(1 + TRS)(1 - TS) &= 1 + TRS - TS - TRSTS \\ &= 1 + TRS - TS - TRS + TS \\ &= 1\end{aligned}$$

A similar computation, using $STR = R - 1$, shows that we have:

$$(1 - TS)(1 + TRS) = 1$$

Thus $1 - TS$ is invertible, with inverse $1 + TRS$, which proves our claim. Now by multiplying by scalars, we deduce from this that for any $\lambda \in \mathbb{C} - \{0\}$ we have:

$$\lambda \notin \sigma(ST) \implies \lambda \notin \sigma(TS)$$

But this leads to the conclusion in the statement.

(3) Regarding now the counterexample to the formula $\sigma(ST) = \sigma(TS)$, in general, let us take S to be the shift on $H = L^2(\mathbb{N})$, given by the following formula:

$$S(e_i) = e_{i+1}$$

As for T, we can take it to be the adjoint of S, and we have:

$$S^*S = 1 \implies 0 \notin \sigma(SS^*)$$

$$SS^* = Proj(e_0^\perp) \implies 0 \in \sigma(SS^*)$$

Thus, the spectra do not match on 0, and so we have our counterexample. □

1b. Spectral radius

Let us develop now some systematic theory for the computation of the spectra, based on what we know about the eigenvalues of the usual complex matrices. As a first result, which is well-known for the usual matrices, and extends well, we have:

THEOREM 1.8. *We have the "polynomial functional calculus" formula*

$$\sigma(P(T)) = P(\sigma(T))$$

valid for any polynomial $P \in \mathbb{C}[X]$, and any operator $T \in B(H)$.

PROOF. We pick a scalar $\lambda \in \mathbb{C}$, and we decompose the polynomial $P - \lambda$:

$$P(X) - \lambda = c(X - r_1) \ldots (X - r_n)$$

We have then the following equivalences:

$$\begin{aligned} \lambda \notin \sigma(P(T)) &\iff P(T) - \lambda \in B(H)^{-1} \\ &\iff c(T - r_1) \ldots (T - r_n) \in B(H)^{-1} \\ &\iff T - r_1, \ldots, T - r_n \in B(H)^{-1} \\ &\iff r_1, \ldots, r_n \notin \sigma(T) \\ &\iff \lambda \notin P(\sigma(T)) \end{aligned}$$

Thus, we are led to the formula in the statement. □

The above result is something very useful, and generalizing it will be our next task. As a first ingredient here, assuming that $A \in M_N(\mathbb{C})$ is invertible, we have:

$$\sigma(A^{-1}) = \sigma(A)^{-1}$$

It is possible to extend this formula to the arbitrary operators, and we will do this in a moment. Before starting, however, we have to find a class of functions generalizing both the polynomials $P \in \mathbb{C}[X]$ and the inverse function $x \to x^{-1}$. The answer to this question is provided by the rational functions, which are as follows:

DEFINITION 1.9. *A rational function $f \in \mathbb{C}(X)$ is a quotient of polynomials:*

$$f = \frac{P}{Q}$$

Assuming that P, Q are prime to each other, we can regard f as a usual function,

$$f : \mathbb{C} - X \to \mathbb{C}$$

with X being the set of zeros of Q, also called poles of f.

Now that we have our class of functions, the next step consists in applying them to operators. Here we cannot expect $f(T)$ to make sense for any f and any T, for instance because T^{-1} is defined only when T is invertible. We are led in this way to:

Definition 1.10. *Given an operator $T \in B(H)$, and a rational function $f = P/Q$ having poles outside $\sigma(T)$, we can construct the following operator,*

$$f(T) = P(T)Q(T)^{-1}$$

that we can denote as a usual fraction, as follows,

$$f(T) = \frac{P(T)}{Q(T)}$$

due to the fact that $P(T), Q(T)$ commute, so that the order is irrelevant.

To be more precise, $f(T)$ is indeed well-defined, and the fraction notation is justified too. In more formal terms, we can say that we have a morphism of complex algebras as follows, with $\mathbb{C}(X)^T$ standing for the rational functions having poles outside $\sigma(T)$:

$$\mathbb{C}(X)^T \to B(H) \quad , \quad f \to f(T)$$

Summarizing, we have now a good class of functions, generalizing both the polynomials and the inverse map $x \to x^{-1}$. We can now extend Theorem 1.8, as follows:

Theorem 1.11. *We have the "rational functional calculus" formula*

$$\sigma(f(T)) = f(\sigma(T))$$

valid for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(T)$.

Proof. We pick a scalar $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set:

$$F = P - \lambda Q$$

By using now Theorem 1.9, for this polynomial, we obtain:

$$\begin{aligned}
\lambda \in \sigma(f(T)) &\iff F(T) \notin B(H)^{-1} \\
&\iff 0 \in \sigma(F(T)) \\
&\iff 0 \in F(\sigma(T)) \\
&\iff \exists \mu \in \sigma(T), F(\mu) = 0 \\
&\iff \lambda \in f(\sigma(T))
\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an application of the above methods, we can investigate certain special classes of operators, such as the self-adjoint ones, and the unitary ones. Let us start with:

Proposition 1.12. *The following happen:*

(1) *We have $\sigma(T^*) = \overline{\sigma(T)}$, for any $T \in B(H)$.*
(2) *If $T = T^*$ then $X = \sigma(T)$ satisfies $X = \overline{X}$.*
(3) *If $U^* = U^{-1}$ then $X = \sigma(U)$ satisfies $X^{-1} = \overline{X}$.*

PROOF. We have several assertions here, the idea being as follows:

(1) The spectrum of the adjoint operator T^* can be computed as follows:

$$\begin{aligned}\sigma(T^*) &= \left\{\lambda \in \mathbb{C}\Big|T^* - \lambda \notin B(H)^{-1}\right\}\\ &= \left\{\lambda \in \mathbb{C}\Big|T - \bar{\lambda} \notin B(H)^{-1}\right\}\\ &= \overline{\sigma(T)}\end{aligned}$$

(2) This is clear indeed from (1).

(3) For a unitary operator, $U^* = U^{-1}$, Theorem 1.11 and (1) give:

$$\sigma(U)^{-1} = \sigma(U^{-1}) = \sigma(U^*) = \overline{\sigma(U)}$$

Thus, we are led to the conclusion in the statement. □

In analogy with what happens for the usual matrices, we would like to improve now (2,3) above, with results stating that the spectrum $X = \sigma(T)$ satisfies $X \subset \mathbb{R}$ for self-adjoints, and $X \subset \mathbb{T}$ for unitaries. This will be tricky. Let us start with:

THEOREM 1.13. *The spectrum of a unitary operator*

$$U^* = U^{-1}$$

is on the unit circle, $\sigma(U) \subset \mathbb{T}$.

PROOF. Assuming $U^* = U^{-1}$, we have the following norm computation:

$$||U|| = \sqrt{||UU^*||} = \sqrt{1} = 1$$

Now if we denote by D the unit disk, we obtain from this:

$$\sigma(U) \subset D$$

On the other hand, once again by using $U^* = U^{-1}$, we have as well:

$$||U^{-1}|| = ||U^*|| = ||U|| = 1$$

Thus, as before with D being the unit disk in the complex plane, we have:

$$\sigma(U^{-1}) \subset D$$

Now by using Theorem 1.11, we obtain $\sigma(U) \subset D \cap D^{-1} = \mathbb{T}$, as desired. □

We have as well a similar result for the self-adjoints, as follows:

THEOREM 1.14. *The spectrum of a self-adjoint operator*

$$T = T^*$$

consists of real numbers, $\sigma(T) \subset \mathbb{R}$.

PROOF. The idea is that we can deduce the result from Theorem 1.13, by using the following remarkable rational function, depending on a parameter $r \in \mathbb{R}$:

$$f(z) = \frac{z + ir}{z - ir}$$

Indeed, for $r >> 0$ the operator $f(T)$ is well-defined, and we have:

$$\left(\frac{T + ir}{T - ir}\right)^* = \frac{T - ir}{T + ir} = \left(\frac{T + ir}{T - ir}\right)^{-1}$$

Thus $f(T)$ is unitary, and by using Theorem 1.13 we obtain:

$$\begin{aligned} \sigma(T) &\subset f^{-1}(f(\sigma(T))) \\ &= f^{-1}(\sigma(f(T))) \\ &\subset f^{-1}(\mathbb{T}) \\ &= \mathbb{R} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

One key thing that we know about matrices, which is clear for the diagonalizable matrices, and then in general follows by density, is the following formula:

$$\sigma(e^A) = e^{\sigma(A)}$$

We would like to have such formulae for the general operators $T \in B(H)$, but this is something quite technical. Consider the rational calculus morphism from Definition 1.10, which is as follows, with the exponent standing for "having poles outside $\sigma(T)$":

$$\mathbb{C}(X)^T \to B(H) \quad , \quad f \to f(T)$$

As mentioned before, the rational functions are holomorphic outside their poles, and this raises the question of extending this morphism, as follows:

$$Hol(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

But for this, we can use the Cauchy formula. Indeed, given a function $f \in \mathbb{C}(X)^T$, the operator $f(T) \in B(H)$ from Definition 1.10 can be recaptured as follows:

$$f(T) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - T}\, dz$$

Now given an arbitrary function $f \in Hol(\sigma(T))$, we can define $f(T) \in B(H)$ by the exactly same formula, and we obtain in this way the desired correspondence:

$$Hol(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

This was for the plan. In practice now, all this needs a bit of care, with many verifications needed, and with the technical remark that a winding number must be added to the above Cauchy formulae, for things to be correct. The result is as follows:

THEOREM 1.15. *Given $T \in B(H)$, we have a morphism of algebras as follows, where $Hol(\sigma(T))$ is the algebra of functions which are holomorphic around $\sigma(T)$,*

$$Hol(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

which extends the previous rational functional calculus $f \to f(T)$. We have:

$$\sigma(f(T)) = f(\sigma(T))$$

Moreover, if $\sigma(T)$ is contained in an open set U and $f_n, f : U \to \mathbb{C}$ are holomorphic functions such that $f_n \to f$ uniformly on compact subsets of U then $f_n(T) \to f(T)$.

PROOF. This follows indeed by reasoning along the above lines, by making a heavy use of the Cauchy formula, and for full details here, we refer to any specialized operator theory book. In what follows, we will not really need this result. □

In order to formulate now our next result, we will need the following notion:

DEFINITION 1.16. *Given an operator $T \in B(H)$, its spectral radius*

$$\rho(T) \in \big[0, ||T||\big]$$

is the radius of the smallest disk centered at 0 containing $\sigma(T)$.

Now with this notion in hand, we have the following key result, improving our key theoretical result so far about spectra, namely $\sigma(T) \neq \emptyset$, from Theorem 1.6:

THEOREM 1.17. *The spectral radius of an operator $T \in B(H)$ is given by*

$$\rho(T) = \lim_{n\to\infty} ||T^n||^{1/n}$$

and in this formula, we can replace the limit by an inf.

PROOF. We have several things to be proved, the idea being as follows:

(1) Our first claim is that the numbers $u_n = ||T^n||^{1/n}$ satisfy:

$$(n+m)u_{n+m} \leq nu_n + mu_m$$

Indeed, we have the following estimate, using the Young inequality $ab \leq a^p/p + b^q/q$, with exponents $p = (n+m)/n$ and $q = (n+m)/m$:

$$\begin{aligned} u_{n+m} &= ||T^{n+m}||^{1/(n+m)} \\ &\leq ||T^n||^{1/(n+m)}||T^m||^{1/(n+m)} \\ &\leq ||T^n||^{1/n} \cdot \frac{n}{n+m} + ||T^m||^{1/m} \cdot \frac{m}{n+m} \\ &= \frac{nu_n + mu_m}{n+m} \end{aligned}$$

(2) Our second claim is that the second assertion holds, namely:

$$\lim_{n\to\infty} ||T^n||^{1/n} = \inf_n ||T^n||^{1/n}$$

For this purpose, we just need the inequality found in (1). Indeed, fix $m \geq 1$, let $n \geq 1$, and write $n = lm + r$ with $0 \leq r \leq m-1$. By using twice $u_{ab} \leq u_b$, we get:

$$\begin{aligned} u_n &\leq \frac{1}{n}(lmu_{lm} + ru_r) \\ &\leq \frac{1}{n}(lmu_m + ru_1) \\ &\leq u_m + \frac{r}{n}u_1 \end{aligned}$$

It follows that we have $\limsup_n u_n \leq u_m$, which proves our claim.

(3) Summarizing, we are left with proving the main formula, which is as follows, and with the remark that we already know that the sequence on the right converges:

$$\rho(T) = \lim_{n\to\infty} ||T^n||^{1/n}$$

In one sense, we can use the polynomial calculus formula $\sigma(T^n) = \sigma(T)^n$. Indeed, this gives the following estimate, valid for any n, as desired:

$$\begin{aligned} \rho(T) &= \sup_{\lambda\in\sigma(T)} |\lambda| \\ &= \sup_{\rho\in\sigma(T)^n} |\rho|^{1/n} \\ &= \sup_{\rho\in\sigma(T^n)} |\rho|^{1/n} \\ &= \rho(T^n)^{1/n} \\ &\leq ||T^n||^{1/n} \end{aligned}$$

(4) For the reverse inequality, we fix a number $\rho > \rho(T)$, and we want to prove that we have $\rho \geq \lim_{n\to\infty} ||T^n||^{1/n}$. By using the Cauchy formula, we have:

$$\begin{aligned} \frac{1}{2\pi i}\int_{|z|=\rho} \frac{z^n}{z-T}\,dz &= \frac{1}{2\pi i}\int_{|z|=\rho} \sum_{k=0}^{\infty} z^{n-k-1}T^k\,dz \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi i}\left(\int_{|z|=\rho} z^{n-k-1}dz\right)T^k \\ &= \sum_{k=0}^{\infty} \delta_{n,k+1}T^k \\ &= T^{n-1} \end{aligned}$$

By applying the norm we obtain from this formula:

$$||T^{n-1}|| \leq \frac{1}{2\pi}\int_{|z|=\rho} \left|\left|\frac{z^n}{z-T}\right|\right| dz \leq \rho^n \cdot \sup_{|z|=\rho} \left|\left|\frac{1}{z-T}\right|\right|$$

Since the sup does not depend on n, by taking n-th roots, we obtain in the limit:

$$\rho \geq \lim_{n\to\infty} ||T^n||^{1/n}$$

Now recall that ρ was by definition an arbitrary number satisfying $\rho > \rho(T)$. Thus, we have obtained the following estimate, valid for any $T \in B(H)$:

$$\rho(T) \geq \lim_{n\to\infty} ||T^n||^{1/n}$$

Thus, we are led to the conclusion in the statement. □

In the case of the normal elements, we have the following finer result:

THEOREM 1.18. *The spectral radius of a normal element,*

$$TT^* = T^*T$$

is equal to its norm.

PROOF. We can proceed in two steps, as follows:

Step 1. In the case $T = T^*$ we have $||T^n|| = ||T||^n$ for any exponent of the form $n = 2^k$, by using the formula $||TT^*|| = ||T||^2$, and by taking n-th roots we get:

$$\rho(T) \geq ||T||$$

Thus, we are done with the self-adjoint case, with the result $\rho(T) = ||T||$.

Step 2. In the general normal case $TT^* = T^*T$ we have $T^n(T^n)^* = (TT^*)^n$, and by using this, along with the result from Step 1, applied to TT^*, we obtain:

$$\begin{aligned}
\rho(T) &= \lim_{n\to\infty} ||T^n||^{1/n} \\
&= \sqrt{\lim_{n\to\infty} ||T^n(T^n)^*||^{1/n}} \\
&= \sqrt{\lim_{n\to\infty} ||(TT^*)^n||^{1/n}} \\
&= \sqrt{\rho(TT^*)} \\
&= \sqrt{||T||^2} \\
&= ||T||
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

1c. Normal operators

By using Theorem 1.18 we can say a number of non-trivial things about the normal operators, commonly known as "spectral theorem for normal operators". As a first result here, we can improve the polynomial functional calculus formula, as follows:

THEOREM 1.19. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$\mathbb C[X]\to B(H)\quad,\quad P\to P(T)$$

having the properties $||P(T)||=||P_{|\sigma(T)}||$, and $\sigma(P(T))=P(\sigma(T))$.

PROOF. This is an improvement of Theorem 1.8 in the normal case, with the extra assertion being the norm estimate. But the element $P(T)$ being normal, we can apply to it the spectral radius formula for normal elements, and we obtain:

$$\begin{aligned}||P(T)|| &= \rho(P(T))\\ &= \sup_{\lambda\in\sigma(P(T))}|\lambda|\\ &= \sup_{\lambda\in P(\sigma(T))}|\lambda|\\ &= ||P_{|\sigma(T)}||\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

We can improve as well the rational calculus formula, and the holomorphic calculus formula, in the same way. Importantly now, at a more advanced level, we have:

THEOREM 1.20. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$C(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

which is isometric, $||f(T)||=||f||$, and has the property $\sigma(f(T))=f(\sigma(T))$.

PROOF. The idea here is to "complete" the morphism in Theorem 1.19, namely:

$$\mathbb C[X]\to B(H)\quad,\quad P\to P(T)$$

Indeed, we know from Theorem 1.19 that this morphism is continuous, and is in fact isometric, when regarding the polynomials $P\in\mathbb C[X]$ as functions on $\sigma(T)$:

$$||P(T)||=||P_{|\sigma(T)}||$$

Thus, by Stone-Weierstrass, we have a unique isometric extension, as follows:

$$C(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

It remains to prove $\sigma(f(T))=f(\sigma(T))$, and we can do this by double inclusion:

"$\subset$" Given a continuous function $f\in C(\sigma(T))$, we must prove that we have:

$$\lambda\notin f(\sigma(T))\implies\lambda\notin\sigma(f(T))$$

For this purpose, consider the following function, which is well-defined:

$$\frac{1}{f-\lambda}\in C(\sigma(T))$$

We can therefore apply this function to T, and we obtain:

$$\left(\frac{1}{f-\lambda}\right)T=\frac{1}{f(T)-\lambda}$$

In particular $f(T)-\lambda$ is invertible, so $\lambda\notin\sigma(f(T))$, as desired.

"$\supset$" Given a continuous function $f\in C(\sigma(T))$, we must prove that we have:

$$\lambda\in f(\sigma(T))\implies\lambda\in\sigma(f(T))$$

But this is the same as proving that we have:

$$\mu\in\sigma(T)\implies f(\mu)\in\sigma(f(T))$$

For this purpose, we approximate our function by polynomials, $P_n\to f$, and we examine the following convergence, which follows from $P_n\to f$:

$$P_n(T)-P_n(\mu)\to f(T)-f(\mu)$$

We know from polynomial functional calculus that we have:

$$P_n(\mu)\in P_n(\sigma(T))=\sigma(P_n(T))$$

Thus, the operators $P_n(T)-P_n(\mu)$ are not invertible. On the other hand, we know that the set formed by the invertible operators is open, so its complement is closed. Thus the limit $f(T)-f(\mu)$ is not invertible either, and so $f(\mu)\in\sigma(f(T))$, as desired. □

As an important comment, Theorem 1.20 is not exactly in final form, because it misses an important point, namely that our correspondence maps:

$$\bar{z}\to T^*$$

However, this is something non-trivial, and we will be back to this later. Observe however that Theorem 1.20 is fully powerful for the self-adjoint operators, $T=T^*$, where the spectrum is real, so where $z=\bar{z}$ on the spectrum. We will be back to this.

As a second result now, along the same lines, we can further extend Theorem 1.20 into a measurable functional calculus theorem, as follows:

THEOREM 1.21. *Given $T\in B(H)$ normal, we have a morphism of algebras as follows, with L^∞ standing for abstract measurable functions, or Borel functions,*

$$L^\infty(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

which is isometric, $||f(T)||=||f||$, and has the property $\sigma(f(T))=f(\sigma(T))$.

PROOF. As before, the idea will be that of "completing" what we have. To be more precise, we can use the Riesz theorem and a polarization trick, as follows:

(1) Given a vector $x\in H$, consider the following functional:

$$C(\sigma(T))\to\mathbb{C}\quad,\quad g\to<g(T)x,x>$$

By the Riesz theorem, this functional must be the integration with respect to a certain measure μ on the space $\sigma(T)$. Thus, we have a formula as follows:

$$< g(T)x, x >= \int_{\sigma(T)} g(z) d\mu(z)$$

Now given an arbitrary Borel function $f \in L^\infty(\sigma(T))$, as in the statement, we can define a number $< f(T)x, x >\in \mathbb{C}$, by using exactly the same formula, namely:

$$< f(T)x, x >= \int_{\sigma(T)} f(z) d\mu(z)$$

Thus, we have managed to define numbers $< f(T)x, x >\in \mathbb{C}$, for all vectors $x \in H$, and in addition we can recover these numbers as follows, with $g_n \in C(\sigma(T))$:

$$< f(T)x, x >= \lim_{g_n \to f} < g_n(T)x, x >$$

(2) In order to define now numbers $< f(T)x, y >\in \mathbb{C}$, for all vectors $x, y \in H$, we can use a polarization trick. Indeed, for any operator $S \in B(H)$ we have:

$$\begin{aligned} < S(x+y), x+y > &= < Sx, x > + < Sy, y > \\ &\quad + < Sx, y > + < Sy, x > \end{aligned}$$

By replacing $y \to iy$, we have as well the following formula:

$$\begin{aligned} < S(x+iy), x+iy > &= < Sx, x > + < Sy, y > \\ &\quad -i < Sx, y > +i < Sy, x > \end{aligned}$$

By multiplying this latter formula by i, we obtain the following formula:

$$\begin{aligned} i < S(x+iy), x+iy > &= i < Sx, x > +i < Sy, y > \\ &\quad + < Sx, y > - < Sy, x > \end{aligned}$$

Now by summing this latter formula with the first one, we obtain:

$$\begin{aligned} < S(x+y), x+y > +i < S(x+iy), x+iy > &= (1+i)[< Sx, x > + < Sy, y >] \\ &\quad +2 < Sx, y > \end{aligned}$$

(3) But with this, we can now finish. Indeed, by combining (1,2), given a Borel function $f \in L^\infty(\sigma(T))$, we can define numbers $< f(T)x, y >\in \mathbb{C}$ for any $x, y \in H$, and it is routine to check, by using approximation by continuous functions $g_n \to f$ as in (1), that we obtain in this way an operator $f(T) \in B(H)$, having all the desired properties. □

As a comment here, the above result and its proof provide us with more than a Borel functional calculus, because what we got is a certain measure on the spectrum $\sigma(T)$, along with a functional calculus for the L^∞ functions with respect to this measure. We will be back to this later, and for the moment we will only need Theorem 1.21 as formulated, with $L^\infty(\sigma(T))$ standing, a bit abusively, for the Borel functions on $\sigma(T)$.

1d. Diagonalization

Let us discuss now some useful decomposition results for the bounded linear operators $T \in B(H)$, that we can now establish, by using the above measurable calculus technology. We know that any $z \in \mathbb{C}$ can be written as follows, with $a, b \in \mathbb{R}$:

$$z = a + ib$$

Also, we know that both the real and imaginary parts $a, b \in \mathbb{R}$, and more generally any real number $c \in \mathbb{R}$, can be written as follows, with $r, s \geq 0$:

$$c = r - s$$

In order to discuss now the operator theoretic generalizations of these results, which by the way covers the usual matrix case too, let us start with the following basic fact:

THEOREM 1.22. *Any operator $T \in B(H)$ can be written as*

$$T = Re(T) + iIm(T)$$

with $Re(T), Im(T) \in B(H)$ being self-adjoint, and this decomposition is unique.

PROOF. This is something elementary, the idea being as follows:

(1) As a first observation, in the case $H = \mathbb{C}$ our operators are usual complex numbers, and the formula in the statement corresponds to the following basic fact:

$$z = Re(z) + iIm(z)$$

(2) In general now, we can use the same formulae for the real and imaginary part as in the complex number case, the decomposition formula being as follows:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

To be more precise, both the operators on the right are self-adjoint, and the summing formula holds indeed, and so we have our decomposition result, as desired.

(3) Regarding now the uniqueness, by linearity it is enough to show that $R + iS = 0$ with R, S both self-adjoint implies $R = S = 0$. But this follows by applying the adjoint to $R + iS = 0$, which gives $R - iS = 0$, and so $R = S = 0$, as desired. □

More generally now, as a continuation of this, and as an answer to some of the questions raised above, in relation with the complex numbers, we have the following result:

THEOREM 1.23. *Given an operator $T \in B(H)$, the following happen:*

1. *We can write $T = A + iB$, with $A, B \in B(H)$ being self-adjoint.*
2. *When $T = T^*$, we can write $T = R - S$, with $R, S \in B(H)$ being positive.*
3. *Thus, we can write any T as a linear combination of 4 positive elements.*

Proof. All this follows from basic spectral theory, as follows:

(1) This is something that we already know, from Theorem 1.22, with the decomposition formula there being something straightforward, as follows:

$$T=\frac{T+T^*}{2}+i\cdot\frac{T-T^*}{2i}$$

(2) This follows from the measurable functional calculus. Indeed, assuming $T=T^*$ we have $\sigma(T)\subset\mathbb{R}$, so we can use the following decomposition formula on $\mathbb{R}$:

$$1=\chi_{[0,\infty)}+\chi_{(-\infty,0)}$$

To be more precise, let us multiply by z, and rewrite this formula as follows:

$$z=\chi_{[0,\infty)}z-\chi_{(-\infty,0)}(-z)$$

Now by applying these measurable functions to T, we obtain as formula as follows, with both the operators $T_+,T_-\in B(H)$ being positive, as desired:

$$T=T_+-T_-$$

(3) This follows indeed by combining the results in (1) and (2) above. □

Going ahead with our decomposition results, another basic thing that we know about complex numbers is that any $z\in\mathbb{C}$ appears as a real multiple of a unitary:

$$z=re^{it}$$

Finding the correct operator theoretic analogue of this is quite tricky, and this even for the usual matrices $A\in M_N(\mathbb{C})$. As a basic result here, we have:

Theorem 1.24. *Given an operator $T\in B(H)$, the following happen:*

(1) *When $T=T^*$ and $||T||\leq 1$, we can write T as an average of 2 unitaries:*

$$T=\frac{U+V}{2}$$

(2) *In the general $T=T^*$ case, we can write T as a rescaled sum of unitaries:*

$$T=\lambda(U+V)$$

(3) *Thus, in general, we can write T as a rescaled sum of 4 unitaries.*

Proof. This follows from the results that we have, as follows:

(1) Assuming $T=T^*$ and $||T||\leq 1$ we have $1-T^2\geq 0$, and the decomposition that we are looking for is as follows, with both the components being unitaries:

$$T=\frac{T+i\sqrt{1-T^2}}{2}+\frac{T-i\sqrt{1-T^2}}{2}$$

To be more precise, the square root can be extracted by using the continuous functional calculus, and the check of the unitarity of the components goes as follows:

$$\begin{aligned}(T+i\sqrt{1-T^2})(T-i\sqrt{1-T^2}) &= T^2+(1-T^2)\\ &= 1\end{aligned}$$

(2) This simply follows by applying (1) to the operator $T/||T||$.

(3) Assuming first that we have $||T||\leq 1$, we know from Theorem 1.23 (1) that we can write $T=A+iB$, with A,B being self-adjoint, and satisfying $||A||,||B||\leq 1$. Now by applying (1) to both A and B, we obtain a decomposition of T as follows:

$$T=\frac{U+V+W+X}{2}$$

In general, we can apply this to the operator $T/||T||$, and we obtain the result. □

Good news, we can now diagonalize the normal operators. We will do this in 3 steps, first for the self-adjoint operators, then for the families of commuting self-adjoint operators, and finally for the general normal operators, by using the following trick:

$$T=Re(T)+iIm(T)$$

However, and coming somehow as bad news, all this will be quite technical. Indeed, the diagonalization in infinite dimensions is more tricky than in finite dimensions, and instead of writing a formula of type $T=UDU^*$, with $U,D\in B(H)$ being respectively unitary and diagonal, we will express our operator as $T=U^*MU$, with $U:H\to K$ being a certain unitary, and $M\in B(K)$ being a certain diagonal operator. The point indeed is that this is how the spectral theorem is used in practice, for concrete applications.

But probably too much talking, let us get to work. We first have:

Theorem 1.25. *Any self-adjoint operator $T\in B(H)$ can be diagonalized,*

$$T=U^*M_fU$$

with $U:H\to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to T, with $f:X\to\mathbb R$ being a certain function, once again associated to T, and with

$$M_f(g)=fg$$

being the usual multiplication operator by f, on the Hilbert space $L^2(X)$.

Proof. The construction of U,f can be done in several steps, as follows:

(1) We first prove the result in the special case where our operator T has a cyclic vector $x\in H$, with this meaning that the following holds:

$$\overline{span\left(T^kx\Big|n\in\mathbb N\right)}=H$$

For this purpose, let us go back to the proof of Theorem 1.21. We will use the following formula from there, with μ being the measure on $X=\sigma(T)$ associated to x:

$$<g(T)x,x>=\int_{\sigma(T)}g(z)d\mu(z)$$

Our claim is that we can define a unitary $U:H\to L^2(X)$, first on the dense part spanned by the vectors T^kx, by the following formula, and then by continuity:

$$U[g(T)x]=g$$

Indeed, the following computation shows that U is well-defined, and isometric:

$$\begin{aligned}||g(T)x||^2&=<g(T)x,g(T)x>\\&=<g(T)^*g(T)x,x>\\&=<|g|^2(T)x,x>\\&=\int_{\sigma(T)}|g(z)|^2d\mu(z)\\&=||g||_2^2\end{aligned}$$

We can then extend U by continuity into a unitary $U:H\to L^2(X)$, as claimed. Now observe that we have the following formula:

$$\begin{aligned}UTU^*g&=U[Tg(T)x]\\&=U[(zg)(T)x]\\&=zg\end{aligned}$$

Thus our result is proved in the present case, with U as above, and with $f(z)=z$.

(2) We discuss now the general case. Our first claim is that H has a decomposition as follows, with each H_i being invariant under T, and admitting a cyclic vector x_i:

$$H=\bigoplus_i H_i$$

Indeed, this is something elementary, the construction being by recurrence in finite dimensions, in the obvious way, and by using the Zorn lemma in general. Now with this decomposition in hand, we can make a direct sum of the diagonalizations obtained in (1), for each of the restrictions $T_{|H_i}$, and we obtain the formula in the statement. □

The above result is very nice, closing more or less the discussion regarding the self-adjoint operators. At the theoretical level, however, there are still a number of comments that can be made, about this, and we will be back to this, at the end of this chapter.

We have the following technical generalization of the above result:

THEOREM 1.26. *Any family of commuting self-adjoint operators $T_i \in B(H)$ can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to $\{T_i\}$, with $f_i : X \to \mathbb{R}$ being certain functions, once again associated to T_i, and with

$$M_{f_i}(g) = f_i g$$

being the usual multiplication operator by f_i, on the Hilbert space $L^2(X)$.

PROOF. This is similar to the proof of Theorem 1.25, by suitably modifying the measurable calculus formula, and μ itself, as to have this working for all operators T_i. □

We can now discuss the case of the arbitrary normal operators, as follows:

THEOREM 1.27. *Any normal operator $T \in B(H)$ can be diagonalized,*

$$T = U^* M_f U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to T, with $f : X \to \mathbb{C}$ being a certain function, once again associated to T, and with

$$M_f(g) = fg$$

being the usual multiplication operator by f, on the Hilbert space $L^2(X)$.

PROOF. This is our main diagonalization theorem, the idea being as follows:

(1) Consider the decomposition of T into its real and imaginary parts, namely:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

We know that the real and imaginary parts are self-adjoint operators. Now since T was assumed to be normal, $TT^* = T^*T$, these real and imaginary parts commute:

$$\left[\frac{T + T^*}{2}, \frac{T - T^*}{2i}\right] = 0$$

Thus Theorem 1.26 applies to these real and imaginary parts, and gives the result. □

This was for our series of diagonalization theorems. There is of course one more result here, regarding the families of commuting normal operators, as follows:

THEOREM 1.28. *Any family of commuting normal operators $T_i \in B(H)$ can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to $\{T_i\}$, with $f_i : X \to \mathbb{C}$ being certain functions, once again associated to T_i, and with

$$M_{f_i}(g) = f_i g$$

being the usual multiplication operator by f_i, on the Hilbert space $L^2(X)$.

PROOF. This is similar to the proof of Theorem 1.26 and Theorem 1.27, by combining the arguments there. To be more precise, this follows as Theorem 1.26, by using the decomposition trick from the proof of Theorem 1.27. □

With the above diagonalization results in hand, we can now "fix" the continuous and measurable functional calculus theorems, with a key complement, as follows:

THEOREM 1.29. *Given a normal operator $T \in B(H)$, the following hold, for both the functional calculus and the measurable calculus morphisms:*

(1) *These morphisms are $*$-morphisms.*
(2) *The function $\bar{z}$ gets mapped to T^*.*
(3) *The functions $Re(z), Im(z)$ get mapped to $Re(T), Im(T)$.*
(4) *The function $|z|^2$ gets mapped to $TT^* = T^*T$.*
(5) *If f is real, then $f(T)$ is self-adjoint.*

PROOF. These assertions are more or less equivalent, with (1) being the main one, which obviously implies everything else. But this assertion (1) follows from the diagonalization result for normal operators, from Theorem 1.27. □

1e. Exercises

Exercises:

EXERCISE 1.30.

EXERCISE 1.31.

EXERCISE 1.32.

EXERCISE 1.33.

EXERCISE 1.34.

EXERCISE 1.35.

EXERCISE 1.36.

EXERCISE 1.37.

Bonus exercise.

CHAPTER 2

C*-algebras

2a. Operator algebras

Good news, we can now talk about operator algebras. Let us start with the following broad definition, obtained by imposing the "minimal" set of reasonable axioms:

Definition 2.1. *An operator algebra is an algebra of bounded operators $A \subset B(H)$ which contains the unit, is closed under taking adjoints,*

$$T \in A \implies T^* \in A$$

and is closed as well under the norm.

Here, as before, H is an arbitrary Hilbert space, with the case that we are mostly interested in being the separable one. Also as before, $B(H)$ is the algebra of linear operators $T : H \to H$ which are bounded, in the sense that $||T|| = \sup_{||x||=1} ||Tx||$ is finite. This algebra has an involution $T \to T^*$, with the adjoint operator $T^* \in B(H)$ being defined by the formula $< Tx, y >=< x, T^*y >$, and in the above definition, the assumption $T \in A \implies T^* \in A$ refers to this involution. Thus, A must be a $*$-algebra.

As a first result now regarding the operator algebras, in relation with the normal operators, where most of the non-trivial results that we have so far are, we have:

Theorem 2.2. *The operator algebra $< T >\subset B(H)$ generated by a normal operator $T \in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(\sigma(T))$$

where $\sigma(T) \subset \mathbb{C}$ denotes as usual the spectrum of T.

Proof. We know that we have a continuous morphism of $*$-algebras, as follows:

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the continuous calculus, also established in the above, this morphism is injective, and its image is the norm closed algebra $< T >$ generated by T, T^*. Thus, we obtain the isomorphism in the statement. □

The above result is very nice, and it is possible to further build on it, as follows:

THEOREM 2.3. *The operator algebra $< T_i >\subset B(H)$ generated by a family of normal operators $T_i\in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(X)$$

where $X\subset\mathbb C$ is a certain compact space associated to the family $\{T_i\}$. Equivalently, any commutative operator algebra $A\subset B(H)$ is of the form $A=C(X)$.

PROOF. We have two assertions here, the idea being as follows:

(1) Regarding the first assertion, this follows exactly as in the proof of Theorem 2.2, by using this time the spectral theorem for families of normal operators.

(2) As for the second assertion, this is clear from the first one, because any commutative algebra $A\subset B(H)$ is generated by its elements $T\in A$, which are all normal. □

All this is good to know, but Theorem 2.2 and Theorem 2.3 remain something quite heavy, based on the spectral theorem. We would like to present now an alternative proof for these results, which is rather elementary, and has the advantage of reconstructing the compact space X directly from the knowledge of the algebra A. We will need:

THEOREM 2.4. *Given an operator $T\in A\subset B(H)$, define its spectrum as:*

$$\sigma(T)=\left\{\lambda\in\mathbb C\Big|T-\lambda\notin A^{-1}\right\}$$

The following spectral theory results hold, exactly as in the $A=B(H)$ case:
1. *We have $\sigma(ST)\cup\{0\}=\sigma(TS)\cup\{0\}$.*
2. *We have polynomial, rational and holomorphic calculus.*
3. *As a consequence, the spectra are compact and non-empty.*
4. *The spectra of unitaries ($U^*=U^{-1}$) and self-adjoints ($T=T^*$) are on $\mathbb T,\mathbb R$.*
5. *The spectral radius of normal elements ($TT^*=T^*T$) is given by $\rho(T)=||T||$.*

In addition, assuming $T\in A\subset B$, the spectra of T with respect to A and to B coincide.

PROOF. This is something that we know well from chapter 1, in the case $A=B(H)$. In general the proof is similar, the idea being as follows:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra $A=B(H)$. All this is standard material, and in fact, things in chapter 1 were written in such a way as for their extension now, to the general operator algebra setting, to be obvious.

(2) Regarding the last assertion, the inclusion $\sigma_B(T)\subset\sigma_A(T)$ is clear. For the converse, assume $T-\lambda\in B^{-1}$, and consider the following self-adjoint element:

$$S=(T-\lambda)^*(T-\lambda)$$

The difference between the two spectra of $S\in A\subset B$ is then given by:

$$\sigma_A(S)-\sigma_B(S)=\left\{\mu\in\mathbb C-\sigma_B(S)\Big|(S-\mu)^{-1}\in B-A\right\}$$

Thus this difference in an open subset of $\mathbb{C}$. On the other hand S being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of S are equal, and in particular S is invertible in A, and so $T - \lambda \in A^{-1}$, as desired.

(3) As an observation, the last assertion applied with $B = B(H)$ shows that the spectrum $\sigma(T)$ as constructed in the statement coincides with the spectrum $\sigma(T)$ as constructed and studied before, so the fact that (1-5) hold indeed is no surprise.

(4) Finally, I can hear you screaming that I should have concieved this book differently, matter of not proving the same things twice. Good point, with my distinguished colleague Bourbaki saying the same, and in answer, wait for the next section, where we will prove exactly the same things a third time. We can discuss pedagogy at that time. □

We can now get back to the commutative algebras, and we have the following result, due to Gelfand, which provides an alternative to Theorem 2.2 and Theorem 2.3:

Theorem 2.5. *Any commutative operator algebra $A \subset B(H)$ is of the form*

$$A = C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi : A \to \mathbb{C}$, with topology making continuous the evaluation maps $ev_T : \chi \to \chi(T)$.

Proof. Given a commutative operator algebra A, we can define X as in the statement. Then X is compact, and $T \to ev_T$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for the usual complex numbers:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

Thus it is enough to prove the equality $ev_{T^*} = ev_T^*$ for self-adjoint elements T. But this is the same as proving that $T = T^*$ implies that ev_T is a real function, which is in turn true, because $ev_T(\chi) = \chi(T)$ is an element of $\sigma(T)$, contained in $\mathbb{R}$.

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_T|| = \rho(T) = ||T||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The above theorem of Gelfand is something very beautiful, and far-reaching. It is possible to further build on it, indefinitely high. We will be back to this, later.

2b. C*-algebras

We have been talking so far about the general operator $*$-algebras $A \subset B(H)$, closed with respect to the norm. But this suggests formulating the following definition:

Definition 2.6. *A C^*-algebra is an complex algebra A, given with:*

(1) *A norm $a \to ||a||$, making it into a Banach algebra.*
(2) *An involution $a \to a^*$, related to the norm by the formula $||aa^*|| = ||a||^2$.*

Here by Banach algebra we mean a complex algebra with a norm satisfying all the conditions for a vector space norm, along with $||ab|| \leq ||a|| \cdot ||b||$ and $||1|| = 1$, and which is such that our algebra is complete, in the sense that the Cauchy sequences converge. As for the involution, this must be antilinear, antimultiplicative, and satisfying $a^{**} = a$.

As basic examples, we have the operator algebra $B(H)$, for any Hilbert space H, and more generally, the norm closed $*$-subalgebras $A \subset B(H)$. It is possible to prove that any C^*-algebra appears in this way, but this is a non-trivial result, called GNS theorem, and more on this later. Note in passing that this result tells us that there is no need to memorize the above axioms for the C^*-algebras, because these are simply the obvious things that can be said about $B(H)$, and its norm closed $*$-subalgebras $A \subset B(H)$.

As a second class of basic examples, which are of great interest for us, we have:

Proposition 2.7. *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a C^*-algebra, with the usual norm and involution, namely:*

$$||f|| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

This algebra is commutative, in the sense that $fg = gf$, for any $f, g \in C(X)$.

Proof. All this is clear from definitions. Observe that we have indeed:

$$||ff^*|| = \sup_{x \in X} |f(x)|^2 = ||f||^2$$

Thus, the axioms are satisfied, and finally $fg = gf$ is clear. □

In general, the C^*-algebras can be thought of as being algebras of operators, over some Hilbert space which is not present. By using this philosophy, one can emulate spectral theory in this setting, with extensions of our previous results. Let us start with:

Definition 2.8. *Given element $a \in A$ of a C^*-algebra, define its spectrum as:*

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \Big| a - \lambda \notin A^{-1} \right\}$$

Also, we call spectral radius of $a \in A$ the number $\rho(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$.

In what regards the examples, for $A=B(H)$ what we have here is the usual notion of spectrum, from chapter 1. More generally, as explained in Theorem 2.4, in the case $A\subset B(H)$ we obtain the same spectra as those in the case $A=B(H)$. Finally, in the case $A=C(X)$, as in Proposition 2.7, the spectrum of a function is its image:

$$\sigma(f)=Im(f)$$

Now with the above notion of spectrum in hand, we have the following result:

Theorem 2.9. *The following results hold, exactly as in the $A\subset B(H)$ case:*

(1) *We have $\sigma(ab)\cup\{0\}=\sigma(ba)\cup\{0\}$.*
(2) *We have polynomial, rational and holomorphic calculus.*
(3) *As a consequence, the spectra are compact and non-empty.*
(4) *The spectra of unitaries ($u^*=u^{-1}$) and self-adjoints ($a=a^*$) are on $\mathbb{T},\mathbb{R}$.*
(5) *The spectral radius of normal elements ($aa^*=a^*a$) is given by $\rho(a)=||a||$.*

In addition, assuming $a\in A\subset B$, the spectra of a with respect to A and to B coincide.

Proof. This is something that we know from chapter 1, in the case $A=B(H)$, and then from this chapter, in the case $A\subset B(H)$. In general, the proof is similar:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra $A=B(H)$. All this is standard material, and in fact, things before were written in such a way as for their extension now, to the general C^*-algebra setting, to be obvious.

(2) Regarding the last assertion, we know this from before for $A\subset B\subset B(H)$, and the proof in general is similar. Indeed, the inclusion $\sigma_B(a)\subset\sigma_A(a)$ is clear. For the converse, assume $a-\lambda\in B^{-1}$, and consider the following self-adjoint element:

$$b=(a-\lambda)^*(a-\lambda)$$

The difference between the two spectra of $b\in A\subset B$ is then given by:

$$\sigma_A(b)-\sigma_B(b)=\left\{\mu\in\mathbb{C}-\sigma_B(b)\Big|(b-\mu)^{-1}\in B-A\right\}$$

Thus this difference in an open subset of $\mathbb{C}$. On the other hand b being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of b are equal, and in particular b is invertible in A, and so $a-\lambda\in A^{-1}$, as desired. □

We can get back now to the commutative algebras, and we have the following result, due to Gelfand, which will be of crucial importance for us:

Theorem 2.10. *The commutative C^*-algebras are exactly the algebras of the form*

$$A=C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi:A\to\mathbb{C}$, with topology making continuous the evaluation maps $ev_a:\chi\to\chi(a)$.

PROOF. This is something that we basically know from before, but always good to talk about it again. Given a commutative C^*-algebra A, we can define X as in the statement. Then X is compact, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for the usual complex numbers:

$$a = \frac{a + a^*}{2} + i \cdot \frac{a - a^*}{2i}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

In view of the Gelfand theorem, we can formulate the following key definition:

DEFINITION 2.11. *Given an arbitrary C^*-algebra A, we write*

$$A = C(X)$$

and call X a compact quantum space.

This might look like something informal, but it is not. Indeed, we can define the category of compact quantum spaces to be the category of the C^*-algebras, with the arrows reversed. When A is commutative, the above space X exists indeed, as a Gelfand spectrum, $X = Spec(A)$. In general, X is something rather abstract, and our philosophy here will be that of studying of course A, but formulating our results in terms of X. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

Let us also mention that, technically speaking, we will see later that the above formalism has its limitations, and needs a fix. But more on this later.

As a first concrete consequence now of the Gelfand theorem, we have:

THEOREM 2.12. *Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.*

1. *We can define $f(a) \in A$, with $f \to f(a)$ being a morphism of C^*-algebras.*
2. *We have the "continuous functional calculus" formula $\sigma(f(a)) = f(\sigma(a))$.*

PROOF. Since a is normal, the C^*-algebra $< a >$ that is generates is commutative, so if we denote by X the space formed by the characters $\chi :< a >\to \mathbb{C}$, we have:

$$< a >= C(X)$$

Now since the map $X \to \sigma(a)$ given by evaluation at a is bijective, we obtain:

$$< a >= C(\sigma(a))$$

Thus, we are dealing with usual functions, and this gives all the assertions. □

As another consequence of the Gelfand theorem, we have:

THEOREM 2.13. *For a normal element $a \in A$, the following are equivalent:*
1. *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
2. *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
3. *$a = cc^*$, for some $c \in A$.*

PROOF. This is very standard, exactly as in $A = B(H)$ case, as follows:

(1) $\Longrightarrow$ (2) Since $f(z) = \sqrt{z}$ is well-defined on $\sigma(a) \subset [0, \infty)$, we can set $b = \sqrt{a}$.

(2) $\Longrightarrow$ (3) This is trivial, because we can set $c = b$.

(3) $\Longrightarrow$ (1) We proceed by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting the fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, that we know to hold for $A = B(H)$, and whose proof in general is similar. □

2c. Basic results

In order to develop some general theory, let us start by investigating the finite dimensional case. Here the ambient algebra is $B(H) = M_N(\mathbb{C})$, any linear subspace $A \subset B(H)$ is automatically closed, for the norm topology, and we have the following result:

THEOREM 2.14. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $n_1, \ldots, n_k \in \mathbb{N}$ satisfying $n_1 + \ldots + n_k = N$, we have indeed an obvious embedding of $*$-algebras, via matrix blocks, as follows:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \to UMU^*$$

(2) In the other sense now, consider a $*$-algebra $A \subset M_N(\mathbb{C})$. It is elementary to prove that the center $Z(A) = A \cap A'$, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \ldots, e_k \in \mathbb{C}^k$, and let $p_1, \ldots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \ldots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \ldots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2 = p_i^* = p_i$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \ldots + p_k = 1$ and the projection equations $p_i^2 = p_i^* = p_i$, we conclude that we have the needed generation property, namely:

$$A_1 + \ldots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \ldots + p_k = 1$, and from the projection equations $p_i^2 = p_i^* = p_i$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i = rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{n_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Moreover, a more careful look at the isomorphisms established in Step 3 shows that at the global level, that of the algebra A itself, the above isomorphism simply comes by twisting the following standard multimatrix embedding, discussed in the beginning of the proof, (1) above, by a certain unitary matrix $U \in U_N$:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

Now by putting everything together, we obtain the result. □

In terms of our usual C^*-algebra formalism, the above result tells us that we have:

THEOREM 2.15. *The finite dimensional C^*-algebras are exactly the algebras*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with norm $||(a_1, \ldots, a_k)|| = \sup_i ||a_i||$, and involution $(a_1, \ldots, a_k)^* = (a_1^*, \ldots, a_k^*)$.

PROOF. This is indeed a reformulation of what we know from Theorem 2.14, in terms of our usual C^*-algebra formalism, from Definition 2.6. □

Let us record as well the quantum space formulation of our result:

THEOREM 2.16. *The finite quantum spaces are exactly the disjoint unions of type*

$$X = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

where M_n is the finite quantum space given by $C(M_n) = M_n(\mathbb{C})$.

PROOF. This is a reformulation of Theorem 2.15, by using the quantum space philosophy. Indeed, for a compact quantum space X, coming from a C^*-algebra A via the formula $A = C(X)$, being finite can only mean that the following number is finite:

$$|X| = \dim_{\mathbb{C}} A < \infty$$

Thus, by using Theorem 2.15, we are led to the conclusion that we must have:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

But since direct sums of algebras A correspond to disjoint unions of quantum spaces X, via the correspondence $A = C(X)$, this leads to the conclusion in the statement. □

As a first application now of Theorem 2.15, we have the following result:

THEOREM 2.17. *Consider a $*$-algebra $A \subset M_N(\mathbb{C})$, written as above:*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The commutant of this algebra is then, with respect with the block decomposition used,

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

and by taking one more time the commutant we obtain A itself, $A = A''$.

PROOF. Let us decompose indeed our algebra A as in Theorem 2.15:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb{C}$ corresponding to a matrix block:

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain A itself, and we are done. $\square$

As another interesting application of Theorem 2.15, clarifying this time the relation with operator theory, in finite dimensions, we have the following result:

THEOREM 2.18. *Given an operator $T \in B(H)$ in finite dimensions, $H = \mathbb{C}^N$, the operator algebra $A = < T >$ that it generates inside $B(H) = M_N(\mathbb{C})$ is*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with the sizes of the blocks $n_1, \ldots, n_k \in \mathbb{N}$ coming from the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$. In the normal case $TT^* = T^*T$, this decomposition comes from

$$T = UDU^*$$

with $D \in M_N(\mathbb{C})$ diagonal, and with $U \in U_N$ unitary.

PROOF. This is something which is routine, by using basic linear algebra:

(1) The fact that $A = < T >$ decomposes into a direct sum of matrix algebras is something that we already know, coming from Theorem 2.15.

(2) By using standard linear algebra, we can compute the block sizes $n_1, \ldots, n_k \in \mathbb{N}$, from the knowledge of the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$.

(3) In the normal case, $TT^* = T^*T$, we can simply invoke the spectral theorem, and by suitably changing the basis, we are led to the conclusion in the statement. $\square$

Let us discuss now a key result, called GNS representation theorem, stating that any C^*-algebra appears as an operator algebra. As a first result here, we have:

PROPOSITION 2.19. *Let A be a commutative C^*-algebra, write $A = C(X)$, with X being a compact space, and let μ be a positive measure on X. We have then*

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to the operator $g \to fg$.

PROOF. Given a continuous function $f \in C(X)$, consider the operator $T_f(g) = fg$, on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^*-algebra embedding $A \subset B(H)$, as claimed. □

In order to prove the GNS representation theorem, we must extend the above construction, to the case where A is not necessarily commutative. Let us start with:

DEFINITION 2.20. *Consider a C^*-algebra A.*

(1) *$\varphi : A \to \mathbb{C}$ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.*
(2) *$\varphi : A \to \mathbb{C}$ is called faithful and positive when $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.*

In the commutative case, $A = C(X)$, the positive elements are the positive functions, $f : X \to [0, \infty)$. As for the positive linear forms $\varphi : A \to \mathbb{C}$, these appear as follows, with μ being positive, and strictly positive if we want φ to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying "positive measures". We can use them as follows:

PROPOSITION 2.21. *Let $\varphi : A \to \mathbb{C}$ be a positive linear form.*

(1) *$< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.*
(2) *By separating and completing we obtain a Hilbert space H.*
(3) *$\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.*
(4) *If φ is faithful in the above sense, then π is faithful.*

PROOF. Almost everything here is straightforward, as follows:

(1) This is clear from definitions, and from the basic properties of the positive elements $a \geq 0$, which can be established exactly as in the $A = B(H)$ case.

(2) This is a standard procedure, which works for any scalar product, the idea being that of dividing by the vectors satisfying $< x, x >= 0$, then completing.

(3) All the verifications here are standard algebraic computations, in analogy with what we have seen many times, for multiplication operators, or group algebras.

(4) Assuming that we have $a \neq 0$, we have then $\pi(aa^*) \neq 0$, which in turn implies by faithfulness that we have $\pi(a) \neq 0$, which gives the result. □

In order to establish the embedding theorem, it remains to prove that any C^*-algebra has a faithful positive linear form $\varphi : A \to \mathbb{C}$. This is something more technical:

PROPOSITION 2.22. *Let A be a C^*-algebra.*

1. *Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.*
2. *A linear form φ is positive iff there is a norm one $h \in A_+$ such that $||\varphi|| = \varphi(h)$.*
3. *For any $a \in A$ there exists a positive norm one form φ such that $\varphi(aa^*) = ||a||^2$.*
4. *If A is separable there is a faithful positive form $\varphi : A \to \mathbb{C}$.*

PROOF. The proof here is quite technical, inspired from the existence proof of the probability measures on abstract compact spaces, the idea being as follows:

(1) This follows from Proposition 2.21, via the following estimate:

$$|\varphi(a)| \leq ||\pi(a)||\varphi(1) \leq ||a||\varphi(1)$$

(2) In one sense we can take $h = 1$. Conversely, let $a \in A_+$, $||a|| \leq 1$. We have:

$$|\varphi(h) - \varphi(a)| \leq ||\varphi|| \cdot ||h - a|| \leq \varphi(h)$$

Thus we have $Re(\varphi(a)) \geq 0$, and with $a = 1 - h$ we obtain:

$$Re(\varphi(1 - h)) \geq 0$$

Thus $Re(\varphi(1)) \geq ||\varphi||$, and so $\varphi(1) = ||\varphi||$, so we can assume $h = 1$. Now observe that for any self-adjoint element a, and any $t \in \mathbb{R}$ we have, with $\varphi(a) = x + iy$:

$$\begin{aligned}
\varphi(1)^2(1 + t^2||a||^2) &\geq \varphi(1)^2||1 + t^2a^2|| \\
&= ||\varphi||^2 \cdot ||1 + ita||^2 \\
&\geq |\varphi(1 + ita)|^2 \\
&= |\varphi(1) - ty + itx| \\
&\geq (\varphi(1) - ty)^2
\end{aligned}$$

Thus we have $y = 0$, and this finishes the proof of our remaining claim.

(3) We can set $\varphi(\lambda aa^*) = \lambda||a||^2$ on the linear space spanned by aa^*, then extend this functional by Hahn-Banach, to the whole A. The positivity follows from (2).

(4) This is standard, by starting with a dense sequence (a_n), and taking the Cesàro limit of the functionals constructed in (3). We have $\varphi(aa^*) > 0$, and we are done. □

With these ingredients in hand, we can now state and prove:

THEOREM 2.23. *Any C^*-algebra appears as a norm closed $*$-algebra of operators*

$$A \subset B(H)$$

over a certain Hilbert space H. When A is separable, H can be taken to be separable.

PROOF. This result, called called GNS representation theorem after Gelfand, Naimark and Segal, follows indeed by combining Proposition 2.21 with Proposition 2.22. □

2d. Weak closures

Instead of further building on the above results, which are already quite non-trivial, let us return to our modest status of apprentice operator algebraists, and declare ourselves rather unsatisfied with what we have, on the following intuitive grounds:

THOUGHT 2.24. *Our assumption that $A\subset B(H)$ is norm closed is not satisfying, because we would like A to be stable under polar decomposition, under taking spectral projections, and more generally, under measurable functional calculus.*

So, let us get now into this, topologies on $B(H)$, and fine-tunings of what we have, based on them. The result that we will need, which is elementary, is as follows:

PROPOSITION 2.25. *For an operator algebra $A\subset B(H)$, the following are equivalent:*

1. *A is closed under the weak operator topology, making each of the linear maps $T\to<Tx,y>$ continuous.*
2. *A is closed under the strong operator topology, making each of the linear maps $T\to Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

PROOF. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S\subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A\subset B(H)$, strongly closed implies weakly closed.

(2) But this latter fact is something standard, which can be proved via an amplification trick. Consider the Hilbert space obtained by summing n times H with itself:

$$K=H\oplus\ldots\oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi:B(H)\to B(K)$, as follows:

$$\pi(T)=\begin{pmatrix}T&&\\&\ddots&\\&&T\end{pmatrix}$$

Assume now that we are given an operator $T\in\bar{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x\in K$:

$$\begin{aligned}T\in\bar{A} &\implies \pi(T)\in\overline{\pi(A)}\\ &\implies \pi(T)x\in\overline{\pi(A)x}\\ &\implies \pi(T)x\in\overline{\pi(A)x}^{||.||}\end{aligned}$$

Now observe that the last formula tells us that for any $x=(x_1,\ldots,x_n)$, and any $\varepsilon>0$, we can find $S\in A$ such that the following holds, for any i:

$$||Sx_i-Tx_i||<\varepsilon$$

Thus T belongs to the strong operator closure of A, as desired. □

In the above the terminology, while standard, is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution, we agree in what follows to call the norm topology "strong", and the weak and strong operator topologies "weak", whenever these two topologies coincide. With this convention, the algebras from Proposition 2.25 are those which are weakly closed, and we can formulate:

DEFINITION 2.26. *A von Neumann algebra is a $*$-algebra of operators*

$$A\subset B(H)$$

which is closed under the weak topology.

As basic examples, we have the algebra $B(H)$ itself, then the singly generated von Neumann algebras, $A=<T>$, with $T\in B(H)$, and then the multiply generated von Neumann algebras, namely $A=<T_i>$, with $T_i\in B(H)$. At the level of the general results, we first have the bicommutant theorem of von Neumann, as follows:

THEOREM 2.27. *For a $*$-algebra $A\subset B(H)$, the following are equivalent:*

1. *A is weakly closed, so it is a von Neumann algebra.*
2. *A equals its algebraic bicommutant A'', taken inside $B(H)$.*

PROOF. Since the commutants are automatically weakly closed, it is enough to show that weakly closed implies $A=A''$. For this purpose, we will prove something a bit more general, stating that given a $*$-algebra of operators $A\subset B(H)$, the following holds, with A'' being the bicommutant inside $B(H)$, and with $\bar{A}$ being the weak closure:

$$A''=\bar{A}$$

We prove this equality by double inclusion, as follows:

"$\supset$" Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S\subset S''$, valid for any set $S\subset B(H)$. In particular, we have:

$$A\subset A''$$

Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \to T$ in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS,\ \forall S \in A' \\ &\implies ST = TS,\ \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 2.25, which allows us to pass from the strong to the weak operator topology, this gives the desired inclusion:

$$\bar{A} \subset A''$$

"$\subset$" Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 2.25. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \Bigg| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \ldots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \ldots & S_{nn} \end{pmatrix} \Bigg| S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Now since this holds for any $x \in K$, we conclude that any $T \in A''$ belongs to the strong operator closure of A. By using now Proposition 2.25, which allows us to pass from the strong to the weak operator closure, we conclude that we have $A'' \subset \bar{A}$, as desired. □

In order to develop now some general theory, let us start by investigating the commutative case. A first result here, that we basically already know, is as follows:

THEOREM 2.28. *Given an operator $T \in B(H)$ which is normal,*

$$TT^* = T^*T$$

the von Neumann algebra $A =< T >$ that it generates inside $B(H)$ is

$$< T >= L^\infty(\sigma(T))$$

with $\sigma(T)$ being its spectrum, formed of numbers $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not invertible.

PROOF. This is something which is very standard, by using the spectral theory for the normal operators $T \in B(H)$, coming from chapter 1. □

More generally now, along the same lines, we have the following result:

THEOREM 2.29. *Given operators $T_i \in B(H)$ which are normal, and which commute, the von Neumann algebra $A =< T_i >$ that these operators generates inside $B(H)$ is*

$$< T_i >= L^\infty(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

PROOF. This is again routine, by using this time the spectral theory for the families of commuting normal operators $T_i \in B(H)$, that we know from chapter 1 too. □

As an interesting abstract consequence of this, we have:

THEOREM 2.30. *The commutative von Neumann algebras are the algebras of type*

$$A = L^\infty(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X, we can realize the commutative algebra $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H. But this is something that we already know, coming from the multiplicity operators $T_f(g) = fg$ from the proof of the GNS theorem, the representation being as follows:

$$L^\infty(X) \subset B(L^2(X))$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X, and an identification $A = L^\infty(X)$. But this follows from Theorem 2.29, because we can write our algebra as follows:

$$A =< T_i >$$

To be more precise, A being commutative, any element $T \in A$ is normal. Thus, we can pick a basis $\{T_i\} \subset A$, and then we have $A =< T_i >$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 2.29 applies, and gives the result. □

In relation now with our noncommutative geometry questions, as a first application of the above, we can extend our noncommutative space setting, as follows:

THEOREM 2.31. *Consider the category of "noncommutative measure spaces", having as objects the pairs (A, tr) consisting of a von Neumann algebra with a faithful trace, and with the arrows reversed, which amounts in writing $A = L^\infty(X)$ and $tr = \int_X$.*

1. *The category of usual measured spaces embeds into this category, and we obtain in this way the objects whose associated von Neumann algebra is commutative.*
2. *Each C^*-algebra given with a trace produces as well a noncommutative measure space, by performing the GNS construction, and taking the weak closure.*

PROOF. This is clear indeed from the basic properties of the GNS construction for the C^*-algebras, and from the basic properties of the von Neumann algebras. □

Moving ahead now with more theory for the von Neumann algebras, there is a long story here, and we are led in this way to the following statement:

THEOREM 2.32. *Given a von Neumann algebra $A \subset B(H)$, if we write its center as*

$$Z(A) = L^\infty(X)$$

then we have a decomposition as follows, with the fibers A_x having trivial center:

$$A = \int_X A_x \, dx$$

Moreover, the factors, $Z(A) = \mathbb{C}$, can be basically classified in terms of the II_1 factors, which are those satisfying $\dim A = \infty$, and having a faithful trace $tr : A \to \mathbb{C}$.

PROOF. This is something that we know to hold in finite dimensions, as a consequence of Theorem 2.14. In general, this is something heavy, the idea being as follows:

(1) This is von Neumann's reduction theory main result, whose statement is already quite hard to understand, and whose proof uses advanced functional analysis.

(2) This is heavy, due to Murray-von Neumann and Connes, the idea being that the other factors can be basically obtained via crossed product constructions. □

We will be back to this in chapter 4, when systematically doing functional analysis.

2e. Exercises

Exercises:

EXERCISE 2.33.

EXERCISE 2.34.

EXERCISE 2.35.

EXERCISE 2.36.

EXERCISE 2.37.

EXERCISE 2.38.

EXERCISE 2.39.

EXERCISE 2.40.

Bonus exercise.

CHAPTER 3

Basic examples

3a. Group algebras

Let us discuss now some basic examples of C^*-algebras. We first have:

THEOREM 3.1. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^* = g^{-1}$.*

1. *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
2. *When Γ is abelian, we have an isomorphism $C^*(\Gamma) \simeq C(G)$, where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.*

PROOF. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma] \subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h) = gh$ that $g \to \pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g \in \Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A = C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A = C(X)$, with:

$$X = Spec(A)$$

But the spectrum $X = Spec(A)$, consisting of the characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result. □

The above result suggests the following definition:

DEFINITION 3.2. *Given a discrete group Γ, the compact quantum space G given by*

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

With this definition in hand, we can now talk about quantum tori, as follows:

THEOREM 3.3. *The basic tori are all group duals, as follows,*

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix} \qquad = \qquad \begin{matrix} \widehat{L_N} & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N \end{matrix}$$

where $F_N = \mathbb{Z}^{*N}$ is the free group on N generators, and $L_N = \mathbb{Z}_2^{*N}$ is its real version.

PROOF. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 3.1 (2), this gives the result. □

Moving ahead, now that we have our formalism, we can start developing free geometry. As a first objective, we would like to better understand the relation between the classical and free tori. In order to discuss this, let us introduce the following notion:

DEFINITION 3.4. *Given a compact quantum space X, its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I =< [a,b] >$$

In this situation, we also say that X appears as a "liberation" of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space.

In relation now with our tori, we have the following result:

THEOREM 3.5. *We have inclusions between the various tori, as follows,*

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

and the free tori on top appear as liberations of the tori on the bottom.

PROOF. This is indeed clear from definitions, because commutativity of a group algebra means precisely that the group in question is abelian. □

As a conclusion now to all this, we have a beginning of free geometry, both real and complex, by knowing at least what the torus of each theory is. And with our construction being definitely the good one, for the simple reason that the main problems in the analysis of the free groups correspond in this way the main questions in our free geometry.

3b. Quantum spheres

In order to extend now the free geometries that we have, real and complex, let us begin with the spheres. We have the following notions:

Definition 3.6. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

Here the fact that these algebras are indeed well-defined comes from the following estimate, which shows that the biggest C^*-norms on these $*$-algebras are bounded:

$$||x_i||^2=||x_ix_i^*||\leq\left|\left|\sum_ix_ix_i^*\right|\right|=1$$

As a first result now, regarding the above free spheres, we have:

Theorem 3.7. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc}S^{N-1}_{\mathbb{R},+}&\rightarrow&S^{N-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb{R}}&\rightarrow&S^{N-1}_{\mathbb{C}}\end{array}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{array}{ccc}C(S^{N-1}_{\mathbb{R},+})&\leftarrow&C(S^{N-1}_{\mathbb{C},+})\\ \downarrow&&\downarrow\\ C(S^{N-1}_{\mathbb{R}})&\leftarrow&C(S^{N-1}_{\mathbb{C}})\end{array}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_\mathbb{R})=C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_\mathbb{C})=C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

It is enough to establish the second isomorphism. So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_\mathbb{C}$ satisfy the defining relations for A, we have a quotient map of as follows:

$$A\to C(S^{N-1}_\mathbb{C})$$

Conversely, let us write $A=C(S)$, by using the Gelfand theorem. The variables $x_1,\ldots,x_N$ become in this way true coordinates, providing us with an embedding $S\subset\mathbb{C}^N$. Also, the quadratic relations become $\sum_i|x_i|^2=1$, so we have $S\subset S^{N-1}_\mathbb{C}$. Thus, we have a quotient map $C(S^{N-1}_\mathbb{C})\to A$, as desired, and this gives all the results. □

By using the free spheres constructed above, we can now formulate:

Definition 3.8. *A real algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i\in\mathbb{C}<x_1,\ldots,x_N>$$

We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinates $x_1,\ldots,x_N$.

As a basic example here, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_\mathbb{C},S^{N-1}_\mathbb{R}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem:

Theorem 3.9. *If $X\subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class}=\left\{x\in S^{N-1}_\mathbb{C}\Big|f_i(x_1,\ldots,x_N)=0\right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class})\to C(X'_{class})$$

Conversely now, from $X\subset S^{N-1}_{\mathbb C,+}$ we obtain $X_{class}\subset S^{N-1}_{\mathbb C}$. Now since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion $X_{class}\subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class})\to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

DEFINITION 3.10. *We agree to identify two real algebraic submanifolds $X,Y\subset S^{N-1}_{\mathbb C,+}$ when we have a $*$-algebra isomorphism between $*$-algebras of coordinates*

$$f:\mathcal C(Y)\to\mathcal C(X)$$

mapping standard coordinates to standard coordinates.

We will see later the reasons for making this convention, coming from amenability. Now back to the tori, as constructed before, we can see that these are examples of algebraic manifolds, in the sense of Definition 3.8. In fact, we have the following result:

THEOREM 3.11. *The four main quantum spheres produce the main quantum tori*

$$\begin{matrix} S^{N-1}_{\mathbb R,+} & \to & S^{N-1}_{\mathbb C,+}\\ \uparrow & & \uparrow\\ S^{N-1}_{\mathbb R} & \to & S^{N-1}_{\mathbb C}\end{matrix}\qquad\to\qquad\begin{matrix} T_N^+ & \to & \mathbb T_N^+\\ \uparrow & & \uparrow\\ T_N & \to & \mathbb T_N\end{matrix}$$

via the formula $T=S\cap\mathbb T_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb C,+}$.

PROOF. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb T_N^+=S^{N-1}_{\mathbb C,+}\cap\mathbb T_N^+$. But this is something trivial, because we have $\mathbb T_N^+\subset S^{N-1}_{\mathbb C,+}$.

(2) Free real case. Here the formula in the statement reads $T_N^+=S^{N-1}_{\mathbb R,+}\cap\mathbb T_N^+$. But this is clear as well, the real version of $\mathbb T_N^+$ being T_N^+.

(3) Classical complex case. Here the formula in the statement reads $\mathbb T_N=S^{N-1}_{\mathbb C}\cap\mathbb T_N^+$. But this is clear as well, the classical version of $\mathbb T_N^+$ being $\mathbb T_N$.

(4) Classical real case. Here the formula in the statement reads $T_N=S^{N-1}_{\mathbb R}\cap\mathbb T_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). □

We will be back to free geometry, later in this book.

3c. Random matrices

Back to the von Neumann algebras, our main results so far concern the finite dimensional case, where the algebra is of the form $A=\oplus_iM_{n_i}(\mathbb C)$, and the commutative case, where the algebra is of the form $A=L^\infty(X)$. In order to advance, we must solve:

QUESTION 3.12. *What are the next simplest von Neumann algebras, generalizing at the same time the finite dimensional ones, $A=\oplus_iM_{n_i}(\mathbb C)$, and the commutative ones, $A=L^\infty(X)$, that we can use as input for our study?*

In this formulation, our question is a no-brainer, the answer to it being that of looking at the direct integrals of matrix algebras, over an arbitrary measured space X:

$$A=\int_XM_{n_x}(\mathbb C)dx$$

However, when thinking a bit, all this looks quite tricky, with most likely lots of technical functional analysis and measure theory involved. So, we will leave the investigation of such algebras, which are indeed quite basic, and called of type I, for later.

Nevermind. Let us replace Question 3.12 with something more modest, as follows:

QUESTION 3.13 (update). *What are the next simplest von Neumann algebras, generalizing at the same time the usual matrix algebras, $A=M_N(\mathbb C)$, and the commutative ones, $A=L^\infty(X)$, that we can use as input for our study?*

But here, what we have is again a no-brainer, because in relation to what has been said above, we just have to restrict the attention to the "isotypic" case, where all fibers are isomorphic. And in this case our algebra is a random matrix algebra:

$$A=\int_XM_N(\mathbb C)dx$$

Which looks quite nice, and so good news, we have our algebras. In practice now, although there is some functional analysis to be done with these algebras, the main questions regard the individual operators $T\in A$, called random matrices. Thus, we are basically back to good old operator theory. Let us begin our discussion with:

DEFINITION 3.14. *A random matrix algebra is a von Neumann algebra of the following type, with X being a probability space, and with $N\in\mathbb N$ being an integer:*

$$A=M_N(L^\infty(X))$$

In other words, A appears as a tensor product, as follows,

$$A=M_N(\mathbb C)\otimes L^\infty(X)$$

of a matrix algebra and a commutative von Neumann algebra.

As a first observation, our algebra can be written as well as follows, with this latter convention being quite standard in the probability literature:

$$A = L^\infty(X, M_N(\mathbb{C}))$$

In connection with the tensor product notation, which is often the most useful one for computations, we have as well the following possible writing, also used in probability:

$$A = L^\infty(X) \otimes M_N(\mathbb{C})$$

Importantly now, each random matrix algebra A is naturally endowed with a canonical von Neumann algebra trace $tr : A \to \mathbb{C}$, which appears as follows:

Proposition 3.15. *Given a random matrix algebra $A = M_N(L^\infty(X))$, consider the linear form $tr : A \to \mathbb{C}$ given by:*

$$tr(T) = \frac{1}{N}\sum_{i=1}^N \int_X T_{ii}^x dx$$

In tensor product notation, $A = M_N(\mathbb{C}) \otimes L^\infty(X)$, we have then the formula

$$tr = \frac{1}{N}Tr \otimes \int_X$$

and this functional $tr : A \to \mathbb{C}$ is a faithful positive unital trace.

Proof. The first assertion, regarding the tensor product writing of tr, is clear from definitions. As for the second assertion, regarding the various properties of tr, this follows from this, because these properties are stable under taking tensor products. □

As before, there is a discussion here in connection with the other possible writings of A. With the probabilistic notation $A = L^\infty(X, M_N(\mathbb{C}))$, the trace appears as:

$$tr(T) = \int_X \frac{1}{N} Tr(T^x)\, dx$$

Also, with the probabilistic tensor notation $A = L^\infty(X) \otimes M_N(\mathbb{C})$, the trace appears exactly as in the second part of Proposition 3.15, with the order inverted:

$$tr = \int_X \otimes \frac{1}{N} Tr$$

To summarize, the random matrix algebras appear to be very basic objects, and the only difficulty, in the beginning, lies in getting familiar with the 4 possible notations for them. Or perhaps 5 possible notations, because we have $A = \int_X M_N(\mathbb{C}) dx$ as well.

Getting to work now, as already said, the main questions about random matrix algebras regard the individual operators $T \in A$, called random matrices. To be more precise, we are interested in computing the laws of such matrices, constructed according to:

THEOREM 3.16. *Given an operator algebra $A \subset B(H)$ with a faithful trace $tr : A \to \mathbb{C}$, any normal element $T \in A$ has a law, namely a probability measure μ satisfying*

$$tr(T^k) = \int_{\mathbb{C}} z^k d\mu(z)$$

with the powers being with respect to colored exponents $k = \circ \bullet \bullet \circ \ldots$, defined via

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

and multiplicativity. This law is unique, and is supported by the spectrum $\sigma(T) \subset \mathbb{C}$. In the non-normal case, $TT^* \neq T^*T$, such a law does not exist.

PROOF. We have two assertions here, the idea being as follows:

(1) In the normal case, $TT^* = T^*T$, we know from the Gelfand theorm, or from the continuous functional calculus theorem, that we have:

$$< T >= C(\sigma(T))$$

Thus the functional $f(T) \to tr(f(T))$ can be regarded as an integration functional on the algebra $C(\sigma(T))$, and by the Riesz theorem, this latter functional must come from a probability measure μ on the spectrum $\sigma(T)$, in the sense that we must have:

$$tr(f(T)) = \int_{\sigma(T)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the operators T^k, taken as usual with respect to colored integer exponents, $k = \circ \bullet \bullet \circ \ldots$, generate the whole operator algebra $C(\sigma(T))$.

(2) In the non-normal case now, $TT^* \neq T^*T$, we must show that such a law does not exist. For this purpose, we can use a positivity trick, as follows:

$$\begin{aligned}
TT^* - T^*T \neq 0 \quad &\implies \quad (TT^* - T^*T)^2 > 0 \\
&\implies \quad TT^*TT^* - TT^*T^*T - T^*TTT^* + T^*TT^*T > 0 \\
&\implies \quad tr(TT^*TT^* - TT^*T^*T - T^*TTT^* + T^*TT^*T) > 0 \\
&\implies \quad tr(TT^*TT^* + T^*TT^*T) > tr(TT^*T^*T + T^*TTT^*) \\
&\implies \quad tr(TT^*TT^*) > tr(TTT^*T^*)
\end{aligned}$$

Now assuming that T has a law $\mu \in \mathcal{P}(\mathbb{C})$, in the sense that the moment formula in the statement holds, the above two different numbers would have to both appear by integrating $|z|^2$ with respect to this law μ, which is contradictory, as desired. □

Back now to the random matrices, as a basic example, assume $X = \{.\}$, so that we are dealing with a usual scalar matrix, $T \in M_N(\mathbb{C})$. By changing the basis of $\mathbb{C}^N$, which won't affect our trace computations, we can assume that T is diagonal:

$$T \sim \begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_N\end{pmatrix}$$

But for such a diagonal matrix, we have the following formula:

$$tr(T^k) = \frac{1}{N}(\lambda_1^k + \ldots + \lambda_N^k)$$

Thus, the law of T is the average of the Dirac masses at the eigenvalues:

$$\mu = \frac{1}{N}\left(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N}\right)$$

As a second example now, assume $N = 1$, and so $T \in L^\infty(X)$. In this case we obtain the usual law of T, because the equation to be satisfied by μ is:

$$\int_X \varphi(T) = \int_{\mathbb{C}} \varphi(x)d\mu(x)$$

At a more advanced level, the main problem regarding the random matrices is that of computing the law of various classes of such matrices, coming in series:

Question 3.17. *What is the law of random matrices coming in series*

$$T_N \in M_N(L^\infty(X))$$

in the $N >> 0$ regime?

The general strategy here, coming from physicists, is that of computing first the asymptotic law μ^0, in the $N \to \infty$ limit, and then looking for the higher order terms as well, as to finally reach to a series in N^{-1} giving the law of T_N, as follows:

$$\mu_N = \mu^0 + N^{-1}\mu^1 + N^{-2}\mu^2 + \ldots$$

As a basic example here, of particular interest are the random matrices having i.i.d. complex normal entries, under the constraint $T = T^*$. Here the asymptotic law μ^0 is the Wigner semicircle law on $[-2, 2]$. We will discuss this, later in this book.

3d. Cuntz algebras

We would like to end this chapter with an interesting class of C^*-algebras, discovered by Cuntz in [**23**], and heavily used since then, for various technical purposes:

Definition 3.18. *The Cuntz algebra O_n is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ satisfying the following condition:*

$$S_1S_1^* + \ldots + S_nS_n^* = 1$$

That is, $O_n \subset B(H)$ is generated by n isometries whose ranges sum up to H.

Observe that H must be infinite dimensional, in order to have isometries as above. In what follows we will prove that O_n is independent on the choice of such isometries, and also that this algebra is simple. We will restrict the attention to the case $n=2$, the proof in general being similar. Let us start with some simple computations, as follows:

PROPOSITION 3.19. *Given a word $i=i_1\ldots i_k$ with $i_l\in\{1,2\}$, we associate to it the element $S_i=S_{i_1}\ldots S_{i_k}$ of the algebra O_2. Then S_i are isometries, and we have*

$$S_i^*S_j=\delta_{ij}1$$

for any two words i,j having the same lenght.

PROOF. We use the relations defining the algebra O_2, namely:

$$S_1^*S_1=S_2^*S_2=1\quad,\quad S_1S_1^*+S_2S_2^*=1$$

The fact that S_i are isometries is clear, here being the check for $i=12$:

$$\begin{aligned}S_{12}^*S_{12}&=(S_1S_2)^*(S_1S_2)\\&=S_2^*S_1^*S_1S_2\\&=S_2^*S_2\\&=1\end{aligned}$$

Regarding the last assertion, by recurrence we just have to establish the formula there for the words of length 1. That is, we want to prove the following formulae:

$$S_1^*S_2=S_2^*S_1=0$$

But these two formulae follow from the fact that the projections $P_i=S_iS_i^*$ satisfy by definition $P_1+P_2=1$. Indeed, we have the following computation:

$$\begin{aligned}P_1+P_2=1&\implies P_1P_2=0\\&\implies S_1S_1^*S_2S_2^*=0\\&\implies S_1^*S_2=S_1^*S_1S_1^*S_2S_2^*S_2=0\end{aligned}$$

Thus, we have the first formula, and the proof of the second one is similar. □

We can use the formulae in Proposition 3.19 as follows:

PROPOSITION 3.20. *Consider words in O_2, meaning products of S_1,S_1^*,S_2,S_2^*.*

1. *Each word in O_2 is of form 0 or $S_iS_j^*$ for some words i,j.*
2. *Words of type $S_iS_j^*$ with $l(i)=l(j)=k$ form a system of $2^k\times2^k$ matrix units.*
3. *The algebra A_k generated by matrix units in (2) is a subalgebra of A_{k+1}.*

PROOF. Here the first two assertions follow from the formulae in Proposition 3.19, and for the last assertion, we can use the following formula:

$$S_iS_j^*=S_i1S_j^*=S_i(S_1S_1^*+S_2S_2^*)S_j^*$$

Thus, we obtain an embedding of algebras A_k, as in the statement. □

Observe now that the embedding constructed in (3) above is compatible with the matrix unit systems in (2). Consider indeed the following diagram:

$$\begin{matrix} A_{k+1} & \simeq & M_{2^{k+1}}(\mathbb{C}) \\ \cup & & \cup \\ A_k & \simeq & M_{2^k}(\mathbb{C}) \end{matrix}$$

With the notation $e_{ix,yj} = e_{ij} \otimes e_{xy}$, the inclusion on the right is given by:

$$\begin{aligned} e_{ij} &\to e_{i1,1h} + e_{i2,2j} \\ &= e_{ij} \otimes e_{11} + e_{ij} \otimes e_{22} \\ &= e_{ij} \otimes 1 \end{aligned}$$

Thus, with standard tensor product notations, the inclusion on the right is the canonical inclusion $m \to m \otimes 1$, and so the above diagram becomes:

$$\begin{matrix} A_{k+1} & \simeq & M_2(\mathbb{C})^{\otimes k+1} \\ \cup & & \cup \\ A_k & \simeq & M_2(\mathbb{C})^{\otimes k} \end{matrix}$$

The passage from the algebra $A = \cup_k A_k \simeq M_2(\mathbb{C})^{\otimes\infty}$ coming from this observation to the full the algebra O_2 that we are interested in can be done by using:

Proposition 3.21. *Each element $X \in < S_1, S_2 > \subset O_2$ decomposes as a finite sum*

$$X = \sum_{i>0} S_1^{*i} X_{-i} + X_0 + \sum_{i>0} X_i S_1^i$$

where each X_i is in the union A of algebras A_k.

Proof. By linearity and by using Proposition 3.20 we may assume that X is a nonzero word, say $X = S_i S_j^*$. In the case $l(i) = l(j)$ we can set $X_0 = X$ and we are done. Otherwise, we just have to add at left or at right terms of the form $1 = S_1^* S_1$. For instance $X = S_2$ is equal to $S_2 S_1^* S_1$, and we can take $X_1 = S_2 S_1^* \in A_1$. □

We must show now that the decomposition $X \to (X_i)$ found above is unique, and then prove that each application $X \to X_i$ has good continuity properties. The following formulae show that in both problems we may restrict attention to the case $i = 0$:

$$X_{i+1} = (X S_1^*)_i \qquad\qquad X_{-i-1} = (S_1 X)_i$$

In order to solve these questions, we use the following fact:

Proposition 3.22. *If P is a nonzero projection in $\mathcal{O}_2 =< S_1, S_2 >\subset O_2$, its k-th average, given by the formula*

$$Q = \sum_{l(i)=k} S_i P S_i^*$$

is a nonzero projection in $\mathcal{O}_2$ having the property that the linear subspace QA_kQ is isomorphic to a matrix algebra, and $Y \to QYQ$ is an isomorphism of A_k onto it.

Proof. We know that the words of form $S_i S_j^*$ with $l(i) = l(j) = k$ are a system of matrix units in A_k. We apply to them the map $Y \to QYQ$, and we obtain:

$$\begin{aligned} QS_iS_j^*Q &= \sum_{pq} S_pPS_p^*S_iS_j^*S_qPS_q^* \\ &= \sum_{pq} \delta_{ip}\delta_{jq} S_pP^2S_q^* \\ &= S_iPS_j^* \end{aligned}$$

The output being a system of matrix units, $Y \to QYQ$ is an isomorphism from the algebra of matrices A_k to another algebra of matrices QA_kQ, and this gives the result. □

Thus any map $Y \to QYQ$ behaves well on the $i = 0$ part of the decomposition on X. It remains to find P such that $Y \to QYQ$ destroys all $i \neq 0$ terms, and we have here:

Proposition 3.23. *Assuming $X_0 \in A_k$, there is a nonzero projection $P \in A$ such that $QXQ = QX_0Q$, where Q is the k-th average of P.*

Proof. We want $Y \to QYQ$ to map to zero all terms in the decomposition of X, except for X_0. Let us call $M_1, \ldots, M_t \in \mathcal{O}_2 - A$ the terms to be destroyed. We want the following equalities to hold, with the sum over all pairs of length k indices:

$$\sum_{ij} S_iPS_i^*M_qS_jPS_j^* = 0$$

The simplest way is to look for P such that all terms of all sums are 0:

$$S_iPS_i^*M_qS_jPS_j^* = 0$$

By multiplying to the left by S_i^* and to the right by S_j, we want to have:

$$PS_i^*M_qS_jP = 0$$

With $N_z = S_i^*M_qS_j$, where z belongs to some new index set, we want to have:

$$PN_zP = 0$$

Since $N_z \in \mathcal{O}_2 - A$, we can write $N_z = S_{m_z}S_{n_z}^*$ with $l(m_z) \neq l(n_z)$, and we want:

$$PS_{m_z}S_{n_z}^*P = 0$$

In order to do this, we can the projections of form $P = S_r S_r^*$. We want:

$$S_r S_r^* S_{m_z} S_{n_z}^* S_r S_r^* = 0$$

Let K be the biggest length of all m_z, n_z. Assume that we have fixed r, of length bigger than K. If the above product is nonzero then both $S_r^* S_{m_z}$ and $S_{n_z}^* S_r$ must be nonzero, which gives the following equalities of words:

$$r_1 \ldots r_{l(m_z)} = m_z \quad , \quad r_1 \ldots r_{l(n_z)} = n_z$$

Assuming that these equalities hold indeed, the above product reduces as follows:

$$S_r S_{r_{l(r)}}^* \ldots S_{r_{l(m_z)+1}}^* S_{r_{l(n_z)+1}} \ldots S_{r_{l(r)}} S_r^*$$

Now if this product is nonzero, the middle term must be nonzero:

$$S_{r_{l(r)}}^* \ldots S_{r_{l(m_z)+1}}^* S_{r_{l(n_z)+1}} \ldots S_{r_{l(r)}} \neq 0$$

In order for this for hold, the indices starting from the middle to the right must be equal to the indices starting from the middle to the left. Thus r must be periodic, of period $|l(m_z) - l(n_z)| > 0$. But this is certainly possible, because we can take any aperiodic infinite word, and let r be the sequence of first M letters, with M big enough. □

We can now start solving our problems. We first have:

Proposition 3.24. *The decomposition of X is unique, and we have*

$$||X_i|| \leq ||X||$$

for any i.

Proof. It is enough to do this for $i = 0$. But this follows from the previous result, via the following sequence of equalities and inequalities:

$$\begin{aligned} ||X_0|| &= ||QX_0Q|| \\ &= ||QXQ|| \\ &\leq ||X|| \end{aligned}$$

Thus we got the inequality in the statement. As for the uniqueness part, this follows from the fact that $X_0 \to QX_0Q = QXQ$ is an isomorphism. □

Remember now we want to prove that the Cuntz algebra O_2 does not depend on the choice of the isometries S_1, S_2. In order to do so, let $\overline{\mathcal{O}}_2$ be the completion of the $*$-algebra $\mathcal{O}_2 = < S_1, S_2 > \subset O_2$ with respect to the biggest C^*-norm. We have:

Proposition 3.25. *We have the equivalence*

$$X = 0 \iff X_i = 0, \forall i$$

valid for any element $X \in \overline{\mathcal{O}}_2$.

PROOF. Assume $X_i = 0$ for any i, and choose a sequence $X^k \to X$ with $X^k \in \mathcal{O}_2$. For $\lambda \in \mathbb{T}$ we define a representation ρ_λ in the following way:

$$\rho_\lambda : S_i \to \lambda S_i$$

We have then $\rho_\lambda(Y) = Y$ for any element $Y \in A$. We fix norm one vectors ξ, η and we consider the following continuous functions $f : \mathbb{T} \to \mathbb{C}$:

$$f^k(\lambda) =< \rho_\lambda(X^k)\xi, \eta >$$

From $X^k \to X$ we get, with respect to the usual sup norm of $C(\mathbb{T})$:

$$f^k \to f$$

Each $X^k \in \mathcal{O}_2$ can be decomposed, and f^k is given by the following formula:

$$f^k(\lambda) = \sum_{i>0} \lambda^{-i} < S_1^{*i} X_{-i}^k \xi, \eta > + < X_0 \xi, \eta > + \sum_{i>0} \lambda^i < X_i^k S_1^i \xi, \eta >$$

This is a Fourier type expansion of f^k, that can we write in the following way:

$$f^k(\lambda) = \sum_{j=-\infty}^{\infty} a_j^k \lambda^j$$

By using Proposition 3.24 we obtain that with $k \to \infty$, we have:

$$|a_j^k| \leq ||X_j^k|| \to ||X_j^\infty|| = 0$$

On the other hand we have $a_j^k \to a_j$ with $k \to \infty$. Thus all Fourier coefficients a_j of f are zero, so $f = 0$. With $\lambda = 1$ this gives the following equality:

$$< X\xi, \eta >= 0$$

This is true for arbitrary norm one vectors ξ, η, so $X = 0$ and we are done. □

We can now formulate the Cuntz theorem, from [23], as follows:

THEOREM 3.26 (Cuntz). *Let S_1, S_2 be isometries satisfying $S_1 S_1^* + S_2 S_2^* = 1$.*

1. *The C^*-algebra O_2 generated by S_1, S_2 does not depend on the choice of S_1, S_2.*
2. *For any nonzero $X \in O_2$ there are $A, B \in O_2$ with $AXB = 1$.*
3. *In particular O_2 is simple.*

PROOF. This basically follows from the various results established above:

(1) Consider the canonical projection map $\pi : \overline{O}_2 \to O_2$. We know that π is surjective, and we will prove now that π is injective. Indeed, if $\pi(X) = 0$ then $\pi(X)_i = 0$ for any i. But $\pi(X)_i$ is in the dense $*$-algebra A, so it can be regarded as an element of $\overline{O}_2$, and with this identification, we have $\pi(X)_i = X_i$ in $\overline{O}_2$. Thus $X_i = 0$ for any i, so $X = 0$. Thus π is an isomorphism. On the other hand $\overline{O}_2$ depends only on $\mathcal{O}_2$, and the above formulae in $\mathcal{O}_2$, for algebraic calculus and for decomposition of an arbitrary $X \in \mathcal{O}_2$, show that $\mathcal{O}_2$ does not depend on the choice of S_1, S_2. Thus, we obtain the result.

(2) Choose a sequence $X^k \to X$ with $X^k \in \mathcal{O}_2$. We have the following formula:

$$(X^*X)_0 = \lim_{k\to\infty}\left(\sum_{i>0} X^{k*}_{-i}X^k_{-i} + X^{k*}_0 X^k_0 + \sum_{i>0} S_1^{*i}X_i^{k*}X_i^k S_1^i\right)$$

Thus $X \neq 0$ implies $(X^*X)_0 \neq 0$. By linearity we can assume that we have:

$$||(X^*X)_0|| = 1$$

Now choose a positive element $Y \in \mathcal{O}_2$ which is close enough to X^*X:

$$||X^*X - Y|| < \varepsilon$$

Since $Z \to Z_0$ is norm decreasing, we have the following estimate:

$$||Y_0|| > 1 - \varepsilon$$

We apply Proposition 3.23 to our positive element $Y \in \mathcal{O}_2$. We obtain in this way a certain projection Q such that $QY_0Q = QYQ$ belongs to a certain matrix algebra. We have $QYQ > 0$, so we can diagonalize this latter element, as follows:

$$QYQ = \sum \lambda_i R_i$$

Here λ_i are positive numbers and R_i are minimal projections in the matrix algebra. Now since $||QYQ|| = ||Y_0||$, there must be an eigenvalue greater that $1-\varepsilon$:

$$\lambda_0 > 1 - \varepsilon$$

By linear algebra, we can pass from a minimal projection to another:

$$U^*U = R_i \quad , \quad UU^* = S_1^k S_1^{*k}$$

The element $B = QU^*S_1^k$ has norm ≤ 1, and we get the following inequality:

$$\begin{aligned}||1 - B^*X^*XB|| &\leq ||1 - B^*YB|| + ||B^*YB - B^*X^*XB||\\ &< ||1 - B^*YB|| + \varepsilon\end{aligned}$$

The last term can be computed by using the diagonalization of QYQ, as follows:

$$\begin{aligned}B^*YB &= S_1^{*k}UQYQU^*S_1^k\\ &= S_1^{*k}\left(\sum \lambda_i U R_i U^*\right)S_1^k\\ &= \lambda_0 S_1^{*k}S_1^kS_1^{*k}S_1^k\\ &= \lambda_0\end{aligned}$$

From $\lambda_0 > 1-\varepsilon$ we get $||1 - B^*YB|| < \varepsilon$, and we obtain the following estimate:

$$||1 - B^*X^*XB|| < 2\varepsilon$$

Thus B^*X^*XB is invertible, say with inverse C, and we have $(B^*X^*)X(BC) = 1$.

(3) This is clear from the formula $AXB = 1$ established in (2). □

3e. Exercises

Exercises:

Exercise 3.27.

Exercise 3.28.

Exercise 3.29.

Exercise 3.30.

Exercise 3.31.

Exercise 3.32.

Exercise 3.33.

Exercise 3.34.

Bonus exercise.

CHAPTER 4

Analytic aspects

4a. Density results

Time now for some more advanced operator algebra theory, and hang on, all this will be quite technical. Let us begin our study with some generalities. We first have:

PROPOSITION 4.1. *The weak operator topology on $B(H)$ is the topology having the following equivalent properties:*

(1) *It makes $T \to < Tx, y >$ continuous, for any $x, y \in H$.*
(2) *It makes $T_n \to T$ when $< T_n x, y > \to < Tx, y >$, for any $x, y \in H$.*
(3) *Has as subbase the sets $U_T(x, y, \varepsilon) = \{S : | < (S - T)x, y > | < \varepsilon\}$.*
(4) *Has as base $U_T(x_1, \ldots, x_n, y_1, \ldots, y_n, \varepsilon) = \{S : | < (S - T)x_i, y_i > | < \varepsilon, \forall i\}$.*

PROOF. The equivalences (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) all follow from definitions, with of course (1,2) referring to the coarsest topology making that things happen. □

Similarly, in what regards the strong operator topology, we have:

PROPOSITION 4.2. *The strong operator topology on $B(H)$ is the topology having the following equivalent properties:*

(1) *It makes $T \to Tx$ continuous, for any $x \in H$.*
(2) *It makes $T_n \to T$ when $T_n x \to Tx$, for any $x \in H$.*
(3) *Has as subbase the sets $V_T(x, \varepsilon) = \{S : ||(S - T)x|| < \varepsilon\}$.*
(4) *Has as base the sets $V_T(x_1, \ldots, x_n, \varepsilon) = \{S : ||(S - T)x_i|| < \varepsilon, \forall i\}$.*

PROOF. Again, the equivalences (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) are all clear, and with (1,2) referring to the coarsest topology making that things happen. □

We know from before that an operator algebra $A \subset B(H)$ is weakly closed if and only if it is strongly closed. Here is a useful generalization of this fact:

THEOREM 4.3. *Given a convex set of bounded operators*

$$C \subset B(H)$$

its weak operator closure and strong operator closure coincide.

PROOF. Since the weak operator topology on $B(H)$ is weaker by definition than the strong operator topology on $B(H)$, we have, for any subset $C\subset B(H)$:

$$\overline{C}^{strong}\subset\overline{C}^{weak}$$

Now by assuming that $C\subset B(H)$ is convex, we must prove that:

$$T\in\overline{C}^{weak}\implies T\in\overline{C}^{strong}$$

In order to do so, let us pick vectors $x_1,\ldots,x_n\in H$ and $\varepsilon>0$. We let $K=H^{\oplus n}$, and we consider the standard embedding $i:B(H)\subset B(K)$, given by:

$$iT(y_1,\ldots,y_n)=(Ty_1,\ldots,Ty_n)$$

We have then the following implications, which are all trivial:

$$T\in\overline{C}^{weak}\implies iT\in\overline{iC}^{weak}\implies iT(x)\in\overline{iC(x)}^{weak}$$

Now since the set $C\subset B(H)$ was assumed to be convex, the set $iC(x)\subset K$ is convex too, and by the Hahn-Banach theorem, for compact sets, it follows that we have:

$$iT(x)\in\overline{iC(x)}^{||.||}$$

Thus, there exists an operator $S\in C$ such that we have, for any i:

$$||Sx_i-Tx_i||<\varepsilon$$

But this shows that we have $S\in V_T(x_1,\ldots,x_n,\varepsilon)$, and since $x_1,\ldots,x_n\in H$ and $\varepsilon>0$ were arbitrary, by Proposition 4.2 it follows that we have $T\in\overline{C}^{strong}$, as desired. □

We will need as well the following standard result:

PROPOSITION 4.4. *Given a vector space $E\subset B(H)$, and a linear form $f:E\to\mathbb{C}$, the following conditions are equivalent:*

(1) *f is weakly continuous.*
(2) *f is strongly continuous.*
(3) *$f(T)=\sum_{i=1}^n<Tx_i,y_i>$, for certain vectors $x_i,y_i\in H$.*

PROOF. This is something standard, using the same tools at those already used in chapter 5, namely basic functional analysis, and amplification tricks:

(1) $\implies$ (2) Since the weak operator topology on $B(H)$ is weaker than the strong operator topology on $B(H)$, weakly continuous implies strongly continuous. To be more precise, assume $T_n\to T$ strongly. Then $T_n\to T$ weakly, and since f was assumed to be weakly continuous, we have $f(T_n)\to f(T)$. Thus f is strongly continuous, as desired.

(2) $\implies$ (3) Assume indeed that our linear form $f:E\to\mathbb{C}$ is strongly continuous. In particular f is strongly continuous at 0, and Proposition 4.2 provides us with vectors $x_1,\ldots,x_n\in H$ and a number $\varepsilon>0$ such that, with the notations there:

$$f(V_0(x_1,\ldots,x_n,\varepsilon))\subset D_0(1)$$

That is, we can find vectors $x_1,\ldots,x_n\in H$ and a number $\varepsilon>0$ such that:

$$||Tx_i||<\varepsilon,\forall i\implies|f(T)|<1$$

But this shows that we have the following estimate:

$$\sum_{i=1}^n||Tx_i||^2<\varepsilon^2\implies|f(T)|<1$$

By linearity, it follows from this that we have the following estimate:

$$|f(T)|<\frac{1}{\varepsilon}\sqrt{\sum_{i=1}^n||Tx_i||^2}$$

Consider now the direct sum $H^{\oplus n}$, and inside it, the following vector:

$$x=(x_1,\ldots,x_n)\in H^{\oplus n}$$

Consider also the following linear space, written in tensor product notation:

$$K=\overline{(E\otimes 1)x}\subset H^{\oplus n}$$

We can define a linear form $f':K\to\mathbb{C}$ by the following formula, and continuity:

$$f'(Tx_1,\ldots,Tx_n)=f(T)$$

We conclude that there exists a vector $y\in K$ such that the following happens:

$$f'\big((T\otimes 1)y\big)=<(T\otimes 1)x,y>$$

But in terms of the original linear form $f:E\to\mathbb{C}$, this means that we have:

$$f(T)=\sum_{i=1}^n<Tx_i,y_i>$$

(3) $\implies$ (1) This is clear, because we have, with respect to the weak topology:

$$\begin{aligned}T_n\to T&\implies\ <T_nx_i,y_i>\to<Tx_i,y_i>,\forall i\\&\implies\ \sum_{i=1}^n<T_nx_i,y_i>\to\sum_{i=1}^n<Tx_i,y_i>\\&\implies\ f(T_n)\to f(T)\end{aligned}$$

Thus, our linear form f is weakly continuous, as desired. □

Here is one more well-known result, that we will need as well:

THEOREM 4.5. *The unit ball of $B(H)$ is weakly compact.*

Proof. If we denote by $B_1 \subset B(H)$ the unit ball, and by $D_1 \subset \mathbb{C}$ the unit disk, we have a morphism as follows, which is continuous with respect to the weak topology on B_1, and with respect to the product topology on the set on the right:

$$B_1 \subset \prod_{||x||,||y||\leq 1} D_1 \quad , \quad T \to (< Tx, y >)_{x,y}$$

Since the set on the right is compact, by Tychonoff, it is enough to show that the image of B_1 is closed. So, let $(c_{xy}) \in \bar{B}_1$. We can then find $T_i \in B_1$ such that:

$$< T_i x, y > \to c_{xy} \quad , \quad \forall x, y$$

But this shows that the following map is a bounded sesquilinear form:

$$H \times H \to \mathbb{C} \quad , \quad (x, y) \to c_{xy}$$

Thus, we can find an operator $T \in B(H)$, and so $T \in B_1$, such that $< Tx, y >= c_{xy}$ for any $x, y \in H$, and this shows that we have $(c_{xy}) \in B_1$, as desired. □

Getting back to operator algebras, we have the following result, due to Kaplansky, which is something very useful, and of independent interest as well:

Theorem 4.6. *Given an operator algebra $A \subset B(H)$, the following happen:*

(1) *The unit ball of A is strongly dense in the unit ball of A''.*
(2) *The same happens for the self-adjoint parts of the above unit balls.*

Proof. This is something quite tricky, the idea being as follows:

(1) Consider the self-adjoint part $A_{sa} \subset A$. By taking real parts of operators, and using the fact that $T \to T^*$ is weakly continuous, we have then:

$$\overline{A_{sa}}^w \subset \left(\overline{A}^w\right)_{sa}$$

Now since the set A_{sa} is convex, and by Theorem 4.3 all weak operator topologies coincide on the convex sets, we conclude that we have in fact equality:

$$\overline{A_{sa}}^w = \left(\overline{A}^w\right)_{sa}$$

(2) With this result in hand, let us prove now the second assertion of the theorem. For this purpose, consider an element $T \in \overline{A}^w$, satisfying $T = T^*$ and $||T|| \leq 1$. Consider as well the following function, going from the interval $[-1, 1]$ to itself:

$$f(t) = \frac{2t}{1 + t^2}$$

By functional calculus we can find an element $S \in \left(\overline{A}^w\right)_{sa}$ such that:

$$f(S) = T$$

In other words, we can find an element $S \in \left(\overline{A}^w\right)_{sa}$ such that:

$$T = \frac{2S}{1+S^2}$$

Now given arbitrary vectors $x_1, \ldots, x_n \in H$ and an arbitrary number $\varepsilon > 0$, let us pick an element $R \in A_{sa}$, subject to the following two inequalities:

$$||RTx_i - STx_i|| \leq \varepsilon \quad , \quad \left|\left|\frac{R}{1+S^2}x_i - \frac{S}{1+S^2}x_i\right|\right| \leq \varepsilon$$

Finally, consider the following element, which has norm ≤ 1:

$$L = \frac{2R}{1+R^2}$$

We have then the following computation, using the above formulae:

$$\begin{aligned}
L - T &= \frac{2R}{1+R^2} - \frac{2S}{1+S^2} \\
&= 2\left(\frac{1}{1+R^2}\left(R(1+S^2) - (1+S^2)R\right)\frac{1}{1+S^2}\right) \\
&= 2\left(\frac{1}{1+R^2}(R-S)\frac{1}{1+S^2} + \frac{R}{1+R^2}(S-R)\frac{S}{1+S^2}\right) \\
&= \frac{2}{1+R^2}(R-S)\frac{1}{1+S^2} + \frac{L}{2}(S-R)T
\end{aligned}$$

Thus, we have the following estimate, for any $i \in \{1, \ldots, n\}$:

$$||(L-T)x_i|| \leq \varepsilon$$

But this gives the second assertion of the theorem, as desired.

(3) Let us prove now the first assertion of the theorem. Given an arbitrary element $T \in \overline{A}^w$, satisfying $||T|| \leq 1$, let us look at the following element:

$$T' = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \in M_2(\overline{A}^w)$$

This element is then self-adjoint, and we can use what we proved in the above, and we are led in this way to the first assertion in the statement, as desired. □

We can go back now to our original question, from the beginning of the present chapter, namely that of abstractly characterizing the von Neumann algebras, and we have:

Theorem 4.7. *A norm closed operator $*$-algebra*

$$A \subset B(H)$$

is a von Neumann algebra precisely when its unit ball is weakly compact.

PROOF. This is something which is now clear, coming from the Kaplansky density results established in Theorem 4.6. To be more precise:

(1) In one sense, assuming that $A \subset B(H)$ is a von Neumann algebra, this algebra is weakly closed. But since the unit ball of $B(H)$ is weakly compact, we are led to the conclusion that the unit ball of A is weakly compact too.

(2) Conversely, assume that an operator algebra $A \subset B(H)$ is such that its unit ball is weakly compact. In particular, the unit ball of A is weakly closed. Now if T satisfying $||T|| \leq 1$ belongs to the weak closure of A, by Kaplansky density we conclude that we have $T \in A$. Thus our algebra A must be a von Neumann algebra, as claimed. $\square$

Many other things can be said, as a continuation of the above, notably with some even more advanced results, of the same type, due to Sakai. We will be back to this.

4b. Tensor products

Tensor products of C^*-algebras. Various norm constructions.

Many things can be said here, by using advanced functional analysis.

4c. Amenability, nuclearity

Let us discuss now amenability questions. Let us start our discussion in the von Neumann algebra setting, where things are particularly simple. We have here:

THEOREM 4.8. *Given a discrete group Γ, we can construct its von Neumann algebra*

$$L(\Gamma) \subset B(l^2(\Gamma))$$

by using the left regular representation. This algebra has a faithful positive trace, $tr(g) = \delta_{g,1}$, and when Γ is abelian we have an isomorphism of tracial von Neumann algebras

$$L(\Gamma) \simeq L^\infty(G)$$

given by a Fourier type transform, where $G = \widehat{\Gamma}$ is the compact dual of Γ.

PROOF. There are many assertions here, the idea being as follows:

(1) The first part is standard, with the left regular representation of Γ working as expected, and being a unitary representation, as follows:

$$\Gamma \subset B(l^2(\Gamma)) \quad , \quad \pi(g) : h \to gh$$

(2) The positivity of the trace comes from the following alternative formula for it, with the equivalence with the definition in the statement being clear:

$$tr(T) = < T1, 1 >$$

(3) The third part is standard as well, because when Γ is abelian the algebra $L(\Gamma)$ is commutative, and its spectral decomposition leads by delinearization to the group characters $\chi:\Gamma\to\mathbb T$, and so the dual group $G=\widehat{\Gamma}$, as indicated.

(4) Finally, the fact that our isomorphism transforms the trace of $L(\Gamma)$ into the Haar integration functional of $L^\infty(G)$ is clear. Moreover, the study of various examples show that what we constructed is in fact the Fourier transform, in its various incarnations. □

Let us record as well the following result, in relation with the above:

Theorem 4.9. *The center of a group von Neumann algebra $L(\Gamma)$ is*

$$Z(L(\Gamma))=\left\{\sum_g\lambda_g g\Big|\lambda_{gh}=\lambda_{hg}\right\}''$$

and if $\Gamma\neq\{1\}$ has infinite conjugacy classes, in the sense that

$$\left|\{ghg^{-1}|g\in G\}\right|=\infty\quad,\quad\forall h\neq 1$$

with this being called ICC property, the algebra $L(\Gamma)$ is a II_1 factor.

Proof. There are two assertions here, the idea being as follows:

(1) Consider a linear combination of group elements, which is in the weak closure of $\mathbb C[\Gamma]$, and so defines an element of the group von Neumann algebra $L(\Gamma)$:

$$a=\sum_g\lambda_g g$$

By linearity, this element $a\in L(\Gamma)$ belongs to the center of $L(\Gamma)$ precisely when it commutes with all the group elements $h\in\Gamma$, and this gives:

$$\begin{aligned}a\in Z(A)\quad&\iff\quad ah=ha\\&\iff\quad\sum_g\lambda_g gh=\sum_g\lambda_g hg\\&\iff\quad\sum_k\lambda_{kh^{-1}}k=\sum_k\lambda_{h^{-1}k}k\\&\iff\quad\lambda_{kh^{-1}}=\lambda_{h^{-1}k}\end{aligned}$$

Thus, we obtain the formula for $Z(L(\Gamma))$ in the statement.

(2) We have to examine the 3 conditions defining the II_1 factors. We already know from Theorem 4.8 that the group algebra $L(G)$ has a trace, given by:

$$tr(g)=\delta_{g,1}$$

Regarding now the center, the condition $\lambda_{gh}=\lambda_{hg}$ that we found is equivalent to the fact that $g\to\lambda_g$ is constant on the conjugacy classes, and we obtain:

$$Z(L(\Gamma))=\mathbb{C}\iff\Gamma=\mathrm{ICC}$$

Finally, assuming that this ICC condition is satisfied, with $\Gamma\neq\{1\}$, then our group Γ is infinite, and so the algebra $L(\Gamma)$ is infinite dimensional, as desired. □

Getting back now to our quantum space questions, we have a beginning of answer, because based on the above, we can formulate the following definition:

Definition 4.10. *Given a discrete group Γ, not necessarily abelian, we can construct its abstract dual $G=\widehat{\Gamma}$ as a quantum measured space, via the following formula:*

$$L^\infty(G)=L(\Gamma)$$

In the case where Γ happens to be abelian, this quantum space $G=\widehat{\Gamma}$ is a classical space, namely the usual Pontrjagin dual of Γ, endowed with its Haar measure.

Let us discuss now the same questions, in the C^*-algebra setting. The situation here is more complicated than in the von Neumann algebra setting, as follows:

Theorem 4.11. *Associated to any discrete group Γ are several group C^*-algebras,*

$$C^*(\Gamma)\to C^*_\pi(\Gamma)\to C^*_{red}(\Gamma)$$

which are constructed as follows:

1. *$C^*(\Gamma)$ is the closure of the group algebra $\mathbb{C}[\Gamma]$, with involution $g^*=g^{-1}$, with respect to the maximal C^*-seminorm on this $*$-algebra, which is a C^*-norm.*
2. *$C^*_{red}(\Gamma)$ is the norm closure of the group algebra $\mathbb{C}[\Gamma]$ in the left regular representation, on the Hilbert space $l^2(\Gamma)$, given by $\lambda(g)(h)=gh$ and linearity.*
3. *$C^*_\pi(\Gamma)$ can be any intermediate C^*-algebra, but for best results, the indexing object π must be a unitary group representation, satisfying $\pi\otimes\pi\subset\pi$.*

Proof. This is something quite technical, with (2) being very similar to the standard von Neumann algebra construction, with (1) being something new, with the norm property there coming from (2), and finally with (3) being an informal statement, that we will comment on later, once we will know about compact quantum groups. □

When Γ is finite, or abelian, or more generally amenable, all the above group algebras coincide. In the abelian case, that we are particularly interested in here, we have:

Theorem 4.12. *When Γ is abelian all its group C^*-algebras coincide, and we have an isomorphism as follows, given by a Fourier type transform,*

$$C^*(\Gamma)\simeq C(G)$$

where $G=\widehat{\Gamma}$ is the compact dual of Γ. Moreover, this isomorphism transforms the standard group algebra trace $tr(g)=\delta_{g,1}$ into the Haar integration of G.

Proof. Since Γ is abelian, any of its group C^*-algebras $A=C^*_\pi(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A=C(X)$, with $X=Spec(A)$. But the spectrum $X=Spec(A)$, consisting of the characters $\chi:A\to\mathbb C$, can be identified by delinearizing with the Pontrjagin dual $G=\widehat{\Gamma}$, and this gives the results. $\square$

At a more advanced level now, we have the following result:

Theorem 4.13. *For a discrete group $\Gamma=<g_1,\ldots,g_N>$, the following conditions are equivalent, and if they are satisfied, we say that Γ is amenable:*

1. *The projection map $C^*(\Gamma)\to C^*_{red}(\Gamma)$ is an isomorphism.*
2. *The morphism $\varepsilon:C^*(\Gamma)\to\mathbb C$ given by $g\to 1$ factorizes through $C^*_{red}(\Gamma)$.*
3. *We have $N\in\sigma(Re(g_1+\ldots+g_N))$, the spectrum being taken inside $C^*_{red}(\Gamma)$.*

The amenable groups include all finite groups, and all abelian groups. As a basic example of a non-amenable group, we have the free group F_N, with $N\geq 2$.

Proof. There are several things to be proved, the idea being as follows:

(1) The implication (1) $\implies$ (2) is trivial, and (2) $\implies$ (3) comes from the following computation, which shows that $N-Re(g_1+\ldots+g_N)$ is not invertible inside $C^*_{red}(\Gamma)$:

$$\begin{aligned}\varepsilon[N-Re(g_1+\ldots+g_N)] &= N-Re[\varepsilon(g_1)+\ldots+\varepsilon(g_n)]\\ &= N-N\\ &= 0\end{aligned}$$

As for (3) $\implies$ (1), this is something more advanced, that we will not need for the moment. We will be back to this later, directly in a more general setting.

(2) The fact that any finite group G is amenable is clear, because all the group C^*-algebras are equal to the usual group $*$-algebra $\mathbb C[G]$, in this case. As for the case of the abelian groups, these are all amenable as well, as shown by Theorem 4.12.

(3) It remains to prove that F_N with $N\geq 2$ is not amenable. By using $F_2\subset F_N$, it is enough to do this at $N=2$. So, consider the free group $F_2=<g,h>$. In order to prove that F_2 is not amenable, we use (1) $\implies$ (3). To be more precise, it is enough to show that 4 is not in the spectrum of the following operator:

$$T=\lambda(g)+\lambda(g^{-1})+\lambda(h)+\lambda(h^{-1})$$

This is a sum of four terms, each of them acting via $\delta_w\to\delta_{ew}$, with e being a certain length one word. Thus if $w\neq 1$ has length n then $T(\delta_w)$ is a sum of four Dirac masses, three of them at words of length $n+1$ and the remaining one at a length $n-1$ word. We can therefore decompose T as a sum T_++T_-, where T_+ adds and T_- cuts:

$$T=T_++T_-$$

That is, if $w\neq 1$ is a word, say beginning with h, then $T_\pm$ act on δ_w as follows:

$$T_+(\delta_w)=\delta_{gw}+\delta_{g^{-1}w}+\delta_{hw}\quad,\quad T_-(\delta_w)=\delta_{h^{-1}w}$$

It follows from definitions that we have $T_+^*=T_-$. We can use the following trick:

$$(T_++T_-)^2+(i(T_+-T_-))^2=2(T_+T_-+T_-T_+)$$

Indeed, this gives $(T_++T_-)^2\leq 2(T_+T_-+T_-T_+)$, and we obtain in this way:

$$||T||^2=||T_++T_-||^2\leq 2||T_+T_-+T_-T_+||$$

Let $w\neq 1$ be a word, say beginning with h. We have then:

$$T_-T_+(\delta_w)=T_-(\delta_{gw}+\delta_{g^{-1}w}+\delta_{hw})=3\delta_w$$

The action of T_-T_+ on the remaining vector δ_1 is computed as follows:

$$T_-T_+(\delta_1)=T_-(\delta_g+\delta_{g^{-1}}+\delta_h+\delta_{h^{-1}})=4\delta_1$$

Summing up, with $P:\delta_w\to\delta_1$ being the projection onto $\mathbb{C}\delta_1$, we have:

$$T_-T_+=3+P$$

On the other hand we have $T_+T_-(\delta_1)=T_+(0)=0$, so the subspace $\mathbb{C}\delta_1$ is invariant under the operator $T_+T_-+T_-T_+$. We have the following norm estimate:

$$||T||^2\leq 2||T_+T_-+T_-T_+||\leq 2\cdot\max\{||3+P||,\ \ ||(T_+T_-+T_-T_+)(1-P)||\}$$

The norm of $3+P$ is equal to 4, and the other norm is estimated as follows:

$$\begin{aligned}||(T_+T_-+T_-T_+)(1-P)|| &\leq ||T_+T_-||+||(3+P)(1-P)||\\ &= ||T_-T_+||+3\\ &= 7\end{aligned}$$

Thus we have $||T||\leq\sqrt{14}<4$, and this finishes the proof. □

Nuclearity and exactness of C^*-algebras. Many things can be said here.

4d. Simplicity, factoriality

In order to prove simplicity and factoriality results, we will need:

THEOREM 4.14 (Dixmier). *Let (A,tr) be a C^*-algebra with a faithful trace. If there is $\varepsilon>0$ such that for $a=a^*$ with $tr(a)=0$ there are unitaries $u_1,u_2,\ldots,u_n$ with*

$$\left|\left|\frac{1}{n}\sum_k u_kau_k^*\right|\right|\leq(1-\varepsilon)||a||$$

then A is simple, meaning that it has no non-trivial ideal, and its trace is unique.

PROOF. This is something very standard, the idea being as follows:

(1) We know that we have an inequality of type $||a'|| \leq (1-\varepsilon)||a||$. But the element a' is also self-adjoint, has trace 0, and its norm is smaller than the norm of a. Thus we can average a' too, and we get an inequality of the following type:

$$||a''|| \leq (1-\varepsilon)||a'||$$

Thus, we can replace in the statement the number $(1-\varepsilon)$ by the smaller number $(1-\varepsilon)^2$. By making this replacement enough times, we conclude that for any $\varepsilon > 0$ and any $a = a^*$ with $tr(a) = 0$ there are unitaries $u_1, u_2, \ldots, u_n$ such that:

$$\left|\left|\frac{1}{n}\sum_k u_k a u_k^*\right|\right| \leq \varepsilon||a||$$

Let I be an ideal, and choose a nonzero $b \in I$. We make the following replacement:

$$b \to \frac{b^*b}{tr(b^*b)}$$

Then our new element $b \in I$ is self-adjoint, and has trace one. Thus the above inequality applies to $a = 1 - b$, and gives the following estimate:

$$\left|\left|1 - \frac{1}{n}\sum_k u_k b u_k^*\right|\right| \leq \varepsilon||1-b||$$

But with ε small enough this gives $||1-b|| < 1$, so the element b' must be invertible. Since from $b' \in I$ we get $I = A$, this ends the proof of first assertion.

(2) In order to prove now the second assertion, let φ be another trace, and let a be as above. By using the trace property of φ and tr, we have:

$$(\varphi - tr)\left(\frac{1}{n}\sum_k u_k a u_k^*\right) = \varphi(a) - tr(a)$$

With ε small, we get by continuity that $\varphi - tr$ vanishes on a. Now observe that this is true under our assumptions $a = a^*$ and $tr(a) = 0$, but by linearity we can suppress the $tr(a) = 0$ assumption, and by using the standard $a = b + ic$ decomposition, with b, c self-adjoints, we can suppress the $a = a^*$ assumption too. Thus $\varphi = tr$, as claimed. □

In fact the Dixmier property is a bit more complicated. A pair (A, tr) has it when for any a various averages of type a' get as close as needed to scalar multiples of 1. In other words, the number ε is not the same for all elements a. This can be stated as follows:

$$\overline{\text{conv}}\left\{uau^*\Big|u \in U(A)\right\} \cap \mathbb{C}1 \neq \emptyset$$

The same argument as in the above proof shows that this general form of the Dixmier property implies simplicity, plus uniqueness of the trace as a bonus. Moreover, it is known that the converse holds. Thus when trying to prove that $C^*_{red}(F_2)$ is simple, which is an a priori quite abstract problem, the thing to do, which is both down-to-earth and not supposed to fail, is to consider averages $a \to a'$ and to estimate their norms.

Back now to $C^*_{red}(F_2)$, we have a lot of unitaries $g \in F_2$ which can be used for making averages. We want to estimate norms of elements of following type, with $z_g = z_{g^{-1}}$:

$$a' = \frac{1}{n}\sum_i g_i \left(\sum_{g\neq 1} z_g g\right) g_i^{-1}$$

Now observe that computing scalar products of type $< a'x, x >$, with the vector $x \in l^2(G)$ written in terms of the standard orthonormal basis, is a matter of describing how various products of group elements multiply up to 1 or not. In other words, our problem can be delinearized, and is in fact a problem about multiplication in F_2.

We need to find the relevant combinatorial property of F_2, and then the Dixmier type estimate will follow by translating everything in terms of $l^2(F_2)$. We have here:

PROPOSITION 4.15 (de la Harpe). *For any finite subset $S \subset F_2 - \{1\}$ there is a certain partition*

$$F_2 = D \sqcup E$$

and three elements g_1, g_2, g_3 such that $SD \cap D = \emptyset$ and $g_iE \cap g_jE = \emptyset$ for $i \neq j$.

PROOF. This is something quite straightforward, as follows:

(1) We write $F_2 =< h, c >$. For any $f \in F_2 - \{1\}$ consider the words $c^m f c^{-m}$ with m big. There are two cases. Either $f = c^s$ and all these words are equal to c^s, or f contains at least one h, in which case this h won't be affected by simplification of $c^m f c^{-m}$. Thus when simplifying h stays in the middle, and for m big this word will begin with a positive power of c and end with a negative power of c. The conclusion is that in both cases for big m the word $c^m f c^{-m}$ begins and ends with a power of c. This is true for any $f \neq 1$, and since S is finite we can take m big enough as to work for all its elements.

(2) In other words, we choose a number m such that $c^m f c^{-m}$ begins and ends with a power of c for any $f \in S$. Let D be the set of words which begin with c^{-m}. Let E to be the rest of F_2. For $i = 1, 2, 3$ define $g_i = h^i c^m$. We claim that this works.

(3) Indeed, the set g_iE is formed by elements of type $h^i c^m e$ with $e \in E$. Since e is known to begin with something different from c^{-m}, there is no simplification at left when making the product $h^i c^m e$, in the sense that the reduced word begins with h^i. This i value distinguishes the sets g_iE. In other words, we have $g_iE \cap g_jE = \emptyset$ for $i \neq j$.

(4) Consider an element of $SD \cap D$. This is at the same time of form $fc^{-m}h^\alpha \ldots$ with $f \in F$ and of form $c^{-m}h^\beta \ldots$ Thus we have an equality of type $c^m f c^{-m} h^\alpha \ldots = h^\beta \ldots$ But $c^m f c^{-m}$ is known to begin and to end with a power of c, and this shows first that there is

no simplification in $c^m f c^{-m} h^\alpha \ldots$ and second that $c^m f c^{-m} h^\alpha \ldots$ begins with a power of c. But this is a contradiction, and the proof of the result is now complete. □

By combining now the above ingredients, we are led to:

THEOREM 4.16 (Powers). *$C^*_{red}(F_2)$ is simple and has unique trace.*

PROOF. Let $A = C^*_{red}(F_2)$ and consider the faithful trace $tr(a) =< \delta_1, a\delta_1 >$. We use the Dixmier property. It is enough to prove that for $a = a^*$ with $tr(a) = 0$ there are unitaries u_1, u_2, u_3 giving the following estimate:

$$\left|\left|\frac{1}{3}\sum_i u_i a u_i^*\right|\right| \leq 0.98||a||$$

It is enough to do this for a in the dense $*$-subalgebra $\mathbb{C}[F_2]$. So, let us write:

$$a = \sum z_g g$$

We know that $tr(a)$ is equal to the number z_1, so this number z_1 is 0. Consider now the support of a, given as usual by the following formula:

$$S(a) = \left\{g \in F_2 \Big| z_g \neq 0\right\}$$

This is a finite subset of F_2 not containing 1, so we can apply the above result, and we obtain a certain partition $F_2 = D \sqcup E$, and certain elements g_1, g_2, g_3. The partition condition $F_2 = D \sqcup E$ translates into a direct sum decomposition, as follows:

$$l^2(F_2) = l^2(D) \oplus l^2(E)$$

Thus the orthogonal projections p and q onto the subspaces $l^2(D)$ and $l^2(E)$ are orthogonal, and sum up to the identity:

$$p \perp q \quad p + q = 1$$

Now observe that $S(a)D \cap D = \emptyset$ translates into the following equality:

$$pap = 0$$

With $u_i = g_i$, the orthogonal projection onto $l^2(g_i E)$ is $u_i q u_i^*$. The last condition $g_i E \cap g_j E = \emptyset$ for $i \neq j$ says that projections $u_i q u_i^*$ are pairwise orthogonal:

$$u_i q u_i^* \perp u_j q u_j^*$$

By linearity we can assume $||a|| = 1$. Now let ξ be a norm one vector. We want to estimate the products $< u_i a u_i^* \xi, \xi >$. The projections $u_i q u_i^*$ being orthogonal, at least one of them, say the one corresponding to $i = 1$, projects ξ to a vector of norm $\leq 1/3$:

$$||u_1 q u_1^* \xi|| \leq \frac{1}{3}$$

With $\xi_1 = u_1^*\xi$ we have $||q\xi_1||^2 \leq 1/3$, and it follows that we have:

$$||p\xi_1||^2 \geq \frac{2}{3} \quad , \quad ||paq\xi_1||^2 \leq \frac{1}{3}$$

On the other hand from $pap = 0$ and $p + q = 1$ we get:

$$pa = paq$$

But this latter formula gives the following estimate:

$$||a\xi_1 - \xi_1|| \geq ||pa\xi_1 - p\xi_1|| \geq ||p\xi_1|| - ||paq\xi_1|| \geq \frac{\sqrt{2}-1}{\sqrt{3}} = \delta_1 > 0$$

We can estimate in this way the scalar product $< a\xi_1, \xi_1 >$, and we get:

$$< a\xi_1, \xi_1 >= \frac{1}{2}\left(||a\xi_1||^2 + ||\xi_1||^2 - ||a\xi_1 - \xi_1||^2\right) \leq \frac{2-\delta_1^2}{2} = \delta_2 < 1$$

We have all ingredients for doing a final estimate, as follows:

$$< b\xi, \xi >\leq \frac{1}{3}\left(< a\xi_1, \xi_1 > + \sum_{i=2}^{3} ||u_i a u_i^*|| \cdot ||\xi_1||\right) \leq \frac{\delta_2 + 2}{3} = \delta_3 < 1$$

Replacing $a \to -a$ gives the same estimate for $- < b\xi, \xi >$ and we are done. □

4e. Exercises

Exercises:

Exercise 4.17.

Exercise 4.18.

Exercise 4.19.

Exercise 4.20.

Exercise 4.21.

Exercise 4.22.

Exercise 4.23.

Exercise 4.24.

Bonus exercise.

Part II

Quantum spaces

Acid green, aquamarine
The girls are dancing in the sand
And I throw my cellular device in the water
Can you reach me? No, you can't

CHAPTER 5

Finite spaces

5a. Finite spaces

Welcome to quantum spaces. Let us start this preliminary chapter on them with some philosophy, a bit a la Heisenberg. Based on what we know, we can formulate:

DEFINITION 5.1. *Given a von Neumann algebra $A \subset B(H)$, we write*

$$A = L^\infty(X)$$

and call X a quantum measured space.

As an example here, for the simplest noncommutative von Neumann algebra that we know, namely the usual matrix algebra $A = M_N(\mathbb{C})$, the formula that we want to write is as follows, with M_N being a certain mysterious quantum space:

$$M_N(\mathbb{C}) = L^\infty(M_N)$$

So, what can we say about this space M_N? As a first observation, this is a finite space, with its cardinality being defined and computed as follows:

$$|M_N| = \dim_{\mathbb{C}} M_N(\mathbb{C}) = N^2$$

Now since this is the same as the cardinality of the set $\{1, \ldots, N^2\}$, we are led to the conclusion that we should have a twisting result as follows, with the twisting operation $X \to X^\sigma$ being something that destroys the points, but keeps the cardinality:

$$M_N = \{1, \ldots, N^2\}^\sigma$$

From an analytic viewpoint now, we would like to understand what is the integration over M_N, giving rise to the corresponding L^∞ functions. And here, we can set:

$$\int_{M_N} A = tr(A)$$

To be more precise, on the left we have the integral of an arbitrary function on M_N, which according to our conventions, should be a usual matrix:

$$A \in L^\infty(M_N) = M_N(\mathbb{C})$$

As for the quantity on the right, the outcome of the computation, this can only be the trace of A. In addition, it is better to choose this trace to be normalized, by $tr(1) = 1$, and this in order for our measure on M_N to have mass 1, as it is ideal:

$$tr(A)=\frac{1}{N}Tr(A)$$

We can say even more about this. Indeed, since the traces of positive matrices are positive, we are led to the following formula, to be taken with the above conventions, which shows that the measure on M_N that we constructed is a probability measure:

$$A>0\implies\int_{M_N}A>0$$

Before going further, let us record what we found, for future reference:

THEOREM 5.2. *The quantum measured space M_N formally given by*

$$M_N(\mathbb{C})=L^\infty(M_N)$$

has cardinality N^2, appears as a twist, in a purely algebraic sense,

$$M_N=\{1,\ldots,N^2\}^\sigma$$

and is a probability space, its uniform integration being given by

$$\int_{M_N}A=tr(A)$$

where at right we have the normalized trace of matrices, $tr=Tr/N$.

PROOF. This is something half-informal, mostly for fun, which basically follows from the above discussion, the details and missing details being as follows:

(1) In what regards the formula $|M_N|=N^2$, coming by computing the complex vector space dimension, as explained above, this is obviously something rock-solid.

(2) Regarding twisting, we would like to have a formula as follows, with the operation $A\to A^\sigma$ being something that destroys the commutativity of the multiplication:

$$L^\infty(M_N)=L^\infty(1,\ldots,N^2)^\sigma$$

In more familiar terms, with usual complex matrices on the left, and with a better-looking product of sets being used on the right, this formula reads:

$$M_N(\mathbb{C})=L^\infty\Big(\{1,\ldots,N\}\times\{1,\ldots,N\}\Big)^\sigma$$

In order to establish this formula, consider the algebra on the right. As a complex vector space, this algebra has the standard basis $\{f_{ij}\}$ formed by the Dirac masses at the points (i,j), and the multiplicative structure of this algebra is given by:

$$f_{ij}f_{kl}=\delta_{ij,kl}$$

Now let us twist this multiplication, according to the formula $e_{ij}e_{kl} = \delta_{jk}e_{il}$. We obtain in this way the usual combination formulae for the standard matrix units $e_{ij} : e_j \to e_i$ of the algebra $M_N(\mathbb{C})$, and so we have our twisting result, as claimed.

(3) In what regards the integration formula in the statement, with the conclusion that the underlying measure on M_N is a probability one, this is something that we fully explained before, and as for the result (1) above, it is something rock-solid.

(4) As a last technical comment, observe that the twisting operation performed in (2) destroys both the involution, and the trace of the algebra. This is something quite interesting, which cannot be fixed, and we will back to it, later on. □

In order to advance now, based on the above result, the key point there is the construction and interpretation of the trace $tr : M_N(\mathbb{C}) \to \mathbb{C}$, as an integration functional. But this leads us into the following natural, and quite puzzling question:

QUESTION 5.3. *In the general context of Definition 5.1, where we formally wrote $A = L^\infty(X)$, what is the underlying integration functional $tr : A \to \mathbb{C}$?*

This is a quite subtle question, and there are several possible answers here. For instance, we would like the integration functional to have the following property:

$$tr(ab) = tr(ba)$$

And the problem is that certain von Neumann algebras do not possess such traces. This is actually something quite advanced, that we do not know yet, but by anticipating a bit, we are in trouble, and we must modify Definition 5.1, as follows:

DEFINITION 5.4 (update). *Given a von Neumann algebra $A \subset B(H)$, coming with a faithful positive unital trace $tr : A \to \mathbb{C}$, we write*

$$A = L^\infty(X)$$

and call X a quantum probability space. We also write the trace as $tr = \int_X$, and call it integration with respect to the uniform measure on X.

At the level of examples, passed the classical probability spaces X, we know from Theorem 5.2 that the quantum space M_N is a finite quantum probability space. But this raises the question of understanding what the finite quantum probability spaces are, in general. And the result here, extending what we know from chapter 2, is as follows:

THEOREM 5.5. *The finite dimensional von Neumann algebras $A \subset B(H)$ over an arbitrary Hilbert space H are exactly the direct sums of matrix algebras,*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

embedded into $B(H)$ by using a partition of unity of $B(H)$ with rank 1 projections

$$1 = P_1 + \ldots + P_k$$

with the "factors" $M_{n_i}(\mathbb{C})$ being each embedded into the algebra $P_iB(H)P_i$.

Proof. This is standard, as in the case $A\subset M_N(\mathbb C)$. Consider the center of A, which is a finite dimensional commutative von Neumann algebra, of the following form:

$$Z(A)=\mathbb C^k$$

Now let P_i be the Dirac mass at $i\in\{1,\ldots,k\}$. Then $P_i\in B(H)$ is an orthogonal projection, and these projections form a partition of unity, as follows:

$$1=P_1+\ldots+P_k$$

With $A_i=P_iAP_i$, we have then a non-unital $*$-algebra decomposition, as follows:

$$A=A_1\oplus\ldots\oplus A_k$$

On the other hand, it follows from the minimality of each of the projections $P_i\in Z(A)$ that we have unital $*$-algebra isomorphisms $A_i\simeq M_{n_i}(\mathbb C)$, and this gives the result. □

In order now to finish, we must solve the following equation:

$$L^\infty(X)=M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$$

But since the direct unions of sets correspond to direct sums at the level of the associated algebras of functions, in the classical case, we can take the following formula as a definition for a direct union of sets, in the general, noncommutative case:

$$L^\infty(X_1\sqcup\ldots\sqcup X_k)=L^\infty(X_1)\oplus\ldots\oplus L^\infty(X_k)$$

With this, and by remembering the definition of M_N, we are led to the conclusion that the solution to our quantum measured space equation above is as follows:

$$X=M_{n_1}\sqcup\ldots\sqcup M_{n_k}$$

For fully solving our problem, in the spirit of the new Definition 5.4, we still have to discuss the traces on $L^\infty(X)$. We are led in this way to the following statement:

Theorem 5.6. *The finite quantum measured spaces are the spaces*

$$X=M_{n_1}\sqcup\ldots\sqcup M_{n_k}$$

according to the following formula, for the associated algebras of functions:

$$L^\infty(X)=M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$$

The cardinality $|X|$ of such a space is the following number,

$$N=n_1^2+\ldots+n_k^2$$

and the possible traces are as follows, with $\lambda_i>0$ summing up to 1:

$$tr=\lambda_1tr_1\oplus\ldots\oplus\lambda_ktr_k$$

Among these traces, we have the canonical trace, appearing as

$$tr:L^\infty(X)\subset\mathcal L(L^\infty(X))\to\mathbb C$$

via the left regular representation, having weights $\lambda_i=n_i^2/N$.

PROOF. We have many assertions here, basically coming from the above discussion, with only the last one needing some explanations. Consider the left regular representation of our algebra $A = L^\infty(X)$, which is given by the following formula:

$$\pi : A \subset \mathcal{L}(A) \quad , \quad \pi(a) : b \to ab$$

We know that the algebra $\mathcal{L}(A)$ of linear operators $T : A \to A$ is isomorphic to a matrix algebra, and more specifically to $M_N(\mathbb{C})$, with $N = |X|$ being as before:

$$\mathcal{L}(A) \simeq M_N(\mathbb{C})$$

Thus, this algebra has a trace $tr : \mathcal{L}(A) \to \mathbb{C}$, and by composing this trace with the representation π, we obtain a certain trace $tr : A \to \mathbb{C}$, that we can call "canonical":

$$tr : A \subset \mathcal{L}(A) \to \mathbb{C}$$

We can compute the weights of this trace by using a multimatrix basis of A, formed by matrix units e^i_{ab}, with $i \in \{1, \ldots, k\}$ and with $a, b \in \{1, \ldots, n_i\}$, and we obtain:

$$\lambda_i = \frac{n_i^2}{N}$$

Thus, we are led to the conclusion in the statement. □

We will be back to quantum spaces on several occasions, in what follows. In fact, the present book is as much on operator algebras as it is on quantum spaces, and this because these two points of view are both useful, and complementary to each other.

5b. Quantum graphs

Time now to do some fancy quantum graph work, by adding some edges to the finite quantum spaces X constructed above. We have indeed the following straightforward extension of the usual notion of finite graph, obtained by using a finite quantum space X as set of vertices, and something quite general as adjacency matrix:

DEFINITION 5.7. *We call "finite quantum graph" a pair of type*

$$Z = (X, d)$$

with X being a finite quantum space, and $d \in M_N(\mathbb{C})$ being a matrix, where $N = |X|$.

This is of course something quite general and tricky, as we will soon discover, with our first observations about this notion being as follows:

(1) In the simplest case, that where $X = \{1, \ldots, N\}$ is a usual finite space, what we have here is a directed graph, with the edges $i \to j$ colored by complex numbers $d_{ij} \in \mathbb{C}$, and with self-edges $i \to i$ allowed too, again colored by numbers $d_{ii} \in \mathbb{C}$.

(2) In the general case, however, where X is arbitrary, the need for extra conditions of type $d = d^*$, or $d_{ii} = 0$, or $d \in M_N(\mathbb{R})$, or $d \in M_N(0,1)$ and so on, is not very natural, and it is best to use Definition 5.7 as such, with no restrictions on d.

In general, a quantum graph can be represented as a colored oriented graph on $\{1,\ldots,N\}$, where $N = |X|$, with the vertices being decorated by single indices i, and with the colors being complex numbers, namely the entries of d. This is similar to the formalism from before, but there is a discussion here in what regards the exact choice of the colors, which are usually irrelevant in connection with our symmetry problematics, and so can be true colors instead of complex numbers. More on this later.

There are many interesting examples of quantum graphs, and some theory too.

5c. Inclusions, diagrams

Another interesting thing that we can do with the finite quantum spaces is to look at inclusions between them, $X \subset Y$. Which might sound quite trivial, but in practice this is something quite tricky, and we are led in this way to the notion of Bratteli diagram.

5d. Basic construction

Following Jones, given an inclusion of finite quantum spaces $X_0 \subset X_1$, we can apply to it some sort of reflection procedure, called basic construction, as to get a second such inclusion, $X_0 \subset X_1 \subset X_2$. Moreover, we can iterate this construction, and we get a tower of such finite quantum spaces $X_0 \subset X_1 \subset X_2 \subset X_3 \subset \ldots$, called Jones tower. Many interesting things can be said about these Jones towers, and their combinatorics.

5e. Exercises

Exercises:

Exercise 5.8.

Exercise 5.9.

Exercise 5.10.

Exercise 5.11.

Exercise 5.12.

Exercise 5.13.

Exercise 5.14.

Exercise 5.15.

Bonus exercise.

CHAPTER 6

Amenability, again

6a. Products, revised

Products, revised.

6b. Inductive limits

Inductive limits.

6c. Hyperfiniteness

In order to get started, let us formulate the following definition:

DEFINITION 6.1. *A von Neumann algebra $A \subset B(H)$ is called hyperfinite when it appears as the weak closure of an increasing limit of finite dimensional algebras:*

$$A = \overline{\bigcup_i A_i}^w$$

When A is a II_1 factor, we call it hyperfinite II_1 factor, and we denote it by R.

As a first observation, there are many hyperfinite von Neumann algebras, for instance because any finite dimensional von Neumann algebra $A = \oplus_i M_{n_i}(\mathbb{C})$ is such an algebra, as one can see simply by taking $A_i = A$ for any i, in the above definition.

Also, given a measured space X, by using a dense sequence of points inside it, we can write $X = \bigcup_i X_i$ with $X_i \subset X$ being an increasing sequence of finite subspaces, and at the level of the corresponding algebras of functions this gives a decomposition as follows, which shows that the algebra $A = L^\infty(X)$ is hyperfinite, in the above sense:

$$L^\infty(X) = \overline{\bigcup_i L^\infty(X_i)}^w$$

The interesting point, however, is that when trying to construct II_1 factors which are hyperfinite, all the possible constructions lead in fact to the same factor, denoted R. This is an old theorem of Murray and von Neumann, that we will explain now.

In order to get started, we will need a number of technical ingredients. Generally speaking, out main tool will be the expectation $E_i : A \to A_i$ from a hyperfinite von

Neumann algebra A onto its finite dimensional subalgebras $A_i \subset A$, so talking about such conditional expectations will be our first task. Let us start with:

Proposition 6.2. *Given an inclusion of finite von Neumann algebras $A \subset B$, there is a unique linear map*

$$E : B \to A$$

which is positive, unital, trace-preserving and satisfies the following condition:

$$E(b_1 a b_2) = b_1 E(a) b_2$$

This map is called conditional expectation from B onto A.

Proof. We make use of the standard representation of the finite von Neumann algebra B, with respect to its trace $tr : B \to \mathbb{C}$, as constructed in chapter 10:

$$B \subset L^2(B)$$

If we denote by Ω the cyclic and separating vector of $L^2(B)$, we have an identification of vector spaces $A\Omega = L^2(A)$. Consider now the following orthogonal projection:

$$e : L^2(B) \to L^2(A)$$

It follows from definitions that we have an inclusion $e(B\Omega) \subset A\Omega$, and so our projection e induces by restriction a certain linear map, as follows:

$$E : B \to A$$

This linear map E and the orthogonal projection e are then related by:

$$exe = E(x)e$$

But this shows that the linear map E satisfies the various conditions in the statement, namely positivity, unitality, trace preservation and bimodule property. As for the uniqueness assertion, this follows by using the same argument, applied backwards, the idea being that a map E as in the statement must come from the projection e. □

Following Jones [51], who was a heavy user of such expectations, we will be often interested in what follows in the orthogonal projection $e : L^2(B) \to L^2(A)$ producing the expectation $E : B \to A$, rather than in E itself. So, let us formulate:

Definition 6.3. *Associated to any inclusion of finite von Neumann algebras $A \subset B$, as above, is the orthogonal projection*

$$e : L^2(B) \to L^2(A)$$

producing the conditional expectation $E : B \to A$ via the following formula:

$$exe = E(x)e$$

This projection is called Jones projection for the inclusion $A \subset B$.

We will heavily use Jones projections later, in the context where both the algebras A, B are II_1 factors, when systematically studying the inclusions of such II_1 factors $A \subset B$, called subfactors. In connection with our present hyperfiniteness questions, the idea, already mentioned above, will be that of using the conditional expectation $E_i : A \to A_i$ from a hyperfinite von Neumann algebra A onto its finite dimensional subalgebras $A_i \subset A$, as well as its Jones projection versions $e_i : L^2(A) \to L^2(A_i)$.

Let us start with a technical approximation result, as follows:

PROPOSITION 6.4. *Assume that a von Neumann algebra $A \subset B(H)$ appears as an increasing limit of von Neumann subalgebras*

$$A = \overline{\bigcup_i A_i}^w$$

and denote by $E_i : A \to A_i$ the corresponding conditional expectations.

1. *We have $||E_i(x) - x|| \to 0$, for any $x \in A$.*
2. *If $x_i \in A_i$ is a bounded sequence, satisfying $x_i = E_i(x_{i+1})$ for any i, then this sequence has a norm limit $x \in A$, satisfying $x_i = E_i(x)$ for any i.*

PROOF. Both the assertions are elementary, as follows:

(1) In terms of the Jones projections $e_i : L^2(A) \to L^2(A_i)$ associated to the expectations $E_i : A \to A_i$, the fact that the algebra A appears as the increasing union of its subalgebras A_i translates into the fact that the e_i are increasing, and converging to 1:

$$e_i \nearrow 1$$

But this gives $||E_i(x) - x|| \to 0$, for any $x \in A$, as desired.

(2) Let $\{x_i\} \subset A$ be a sequence as in the statement. Since this sequence was assumed to be bounded, we can pick a weak limit $x \in A$ for it, and we have then, for any i:

$$E_i(x) = x_i$$

Now by (1) we obtain from this $||x - x_n|| \to 0$, which gives the result. □

We have now all the needed ingredients for formulating a first key result, in connection with the hyperfinite II_1 factors, due to Murray-von Neumann, as follows:

PROPOSITION 6.5. *Given an increasing union on matrix algebras, the following construction produces a hyperfinite II_1 factor*

$$R = \overline{\bigcup_{n_i} M_{n_i}(\mathbb{C})}^w$$

called Murray-von Neumann hyperfinite factor.

PROOF. This basically follows from the above, in two steps, as follows:

(1) The von Neumann algebra R constructed in the statement is hyperfinite by definition, with the remark here that the trace on it $tr : R \to \mathbb{C}$ comes as the increasing union of the traces on the matrix components $tr : M_{n_i}(\mathbb{C}) \to \mathbb{C}$, and with all the details here being elementary to check, by using the usual standard form technology.

(2) Thus, it remains to prove that R is a factor. For this purpose, pick an element belonging to its center, $x \in Z(R)$, and consider its expectation on $A_i = M_{n_i}(\mathbb{C})$:

$$x_i = E_i(x)$$

We have then $x_i \in Z(A_i)$, and since the matrix algebra $A_i = M_{n_i}(\mathbb{C})$ is a factor, we deduce from this that this expected value $x_i \in A_i$ is given by:

$$x_i = tr(x_i)1 = tr(x)1$$

On the other hand, Proposition 6.4 applies, and shows that we have:

$$||x_i - x|| = ||E_i(x) - x|| \to 0$$

Thus our element is a scalar, $x = tr(x)1$, and so R is a factor, as desired. □

Next, we have the following substantial improvement of the above result, also due to Murray-von Neumann, which will be our final saying on the subject:

THEOREM 6.6. *There is a unique hyperfinite* II_1 *factor, called Murray-von Neumann hyperfinite factor R, which appears as an increasing union on matrix algebras,*

$$R = \overline{\bigcup_{n_i} M_{n_i}(\mathbb{C})}^w$$

with the isomorphism class of this union not depending on the exact sizes of the matrix algebras involved, nor on the particular inclusions between them.

PROOF. We already know from Proposition 6.5 that the union in the statement is a hyperfinite II_1 factor, for any choice of the matrix algebras involved, and of the inclusions between them. Thus, in order to prove the result, it all comes down in proving the uniqueness of the hyperfinite II_1 factor. But this can be proved as follows:

(1) Given a II_1 factor A, a von Neumann subalgebra $B \subset A$, and a subset $S \subset A$, let us write $S \subset_\varepsilon B$ when the following condition is satisfied, with $||x||_2 = \sqrt{tr(x^*x)}$:

$$\forall x \in S, \exists y \in B, ||x - y||_2 \leq \varepsilon$$

With this convention made, given a II_1 factor A, the fact that this factor is hyperfinite in the sense of Definition 6.1 tells us that for any finite subset $S \subset A$, and any $\varepsilon > 0$, we can find a finite dimensional von Neumann subalgebra $B \subset A$ such that:

$$S \subset_\varepsilon B$$

(2) With this observation made, assume that we are given a hyperfinite II_1 factor A. Let us pick a dense sequence $\{x_k\} \subset A$, and let us set:

$$S_k = \{x_1, \ldots, x_k\}$$

By choosing $\varepsilon = 1/k$ in the above, we can find, for any $k \in \mathbb{N}$, a finite dimensional von Neumann subalgebra $B_k \subset A$ such that the following condition is satisfied:

$$S_k \subset_{1/k} B_k$$

(3) Our first claim is that, by suitably choosing our subalgebra $B_k \subset A$, we can always assume that this is a matrix algebra, of the following special type:

$$B_k = M_{2^{n_k}}(\mathbb{C})$$

But this is something which is quite routine, which can be proved by starting with a finite dimensional subalgebra $B_k \subset A$ as above, and then perturbing its set of minimal projections $\{e_i\}$ into a set of projections $\{e_i'\}$ which are close in norm, and have as traces multiples of 2^n, with $n >> 0$. Indeed, the algebra $B_k' \subset A$ having these new projections $\{e_i'\}$ as minimal projections will be then arbitrarily close to the algebra B_k, and so will still contain the subset S_k in the above approximate sense, and due to our trace condition, will be contained in a subalgebra of type $B_k'' \simeq M_{2^{n_k}}(\mathbb{C})$, as desired.

(4) Our next claim, whose proof is similar, by using standard perturbation arguments for the corresponding sets of minimal projections, is that in the above the sequence of subalgebras $\{B_k\}$ can be chosen increasing. Thus, up to a rescaling of everything, we can assume that our sequence of subalgebras $\{B_k\}$ is as follows:

$$B_k = M_{2^k}(\mathbb{C})$$

(5) But this finishes the proof. Indeed, according to the above, we have managed to write our arbitrary hyperfinite II_1 factor A as a weak limit of the following type:

$$A = \overline{\bigcup_k M_{2^k}(\mathbb{C})}^{\,w}$$

Thus we have uniqueness indeed, and our result is proved. □

The above result is something quite fundamental, and adds to a series of similar results, or rather philosophical conclusions, which are quite surprising, as follows:

(1) We have seen early on in this book that, up to isomorphism, there is only one Hilbert to be studied, namely the infinite dimensional separable Hilbert space, which can be taken to be, according to knowledge and taste, either $H = L^2(\mathbb{R})$, or $H = l^2(\mathbb{N})$.

(2) Regarding now the study of the operator algebras $A \subset B(H)$ over this unique Hilbert space, another somewhat surprising conclusion, from the above, is that we won't miss much by assuming that $A = M_N(L^\infty(X))$ is a random matrix algebra.

(3) And now, guess what, what we just found is that when trying to get beyond random matrices, and what can be done with them, we are led to yet another unique von Neumann algebra, namely the above Murray-von Neumann hyperfinite II_1 factor R.

(4) And for things to be complete, we will see later that when getting beyond type II_1, things won't change, because the other types of hyperfinite factors, not necessarily of type II_1, can be all shown to ultimately come from R, via various constructions.

All this is certainly quite interesting, philosophically speaking. All in all, always the same conclusion, no need to go far to get to interesting algebras and questions: these interesting algebras and questions are just there, the most obvious ones.

Now back to more concrete things, one question is about how to best think of R, with Theorem 6.6 as stated not providing us with an answer. To be more precise, we would like to know what is the "best model" for R, that is, what exact matrix algebras should we use in practice, and with which inclusions between them. And here, a look at the proof of Theorem 6.6 suggests that the "best writing" of R is as follows:

$$R=\overline{\bigcup_k M_{2^k}(\mathbb{C})}^{\,w}$$

And we can in fact do even better, by observing that the inclusions between matrix algebras of size 2^k appear via tensor products, and formulating things as follows:

Proposition 6.7. *The hyperfinite* II_1 *factor* R *appears as*

$$R=\overline{\bigotimes_{r\in\mathbb{N}} M_2(\mathbb{C})}^{\,w}$$

with the infinite tensor product being defined as an inductive limit, in the obvious way.

Proof. This follows from the above discussion, and with the remark that there is a binary choice there, of left/right type, to be made when constructing the inductive limit. And we prefer here not to make any choice, and leave things like this, because the best choice here always depends on the precise applications that you have in mind. □

Along the same lines, we can ask as well for precise group algebra models for the hyperfinite II_1 factor, $R=L(\Gamma)$, and the canonical choice here is as follows:

Proposition 6.8. *The hyperfinite* II_1 *factor* R *appears as*

$$R=L(S_\infty)$$

with $S_\infty=\bigcup_{r\in\mathbb{N}}S_r$ *being the infinite symmetric group.*

Proof. Consider indeed the infinite symmetric group S_∞, which is by definition the group of permutations of $\{1,2,3,\ldots\}$ having finite support. Since such an infinite permutation with finite support must appear by extending a certain finite permutation $\sigma\in S_r$, with fixed points outside $\{1,\ldots,r\}$, we have then, as stated:

$$S_\infty=\bigcup_{r\in\mathbb N}S_r$$

But this shows that the von Neumann algebra $L(S_\infty)$ is hyperfinite. On the other hand S_∞ has the ICC property, and so $L(S_\infty)$ is a II_1 factor. Thus, $L(S_\infty)=R$. □

There are of course some more things that can be said here, because other groups of the same type as S_∞, namely appearing as increasing limits of finite subgroups, and having the ICC property, will produce as well the hyperfinite factor, $L(\Gamma)=R$, and so there is some group theory to be done here, in order to fully understand such groups.

However, we prefer to defer the discussion for later, after learning about amenability, which will lead to a substantial update of our theory, making such things obsolete.

As an interesting consequence of all this, however, let us formulate:

Proposition 6.9. *Given two groups Γ,Γ', each having the ICC property, and each appearing as an increasing union of finite subgroups, we have*

$$L(\Gamma)\simeq L(\Gamma')$$

while the corresponding group algebras might not be isomorphic, $\mathbb C[\Gamma]\neq\mathbb C[\Gamma']$.

Proof. Here the first assertion follows from the above discusssion, the von Neumann algebra in question being the hyperfinite II_1 factor R. As for the last assertion, there are countless counterexamples here, all coming from basic group theory. □

The point with the above result is that the isomorphisms of type $L(\Gamma)\simeq L(\Gamma')$ are in general impossible to prove with bare hands. Thus, we can see here the power of the Murray-von Neumann results. And we can also see the magic of the weak topology, which by some kind of miracle, makes everyone equal in the end.

6d. Amenability

The hyperfinite II_1 factor R, which is a quite fascinating object, was heavily investigated by Murray-von Neumann, and then by Connes. There are many things that can be said about it, which all interesting, but are usually quite technical as well.

As a central result here, in what regards advanced hyperfiniteness theory, we have the following theorem of Connes, whose proof is something remarkably heavy, and which is arguably the deepest result in operator algebra related functional analysis:

THEOREM 6.10. *For a finite von Neumann algebra A, the following are equivalent:*

(1) *A is hyperfinite in the usual sense, namely it appears as the weak closure of an increasing limit of finite dimensional algebras:*

$$A=\overline{\bigcup_i A_i}^{\,w}$$

(2) *A amenable, in the sense that the standard inclusion $A\subset B(H)$, with $H=L^2(A)$, admits a conditional expectation $E:B(H)\to A$.*

PROOF. This result, due to Connes, is something fairly heavy, that only a handful of people have really managed to understand, the idea being as follows:

(1) $\implies$ (2) Assuming that the algebra A is hyperfinite, let us write it as the weak closure of an increasing limit of finite dimensional subalgebras:

$$A=\overline{\bigcup_i A_i}^{\,w}$$

Consider the inclusion $A\subset B(H)$, with $H=L^2(A)$. In order to construct an expectation $E:B(H)\to A$, let us pick an ultrafilter ω on $\mathbb N$. Given $T\in B(H)$, we can define the following quantity, with μ_i being the Haar measure on the unitary group $U(A_i)$:

$$\psi(T)=\lim_{i\to\omega}\int_{U(A_i)}UTU^*\,d\mu_i(U)$$

With this construction made, by using now the standard involution $J:H\to H$, given by the formula $T\to T^*$, we can further define a map as follows:

$$E:B(H)\to A\quad,\quad E(T)=J\psi(T)J$$

But this is the expectation that we are looking for, with its left and right invariance properties coming from the left and right invariance of each Haar measure μ_i.

(2) $\implies$ (1) This is something heavy, using lots of advanced functional analysis, and for details here, we refer to Connes' original paper. □

So, this was for the story with Theorem 6.10, very simplified as to fit here in 1 page. Of course, do not hesitate to get tp the original paper of Connes, and learn things from there in detail, all this is first-class functional analysis, definitely worth learning.

We should also mention, in relation with this, that Connes' results, from his original paper, besides proving the above implication (2) $\implies$ (1), provide also a considerable extension of Theorem 6.10, with a number of further equivalent formulations of the notion of amenability, which are a bit more technical, but all good to know.

The story here, still a bit simplified, is as follows:

Fact 6.11 (Connes). *For a finite von Neumann algebra A, the following conditions are in fact equivalent:*

(1) *A is hyperfinite, in the sense that it appears as the weak closure of an increasing limit of finite dimensional algebras:*
$$A=\overline{\bigcup_i A_i}^{\,w}$$
(2) *A amenable, in the sense that the standard inclusion $A\subset B(H)$, with $H=L^2(A)$, admits a conditional expectation:*
$$E:B(H)\to A$$
(3) *There exist unit vectors $\xi_n\in L^2(A)\otimes L^2(A)$ such that, for any $x\in A$:*
$$||x\xi_n-\xi_n x||_2\to 0\quad,\quad <x\xi_n,\xi_n>\to tr(x)$$
(4) *For any $x_1,\ldots,x_k\in A$ and $y_1,\ldots,y_k\in A$ we have:*
$$\left|tr\left(\sum_i x_iy_i\right)\right|\leq\left|\left|\sum_i x_i\otimes y_i^{opp}\right|\right|_{min}$$

Again, this is something technical and advanced, that we won't get into, in this book. Let us mention however that the idea with all this is as follows:

(1) $\implies$ (2) is elementary, as explained above.

(2) $\implies$ (3) can be proved by using an inequality due to Powers-Størmer.

(3) $\implies$ (4) is something quite technical, but doable as well.

(4) $\implies$ (2) is again something technical, but doable as well.

(2) $\implies$ (1) is, as before in Theorem 6.10, the difficult implication.

Regarding the difficult implication, (2) $\implies$ (1), the difficulty here comes of course from the fact that, no matter what beautiful abstract functional analysis things you know about A, at some point you will have to get to work, and construct that finite dimensional subalgebras $A_i\subset A$, and it is not even clear where to start from. For a solution to this problem, and for more, we refer to Connes's article, and also to his book [**19**].

Getting back now to more everyday mathematics, the above results as stated remain something quite abstract, and advanced, and understanding their concrete implications will be our next task. In the case of the II_1 factors, we have the following result:

Theorem 6.12. *For a II_1 factor R, the following are equivalent:*

(1) *R amenable, in the sense that we have an expectation, as follows:*
$$E:B(L^2(R))\to R$$
(2) *R is the Murray-von Neumann hyperfinite II_1 factor.*

PROOF. This follows indeed from Theorem 6.10, when coupled with the Murray-von Neumann uniqueness result for the hyperfinite II_1 factor, from Theorem 6.6. □

As another application, getting back now to the general case, that of the finite von Neumann algebras, from Theorem 6.10 as stated, a first question is about how all this applies to the group von Neumann algebras, and more generally to the quantum group von Neumann algebras $L(\Gamma)$. In order to discuss this, let us start with the case of the usual discrete groups Γ. We will need the following result, which is standard:

THEOREM 6.13. *For a discrete group Γ, the following two conditions are equivalent, and if they are satisfied, we say that Γ is amenable:*

(1) *Γ admits an invariant mean $m : l^\infty(\Gamma) \to \mathbb{C}$.*
(2) *The projection map $C^*(\Gamma) \to C^*_{red}(\Gamma)$ is an isomorphism.*

Moreover, the class of amenable groups contains all the finite groups, all the abelian groups, and is stable under taking subgroups, quotients and products.

PROOF. This is something very standard, the idea being as follows:

(1) The equivalence (1) $\iff$ (2) is standard, with the amenability conditions (1,2) being in fact part of a much longer list of amenability conditions, including well-known criteria of Følner, Kesten and others. We will be back to this, with details, in a moment, directly in a more general setting, that of the discrete quantum groups.

(2) As for the last assertion, regarding the finite groups, the abelian groups, and then the stability under taking subgroups, quotients and products, this is something elementary, which follows by using either of the above definitions of the amenability. □

Getting back now to operator algebras, we can complement Theorem 6.10 with:

THEOREM 6.14. *For a group von Neumann algebra $A = L(\Gamma)$, the following conditions are equivalent:*

(1) *A is hyperfinite.*
(2) *A amenable.*
(3) *Γ is amenable.*

PROOF. The group von Neumann algebras $A = L(\Gamma)$ being by definition finite, Theorem 6.10 applies, and gives the equivalence (1) $\iff$ (2). Thus, it remains to prove that we have (2) $\iff$ (3), and we can prove this as follows:

(2) $\implies$ (3) This is something clear, because if we assume that $A = L(\Gamma)$ is amenable, we have by definition a conditional expectation $E : B(L^2(A)) \to A$, and the restriction of this conditional expectation is the desired invariant mean $m : l^\infty(\Gamma) \to \mathbb{C}$.

(3) $\implies$ (2) Assume that we are given a discrete amenable group Γ. In view of Theorem 6.13, this means that Γ has an invariant mean, as follows:

$$m : l^\infty(\Gamma) \to \mathbb{C}$$

Consider now the Hilbert space $H = l^2(\Gamma)$, and for any operator $T \in B(H)$ consider the following map, which is a bounded sesquilinear form:

$$\varphi_T : H \times H \to \mathbb{C}$$
$$(\xi, \eta) \to m\left[\gamma \to < \rho_\gamma T \rho_\gamma^* \xi, \eta >\right]$$

By using the Riesz representation theorem, we conclude that there exists a certain operator $E(T) \in B(H)$, such that the following holds, for any two vectors ξ, η:

$$\varphi_T(\xi, \eta) = < E(T)\xi, \eta >$$

Summarizing, to any operator $T \in B(H)$ we have associated another operator, denoted $E(T) \in B(H)$, such that the following formula holds, for any two vectors ξ, η:

$$< E(T)\xi, \eta > = m\left[\gamma \to < \rho_\gamma T \rho_\gamma^* \xi, \eta >\right]$$

In order to prove now that this linear map E is the desired expectation, observe that for any group element $g \in \Gamma$, and any two vectors $\xi, \eta \in H$, we have:

$$\begin{aligned} < \rho_g E(T) \rho_g^* \xi, \eta > &= < E(T) \rho_g^* \xi, \rho_g^* \eta > \\ &= m\left[\gamma \to < \rho_\gamma T \rho_\gamma^* \rho_g^* \xi, \rho_g^* \eta >\right] \\ &= m\left[\gamma \to < \rho_{g\gamma} T \rho_{g\gamma}^* \xi, \eta >\right] \\ &= m\left[\gamma \to < \rho_\gamma T \rho_\gamma^* \xi, \eta >\right] \\ &= < E(T)\xi, \eta > \end{aligned}$$

Since this is valid for any $\xi, \eta \in H$, we conclude that we have, for any $g \in \Gamma$:

$$\rho_g E(T) \rho_g^* = E(T)$$

But this shows that the element $E(T) \in B(H)$ is in the commutant of the right regular representation of Γ, and so belongs to the left regular group algebra of Γ:

$$E(T) \in L(\Gamma)$$

Summarizing, we have constructed a certain linear map $E : B(H) \to L(\Gamma)$. Now by using the above explicit formula of it, in terms of $m : l^\infty(\Gamma) \to \mathbb{C}$, which was assumed to be an invariant mean, we conclude that E is indeed an expectation, as desired. □

As a very concrete application of all this technology, in relation now with the discrete group algebras which are II_1 factors, the results that we have lead to:

THEOREM 6.15. *For a discrete group Γ, the following conditions are equivalent:*

(1) *Γ is amenable, and has the ICC property.*
(2) *$A = L(\Gamma)$ is the hyperfinite II_1 factor R.*

PROOF. This follows indeed from Theorem 6.14, coupled with the standard fact, that we know well from chapter 4, that a group algebra $A = L(\Gamma)$ is a factor, and so a II_1 factor, precisely when the group Γ has the ICC property. □

6e. Exercises

Exercises:

Exercise 6.16.

Exercise 6.17.

Exercise 6.18.

Exercise 6.19.

Exercise 6.20.

Exercise 6.21.

Exercise 6.22.

Exercise 6.23.

Bonus exercise.

CHAPTER 7

Spheres, revised

7a. Quantum groups

In order to better understand the structure of $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}$, we need to talk about free rotations. Following Woronowicz [99], let us start with:

DEFINITION 7.1. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras as follows,

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to\mathbb{C}\quad,\quad S:A\to A^{opp}$$

called comultiplication, counit and antipode.

Obviously, this is something tricky, and we will see details in a moment, the idea being that these are the axioms which best fit with what we want to do, in this book. Let us also mention, technically, that $\otimes$ in the above can be any topological tensor product, and with the choice of $\otimes$ being irrelevant, but more on this later. Also, A^{opp} is the opposite algebra, with multiplication $a\cdot b=ba$, and more on this later too.

We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. With this convention, we have the following key result, from Woronowicz [99]:

PROPOSITION 7.2. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi)=(g,h)\to\varphi(gh)\quad,\quad\varepsilon(\varphi)=\varphi(1)\quad,\quad S(\varphi)=g\to\varphi(g^{-1})$$

(2) *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g)=g\otimes g\quad,\quad\varepsilon(g)=1\quad,\quad S(g)=g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. This is something very standard, the idea being as follows:

(1) Given $G \subset U_N$, we can set $A = C(G)$, which is a Woronowicz algebra, together with the matrix $u = (u_{ij})$ formed by coordinates of G, given by:

$$g = \begin{pmatrix} u_{11}(g) & \ldots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \ldots & u_{NN}(g) \end{pmatrix}$$

Conversely, if (A, u) is a commutative Woronowicz algebra, by using the Gelfand theorem we can write $A = C(X)$, with X being a certain compact space. The coordinates u_{ij} give then an embedding $X \subset M_N(\mathbb{C})$, and since the matrix $u = (u_{ij})$ is unitary we actually obtain an embedding $X \subset U_N$, and finally by using the maps Δ, ε, S we conclude that our compact subspace $X \subset U_N$ is in fact a compact Lie group, as desired.

(2) Consider a finitely generated group $F_N \to \Gamma$. We can set $A = C^*(\Gamma)$, which is by definition the completion of the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by $g^* = g^{-1}$, for any $g \in \Gamma$, with respect to the biggest C^*-norm, and we obtain a Woronowicz algebra, together with the diagonal matrix formed by the generators of Γ:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Conversely, if (A, u) is a cocommutative Woronowicz algebra, the Peter-Weyl theory of Woronowicz, to be explained below, shows that the irreducible corepresentations of A are all 1-dimensional, and form a group Γ, and so we have $A = C^*(\Gamma)$, as desired. Thus, theorem proved, modulo a representation theory discussion, to come soon. □

In general now, the structural maps Δ, ε, S have the following properties:

Proposition 7.3. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$
$$(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$$

(2) *S satisfies the antipode axiom, on the $*$-subalgebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

Proof. Observe first that the result holds in the case where A is commutative. Indeed, by using Proposition 7.2 (1) we can write:

$$\Delta = m^t \quad , \quad \varepsilon = u^t \quad , \quad S = i^t$$

The 3 conditions in the statement come then by transposition from the basic 3 group theory conditions satisfied by m, u, i, which are as follows, with $\delta(g) = (g, g)$:

$$m(m \times id) = m(id \times m)$$
$$m(id \times u) = m(u \times id) = id$$
$$m(id \times i)\delta = m(i \times id)\delta = 1$$

Observe also that the result holds as well in the case where A is cocommutative, by using Proposition 7.2 (1). In the general case now, the proof goes as follows:

(1) We have the following computation:

$$(\Delta \otimes id)\Delta(u_{ij}) = \sum_l \Delta(u_{il}) \otimes u_{lj} = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

We have as well the following computation, which gives the first formula:

$$(id \otimes \Delta)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \Delta(u_{kj}) = \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}$$

On the other hand, we have the following computation:

$$(id \otimes \varepsilon)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \varepsilon(u_{kj}) = u_{ij}$$

We have as well the following computation, which gives the second formula:

$$(\varepsilon \otimes id)\Delta(u_{ij}) = \sum_k \varepsilon(u_{ik}) \otimes u_{kj} = u_{ij}$$

(2) By using the fact that the matrix $u = (u_{ij})$ is unitary, we obtain:

$$\begin{aligned} m(id \otimes S)\Delta(u_{ij}) &= \sum_k u_{ik} S(u_{kj}) \\ &= \sum_k u_{ik} u_{jk}^* \\ &= (uu^*)_{ij} \\ &= \delta_{ij} \end{aligned}$$

We have as well the following computation, which gives the result:

$$\begin{aligned} m(S \otimes id)\Delta(u_{ij}) &= \sum_k S(u_{ik}) u_{kj} \\ &= \sum_k u_{ki}^* u_{kj} \\ &= (u^*u)_{ij} \\ &= \delta_{ij} \end{aligned}$$

(3) Finally, the formula $S^2 = id$ holds as well on generators, and so in general too. □

Let us record as well the following technical result:

Proposition 7.4. *Given a Woronowicz algebra (A,u), we have $u^t=\bar{u}^{-1}$, so u is biunitary, in the sense that it is unitary, with unitary transpose.*

Proof. We have the following computation, based on the fact that u is unitary:

$$\begin{aligned}(uu^*)_{ij}=\delta_{ij}\quad&\implies\quad\sum_kS(u_{ik}u_{jk}^*)=\delta_{ij}\\&\implies\quad\sum_ku_{kj}u_{ki}^*=\delta_{ij}\\&\implies\quad(u^t\bar{u})_{ji}=\delta_{ij}\end{aligned}$$

Similarly, we have the following computation, once agan using the unitarity of u:

$$\begin{aligned}(u^*u)_{ij}=\delta_{ij}\quad&\implies\quad\sum_kS(u_{ki}^*u_{kj})=\delta_{ij}\\&\implies\quad\sum_ku_{jk}^*u_{ik}=\delta_{ij}\\&\implies\quad(\bar{u}u^t)_{ji}=\delta_{ij}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, the Woronowicz algebras appear to have nice properties. In view of Proposition 7.2 and Proposition 7.3, we can formulate the following definition:

Definition 7.5. *Given a Woronowicz algebra A, we formally write*

$$A=C(G)=C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is commutative and cocommutative, G and Γ are usual abelian groups, dual to each other. In general, we still agree to write $G=\widehat{\Gamma},\Gamma=\widehat{G}$, but in a formal sense. As a final piece of general theory now, let us complement Definition 7.1 with:

Definition 7.6. *Given two Woronowicz algebras (A,u) and (B,v), we write*

$$A\simeq B$$

and identify the corresponding quantum groups, when we have an isomorphism

$$<u_{ij}>\simeq<v_{ij}>$$

of $*$-algebras, mapping standard coordinates to standard coordinates.

With this convention, which is in tune with our conventions for algebraic manifolds from chapter 1, and more on this later, any compact or discrete quantum group corresponds to a unique Woronowicz algebra, up to equivalence. Also, we can see now why in

Definition 7.1 the choice of the exact topological tensor product $\otimes$ is irrelevant. Indeed, no matter what tensor product $\otimes$ we use there, we end up with the same Woronowicz algebra, and the same compact and discrete quantum groups, up to equivalence.

In practice, we will use in what follows the simplest such tensor product $\otimes$, which is the maximal one, obtained as completion of the usual algebraic tensor product with respect to the biggest C^*-norm. With the remark that this product is something rather abstract, and so can be treated, in practice, as a usual algebraic tensor product.

Moving ahead now, let us call corepresentation of A any unitary matrix $v \in M_n(\mathcal{A})$, where $\mathcal{A} =< u_{ij} >$, satisfying the same conditions are those satisfied by u, namely:

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

These corepresentations can be then thought of as corresponding to the finite dimensional unitary smooth representations of the underlying compact quantum group G. Following Woronowicz [99], we have the following key result:

Theorem 7.7. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

which can be constructed by starting with any faithful positive form $\varphi \in A^*$, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation $v \in M_n(\mathbb{C}) \otimes A$ we have

$$\left(id \otimes \int_G\right) v = P$$

where P is the orthogonal projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Proof. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a = (\tau \otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v)=\left\{\xi\in\mathbb C^n\Big|v\xi=\xi\right\}$$

The point now is that when $\varphi\in A^*$ is faithful, by using a standard positivity trick, one can prove that we have $P_\varphi=P$. Assume indeed $P_\varphi\xi=\xi$, and let us set:

$$a=\sum_i\left(\sum_j v_{ij}\xi_j-\xi_i\right)\left(\sum_k v_{ik}\xi_k-\xi_i\right)^*$$

We must prove that we have $a=0$. Since v is biunitary, we have:

$$\begin{aligned}
a&=\sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar{\xi}_k-\frac{1}{N}\bar{\xi}_i\right)\right)\\
&=\sum_{ijk}v_{ij}v_{ik}^*\xi_j\bar{\xi}_k-\frac{1}{N}v_{ij}\xi_j\bar{\xi}_i-\frac{1}{N}v_{ik}^*\xi_i\bar{\xi}_k+\frac{1}{N^2}\xi_i\bar{\xi}_i\\
&=\sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar{\xi}_i-\sum_{ik}v_{ik}^*\xi_i\bar{\xi}_k+\sum_i|\xi_i|^2\\
&=||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&=2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $P_\varphi\xi=\xi$, we obtain from this:

$$\begin{aligned}
\varphi(a)&=2\varphi(||\xi||^2-Re(<v\xi,\xi>))\\
&=2(||\xi||^2-Re(<P_\varphi\xi,\xi>))\\
&=2(||\xi||^2-||\xi||^2)\\
&=0
\end{aligned}$$

Now since φ is faithful, this gives $a=0$, and so $v\xi=\xi$. Thus $\int_\varphi$ is independent of φ, and is given on coefficients $a=(\tau\otimes id)v$ by the following formula:

$$\left(id\otimes\int_\varphi\right)v=P$$

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

Consider the dense $*$-subalgebra $\mathcal{A}\subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$<a,b>=\int_G ab^*$$

We have then the following result, also due to Woronowicz [99]:

THEOREM 7.8. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
3. *$\mathcal{A}=\bigoplus_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [99], the idea being as follows:

(1) Given a corepresentation $v\in M_n(A)$, consider its interwiner algebra:

$$End(v)=\left\{T\in M_n(\mathbb{C})\Big|Tv=vT\right\}$$

It is elementary to see that this is a finite dimensional C^*-algebra, and we conclude from this that we have a decomposition as follows:

$$End(v)=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1=p_1+\ldots+p_k$$

But this decomposition allows us to define subcorepresentations $v_i\subset v$, which are irreducible, so we obtain, as desired, a decomposition $v=v_1+\ldots+v_k$.

(2) To any corepresentation $v\in M_n(A)$ we associate its space of coefficients, given by $C(v)=span(v_{ij})$. The construction $v\to C(v)$ is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A}=\sum_{k\in\mathbb{N}*\mathbb{N}}C(u^{\otimes k})$$

Now given an arbitrary corepresentation $v\in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v)\subset\mathcal{A}$, and so we must have, for certain positive integers $k_1,\ldots,k_p$, an inclusion of vector spaces, as follows:

$$C(v)\subset C(u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v\subset u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p}$$

Thus, by using (1), we are led to the conclusion in the statement.

(3) By using (1) and (2), we obtain a linear space decomposition as follows:

$$\mathcal{A}=\sum_{v\in Irr(A)}C(v)=\sum_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$$

In order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C(v)\perp C(w)$$

As a first observation, which follows from an elementary computation, for any two corepresentations v,w we have a Frobenius type isomorphism, as follows:

$$Hom(v,w)\simeq Fix(\bar{v}\otimes w)$$

Now let us set $P_{ia,jb}=\int_G v_{ij}w_{ab}^*$. According to Theorem 7.7, the matrix P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(\bar{v},\bar{w})=\{0\}$$

Thus we have $P=0$, and so $C(v)\perp C(w)$, which gives the result.

(4) The algebra $\mathcal{A}_{central}$ contains indeed all the characters, because we have:

$$\Sigma\Delta(\chi_v)=\sum_{ij}v_{ji}\otimes v_{ij}=\Delta(\chi_v)$$

The fact that the characters span $\mathcal{A}_{central}$, and form an orthogonal basis of it, follow from (3). Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl}=\int_G v_{ij}v_{kl}^*$$

We know from Theorem 7.7 that these integrals form the orthogonal projection onto $Fix(v\otimes\bar{v})\simeq End(\bar{v})=\mathbb{C}1$. By using this fact, we obtain the following formula:

$$\int_G\chi_v\chi_v^*=\sum_{ij}\int_G v_{ii}v_{jj}^*=\sum_i\frac{1}{N}=1$$

Thus the characters have indeed norm 1, and we are done. □

We refer to Woronowicz [99] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A=C^*(\Gamma)$ for some discrete group Γ, as mentioned in Proposition 7.2.

At a more technical level now, we have a number of more advanced results, from Woronowicz [99], [100] and other papers, that must be known as well. We will present them quickly, and for details you check my book [8]. First we have:

THEOREM 7.9. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some standard functional analysis. □

Yet another important result is Tannakian duality, as follows:

THEOREM 7.10. *The following operations are inverse to each other:*

1. *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
2. *The construction $C \to A$, which associates to a tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. This is something quite deep, the idea being as follows:

(1) We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

(2) We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, after some elementary algebra we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [100]. □

7b. Free rotations

Good news, with the above general theory in hand, we can go back now to our free geometry program, as developed in chapter 3, and substantially build on that. Indeed, the point is that we can talk now about free rotations. Following Wang, we have:

Theorem 7.11. *The following constructions produce compact quantum groups,*

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)$$

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

which appear respectively as liberations of the groups O_N and U_N.

Proof. This first assertion follows from the elementary fact that if a matrix $u=(u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Indeed, the biunitarity of u^Δ can be checked by a direct computation. Regarding now the matrix $u^\varepsilon=1_N$, this is clearly biunitary. Also, regarding the matrix u^S, there is nothing to prove here either, because its unitarity its clear too. And finally, observe that if u has self-adjoint entries, then so do the above matrices $u^\Delta,u^\varepsilon,u^S$.

Thus our claim is proved, and we can define morphisms Δ,ε,S as in Definition 7.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, this follows exactly as for the free spheres, by adapting the sphere proof from chapter 3. □

The basic properties of O_N^+,U_N^+ can be summarized as follows:

Theorem 7.12. *The quantum groups O_N^+,U_N^+ have the following properties:*

1. *The closed subgroups $G\subset U_N^+$ are exactly the $N\times N$ compact quantum groups. As for the closed subgroups $G\subset O_N^+$, these are those satisfying $u=\bar{u}$.*
2. *We have liberation embeddings $O_N\subset O_N^+$ and $U_N\subset U_N^+$, obtained by dividing the algebras $C(O_N^+),C(U_N^+)$ by their respective commutator ideals.*
3. *We have as well embeddings $\widehat{L}_N\subset O_N^+$ and $\widehat{F}_N\subset U_N^+$, where L_N is the free product of N copies of $\mathbb{Z}_2$, and where F_N is the free group on N generators.*

Proof. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 7.1, the formula $S(u_{ij})=u_{ji}^*$ shows that the matrix $\bar{u}$ must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N),C(U_N)$, similar to those in Theorem 7.11, but with the commutativity between the standard coordinates and their adjoints added:

$$C(O_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)$$

$$C(U_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

Thus, we are led to the conclusion in the statement.

(3) This follows indeed from (1) and from Proposition 7.2, with the remark that with $u = diag(g_1, \ldots, g_N)$, the condition $u = \bar{u}$ is equivalent to $g_i^2 = 1$, for any i. □

The last assertion in Theorem 7.12 suggests the following construction:

Proposition 7.13. *Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \middle| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda = < g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

With this notion in hand, Theorem 7.12 (3) reformulates as follows:

Theorem 7.14. *The diagonal tori of the basic unitary groups are the basic tori:*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

In particular, the basic unitary groups are all distinct.

Proof. This is something clear and well-known in the classical case, and in the free case, this is a reformulation of Theorem 7.12 (3), which tells us that the diagonal tori of O_N^+, U_N^+, in the sense of Proposition 7.13, are the group duals $\widehat{L}_N, \widehat{F}_N$. □

There is an obvious relation here with the considerations from chapter 3, that we will analyse later on. As a second result now regarding our free quantum groups, relating them this time to the free spheres constructed in chapter 3, we have:

THEOREM 7.15. *We have embeddings of algebraic manifolds as follows, obtained in double indices by rescaling the coordinates, $x_{ij} = u_{ij}/\sqrt{N}$:*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad \to \quad \begin{matrix} S^{N^2-1}_{\mathbb{R},+} & \longrightarrow & S^{N^2-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N^2-1}_{\mathbb{R}} & \longrightarrow & S^{N^2-1}_{\mathbb{C}} \end{matrix}$$

Moreover, the quantum groups appear from the quantum spheres via

$$G = S \cap U_N^+$$

with the intersection being computed inside the free sphere $S^{N^2-1}_{\mathbb{C},+}$.

PROOF. As explained in Theorem 7.12, the biunitarity of the matrix $u = (u_{ij})$ gives an embedding of algebraic manifolds, as follows:

$$U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$$

Now since the relations defining $O_N, O_N^+, U_N \subset U_N^+$ are the same as those defining $S^{N^2-1}_{\mathbb{R}}, S^{N^2-1}_{\mathbb{R},+}, S^{N^2-1}_{\mathbb{C}} \subset S^{N^2-1}_{\mathbb{C},+}$, this gives the result. □

Summarizing, we have now up and working some free rotation groups, which are closely related to the free spheres and tori constructed in chapter 3.

7c. Quantum isometries

In order to further discuss now the relation with the spheres, which can only come via some sort of "isometric actions", let us start with the following standard fact:

PROPOSITION 7.16. *Given a closed subset $X \subset S^{N-1}_{\mathbb{C}}$, the formula*

$$G(X) = \left\{ U \in U_N \Big| U(X) = X \right\}$$

defines a compact group of unitary matrices, or isometries, called affine isometry group of X. For the spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain in this way the groups O_N, U_N.

PROOF. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X \subset \mathbb{C}^N$, but we are not interested here in such general spaces. □

Observe that in the case of the real and complex spheres, the affine isometry group $G(X)$ leaves invariant the Riemannian metric, because this metric is equivalent to the one inherited from $\mathbb{C}^N$, which is preserved by our isometries $U \in U_N$.

Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X, with of course some extra care in relation with the complex structure, as for the complex sphere $X = S^{N-1}_\mathbb{C}$ to produce $G(X) = U_N$ instead of $G(X) = O_{2N}$. But, such things won't really work for the free spheres, and so are to be avoided.

The point now is that we have the following quantum analogue of Proposition 7.16, which is a perfect analogue, save for the fact that X is now assumed to be algebraic, for some technical reasons, which allows us to talk about quantum isometry groups:

THEOREM 7.17. *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

defines a morphism of C^*-algebras $\Phi : C(X) \to C(X) \otimes C(G)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

PROOF. Assume indeed that our manifold $X \subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

In order to prove the result, consider the following variables:

$$X_i = \sum_j x_j \otimes u_{ji} \in C(X) \otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G = G^+(X)$ appears as:

$$C(G) = C(U_N^+) \Big/ \Big\langle f_\alpha(X_1, \ldots, X_N) = 0 \Big\rangle$$

In order to prove this, pick one of the defining polynomials, and write it as follows:

$$f_\alpha(x_1, \ldots, x_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \cdot x_{i_1^r} \ldots x_{i_{s_r}^r}$$

With $X_i = \sum_j x_j \otimes u_{ji}$ as above, we have the following formula:

$$f_\alpha(X_1, \ldots, X_N) = \sum_r \sum_{i_1^r \ldots i_{s_r}^r} \lambda_r \sum_{j_1^r \ldots j_{s_r}^r} x_{j_1^r} \ldots x_{j_{s_r}^r} \otimes u_{j_1^r i_1^r} \ldots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f_\alpha(X_1, \ldots, X_N) = 0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a closed subspace $G \subset U_N^+$, with a universal map, as follows:

$$\Phi : C(X) \to C(X) \otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider as well the following elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f_\alpha(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f_\alpha(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f_\alpha(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. $\square$

We can now formulate a result about spheres and rotations, as follows:

THEOREM 7.18. *The quantum isometry groups of the basic spheres are*

$$\begin{matrix}S^{N-1}_{\mathbb R,+}&\longrightarrow&S^{N-1}_{\mathbb C,+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb R}&\longrightarrow&S^{N-1}_{\mathbb C}\end{matrix}\qquad\to\qquad\begin{matrix}O_N^+&\longrightarrow&U_N^+\\ \uparrow&&\uparrow\\ O_N&\longrightarrow&U_N\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

PROOF. We have 4 results to be proved, the idea being as follows:

$\underline{S^{N-1}_{\mathbb C,+}}$. Let us first construct an action $U_N^+\curvearrowright S^{N-1}_{\mathbb C,+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb C,+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

By using the biunitarity of u, we have the following computation:

$$\sum_i X_iX_i^*=\sum_{ijk}x_jx_k^*\otimes u_{ji}u_{ki}^*=\sum_j x_jx_j^*\otimes 1=1\otimes 1$$

Once again by using the biunitarity of u, we have as well:

$$\sum_i X_i^*X_i=\sum_{ijk}x_j^*x_k\otimes u_{ji}^*u_{ki}=\sum_j x_j^*x_j\otimes 1=1\otimes 1$$

Thus we have an action $U_N^+\curvearrowright S^{N-1}_{\mathbb C,+}$, which gives $G^+(S^{N-1}_{\mathbb C,+})=U_N^+$, as desired.

<u>$S^{N-1}_{\mathbb{R},+}$</u>. Let us first construct an action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$. We already know that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, so we just have to check that these variables are self-adjoint. But this is clear from $u=\bar{u}$, as follows:

$$X_i^*=\sum_j x_j^*\otimes u_{ji}^*=\sum_j x_j\otimes u_{ji}=X_i$$

Conversely, assume that we have an action $G\curvearrowright S^{N-1}_{\mathbb{R},+}$, with $G\subset U_N^+$. The variables $X_i=\sum_j x_j\otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u=\bar{u}$. Thus our quantum group must satisfy $G\subset O_N^+$, as desired.

<u>$S^{N-1}_{\mathbb{C}}$</u>. The fact that we have an action $U_N\curvearrowright S^{N-1}_{\mathbb{C}}$ is clear. Conversely, assume that we have an action $G\curvearrowright S^{N-1}_{\mathbb{C}}$, with $G\subset U_N^+$. We must prove that this implies $G\subset U_N$, and we will use a standard trick of Bhowmick-Goswami. We have:

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_ix_k)=\sum_{jl}x_jx_l\otimes u_{ji}u_{lk}$$

$$\Phi(x_kx_i)=\sum_{jl}x_lx_j\otimes u_{lk}u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_ix_k)=\sum_{j<l}x_jx_l\otimes(u_{ji}u_{lk}+u_{li}u_{jk})+\sum_j x_j^2\otimes u_{ji}u_{jk}$$

$$\Phi(x_ix_k)=\sum_{j<l}x_jx_l\otimes(u_{lk}u_{ji}+u_{jk}u_{li})+\sum_j x_j^2\otimes u_{jk}u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji}u_{lk}+u_{li}u_{jk}=u_{lk}u_{ji}+u_{jk}u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^*u_{ij}^*+u_{kj}^*u_{il}^*=u_{ij}^*u_{kl}^*+u_{il}^*u_{kj}^*$$

$$u_{ij}u_{kl}+u_{il}u_{kj}=u_{kl}u_{ij}+u_{kj}u_{il}$$

$$u_{ji}u_{lk}+u_{jk}u_{li}=u_{lk}u_{ji}+u_{li}u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li}u_{jk}=u_{jk}u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_ix_k^*)=\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*$$

$$\Phi(x_k^*x_i)=\sum_{jl}x_l^*x_j\otimes u_{lk}^*u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*=\sum_{jl}x_jx_l^*\otimes u_{lk}^*u_{ji}$$

Thus we have $u_{ji}u_{lk}^*=u_{lk}^*u_{ji}$, and so $G\subset U_N$, as claimed.

$\underline{S^{N-1}_\mathbb{R}}$. The fact that we have an action $O_N\curvearrowright S^{N-1}_\mathbb{R}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned}G\curvearrowright S^{N-1}_\mathbb{R}\quad&\implies\quad G\curvearrowright S^{N-1}_{\mathbb{R},+},S^{N-1}_\mathbb{C}\\&\implies\quad G\subset O_N^+,U_N\\&\implies\quad G\subset O_N^+\cap U_N=O_N\end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_\mathbb{R})=O_N$, as desired. □

7d. Haar integration

Let us discuss now the correspondence $U\to S$. In the classical case the situation is very simple, because the sphere $S=S^{N-1}$ appears by rotating the point $x=(1,0,\ldots,0)$ by the isometries in $U=U_N$. Moreover, the stabilizer of this action is the subgroup $U_{N-1}\subset U_N$ acting on the last $N-1$ coordinates, and so the sphere $S=S^{N-1}$ appears from the corresponding rotation group $U=U_N$ as an homogeneous space, as follows:

$$S^{N-1}=U_N/U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U\to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1})\subset C(U_N)\quad,\quad x_i\to u_{1i}$$

In general now, the straightforward homogeneous space interpretation of S as above fails. However, we can have some theory going by using the functional analytic viewpoint, with an embedding $x_i\to u_{1i}$ as above. Let us start with the following result:

Theorem 7.19. *For the basic spheres, we have a diagram as follows,*

$$\begin{array}{ccc} C(S) & \xrightarrow{\Phi} & C(S)\otimes C(U)\\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id}\\ C(U) & \xrightarrow{\Delta} & C(U)\otimes C(U) \end{array}$$

where on top $\Phi(x_i)=\sum_j x_j\otimes u_{ji}$, and on the left $\alpha(x_i)=u_{1i}$.

Proof. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows, on these coordinates:

$$\begin{array}{ccc} x_i & \longrightarrow & \sum_j x_j\otimes u_{ji}\\ \downarrow & & \downarrow\\ u_{1i} & \longrightarrow & \sum_j u_{1j}\otimes u_{ji} \end{array}$$

Thus by linearity and multiplicativity, the whole the diagram commutes. □

As a consequence of the above result, we can now formulate:

Proposition 7.20. *We have a quotient map and an inclusion as follows,*

$$U\to S_U\subset S$$

with S_U being the first row space of U, given by

$$C(S_U)=<u_{1i}>\subset C(U)$$

at the level of the corresponding algebras of functions.

Proof. At the algebra level, we have an inclusion and a quotient map as follows:

$$C(S)\to C(S_U)\subset C(U)$$

Thus, we obtain the result, by transposing. □

The above result is all that we need, for getting started with our study, and we will prove in what follows that the inclusion $S_U\subset S$ constructed above is an isomorphism. This will produce the correspondence $U\to S$ that we are currently looking for.

In order to do so, we will use the uniform integration over S, which can be introduced, in analogy with what happens in the classical case, in the following way:

Definition 7.21. *We endow each of the algebras $C(S)$ with its integration functional*

$$\int_S : C(S) \to C(U) \to \mathbb{C}$$

obtained by composing the morphism $x_i \to u_{1i}$ with the Haar integration of $C(U)$.

In order to efficiently integrate over the sphere S, and in the lack of some trick like spherical coordinates, we need to know how to efficiently integrate over the corresponding quantum isometry group U. As before in the classical case, we have:

Theorem 7.22. *Assuming that a compact quantum group $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

for any indices $i_r, j_r \in \{1,\ldots,N\}$ and any exponents $e_r \in \{\emptyset, *\}$, where δ are the usual Kronecker type symbols, and where

$$W_{kN} = G_{kN}^{-1}$$

is the inverse of the matrix $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. Let us arrange indeed all the integrals to be computed, at a fixed value of the exponent $k = (e_1 \ldots e_k)$, into a single matrix, of size $N^k \times N^k$, as follows:

$$P_{i_1\ldots i_k, j_1 \ldots j_k} = \int_G u_{i_1 j_1}^{e_1} \ldots u_{i_k j_k}^{e_k}$$

According to the construction of the Haar measure of Woronowicz, explained in the above, this matrix P is the orthogonal projection onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi \Big| \pi \in D(k)\right)$$

In order to compute this projection, consider the following linear map:

$$E(x) = \sum_{\pi \in D(k)} < x, \xi_\pi > \xi_\pi$$

Consider as well the inverse W of the restriction of E to the following space:

$$span\left(T_\pi \Big| \pi \in D(k)\right)$$

By a standard linear algebra computation, it follows that we have:

$$P = WE$$

But the restriction of E is the linear map corresponding to G_{kN}, so W is the linear map corresponding to W_{kN}, and this gives the result. □

With this in hand, we can now integrate over the spheres S, as follows:

THEOREM 7.23. *The integration over the basic spheres is given by*

$$\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_\pi\sum_{\sigma\leq\ker i}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, *where* $W_{kN}=G_{kN}^{-1}$ *is the inverse of* $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

PROOF. According to our conventions, the integration over S is a particular case of the integration over U, via $x_i=u_{1i}$. By using now Theorem 7.22, we obtain:

$$\begin{aligned}\int_S x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}&=\int_U u_{1i_1}^{e_1}\ldots u_{1i_k}^{e_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Again with some inspiration from the classical case, we have the following key result:

THEOREM 7.24. *The integration functional of S has the ergodicity property*

$$\left(id\otimes\int_U\right)\Phi(x)=\int_S x$$

where $\Phi:C(S)\to C(S)\otimes C(U)$ *is the universal affine coaction map.*

PROOF. In the real case, $x_i=x_i^*$, it is enough to check the equality in the statement on an arbitrary product of coordinates, $x_{i_1}\ldots x_{i_k}$. The left term is as follows:

$$\begin{aligned}\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})&=\sum_{j_1\ldots j_k}x_{j_1}\ldots x_{j_k}\int_U u_{j_1i_1}\ldots u_{j_ki_k}\\&=\sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D(k)}\delta_\pi(j)\delta_\sigma(i)W_{kN}(\pi,\sigma)x_{j_1}\ldots x_{j_k}\\&=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\sum_{j_1\ldots j_k}\delta_\pi(j)x_{j_1}\ldots x_{j_k}\end{aligned}$$

Let us look now at the last sum on the right. The situation is as follows:

– In the free case we have to sum quantities of type $x_{j_1}\ldots x_{j_k}$, over all choices of multi-indices $j=(j_1,\ldots,j_k)$ which fit into our given noncrossing pairing π, and just by using the condition $\sum_i x_i^2=1$, we conclude that the sum is 1.

– The same happens in the classical case. Indeed, our pairing π can now be crossing, but we can use the commutation relations $x_ix_j=x_jx_i$, and the sum is again 1.

Thus the sum on the right is 1, in all cases, and we obtain:

$$\left(id\otimes\int_U\right)\Phi(x_{i_1}\ldots x_{i_k})=\sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)$$

On the other hand, another application of the Weingarten formula gives:

$$\begin{aligned}\int_S x_{i_1}\ldots x_{i_k} &= \int_U u_{1i_1}\ldots u_{1i_k}\\ &= \sum_{\pi,\sigma\in D(k)}\delta_\pi(1)\delta_\sigma(i)W_{kN}(\pi,\sigma)\\ &= \sum_{\pi,\sigma\in D(k)}\delta_\sigma(i)W_{kN}(\pi,\sigma)\end{aligned}$$

Thus, we are done with the proof of the result, in the real case. In the complex case the proof is similar, by adding exponents everywhere. □

We can now deduce a useful characterization of the integration, as follows:

THEOREM 7.25. *There is a unique positive unital trace $tr:C(S)\to\mathbb{C}$ satisfying*

$$(tr\otimes id)\Phi(x)=tr(x)1$$

where Φ is the coaction map of the corresponding quantum isometry group,

$$\Phi:C(S)\to C(S)\otimes C(U)$$

and this is the canonical integration, as constructed in Definition 7.21.

PROOF. This follows indeed by using Theorem 7.24. □

7e. Exercises

Exercises:

EXERCISE 7.26.

EXERCISE 7.27.

EXERCISE 7.28.

EXERCISE 7.29.

EXERCISE 7.30.

EXERCISE 7.31.

EXERCISE 7.32.

EXERCISE 7.33.

Bonus exercise.

CHAPTER 8

Crossed products

8a. Crossed products

Crossed products.

8b. Basic examples

Basic examples.

8c. Analytic aspects

Analytic aspects.

8d. Generalizations

Generalizations.

8e. Exercises

Exercises:

Exercise 8.1.

Exercise 8.2.

Exercise 8.3.

Exercise 8.4.

Exercise 8.5.

Exercise 8.6.

Exercise 8.7.

Exercise 8.8.

Bonus exercise.

Part III

Geometry, topology

In the shape of things to come
Too much poison come undone
Cause there's nothing else to do
Every me and every you

CHAPTER 9

K-theory

9a. K-theory

Let us first look at the classical case, where X is a usual compact space. You might say right away that wrong way, what we need for doing geometry is a manifold. But my answer here is modesty, and no hurry. It is right that you cannot do much geometry with a compact space X, but you can do some, and we have here, for instance:

Definition 9.1. *Given a compact space X, its first K-theory group $K_0(X)$ is the group of formal differences of complex vector bundles over X.*

This notion is quite interesting, and we can talk in fact about higher K-theory groups $K_n(X)$ as well, and all this is related to the homotopy groups $\pi_n(X)$ too. There are many non-trivial results on the subject, the end of the game being of course that of understanding the "shape" of X, that you need to know a bit about, before getting into serious geometry, in the case where X happens to be a manifold.

As a question for us now, operator algebra theorists, we have:

Question 9.2. *Can we talk about the first K-theory group $K_0(X)$ of a compact quantum space X?*

We will see that this is a quite subtle question. To be more precise, we will see that we can talk, in a quite straightforward way, of the group $K_0(A)$ of an arbitrary C^*-algebra A, which is constructed as to have $K_0(A) = K_0(X)$ in the commutative case, where $A = C(X)$, with X being a usual compact space. In the noncommutative case, however, $K_0(A)$ will sometimes depend on the choice of A satisfying $A = C(X)$, and so all this will eventually lead to a sort of dead end, and to a rather "no" answer to Question 9.2.

Getting started now, in order to talk about the first K-theory group $K_0(A)$ of an arbitrary C^*-algebra A, we will need the following simple fact:

Proposition 9.3. *Given a C^*-algebra A, the finitely generated projective A-modules E appear via quotient maps $f : A^n \to E$, so are of the form*

$$E = pA^n$$

with $p \in M_n(A)$ being an idempotent. In the commutative case, $A = C(X)$ with X classical, these A-modules consist of sections of the complex vector bundles over X.

PROOF. Here the first assertion is clear from definitions, via some standard algebra, and the second assertion is clear from definitions too, again via some algebra. □

With this in hand, let us go back to Definition 9.1. Given a compact space X, it is now clear that its K-theory group $K_0(X)$ can be recaptured from the knowledge of the associated C^*-algebra $A = C(X)$, and to be more precise we have $K_0(X) = K_0(A)$, when the first K-theory group of an arbitrary C^*-algebra is constructed as follows:

DEFINITION 9.4. *The first K-theory group of a C^*-algebra A is the group of formal differences*

$$K_0(A) = \{p - q\}$$

of equivalence classes of projections $p \in M_n(A)$, with the equivalence being given by

$$p \sim q \iff \exists u, uu^* = p, u^*u = q$$

and with the additive structure being the obvious one, by diagonal concatenation.

This is very nice, and as a first example, we have $K_0(\mathbb{C}) = \mathbb{Z}$. More generally, as already mentioned above, it follows from Proposition 9.3 that in the commutative case, where $A = C(X)$ with X being a compact space, we have $K_0(A) = K_0(X)$. Observe also that we have, by definition, the following formula, valid for any $n \in \mathbb{N}$:

$$K_0(A) = K_0(M_n(A))$$

Some further elementary observations include the fact that K_0 behaves well with respect to direct sums and with inductive limits, and also that K_0 is a homotopy invariant, and for details here, we refer to any introductory book on the subject, such as [**15**].

In what concerns us, back to our Question 9.2, what has been said above is certainly not enough for investigating our question, and we need more examples. However, these examples are not easy to find, and for getting them, we need more theory. We have:

DEFINITION 9.5. *The second K-theory group of a C^*-algebra A is the group of connected components of the unitary group of $GL_\infty(A)$, with*

$$GL_n(A) \subset GL_{n+1}(A) \quad , \quad a \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

being the embeddings producing the inductive limit $GL_\infty(A)$.

Again, for a basic example we can take $A = \mathbb{C}$, and we have here $K_1(\mathbb{C}) = \{1\}$, trivially. In fact, in the commutative case, where $A = C(X)$, with X being a usual compact space, it is possible to establish a formula of type $K_1(A) = K_1(X)$. Further elementary observations include the fact that K_1 behaves well with respect to direct sums and with inductive limits, and also that K_1 is a homotopy invariant.

Importantly, the first and second K-theory groups are related, as follows:

THEOREM 9.6. *Given a C^*-algebra A, we have isomorphisms as follows, with*

$$SA = \left\{ f \in C([0,1], A) \Big| f(0) = 0 \right\}$$

standing for the suspension operation for the C^*-algebras:

(1) $K_1(A) = K_0(SA)$.
(2) $K_0(A) = K_1(SA)$.

PROOF. Here the isomorphism in (1) is something rather elementary, and the isomorphism in (2) is something more complicated. In both cases, the idea is to start first with the commutative case, where $A = C(X)$ with X being a compact space, and understand there the isomorphisms (1,2), called Bott periodicity isomorphisms. Then, with this understood, the extension to the general C^*-algebra case is quite straightforward. □

The above result is quite interesting, making it clear that the groups K_0, K_1 are of the same nature. In fact, it is possible to be a bit more abstract here, and talk in various clever ways about the higher K-theory groups, $K_n(A)$ with $n \in \mathbb{N}$, of an arbitrary C^*-algebra, with the result that these higher K-theory groups are subject to Bott periodicity:

$$K_n(A) = K_{n+2}(A)$$

Going ahead with examples, following Cuntz [23] and related papers, we have:

THEOREM 9.7. *The K-theory groups of the Cuntz algebra O_n are given by*

$$K_0(O_n) = \mathbb{Z}_{n-1} \quad , \quad K_1(O_n) = \{1\}$$

with the equivalent projections $P_i = S_i S_i^*$ standing for the standard generator of $\mathbb{Z}_{n-1}$.

PROOF. We recall that the Cuntz algebra O_n is generated by isometries $S_1, \ldots, S_n$ satisfying $S_1 S_1^* + \ldots + S_n S_n^* = 1$. Since we have $S_i^* S_i = 1$, with $P_i = S_i S_i^*$, we have:

$$P_1 \sim \ldots \sim P_n \sim 1$$

On the other hand, we also know that we have $P_1 + \ldots + P_n = 1$, and the conclusion is that, in the first K-theory group $K_1(O_n)$, the following equality happens:

$$n[1] = [1]$$

Thus $(n-1)[1] = 0$, and it is quite elementary to prove that $k[1] = 0$ happens in fact precisely when k is a multiple of $n - 1$. Thus, we have a group embedding, as follows:

$$\mathbb{Z}_{n-1} \subset K_0(O_n)$$

The whole point now is that of proving that this group embedding is an isomorphism, which in practice amounts in proving that any projection in O_n is equivalent to a sum of the form $P_1 + \ldots + P_k$, with $P_i = S_i S_i^*$ as above. Which is something non-trivial, requiring the use of Bott periodicity, and the consideration of the second K-theory group $K_1(O_n)$ as well, and for details here, we refer to Cuntz [23] and related papers. □

The point now is that what we have in Theorem 9.7 is a true noncommutative computation, dealing with an algebra which is rather of "free" type, and this suggests that the answer to Question 9.2 might be "yes". However, as bad news, we have:

THEOREM 9.8. *There are discrete groups Γ having the property that the projection*

$$\pi : C^*(\Gamma) \to C^*_{red}(\Gamma)$$

is not an isomorphism, at the level of K-theory groups.

PROOF. For constructing such a counterexample, the group Γ must be definitely nonamenable, and the first thought goes to the free group F_2. But it is possible to prove that F_2 is K-amenable, in the sense that π is an isomorphism at the K-theory level. However, counterexamples do exist, such as the infinite groups Γ having Kazhdan's property (T). Indeed, for such a group the asssociated Kazhdan projection $p \in K_0(C^*(\Gamma))$ is nonzero, while mapping to the zero element $0 \in K_0(C^*_{red}(\Gamma))$, so we have our counterexample. □

As a conclusion to all this, which might seem a bit dissapointing, we have:

CONCLUSION 9.9. *The answer to Question 9.2 is no.*

Of course, the answer to Question 9.2 remains "yes" in many cases, the general idea being that, as long as we don't get too far away from the classical case, the answer remains "yes", so we can talk about the K-theory groups of our compact quantum spaces X, and also, about countless other invariants inspired from the classical theory. For a survey of what can be done here, including applications too, we refer to Connes' book [19].

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.10.

EXERCISE 9.11.

EXERCISE 9.12.

EXERCISE 9.13.

EXERCISE 9.14.

EXERCISE 9.15.

EXERCISE 9.16.

EXERCISE 9.17.

Bonus exercise.

CHAPTER 10

Smooth structure

10a. Smooth structure

10b.

10c.

10d.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

Differential geometry

11a. Differential geometry

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.

CHAPTER 12

Algebraic geometry

12a. Algebraic geometry

We discuss now an abstract extension of the various constructions of algebraic manifolds that we have so far. The idea will be that of looking at certain classes of algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$, which are homogeneous spaces, of a certain special type.

Let us start with the following definition, which is something quite general:

Definition 12.1. *An affine homogeneous space over a closed subgroup $G \subset U_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$, such that there exists an index set $I \subset \{1,\ldots,N\}$ such that*

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j\in I} u_{ji} \quad , \quad \Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

define morphisms of C^*-algebras, satisfying the following condition,

$$\left(id \otimes \int_G\right)\Phi = \int_G \alpha(.)1$$

called ergodicity condition for the action.

Let us mention right away that this definition is something quite tricky, based on the explicit examples of homogeneous spaces that we have in mind, rather than on whatever abstract considerations, and that will take us some time to understand.

To start with, as a basic example, $O_N^+ \to S^{N-1}_{\mathbb{R},+}$ is indeed affine in our sense, with $I = \{1\}$. The same goes for $U_N^+ \to S^{N-1}_{\mathbb{C},+}$, which is affine as well, also with $I = \{1\}$.

Observe that the $1/\sqrt{|I|}$ constant appearing above is the correct one, because:

$$\begin{aligned}\sum_i \left(\sum_{j\in I} u_{ji}\right)\left(\sum_{k\in I} u_{ki}\right)^* &= \sum_i \sum_{j,k\in I} u_{ji}u_{ki}^* \\ &= \sum_{j,k\in I}(uu^*)_{jk} \\ &= |I|\end{aligned}$$

As a first general result about such spaces, we have:

Proposition 12.2. *Consider an affine homogeneous space X, as above.*

(1) *The coaction condition $(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$ is satisfied.*
(2) *We have as well the formula $(\alpha\otimes id)\Phi=\Delta\alpha$.*

Proof. The coaction condition is clear. For the second formula, we first have:

$$\begin{aligned}(\alpha\otimes id)\Phi(x_i) &= \sum_k\alpha(x_k)\otimes u_{ki}\\ &= \frac{1}{\sqrt{|I|}}\sum_k\sum_{j\in I}u_{jk}\otimes u_{ki}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\Delta\alpha(x_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\Delta(u_{ji})\\ &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\end{aligned}$$

Thus, by linearity, multiplicativity and continuity, we obtain the result. □

Summarizing, the terminology in Definition 12.1 is justified, in the sense that what we have there are indeed certain homogeneous spaces, of very special, "affine" type. As a second result regarding such spaces, which closes the discussion in the case where α is injective, which is something that happens in many cases, we have:

Theorem 12.3. *When α is injective we must have $X=X_{G,I}^{min}$, where:*

$$C(X_{G,I}^{min})=\left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)$$

Moreover, $X_{G,I}^{min}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

Proof. The first assertion is clear from definitions. Regarding now the second assertion, consider the variables in the statement:

$$X_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\in C(G)$$

In order to prove that we have $X_{G,I}^{min}\subset S^{N-1}_{\mathbb{C},+}$, observe first that we have:

$$\begin{aligned}\sum_i X_iX_i^* &= \frac{1}{|I|}\sum_i\sum_{j,k\in I}u_{ji}u_{ki}^*\\ &= \frac{1}{|I|}\sum_{j,k\in I}(uu^*)_{jk}\\ &= 1\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\sum_i X_i^* X_i &= \frac{1}{|I|}\sum_i \sum_{j,k\in I} u_{ji}^* u_{ki}\\ &= \frac{1}{|I|}\sum_{j,k\in I}(\bar{u}u^t)_{jk}\\ &= 1\end{aligned}$$

Thus $X_{G,I}^{min}\subset S^{N-1}_{\mathbb{C},+}$. Finally, observe that we have:

$$\begin{aligned}\Delta(X_i) &= \frac{1}{\sqrt{|I|}}\sum_{j\in I}\sum_k u_{jk}\otimes u_{ki}\\ &= \sum_k X_k\otimes u_{ki}\end{aligned}$$

Thus we have indeed a coaction map, given by $\Phi=\Delta$. As for the ergodicity condition, namely $(id\otimes\int_G)\Delta=\int_G(.)1$, this holds as well, by definition of the integration functional $\int_G$. Thus, our axioms for affine homogeneous spaces are indeed satisfied. □

Our purpose now will be to show that the affine homogeneous spaces appear as follows, a bit in the same way as the discrete group algebras:

$$X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$$

We make the standard convention that all the tensor exponents k are "colored integers", that is, $k=e_1\ldots e_k$ with $e_i\in\{\circ,\bullet\}$, with $\circ$ corresponding to the usual variables, and with $\bullet$ corresponding to their adjoints. With this convention, we have:

Proposition 12.4. *The ergodicity condition, namely*

$$\left(id\otimes\int_G\right)\Phi=\int_G\alpha(.)1$$

is equivalent to the condition

$$(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\quad,\quad\forall k,\forall i_1,\ldots,i_k$$

where P is the matrix formed by the Peter-Weyl integrals of exponent k,

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

and where $(x^{\otimes k})_{i_1\ldots i_k}=x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}$.

PROOF. We have the following computation:

$$\begin{aligned}\left(id\otimes\int_G\right)\Phi(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \sum_{j_1\ldots j_k} x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \sum_{j_1\ldots j_k} P_{i_1\ldots i_k,j_1\ldots j_k}(x^{\otimes k})_{j_1\ldots j_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}\int_G\alpha(x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}) &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I} P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a consequence, we have the following result:

THEOREM 12.5. *We must have $X\subset X_{G,I}^{max}$, as subsets of $S^{N-1}_{\mathbb{C},+}$, where:*

$$C(X_{G,I}^{max})=C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle$$

Moreover, $X_{G,I}^{max}$ is affine homogeneous, for any $G\subset U_N^+$, and any $I\subset\{1,\ldots,N\}$.

PROOF. Let us first prove that we have an action $G\curvearrowright X_{G,I}^{max}$. We must show here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $X_{G,I}^{max}$. We have:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{l_1\ldots l_k} P_{i_1\ldots i_k,l_1\ldots l_k}(X^{\otimes k})_{l_1\ldots l_k}\\ &= \sum_{l_1\ldots l_k} P_{i_1\ldots i_k,l_1\ldots l_k}\sum_{j_1\ldots j_k} x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes u_{j_1l_1}^{e_1}\ldots u_{j_kl_k}^{e_k}\\ &= \sum_{j_1\ldots j_k} x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\otimes(u^{\otimes k}P^t)_{j_1\ldots j_k,i_1\ldots i_k}\end{aligned}$$

Since by Peter-Weyl the transpose of $P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$ is the orthogonal projection onto $Fix(u^{\otimes k})$, we have $u^{\otimes k}P^t=P^t$. We therefore obtain:

$$\begin{aligned}(PX^{\otimes k})_{i_1\ldots i_k} &= \sum_{j_1\ldots j_k}P_{i_1\ldots i_k,j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ &= (Px^{\otimes k})_{i_1\ldots i_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\end{aligned}$$

Thus we have an action $G\curvearrowright X_{G,I}^{max}$, and since this action is ergodic by Proposition 12.4, we have an affine homogeneous space, as claimed. □

We can now merge the results that we have, and we obtain:

THEOREM 12.6. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following C^*-subalgebra and the following quotient C^*-algebra,*

$$\begin{aligned}C(X_{G,I}^{min}) &= \left\langle\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ji}\Big|i=1,\ldots,N\right\rangle\subset C(G)\\ C(X_{G,I}^{max}) &= C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle(Px^{\otimes k})_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}\Big|\forall k,\forall i_1,\ldots i_k\right\rangle\end{aligned}$$

then we have maps as follows,

$$G\to X_{G,I}^{min}\subset X_{G,I}^{max}\subset S^{N-1}_{\mathbb{C},+}$$

the space $G\to X_{G,I}^{max}$ is affine homogeneous, and any affine homogeneous space $G\to X$ appears as an intermediate space $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$.

PROOF. This follows indeed from the various results that we have, namely Theorem 12.3 and Theorem 12.5, regarding the minimal and maximal constructions. □

At the level of the general theory, we will need one more result, as follows:

THEOREM 12.7. *Assuming that $G\to X$ is an affine homogeneous space, with index set $I\subset\{1,\ldots,N\}$, the Haar integration functional $\int_X=\int_G\alpha$ is given by*

$$\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=\sum_{\pi,\sigma\in D}K_I(\pi)\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)$$

where $\{\xi_\pi|\pi\in D\}$ is a basis of $Fix(u^{\otimes k})$, and where $W_{kN}=G_{kN}^{-1}$ with

$$G_{kN}(\pi,\sigma)=<\xi_\pi,\xi_\sigma>$$

is the associated Weingarten matrix, and $K_I(\pi)=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}(\xi_\pi)_{j_1\ldots j_k}$.

PROOF. By using the Weingarten formula for the quantum group G, in its abstract form, coming from Peter-Weyl theory, as discussed before, we have:

$$\begin{aligned}\int_X x_{i_1}^{e_1}\ldots x_{i_k}^{e_k} &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}\\ &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

But this gives the formula in the statement, and we are done. □

As a conclusion now, in view of Theorem 12.6, the situation with our affine homogeneous spaces is, from a point of view of abstract functional analysis, a bit similar to that of the full and reduced group algebras, with intermediate objects between them.

However, in addition to this, here is a natural example of an intermediate space $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$, which will be of interest for us, in what follows:

THEOREM 12.8. *Given a closed quantum subgroup $G\subset U_N^+$, and a set $I\subset\{1,\ldots,N\}$, if we consider the following quotient algebra*

$$C(X_{G,I}^{med})=C(S^{N-1}_{\mathbb{C},+})\Big/\left\langle\sum_{j_1\ldots j_k}\xi_{j_1\ldots j_k}x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\xi_{j_1\ldots j_k}\Big|\forall k,\forall\xi\in Fix(u^{\otimes k})\right\rangle$$

we obtain in this way an affine homogeneous space $G\to X_{G,I}$.

PROOF. We know from Theorem 12.5 that $X_{G,I}^{max}\subset S^{N-1}_{\mathbb{C},+}$ is constructed by imposing to the standard coordinates the conditions $Px^{\otimes k}=P^I$, where:

$$P_{i_1\ldots i_k,j_1\ldots j_k}=\int_G u_{j_1i_1}^{e_1}\ldots u_{j_ki_k}^{e_k}$$

$$P^I_{i_1\ldots i_k}=\frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}P_{i_1\ldots i_k,j_1\ldots j_k}$$

According to the Weingarten integration formula for G, we have:

$$\begin{aligned}(Px^{\otimes k})_{i_1\ldots i_k} &= \sum_{j_1\ldots j_k}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)x_{j_1}^{e_1}\ldots x_{j_k}^{e_k}\\ P^I_{i_1\ldots i_k} &= \frac{1}{\sqrt{|I|^k}}\sum_{j_1\ldots j_k\in I}\sum_{\pi,\sigma\in D}(\xi_\pi)_{j_1\ldots j_k}\overline{(\xi_\sigma)}_{i_1\ldots i_k}W_{kN}(\pi,\sigma)\end{aligned}$$

Thus $X_{G,I}^{med}\subset X_{G,I}^{max}$, and the other assertions are standard as well. □

We can now put everything together, and we obtain the following result, summarizing what we know so far from the above, regarding the affine homogeneous spaces:

THEOREM 12.9. *Given a closed subgroup $G\subset U_N^+$, and a subset $I\subset\{1,\ldots,N\}$, the affine homogeneous spaces over G, with index set I, have the following properties:*

1. *These are exactly the intermediate subspaces $X_{G,I}^{min}\subset X\subset X_{G,I}^{max}$ on which G acts affinely, with the action being ergodic.*
2. *For the minimal and maximal spaces $X_{G,I}^{min}$ and $X_{G,I}^{max}$, as well as for the intermediate space $X_{G,I}^{med}$ constructed above, these conditions are satisfied.*
3. *By performing the GNS construction with respect to the Haar integration functional $\int_X=\int_G\alpha$ we obtain the minimal space $X_{G,I}^{min}$.*

We agree to identify all these spaces, via the GNS construction, and denote them $X_{G,I}$.

PROOF. This is indeed something quite self-explanatory, which follows by combining the various results and observations formulated above. □

As an illustration, let us discuss now the group dual case. For simplifying, we will discuss the case of the "diagonal" embeddings only. Given a finitely generated discrete group $\Gamma=<g_1,\ldots,g_N>$, we can consider the following "diagonal" embedding:

$$\widehat{\Gamma}\subset U_N^+\quad,\quad u_{ij}=\delta_{ij}g_i$$

With this convention, we have the following result:

THEOREM 12.10. *In the group dual case, $G=\widehat{\Gamma}$ with $\Gamma=<g_1,\ldots,g_N>$, we have*

$$X=\widehat{\Gamma}_I\quad,\quad \Gamma_I=<g_i|i\in I>\subset\Gamma$$

for any affine homogeneous space X, when identifying full and reduced group algebras.

PROOF. Assume indeed that we have an affine homogeneous space $G\to X$. In terms of the rescaled coordinates $h_i=\sqrt{|I|}x_i$, our axioms for α,Φ read:

$$\alpha(h_i)=\delta_{i\in I}g_i\quad,\quad\Phi(h_i)=h_i\otimes g_i$$

As for the ergodicity condition, this translates as follows:

$$\begin{aligned}
&\left(id\otimes\int_G\right)\Phi(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p})=\int_G\alpha(h_{i_1}^{e_p}\ldots h_{i_p}^{e_p})\\
\iff\quad&\left(id\otimes\int_G\right)(h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}\otimes g_{i_1}^{e_1}\ldots g_{i_p}^{e_p})=\int_G\delta_{i_1\in I}\ldots\delta_{i_p\in I}g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}\\
\iff\quad&\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{g_{i_1}^{e_1}\ldots g_{i_p}^{e_p},1}\delta_{i_1\in I}\ldots\delta_{i_p\in I}\\
\iff\quad&\left[g_{i_1}^{e_1}\ldots g_{i_p}^{e_p}=1\implies h_{i_1}^{e_1}\ldots h_{i_p}^{e_p}=\delta_{i_1\in I}\ldots\delta_{i_p\in I}\right]
\end{aligned}$$

Now observe that from $g_ig_i^*=g_i^*g_i=1$ we obtain in this way:

$$h_ih_i^*=h_i^*h_i=\delta_{i\in I}$$

Thus the elements h_i vanish for $i \notin I$, and are unitaries for $i \in I$. We conclude that we have $X = \widehat{\Lambda}$, where $\Lambda =< h_i | i \in I >$ is the group generated by these unitaries. In order to finish now the proof, our claim is that for indices $i_x \in I$ we have:

$$g_{i_1}^{e_1} \ldots g_{i_p}^{e_p} = 1 \iff h_{i_1}^{e_1} \ldots h_{i_p}^{e_p} = 1$$

Indeed, $\Longrightarrow$ comes from the ergodicity condition, as processed above, and $\Longleftarrow$ comes from the existence of the morphism α, which is given by $\alpha(h_i) = g_i$, for $i \in I$. □

12b.

12c.

12d.

12e. Exercises

Exercises:

Exercise 12.11.

Exercise 12.12.

Exercise 12.13.

Exercise 12.14.

Exercise 12.15.

Exercise 12.16.

Exercise 12.17.

Exercise 12.18.

Bonus exercise.

Part IV

Analytic aspects

Who knows what stands in front of our lives?
I fashion my future on films in space
Silence tells me secretly
Everything, everything

CHAPTER 13

Integration theory

13a. Integration theory

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

Advanced calculus

14a. Advanced calculus

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

Representations

15a. Representations

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

Matrix models

16a. Models, level

You can model everything with random matrices, the saying in analysis goes. In this chapter we discuss modelling questions for the affine manifolds $X\subset S^{N-1}_{\mathbb C,+}$, and then for the projective manifolds $X\subset P^{N-1}_+$. Let us start with a key definition, as follows:

Definition 16.1. *A matrix model for a noncommutative algebraic manifold $X\subset S^{N-1}_{\mathbb C,+}$ is a morphism of C^*-algebras of the following type,*

$$\pi:C(X)\to M_K(C(T))$$

with T being a compact space, and $K\in\mathbb N$ being an integer.

As a first observation, when X happens to be classical, we can take $K=1$ and $T=X$, and we have a faithful model for our manifold, namely:

$$id:C(X)\to M_1(C(X))$$

In general, we cannot use $K=1$, and the smallest value $K\in\mathbb N$ doing the job, if any, will correspond somehow to the "degree of noncommutativity" of our manifold.

With the help of some von Neumann algebra theory, we can now go ahead with our program, and discuss von Neumann algebraic extensions. We have the following result:

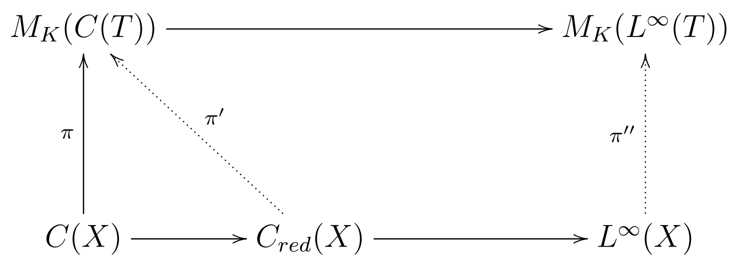
Theorem 16.2. *Given a matrix model $\pi:C(X)\to M_K(C(T))$, with both X,T being assumed to have integration functionals, the following are equivalent:*

(1) *π is stationary, in the sense that $\int_X=(tr\otimes\int_T)\pi$.*
(2) *π produces an inclusion $\pi':C_{red}(X)\subset M_K(X(T))$.*
(3) *π produces an inclusion $\pi'':L^\infty(X)\subset M_K(L^\infty(T))$.*

Moreover, in the quantum group case, these conditions imply that π is faithful.

Proof. This is standard functional analysis, as follows:

(1) Consider the following diagram, with all the solid arrows being by definition the canonical maps between the algebras concerned:



(2) With this picture in hand, the implications (1) $\iff$ (2) $\iff$ (3) between the conditions (1,2,3) in the statement are all clear, coming from the basic properties of the GNS construction, and of the von Neumann algebras, explained in the above.

(3) As for the last assertion, this is something more subtle, coming from the fact that if $L^\infty(G)$ is of type I, as required by (3), then G must be coamenable. □

Let us go back now to our basic notion of a matrix model, from Definition 16.1, and develop some more general theory, in that setting. We first have:

PROPOSITION 16.3. *A 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ must come from a map*

$$p : T \to X_{class} \subset X$$

and π is faithful precisely when $X = X_{class}$, and when p is surjective.

PROOF. According to our conventions, a 1×1 model for a manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is simply a morphism of algebras $\pi : C(X) \to C(T)$. Now since $C(T)$ is commutative, this morphism must factorize through the abelianization of $C(X)$, as follows:

$$\pi : C(X) \to C(X_{class}) \to C(T)$$

Thus, our morphism π must come by transposition from a map p, as claimed. □

In order to generalize the above trivial fact, we can use:

DEFINITION 16.4. *Let $X \subset S^{N-1}_{\mathbb{C},+}$. We define a closed subspace $X^{(K)} \subset X$ by*

$$C(X^{(K)}) = C(X)/J_K$$

where J_K is the common null space of matrix representations of $C(X)$, of size $L \leq K$,

$$J_K = \bigcap_{L\leq K} \ \bigcap_{\pi : C(X)\to M_L(\mathbb{C})} \ker(\pi)$$

and we call $X^{(K)}$ the "part of X which is realizable with $K \times K$ models".

As a basic example here, the first such space, at $K = 1$, is the classical version:

$$X^{(1)} = X_{class}$$

Observe that we have embeddings of quantum spaces, as follows:

$$X^{(1)} \subset X^{(2)} \subset X^{(3)} \subset \ldots\ldots \subset X^{(\infty)} \subset X$$

Getting back now to the case $K < \infty$, we first have the following result:

Proposition 16.5. *Consider an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$.*

(1) *Given a closed subspace $Y \subset X \subset S^{N-1}_{\mathbb{C},+}$, we have $Y \subset X^{(K)}$ precisely when any irreducible representation of $C(Y)$ has dimension $\leq K$.*
(2) *In particular, we have $X^{(K)} = X$ precisely when any irreducible representation of $C(X)$ has dimension $\leq K$.*

Proof. This follows from general C^*-algebra theory, as follows:

(1) If any irreducible representation of $C(Y)$ has dimension $\leq K$, then we have $Y \subset X^{(K)}$, because the irreducible representations of a C^*-algebra separate its points. Conversely, assuming $Y \subset X^{(K)}$, it is enough to show that any irreducible representation of the algebra $C(X^{(K)})$ has dimension $\leq K$. But this is once again well-known.

(2) This follows indeed from (1). □

The connection with the previous considerations comes from:

Theorem 16.6. *If $X \subset S^{N-1}_{\mathbb{C},+}$ has a faithful matrix model*

$$C(X) \to M_K(C(T))$$

then we have $X = X^{(K)}$.

Proof. This follows from the above, via standard theory for the C^*-algebras. □

We can now discuss the universal $K \times K$-matrix model, constructed as follows:

Theorem 16.7. *Given $X \subset S^{N-1}_{\mathbb{C},+}$ algebraic, the category of its $K \times K$ matrix models, with $K \geq 1$ being fixed, has a universal object as follows:*

$$\pi_K : C(X) \to M_K(C(T_K))$$

That is, given a model $\rho : C(X) \to M_K(C(T))$, we have a diagram of type

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \ \pi\ \ } & M_K(C(T_K)) \\ & {\scriptstyle\rho}\searrow \quad \swarrow & \\ & M_K(C(T)) & \end{array}$$

where the map on the right is unique, and arises from a continuous map $T \to T_K$.

Proof. Consider the universal commutative C^*-algebra generated by elements $x_{ij}(a)$, with $1 \leq i,j \leq K$ and $a \in \mathcal{O}(X)$, subject to the following relations:

$$x_{ij}(a+\lambda b) = x_{ij}(a) + \lambda x_{ij}(b)$$
$$x_{ij}(ab) = \sum_k x_{ik}(a)x_{kj}(b)$$
$$x_{ij}(1) = \delta_{ij}$$
$$x_{ij}(a)^* = x_{ji}(a^*)$$

This algebra is indeed well-defined because of the following relations:

$$\sum_l \sum_k x_{ik}(z_l^*)x_{ki}(z_l) = 1$$

Now let T_K be the spectrum of this algebra. Since X is algebraic, we have:

$$\pi : C(X) \to M_K(C(T_K)) \quad , \quad \pi(z_k) = (x_{ij}(z_k))$$

By construction of T_K and π, we have the universal matrix model. □

Still following [8], as an illustration for the above, we have:

Proposition 16.8. *Let $X \subset S^{N-1}_{\mathbb{C},+}$ with X algebraic and $X_{class} \neq \emptyset$, and let*

$$\pi : C(X) \to M_K(C(T_K))$$

be the universal matrix model. Then we have

$$C(X^{(K)}) = C(X)/Ker(\pi)$$

and hence $X = X^{(K)}$ if and only if X has a faithful $K \times K$-matrix model.

Proof. We have to prove that $Ker(\pi) = J_K$, the latter ideal being the intersection of the kernels of all matrix representations as follows, with $L \leq K$:

$$C(X) \to M_L(\mathbb{C})$$

For $a \notin Ker(\pi)$, we see that $a \notin J_K$ by evaluating at an appropriate element of T_K. Conversely, assume that we are given $a \in Ker(\pi)$. Let $\rho : C(X) \to M_L(\mathbb{C})$ be a representation with $L \leq K$, and let $\varepsilon : C(X) \to \mathbb{C}$ be a representation. We can extend ρ to a representation $\rho' : C(X) \to M_K(\mathbb{C})$ by letting, for any $b \in C(X)$:

$$\rho'(b) = \begin{pmatrix} \rho(b) & 0 \\ 0 & \varepsilon(b)I_{K-L} \end{pmatrix}$$

The universal property of the universal matrix model yields that $\rho'(a) = 0$, since $\pi(a) = 0$. Thus $\rho(a) = 0$. We therefore have $a \in J_K$, and $Ker(\pi) \subset J_K$, and the first statement is proved. The last statement follows from the first one. □

Next, we have the following result, also from [8]:

Proposition 16.9. *Let $X\subset S^{N-1}_{\mathbb{C},+}$ be algebraic, and satisfying:*

$$X_{class}\neq\emptyset$$

Then $X^{(K)}$ is algebraic as well.

Proof. We keep the notations above, and consider the following map:

$$\pi_0:\mathcal{O}(X)\to M_K(C(T_K))\quad,\quad z_l\to(x_{ij}(z_l))$$

This induces a $*$-algebra map, as follows:

$$\tilde{\pi_0}:C^*(\mathcal{O}(X)/Ker(\pi_0))\to M_K(C(T_K))$$

We need to show that $\tilde{\pi}_0$ is injective. For this purpose, observe that the universal model factorizes as follows, where p is canonical surjection:

$$\pi:C(X)\stackrel{p}{\to}C^*(\mathcal{O}(X)/Ker(\pi_0))\stackrel{\tilde{\pi}_0}{\to}M_K(C(T_K))$$

We therefore obtain $Ker(\pi)=Ker(p)$, and we conclude that:

$$C(X^{(K)})=C(X)/Ker(p)=C^*(\mathcal{O}(X)/Ker(\pi_0))$$

Thus $X^{(K)}$ is indeed algebraic. Since $\mathcal{O}(X)/Ker(\pi_0)$ is isomorphic to a $*$-subalgebra of $M_K(C(T_K))$, it satisfies the standard Amitsur-Levitski polynomial identity:

$$S_{2K}(x_1,\ldots,x_{2K})=0$$

By density, so does $C^*(\mathcal{O}(X)/Ker(\pi_0))$. Thus any irreducible representation of the algebra $C^*(\mathcal{O}(X)/Ker(\pi_0))$ has dimension $\leq K$. Consider now an element as follows:

$$a\in C^*(\mathcal{O}(X)/Ker(\pi_0))$$

Assuming $a\neq 0$ we can, by the same reasoning as in the previous proof, find a representation as follows, such that $\rho(a)\neq 0$:

$$\rho:C^*(\mathcal{O}(X)/Ker(\pi_0))\to M_K(\mathbb{C})$$

Indeed, a given algebra map $\varepsilon:C(X)\to\mathbb{C}$ induces an algebra map as follows:

$$C(T_K)\to\mathbb{C}\quad,\quad x_{ij}(a)\to\delta_{ij}\varepsilon(a)$$

But this map enables us to extend representations, as before. By construction the universal model space yields an algebra map as follows:

$$M_K(C(T_K))\to M_K(\mathbb{C})$$

The composition with $\tilde{\pi}_0p=\pi$ is then ρp, so $\tilde{\pi}_0(a)\neq 0$, and $\tilde{\pi}_0$ is injective. □

Summarizing, we have proved the following result:

THEOREM 16.10. *Let $X \subset S^{N-1}_{\mathbb{C},+}$ be algebraic, satisfying $X_{class} \neq \emptyset$. Then we have an increasing sequence of algebraic submanifolds*

$$X_{class} = X^{(1)} \subset X^{(2)} \subset X^{(3)} \subset \ldots\ldots \subset X$$

with $C(X^{(K)}) \subset M_K(C(T_K))$ being obtained by factorizing the universal model.

PROOF. This follows indeed from the above results. □

All the above is quite interesting, and we can say that $X \subset S^{N-1}_{\mathbb{C},+}$ has matrix level $K \in \mathbb{N} \cup \{\infty\}$ when the inclusion $X^{(K)} \subset X$ is an equality, with K being minimal.

16b. Stationary models

As a key illustration for the above modeling theory, let us discuss now the half-liberation operation, which is connected to $X^{(2)}$. Let us start with:

THEOREM 16.11. *Given a conjugation-stable closed subgroup $H \subset U_N$, consider the algebra $C([H]) \subset M_2(C(H))$ generated by the following variables:*

$$u_{ij} = \begin{pmatrix}0&v_{ij}\\ \bar{v}_{ij}&0\end{pmatrix}$$

Then $[H]$ is a compact quantum group, we have $[H] \subset O_N^*$, and any non-classical subgroup $G \subset O_N^*$ appears in this way, with $G = O_N^*$ itself appearing from $H = U_N$.

PROOF. We have several things to be proved, the idea being as follows:

(1) As a first observation, the matrices in the statement are self-adjoint. Let us prove now that these matrices are orthogonal. We have:

$$\sum_k u_{ik}u_{jk} = \sum_k \begin{pmatrix}0&v_{ik}\\ \bar{v}_{ik}&0\end{pmatrix}\begin{pmatrix}0&v_{jk}\\ \bar{v}_{jk}&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

In the other sense, the computation is similar, as follows:

$$\sum_k u_{ki}u_{kj} = \sum_k \begin{pmatrix}0&v_{ki}\\ \bar{v}_{ki}&0\end{pmatrix}\begin{pmatrix}0&v_{kj}\\ \bar{v}_{kj}&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

(2) Our second claim is that the matrices in the statement half-commute. Consider indeed arbitrary antidiagonal 2×2 matrices, with commuting entries, as follows:

$$X_i = \begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}$$

We have then the following computation:

$$X_iX_jX_k = \begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&x_j\\ y_j&0\end{pmatrix}\begin{pmatrix}0&x_k\\ y_k&0\end{pmatrix} = \begin{pmatrix}0&x_iy_jx_k\\ y_ix_jy_k&0\end{pmatrix}$$

Since this quantity is symmetric in i,k, we obtain, as desired:

$$X_iX_jX_k=X_kX_jX_i$$

(3) According now to the definition of the quantum group O_N^*, we have a representation of algebras, as follows where w is the fundamental corepresentation of $C(O_N^*)$:

$$\pi:C(O_N^*)\to M_2(C(H))\quad,\quad w_{ij}\to u_{ij}$$

Thus, with the compact quantum space $[H]$ being constructed as in the statement, we have a representation of algebras, as follows:

$$\rho:C(O_N^*)\to C([H])\quad,\quad w_{ij}\to u_{ij}$$

(4) With this in hand, it is routine to check that the compact quantum space $[H]$ constructed in the statement is indeed a compact quantum group, with this being best viewed via an equivalent construction, with a quantum group embedding as follows:

$$C([H])\subset C(H)\rtimes\mathbb Z_2$$

(5) As for the proof of the converse, stating that any non-classical subgroup $G\subset O_N^*$ appears in this way, this is something more tricky.

(6) Finally, we have $O_N^*=[U_N]$, and we will be back to this later. □

Getting now to the manifold case, we have here:

Definition 16.12. *The half-classical version of a manifold $X\subset S^{N-1}_{\mathbb R,+}$ is given by:*

$$C(X^*)=C(X)\Big/\left\langle abc=cba\Big|\forall a,b,c\in\{x_i\}\right\rangle$$

We say that X is half-classical when $X=X^*$.

In order to understand now the structure of X^*, we can use an old matrix model method, which goes back to Bichon-Dubois-Violette, and then to Bichon.

This is based on the following observation, that we already met in the above:

Proposition 16.13. *For any $z\in\mathbb C^N$, the matrices*

$$X_i=\begin{pmatrix}0&z_i\\ \bar z_i&0\end{pmatrix}$$

are self-adjoint, and half-commute.

Proof. This is indeed something that we know from the above. □

We will need an abstract definition, as follows:

Definition 16.14. *Given a noncommutative polynomial $f \in \mathbb{R} < x_1, \ldots, x_N >$ in N variables, we define a usual polynomial in $2N$ variables*

$$f^\circ \in \mathbb{R}[z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N]$$

according to the formula

$$f = x_{i_1} x_{i_2} x_{i_3} x_{i_4} \ldots \implies f^\circ = z_{i_1} \bar{z}_{i_2} z_{i_3} \bar{z}_{i_4} \ldots$$

in the monomial case, and then by extending this correspondence, by linearity.

As a basic example here, the polynomial defining the free real sphere $S^{N-1}_{\mathbb{R},+}$ produces in this way the polynomial defining the complex sphere $S^{N-1}_{\mathbb{C}}$:

$$f = x_1^2 + \ldots + x_N^2 \implies f^\circ = |z_1|^2 + \ldots + |z_N|^2$$

Also, given a polynomial $f \in \mathbb{R} < x_1, \ldots, x_N >$, we can decompose it into its even and odd parts, $f = g + h$, by putting into g/h the monomials of even/odd length. Observe that with $z = (z_1, \ldots, z_N)$, these odd and even parts are given by:

$$g(z) = \frac{f(z) + f(-z)}{2} \quad , \quad h(z) = \frac{f(z) - f(-z)}{2}$$

With these conventions, we have the following result:

Proposition 16.15. *Given a manifold X, coming from a family of noncommutative polynomials $\{f_\alpha\} \subset \mathbb{R} < x_1, \ldots, x_N >$, we have a morphism algebras*

$$\pi : C(X) \to M_2(\mathbb{C}) \quad , \quad \pi(x_i) = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

precisely when $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ belongs to the real algebraic manifold

$$Y = \left\{ z \in \mathbb{C}^N \middle| g^\circ_\alpha(z_1, \ldots, z_N) = h^\circ_\alpha(z_1, \ldots, z_N) = 0, \forall \alpha \right\}$$

where $f_\alpha = g_\alpha + h_\alpha$ is the even/odd decomposition of f_α.

Proof. Let X_i be the matrices in the statement. In order for $x_i \to X_i$ to define a morphism of algebras, these matrices must satisfy the equations defining X. Thus, the space Y in the statement consists of the points $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$ satisfying:

$$f_\alpha(X_1, \ldots, X_N) = 0 \quad , \quad \forall \alpha$$

Now observe that the matrices X_i in the statement multiply as follows:

$$X_{i_1} X_{j_1} \ldots X_{i_k} X_{j_k} = \begin{pmatrix} z_{i_1} \bar{z}_{j_1} \ldots z_{i_k} \bar{z}_{j_k} & 0 \\ 0 & \bar{z}_{i_1} z_{j_1} \ldots \bar{z}_{i_k} z_{j_k} \end{pmatrix}$$

$$X_{i_1} X_{j_1} \ldots X_{i_k} X_{j_k} X_{i_{k+1}} = \begin{pmatrix} 0 & z_{i_1} \bar{z}_{j_1} \ldots z_{i_k} \bar{z}_{j_k} z_{i_{k+1}} \\ \bar{z}_{i_1} z_{j_1} \ldots \bar{z}_{i_k} z_{j_k} \bar{z}_{i_{k+1}} & 0 \end{pmatrix}$$

We therefore obtain, in terms of the even/odd decomposition $f_\alpha = g_\alpha + h_\alpha$:

$$f_\alpha(X_1,\ldots,X_N)=\begin{pmatrix}g_\alpha^\circ(z_1,\ldots,z_N)&h_\alpha^\circ(z_1,\ldots,z_N)\\ \overline{h_\alpha^\circ(z_1,\ldots,z_N)}&\overline{g_\alpha^\circ(z_1,\ldots,z_N)}\end{pmatrix}$$

Thus, we obtain the equations for Y from the statement. □

As a first consequence, of theoretical interest, a necessary condition for X to exist is that the manifold $Y\subset\mathbb{C}^N$ constructed above must be compact, and we will be back to this later. In order to discuss now modeling questions, we will need as well:

Definition 16.16. *Assuming that we are given a manifold Z, appearing via*

$$C(Z)=C^*\left(z_1,\ldots,z_N\Big|f_\alpha(z_1,\ldots,z_N)=0\right)$$

we define the projective version of Z to be the quotient space $Z\to PZ$ corresponding to the subalgebra $C(PZ)\subset C(Z)$ generated by the variables $x_{ij}=z_iz_j^*$.

The relation with the half-classical manifolds comes from the fact that the projective version of a half-classical manifold is classical. Indeed, from $abc=cba$ we obtain:

$$\begin{aligned}ab\cdot cd&=(abc)d\\&=(cba)d\\&=c(bad)\\&=c(dab)\\&=cd\cdot ab\end{aligned}$$

Finally, let us call as before "matrix model" any morphism of unital C^*-algebras $f:A\to B$, with target algebra $B=M_K(C(Y))$, with $K\in\mathbb{N}$, and Y being a compact space. With these conventions, following Bichon, we have the following result:

Theorem 16.17. *Given a half-classical manifold X which is symmetric, in the sense that all its defining polynomials f_α are even, its universal 2×2 antidiagonal model,*

$$\pi:C(X)\to M_2(C(Y))$$

where Y is the manifold constructed in Proposition 16.15, is faithful. In addition, the construction $X\to Y$ is such that X exists precisely when Y is compact.

Proof. We use a standard trick. Indeed, the universal model π in the statement induces, at the level of projective versions, a certain representation:

$$C(PX)\to M_2(C(PY))$$

By using the multiplication formulae from the proof of Proposition 16.15, the image of this representation consists of diagonal matrices, and the upper left components of these matrices are the standard coordinates of PY. Thus, we have an isomorphism:

$$PX \simeq PY$$

We can conclude then by using a grading trick. □

The above result shows that when X is symmetric, we have $X^* \subset X^{(2)}$. Going beyond this observation is an interesting problem. In what follows, we will rather need a more detailed version of the above result. For this purpose, we can use:

Definition 16.18. *Associated to any compact manifold $Y \subset \mathbb{C}^N$ is the real compact half-classical manifold $[Y]$, having as coordinates the following variables,*

$$X_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

where $z_1, \ldots, z_N$ are the standard coordinates on Y. In other words, $[Y]$ is given by the fact that $C([Y]) \subset M_2(C(Y))$ is the algebra generated by these matrices.

Here the fact that the manifold $[Y]$ is indeed half-classical follows from the results above. As for the fact that $[Y]$ is indeed algebraic, this follows from Theorem 16.17. Now with this notion in hand, we can reformulate Theorem 16.17, as follows:

Theorem 16.19. *The symmetric half-classical manifolds X appear as follows:*

1. *We have $X = [Y]$, for a certain conjugation-invariant subspace $Y \subset \mathbb{C}^N$.*
2. *$PX = P[Y]$, and X is maximal with this property.*
3. *In addition, we have an embedding $C([X]) \subset C(X) \rtimes \mathbb{Z}_2$.*

Proof. This follows from Theorem 16.17, with the embedding in (3) being constructed via $x_i = z_i \otimes \tau$, where τ is the standard generator of $\mathbb{Z}_2$. □

Many other things can be said, as a continuation of the above, for instance by using cyclic $N \times N$ matrices, instead of antidiagonal 2×2 matrices.

16c. Inner faithfulness

Let us discuss now some more subtle examples of stationary models, related to the Pauli matrices, and Weyl matrices, and physics. We first have:

Definition 16.20. *Given a finite abelian group H, the associated Weyl matrices are*

$$W_{ia} : e_b \to < i, b > e_{a+b}$$

where $i \in H$, $a, b \in \widehat{H}$, and where $(i, b) \to < i, b >$ is the Fourier coupling $H \times \widehat{H} \to \mathbb{T}$.

As a basic example, consider the simplest cyclic group, namely:

$$H = \mathbb{Z}_2 = \{0, 1\}$$

Here the Fourier coupling is $< i,b >= (-1)^{ib}$, and the Weyl matrices act as follows:

$$W_{00}: e_b \to e_b \quad , \quad W_{10}: e_b \to (-1)^b e_b$$

$$W_{11}: e_b \to (-1)^b e_{b+1} \quad , \quad W_{01}: e_b \to e_{b+1}$$

Thus, we have the following formulae for the Weyl matrices:

$$W_{00} = \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad , \quad W_{10} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$W_{11} = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \quad , \quad W_{01} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices. Now back to the general case, we have the following well-known result:

PROPOSITION 16.21. *The Weyl matrices are unitaries, and satisfy:*

(1) $W_{ia}^* =< i,a > W_{-i,-a}$.
(2) $W_{ia}W_{jb} =< i,b > W_{i+j,a+b}$.
(3) $W_{ia}W_{jb}^* =< j-i,b > W_{i-j,a-b}$.
(4) $W_{ia}^*W_{jb} =< i,a-b > W_{j-i,b-a}$.

PROOF. The unitarity follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned} W_{ia}^* &= \left(\sum_b < i,b > E_{a+b,b}\right)^* \\ &= \sum_b < -i,b > E_{b,a+b} \\ &= \sum_b < -i,b-a > E_{b-a,b} \\ &= < i,a > W_{-i,-a} \end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned} W_{ia}W_{jb} &= \left(\sum_d < i,b+d > E_{a+b+d,b+d}\right)\left(\sum_d < j,d > E_{b+d,d}\right) \\ &= \sum_d < i,b >< i+j,d > E_{a+b+d,d} \\ &= < i,b > W_{i+j,a+b} \end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned} W_{ia}W_{jb}^* &= < j,b > W_{ia}W_{-j,-b} \\ &= < j,b >< i,-b > W_{i-j,a-b} \end{aligned}$$

We obtain as well the following formula:

$$\begin{aligned} W_{ia}^*W_{jb} &= < i,a > W_{-i,-a}W_{jb} \\ &= < i,a >< -i,b > W_{j-i,b-a} \end{aligned}$$

But this gives the formulae in the statement, and we are done. □

With $n = |H|$, we can use an isomorphism $l^2(\widehat{H}) \simeq \mathbb{C}^n$ as to view each W_{ia} as a usual matrix, $W_{ia} \in M_n(\mathbb{C})$, and hence as a usual unitary, $W_{ia} \in U_n$. Also, given a vector ξ, we denote by $Proj(\xi)$ the orthogonal projection onto $\mathbb{C}\xi$. Following [8], we have:

PROPOSITION 16.22. *Given a closed subgroup $E \subset U_n$, we have a representation*

$$\pi_H : C(S_N^+) \to M_N(C(E))$$

$$w_{ia,jb} \to [U \to Proj(W_{ia}UW_{jb}^*)]$$

where $n = |H|, N = n^2$, and where W_{ia} are the Weyl matrices associated to H.

PROOF. The Weyl matrices being given by $W_{ia} : e_b \to < i,b > e_{a+b}$, we have:

$$tr(W_{ia}) = \begin{cases} 1 & \text{if } (i,a) = (0,0) \\ 0 & \text{if } (i,a) \neq (0,0) \end{cases}$$

Together with the formulae in Proposition 16.21, this shows that the Weyl matrices are pairwise orthogonal with respect to the following scalar product on $M_n(\mathbb{C})$:

$$< x,y >= tr(xy^*)$$

Thus, these matrices form an orthogonal basis of $M_n(\mathbb{C})$, consisting of unitaries:

$$W = \left\{ W_{ia} \Big| i \in H, a \in \widehat{H} \right\}$$

Thus, each row and each column of the matrix $\xi_{ia,jb} = W_{ia}UW_{jb}^*$ is an orthogonal basis of $M_n(\mathbb{C})$, and so the corresponding projections form a magic unitary, as claimed. □

We will need the following well-known result:

PROPOSITION 16.23. *With $T = Proj(x_1)\ldots Proj(x_p)$ and $||x_i|| = 1$ we have*

$$< T\xi,\eta >=< \xi,x_p >< x_p,x_{p-1} > \ldots < x_2,x_1 >< x_1,\eta >$$

for any ξ,η. In particular, we have:

$$Tr(T) =< x_1,x_p >< x_p,x_{p-1} > \ldots < x_2,x_1 >$$

PROOF. For $||x||=1$ we have $Proj(x)\xi=<\xi,x>x$. This gives:

$$\begin{aligned}T\xi&=Proj(x_1)\ldots Proj(x_p)\xi\\&=Proj(x_1)\ldots Proj(x_{p-1})<\xi,x_p>x_p\\&=Proj(x_1)\ldots Proj(x_{p-2})<\xi,x_p><x_p,x_{p-1}>x_{p-1}\\&=\ldots\\&=<\xi,x_p><x_p,x_{p-1}>\ldots<x_2,x_1>x_1\end{aligned}$$

Now by taking the scalar product with η, this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over $\xi=\eta=e_i$. □

Now back to the Weyl matrix models, let us first compute T_p. We have:

PROPOSITION 16.24. *We have the formula*

$$\begin{aligned}(T_p)_{ia,jb}&=\frac{1}{N}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\&\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{j_2-j_1,b_2-b_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{j_1-j_p,b_1-b_p}U^*)dU\end{aligned}$$

with all the indices varying in a cyclic way.

PROOF. By using the trace formula in Proposition 16.23, we obtain:

$$\begin{aligned}&(T_p)_{ia,jb}\\&=\left(tr\otimes\int_E\right)\left(Proj(W_{i_1a_1}UW^*_{j_1b_1})\ldots Proj(W_{i_pa_p}UW^*_{j_pb_p})\right)\\&=\frac{1}{N}\int_E<W_{i_1a_1}UW^*_{j_1b_1},W_{i_pa_p}UW^*_{j_pb_p}>\ldots<W_{i_2a_2}UW^*_{j_2b_2},W_{i_1a_1}UW^*_{j_1b_1}>dU\end{aligned}$$

In order to compute now the scalar products, observe that we have:

$$\begin{aligned}<W_{ia}UW_{jb}^*,W_{kc}UW_{ld}^*>&=tr(W_{jb}U^*W_{ia}^*W_{kc}UW_{ld}^*)\\&=tr(W_{ia}^*W_{kc}UW_{ld}^*W_{jb}U^*)\\&=<i,a-c><l,d-b>tr(W_{k-i,c-a}UW_{j-l,b-d}U^*)\end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. □

Consider now the Weyl group $W=\{W_{ia}\}\subset U_n$, that we already met in the proof of Proposition 16.23. We have the following result, from [8]:

THEOREM 16.25. *For any compact group $W\subset E\subset U_n$, the model*

$$\pi_H:C(S_N^+)\to M_N(C(E))$$

$$w_{ia,jb}\to[U\to Proj(W_{ia}UW_{jb}^*)]$$

constructed above is stationary on its image.

PROOF. We must prove that we have $T_p^2=T_p$. We have:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
=&\sum_{kc}(T_p)_{ia,kc}(T_p)_{kc,jb}\\
=&\frac{1}{N^2}\sum_{kc}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><k_1,c_1-c_2>\ldots<k_p,c_p-c_1>\\
&<k_1,c_1-c_p>\ldots<k_p,c_p-c_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\
&\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p}U^*)dU\\
&\int_E tr(W_{k_1-k_2,c_1-c_2}VW_{j_2-j_1,b_2-b_1}V^*)\ldots tr(W_{k_p-k_1,c_p-c_1}VW_{j_1-j_p,b_1-b_p}V^*)dV
\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
=&\frac{1}{N^2}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\
&\int_E\int_E\sum_{kc}<k_1-k_p,c_1-c_p>\ldots<k_p-k_{p-1},c_p-c_{p-1}>\\
&tr(W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1}U^*)tr(W_{k_1-k_2,c_1-c_2}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\qquad\qquad\ldots\ldots\\
&tr(W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p}U^*)tr(W_{k_p-k_1,c_p-c_1}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

Let us denote by I the above double integral. By using $W_{kc}^*=<k,c>W_{-k,-c}$ for each of the couplings, and by moving as well all the U^* variables to the left, we obtain:

$$\begin{aligned}
I\ =&\int_E\int_E\sum_{kc}tr(U^*W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1})tr(W^*_{k_2-k_1,c_2-c_1}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\qquad\qquad\ldots\ldots\\
&tr(U^*W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p})tr(W^*_{k_1-k_p,c_1-c_p}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned}
tr(AW_{kc})tr(W_{kc}^*B)&=\frac{1}{N}\sum_{qrst}A_{qr}(W_{kc})_{rq}(W_{kc}^*)_{st}B_{ts}\\
&=\frac{1}{N}\sum_{qrst}A_{qr}<k,q>\delta_{r-q,c}<k,-s>\delta_{t-s,c}B_{ts}\\
&=\frac{1}{N}\sum_{qs}<k,q-s>A_{q,q+c}B_{s+c,s}
\end{aligned}$$

If we denote by A_x, B_x the variables which appear in the formula of I, we have:

$$\begin{aligned}
& I \\
= & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_2-k_1, q_1-s_1 > \ldots < k_1-k_p, q_p-s_p > \\
& (A_1)_{q_1,q_1+c_2-c_1}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{q_p,q_p+c_1-c_p}(B_p)_{s_p+c_1-c_p,s_p} \\
= & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1, q_p-s_p-q_1+s_1 > \ldots < k_p, q_{p-1}-s_{p-1}-q_p+s_p > \\
& (A_1)_{q_1,q_1+c_2-c_1}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{q_p,q_p+c_1-c_p}(B_p)_{s_p+c_1-c_p,s_p}
\end{aligned}$$

Now observe that we can perform the sums over $k_1,\ldots,k_p$. We obtain in this way a multiplicative factor n^p, along with the condition:

$$q_1-s_1=\ldots=q_p-s_p$$

Thus we must have $q_x=s_x+a$ for a certain a, and the above formula becomes:

$$I=\frac{1}{n^p}\int_E\int_E\sum_{csa}(A_1)_{s_1+a,s_1+c_2-c_1+a}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{s_p+a,s_p+c_1-c_p+a}(B_p)_{s_p+c_1-c_p,s_p}$$

Consider now the variables $r_x=c_{x+1}-c_x$, which altogether range over the set Z of multi-indices having sum 0. By replacing the sum over c_x with the sum over r_x, which creates a multiplicative n factor, we obtain the following formula:

$$I=\frac{1}{n^{p-1}}\int_E\int_E\sum_{r\in Z}\sum_{sa}(A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1}\ldots(A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}$$

For an arbitrary multi-index r, we have the following formula:

$$\delta_{\sum_i r_i,0}=\frac{1}{n}\sum_i < i,r_1 > \ldots < i,r_p >$$

Thus, we can replace the sum over $r\in Z$ by a full sum, as follows:

$$\begin{aligned}
I = & \frac{1}{n^p}\int_E\int_E\sum_{rsia} < i,r_1 > (A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1} \\
& \ldots\ldots \\
& < i,r_p > (A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}
\end{aligned}$$

In order to "absorb" now the indices i,a, we can use the following formula:

$$\begin{aligned}
& W_{ia}^*AW_{ia} \\
= & \left(\sum_b < i,-b > E_{b,a+b}\right)\left(\sum_{bc}E_{a+b,a+c}A_{a+b,a+c}\right)\left(\sum_c < i,c > E_{a+c,c}\right) \\
= & \sum_{bc} < i,c-b > E_{bc}A_{a+b,a+c}
\end{aligned}$$

Thus we have the following formula:

$$(W_{ia}^*AW_{ia})_{bc} =< i, c-b > A_{a+b,a+c}$$

With this in hand, our formula becomes:

$$\begin{aligned} & I \\ = & \frac{1}{n^p}\int_E\int_E\sum_{rsia}(W_{ia}^*A_1W_{ia})_{s_1,s_1+r_1}(B_1)_{s_1+r_1,s_1}\ldots(W_{ia}^*A_pW_{ia})_{s_p,s_p+r_p}(B_p)_{s_p+r_p,s_p} \\ = & \int_E\int_E\sum_{ia}tr(W_{ia}^*A_1W_{ia}B_1)\ldots\ldots tr(W_{ia}^*A_pW_{ia}B_p) \end{aligned}$$

Now by replacing A_x, B_x with their respective values, we obtain:

$$\begin{aligned} I &= \int_E\int_E\sum_{ia}tr(W_{ia}^*U^*W_{i_1-i_2,a_1-a_2}UW_{ia}VW_{j_2-j_1,b_2-b_1}V^*) \\ & \qquad\qquad\ldots\ldots \\ & \quad tr(W_{ia}^*U^*W_{i_p-i_1,a_p-a_1}UW_{ia}VW_{j_1-j_p,b_1-b_p}V^*)dUdV \end{aligned}$$

By moving the $W_{ia}^*U^*$ variables at right, we obtain, with $S_{ia}=UW_{ia}V$:

$$\begin{aligned} I &= \sum_{ia}\int_E\int_E tr(W_{i_1-i_2,a_1-a_2}S_{ia}W_{j_2-j_1,b_2-b_1}S_{ia}^*) \\ & \qquad\qquad\ldots\ldots \\ & \quad tr(W_{i_p-i_1,a_p-a_1}S_{ia}W_{j_1-j_p,b_1-b_p}S_{ia}^*)dUdV \end{aligned}$$

Now since S_{ia} is Haar distributed when U, V are Haar distributed, we obtain:

$$I = N\int_E\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{j_2-j_1,b_2-b_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{j_1-j_p,b_1-b_p}U^*)dU$$

But this is exactly N times the integral in the formula of $(T_p)_{ia,jb}$, from Proposition 16.24. Since the N factor cancels with one of the two N factors that we found in the beginning of the proof, when first computing $(T_p^2)_{ia,jb}$, we are done. □

As an illustration for the above result, going back to [8], we have:

Theorem 16.26. *We have a stationary matrix model*

$$\pi : C(S_4^+) \subset M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_ixc_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices.

PROOF. As already explained in the comments following Definition 16.20, the Pauli matrices appear as particular cases of the Weyl matrices:

$$W_{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad,\quad W_{10}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$W_{11}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\quad,\quad W_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

Thus, Theorem 16.25 produces in this case the model in the statement. □

Observe that, since the projection $Proj(c_ixc_j)$ depends only on the image of x in the quotient $SU_2\to SO_3$, we can replace the model space SU_2 by the smaller space SO_3. This can be used in conjunction with the isomorphism $S_4^+\simeq SO_3^{-1}$, and as explained in [8], our model becomes in this way something more conceptual, as follows:

$$\pi:C(SO_3^{-1})\subset M_4(C(SO_3))$$

As a philosophical conclusion, to this and to some previous findings as well, no matter what we do, we always end up getting back to SU_2,SO_3. Thus, we are probably doing some physics here. This is indeed the case, the above computations being closely related to the standard computations for the Ising and Potts models. The general relation, however, between quantum permutations and lattice models, is not axiomatixed yet.

We know from the above that we have a stationary matrix model for the algebra $C(S_4^+)$. In view of [8], this suggests the following conjecture:

CONJECTURE 16.27. *Given a quantum permutation group of* 4 *points,*

$$G\subset S_4^+\simeq SO_3^{-1}$$

coming by twisting a usual ADE subgroup of the group SO_3*,*

$$H\subset SO_3$$

the restriction of the Pauli model for $C(S_4^+)$*, with fibers coming from the elements of* $H\subset SO_3$*, has the algebra* $C(G)$ *as Hopf image.*

To be more precise, the main result from the previous section tells us that the conjecture holds for $G=S_4^+$ itself. Indeed, here we have $H=SO_3$, so the corresponding restriction of the Pauli model for $C(S_4^+)$ is the Pauli model itself, and this model being stationary, its Hopf image is the algebra $C(S_4^+)$ itself, as stated.

In general, the above conjecture does not look that scary, because the same methods used for S_4^+ can be used for any subgroup $G\subset S_4^+$. However, the problem is that, unless a global method in order to uniformly deal with the problem is found, this would need a case-by-case study depending on $G\subset S_4^+$, which looks quite time-consuming.

16d. Universal models

Following [8], let us study now the universal flat model for $C(S_N^+)$. Given a flat magic unitary $u=(u_{ij})$, we can write it, in a non-unique way, as follows:

$$u_{ij}=Proj(\xi_{ij})$$

The array $\xi=(\xi_{ij})$ is then a "magic basis", in the sense that each of its rows and columns is an orthonormal basis of $\mathbb C^N$, and with this being an interesting combinatorial notion, mixing linear algebra and design theory. More on such magic bases later.

In relation now with universal models, we are led to two spaces, as follows:

Definition 16.28. *Associated to any $N\in\mathbb N$ are the following spaces:*

(1) *X_N, the space of all $N\times N$ flat magic unitaries $u=(u_{ij})$.*
(2) *K_N, the space of all $N\times N$ magic bases $\xi=(\xi_{ij})$.*

Let us recall now that the rank 1 projections $p\in M_N(\mathbb C)$ can be identified with the corresponding 1-dimensional subspaces $E\subset\mathbb C^N$, which are by definition the elements of the complex projective space $P^{N-1}_{\mathbb C}$. In addition, if we consider the complex sphere, $S^{N-1}_{\mathbb C}=\{z\in\mathbb C^N|\sum_i|z_i|^2=1\}$, we have a quotient map as follows:

$$\pi:S^{N-1}_{\mathbb C}\to P^{N-1}_{\mathbb C}\quad,\quad z\to Proj(z)$$

Observe that we have $\pi(z)=\pi(z')$ precisely when $z'=wz$, for some $w\in\mathbb T$. Consider as well the embedding $U_N\subset(S^{N-1}_{\mathbb C})^N$ given by $x\to(x_1,\ldots,x_N)$, where $x_1,\ldots,x_N$ are the rows of x. Finally, as before, let us call an abstract matrix stochastic/bistochastic when the entries on each row/each row and column sum up to 1.

With these notations and conventions, the abstract model spaces X_N,K_N from Definition 16.28 that we are interested in, and some related spaces, are as follows:

Proposition 16.29. *We have inclusions and surjections as follows,*

$$\begin{matrix}K_N&\subset&U_N^N&\subset&M_N(S^{N-1}_{\mathbb C})\\ \\ \downarrow&&\downarrow&&\downarrow\\ \\ X_N&\subset&Y_N&\subset&M_N(P^{N-1}_{\mathbb C})\end{matrix}$$

where X_N,Y_N consist of bistochastic/stochastic matrices, and K_N is the lift of X_N.

Proof. This follows from the above discussion. Indeed, the quotient map $S^{N-1}_{\mathbb C}\to P^{N-1}_{\mathbb C}$ induces the quotient map $M_N(S^{N-1}_{\mathbb C})\to M_N(P^{N-1}_{\mathbb C})$ at right, and the lift of the space of stochastic matrices $Y_N\subset M_N(P^{N-1}_{\mathbb C})$ is then the rescaled group U_N^N, as claimed. □

In order to get some insight into the structure of the above spaces X_N, K_N, we can use some inspiration from the well-known Sinkhorn algorithm from linear algebra. Indeed, this algorithm starts with a $N \times N$ matrix having positive entries and produces, via successive averagings over rows/columns, a bistochastic matrix.

In our situation, we would like to have an "averaging" map $Y_N \to Y_N$, whose infinite iteration lands in the model space X_N. Equivalently, we would like to have an "averaging" map $U_N^N \to U_N^N$, whose infinite iteration lands in the space K_N.

In order to construct such averaging maps, we use the orthogonalization procedure coming from the polar decomposition, the result that we need being as follows:

Proposition 16.30. *We have orthogonalization maps as follows,*

$$\begin{array}{ccc}
(S_\mathbb{C}^{N-1})^N & \xrightarrow{\ \alpha\ } & (S_\mathbb{C}^{N-1})^N \\
\downarrow & & \downarrow \\
(P_\mathbb{C}^{N-1})^N & \xrightarrow{\ \beta\ } & (P_\mathbb{C}^{N-1})^N
\end{array}$$

where $\alpha(x)_i = Pol([(x_i)_j]_{ij})$*, and* $\beta(p) = (P^{-1/2}p_iP^{-1/2})_i$*, with* $P = \sum_i p_i$*.*

Proof. This is something which is routine, the idea being as follows:

(1) Our first claim is that we have a factorization as in the statement. Indeed, pick $p_1, \ldots, p_N \in P_\mathbb{C}^{N-1}$, and write $p_i = Proj(x_i)$, with $||x_i|| = 1$. We can then apply α, as to obtain a vector $\alpha(x) = (x_i')_i$, and then set $\beta(p) = (p_i')$, where $p_i' = Proj(x_i')$.

(2) Our first task is to prove that β is well-defined. Consider indeed vectors $\tilde{x}_i$, satisfying $Proj(\tilde{x}_i) = Proj(x_i)$. We have then $\tilde{x}_i = \lambda_i x_i$, for certain scalars $\lambda_i \in \mathbb{T}$, and so the matrix formed by these vectors is $\widetilde{M} = \Lambda M$, with $\Lambda = diag(\lambda_i)$. It follows that $Pol(\widetilde{M}) = \Lambda Pol(M)$, and so $\tilde{x}_i' = \lambda_i x_i$, and finally $Proj(\tilde{x}_i') = Proj(x_i')$, as desired.

(3) It remains to prove that β is given by the formula in the statement. For this purpose, observe first that, given $x_1, \ldots, x_N \in S_\mathbb{C}^{N-1}$, with $p_i = Proj(x_i)$ we have:

$$\begin{aligned}
\sum_i p_i &= \sum_i [(\bar{x}_i)_k (x_i)_l]_{kl} \\
&= \sum_i (\bar{M}_{ik} M_{il})_{kl} \\
&= ((M^*M)_{kl})_{kl} \\
&= M^*M
\end{aligned}$$

(4) We can now compute the projections $p_i'=Proj(x_i')$. Indeed, the coefficients of these projections are given by $(p_i')_{kl}=\bar{U}_{ik}U_{il}$ with $U=MP^{-1/2}$, and we obtain, as desired:

$$\begin{aligned}(p_i')_{kl} &= \sum_{ab}\bar{M}_{ia}P_{ak}^{-1/2}M_{ib}P_{bl}^{-1/2}\\ &= \sum_{ab}P_{ka}^{-1/2}\bar{M}_{ia}M_{ib}P_{bl}^{-1/2}\\ &= \sum_{ab}P_{ka}^{-1/2}(p_i)_{ab}P_{bl}^{-1/2}\\ &= (P^{-1/2}p_iP^{-1/2})_{kl}\end{aligned}$$

(5) An alternative proof uses the fact that the elements $p_i'=P^{-1/2}p_iP^{-1/2}$ are self-adjoint, and sum up to 1. The fact that these elements are indeed idempotents can be checked directly, via $p_iP^{-1}p_i=p_i$, because this equality holds on $\ker p_i$, and also on x_i. □

As an illustration, here is how the orthogonalization works at $N=2$:

PROPOSITION 16.31. *At $N=2$ the orthogonalization procedure for*

$$(Proj(x),Proj(y))$$

amounts in considering the vectors

$$\frac{x+y}{\sqrt{2}}\quad,\quad\frac{x-y}{\sqrt{2}}$$

and then rotating by $45°$.

PROOF. By performing a rotation, we can restrict attention to the case $x=(\cos t,\sin t)$ and $y=(\cos t,-\sin t)$, with $t\in(0,\pi/2)$. Here the computations are as follows:

$$\begin{aligned}M=\begin{pmatrix}\cos t&\sin t\\ \cos t&-\sin t\end{pmatrix}\quad&\implies\quad P=M^*M=\begin{pmatrix}2\cos^2t&0\\ 0&2\sin^2t\end{pmatrix}\\ &\implies\quad P^{-1/2}=|M|^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{1}{\cos t}&0\\ 0&\frac{1}{\sin t}\end{pmatrix}\\ &\implies\quad U=M|M|^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\ 1&-1\end{pmatrix}\end{aligned}$$

Thus the orthogonalization procedure replaces $(Proj(x),Proj(y))$ by the orthogonal projections on the vectors $(\frac{1}{\sqrt{2}}(1,1),\frac{1}{\sqrt{2}}(-1,1))$, and this gives the result. □

With these preliminaries in hand, let us discuss now the version that we need of the Sinkhorn algorithm. The orthogonalization procedure is as follows:

THEOREM 16.32. *The orthogonalization maps α, β induce maps as follows,*

$$\begin{array}{ccc} U_N^N & \xrightarrow{\Phi} & U_N^N \\ \downarrow & & \downarrow \\ Y_N & \xrightarrow{\Psi} & Y_N \end{array}$$

which are the transposition maps on K_N, X_N, and which are projections at $N = 2$.

PROOF. It follows from definitions that $\Phi(x)$ is obtained by putting the components of $x = (x_i)$ in a row, then picking the j-th column vectors of each x_i, calling M_j this matrix, then taking the polar part $x'_j = Pol(M_j)$, and finally setting $\Phi(x) = x'$. Thus:

$$\Phi(x) = Pol((x_{ij})_i)_j$$

$$\Psi(u) = (P_i^{-1/2} u_{ji} P_i^{-1/2})_{ij}$$

Thus, the first assertion is clear, and the second assertion is clear too. □

Our claim is that the algorithm converges, as follows:

CONJECTURE 16.33. *The above maps Φ, Ψ increase the volume,*

$$vol : U_N^N \to Y_N \to [0,1], \quad vol(u) = \prod_j |\det((u_{ij})_i)|$$

and respectively land, after an infinite number of steps, in K_N/X_N.

Observe that the quantities of type $|\det(p_1, \ldots, p_N)|$ are indeed well-defined, for any $p_1, \ldots, p_N \in P^{N-1}_{\mathbb{C}}$, because multiplying by scalars $\lambda_i \in \mathbb{T}$ doesn't change the volume. Thus, the volume map $vol : U_N^N \to [0,1]$ factorizes through Y_N, as stated above.

As a main application of the above conjecture, the infinite iteration $(\Phi^2)^\infty : U_N^N \to K_N$ would provide us with an integration on K_N, and hence on the quotient space $K_N \to X_N$ as well, by taking the push-forward measures, coming from the Haar measure on U_N^N.

In relation now with the matrix model problematics, we have:

CONJECTURE 16.34. *The universal $N \times N$ flat matrix representation*

$$\pi_N : C(S_N^+) \to M_N(C(X_N)), \quad \pi_N(w_{ij}) = (u \to u_{ij})$$

is inner faithful at any $N \geq 4$.

Regarding the $N = 4$ conjecture, the problem here is that of proving that the composition $C(S_4^+) \to M_4(C(X_4)) \to \mathbb{C}$ equals the Haar integration on S_4^+. As for the $N \geq 5$ conjecture, the problem here is that of proving that the truncated moments c_p^r converge with $r \to \infty$ to the Catalan numbers. None of these questions is trivial.

Still following [8], our purpose now will be to advance towards a unification of the two conjectures formulated above. The point indeed is that when trying to approach Conjecture 16.34 with standard probabilistic tools, the estimates that are needed seem to be related to those required for approaching Conjecture 16.33.

We first have the following definition, inspired by the above results:

Definition 16.35. *Associated to $x \in M_N(S^{N-1}_{\mathbb{C}})$ is the $N^p \times N^p$ matrix*

$$(T_p^x)_{i_1\ldots i_p, j_1\ldots j_p} = \frac{1}{N} < x_{i_1j_1}, x_{i_pj_p} >< x_{i_pj_p}, x_{i_{p-1}j_{p-1}} > \ldots\ldots < x_{i_2j_2}, x_{i_1j_1} >$$

where the scalar products are the usual ones on $S^{N-1}_{\mathbb{C}} \subset \mathbb{C}^N$.

The first few values of these matrices, at $p = 1, 2, 3$, are as follows:

$$\begin{aligned}(T_1^x)_{ia} &= \frac{1}{N} < x_{ia}, x_{ia} >= \frac{1}{N}\\ (T_2^x)_{ij,ab} &= \frac{1}{N} < x_{ia}, x_{jb} >< x_{jb}, x_{ia} >= \frac{1}{N} | < x_{ia}, x_{jb} > |^2\\ (T_3^x)_{ijk,abc} &= \frac{1}{N} < x_{ia}, x_{kc} >< x_{kc}, x_{jb} >< x_{jb}, x_{ia} >\end{aligned}$$

The interest in these matrices, in connection with Conjecture 16.33, comes from:

Proposition 16.36. *For the universal model, the matrices T_p are*

$$T_p = \int_{K_N} T_p^x dx$$

where dx is the measure on the model space K_N coming from Conjecture 16.33.

Proof. This is a trivial statement, because by definition of T_p, we have:

$$\begin{aligned}(T_p)_{i_1\ldots i_p, j_1\ldots j_p} &= tr(u_{i_1j_1}\ldots u_{i_pj_p}) = \int_{K_N} tr(u^x_{i_1j_1}\ldots u^x_{i_pj_p})dx\\ &= \int_{K_N} tr(Proj(x_{i_1j_1})\ldots Proj(x_{i_px_p}))dx\\ &= \frac{1}{N}\int_{K_N} < x_{i_1j_1}, x_{i_pj_p} > \ldots\ldots < x_{i_2j_2}, x_{i_1j_1} > dx\\ &= \int_{K_N} (T_p^x)_{i_1\ldots i_p, j_1\ldots j_p} dx\end{aligned}$$

Thus the formula in the statement holds indeed. ∎

In fact, the matrices T_p^x are related to Conjecture 16.34 as well. Indeed, to any noncrossing partition $\pi\in NC(p)$ let us associate the following vector of $(\mathbb C^N)^{\otimes p}$:

$$\xi_\pi=\sum_{\ker i\leq\pi}e_{i_1}\otimes\ldots\otimes e_{i_p}$$

With this convention, we have the following result, again from [8]:

PROPOSITION 16.37. *For any $x\in M_N(S^{N-1}_\mathbb C)$, the following hold:*

1. *If $\{x_{ij}\}_i$ are pairwise orthogonal then $(T_p^x)^*\xi_{||\ldots|}=\xi_{||\ldots|}$ and $T_p^x\xi_{\sqcap\sqcap\ldots\sqcap}=\xi_{\sqcap\sqcap\ldots\sqcap}$.*
2. *If $\{x_{ij}\}_j$ are pairwise orthogonal then $T_p^x\xi_{||\ldots|}=\xi_{||\ldots|}$ and $(T_p^x)^*\xi_{\sqcap\sqcap\ldots\sqcap}=\xi_{\sqcap\sqcap\ldots\sqcap}$.*
3. *If $\{x_{ij}\}_i$ or $\{x_{ij}\}_j$ are pairwise orthogonal then $<T_p^x\xi_{||\ldots|},\xi_{||\ldots|}>=N^p$.*
4. *We have $<T_p^x\xi_{\sqcap\sqcap\ldots\sqcap},\xi_{\sqcap\sqcap\ldots\sqcap}>=N$, without assumptions on x.*

PROOF. Assuming that $\{x_{ij}\}_i$ are pairwise orthogonal, we have indeed:

$$\begin{aligned}(T_p^x\xi_{\sqcap\sqcap\ldots\sqcap})_{i_1\ldots i_p}&=\sum_j(T_p^x)_{i_1\ldots i_p,j\ldots j}\\&=\frac{1}{N}\sum_j<x_{i_1j},x_{i_pj}>\ldots\ldots<x_{i_2j},x_{i_1j}>\\&=\delta_{i_1,\ldots,i_p}\end{aligned}$$

Thus we have proved (1), and the proof of (2,3,4) is similar. See [8]. □

We have the following statement, supported by computer calculations:

CONJECTURE 16.38. *Consider the following function, with $x\in M_N(S^{N-1}_\mathbb C)$,*

$$F_p(x)=\frac{1}{N^p}||T_p^x\xi_{\sqcap\sqcap\ldots\sqcap}||^2$$

depending on a fixed integer $p\geq2$. Then, for any $x\in U_N^N$ we have

$$F_p(x)\geq F_p(\Psi^2(x))$$

with equality precisely when $x\in K_N$, in which case $F_p(x)=1$.

This conjecture is quite interesting, in relation with the above, because by a compacity argument, this would prove that our Sinkhorn type algorithm converges. Thus, what we have here is a first step towards unifying Conjectures 16.33 and 16.34.

We refer to [8] and the related literature for more on these questions.

Along the same lines, universal models for quantum groups, there are of course some easier questions too, regarding the modeling of the various possible subgroups $G\subset S_N^+$, as for instance the group duals $\widehat\Gamma\subset S_N^+$, and again, we refer here to the literature.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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Index