Initiation to subfactors

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ABSTRACT. This is an introduction to the theory of subfactors, as developed by Jones and others. A von Neumann algebra is a weakly closed operator *-algebra $A \subset B(H)$, a factor is such an algebra having trivial center, $Z(A) = \mathbb{C}$, and a subfactor is an inclusion of such factors, $A \subset B$. We first discuss the basics of the theory, notably with Jones' discovery that any subfactor $A \subset B$ produces a representation of the Temperley-Lieb algebra, and in fact, produces a planar algebra, which can be thought of as being the algebra of invariants of a certain quantum group type object G. Then we discuss more specialized aspects, dealing with the case where G is finite, or more generally amenable. Finally, we provide an introduction to the most interesting case, where both factors A, Bare isomorphic to the Murray-von Neumann hyperfinite factor R.

Preface

The algebras of bounded linear operators $A \subset B(H)$ on a complex Hilbert space H, typically taken separable, were studied starting with the work of von Neumann in the 1930s. Von Neumann was interested in quantum mechanics, a new discipline at that time, developed in the 1920s by Heisenberg, Schrödinger, Dirac and others. One of the conclusions of quantum mechanics was the fact that the states of a quantum system are described by the vectors of a complex Hilbert space H, and the observables are described by certain linear operators, which are possibly unbounded, $T : H \to H$. By looking at the commutants of such observables, $A = \{T\}'$, von Neumann was led into the study of the weakly closed operator *-algebras $A \subset B(H)$, now called von Neumann algebras.

There has been a lot of work on the von Neumann algebras, with the fundamentals developed by Murray and von Neumann in the 1930s and 1940s, and with some further fundamentals developed by Connes in the early 1970s. The conclusion of this work, which is something non-trivial, is that that the "building blocks" of the whole theory are the von Neumann algebras $A \subset B(H)$ which are infinite dimensional, $\dim(A) = \infty$, have trivial center, $Z(A) = \mathbb{C}$, and have a trace $tr : A \to \mathbb{C}$. These are called II₁ factors.

In view of this finding, an interesting question, which is something very natural, mathematically speaking, and is motivated as well by various questions from quantum mechanics, is to look at the inclusions $A \subset B$ of such II₁ factors, which are called subfactors. This was done by Jones in the late 1970s, and then all over the 1980s and 1990s, with the remarkable conclusion that any such subfactor $A \subset B$ produces a representation of the Temperley-Lieb algebra, and in fact, produces a planar algebra, which can be thought of as being the algebra of invariants of a certain quantum group type object G.

This book is an introduction to this, subfactor theory, as developed by Jones and others, and its ramifications. The book is organized in four parts, as follows:

(1) We first discuss the basics of the theory, notably with explanations on the abovementioned key finding of Jones, involving the Temperley-Lieb algebra.

(2) We then go on a more advanced discussion, following Jones and Popa, featuring planar algebras, and the associated quantum group type objects G.

PREFACE

(3) We discuss then some more specialized aspects, dealing with the case where the above-mentioned underlying quantum group type object G is finite.

(4) Finally, we provide an introduction to the most interesting case, where both factors A, B are isomorphic to the Murray-von Neumann hyperfinite factor R.

In the hope that you will find this book useful. The material here will be quite old style, basically stopping around 2000, but do not worry, the old-fashioned questions at the end, regarding the subfactors of R, are still open, and waiting for your input.

Many thanks to everyone, having helped me to progress in my subfactor learning, over the years. Thanks as well to my cats, for some help with functional analysis.

Cergy, April 2025 Teo Banica

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Part I

Subfactors

Farewell, Angelina The bells of the crown Are being stolen by bandits I must follow the sound

CHAPTER 1

Operator algebras

1a. Normed algebras

We must first talk about operator algebras. Let us start with the following broad definition, obtained by imposing the "minimal" set of reasonable axioms:

DEFINITION 1.1. An operator algebra is an algebra of bounded operators $A \subset B(H)$ which contains the unit, is closed under taking adjoints,

$$T \in A \implies T^* \in A$$

and is closed as well under the norm.

As a first result now regarding the operator algebras, in relation with the normal operators, where most of the non-trivial results in operator theory are, we have:

THEOREM 1.2. The operator algebra $\langle T \rangle \subset B(H)$ generated by a normal operator $T \in B(H)$ appears as an algebra of continuous functions,

$$< T > = C(\sigma(T))$$

where $\sigma(T) \subset \mathbb{C}$ denotes as usual the spectrum of T.

PROOF. We know that we have a continuous morphism of *-algebras, as follows:

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the continuous calculus for the normal operators, the image of this morphism is the norm closed algebra $\langle T \rangle$ generated by the operators T, T^* . Thus, we obtain the isomorphism in the statement.

The above result is very nice, and it is possible to further build on it, as follows:

THEOREM 1.3. The operator algebra $\langle T_i \rangle \subset B(H)$ generated by a family of normal operators $T_i \in B(H)$ appears as an algebra of continuous functions,

$$\langle T \rangle = C(X)$$

where $X \subset \mathbb{C}$ is a certain compact space associated to the family $\{T_i\}$. Equivalently, any commutative operator algebra $A \subset B(H)$ is of the form A = C(X).

PROOF. We have two assertions here, the idea being as follows:

(1) Regarding the first assertion, this follows exactly as in the proof of Theorem 1.2, by using this time the spectral theorem for families of normal operators.

(2) As for the second assertion, this is clear from the first one, because any commutative algebra $A \subset B(H)$ is generated by its elements $T \in A$, which are all normal.

All this is good to know, but Theorem 1.2 and Theorem 1.3 remain something quite heavy, based on the spectral theorem. We would like to present now an alternative proof for these results, which is rather elementary, and has the advantage of reconstructing the compact space X directly from the knowledge of the algebra A. We will need:

THEOREM 1.4. Given an operator $T \in A \subset B(H)$, define its spectrum as:

$$\sigma(T) = \left\{ \lambda \in \mathbb{C} \middle| T - \lambda \notin A^{-1} \right\}$$

The following spectral theory results hold, exactly as in the A = B(H) case:

(1) We have $\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}.$

(2) We have polynomial, rational and holomorphic calculus.

- (3) As a consequence, the spectra are compact and non-empty.
- (4) The spectra of unitaries $(U^* = U^{-1})$ and self-adjoints $(T = T^*)$ are on \mathbb{T}, \mathbb{R} .
- (5) The spectral radius of normal elements $(TT^* = T^*T)$ is given by $\rho(T) = ||T||$.

In addition, assuming $T \in A \subset B$, the spectra of T with respect to A and to B coincide.

PROOF. This is something that we know well, in the case A = B(H). In general the proof is similar, the idea being as follows:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra A = B(H).

(2) Regarding the last assertion, the inclusion $\sigma_B(T) \subset \sigma_A(T)$ is clear. For the converse, assume $T - \lambda \in B^{-1}$, and consider the following self-adjoint element:

$$S = (T - \lambda)^* (T - \lambda)$$

The difference between the two spectra of $S \in A \subset B$ is then given by:

$$\sigma_A(S) - \sigma_B(S) = \left\{ \mu \in \mathbb{C} - \sigma_B(S) \middle| (S - \mu)^{-1} \in B - A \right\}$$

Thus this difference in an open subset of \mathbb{C} . On the other hand S being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of S are equal, and in particular S is invertible in A, and so $T - \lambda \in A^{-1}$, as desired.

(3) As an observation, the last assertion applied with B = B(H) shows that the spectrum $\sigma(T)$ as constructed in the statement coincides with the spectrum $\sigma(T)$ as constructed and studied before, so the fact that (1-5) hold indeed is no surprise.

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(4) Finally, I can hear you screaming that I should have concieved this book differently, matter of not proving the same things twice. Good point, with my distinguished colleague Bourbaki saying the same, and in answer, wait for the end of this section, where we will prove exactly the same things a third time. We can discuss pedagogy at that time. \Box

We can now get back to the commutative algebras, and we have the following result, due to Gelfand, which provides an alternative to Theorem 1.2 and Theorem 1.3:

THEOREM 1.5. Any commutative operator algebra $A \subset B(H)$ is of the form

$$A = C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi : A \to \mathbb{C}$, with topology making continuous the evaluation maps $ev_T : \chi \to \chi(T)$.

PROOF. Given a commutative operator algebra A, we can define X as in the statement. Then X is compact, and $T \to ev_T$ is a morphism of algebras, as follows:

$$ev: A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the z = Re(z) + iIm(z) formula for the usual complex numbers:

$$T = \frac{T+T^*}{2} + i \cdot \frac{T-T^*}{2i}$$

Thus it is enough to prove the equality $ev_{T^*} = ev_T^*$ for self-adjoint elements T. But this is the same as proving that $T = T^*$ implies that ev_T is a real function, which is in turn true, because $ev_T(\chi) = \chi(T)$ is an element of $\sigma(T)$, contained in \mathbb{R} .

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_T|| = \rho(T) = ||T||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because ev(A) is a closed subalgebra of C(X), which separates the points. \Box

We have been talking so far about the general operator *-algebras $A \subset B(H)$, closed with respect to the norm. But this suggests formulating the following definition:

DEFINITION 1.6. A C^* -algebra is an complex algebra A, given with:

- (1) A norm $a \to ||a||$, making it into a Banach algebra.
- (2) An involution $a \to a^*$, related to the norm by the formula $||aa^*|| = ||a||^2$.

Here by Banach algebra we mean a complex algebra with a norm satisfying all the conditions for a vector space norm, along with $||ab|| \leq ||a|| \cdot ||b||$ and ||1|| = 1, and which is such that our algebra is complete, in the sense that the Cauchy sequences converge. As for the involution, this must be antilinear, antimultiplicative, and satisfying $a^{**} = a$.

As basic examples, we have the operator algebra B(H), for any Hilbert space H, and more generally, the norm closed *-subalgebras $A \subset B(H)$. More on these later. As a second class of basic examples now, which are of great interest for us, we have:

PROPOSITION 1.7. If X is a compact space, the algebra C(X) of continuous functions $f: X \to \mathbb{C}$ is a C^* -algebra, with the usual norm and involution, namely:

$$||f|| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

This algebra is commutative, in the sense that fg = gf, for any $f, g \in C(X)$.

PROOF. All this is clear from definitions. Observe that we have indeed:

$$||ff^*|| = \sup_{x \in X} |f(x)|^2 = ||f||^2$$

Thus, the axioms are satisfied, and finally fg = gf is clear.

In general, the C^* -algebras can be thought of as being algebras of operators, over some Hilbert space which is not present. By using this philosophy, let us formulate:

DEFINITION 1.8. Given element $a \in A$ of a C^{*}-algebra, define its spectrum as:

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1} \right\}$$

Also, we call spectral radius of $a \in A$ the number $\rho(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$.

In what regards the examples, for A = B(H) what we have here is the usual notion of spectrum, from operator theory. More generally, as explained in Theorem 1.4, in the case $A \subset B(H)$ we obtain the same spectra as those in the case A = B(H). Finally, in the case A = C(X), as in Proposition 1.7, the spectrum of a function is its image:

$$\sigma(f) = Im(f)$$

Now with the above notion of spectrum in hand, we have the following result:

THEOREM 1.9. The following results hold, exactly as in the $A \subset B(H)$ case:

(1) We have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.

- (2) We have polynomial, rational and holomorphic calculus.
- (3) As a consequence, the spectra are compact and non-empty.
- (4) The spectra of unitaries $(u^* = u^{-1})$ and self-adjoints $(a = a^*)$ are on \mathbb{T}, \mathbb{R} .

(5) The spectral radius of normal elements ($aa^* = a^*a$) is given by $\rho(a) = ||a||$.

In addition, assuming $a \in A \subset B$, the spectra of a with respect to A and to B coincide.

PROOF. This is something that we know from operator theory, in the case A = B(H), and from this chapter, in the case $A \subset B(H)$. In general, the proof is similar:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra A = B(H). All

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this is standard material, and in fact, things before were written in such a way as for their extension now, to the general C^* -algebra setting, to be obvious.

(2) Regarding the last assertion, we know this from before for $A \subset B \subset B(H)$, and the proof in general is similar. Indeed, the inclusion $\sigma_B(a) \subset \sigma_A(a)$ is clear. For the converse, assume $a - \lambda \in B^{-1}$, and consider the following self-adjoint element:

$$b = (a - \lambda)^* (a - \lambda)$$

The difference between the two spectra of $b \in A \subset B$ is then given by:

$$\sigma_A(b) - \sigma_B(b) = \left\{ \mu \in \mathbb{C} - \sigma_B(b) \middle| (b - \mu)^{-1} \in B - A \right\}$$

Thus this difference in an open subset of \mathbb{C} . On the other hand b being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of b are equal, and in particular b is invertible in A, and so $a - \lambda \in A^{-1}$, as desired.

And with this, good news, we will never ever talk about such things again.

Moving on, we can get back now to the commutative algebras, and we have the following result, due to Gelfand, which will be of crucial importance for us:

THEOREM 1.10. The commutative C^* -algebras are exactly the algebras of the form

$$A = C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi : A \to \mathbb{C}$, with topology making continuous the evaluation maps $ev_a : \chi \to \chi(a)$.

PROOF. This is something that we basically know from before, but always good to talk about it again. Given a commutative C^* -algebra A, we can define X as in the statement. Then X is compact, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev: A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the z = Re(z) + iIm(z) formula for the usual complex numbers:

$$a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in \mathbb{R} .

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because ev(A) is a closed subalgebra of C(X), which separates the points. \Box

In view of the Gelfand theorem, we can formulate the following key definition:

DEFINITION 1.11. Given an arbitrary C^* -algebra A, we write

$$A = C(X)$$

and call X a compact quantum space.

This might look like something informal, but it is not. Indeed, we can define the category of compact quantum spaces to be the category of the C^* -algebras, with the arrows reversed. When A is commutative, the above space X exists indeed, as a Gelfand spectrum, X = Spec(A). In general, X is something rather abstract, and our philosophy here will be that of studying of course A, but formulating our results in terms of X. For instance whenever we have a morphism $\Phi : A \to B$, we will write A = C(X), B = C(Y), and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

Let us also mention that, technically speaking, we will see later that the above formalism has its limitations, and needs a fix. But more on this later.

As a first concrete consequence now of the Gelfand theorem, we have:

THEOREM 1.12. Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.

- (1) We can define $f(a) \in A$, with $f \to f(a)$ being a morphism of C^* -algebras.
- (2) We have the "continuous functional calculus" formula $\sigma(f(a)) = f(\sigma(a))$.

PROOF. Since a is normal, the C^* -algebra $\langle a \rangle$ that is generates is commutative, so if we denote by X the space formed by the characters $\chi :< a \rightarrow \mathbb{C}$, we have:

 $\langle a \rangle = C(X)$

Now since the map $X \to \sigma(a)$ given by evaluation at a is bijective, we obtain:

$$\langle a \rangle = C(\sigma(a))$$

Thus, we are dealing with usual functions, and this gives all the assertions.

As another consequence of the Gelfand theorem, we have:

THEOREM 1.13. For a normal element $a \in A$, the following are equivalent:

- (1) a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.
- (2) $a = b^2$, for some $b \in A$ satisfying $b = b^*$.
- (3) $a = cc^*$, for some $c \in A$.

PROOF. This is very standard, exactly as in A = B(H) case, as follows:

- (1) \implies (2) Since $f(z) = \sqrt{z}$ is well-defined on $\sigma(a) \subset [0, \infty)$, we can set $b = \sqrt{a}$.
- (2) \implies (3) This is trivial, because we can set c = b.

(3) \implies (1) We can proceed here by contradiction. By multiplying c by a suitable element of $\langle cc^* \rangle$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now d = x + iy with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \ge 0$$

Thus $d^*d \ge 0$, contradicting the fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, that we know to hold for A = B(H), and whose proof in general is similar.

1b. Operator algebras

Instead of further building on the above results, which are already quite non-trivial, let us return to our modest status of apprentice operator algebraists, and declare ourselves unsatisfied with the above formalism, on the following intuitive grounds:

THOUGHT 1.14. Our assumption that $A \subset B(H)$ is norm closed is not satisfying, because we would like A to be stable under polar decomposition, under taking spectral projections, and more generally, under measurable functional calculus.

So, let us get now into this, topologies on B(H), and fine-tunings of our operator algebra formalism, based on them. The result that we will need is as follows:

PROPOSITION 1.15. For a subalgebra $A \subset B(H)$, the following are equivalent:

- (1) A is closed under the weak operator topology, making each of the linear maps $T \rightarrow \langle Tx, y \rangle$ continuous.
- (2) A is closed under the strong operator topology, making each of the linear maps $T \rightarrow Tx$ continuous.

In the case where these conditions are satisfied, A is closed under the norm topology.

PROOF. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S \subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A \subset B(H)$, strongly closed implies weakly closed.

(2) Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in B(H), and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator $T \in \overline{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x \in K$:

$$T \in \overline{A} \implies \pi(T) \in \overline{\pi(A)}$$
$$\implies \pi(T)x \in \overline{\pi(A)x}$$
$$\implies \pi(T)x \in \overline{\pi(A)x}^{||.||}$$

Now observe that the last formula tells us that for any $x = (x_1, \ldots, x_n)$, and any $\varepsilon > 0$, we can find $S \in A$ such that the following holds, for any *i*:

$$||Sx_i - Tx_i|| < \varepsilon$$

Thus T belongs to the strong operator closure of A, as desired.

Observe that in the above the terminology is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution, we agree to call the norm topology "strong", and the weak and strong operator topologies "weak", whenever these two topologies coincide. With this convention made, the algebras $A \subset B(H)$ in Proposition 1.15 are those which are weakly closed. Thus, we can now formulate:

DEFINITION 1.16. A von Neumann algebra is an operator algebra

 $A \subset B(H)$

which is closed under the weak topology.

These algebras will be our main objects of study, in what follows. As basic examples, we have the algebra B(H) itself, then the singly generated algebras, $A = \langle T \rangle$ with $T \in B(H)$, and then the multiply generated algebras, $A = \langle T_i \rangle$ with $T_i \in B(H)$. But for the moment, let us keep things simple, and build directly on Definition 1.16, by using basic functional analysis methods. We will need the following key result:

THEOREM 1.17. For an operator algebra $A \subset B(H)$, we have

 $A''=\bar{A}$

with A'' being the bicommutant inside B(H), and \overline{A} being the weak closure.

PROOF. We can prove this by double inclusion, as follows:

" \supset " Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S \subset S''$, valid for any set $S \subset B(H)$. In particular, we have:

$$A \subset A'$$

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Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \to T$ in this topology, we have:

$$T_i \in A'' \implies ST_i = T_i S, \ \forall S \in A'$$
$$\implies ST = TS, \ \forall S \in A'$$
$$\implies T \in A$$

Thus our claim is proved, and together with Proposition 1.15, which allows us to pass from the strong to the weak operator topology, this gives $\bar{A} \subset A''$, as desired.

" \subset " Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \overline{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 1.15. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in B(H), and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \middle| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & & \vdots \\ S_{n1} & \dots & S_{nn} \end{pmatrix} \middle| S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Since this holds for any $x \in K$, we conclude that any operator $T \in A''$ belongs to the strong operator closure of A. By using now Proposition 1.15, which allows us to pass from the strong to the weak operator closure, we conclude that we have:

$$A'' \subset \bar{A}$$

Thus, we have the desired reverse inclusion, and this finishes the proof.

Now by getting back to the von Neumann algebras, from Definition 1.16, we have the following result, which is a reformulation of Theorem 1.17, by using this notion:

THEOREM 1.18. For an operator algebra $A \subset B(H)$, the following are equivalent:

- (1) A is weakly closed, so it is a von Neumann algebra.
- (2) A equals its algebraic bicommutant A'', taken inside B(H).

PROOF. This follows from the formula $A'' = \overline{A}$ from Theorem 1.17, along with the trivial fact that the commutants are automatically weakly closed.

The above statement, called bicommutant theorem, and due to von Neumann [85], is quite interesting, philosophically speaking. Among others, it shows that the von Neumann algebras are exactly the commutants of the self-adjoint sets of operators:

PROPOSITION 1.19. Given a subset $S \subset B(H)$ which is closed under *, the commutant A = S'

is a von Neumann algebra. Any von Neumann algebra appears in this way.

PROOF. We have two assertions here, the idea being as follows:

(1) Given $S \subset B(H)$ satisfying $S = S^*$, the commutant A = S' satisfies $A = A^*$, and is also weakly closed. Thus, A is a von Neumann algebra. Note that this follows as well from the following "tricommutant formula", which follows from Theorem 1.18:

$$S''' = S'$$

(2) Given a von Neumann algebra $A \subset B(H)$, we can take S = A'. Then S is closed under the involution, and we have S' = A, as desired.

1C. BASIC THEOREMS

Observe that Proposition 1.19 can be regarded as yet another alternative definition for the von Neumann algebras, and with this definition being probably the best one when talking about quantum mechanics, where the self-adjoint operators $T: H \to H$ can be though of as being "observables" of the system, and with the commutants A = S' of the sets of such observables $S = \{T_i\}$ being the algebras $A \subset B(H)$ that we are interested in. And with all this actually needing some discussion about self-adjointness, and about boundedness too, but let us not get into this here, and stay mathematical, as before.

As another interesting consequence of Theorem 1.18, we have:

PROPOSITION 1.20. Given a von Neumann algebra $A \subset B(H)$, its center

$$Z(A) = A \cap A^*$$

regarded as an algebra $Z(A) \subset B(H)$, is a von Neumann algebra too.

PROOF. This follows from the fact that the commutants are weakly closed, that we know from the above, which shows that $A' \subset B(H)$ is a von Neumann algebra. Thus, the intersection $Z(A) = A \cap A'$ must be a von Neumann algebra too, as claimed.

1c. Basic theorems

In order to develop some general theory, let us start by investigating the finite dimensional case. Here the ambient algebra is $B(H) = M_N(\mathbb{C})$, any linear subspace $A \subset B(H)$ is automatically closed, for all 3 topologies in Proposition 1.15, and we have:

THEOREM 1.21. The *-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $n_1, \ldots, n_k \in \mathbb{N}$ satisfying $n_1 + \ldots + n_k = N$, we have indeed an obvious embedding of *-algebras, via matrix blocks, as follows:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \rightarrow UMU^*$$

(2) In the other sense now, consider a *-algebra $A \subset M_N(\mathbb{C})$. It is elementary to prove that the center $Z(A) = A \cap A'$, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \ldots, e_k \in \mathbb{C}^k$, and let $p_1, \ldots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \ldots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \ldots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital *-subalgebras of A:

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital *-subalgebra operations coming from the projection equations $p_i^2 = p_i^* = p_i$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital *-algebra sum decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \ldots + p_k = 1$ and the projection equations $p_i^2 = p_i^* = p_i$, we conclude that we have the needed generation property, namely:

$$A_1 + \ldots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \ldots + p_k = 1$, and from the projection equations $p_i^2 = p_i^* = p_i$.

Step 3. Our claim now, which will finish the proof, is that each of the *-subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i = rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{n_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Moreover, a more careful look at the isomorphisms established in Step 3 shows that at the global level, that of the algebra A itself, the above isomorphism simply comes by

twisting the following standard multimatrix embedding, discussed in the beginning of the proof, (1) above, by a certain unitary matrix $U \in U_N$:

 $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$

Now by putting everything together, we obtain the result.

In relation with the bicommutant theorem, we have the following result, which fully clarifies the situation, with a very explicit proof, in finite dimensions:

PROPOSITION 1.22. Consider a *-algebra $A \subset M_N(\mathbb{C})$, written as above:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The commutant of this algebra is then, with respect with the block decomposition used,

 $A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$

and by taking one more time the commutant we obtain A itself, A = A''.

PROOF. Let us decompose indeed our algebra A as in Theorem 1.21:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of \mathbb{C} corresponding to a matrix block:

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain A itself, and we are done.

As another interesting application of Theorem 1.21, clarifying this time the relation with operator theory, in finite dimensions, we have the following result:

THEOREM 1.23. Given an operator $T \in B(H)$ in finite dimensions, $H = \mathbb{C}^N$, the von Neumann algebra $A = \langle T \rangle$ that it generates inside $B(H) = M_N(\mathbb{C})$ is

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with the sizes of the blocks $n_1, \ldots, n_k \in \mathbb{N}$ coming from the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$. In the normal case $TT^* = T^*T$, this decomposition comes from

$$T = UDU$$

with $D \in M_N(\mathbb{C})$ diagonal, and with $U \in U_N$ unitary.

PROOF. This is something which is routine, by using the linear algebra and spectral theory developed in chapter 1, for the matrices $M \in M_N(\mathbb{C})$. To be more precise:

(1) The fact that $A = \langle T \rangle$ decomposes into a direct sum of matrix algebras is something that we already know, coming from Theorem 1.21.

(2) By using standard linear algebra, we can compute the block sizes $n_1, \ldots, n_k \in \mathbb{N}$, from the knowledge of the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$.

(3) In the normal case, $TT^* = T^*T$, we can simply invoke the spectral theorem, and by suitably changing the basis, we are led to the conclusion in the statement.

Let us get now to infinite dimensions, with Theorem 1.23 as our main source of inspiration. The same argument as there applies, provided that we are in the normal case, and we have the following result, summarizing our basic knowledge here:

THEOREM 1.24. Given a bounded operator $T \in B(H)$ which is normal, $TT^* = T^*T$, the von Neumann algebra $A = \langle T \rangle$ that it generates inside B(H) is

$$< T > = L^{\infty}(\sigma(T))$$

with $\sigma(T) \subset \mathbb{C}$ being as usual its spectrum.

PROOF. The measurable functional calculus theorem for the normal operators tells us that we have a weakly continuous morphism of *-algebras, as follows:

$$L^{\infty}(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the measurable calculus, this morphism is injective, and its image is the weakly closed algebra $\langle T \rangle$ generated by the operators T, T^* . Thus, we obtain the isomorphism in the statement.

More generally now, along the same lines, we have the following result:

THEOREM 1.25. Given operators $T_i \in B(H)$ which are normal, and which commute, the von Neumann algebra $A = \langle T_i \rangle$ that these operators generates inside B(H) is

$$\langle T_i \rangle = L^{\infty}(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

PROOF. This is once again routine, by using this time the spectral theory for the families of commuting normal operators $T_i \in B(H)$.

As a fundamental consequence now of the above results, we have:

THEOREM 1.26. The commutative von Neumann algebras are the algebras

$$A = L^{\infty}(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X, we can realize the $A = L^{\infty}(X)$ as a von Neumann algebra, on a certain Hilbert space H. But this is something that we know well, the representation being as follows:

$$L^{\infty}(X) \subset B(L^2(X))$$
 , $f \to (g \to fg)$

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(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X, and an identification $A = L^{\infty}(X)$. But this follows from Theorem 1.25, because we can write our algebra as follows:

$$A = \langle T_i \rangle$$

To be more precise, A being commutative, any element $T \in A$ is normal, so we can pick a basis $\{T_i\} \subset A$, and then we have $A = \langle T_i \rangle$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 1.25 applies, and gives the result.

(3) Alternatively, and more explicitly, we can deduce this from Theorem 1.24, applied with $T = T^*$. Indeed, by using T = Re(T) + iIm(T), we conclude that any von Neumann algebra $A \subset B(H)$ is generated by its self-adjoint elements $T \in A$. Moreover, by using measurable functional calculus, we conclude that A is linearly generated by its projections. But then, assuming $A = \overline{span}\{p_i\}$, with p_i being projections, we can set:

$$T = \sum_{i=0}^{\infty} \frac{p_i}{3^i}$$

Then $T = T^*$, and by functional calculus we have $p_0 \in T >$, then $p_1 \in T >$, and so on. Thus A = T >, and $A = L^{\infty}(X)$ comes now via Theorem 1.24, as claimed. \Box

The above result is the foundation for all the advanced von Neumann algebra theory, that we will discuss in the remainder of this book, and there are many things that can be said about it. To start with, in relation with the general theory of the normed closed algebras, that we developed in the beginning of this chapter, we have:

WARNING 1.27. Although the von Neumann algebras are norm closed, the theory of norm closed algebras does not always apply well to them. For instance for $A = L^{\infty}(X)$ Gelfand gives $A = C(\widehat{X})$, with \widehat{X} being a certain technical compactification of X.

In short, this would be my advice, do not mess up the two theories that we will be developing in this book, try finding different rooms for them, in your brain. At least at this stage of things, because later, do not worry, we will be playing with both.

Now forgetting about Gelfand, and taking Theorem 1.26 as such, tentative foundation for the theory that we want to develop, as a first consequence of this, we have:

THEOREM 1.28. Given a von Neumann algebra $A \subset B(H)$, we have

$$Z(A) = L^{\infty}(X)$$

with X being a certain measured space.

PROOF. We know from Proposition 1.20 that the center $Z(A) \subset B(H)$ is a von Neumann algebra. Thus Theorem 1.13 applies, and gives the result.

It is possible to further build on this, with a powerful decomposition result as follows, over the measured space X constructed in Theorem 1.28:

$$A = \int_X A_x \, dx$$

But more on this later, after developing the appropriate tools for this program, which is something non-trivial. Among others, before getting into such things, we will have to study the von Neumann algebras A having trivial center, $Z(A) = \mathbb{C}$, called factors, which include the fibers A_x in the above decomposition result. More on this later.

1d. Basic examples

Our main results so far on the von Neumann algebras concern the finite dimensional case, where the algebra is of the form $A = \bigoplus_i M_{n_i}(\mathbb{C})$, and the commutative case, where the algebra is of the form $A = L^{\infty}(X)$. In order to advance, we must solve:

QUESTION 1.29. What are the next simplest von Neumann algebras, generalizing at the same time the finite dimensional ones, $A = \bigoplus_i M_{n_i}(\mathbb{C})$, and the commutative ones, $A = L^{\infty}(X)$, that we can use as input for our study?

In this formulation, our question is a no-brainer, the answer to it being that of looking at the direct integrals of matrix algebras, over an arbitrary measured space X:

$$A = \int_X M_{n_x}(\mathbb{C}) dx$$

However, when thinking a bit, all this looks quite tricky, with most likely lots of technical functional analysis and measure theory involved. So, we will leave the investigation of such algebras, which are indeed quite basic, and called of type I, for later.

Nevermind. Let us replace Question 1.29 with something more modest, as follows:

QUESTION 1.30 (update). What are the next simplest von Neumann algebras, generalizing at the same time the usual matrix algebras, $A = M_N(\mathbb{C})$, and the commutative ones, $A = L^{\infty}(X)$, that we can use as input for our study?

But here, what we have is again a no-brainer, because in relation to what has been said above, we just have to restrict the attention to the "isotypic" case, where all fibers are isomorphic. And in this case our algebra is a random matrix algebra:

$$A = \int_X M_N(\mathbb{C}) dx$$

Which looks quite nice, and so good news, we have our algebras. In practice now, although there is some functional analysis to be done with these algebras, the main questions regard the individual operators $T \in A$, called random matrices. Thus, we are basically back to good old operator theory. Let us begin our discussion with:

DEFINITION 1.31. A random matrix algebra is a von Neumann algebra of the following type, with X being a probability space, and with $N \in \mathbb{N}$ being an integer:

$$A = M_N(L^{\infty}(X))$$

In other words, A appears as a tensor product, as follows,

$$A = M_N(\mathbb{C}) \otimes L^{\infty}(X)$$

of a matrix algebra and a commutative von Neumann algebra.

As a first observation, our algebra can be written as well as follows, with this latter convention being quite standard in the probability literature:

$$A = L^{\infty}(X, M_N(\mathbb{C}))$$

In connection with the tensor product notation, which is often the most useful one for computations, we have as well the following possible writing, also used in probability:

$$A = L^{\infty}(X) \otimes M_N(\mathbb{C})$$

Importantly now, each random matrix algebra A is naturally endowed with a canonical von Neumann algebra trace $tr: A \to \mathbb{C}$, which appears as follows:

PROPOSITION 1.32. Given a random matrix algebra $A = M_N(L^{\infty}(X))$, consider the linear form $tr : A \to \mathbb{C}$ given by:

$$tr(T) = \frac{1}{N} \sum_{i=1}^{N} \int_{X} T_{ii}^{x} dx$$

In tensor product notation, $A = M_N(\mathbb{C}) \otimes L^{\infty}(X)$, we have then the formula

$$tr = \frac{1}{N}Tr \otimes \int_X$$

and this functional $tr: A \to \mathbb{C}$ is a faithful positive unital trace.

PROOF. The first assertion, regarding the tensor product writing of tr, is clear from definitions. As for the second assertion, regarding the various properties of tr, this follows from this, because these properties are stable under taking tensor products.

As before, there is a discussion here in connection with the other possible writings of A. With the probabilistic notation $A = L^{\infty}(X, M_N(\mathbb{C}))$, the trace appears as:

$$tr(T) = \int_X \frac{1}{N} Tr(T^x) \, dx$$

Also, with the probabilistic tensor notation $A = L^{\infty}(X) \otimes M_N(\mathbb{C})$, the trace appears exactly as in the second part of Proposition 1.32, with the order inverted:

$$tr = \int_X \otimes \frac{1}{N} Tr$$

To summarize, the random matrix algebras appear to be very basic objects, and the only difficulty, in the beginning, lies in getting familiar with the 4 possible notations for them. Or perhaps 5 possible notations, because we have $A = \int_X M_N(\mathbb{C}) dx$ as well.

Getting to work now, as already said, the main questions about random matrix algebras regard the individual operators $T \in A$, called random matrices. To be more precise, we are interested in computing the laws of such matrices, constructed according to:

THEOREM 1.33. Given an operator algebra $A \subset B(H)$ with a faithful trace $tr : A \to \mathbb{C}$, any normal element $T \in A$ has a law, namely a probability measure μ satisfying

$$tr(T^k) = \int_{\mathbb{C}} z^k d\mu(z)$$

with the powers being with respect to colored exponents $k = \circ \bullet \circ \ldots$, defined via

$$a^{\emptyset} = 1$$
 , $a^{\circ} = a$, $a^{\bullet} = a^{*}$

and multiplicativity. This law is unique, and is supported by the spectrum $\sigma(T) \subset \mathbb{C}$. In the non-normal case, $TT^* \neq T^*T$, such a law does not exist.

PROOF. We have two assertions here, the idea being as follows:

(1) In the normal case, $TT^* = T^*T$, we know from before, based on the continuous functional calculus theorem, that we have:

$$< T > = C(\sigma(T))$$

Thus the functional $f(T) \to tr(f(T))$ can be regarded as an integration functional on the algebra $C(\sigma(T))$, and by the Riesz theorem, this latter functional must come from a probability measure μ on the spectrum $\sigma(T)$, in the sense that we must have:

$$tr(f(T)) = \int_{\sigma(T)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the operators T^k , taken as usual with respect to colored integer exponents, $k = \circ \bullet \circ \circ \ldots$, generate the whole operator algebra $C(\sigma(T))$.

(2) In the non-normal case now, $TT^* \neq T^*T$, we must show that such a law does not exist. For this purpose, we can use a positivity trick, as follows:

$$TT^* - T^*T \neq 0 \implies (TT^* - T^*T)^2 > 0$$

$$\implies TT^*TT^* - TT^*T^*T - T^*TTT^* + T^*TT^*T > 0$$

$$\implies tr(TT^*TT^* - TT^*T^*T - T^*TTT^* + T^*TT^*T) > 0$$

$$\implies tr(TT^*TT^* + T^*TT^*T) > tr(TT^*T^*T + T^*TTT^*)$$

$$\implies tr(TT^*TT^*) > tr(TTT^*T^*)$$

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Now assuming that T has a law $\mu \in \mathcal{P}(\mathbb{C})$, in the sense that the moment formula in the statement holds, the above two different numbers would have to both appear by integrating $|z|^2$ with respect to this law μ , which is contradictory, as desired.

Back now to the random matrices, as a basic example, assume $X = \{.\}$, so that we are dealing with a usual scalar matrix, $T \in M_N(\mathbb{C})$. By changing the basis of \mathbb{C}^N , which won't affect our trace computations, we can assume that T is diagonal:

$$T \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

But for such a diagonal matrix, we have the following formula:

$$tr(T^k) = \frac{1}{N}(\lambda_1^k + \ldots + \lambda_N^k)$$

Thus, the law of T is the average of the Dirac masses at the eigenvalues:

$$\mu = \frac{1}{N} \left(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N} \right)$$

As a second example now, assume N = 1, and so $T \in L^{\infty}(X)$. In this case we obtain the usual law of T, because the equation to be satisfied by μ is:

$$\int_X \varphi(T) = \int_{\mathbb{C}} \varphi(x) d\mu(x)$$

At a more advanced level, the main problem regarding the random matrices is that of computing the law of various classes of such matrices, coming in series:

QUESTION 1.34. What is the law of random matrices coming in series

$$T_N \in M_N(L^\infty(X))$$

in the N >> 0 regime?

The general strategy here, coming from physicists, is that of computing first the asymptotic law μ^0 , in the $N \to \infty$ limit, and then looking for the higher order terms as well, as to finally reach to a series in N^{-1} giving the law of T_N , as follows:

$$\mu_N = \mu^0 + N^{-1}\mu^1 + N^{-2}\mu^2 + \dots$$

As a basic example here, of particular interest are the random matrices having i.i.d. complex normal entries, under the constraint $T = T^*$. Here the asymptotic law μ^0 is the Wigner semicircle law on [-2, 2]. We will discuss this later in this book.

Let us end this preliminary chapter on operator algebras with some philosophy, a bit a la Heisenberg. In relation with general "quantum space" goals, Theorem 1.26 is something very interesting, philosophically speaking, suggesting us to formulate:

DEFINITION 1.35. Given a von Neumann algebra $A \subset B(H)$, we write

 $A = L^{\infty}(X)$

and call X a quantum measured space.

As an example here, for the simplest noncommutative von Neumann algebra that we know, namely the usual matrix algebra $A = M_N(\mathbb{C})$, the formula that we want to write is as follows, with M_N being a certain mysterious quantum space:

$$M_N(\mathbb{C}) = L^\infty(M_N)$$

So, what can we say about this space M_N ? As a first observation, this is a finite space, with its cardinality being defined and computed as follows:

$$|M_N| = \dim_{\mathbb{C}} M_N(\mathbb{C}) = N^2$$

Now since this is the same as the cardinality of the set $\{1, \ldots, N^2\}$, we are led to the conclusion that we should have a twisting result as follows, with the twisting operation $X \to X^{\sigma}$ being something that destroys the points, but keeps the cardinality:

$$M_N = \{1, \dots, N^2\}^{\sigma}$$

From an analytic viewpoint now, we would like to understand what is the integration over M_N , giving rise to the corresponding L^{∞} functions. And here, we can set:

$$\int_{M_N} A = tr(A)$$

To be more precise, on the left we have the integral of an arbitrary function on M_N , which according to our conventions, should be a usual matrix:

$$A \in L^{\infty}(M_N) = M_N(\mathbb{C})$$

As for the quantity on the right, the outcome of the computation, this can only be the trace of A. In addition, it is better to choose this trace to be normalized, by tr(1) = 1, and this in order for our measure on M_N to have mass 1, as it is ideal:

$$tr(A) = \frac{1}{N}Tr(A)$$

We can say even more about this. Indeed, since the traces of positive matrices are positive, we are led to the following formula, to be taken with the above conventions, which shows that the measure on M_N that we constructed is a probability measure:

$$A > 0 \implies \int_{M_N} A > 0$$

Before going further, let us record what we found, for future reference:

THEOREM 1.36. The quantum measured space M_N formally given by

$$M_N(\mathbb{C}) = L^{\infty}(M_N)$$

has cardinality N^2 , appears as a twist, in a purely algebraic sense,

$$M_N = \{1, \ldots, N^2\}^c$$

and is a probability space, its uniform integration being given by

$$\int_{M_N} A = tr(A)$$

where at right we have the normalized trace of matrices, tr = Tr/N.

PROOF. This is something half-informal, mostly for fun, which basically follows from the above discussion, the details and missing details being as follows:

(1) In what regards the formula $|M_N| = N^2$, coming by computing the complex vector space dimension, as explained above, this is obviously something rock-solid.

(2) Regarding twisting, we would like to have a formula as follows, with the operation $A \to A^{\sigma}$ being something that destroys the commutativity of the multiplication:

$$L^{\infty}(M_N) = L^{\infty}(1, \dots, N^2)^{\sigma}$$

In more familiar terms, with usual complex matrices on the left, and with a betterlooking product of sets being used on the right, this formula reads:

$$M_N(\mathbb{C}) = L^{\infty} \Big(\{1, \dots, N\} \times \{1, \dots, N\} \Big)^c$$

In order to establish this formula, consider the algebra on the right. As a complex vector space, this algebra has the standard basis $\{f_{ij}\}$ formed by the Dirac masses at the points (i, j), and the multiplicative structure of this algebra is given by:

$$f_{ij}f_{kl} = \delta_{ij,kl}$$

Now let us twist this multiplication, according to the formula $e_{ij}e_{kl} = \delta_{jk}e_{il}$. We obtain in this way the usual combination formulae for the standard matrix units $e_{ij} : e_j \to e_i$ of the algebra $M_N(\mathbb{C})$, and so we have our twisting result, as claimed.

(3) In what regards the integration formula in the statement, with the conclusion that the underlying measure on M_N is a probability one, this is something that we fully explained before, and as for the result (1) above, it is something rock-solid.

(4) As a last technical comment, observe that the twisting operation performed in (2) destroys both the involution, and the trace of the algebra. This is something quite interesting, which cannot be fixed, and we will back to it, later on. \Box

1e. Exercises

Exercises:

EXERCISE 1.37.

EXERCISE 1.38.

EXERCISE 1.39.

EXERCISE 1.40.

Exercise 1.41.

EXERCISE 1.42.

EXERCISE 1.43.

Exercise 1.44.

Bonus exercise.

CHAPTER 2

Finite factors

2a. Finite factors

Welcome to subfactors. In this chapter we discuss basic the study of the II_1 factors, following Murray and von Neumann [58], [59], [60], [85], [86], which is the basis for everything advanced and modern, in relation with the operator algebras.

We will only present here the very basic theory of the II_1 factors, and we will come back to them, on a regular basis, later. In fact, as we will soon discover, these II_1 factors are the "building blocks" of the whole von Neumann algebra theory.

Let us first talk about general factors. There are several possible ways of introducing them, and dividing them into several classes, for further study. In what concerns us, we will use a rather intuitive approach. The general idea, which is quite natural, is that among the von Neumann algebras $A \subset B(H)$, of particular interest are the "free" ones, having trivial center, $Z(A) = \mathbb{C}$. These algebras are called factors:

DEFINITION 2.1. A factor is a von Neumann algebra $A \subset B(H)$ whose center

 $Z(A) = A \cap A'$

which is a commutative von Neumann algebra, reduces to the scalars, $Z(A) = \mathbb{C}$.

This notion is something quite subtle, that you probably already met, in your learning of operator algebras, but time now to clarify all this. The idea is that there are two main motivations for the study of factors, with each of them being more than enough, as to serve as a strong motivation. First, at the intuitive level, we have:

PRINCIPLE 2.2 (Freeness). The following happen:

- (1) The condition $Z(A) = \mathbb{C}$ defining the factors is, obviously, opposite to the condition Z(A) = A defining the commutative von Neumann algebras.
- (2) Therefore, the factors are the von Neumann algebras which are "free", meaning as far as possible from the commutative ones.
- (3) Equivalently, with $A = L^{\infty}(X)$, the quantum spaces X coming from factors are those which are "free", meaning as far as possible from the classical spaces.

So, this was for our first principle, which is something reasonable, intuitive, and selfexplanatory, and which can surely serve as a strong motivation for the study of factors.

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In fact, all that has being said above comes straight from the structure theorem for the commutative von Neumann algebras, $A = L^{\infty}(X)$, with X being a measured space, that you probably know well, and the above principle is just a corollary of that theorem.

At a more advanced level, another motivation for the study of factors, which among others justifies the name "factors" for them, comes from the reduction theory of von Neumann [87], which is something non-trivial, that can be summarized as follows:

PRINCIPLE 2.3 (Reduction theory). Given a von Neumann algebra $A \subset B(H)$, if we write its center $Z(A) \subset A$, which is a commutative von Neumann algebra, as

$$Z(A) = L^{\infty}(X)$$

with X being a measured space, then the whole algebra decomposes as

$$A = \int_X A_x \, dx$$

with the fibers A_x being factors, that is, satisfying $Z(A_x) = \mathbb{C}$.

As a first comment, you have already seen an instance of such decomposition results when learning about finite dimensional algebras. Indeed, such algebras decompose, in agreement with the above, as direct sums of matrix algebras, as follows:

$$A = \bigoplus_{x} M_{n_x}(\mathbb{C})$$

In general, however, things are more complicated than this, and technically speaking, and as opposed to Principle 2.2, which was more of a triviality, Principle 2.3 is a tough theorem, due to von Neumann [87]. More on this later, on several occasions.

This was for the story, and let us close this philosophical discussion with:

CONCLUSION 2.4. Regardless of the approach and technical level, be that beginner or advanced, the von Neumann factors are the algebras that matter.

Getting to work now, there are many things that can be said about factors.

In order to get started, let us first study their projections. We will see that many interesting things happen, with everything coming from the following technical result:

PROPOSITION 2.5. Given two projections $p, q \neq 0$ in a factor A, we have

 $puq \neq 0$

for a certain unitary $u \in A$.

PROOF. Assume by contradiction puq = 0, for any unitary $u \in A$. This gives:

$$u^* puq = 0$$

By using this for all the unitaries $u \in A$, we obtain the following formula:

$$\left(\bigvee_{u\in U_A} u^* p u\right)q = 0$$

On the other hand, from $p \neq 0$ we obtain, by factoriality of A:

$$\bigvee_{u \in U_A} u^* p u = 1$$

Thus, our previous formula is in contradiction with $q \neq 0$, as desired.

Getteing back now to the order on projections, and to the whole von Neumann projection philosophy, in the case of factors things simplify, as follows:

THEOREM 2.6. Given two projections $p, q \in A$ in a factor, we have

$$p \preceq q$$
 or $q \preceq p$

and so \leq is a total order on the equivalence classes of projections $p \in A$.

PROOF. This basically follows from Proposition 2.5, and from the Zorn lemma, by using some standard functional analysis arguments. To be more precise:

(1) Consider indeed the following set of partial isometries:

$$S = \left\{ u \middle| uu^* \le p, u^*u \le q \right\}$$

We can then order this set S by saying that we have $u \leq v$ when $u^*u \leq v^*v$, and when u = v holds on the initial domain u^*uH of u. With this convention made, the Zorn lemma applies, and provides us with a maximal element $u \in S$.

(2) In the case where this maximal element $u \in S$ satisfies $uu^* = p$ or $u^*u = q$, we are led to one of the conditions $p \leq q$ or $q \leq p$ in the statement, and we are done.

(3) So, assume that we are in the case left, $uu^* \neq p$ and $u^*u \neq q$. By Proposition 2.5 we obtain a unitary $v \neq 0$ satisfying the following conditions:

$$vv^* \le p - uu^*$$
$$v^*v \le q - u^*u$$

But these conditions show that the element $u + v \in S$ is strictly bigger than $u \in S$, which is a contradiction, and we are done.

2. FINITE FACTORS

Moving ahead now, as explained in the beginning of this book, for a variety of reasons, which can be elementary or advanced, and also mathematical or physical, we are mainly interested in the case where our algebras have traces:

 $tr: A \to \mathbb{C}$

And in relation with the factors, by leaving aside the rather trivial case of the matrix algebras $A = M_N(\mathbb{C})$, we are led in this way to the following key notion:

DEFINITION 2.7. A II₁ factor is a von Neumann algebra $A \subset B(H)$ which:

- (1) Is infinite dimensional, dim $A = \infty$.
- (2) Has trivial center, $Z(A) = \mathbb{C}$.
- (3) Has a trace $tr: A \to \mathbb{C}$.

Here the order of the axioms is a bit random, with any of the possible 3! = 6 choices making sense, and corresponding to a slightly different vision on what the II₁ factors truly are. With the above order, with (1) we are making it clear, right from the beginning, that we are not here for revolutionizing linear algebra. Then with (2) we adhere to Definition 2.1, and to what was said next about it, on freeness and reduction. And finally with (3) we adhere to the above principle, that von Neumann algebras must have traces.

More technically now, and leaving aside anything subjective, the above definition is motivated by the heavy classification work of Murray, von Neumann and Connes [15], [16], [58], [59], [60], [85], [86], [87], whose conclusion is more or less that everything in von Neumann algebras reduces, via some quite complicated procedures, we should mention that, to the study of the II_1 factors. With the mantra here being as follows:

FACT 2.8. The II₁ factors are the building blocks of the whole von Neumann algebra theory.

To be more precise, this statement, that we will get to understand later, is something widely agreed upon, at least among operator algebra experts who are familiar with von Neumann algebras, and with this agreement being something great. What remains controversial, however, is how to start playing with these Lego bricks that we have:

(1) A first option is that of adding the matrix algebras $M_N(\mathbb{C})$, not to be forgotten, and then stacking together such Lego bricks. According to the von Neumann reduction theory, this leads to the von Neumann algebras having traces, $tr: A \to \mathbb{C}$.

(2) A second option, perhaps even more playful, is that of taking crossed products of such Lego bricks by their automorphisms scaling the trace, or performing more general constructions inspired by advanced ergodic theory. This leads to general factors.
(3) And the third option is that of being a bad kid, or perhaps some kind of nerd, engineer in the becoming, and picking such a Lego brick, or a handful of them, and breaking them, see what's inside. Good option too, and more on this later.

Getting to work now, in practice, and forgetting about reduction theory, which raises the possibility of decomposing any tracial von Neumann algebra into factors, in order to obtain explicit examples of II_1 factors, it is not even clear that such beasts exist. Fortunately the group von Neumann algebras are there, and we have the following result, which provides us with some examples of II_1 factors, to start with:

THEOREM 2.9. The center of a group von Neumann algebra $L(\Gamma)$ is

$$Z(L(\Gamma)) = \left\{ \sum_{g} \lambda_{g} g \Big| \lambda_{gh} = \lambda_{hg} \right\}'$$

and if $\Gamma \neq \{1\}$ has infinite conjugacy classes, in the sense that

$$\left| \{ghg^{-1} | g \in G\} \right| = \infty \quad , \quad \forall h \neq 1$$

with this being called ICC property, the algebra $L(\Gamma)$ is a II₁ factor.

PROOF. There are two assertions here, the idea being as follows:

(1) Consider a linear combination of group elements, which is in the weak closure of $\mathbb{C}[\Gamma]$, and so defines an element of the group von Neumann algebra $L(\Gamma)$:

$$a = \sum_g \lambda_g g$$

By linearity, this element $a \in L(\Gamma)$ belongs to the center of $L(\Gamma)$ precisely when it commutes with all the group elements $h \in \Gamma$, and this gives:

$$a \in Z(A) \iff ah = ha$$
$$\iff \sum_{g} \lambda_{g}gh = \sum_{g} \lambda_{g}hg$$
$$\iff \sum_{k} \lambda_{kh^{-1}}k = \sum_{k} \lambda_{h^{-1}k}k$$
$$\iff \lambda_{kh^{-1}} = \lambda_{h^{-1}k}$$

Thus, we obtain the formula for $Z(L(\Gamma))$ in the statement.

(2) We have to examine the 3 conditions defining the II₁ factors. We already know from basic algebra that the group algebra L(G) has a trace, given by:

$$tr(g) = \delta_{g,1}$$

Regarding now the center, the condition $\lambda_{gh} = \lambda_{hg}$ that we found is equivalent to the fact that $g \to \lambda_q$ is constant on the conjugacy classes, and we obtain:

$$Z(L(\Gamma)) = \mathbb{C} \iff \Gamma = \mathrm{ICC}$$

Finally, assuming that this ICC condition is satisfied, with $\Gamma \neq \{1\}$, then our group Γ is infinite, and so the algebra $L(\Gamma)$ is infinite dimensional, as desired.

In order to look now for more examples of II₁ factors, an idea would be that of attempting to decompose into factors the group von Neumann algebras $L(\Gamma)$, but this is something difficult, and in fact we won't really exit the group world in this way. Difficult as well is to investigate the factoriality of the von Neumann algebras of discrete quantum groups $L(\Gamma)$, because the basic computations from the proof of Theorem 2.9 won't extend to this setting, where the group elements $g \in \Gamma$ become corepresentations $g \in M_N(L(\Gamma))$. Despite years of efforts, it is presently not known at all what the "quantum ICC" condition should mean, and the problem comes from this. But more on this later.

In short, we have to stop here the construction of examples, and Theorem 2.9 will be what we have, at least for the moment. With this being actually not a big issue, the group factors $L(\Gamma)$ being known to be quite close to the generic II₁ factors.

2b. Basic results

Getting away now from the above difficulties, let us go back to the abstract II_1 factors, as axiomatized in Definition 2.7. In order to investigate them, the idea will be that of looking at the projections, and their equivalence classes.

In the case of the II_1 factors, as a first interesting remark, the presence of the trace trivializes the proof of the main result that we have about projections, as follows:

THEOREM 2.10. Given two projections $p, q \in A$ in a II₁ factor we have, trivially

$$p \preceq q$$
 or $q \preceq p$

and so \leq is a total order on the equivalence classes of projections $p \in A$.

PROOF. This is something that we already know, from Theorem 2.6, and which actually holds for any factor, with the non-trivial part being the following implication:

$$p \preceq q, \ q \preceq p \implies p \simeq q$$

But this implication is clear in the present II_1 factor setting, by using the trace.

The above theorem and its proof, which are remarkable, are the first in a series of mysteries, in what concerns the special case of the II_1 factors. More such mysteries to follow. In order to study now the trace of the II_1 factors, we will need:

PROPOSITION 2.11. Given a weakly closed left ideal $I \subset A$ in a von Neumann algebra, there exists a unique projection $p \in A$ such that:

$$I = Ap$$

Moreover, if $I \subset A$ is assumed to be a two-sided ideal, then $p \in Z(A)$.

PROOF. We have several things to be proved, the idea being as follows:

(1) Given an ideal $I \subset A$ as in the statement, consider the following intersection:

$$I \cap I^* \subset A$$

This is a weakly closed non-unital *-subalgebra of A, so if we denote by $p \in A$ its largest projection, or unit, then we have an inclusion $Ap \subset I$.

(2) Conversely now, let us pick $x \in I$. By polar decomposition we can write x = u|x|, and we have the following implications, which prove the reverse inclusion $I \subset Ap$:

$$\begin{aligned} x \in I &\implies |x| = u^* x \in I \\ &\implies |x| \in I \cap I^* \\ &\implies |x|p = |x| \\ &\implies x = u|x| = u|x|p \in Ap \end{aligned}$$

(3) The uniqueness assertion is clear from the comparison theorem for projections.

(4) Regarding now the last assertion, assume that $I \subset A$ is a two-sided weakly closed ideal. Then for any unitary $u \in A$ we have:

$$I = uIu^* \implies uIu^* = Ap$$
$$\implies I = Aupu^*$$

Thus by uniqueness we obtain $upu^* = p$, and so $p \in Z(A)$, as desired.

As a first main result now regarding the II_1 factors, following the paper of Murray and von Neumann [60], which by the way is a must-read, we have:

THEOREM 2.12. Given a II_1 factor A, any weakly continuous positive trace

$$tr: A \to \mathbb{C}$$

is automatically faithful.

PROOF. Consider the null space of the trace, which is by definition:

$$I = \left\{ x \in A \middle| tr(x^*x) = 0 \right\}$$

We have the following inequality, which shows that I is a left ideal:

$$x^*a^*ax \le ||a||^2x^*x$$

Now by using the trace condition tr(ab) = tr(ba), we conclude that I is a two-sided ideal. Also, the Cauchy-Schwarz inequality gives:

$$tr(x^*x) = 0 \iff tr(xy) = 0, \forall y \in A$$

We conclude from this that I is an intersection of kernels of weakly closed functionals, which are weakly closed, and so it is weakly closed. Thus the last assertion in Proposition 2.11 applies, and produces a projection $p \in Z(A)$ such that:

$$I = Ap$$

Now since A was assumed to be a factor, we have $Z(A) = \mathbb{C}$. Thus p = 0, and so the null ideal of the trace is $I = \{0\}$, and so our trace tr is faithful, as desired.

Our goal now will be that of proving that the trace on a II_1 factor is unique, and takes on projections any value in [0, 1]. Let us start with a technical result, as follows:

PROPOSITION 2.13. Given a II_1 factor A, the traces of the projections

$$tr(p) \in [0,1]$$

can take arbitrarily small values.

PROOF. Consider the set formed by all values of the trace on the projections:

$$S = \left\{ tr(p) \middle| p^2 = p = p^* \in A \right\}$$

We want to prove that the following number equals 0:

$$c = \inf(S - \{0\})$$

In order to do so, assume by contradiction c > 0, pick $\varepsilon > 0$ small, and pick a projection $p \in A$ such that the following condition is satisfied:

$$tr(p) < c + \varepsilon$$

Since we are in a II₁ factor, this projection $p \in A$ cannot be minimal, and so we can find another projection $q \in A$ satisfying q < p. Now observe that we have:

$$tr(p-q) = tr(p) - tr(q)$$

$$\leq tr(p) - c$$

$$\leq \varepsilon$$

Thus with $\varepsilon < c$ we obtain a contradiction, and so c = 0, as desired.

In order to prove our next main result, we will need as well:

PROPOSITION 2.14. Given a II₁ factor A on a Hilbert space H and a projection $p \in A$, the von Neumann algebra pAp is a II₁ factor on the Hilbert space pH.

2B. BASIC RESULTS

PROOF. We have to prove that the von Neumann algebra pAp has a trace, and is infinite dimensional, and these two properties can be proved as follows:

(1) In what regards the trace, we know that the trace $tr : A \to \mathbb{C}$ restricts to a trace $tr : pAp \to \mathbb{C}$, which must be nonzero, as desired.

(2) In what regards the infinite dimensionality, this follows from the fact that a minimal projection in pAp would be minimal in A, which is impossible.

Still following the fundamental paper of Murray and von Neumann [60], we can now formulate a second main result regarding the II_1 factors, as follows:

THEOREM 2.15. Given a II_1 factor A, the traces of projections

$$tr(p) \in [0,1]$$

can take any values in [0, 1].

PROOF. Given a number $c \in [0, 1]$, consider the following set:

$$S = \left\{ p^2 = p = p^* \in A \middle| tr(p) \le c \right\}$$

This set satisfies the assumptions of the Zorn lemma, and so by this lemma we can find a maximal element $p \in S$. Assume by contradiction that we have:

The point now is that by using Proposition 2.13 and Proposition 2.14, we can slightly enlarge the trace of p, and we obtain a contradiction, as desired.

As a third and last main result regarding the II_1 factors, also from [60], we have:

THEOREM 2.16. The trace of a II_1 factor

$$tr: A \to \mathbb{C}$$

is unique.

PROOF. This can be proved in many ways, a standard one being that of proving that any two traces agree on the projections, as a consequence of the above results:

(1) Assume indeed that we have a second trace $tr' : A \to \mathbb{C}$. Since A is generated by its projections, it is enough to show that we have tr = tr' on projections.

(2) As a first observation, since traces on matrix algebras are unique, we obtain that we have tr = tr' on the projections $p \in A$ having rational trace, $tr(p) \in \mathbb{Q}$.

(3) So, let us pick $p \in A$ having non-rational trace, $tr(p) \notin \mathbb{Q}$, and prove that we have tr(p) = tr'(p). The idea will be that of using the result for the projections having rational traces, applied to an infinite direct sum of projections, converging to p.

(4) To be more precise, assume that we have constructed our sequence $p_i \to p$ up to order $n \in \mathbb{N}$, and let us try to construct p_{n+1} . The idea is to use the following algebra:

$$A_n = (p - p_n)A(p - p_n)$$

(5) Indeed this algebra is a II₁ factor, and we can choose inside it a projection p_{n+1} satisfying $p_n \leq p_{n+1} \leq p$, such that tr = tr' on it, and such that:

$$tr(p - p_{n+1}) \le \frac{1}{2} \cdot tr(p - p_n)$$

(6) According to our choices for these projections p_n , we have:

$$p = \bigvee_{n=1}^{\infty} p_n$$

Thus when evaluating tr, tr' on p we obtain the same result, as desired.

In what regards illustrations for all this, as examples of II₁ factors we have so far the group von Neumann algebras $L(\Gamma)$, with Γ being an ICC group. In certain cases, it is possible to say more about all the above, and in particular about the projections, for instance with quite explicit procedures for constructing projections $p \in L(\Gamma)$ having an arbitrary prescribed trace $x \in [0, 1]$. We will be back to this later, when discussing more in detail the group von Neumann algebras $L(\Gamma)$, and their generalizations.

Back to theory, we have seen that the II₁ factors are very interesting objects, naturally lying above the matrix algebras $M_N(\mathbb{C})$, which are type I factors. From this perspective, a II₁ factor $A \subset B(H)$ is not really in need of the ambient Hilbert space H, and the question of "representing" it appears. We will discuss this question, in two steps:

- (1) A first question is that of understanding the possible embeddings $A \subset B(H)$, with H being a Hilbert space. The main result here will be the construction of a numeric invariant dim_A H, called coupling constant.
- (2) A second question is that of understanding the possible embeddings $A \subset B$, with B being another II₁ factor. By using the coupling constant for both A, B we will construct a numeric invariant [B : A], called index.

We will discuss now (1), and leave (2) for later, towards the end of this chapter. In order to get started, let us formulate the following definition:

DEFINITION 2.17. Given a von Neumann algebra A with a trace $tr : A \to \mathbb{C}$, the emdedding

$$A \subset B(L^2(A))$$

obtained by GNS construction is called standard form of A.

2B. BASIC RESULTS

Here we use the GNS construction, from functional analysis. As the name indicates, the standard representation is something "standard", to be compared with any other representation $A \subset B(H)$, in order to understand this latter representation.

As known from functional analysis, the GNS construction has a number of unique features, that can be exploited. In the present setting, the main result is as follows:

THEOREM 2.18. In the context of the standard representation we have

$$A' = JAJ$$

with $J: L^2(A) \to L^2(A)$ being the antilinear map given by $T \to T^*$.

PROOF. Observe first that any $T \in A$ can be regarded as a vector $T \in L^2(A)$, to which we can associate, in an antilinear way, the vector $T^* \in L^2(A)$. Thus we have indeed an antilinear map J as in the statement. In terms of the standard cyclic and separating vector Ω for the GNS representation, the formula of this formula J is:

$$J(x\Omega) = x^*\Omega$$

(1) Our first claim is that we have the following formula:

$$\langle J\xi, J\eta \rangle = \langle \xi, \eta \rangle$$

Indeed, with $\xi = x\Omega$ and $\eta = y\Omega$, we have the following computation:

$$\langle J\xi, J\eta \rangle = \langle yx^*\Omega, \Omega \rangle$$

= $tr(yx^*)$
= $\langle \xi, \eta \rangle$

(2) Our second claim is that we have the following formula:

$$JxJ(y\Omega) = yx^*\Omega$$

Indeed, this follows from the following computation:

$$JxJ(y\Omega) = J(xy^*\Omega) = yx^*\Omega$$

(3) Our claim now is that we have an inclusion as follows:

$$JAJ \subset A'$$

Indeed, this follows from the formula obtained in (2).

(4) In order to prove the reverse inclusion, our claim is that for $x \in A'$ we have:

$$Jx\Omega = x^*\Omega$$

Indeed, this follows from the following computation, valid for any $y \in A$:

$$\langle Jx\Omega, y\Omega \rangle = \langle Jy\Omega, x\Omega \rangle$$
$$= \langle y^*\Omega, x\Omega \rangle$$
$$= \langle \Omega, xy\Omega \rangle$$
$$= \langle x^*\Omega, y\Omega \rangle$$

(5) Our claim now is that the following formula defines a trace on A':

$$Tr(x) = \langle x\Omega, \Omega \rangle$$

Indeed, for any two elements $x, y \in A'$ we have:

$$\langle xy\Omega, \Omega \rangle = \langle y\Omega, x^*\Omega \rangle$$

$$= \langle y\Omega, Jx\Omega \rangle$$

$$= \langle x\Omega, Jy\Omega \rangle$$

$$= \langle x\Omega, y^*\Omega \rangle$$

$$= \langle yx\Omega, \Omega \rangle$$

(6) We can now finish the proof. Indeed, by using the trace constructed in (5), we can apply our results obtained so far to A', and we obtain $JA'J \subset A$, as desired. \Box

As a basic illustration for the above result, we have:

THEOREM 2.19. The commutant of a von Neumann group algebra $L(\Gamma)$, which is obtained by definition by using the left regular representation, is the von Neumann group algebra $R(\Gamma)$, obtained by using the right regular representation.

PROOF. We recall that the left and the right representations of a discrete group Γ are given by the following formulae, by using the standard identification $\Gamma \subset l^2(\Gamma)$:

$$\lambda_g: h \to gh \quad , \quad \rho_g: h \to hg^{-1}$$

We have $Jg = g^{-1}$ for any group element $g \in \Gamma$, and by using this, we obtain:

$$J\lambda_g Jh = J\lambda_g h^{-1}$$
$$= Jgh^{-1}$$
$$= hg^{-1}$$
$$= \rho_g h$$

Thus, the left and right representations are related by the following formula:

$$J\lambda_g J = \rho_g$$

By using now Theorem 2.18 we can compute commutants, as follows:

$$L(\Gamma)' = JL(\Gamma)J = R(\Gamma)$$

Finally, we have $L(\Gamma) = R(\Gamma)'$ too, by taking the commutant.

2B. BASIC RESULTS

As another application of the standard representation, let us go back to the uniqueness of the trace, that we know from Theorem 2.16. There are several alternative proofs for this fact, which are all instructive. As a first such statement and proof, we have:

THEOREM 2.20. Given a II₁ factor A, and an element $a \in A$, we have the following Dixmier averaging property:

$$\overline{span\left\{uau^* \middle| u \in U_A\right\}}^w \cap \mathbb{C}1 \neq \emptyset$$

In particular, the II₁ factor trace $tr: A \to \mathbb{C}$ is unique.

PROOF. We use the basic theory of the regular representation $A \subset L^2(A)$, with respect to the given trace $tr : A \to \mathbb{C}$, explained above. The proof goes as follows:

(1) Given an element $a \in A$, consider the space in the statement, obtained as the weak closure of the space spanned by the spinned versions of a, namely:

$$K_a = \overline{span\left\{uau^* \middle| u \in U_A\right\}}^*$$

This linear space $K_a \subset A$ is by definition weakly closed, and it follows that the subset $K_a \Omega \subset L^2(A)$, where $\Omega \in L^2(A)$ is the canonical trace vector, is a weakly closed convex subset. In particular, we see that $K_a \Omega \subset L^2(A)$ is a norm closed convex subset.

(2) In view of this, we can consider the unique element $b \in K_a$ having the property that $b\Omega$ has a minimal norm. We have then the following formula, for any unitary $u \in U_A$, where $J: L^2(A) \to L^2(A)$ is the standard antilinear map, given by $T \to T^*$:

$$||uJuJb\Omega|| = ||b\Omega|$$

By uniqueness of b, it follows that for any unitary $u \in U_A$, we have:

$$uJuJb\Omega = b\Omega$$

But this shows that for any unitary $u \in U_A$, we have:

$$ubu^* = b$$

We conclude that we have $b \in \mathbb{C}1$, and this proves the first assertion.

(3) Regarding now the second assertion, consider an arbitrary trace $tr : A \to \mathbb{C}$. By using $tr(uau^*) = tr(a)$, we conclude that this trace is constant on the following set:

$$K_a = \overline{span\left\{uau^* \middle| u \in U_A\right\}}$$

Now by using the first assertion, we conclude that we have the following formula:

$$\overline{span\left\{uau^* \middle| u \in U_A\right\}}^w \cap \mathbb{C}1 = \left\{tr(a)1\right\}$$

Summarizing, we have obtained a purely algebraic formula for our trace $tr: A \to \mathbb{C}$, and it follows that this trace is indeed unique, as claimed.

In relation with the above, let us mention that there is as well a third proof for the uniqueness of the trace, due to Yeadon, based on nothing or almost, meaning the definition of the II₁ factors, along with some abstract functional analysis. For more on all this, basic theory of the II₁ factors, we refer to the standard operator algebra books, with some good choices here being the books of Connes [17], Takesaki [77] and Blackadar [12].

2c. Type II factors

Let us go back now to the general theory of the II₁ factors, with the aim of talking about representations of such II₁ factors, inside the category of the II₁ factors, $A \subset B$. For this purpose we will need a key notion, called coupling constant.

In order to discuss the construction of the coupling constant, we will need some further results on the type II factors, complementing those that we already have. The point indeed is that the class of II factors, to be axiomatized later, and with this being not something urgent, comprises, besides the II₁ factors discussed above, the II_{∞} factors as well:

DEFINITION 2.21. A II_{∞} factor is a von Neumann algebra of the form

 $B = A \otimes B(H)$

with A being a II_1 factor, and with H being an infinite dimensional Hilbert space.

We should mention that there are several possible ways of defining the II_{∞} factors, and the above definition is something rather intuitive, the point being that, once you learn the theory of the II_{∞} factors, as we will do here, what you remember at the end of the day is what has been said above, $B = A \otimes B(H)$, with A being a II_1 factor.

Getting started now, as a useful characterization of such factors, we have:

PROPOSITION 2.22. For an infinite factor B, the following are equivalent:

- (1) There exists a projection $p \in B$ such that pBp is a II₁ factor.
- (2) B is a II_{∞} factor.

PROOF. This is something elementary, as follows:

(1) \implies (2) Assume indeed that $p \in B$ is a projection such that pBp is a II₁ factor. We choose a maximal family of pairwise orthogonal projections $\{p_i\} \subset B$ satisfying $p_i \simeq p$, for any *i*, and we consider the following projection, which satisfies $q \preceq p$:

$$q = 1 - \sum_{i} p_i$$

Since the indexing set for our set of projections $\{p_i\}$ must be infinite, we can use a strict embedding of this index set into itself, as to write a formula as follows:

$$1 = q + \sum_{i} p_{i}$$
$$\preceq p_{0} + \sum_{i \neq 0} p_{i}$$
$$\preceq 1$$

Thus we have $\sum_i p_i \simeq 1$, and we may further suppose that we have in fact:

$$\sum_{i} p_i = 1$$

Thus the family $\{p_i\}$ can be used in order to construct a copy $B(H) \subset B$, with $H = l^2(\mathbb{N})$, and we must have $B = A \otimes B(H)$, with A being a II₁ factor, as desired.

(2) \implies (1) This is clear, because when assuming $B = A \otimes B(H)$, as in Definition 2.21, we can take our projection $p \in B$ to be of the form $p = 1 \otimes q$, with $q \in B(H)$ being a rank 1 projection, and we have then pBp = A, which is a II₁ factor, as desired. \Box

Getting back now to the original interpretation of the II_{∞} factors, from Definition 2.21, the tensor product writing there $B = A \otimes B(H)$ suggests tensoring the trace of the II_1 factor A with the usual operator trace of B(H). We are led in this way to:

DEFINITION 2.23. Given a II_{∞} factor B, written as $B = A \otimes B(H)$, with A being a II_1 factor and with H being an infinite dimensional Hilbert space, we define a map

$$tr: B_+ \to [0,\infty]$$
 , $tr((x_{ij})) = \sum_i tr(x_{ii})$

where we have chosen a basis of H, as to have $H \simeq l^2(\mathbb{N})$, and so $B(H) \subset M_{\infty}(\mathbb{C})$.

As an important observation, to start with, unlike in the II₁ factor case, that of the factor A, or in the I_{∞} factor case, that of the factor B(H), it is not possible to suitably normalize the trace constructed above. This follows indeed from the results below.

On the positive side now, the above trace has many useful properties, as follows:

PROPOSITION 2.24. The II_{∞} factor trace that we constructed above

 $tr: B_+ \to [0,\infty]$

has the following properties:

- (1) tr(x+y) = tr(x) + tr(y), and $tr(\lambda x) = \lambda tr(x)$ for $\lambda \ge 0$.
- (2) If $x_i \nearrow x$ then $tr(x_i) \rightarrow tr(x)$.
- (3) $tr(xx^*) = tr(x^*x)$.
- (4) $tr(uxu^*) = tr(x)$ for any $u \in U_B$.

PROOF. All this is elementary, the idea being as follows:

- (1) This is clear from definitions.
- (2) This is again clear from definitions.
- (3) This is something which is elementary as well.
- (4) This comes from (3), via the formula $uxu^* = u\sqrt{x} \cdot \sqrt{x}u^*$.

As a main result now regarding the II_{∞} factor trace, we have:

THEOREM 2.25. The II_{∞} factor trace $tr : B_+ \to [0, \infty]$ constructed above, when restricted to the projections

$$tr: P(B) \to [0,\infty]$$

induces an isomorphism between the totally ordered set of equivalence classes of projections in B and the interval $[0, \infty]$.

PROOF. We have several things to be checked here, as follows:

(1) Our first claim is that a projection $p \in B$ is finite precisely when $tr(p) < \infty$.

– Indeed, in one sense, assume that we have $tr(p) < \infty$. If our projection p was to be infinite, we would have a subprojection $q \leq p$ having the same trace as p, and so r = p - q would be a projection of trace 0, which is impossible. Thus p is indeed finite.

- In the other sense now, assuming $tr(p) = \infty$, we have to prove that p is infinite. For this purpose, let us pick a projection $q \leq p$ having finite trace. Then r = p - q satisfies $tr(r) = \infty$, and so we can iterate the procedure, and we end up with an infinite sequence of pairwise orthogonal projections, which are all smaller than p. But this shows that pdominates an infinite projection, and so that p itself is infinite, as desired.

(2) Our second claim is that if $p, q \in B$ are projections, with p finite, then:

$$p \preceq q \iff tr(p) = tr(q)$$

But this follows exactly as in the II_1 factor case, discussed above.

(3) Our third and final claim, which will finish the proof, is that any infinite projection is equivalent to the identity. For this purpose, assume that $p \in B$ is infinite. By definition, this means that we can find a unitary $u \in B$ such that:

$$uu^* = p$$
 , $u^*u \le p$, $uu^* \ne p$

But these conditions show that $(u^n)^i u^n$ is a strictly decreasing sequence of equivalent projections, and by using this sequence we conclude that we have $1 \leq p$, as desired. \Box

Moving ahead now, in order to further investigate the II_{∞} factors, we will need:

THEOREM 2.26. Given a II₁ factor $A \subset B(H)$, there exists an isometry

 $u: H \to L^2(A) \otimes l^2(\mathbb{N})$

such that $ux = (x \otimes 1)u$, for any $x \in A$.

PROOF. We use a standard idea, that we used many times before, namely an amplification trick. Given a II₁ factor $A \subset B(H)$, consider the following Hilbert space:

$$K = H \oplus L^2(A) \otimes l^2(\mathbb{N})$$

Consider, as operators over this space K, the following projections:

$$p=id\oplus 0$$
 , $q=0\oplus id$

Both these projections p, q belong then to A', which is a type II_{∞} factor. Now since $q \in A'$ is infinite, by Theorem 2.25 we can find a partial isometry $u \in A'$ such that:

$$u^*u = p \quad , \quad uu^* \le q$$

Now let us represent this partial isometry $u \in B(K)$ as a 2×2 matrix, as follows:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The above conditions $u^*u = p$ and $uu^* \leq q$ reformulate then as follows:

$$b^*b + d^*d = 0$$
 , $aa^* + bb^* = 0$

We conclude that our partial isometry $u \in B(K)$ has the following special form:

$$u = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

But the operator $c: H \to l^2(A) \otimes l^2(\mathbb{N})$ that we found in this way must be an isometry, and from $u \in A'$ we obtain $ux = (x \otimes 1)u$, for any $x \in A$, as desired.

As a basic consequence of the above result, which is something good to know, and that we will use many times in what follows, we have:

THEOREM 2.27. The commutant of a II₁ factor is a II₁ factor, or a II_{∞} factor.

PROOF. This follows indeed from the explicit interpretation of the operator algebra embedding $A \subset B(H)$ of our II₁ factor A, found in Theorem 2.26.

Summarizing, we have an extension of the general theory of the II₁ factors, developed before, to the general case of the type II factors, which comprises by definition the II₁ factors and the II_{∞} factors. All this is of course technically very useful.

2d. Coupling constant

We are now in position of constructing the coupling constant. The idea here, following as usual the key paper of Murray and von Neumann [60], will be that given a representation of a II₁ factor $A \subset B(H)$, we can try to understand how far is this representation from the standard form, where $H = L^2(A)$, from "above" or from "below".

In order to discuss this, which is something quite technical, let us start with:

PROPOSITION 2.28. Given a II₁ factor $A \subset B(H)$, with its embedding into B(H) being represented as above, in terms of an isometry

$$u: H \to L^2(A) \otimes l^2(\mathbb{N})$$
, $ux = (x \otimes 1)u$

the following quantity does not depend on the choice of this isometry u:

$$C = tr(uu^*)$$

Moreover, for the standard form, where $H = L^2(A)$, this constant takes the value 1.

PROOF. Assume indeed that we have an isometry u as in the statement, and that we have as well a second such isometry, of the same type, namely:

$$v: H \to L^2(A) \otimes l^2(\mathbb{N})$$
, $vx = (x \otimes 1)v$

We have then $uu^* = uv^*vu^*$, and by using this, we obtain:

$$C_u = tr(uu^*)$$

= $tr(uv^*vu^*)$
= $tr(vu^*uv^*)$
= $tr(vv^*)$
= C_v

Thus, we are led to the conclusion in the statement. As for the last assertion, regarding the standard form, this is clear from definitions, because here we can take u = 1.

As a conclusion to all this, given a II₁ factor $A \subset B(H)$, we know from Theorem 2.26 that H must appear as an "inflated" version of $L^2(A)$. The corresponding inflation constant is a certain number, that we can call coupling constant, as follows:

DEFINITION 2.29. Given a representation of a II₁ factor $A \subset B(H)$, we can talk about the corresponding coupling constant, as being the number

$$\dim_A H \in (0,\infty]$$

constructed as follows, with $u: H \to L^2(A) \otimes l^2(\mathbb{N})$ isometry satisfying $ux = (x \otimes 1)u$:

$$\dim_A H = tr(uu^*)$$

For the standard form, where $H = L^2(A)$, this coupling constant takes the value 1.

This definition might seem a bit complicated, but things here are quite non-trivial, and there is no way of doing something substantially simpler. Alternatively, we can define the coupling constant via the following formula, after proving first that the number on the right is indeed independent of the choice on a nonzero vector $x \in H$:

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

This latter formula was in fact the original definition of the coupling constant, by Murray and von Neumann [60]. However, technically speaking, things are slightly easier when using the approach in Definition 2.29. We will be back to this key formula of Murray and von Neumann, with full explanations, in a moment.

Let us start our study of the coupling constant with some basic results, coming from definitions and from what we already have, as results, as follows:

PROPOSITION 2.30. The coupling constant dim_A $H \in (0, \infty]$ associated to a II₁ factor representation $A \subset B(H)$ has the following properties:

- (1) For the standard form, $H = L^2(A)$, we have dim_A H = 1.
- (2) For the usual representation on $H = L^2(A) \otimes l^2(\mathbb{N})$, we have $\dim_A H = \infty$.
- (3) We have $\dim_A H < \infty$ precisely when A' is a II₁ factor.
- (4) We have additivity, $\dim_A(\bigoplus_i H_i) = \sum_i \dim_A H_i$.
- (5) We have $\dim_A(L^2(A)p) = tr(p)$, for any projection $p \in A$.
- (6) The coupling constant can take any value in $(0, \infty]$.

PROOF. All these assertions are elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something that comes from definitions too.

- (3) This comes from the general properties of the II_{∞} factors, and their traces.
- (4) Again, this is clear from the definition of the coupling constant.
- (5) This follows by using $u(x) = x \otimes \xi$, with $\xi \in l^2(\mathbb{N})$ being of norm 1.
- (6) This follows by starting with (5), and then making direct sums, as in (4).

At a more advanced level now, in relation with projections and compressions, and getting towards the above-mentioned Murray-von Neumann approach, we have:

PROPOSITION 2.31. We have the compression formula

$$\dim_{pAp}(pH) = \frac{\dim_A H}{tr_A(p)}$$

valid for any projection $p \in A$.

PROOF. We can prove this result in two steps, as follows:

(1) Assume that H is as follows, with $q \in A$ being a projection satisfying $q \leq p$:

 $H = L^2(A)q$

We can use the following unitary, intertwining the left and right actions of pAp:

$$L^2(pAp) \to pL^2(A)p \quad , \quad pxp\Omega \to p(x\Omega)p$$

Indeed, we obtain that the following algebras are unitarily equivalent:

$$pAp \subset B(pL^2(A)q)$$
, $pAp \subset B(L^2(pAp)q)$

Thus, by using the formula (5) in Proposition 2.30 we obtain, as desired:

$$\dim_{pAp}(pH) = tr_{pAp}(q)$$
$$= \frac{tr_A(q)}{tr_A(p)}$$
$$= \frac{\dim_A H}{tr_A(p)}$$

(2) In the general case now, where H is arbitrary, the result follows from what we proved above, and from the additivity property from Proposition 2.30 (4).

With all these properties established, we can now recover, as a theorem, the original definition of the coupling constant, due to Murray and von Neumann, as follows:

THEOREM 2.32. Given a II₁ factor $A \subset B(H)$, with the commutant $A' \subset B(H)$ assumed to be finite, the corresponding coupling constant is finite, given by

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

with the number on the right being independent of the choice on a nonzero vector $x \in H$. In the case where A' is infinite, the corresponding coupling constant is infinite.

PROOF. There are several things to be proved here, the idea being as follows:

(1) We know from Proposition 2.30 (3) that we have $\dim_A H < \infty$ precisely when the commutant $A' \subset B(H)$ is finite. Thus, we may assume that we are in this case.

(2) Assuming so, we have the following formula, valid for any projection $p \in A'$, which follows from the basic properties of the coupling constant, established above:

$$\dim_{Ap}(pH) = tr_{A'}(p)\dim_A H$$

(3) Now with this formula in hand, the formula in the statement follows as well, once again by doing a number of standard amplification and compression manipulations. \Box

As an illustration for all this, given an inclusion of ICC groups $\Lambda \subset \Gamma$, whose group algebras are both II₁ factors, we have the following formula:

$$\dim_{L(\Lambda)} L^2(\Gamma) = [\Gamma : \Lambda]$$

There are many other examples of explicit computations of the coupling constant, all leading into interesting mathematics. We will be back to this.

As a last topic for this chapter, given a II₁ factor A, let us discuss now the representations of type $A \subset B$, with B being another II₁ factor. This is a quite natural notion, perhaps even more natural than the representations $A \subset B(H)$, because we have previously decided that the II₁ factors B, and not the full operator algebras B(H), are the correct infinite dimensional generalization of the usual matrix algebras $M_N(\mathbb{C})$.

This was for the philosophy, and one can of course agree or not with this. Or at least agree or not at the present point of the presentation, because once we will get into the structure of the subfactors $A \subset B$, which is something amazing, there is no way back.

In practice now, given an inclusion of II_1 factors $A \subset B$, a first question is that of defining its index, measuring how big is B compared to A. The first thought here goes into defining the index of $A \subset B$ as being a purely algebraic quantity, as follows:

$N = \dim_A B$

However, this is non-trivial, due to the fact that we are in the "continuous dimension" setting, and so our algebraic intuition, where indices are always integers, will not help us much. We will be back to this question later, with a technical solution to it.

In order to solve our index problem, a much better approach is by using the ambient operator algebra B(H), or rather the ambient Hilbert space H, as follows:

THEOREM 2.33. Given an inclusion of II_1 factors $A \subset B$, the number

$$N = \frac{\dim_A H}{\dim_B H}$$

is independent of the ambient Hilbert space H, and is called index.

PROOF. The fact that the index of the subfactor $A \subset B$, as defined by the above formula, is indeed independent of the ambient Hilbert space H, comes from the various basic properties of the coupling constant, established above.

There are many examples of subfactors coming from groups, and every time we obtain the intuitive index. More suprisingly now, Jones proved in [40] that the index, when small, is in fact "quantized", subject to the following unexpected restriction:

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \left| n \ge 3 \right\} \cup [4, \infty] \right\}$$

This is in fact part of a series of non-trivial results about the subfactors, due to Jones, and also Ocneanu, Popa, Wassermann and others, and involving as well the Temperley-Lieb algebra, and many more. We will be back to this, in a moment.

2e. Exercises

Exercises:

EXERCISE 2.34. EXERCISE 2.35. EXERCISE 2.36. EXERCISE 2.37. EXERCISE 2.38. EXERCISE 2.39. EXERCISE 2.40. EXERCISE 2.41. Bonus exercise.

CHAPTER 3

Subfactors

3a. Subfactors

We recall that a II₁ factor is a von Neumann algebra $A \subset B(H)$ which has trivial center, $Z(A) = \mathbb{C}$, is infinite dimensional, and has a trace $tr : A \to \mathbb{C}$. For a number of reasons, ranging from simple and intuitive to fairly advanced, explained in chapter 1, such algebras are the core at the whole von Neumann algebra theory.

The world of II_1 factors is a bit similar to the world of the usual matrix algebras $M_N(\mathbb{C})$, which are actually called type I factors, in the sense that it is "self-sufficient", with no need to go further than that. In particular, a nice representation theory for such II_1 factors can be obtained by staying inside the class of II_1 factors, and we have the following definition to start with, which will keep us busy for the rest of this book:

DEFINITION 3.1. A subfactor is an inclusion of II₁ factors $A \subset B$.

We will see later some examples of such inclusions, along with motivations for their study. In order to get started now, the first thing to be done with such an inclusion is that of defining its index, as a quantity of the following type:

$$[B:A] = \dim_A B$$

Since both A, B are infinite dimensional algebras, this is not exactly obvious. In addition, in view of our previous experience with the II₁ factors, and notably with their "continuous dimension" features, we can only expect the index to range as follows:

$$[B:A] \in [1,\infty]$$

In order to discuss this, let us recall from chapter 2 that given a representation of a II₁ factor $A \subset B(H)$, we can construct a number as follows, called coupling constant, which for the standard form, where $H = L^2(A)$, takes the value 1, and which in general mesures how far is $A \subset B(H)$ from the standard form:

$$\dim_A H \in (0,\infty)$$

Getting now to the subfactors, in the sense of Definition 3.1, we have the following construction, that we know as well from chapter 2:

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THEOREM 3.2. Given a subfactor $A \subset B$, the number

$$N = \frac{\dim_A H}{\dim_B H} \in [1, \infty]$$

is independent of the ambient Hilbert space H, and is called index.

PROOF. This is something that we know from chapter 2, the idea being that the independence of the index from the choice of the ambient Hilbert space H comes from the various basic properties of the coupling constant.

There are many examples of subfactors, and we will discuss this gradually, in what follows. Following Jones [40], the most basic examples of subfactors are as follows:

PROPOSITION 3.3. Assuming that G is a compact group, acting on a II_1 factor P in a minimal way, in the sense that we have

$$(P^G)' \cap P = \mathbb{C}$$

and that $H \subset G$ is a closed subgroup of finite index, we have a subfactor

$$P^G \subset P^H$$

having index N = [G : H], called Jones subfactor.

PROOF. This is something standard, the idea being that the factoriality of P^G , P^H comes from the minimality of the action, and that the index formula is clear. We will be back with full details about this later, directly in a more general setting.

In order to study the subfactors, let us start with the following standard result:

PROPOSITION 3.4. Given a subfactor $A \subset B$, there is a unique linear map

 $E: B \to A$

which is positive, unital, trace-preserving and satisfies the following condition:

$$E(b_1ab_2) = b_1E(a)b_2$$

This map is called conditional expectation from B onto A.

PROOF. We make use of the standard representation of the II₁ factor B, with respect to its unique trace $tr: B \to \mathbb{C}$, as constructed in chapter 2:

 $B \subset L^2(B)$

If we denote by Ω the standard cyclic and separating vector of $L^2(B)$, we have an identification $A\Omega = L^2(A)$. Consider now the following orthogonal projection:

$$e: L^2(B) \to L^2(A)$$

It follows from definitions that we have an inclusion as follows:

$$e(B\Omega) \subset A\Omega$$

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Thus e induces by restriction a certain linear map $E: B \to A$. This linear map E and the orthogonal projection e are then related by:

$$exe = E(x)e$$

But this shows that the linear map E satisfies the various conditions in the statement, namely positivity, unitality, trace preservation and bimodule property. As for the uniqueness assertion, this follows by using the same argument, applied backwards, the idea being that a map E as in the statement must come from the projection e.

Following Jones [40], we will be interested in what follows in the orthogonal projection $e: L^2(B) \to L^2(A)$ producing the expectation $E: B \to A$, rather than in E itself:

DEFINITION 3.5. Associated to any subfactor $A \subset B$ is the orthogonal projection

$$e: L^2(B) \to L^2(A)$$

producing the conditional expectation $E: B \to A$ via the following formula:

exe = E(x)e

This projection is called Jones projection for the subfactor $A \subset B$.

Quite remarkably, the subfactor $A \subset B$, as well as its commutant, can be recovered from the knowledge of this projection, in the following way:

PROPOSITION 3.6. Given a subfactor $A \subset B$, with Jones projection e, we have

$$A = B \cap \{e\}'$$
$$A' = (B' \cap \{e\})''$$

as equalities of von Neumann algebras, acting on the space $L^2(B)$.

PROOF. These formulae basically follow from exe = E(x)e, as follows:

(1) Let us first prove that we have $A \subset B \cap \{e\}'$. Given $x \in A$, we have:

$$xe = E(x)e = exe$$

$$x^*e = E(x^*)e = ex^*e$$

Thus, we obtain, as desired, that x commutes with e:

$$ex = (x^*e)^* = (ex^*e)^* = exe = xe$$

(2) Let us prove now that $B \cap \{e\}' \subset A$. Assuming ex = xe, we have:

$$E(x)e = exe = xe^2 = xe$$

We conclude from this that we have the following equality:

$$[E(x) - x)\Omega = (E(x) - x)e\Omega = 0$$

Now since Ω is separating for *B* we have, as desired:

$$x = E(x) \in A$$

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(3) In order to prove now $A' = \langle B', e \rangle$, observe that we have:

$$A = B \cap \{e\}' = B'' \cap \{e\}' = (B' \cap \{e\})'$$

Now by taking the commutant, we obtain $A' = (B' \cap \{e\})''$, as desired.

Still following Jones [40], we are now ready to formulate a key definition:

DEFINITION 3.7. Associated to any subfactor $A \subset B$ is the basic construction

 $A \subset_e B \subset C$

with $C = \langle B, e \rangle$ being the algebra generated by B and by the Jones projection

 $e: L^2(B) \to L^2(A)$

acting on the Hilbert space $L^2(B)$.

The idea in what follows will be that $B \subset C$ appears as a kind of "reflection" of $A \subset B$, and also that the basic construction can be iterated, with all this leading to nontrivial results. Let us start by further studying the basic construction:

THEOREM 3.8. Given a subfactor $A \subset B$ having finite index,

 $[B:A] < \infty$

the basic construction $A \subset_e B \subset C$ has the following properties:

(1) C = JA'J.(2) $C = \overline{B + Beb}.$ (3) C is a II₁ factor. (4) [C:B] = [B:A].(5) eCe = Ae.(6) $tr(e) = [B:A]^{-1}.$ (7) $tr(xe) = tr(x)[B:A]^{-1}$, for any $x \in B.$

PROOF. All this is standard, the idea being as follows:

(1) We have JB'J = B and JeJ = e, which gives:

$$JA'J = J < B', e > J$$
$$= < JB'J, JeJ >$$
$$= < B, e >$$
$$= C$$

(2) This follows from the fact that the vector space B + BeB is closed under multiplication, and from the fact that we have exe = E(x)e.

(3) This follows from the fact, that we know from chapter 2, that our finite index assumption $[B:A] < \infty$ is equivalent to the fact that A' is a factor. But this is in turn equivalent to the fact that C = JA'J is a factor, as desired.

(4) We have indeed the following computation:

$$[C:B] = \frac{\dim_B L^2(B)}{\dim_C L^2(B)}$$
$$= \frac{1}{\dim_C L^2(B)}$$
$$= \frac{1}{\dim_{JA'J} L^2(B)}$$
$$= \frac{1}{\dim_{A'} L^2(B)}$$
$$= \dim_A L^2(B)$$
$$= [B:A]$$

- (5) This follows indeed from (2) and from the formula exe = E(x)e.
- (6) We have the following computation:

$$1 = \dim_A L^2(A)$$

= dim_A(eL^2(B))
= tr_{A'}(e) dim_A(L^2(B))
= tr_{A'}(a)[B:A]

Now since C = JA'J and JeJ = e, we obtain from this, as desired:

$$tr(e) = tr_{JA'J}(JeJ) = tr_{A'}(e) = [B:A]^{-1}$$

(7) We already know from (6) that the formula in the statement holds for x = 1. In order to discuss the general case, observe first that for $x, y \in A$ we have:

$$tr(xye) = tr(yex) = tr(yxe)$$

Thus the linear map $x \to tr(xe)$ is a trace on A, and by uniqueness of the trace on A, we must have, for a certain constant c > 0:

$$tr(xe) = c \cdot tr(x)$$

Now by using (6) we obtain $c = [B : A]^{-1}$, so we have proved the formula in the statement for $x \in A$. The passage to the general case $x \in B$ can be done as follows:

$$tr(xe) = tr(exe)$$

= $tr(E(x)e)$
= $tr(E(x))c$
= $tr(x)c$

Thus, we have proved the formula in the statement, in general.

3b. The Jones tower

The above result is quite interesting, so let us perform now twice the basic construction, and see what we get. The result here, which is more technical, is as follows:

PROPOSITION 3.9. Associated to $A \subset B$ is the double basic construction

 $A \subset_e B \subset_f C \subset D$

with e, f being the following orthogonal projections,

$$e: L^2(B) \to L^2(A)$$

 $f: L^2(C) \to L^2(B)$

having the following properties:

$$fef = [B:A]^{-1}f$$
$$efe = [B:A]^{-1}e$$

PROOF. We have two formulae to be proved, the idea being as follows:

(1) The first formula is clear, because we have:

$$fef = E(e)f$$

= $tr(e)f$
= $[B:A]^{-1}f$

(2) Regarding now the second formula, it is enough to check it on the dense subset $(B + BeB)\Omega$. Thus, we must show that for any $x, y, z \in B$, we have:

$$efe(x + yez)\Omega = [B : A]^{-1}e(x + yez)\Omega$$

For this purpose, we will prove that we have, for any $x, y, z \in B$:

$$efex\Omega = [B:A]^{-1}ex\Omega$$

$$efeyez\Omega = [B:A]^{-1}eyez\Omega$$

The first formula can be established as follows:

$$efex\Omega = efexf\Omega$$

= $eE(ex)f\Omega$
= $eE(e)xf\Omega$
= $[B:A]^{-1}exf\Omega$
= $[B:A]^{-1}ex\Omega$

The second formula can be established as follows:

$$efeyez\Omega = efeyezf\Omega$$

= $eE(eyez)f\Omega$
= $eE(eye)zf\Omega$
= $eE(E(y)e)zf\Omega$
= $eE(y)E(e)zf\Omega$
= $[B:A]^{-1}eE(y)zf\Omega$
= $[B:A]^{-1}eyezf\Omega$
= $[B:A]^{-1}eyez\Omega$

Thus, we are led to the conclusion in the statement.

We can in fact perform the basic construction by recurrence, and we obtain:

THEOREM 3.10. Associated to any subfactor $A_0 \subset A_1$ is the Jones tower

 $A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$

with the Jones projections having the following properties:

(1)
$$e_i^2 = e_i = e_i^*$$
.
(2) $e_i e_j = e_j e_i$ for $|i - j| \ge 2$.

(3)
$$e_i e_{i\pm 1} e_i = [B:A]^{-1} e_i$$
.

(4) $tr(we_{n+1}) = [B:A]^{-1}tr(w)$, for any word $w \in \langle e_1, \dots, e_n \rangle$.

PROOF. This follows from Theorem 3.8 and Proposition 3.9, because the triple basic construction does not need in fact any further study. See Jones [40]. \Box

3c. Temperley-Lieb

The relations found in Theorem 3.10 are in fact well-known, from the standard theory of the Temperley-Lieb algebra. This algebra, discovered by Temperley and Lieb in the context of statistical mechanics [79], has a very simple definition, as follows:

DEFINITION 3.11. The Temperley-Lieb algebra of index $N \in [1, \infty)$ is defined as

$$TL_N(k) = span(NC_2(k,k))$$

with product given by vertical concatenation, with the rule

$$\bigcirc = N$$

for the closed circles that might appear when concatenating.

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In other words, the algebra $TL_N(k)$, depending on parameters $k \in \mathbb{N}$ and $N \in [1, \infty)$, is the formal linear span of the pairings $\pi \in NC_2(k, k)$. The product operation is obtained by linearity, for the pairings which span $TL_N(k)$ this being the usual vertical concatenation, with the conventions that things go "from top to bottom", and that each circle that might appear when concatenating is replaced by a scalar factor, equal to N.

In order to make the connection with subfactors, let us start with:

PROPOSITION 3.12. The Temperley-Lieb algebra $TL_N(k)$ is generated by the diagrams

$$\varepsilon_1 = {}^{\cup}_{\cap}$$
, $\varepsilon_2 = |{}^{\cup}_{\cap}$, $\varepsilon_3 = |{}^{\cup}_{\cap}$, ...

which are all multiples of projections, in the sense that their rescaled versions

$$e_i = N^{-1}\varepsilon_i$$

satisfy the abstract projection relations $e_i^2 = e_i = e_i^*$.

PROOF. We have two assertions here, the idea being as follows:

(1) The fact that the algebra $TL_N(k)$ is indeed generated by the sequence of diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ follows by drawing pictures, and more specifically by graphically decomposing each basis element $\pi \in NC_2(k, k)$ as a product of such elements ε_i .

(2) Regarding now the projection assertion, when composing ε_i with itself we obtain ε_i itself, times a circle. Thus, according to our multiplication conventions, we have:

$$\varepsilon_i^2 = N\varepsilon_i$$

Also, when turning upside-down ε_i , we obtain ε_i itself. Thus, according to our involution convention for the Temperley-Lieb algebra, we have:

$$\varepsilon_i^* = \varepsilon_i$$

We conclude that the rescalings $e_i = N^{-1} \varepsilon_i$ satisfy $e_i^2 = e_i = e_i^*$, as desired.

As a second result now, making the link with Theorem 3.10, we have:

PROPOSITION 3.13. The standard generators $e_i = N^{-1}\varepsilon_i$ of the Temperley-Lieb algebra $TL_N(k)$ have the following properties, where tr is the trace obtained by closing:

(1) $e_i e_j = e_j e_i$ for $|i - j| \ge 2$.

(2)
$$e_i e_{i+1} e_i = [B:A]^{-1} e_i.$$

(2) $e_i e_{i\pm 1} e_i = [B : A]^{-1} tr(w)$, for any word $w \in \langle e_1, \dots, e_n \rangle$. (3) $tr(we_{n+1}) = [B : A]^{-1} tr(w)$, for any word $w \in \langle e_1, \dots, e_n \rangle$.

PROOF. This follows indeed by doing some elementary computations with diagrams, in the spirit of those performed in the proof of Proposition 3.12. Indeed:

(1) This is clear from the definition of the diagrams ε_i .

- (2) This is clear as well from the definition of the diagrams ε_i .
- (3) This is something which is clear too, from the definition of ε_{n+1} .

3C. TEMPERLEY-LIEB

With the above results in hand, we can now reformulate our main finding about subfactors, namely Theorem 3.10, into something more conceptual, as follows:

THEOREM 3.14. Given a finite index subfactor $A_0 \subset A_1$, with Jones tower

 $A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \dots$

the rescaled sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ produces a representation

 $TL_N \subset B(H)$

of the Temperley-Lieb algebra of index $N = [A_1 : A_0]$.

PROOF. The idea here is that Theorem 3.10, coming from the study of the basic construction, tells us that the rescaled sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ behaves algebrically exactly as the sequence of diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \in TL_N$ given by:

$$\varepsilon_1 = {\cup \atop \cap}$$
, $\varepsilon_2 = {| \atop \cap}$, $\varepsilon_3 = {| \atop \cap}$, ...

But these diagrams generate TL_N , and so we have an embedding $TL_N \subset B(H)$, where H is the Hilbert space where our subfactor $A_0 \subset A_1$ lives, as claimed.

Before going further, with some examples, more theory, and consequences of Theorem 3.14, let us make the following key observation, also from Jones [40]:

THEOREM 3.15. Given a finite index subfactor $A_0 \subset A_1$, the graded algebra

$$P = (P_k)$$

formed by the sequence of higher relative commutants

 $P_k = A'_0 \cap A_k$

contains the copy of the Temperley-Lieb algebra constructed above:

 $TL_N \subset P$

This graded algebra $P = (P_k)$ is called "planar algebra" of the subfactor.

PROOF. As a first observation, since the Jones projection $e_1 : A_1 \to A_0$ commutes with A_0 , as was previously established in the above, we have:

 $e_1 \in P'_2$

By translation we obtain from this that we have, for any $k \in \mathbb{N}$:

$$e_1,\ldots,e_{k-1}\in P_k$$

Thus we have indeed an inclusion of graded algebras $TL_N \subset P$, as claimed.

The point with the above result, which explains among others the terminology at the end, is that, in the context of Theorem 3.14, the planar algebra structure of TL_N , obtained by composing diagrams, extends into an abstract planar algebra structure of P. See [42]. We will discuss all this, with full details, later in this book.

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3d. The index theorem

As an interesting consequence of the above results, somehow contradicting the "continuous geometry" philosophy that has being going on so far, in relation with the II₁ factors, we have the following surprising result, also from Jones' original paper [40]:

THEOREM 3.16. The index of subfactors $A \subset B$ is "quantized" in the [1,4] range,

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \left| n \ge 3 \right\} \cup [4, \infty] \right\}$$

with the obstruction coming from the existence of the representation $TL_N \subset B(H)$.

PROOF. This comes from the basic construction, and more specifically from the combinatorics of the Jones projections e_1, e_2, e_3, \ldots , the idea being as follows:

(1) In order to best comment on what happens, when iterating the basic construction, let us record the first few values of the numbers in the statement:

$$4\cos^2\left(\frac{\pi}{3}\right) = 1 \quad , \quad 4\cos^2\left(\frac{\pi}{4}\right) = 2$$
$$4\cos^2\left(\frac{\pi}{5}\right) = \frac{3+\sqrt{5}}{2} \quad , \quad 4\cos^2\left(\frac{\pi}{6}\right) = 3$$

(2) When performing a basic construction, we obtain, by trace manipulations on e_1 :

$$N \notin (1,2)$$

With a double basic construction, we obtain, by trace manipulations on $\langle e_1, e_2 \rangle$:

$$N \notin \left(2, \frac{3+\sqrt{5}}{2}\right)$$

With a triple basic construction, we obtain, by trace manipulations on $\langle e_1, e_2, e_3 \rangle$:

$$N \notin \left(\frac{3+\sqrt{5}}{2},3\right)$$

Thus, we are led to the conclusion in the statement, by a kind of recurrence, involving a certain family of orthogonal polynomials.

(3) In practice now, the most elegant way of proving the result is by using the fundamental fact, explained in Theorem 3.14, that that sequence of Jones projections $e_1, e_2, e_3, \ldots \subset B(H)$ generate a copy of the Temperley-Lieb algebra of index N:

$$TL_N \subset B(H)$$

With this result in hand, we must prove that such a representation cannot exist in index N < 4, unless we are in the following special situation:

$$N = 4\cos^2\left(\frac{\pi}{n}\right)$$

But this can be proved by using some suitable trace and positivity manipulations on TL_N , as in (2) above. For full details here, we refer to [28], [40], [46].

The above result raises the question of understanding if there are further restrictions on the index of subfactors $A \subset B$, in the range found there, namely:

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \left| n \ge 3 \right\} \cup [4, \infty] \right\}$$

In the simplest formulation of the question, the answer is generally "no", as follows:

THEOREM 3.17. Consider the Murray-von Neumann hyperfinite II₁ factor R. Its subfactors $R_0 \subset R$ are then as follows:

- (1) They exist for all admissible index values, $N \in \{4 \cos^2\left(\frac{\pi}{n}\right) | n \ge 3\} \cup [4, \infty]$.
- (2) In index $N \leq 4$, they can be realized as irreducible subfactors, $R'_0 \cap R = \mathbb{C}$.
- (3) In index N > 4, they can be realized as arbitrary subfactors.

PROOF. This is something quite tricky, worked out in Jones' original paper [40], and requiring some advanced algebra methods, the idea being as follows:

(1) This basically follows by taking a copy of the Temperley-Lieb algebra TL_N , and then building a subfactor out of it, first by constructing a certain inclusion of inductive limits of finite dimensional algebras, $\mathcal{A} \subset \mathcal{B}$, and then by taking the weak closure, which produces copies of the Murray-von Neumann hyperfinite II₁ factor, $A \simeq B \simeq R$.

(2) This follows by examining and fine-tuning the construction in (1), which can be performed as to have control over the relative commutant.

(3) This follows as well from (1), and with the simplest proof here being in fact quite simple, based on a projection trick. \Box

As another application now, which is more theoretical, let us go back to the question of defining the index of a subfactor in a purely algebraic manner, which was open since chapter 2. The answer here, due to Pimsner and Popa [67], is as follows:

THEOREM 3.18. Any finite index subfactor $A \subset B$ has an algebraic orthonormal basis, called Pimsner-Popa basis, which is constructed as follows:

- (1) In integer index, $N \in \mathbb{N}$, this is a usual basis, of type $\{b_1, \ldots, b_N\}$, whose length is exactly the index.
- (2) In non-integer index, $N \notin \mathbb{N}$, this is something of type $\{b_1, \ldots, b_n, c\}$, having length n + 1, with n = [N], and with $N n \in (0, 1)$ being related to c.

PROOF. This is something quite technical, which follows from the basic theory of the basic construction. We refer here to the paper of Pimsner and Popa [67]. \Box

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3e. Exercises

Exercises:

Exercise 3.19.

EXERCISE 3.20.

EXERCISE 3.21.

Exercise 3.22.

Exercise 3.23.

EXERCISE 3.24.

EXERCISE 3.25.

Exercise 3.26.

Bonus exercise.

CHAPTER 4

Basic examples

4a. Fixed points

Let us discuss now some basic examples of subfactors, with concrete illustrations for all the above notions, constructions, and general theory. These examples will all come from group actions $G \curvearrowright P$, which are assumed to be minimal, in the sense that:

$$(P^G)' \cap P = \mathbb{C}$$

We will not provide proofs for the next few results to follow, the idea being that these constructions can be unified, and that we would like to keep the proofs for the unifications only. As a starting point, we have the following result, that we already know:

PROPOSITION 4.1. Assuming that G is a compact group, acting minimally on a II₁ factor P, and that $H \subset G$ is a subgroup of finite index, we have a subfactor

$$P^G \subset P^H$$

having index N = [G : H], called Jones subfactor.

PROOF. This is something that we know, the idea being that the factoriality of P^G , P^H comes from the minimality of the action, and that the index formula is clear.

Along the same lines, we have the following result:

PROPOSITION 4.2. Assuming that G is a finite group, acting minimally on a II_1 factor P, we have a subfactor as follows,

 $P\subset P\rtimes G$

having index N = |G|, called Ocneanu subfactor.

PROOF. This is standard as well, the idea being that the factoriality of $P \rtimes G$ comes from the minimality of the action, and that the index formula is clear.

We have as well a third result of the same type, as follows:

PROPOSITION 4.3. Assuming that G is a compact group, acting minimally on a II₁ factor P, and that $G \to PU_n$ is a projective representation, we have a subfactor

$$P^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

having index $N = n^2$, called Wassermann subfactor.

4. BASIC EXAMPLES

PROOF. As before, the idea is that the factoriality of P^G , $(M_n(\mathbb{C}) \otimes P)^G$ comes from the minimality of the action, and the index formula is clear.

The above subfactors look quite related, and indeed they are, due to:

THEOREM 4.4. The Jones, Ocneanu and Wassermann subfactors are all of the same nature, and can be written as follows,

$$(P^G \subset P^H) \simeq ((\mathbb{C} \otimes P)^G \subset (l^{\infty}(G/H) \otimes P)^G) (P \subset P \rtimes G) \simeq ((l^{\infty}(G) \otimes P)^G \subset (\mathcal{L}(l^2(G)) \otimes P)^G) (P^G \subset (M_n(\mathbb{C}) \otimes P)^G) \simeq ((\mathbb{C} \otimes P)^G \subset (M_n(\mathbb{C}) \otimes P)^G)$$

with standard identifications for the various tensor products and fixed point algebras.

PROOF. This is something very standard, modulo all kinds of standard identifications. We will explain all this more in detail later, after unifying these subfactors. \Box

In order to unify now the above constructions of subfactors, the idea is quite clear. Given a compact group G, acting minimally on a II₁ factor P, and an inclusion of finite dimensional algebras $B_0 \subset B_1$, endowed as well with an action of G, we would like to construct a kind of generalized Wassermann subfactor, as follows:

$$(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$$

In order to do this, we must talk first about the finite dimensional algebras B, and about inclusions of such algebras $B_0 \subset B_1$. Let us start with the following definition:

DEFINITION 4.5. Associated to any finite dimensional algebra B is its canonical trace, obtained by composing the left regular representation with the trace of $\mathcal{L}(B)$:

$$tr: B \subset \mathcal{L}(B) \to \mathbb{C}$$

We say that an inclusion of finite dimensional algebras $B_0 \subset B_1$ is Markov if it commmutes with the canonical traces of B_0, B_1 .

In what regards the first notion, that of the canonical trace, this is something that we know well, from chapter 1. Indeed, as explained there, we can formally write B = C(X), with X being a finite quantum space, and the canonical trace $tr : B \to \mathbb{C}$ is then precisely the integration with respect to the "counting measure" on X.

In what regards the second notion, that of a Markov inclusion, this is something very natural too, and as a first example here, any inclusion of type $\mathbb{C} \subset B$ is Markov. In general, if we write $B_0 = C(X_0)$ and $B_1 = C(X_1)$, then the inclusion $B_0 \subset B_1$ must come from a certain fibration $X_1 \to X_0$, and the inclusion $B_0 \subset B_1$ is Markov precisely when the fibration $X_1 \to X_0$ commutes with the respective counting measures.

We will be back to Markov inclusions and their various properties on several occasions, in what follows. For our next purposes here, we just need the following result:

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PROPOSITION 4.6. Given a Markov inclusion of finite dimensional algebras $B_0 \subset B_1$ we can perform to it the basic construction, as to obtain a Jones tower

$$B_0 \subset_{e_1} B_1 \subset_{e_2} B_2 \subset_{e_3} B_3 \subset \dots$$

exactly as we did in the above for the inclusions of II_1 factors.

PROOF. This is something quite routine, by following the computations in the above, from the case of the II₁ factors, and with everything extending well. It is of course possible to do something more general here, unifying the constructions for the inclusions of II₁ factors $A_0 \subset A_1$, and for the inclusions of Markov inclusions of finite dimensional algebras $B_0 \subset B_1$, but we will not need this degree of generality, in what follows.

With these ingredients in hand, getting back now to the Jones, Ocneanu and Wassermann subfactors, from Theorem 4.4, the point is that these constructions can be unified, and then further studied, the final result on the subject being as follows:

THEOREM 4.7. Let G be a compact group, and $G \to Aut(P)$ be a minimal action on a II₁ factor. Consider a Markov inclusion of finite dimensional algebras

$$B_0 \subset B_1$$

and let $G \to Aut(B_1)$ be an action which leaves invariant B_0 , and which is such that its restrictions to the centers of B_0 and B_1 are ergodic. We have then a subfactor

$$(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$$

of index $N = [B_1 : B_0]$, called generalized Wassermann subfactor, whose Jones tower is $(B_1 \otimes P)^G \subset (B_2 \otimes P)^G \subset (B_3 \otimes P)^G \subset \dots$

where
$$\{B_i\}_{i\geq 1}$$
 are the algebras in the Jones tower for $B_0 \subset B_1$, with the canonical actions
of G coming from the action $G \to Aut(B_1)$, and whose planar algebra is given by:

$$P_k = (B'_0 \cap B_k)^G$$

These subfactors generalize the Jones, Ocneanu and Wassermann subfactors.

PROOF. There are several things to be proved, the idea being as follows:

(1) As before on various occasions, the idea is that the factoriality of the algebras $(B_i \otimes P)^G$ comes from the minimality of the action $G \to Aut(P)$, and that the index formula is clear as well, from the definition of the coupling constant and of the index.

(2) Regarding the Jones tower assertion, the precise thing to be checked here is that if $A \subset B \subset C$ is a basic construction, then so is the following sequence of inclusions:

$$(A \otimes P)^G \subset (B \otimes P)^G \subset (C \otimes P)^G$$

But this is something standard, which follows by verifying the basic construction conditions. We will be back to this in a moment, directly in a more general setting.

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(3) Next, regarding the planar algebra assertion, we have to prove here that for any indices $i \leq j$, we have the following equality between subalgebras of $B_j \otimes P$:

$$((B_i \otimes P)^G)' \cap (B_j \otimes P)^G = (B'_i \cap B^G_j) \otimes 1$$

But this is something which is routine too, following Wassermann [90], and we will be back to this in a moment, with full details, directly in a more general setting.

(4) Finally, the last assertion, regarding the main examples of such subfactors, which are those of Jones, Ocneanu, Wassermann, follows from Theorem 4.4. \Box

In addition to the Jones, Ocneanu and Wassermann subfactors, discussed and unified in the above, we have the Popa subfactors, which are constructed as follows:

PROPOSITION 4.8. Given a discrete group $\Gamma = \langle g_1, \ldots, g_n \rangle$, acting faithfully via outer automorphisms on a II₁ factor Q, we have the following "diagonal" subfactor

$$\left\{ \begin{pmatrix} g_1(q) & & \\ & \ddots & \\ & & g_n(q) \end{pmatrix} \middle| q \in Q \right\} \subset M_n(Q)$$

having index $N = n^2$, called Popa subfactor.

PROOF. This is something standard, a bit as for the Jones, Ocneanu and Wassermann subfactors, with the result basically coming from the work of Popa, who was the main user of such subfactors. We will come in a moment with a more general result in this direction, involving discrete quantum groups, along with a complete proof. \Box

In order to unify now Theorem 4.4 and Proposition 4.8, observe that the diagonal subfactors can be written in the following way, by using a group dual:

$$(Q \rtimes \Gamma)^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}}$$

Here the group dual $\widehat{\Gamma}$ acts on $P = Q \rtimes \Gamma$ via the dual of the action $\Gamma \subset Aut(Q)$, and on $M_n(\mathbb{C})$ via the adjoint action of the following representation:

$$\oplus g_i:\widehat{\Gamma}\to\mathbb{C}^n$$

Summarizing, we are led into quantum groups. Our plan in what follows will be that of discussing the quantum extension of Theorem 4.4, covering the diagonal subfactors as well, and this time with full details, and with examples and illustrations as well.

4b. Quantum groups

As a starting point, we have the following key definition, due to Woronowicz [99]:

4B. QUANTUM GROUPS

DEFINITION 4.9. A Woronowicz algebra is a C^* -algebra A, given with a unitary matrix $v \in M_N(A)$ whose coefficients generate A, such that the formulae

$$\Delta(v_{ij}) = \sum_{k} v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

define morphisms of C^* -algebras $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$, $S : A \to A^{opp}$.

We say that A is cocommutative when $\Sigma \Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip. We have the following result, which justifies the terminology and axioms:

PROPOSITION 4.10. The following are Woronowicz algebras:

(1) C(G), with $G \subset U_N$ compact Lie group. Here the structural maps are:

$$\Delta(\varphi) = [(g,h) \to \varphi(gh)] \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S(\varphi) = [g \to \varphi(g^{-1})]$$

(2) $C^*(\Gamma)$, with $F_N \to \Gamma$ finitely generated group. Here the structural maps are:

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-\frac{1}{2}}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we have to indicate a certain matrix v. For the first assertion, we can use the matrix $v = (v_{ij})$ formed by matrix coordinates of G, given by:

$$g = \begin{pmatrix} v_{11}(g) & \dots & v_{1N}(g) \\ \vdots & & \vdots \\ v_{N1}(g) & \dots & v_{NN}(g) \end{pmatrix}$$

As for the second assertion, we can use here the diagonal matrix formed by generators:

$$v = \begin{pmatrix} g_1 & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case this follows from the Peter-Weyl theory, explained below. \Box

In view of Proposition 4.10, we can formulate the following definition:

DEFINITION 4.11. Given a Woronowicz algebra A, we formally write

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is both commutative and cocommutative, G is a compact abelian group, Γ is a discrete abelian group, and these groups are dual to each other:

$$G = \widehat{\Gamma} \quad , \quad \Gamma = \widehat{G}$$

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In general, we still agree to write the formulae $G = \widehat{\Gamma}, \Gamma = \widehat{G}$, but in a formal sense. Finally, let us make as well the following convention:

DEFINITION 4.12. We identify two Woronowicz algebras (A, v) and (B, w), as well as the corresponding quantum groups, when we have an isomorphism of *-algebras

 $\langle v_{ij} \rangle \simeq \langle w_{ij} \rangle$

mapping standard coordinates to standard coordinates.

This convention is here for avoiding amenability issues, as for any compact or discrete quantum group to correspond to a unique Woronowicz algebra. More on this later.

Moving ahead now, let us call corepresentation of A any unitary matrix $u \in M_n(\mathcal{A})$, where $\mathcal{A} = \langle v_{ij} \rangle$, satisfying the same conditions as those satisfied by u, namely:

$$\Delta(u_{ij}) = \sum_{k} u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}^*$$

We have the following key result, due to Woronowicz [99]:

THEOREM 4.13. Any Woronowicz algebra has a unique Haar integration functional,

$$\left(\int_{G} \otimes id\right) \Delta = \left(id \otimes \int_{G}\right) \Delta = \int_{G} (.)1$$

which can be constructed by starting with any faithful positive form $\varphi \in A^*$, and setting

$$\int_G = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi) \Delta$. Moreover, for any corepresentation $u \in M_n(\mathbb{C}) \otimes A$ we have

$$\left(id\otimes\int_G\right)u=P$$

where P is the orthogonal projection onto $Fix(u) = \{\xi \in \mathbb{C}^n | u\xi = \xi\}.$

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi \in A^*$, our claim is that the following limit converges, for any $a \in A$:

$$\int_{\varphi} a = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}(a)$$

Indeed, by linearity we can assume that $a \in A$ is the coefficient of certain corepresentation, $a = (\tau \otimes id)u$. But in this case, an elementary computation gives the following formula, with P_{φ} being the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)u$:

$$\left(id\otimes\int_{\varphi}\right)u=P_{\varphi}$$
(2) Since $u\xi = \xi$ implies $[(id \otimes \varphi)u]\xi = \xi$, we have $P_{\varphi} \ge P$, where P is the orthogonal projection onto the fixed point space in the statement, namely:

$$Fix(u) = \left\{ \xi \in \mathbb{C}^n \middle| u\xi = \xi \right\}$$

The point now is that when $\varphi \in A^*$ is faithful, by using a standard positivity trick, we can prove that we have $P_{\varphi} = P$, exactly as in the classical case.

(3) With the above formula in hand, the left and right invariance of $\int_G = \int_{\varphi}$ is clear on coefficients, and so in general, and this gives all the assertions. See [99].

We can now develop, again following [99], the Peter-Weyl theory for the corepresentations of A. Consider the dense subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation v, and endow it with the following scalar product:

$$< a, b > = \int_{G} ab^*$$

With this convention, we have the following result, from [99]:

THEOREM 4.14. We have the following Peter-Weyl type results:

- (1) Any corepresentation decomposes as a sum of irreducible corepresentations.
- (2) Each irreducible corepresentation appears inside a certain $v^{\otimes k}$.
- (3) $\mathcal{A} = \bigoplus_{u \in Irr(A)} M_{\dim(u)}(\mathbb{C})$, the summands being pairwise orthogonal.
- (4) The characters of irreducible corepresentations form an orthonormal system.

PROOF. All these results are from [99], the idea being as follows:

(1) Given $u \in M_n(A)$, the intertwiner algebra $End(u) = \{T \in M_n(\mathbb{C}) | Tu = uT\}$ is a finite dimensional C^* -algebra, and so decomposes as $End(u) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$. But this gives a decomposition of type $u = u_1 + \ldots + u_k$, as desired.

(2) Consider the Peter-Weyl corepresentations, $v^{\otimes k}$ with k colored integer, defined by $v^{\otimes \emptyset} = 1, v^{\otimes \circ} = v, v^{\otimes \bullet} = \bar{v}$ and multiplicativity. The coefficients of these corepresentations span the dense algebra \mathcal{A} , and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is a *-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id \otimes \int_G)u$ is the orthogonal projection P_u onto the space Fix(u), for any corepresentation u.

(4) Let us define indeed the character of $u \in M_n(A)$ to be the trace, $\chi_u = Tr(u)$. Since this character is a coefficient of u, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id \otimes \int_G)u = P_u$.

We can now solve a problem that we left open before, namely:

PROPOSITION 4.15. The cocommutative Woronowicz algebras appear as the quotients

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma)$$

given by $A = C^*_{\pi}(\Gamma)$ with $\pi \otimes \pi \subset \pi$, with Γ being a discrete group.

PROOF. This follows from the Peter-Weyl theory, and clarifies a number of things said before, notably in Proposition 4.10. Indeed, for a cocommutative Woronowicz algebra the irreducible corepresentations are all 1-dimensional, and this gives the results. \Box

As another consequence of the above results, once again by following Woronowicz [99], we have the following statement, dealing with functional analysis aspects, and extending what we already knew about the C^* -algebras of the usual discrete groups:

THEOREM 4.16. Let A_{full} be the enveloping C^* -algebra of \mathcal{A} , and A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:

- (1) The Haar functional of A_{full} is faithful.
- (2) The projection map $A_{full} \rightarrow A_{red}$ is an isomorphism.
- (3) The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red} .
- (4) We have $N \in \sigma(Re(\chi_v))$, the spectrum being taken inside A_{red} .

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) \iff (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red} .

(2) \iff (3) Here \implies is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type ($\varepsilon \otimes id$) Φ . See [99].

(3) \iff (4) Here \implies is clear, coming from $\varepsilon(N - Re(\chi(v))) = 0$, and the converse can be proved by doing some functional analysis. Once again, we refer here to [99]. \Box

Let us discuss now some new examples of quantum groups, which will play a key role in what follows, in relation with subfactors and planar algebras. Following Wang [89], we first have the following result, which is something quite straightforward:

PROPOSITION 4.17. The following universal algebras are Woronowicz algebras,

$$C(O_N^+) = C^* \left((v_{ij})_{i,j=1,\dots,N} \middle| v = \bar{v}, v^t = v^{-1} \right)$$
$$C(U_N^+) = C^* \left((v_{ij})_{i,j=1,\dots,N} \middle| v^* = v^{-1}, v^t = \bar{v}^{-1} \right)$$

so the underlying compact quantum spaces O_N^+, U_N^+ are compact quantum groups.

PROOF. This follows from the elementary fact that if a matrix $v = (v_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$v_{ij}^{\Delta} = \sum_{k} v_{ik} \otimes v_{kj} \quad , \quad v_{ij}^{\varepsilon} = \delta_{ij} \quad , \quad v_{ij}^{S} = v_{ji}^{*}$$

Thus, we can indeed define morphisms Δ, ε, S as in Definition 4.9, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$, and this gives the result.

There is a connection here with group duals, coming from:

PROPOSITION 4.18. Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:

$$C(T) = C(G) \Big/ \left\langle v_{ij} = 0 \middle| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda = \langle g_1, \ldots, g_N \rangle$ is the discrete group generated by the elements $g_i = v_{ii}$, which are unitaries inside C(T).

PROOF. Since u is unitary, its diagonal entries $g_i = v_{ii}$ are unitaries inside C(T). Moreover, from $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^{*}-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed.

With this notion in hand, we have the following result:

THEOREM 4.19. The diagonal tori of the basic rotation groups are as follows,



where F_N is the free group on N generators, and * is a group-theoretical free product.

PROOF. This is clear indeed from U_N^+ , and the other results can be obtained by imposing to the generators of F_N the relations defining the corresponding quantum groups. \Box

Getting now into more examples, we have the following key result:

THEOREM 4.20. The classical and free, real and complex quantum rotation groups can be complemented with quantum reflection groups, as follows,



with $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$ being the hyperoctahedral group and the full complex reflection group, and $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$ and $K_N^+ = \mathbb{T} \wr_* S_N^+$ being their free versions.

PROOF. This is something quite tricky, the idea being as follows:

(1) The first observation is that S_N , regarded as group of permutations of the N coordinate axes of \mathbb{R}^N , is a group of orthogonal matrices, $S_N \subset O_N$. The corresponding coordinate functions $v_{ij} : S_N \to \{0, 1\}$ form a matrix $v = (v_{ij})$ which is "magic", in the sense that its entries are projections, summing up to 1 on each row and each column. In fact, by using the Gelfand theorem, we have the following presentation result:

$$C(S_N) = C^*_{comm} \left((v_{ij})_{i,j=1,\dots,N} \middle| v = \text{magic} \right)$$

(2) Based on the above, and following Wang's paper [89], we can construct the free analogue S_N^+ of the symmetric group S_N via the following formula:

$$C(S_N^+) = C^*\left((v_{ij})_{i,j=1,\dots,N} \middle| v = \text{magic}\right)$$

Here the fact that we have indeed a Woronowicz algebra is standard, exactly as for the free rotation groups in Proposition 4.17, because if a matrix $v = (v_{ij})$ is magic, then so are the matrices $v^{\Delta}, v^{\varepsilon}, v^{S}$ constructed there, and this gives the existence of Δ, u, S .

(3) Consider now the group $H_N^s \subset U_N$ consisting of permutation-like matrices having as entries the *s*-th roots of unity. This group decomposes as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N$$

It is straightforward then to construct a free analogue $H_N^{s+} \subset U_N^+$ of this group, for instance by formulating a definition as follows, with \wr_* being a free wreath product:

$$H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

(4) In order to finish, besides the case s = 1, of particular interest are the cases $s = 2, \infty$. Here the corresponding reflection groups are as follows:

$$H_N = \mathbb{Z}_2 \wr S_N \quad , \quad K_N = \mathbb{T} \wr S_N$$

As for the corresponding quantum groups, these are denoted as follows:

$$H_N^+ = \mathbb{Z}_2 \wr_* S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_* S_N^+$$

Thus, we are led to the conclusions in the statement.

4c. Diagrams, easiness

Getting now towards easiness, let us start with the following definition:

DEFINITION 4.21. The Tannakian category associated to a Woronowicz algebra (A, v)is the collection $C_A = (C_A(k, l))$ of vector spaces

$$C_A(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

where the corepresentations $v^{\otimes k}$ with $k = \circ \bullet \circ \ldots$ colored integer, defined by

$$v^{\otimes \emptyset} = 1$$
 , $v^{\otimes \circ} = v$, $v^{\otimes \bullet} = \bar{v}$

and multiplicativity, $v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$, are the Peter-Weyl corepresentations.

As a key remark, the fact that $v \in M_N(A)$ is biunitary translates into the following conditions, where $R : \mathbb{C} \to \mathbb{C}^N \otimes \mathbb{C}^N$ is the linear map given by $R(1) = \sum_i e_i \otimes e_i$:

$$R \in Hom(1, v \otimes \bar{v}) \quad , \quad R \in Hom(1, \bar{v} \otimes v)$$
$$R^* \in Hom(v \otimes \bar{v}, 1) \quad , \quad R^* \in Hom(\bar{v} \otimes v, 1)$$

We are therefore led to the following abstract definition, summarizing the main properties of the categories appearing from Woronowicz algebras:

DEFINITION 4.22. Let H be a finite dimensional Hilbert space. A tensor category over H is a collection C = (C(k, l)) of subspaces

$$C(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

(1) $S, T \in C$ implies $S \otimes T \in C$.

- (2) If $S, T \in C$ are composable, then $ST \in C$.
- (3) $T \in C$ implies $T^* \in C$.
- (4) Each C(k, k) contains the identity operator.
- (5) $C(\emptyset, \circ \bullet)$ and $C(\emptyset, \bullet \circ)$ contain the operator $R: 1 \to \sum_i e_i \otimes e_i$.

The point now is that conversely, we can associate a Woronowicz algebra to any tensor category in the sense of Definition 4.22, in the following way:

PROPOSITION 4.23. Given a tensor category C = (C(k, l)) over \mathbb{C}^N , as above,

$$A_C = C^*\left((v_{ij})_{i,j=1,\dots,N} \middle| T \in Hom(v^{\otimes k}, v^{\otimes l}), \forall k, l, \forall T \in C(k,l)\right)$$

is a Woronowicz algebra.

PROOF. This is something standard, because the relations $T \in Hom(v^{\otimes k}, v^{\otimes l})$ determine a Hopf ideal, so they allow the construction of Δ, ε, S as in Definition 4.9.

With the above constructions in hand, we have the following result:

THEOREM 4.24. The Tannakian duality constructions

$$C \to A_C \quad , \quad A \to C_A$$

are inverse to each other, modulo identifying full and reduced versions.

PROOF. The idea is that we have $C \subset C_{A_C}$, for any algebra A, and so we are left with proving that we have, for any category C, an inclusion as follows:

 $C_{A_C} \subset C$

But this can be proved indeed, by performing a long series of algebraic manipulations, including a use of von Neumann's bicommutant theorem, and for details we refer to Malacarne [54], and also to Woronowicz [100], where this result was first proved. \Box

In practice now, all this is quite abstract, and we will rather need Brauer type results, for the specific quantum groups that we are interested in. Let us start with:

DEFINITION 4.25. Let P(k, l) be the set of partitions between an upper colored integer k, and a lower colored integer l. A collection of subsets

$$D = \bigsqcup_{k,l} D(k,l)$$

with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:

- (1) Stability under the horizontal concatenation, $(\pi, \sigma) \rightarrow [\pi\sigma]$.
- (2) Stability under vertical concatenation $(\pi, \sigma) \to [\frac{\sigma}{\pi}]$, with matching middle symbols.
- (3) Stability under the upside-down turning *, with switching of colors, $\circ \leftrightarrow \bullet$.
- (4) Each set P(k,k) contains the identity partition $|| \dots ||$.
- (5) The sets $P(\emptyset, \bullet \bullet)$ and $P(\emptyset, \bullet \circ)$ both contain the semicircle \cap .

Observe the similarity with the various axioms from Definition 4.22.

In fact, Definition 4.25 is concieved as to be a delinearized version of Definition 4.22, and the relation with the Tannakian categories comes from:

PROPOSITION 4.26. Given a partition $\pi \in P(k, l)$, consider the linear map

$$T_{\pi}: (\mathbb{C}^N)^{\otimes k} \to (\mathbb{C}^N)^{\otimes l}$$

given by the following formula, where e_1, \ldots, e_N is the standard basis of \mathbb{C}^N ,

$$T_{\pi}(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \dots j_l} \delta_{\pi} \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_{\pi} \in \{0, 1\}$ depending on whether the indices fit or not. The assignment $\pi \to T_{\pi}$ is then categorical, in the sense that we have

$$T_{\pi} \otimes T_{\sigma} = T_{[\pi\sigma]}$$
 , $T_{\pi}T_{\sigma} = N^{c(\pi,\sigma)}T_{[\pi]}$, $T_{\pi}^* = T_{\pi^*}$

where $c(\pi, \sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. The formulae in the statement are all elementary, as follows:

(1) The concatenation axiom follows from the following computation:

$$(T_{\pi} \otimes T_{\sigma})(e_{i_{1}} \otimes \ldots \otimes e_{i_{p}} \otimes e_{k_{1}} \otimes \ldots \otimes e_{k_{r}})$$

$$= \sum_{j_{1} \dots j_{q}} \sum_{l_{1} \dots l_{s}} \delta_{\pi} \begin{pmatrix} i_{1} & \dots & i_{p} \\ j_{1} & \dots & j_{q} \end{pmatrix} \delta_{\sigma} \begin{pmatrix} k_{1} & \dots & k_{r} \\ l_{1} & \dots & l_{s} \end{pmatrix} e_{j_{1}} \otimes \ldots \otimes e_{j_{q}} \otimes e_{l_{1}} \otimes \ldots \otimes e_{l_{s}}$$

$$= \sum_{j_{1} \dots j_{q}} \sum_{l_{1} \dots l_{s}} \delta_{[\pi\sigma]} \begin{pmatrix} i_{1} & \dots & i_{p} & k_{1} & \dots & k_{r} \\ j_{1} & \dots & j_{q} & l_{1} & \dots & l_{s} \end{pmatrix} e_{j_{1}} \otimes \ldots \otimes e_{j_{q}} \otimes e_{l_{1}} \otimes \ldots \otimes e_{l_{s}}$$

$$= T_{[\pi\sigma]}(e_{i_{1}} \otimes \ldots \otimes e_{i_{p}} \otimes e_{k_{1}} \otimes \ldots \otimes e_{k_{r}})$$

(2) The composition axiom follows from the following computation:

$$T_{\pi}T_{\sigma}(e_{i_{1}}\otimes\ldots\otimes e_{i_{p}})$$

$$=\sum_{j_{1}\ldots j_{q}}\delta_{\sigma}\begin{pmatrix}i_{1}&\ldots&i_{p}\\j_{1}&\ldots&j_{q}\end{pmatrix}\sum_{k_{1}\ldots k_{r}}\delta_{\pi}\begin{pmatrix}j_{1}&\ldots&j_{q}\\k_{1}&\ldots&k_{r}\end{pmatrix}e_{k_{1}}\otimes\ldots\otimes e_{k_{r}}$$

$$=\sum_{k_{1}\ldots k_{r}}N^{c(\pi,\sigma)}\delta_{[\frac{\sigma}{\pi}]}\begin{pmatrix}i_{1}&\ldots&i_{p}\\k_{1}&\ldots&k_{r}\end{pmatrix}e_{k_{1}}\otimes\ldots\otimes e_{k_{r}}$$

$$=N^{c(\pi,\sigma)}T_{[\frac{\sigma}{\pi}]}(e_{i_{1}}\otimes\ldots\otimes e_{i_{p}})$$

(3) Finally, the involution axiom follows from the following computation:

$$T_{\pi}^{*}(e_{j_{1}} \otimes \ldots \otimes e_{j_{q}})$$

$$= \sum_{i_{1} \ldots i_{p}} < T_{\pi}^{*}(e_{j_{1}} \otimes \ldots \otimes e_{j_{q}}), e_{i_{1}} \otimes \ldots \otimes e_{i_{p}} > e_{i_{1}} \otimes \ldots \otimes e_{i_{p}}$$

$$= \sum_{i_{1} \ldots i_{p}} \delta_{\pi} \begin{pmatrix} i_{1} & \ldots & i_{p} \\ j_{1} & \ldots & j_{q} \end{pmatrix} e_{i_{1}} \otimes \ldots \otimes e_{i_{p}}$$

$$= T_{\pi^{*}}(e_{j_{1}} \otimes \ldots \otimes e_{j_{q}})$$

Summarizing, our correspondence is indeed categorical.

In relation with quantum groups, we have the following result:

THEOREM 4.27. Each category of partitions D = (D(k, l)) produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the following formula:

$$Hom(v^{\otimes k}, v^{\otimes l}) = span\left(T_{\pi} \middle| \pi \in D(k, l)\right)$$

To be more precise, the spaces on the right form a Tannakian category, and so produce a certain closed subgroup $G_N \subset U_N^+$, via the Tannakian duality correspondence.

PROOF. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from Malacarne [54], as explained in Theorem 4.24. Indeed, let us set:

$$C(k,l) = span\left(T_{\pi} \middle| \pi \in D(k,l)\right)$$

By using the various axioms in Definition 4.25, and the categorical properties of the operation $\pi \to T_{\pi}$, from Proposition 4.26, we deduce that C = (C(k, l)) is a Tannakian category. Thus the Tannakian duality applies, and gives the result.

Philosophically speaking, the quantum groups appearing as in Theorem 4.27 are the simplest, from the perspective of Tannakian duality, so let us formulate:

DEFINITION 4.28. A closed subgroup $G \subset U_N^+$ is called easy when we have

$$Hom(v^{\otimes k}, v^{\otimes l}) = span\left(T_{\pi} \middle| \pi \in D(k, l)\right)$$

for any colored integers k, l, for a certain category of partitions $D \subset P$.

Getting now to examples, we have the following Brauer type result:

THEOREM 4.29. The basic quantum rotation and reflection groups,



are all easy, the corresponding categories of partitions being as follows,



with on top, the symbol NC standing everywhere for noncrossing partitions.

PROOF. This is something well-known and routine, as follows:

(1) Let us first discuss the easiness property of O_N^+, U_N^+ . The quantum group U_N^+ is by definition constructed via the following relations:

$$v^* = v^{-1}$$
 , $v^t = \bar{v}^{-1}$

Thus, the following operators must be in the associated Tannakian category C:

$$T_{\pi}$$
, $\pi = \bigcap_{\circ \bullet}$, T_{π} , $\pi = \bigcap_{\bullet \circ}$

It follows that the associated Tannakian category is $C = span(T_{\pi} | \pi \in D)$, with:

$$D = < \cap_{\circ \bullet} \cap_{\circ} = \mathcal{N}C_2$$

Now by imposing the extra relation $v = \bar{v}$, we obtain the easiness of O_N^+ as well.

(2) In what regards now H_N^+, K_N^+ , the first observation is that the magic condition satisfied by v can be reformulated as follows, with $Y \in P(2, 1)$ being the fork partition:

$$T_Y \in Hom(v^{\otimes 2}, v)$$

Now by proceeding as in the proof for U_N^+ discussed above, we conclude that the quantum group S_N^+ is indeed easy, the associated category of partitions being:

$$D = \langle NC_2, Y \rangle = NC$$

With this in hand, we can pass to the quantum groups H_N^+, K_N^+ in a standard way, and we are led to easiness, and the categories in the statement.

(3) Finally, we can pass from the upper face to the lower face of the cube by adding the basic crossing, and this produces the various categories in the statement. \square

4d. Actions, invariants

Good news, with the above general quantum group theory in hand, we can now go back to the generalized Wassermann subfactors and the Popa subactors, and unify them. Let us start our discussion with some basic action theory. We first have:

DEFINITION 4.30. A coaction of a Woronowicz algebra A on a finite von Neumann algebra P is an injective morphism $\Phi: P \to P \otimes A''$ satisfying the following conditions:

- (1) Coassociativity: $(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$.
- (2) Trace equivariance: $(tr \otimes id)\Phi = tr(.)1.$ (3) Smoothness: $\overline{\mathcal{P}}^w = P$, where $\mathcal{P} = \Phi^{-1}(P \otimes_{alg} \mathcal{A}).$

The above conditions come from what happens in the commutative case, A = C(G), where they correspond to the usual associativity, trace equivariance and smoothness of the corresponding action $G \curvearrowright P$. Along the same lines, we have as well:

DEFINITION 4.31. A coaction $\Phi: P \to P \otimes A''$ as above is called:

- (1) Ergodic, if the algebra $P^{\Phi} = \{p \in P | \Phi(p) = p \otimes 1\}$ reduces to \mathbb{C} .
- (2) Faithful, if the span of $\{(f \otimes id)\Phi(P) | f \in P_*\}$ is dense in A''.
- (3) Minimal, if it is faithful, and satisfies $(P^{\Phi})' \cap P = \mathbb{C}$.

Observe that the minimality of the action implies in particular that the fixed point algebra P^{Φ} is a factor. Thus, we are getting here to the case that we are interested in, actions producing factors, via their fixed point algebras. More on this later.

In order to prove our subfactor results, we need of some general theory regarding the minimal actions. Following Wassermann [90], let us start with the following definition:

DEFINITION 4.32. Let $\Phi: P \to P \otimes A''$ be a coaction. An eigenmatrix for a corepresentation $u \in B(H) \otimes A$ is an element $M \in B(H) \otimes P$ satisfying:

$$(id \otimes \Phi)M = M_{12}u_{13}$$

A coaction is called semidual if each corepresentation has a unitary eigenmatrix.

4D. ACTIONS, INVARIANTS

As a basic example here, the canonical coaction $\Delta : A \to A \otimes A$ is semidual. We will prove in what follows, following the work of Wassermann in the usual compact group case, that the minimal coactions of Woronowicz algebras are semidual. We first have:

PROPOSITION 4.33. If $\Phi: P \to P \otimes A''$ is a minimal coaction and $u \in Irr(A)$ is a corepresentation, then u has a unitary eigenmatrix precisely when $P^u \neq \{0\}$.

PROOF. Given $u \in M_n(A)$, consider the following unitary corepresentation:

$$u^+ = (n \otimes 1) \oplus u = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \in M_2(M_n(\mathbb{C}) \otimes \mathcal{A}) = M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes \mathcal{A}$$

Then, if the following algebra is a factor, u must have a unitary eigenmatrix:

$$X_u = (M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes P)^{\pi_u + 1}$$

So, let us prove that X_u is a factor. For this purpose, let $x \in Z(X_u)$. We have then $1 \otimes 1 \otimes P^{\Phi} \subset X_u$, and from the irreducibility of the inclusion $P^{\pi} \subset P$ we obtain that:

 $x \in M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes 1$

On the other hand, we have the following formula:

$$X_u \cap M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes 1 = End(u^+) \otimes 1$$

Since our corepresentation u was chosen to be irreducible, it follows that x must be of the following form, with $y \in M_n(\mathbb{C})$, and with $\lambda \in \mathbb{C}$:

$$x = \begin{pmatrix} y & 0\\ 0 & \lambda I \end{pmatrix} \otimes 1$$

Now let us pick a nonzero element $p \in P^u$, and write:

$$\Phi(p) = \sum_{ij} p_{ij} \otimes u_{ij}$$

Then $\Phi(p_{ij}) = \sum_k p_{kj} \otimes u_{ki}$ for any i, j, and so each column of $(p_{ij})_{ij}$ is a *u*-eigenvector. Choose such a nonzero column l and let m^i be the matrix having the *i*-th row equal to l, and being zero elsewhere. Then m_i is a *u*-eigenmatrix for any i, and this implies that:

$$\begin{pmatrix} 0 & m^i \\ 0 & 0 \end{pmatrix} \in X_u$$

The commutation relation of this matrix with x is as follows:

$$\begin{pmatrix} y & 0 \\ 0 & \lambda I \end{pmatrix} \begin{pmatrix} 0 & m^i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & m^i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & \lambda I \end{pmatrix}$$

But this gives $(y - \lambda I)m^i = 0$. Now by definition of m^i , this shows that the *i*-th column of $y - \lambda I$ is zero. Thus $y - \lambda I = 0$, and so $x = \lambda 1$, as desired.

We can now prove a main result about minimal coactions, as follows:

THEOREM 4.34. The minimal coactions are semidual.

PROOF. Let K be the set of finite dimensional unitary corepresentations of A which have unitary eigenmatrices. Then, according to the above, the following happen:

(1) K is stable under taking tensor products, because if M, N are eigenmatrices for u, w, then $M_{13}N_{23}$ is an eigenmatrix for $u \otimes w$. Also, K is stable under taking sums, because if M_i are eigenmatrices for u_i , then $diag(M_i)$ is an eigenmatrix for $\oplus u_i$.

(2) K is stable under substractions. Indeed, if M is an eigenmatrix for $U = \bigoplus_{i=1}^{n} u_i$, then the first dim (u_1) columns of M are formed by elements of P^{u_1} , the next dim (u_2) columns of M are formed by elements of P^{u_2} , and so on. Now if M is unitary, it is in particular invertible, so all P^{u_i} are different from $\{0\}$, and we may conclude that we can indeed substract corepresentations from U, by using Proposition 4.33.

(3) K is stable under complex conjugation. Indeed, if $u \in Irr(A)$ has a nonzero eigenmatrix M then \overline{M} is an eigenmatrix for \overline{u} . By Proposition 4.33 we obtain from this that $P^{\overline{u}} \neq \{0\}$, and we may conclude by using again Proposition 4.33.

Thus, the set K of corepresentations which have unitary eigenmatrices is stable by all the standard operations that can be performed on the finite dimensional unitary corepresentations, and with this in hand, by using Peter-Weyl, we obtain the result.

Let us construct now the fixed point subfactors. We first have:

PROPOSITION 4.35. Consider a Woronowicz algebra $A = (A, \Delta, S)$, and denote by A_{σ} the Woronowicz algebra $(A, \sigma \Delta, S)$, where σ is the flip. Given coactions

$$\beta: B \to B \otimes A$$
$$\pi: P \to P \otimes A_{\sigma}$$

with B being finite dimensional, the following linear map, while not being multiplicative in general, is coassociative with respect to the comultiplication $\sigma\Delta$ of A_{σ} ,

$$\beta \odot \pi : B \otimes P \to B \otimes P \otimes A_{\sigma}$$

$$b \otimes p \to \pi(p)_{23}((id \otimes S)\beta(b))_{13}$$

and its fixed point space, which is by definition the following linear space,

$$(B \otimes P)^{\beta \odot \pi} = \left\{ x \in B \otimes P \middle| (\beta \odot \pi) x = x \otimes 1 \right\}$$

is then a von Neumann subalgebra of $B \otimes P$.

PROOF. This is something standard, which follows from a straightforward algebraic verification, explained in [6]. As mentioned in the statement, to be noted is that the tensor product coaction $\beta \odot \pi$ is not multiplicative in general. See [6].

In order to construct now fixed point subfactors, our first task is to investigate the factoriality of such algebras, and we have here the following result:

THEOREM 4.36. If $\beta : B \to B \otimes A$ is a coaction and $\pi : P \to P \otimes A_{\sigma}$ is a minimal coaction, then the following conditions are equivalent:

- (1) The von Neumann algebra $(B \otimes P)^{\beta \odot \pi}$ is a factor.
- (2) The coaction β is centrally ergodic, $Z(B) \cap B^{\beta} = \mathbb{C}$.

PROOF. The first observation is that, according to Theorem 4.34, the following diagram is a non-degenerate commuting square:

$$\begin{array}{rcl} P & \subset & B \otimes P \\ \cup & & \cup \\ P^{\pi} & \subset & (B \otimes P)^{\beta \odot \pi} \end{array}$$

Thus, it is enough to check the following equality, between subalgebras of $B \otimes P$:

$$Z((B \otimes P)^{\beta \odot \pi}) = (Z(B) \cap B^{\beta}) \otimes 1$$

So, let x be in the algebra on the left. Then x commutes with $1 \otimes P^{\pi}$, so it has to be of the form $b \otimes 1$. Thus x commutes with $1 \otimes P$. But x commutes with $(B \otimes P)^{\beta \odot \pi}$, and from the non-degeneracy of the above square, x commutes with $B \otimes P$, and in particular with $B \otimes 1$. Thus we have $b \in Z(B) \cap B^{\beta}$. As for the other inclusion, this is obvious. \Box

We will need in what follows the following technical result:

PROPOSITION 4.37. Consider two commuting squares, as follows:

$$\begin{array}{ccccc} F & \subset & E & \subset & D \\ \cup & & \cup & & \cup \\ A & \subset & B & \subset & C \end{array}$$

- (1) If the square on the left and the big square are non-degenerate, then so is the square on the right.
- (2) If both squares are non-degenerate, $F \subset E \subset D$ is a basic construction, and the Jones projection $e \in D$ for this basic construction belongs to C, then the square on the right is the basic construction for the square on the left.

PROOF. The first assertion is clear from the following computation:

$$D = \overline{sp}^{w} CF = \overline{sp}^{w} CBF = \overline{sp}^{w} CE$$

Let $\Psi: D \to C$ be the expectation. By non-degeneracy, we have that:

$$E = \overline{sp}^w FB = \overline{sp}^w BF$$

We also have $D = \overline{sp}^w EeE$ by the basic construction, so we get that:

$$C = \Psi(D)$$

= $\Psi(\overline{sp}^w EeE)$
= $\Psi(\overline{sp}^w BFeFB)$
= $\Psi(\overline{sp}^w BeFB)$
= $\overline{sp}^w Be\Psi(F)B$
= $\overline{sp}^w BeAB$
= $\overline{sp}^w BeB$

Thus the algebra C is generated by B and e, and this gives the result.

Next in line, we have the following key technical result:

PROPOSITION 4.38. If $\beta : B \to B \otimes A$ is a coaction then

$$\begin{array}{rcl} A & \subset & B \otimes A \\ \cup & & \uparrow \beta \\ \mathbb{C} & \subset & B \end{array}$$

is a non-degenerate commuting square.

PROOF. From the β -equivariance of the trace we get that the inclusion on the left commutes with the traces. Then, from the formula $E_{\beta} = (id \otimes \int_{A})\beta$ we get that the above diagram is a commuting square. Choose now an orthonormal basis $\{b_i\}$ of B, write $\beta: b_i \to \sum_j b_j \otimes u_{ji}$, and consider the corresponding unitary corepresentation:

$$u_{\beta} = \sum e_{ij} \otimes u_{ij}$$

Then for any k and any $a \in A$ we have the following computation:

$$\sum_{i} \beta(b_i)(1 \otimes u_{ki}^* a) = \sum_{ij} b_j \otimes u_{ji} u_{ki}^* a = b_k \otimes a$$

Thus our commuting square is non-degenerate, as claimed.

Getting now to the generalized Wassermann subfactors, we first have:

PROPOSITION 4.39. Given a Markov inclusion of finite dimensional algebras $B_0 \subset B_1$, construct its Jones tower, and denote it as follows:

$$B_0 \subset B_1 \subset_{e_1} B_2 = \langle B_1, e_1 \rangle \subset_{e_2} B_3 = \langle B_2, e_2 \rangle \subset_{e_3} \dots$$

If $\beta_1 : B_1 \to B_1 \otimes A$ is a coaction/anticoaction leaving B_0 invariant then there exists a unique sequence $\{\beta_i\}_{i\geq 0}$ of coactions/anticoactions

$$\beta_i: B_i \to B_i \otimes A$$

such that each β_i extends β_{i-1} and leaves invariant the Jones projection e_{i-1} .

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PROOF. By taking opposite inclusions we see that the assertion for anticoactions is equivalent to the one for coactions, that we will prove now. The uniqueness is clear from $B_i = \langle B_{i-1}, e_{i-1} \rangle$. For the existence, we can apply Proposition 4.38 to:

$$\begin{array}{rcccc} A & \subset & B_0 \otimes A & \subset & B_1 \otimes A \\ \cup & & \uparrow \beta_0 & & \uparrow \beta_1 \\ \mathbb{C} & \subset & B_0 & \subset & B_1 \end{array}$$

Indeed, we get in this way that the square on the right is a non-degenerate. Now by performing basic constructions to it, we get a sequence as follows:

It is easy to see from definitions that the β_i are coactions, that they extend each other, and that they leave invariant the Jones projections. But this gives the result.

With the above technical results in hand, we can now formulate our main theorem regarding the fixed point subfactors, of the most possible general type, as follows:

THEOREM 4.40. Let G be a compact quantum group, and $G \to Aut(P)$ be a minimal action on a II₁ factor. Consider a Markov inclusion of finite dimensional algebras

 $B_0 \subset B_1$

and let $G \to Aut(B_1)$ be an action which leaves invariant B_0 and which is such that its restrictions to the centers of B_0 and B_1 are ergodic. We have then a subfactor

$$(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$$

of index $N = [B_1 : B_0]$, called generalized Wassermann subfactor, whose Jones tower is

$$(B_1 \otimes P)^G \subset (B_2 \otimes P)^G \subset (B_3 \otimes P)^G \subset \dots$$

where $\{B_i\}_{i\geq 1}$ are the algebras in the Jones tower for $B_0 \subset B_1$, with the canonical actions of G coming from the action $G \to Aut(B_1)$, and whose planar algebra is given by:

$$P_k = (B'_0 \cap B_k)^G$$

These subfactors generalize the Jones, Ocneanu, Wassermann and Popa subfactors.

PROOF. We have several things to be proved, the idea being as follows:

(1) The first part of the statement, regarding the factoriality, the index and the Jones tower assertions, is something that follows exactly as in the classical group case.

(2) In order to prove now the planar algebra assertion, consider the following diagram, with i < j being arbitrary integers:

$$\begin{array}{ccccc} P & \subset & B_i \otimes P & \subset & B_j \otimes P \\ \cup & & \cup & & \cup \\ P^{\pi} & \subset & (B_i \otimes P)^{\beta_i \otimes \pi} & \subset & (B_j \otimes P)^{\beta_j \otimes \pi} \end{array}$$

We know from Proposition 4.39 that the big square and the square on the left are both non-degenerate commuting squares. Thus Proposition 4.38 applies, and shows that the square on the right is a non-degenerate commuting square.

(3) Consider now the following sequence of non-degenerate commuting squares:

Since the Jones projections live in the lower line, Proposition 6.39 applies and shows that this is a sequence of basic constructions for non-degenerate commuting squares. In particular the lower line is a sequence of basic constructions, as desired. \Box

We discuss now some converses to the above results, which are rather specialized results, of Tannakian nature. Let us start with the following result:

THEOREM 4.41. Given a quantum permutation group $G \subset S_N^+$, consider the associated coaction map on C(X), where $X = \{1, \ldots, N\}$,

$$\Phi: C(X) \to C(X) \otimes C(G) \quad , \quad e_j \to \sum_j e_j \otimes u_{ji}$$

and then consider the tensor powers of this coaction, which are the following linear maps:

$$\Phi^k: C(X^k) \to C(X^k) \otimes C(G) \quad , \quad e_{i_1 \dots i_k} \to \sum_{j_1 \dots j_k} e_{j_1 \dots j_k} \otimes u_{j_1 i_1} \dots u_{j_k i_k}$$

The fixed point spaces of these latter coactions are then given by the formula

$$P_k = Fix(u^{\otimes k})$$

and form a planar subalgebra of the spin planar algebra \mathcal{S}_N .

PROOF. This is something which certainly follows from the above results regarding the Wassermann type subfactors, but can be deduced as well directly. \Box

As a second result now, completing our study, we have:

THEOREM 4.42. Given a subalgebra $Q \subset S_N$, there is a unique quantum group

$$G \subset S_N^+$$

whose associated planar algebra is Q.

PROOF. The idea is that this will follow by applying Tannakian duality to the annular category over Q. Let n, m be positive integers. To any element $T_{n+m} \in Q_{n+m}$ we can associate a linear map $L_{nm}(T_{n+m}) : P_n(X) \to P_m(X)$ in the following way:

$$L_{nm}\begin{pmatrix} | & | & | \\ T_{n+m} \\ | & | & | \end{pmatrix} : \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \rightarrow \begin{pmatrix} | & | & \cap \\ T_{n+m} | \\ | & | & | \\ | & | & | \\ a_n | & | & | \\ \cup & | & | \end{pmatrix}$$

That is, we consider the planar (n, n + m, m)-tangle having an small input *n*-box, a big input n + m-box and an output *m*-box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X) \otimes P_{n+m}(X) \to P_m(X)$$

Now let us put the element T_{n+m} in the big input box. We obtain in this way a certain linear map $P_n(X) \to P_m(X)$, that we call L_{nm} . Now let us set:

$$Q_{nm} = \left\{ L_{nm}(T_{n+m}) : P_n(X) \to P_m(X) \middle| T_{n+m} \in Q_{n+m} \right\}$$

These spaces form a Tannakian category, and so by [100] we obtain a Woronowicz algebra (A, u), such that the following equalities hold, for any m, n:

$$Hom(u^{\otimes m}, u^{\otimes n}) = Q_{mn}$$

We prove that u is a magic unitary. We have $Hom(1, u^{\otimes 2}) = Q_{02} = Q_2$, so the unit of Q_2 must be a fixed vector of $u^{\otimes 2}$. But $u^{\otimes 2}$ acts on the unit of Q_2 as follows:

$$u^{\otimes 2}(1) = \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (uu^t)_{kl}$$

From $u^{\otimes 2}(1) = 1 \otimes 1$ ve get that uu^t is the identity matrix, and together with the unitarity of u, this gives $u^t = u^* = u^{-1}$. Consider now the Jones projection $E_1 \in Q_3$. The linear map $M = L_{21}(E_1)$ is the multiplication $\delta_i \otimes \delta_j \to \delta_{ij}\delta_i$, and we have:

$$(M \otimes id)u^{\otimes 2} \left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) = \sum_{k} \begin{pmatrix} k \\ k \end{pmatrix} \delta_{k} \otimes u_{ki}u_{kj}$$
$$u(M \otimes id) \left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) = \sum_{k} \begin{pmatrix} k \\ k \end{pmatrix} \delta_{k} \otimes \delta_{ij}u_{ki}$$

Thus $u_{ki}u_{kj} = \delta_{ij}u_{ki}$ for any i, j, k, and we deduce from this that u is a magic unitary. Now if P is the planar algebra associated to u, we have $Hom(1, v^{\otimes n}) = P_n = Q_n$, as desired. As for the uniqueness, this is clear from the Peter-Weyl theory from [99]. \Box

The above results, following old papers from the early 00s, explained in [6], regarding the subgroups $G \subset S_N^+$, have several generalizations, to the subgroups $G \subset O_N^+$ and $G \subset U_N^+$, as well as subfactor versions, going beyond the purely combinatorial level. For the modern story, we refer here to Tarrago-Wahl [78] and related papers.

4e. Exercises

Exercises:

EXERCISE 4.43.

EXERCISE 4.44. EXERCISE 4.45.

Exercise 4.46.

Exercise 4.47.

EXERCISE 4.48.

Exercise 4.49.

EXERCISE 4.50.

Bonus exercise.

Part II

Invariants

I was standing by the Nile When I saw the lady smile I would take her out for a while For a while

CHAPTER 5

Higher commutants

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

Exercise 5.1.

EXERCISE 5.2.

Exercise 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

Exercise 5.6.

EXERCISE 5.7.

EXERCISE 5.8.

Bonus exercise.

CHAPTER 6

Planar algebras

6a. Planar algebras

The Temperley-Lieb category that we met in chapter 1 is more than a category, it is a planar algebra. In order to explain this fact, which will be of key importance in what follows, following Jones [42], let us start with the following general definition:

DEFINITION 6.1. The planar algebras are defined as follows:

- (1) We consider rectangles in the plane, with the sides parallel to the coordinate axes, and taken up to planar isotopy, and we call such rectangles boxes.
- (2) A labeled box is a box with 2n marked points on its boundary, n on its upper side, and n on its lower side, for some integer $n \in \mathbb{N}$.
- (3) A tangle is labeled box, containing a number of labeled boxes, with all marked points, on the big and small boxes, being connected by noncrossing strings.
- (4) A planar algebra is a sequence of finite dimensional vector spaces $P = (P_n)$, together with linear maps $P_{n_1} \otimes \ldots \otimes P_{n_k} \to P_n$, one for each tangle, such that the gluing of tangles corresponds to the composition of linear maps.

In this definition we are using rectangles, but everything being up to isotopy, we could have used instead circles with marked points, as in [42]. Our choice for using rectangles comes from the main examples that we have in mind, to be discussed below, where the planar algebra structure is best viewed by using rectangles, as above.

This being said, when convenient, we agree to use circles with marked points for the outer box, or for the inner boxes, or for both, with the convention that the marked point is the lower left corner of the rectangle. Here is a planar tangle, drawn in this way, with the marked points on both circles being by definition those at South-West:



And, exercise for you to see what this tangle becomes, in rectangular notation.

Getting back now to what Definition 6.1 says, in relation with the tangle pictured above, that tangle has two inner boxes, having respectively $2 \times 2 = 4$ and $2 \times 3 = 6$ marked points on their boundaries, and the outer box has $2 \times 4 = 8$ marked points on its boundary. Thus, that tangle π must produce a linear map as follows:

$$T_{\pi}: P_2 \otimes P_3 \to P_4$$

You get the point, I hope, Definition 6.1 is something very useful in the context of algebra, in order to index various possible operations on a sequence of finite dimensional vector spaces $P = (P_n)$, by diagrams as above. Of course, all this remains very vague for the moment, but we will see many examples and illustrations, in what follows.

Getting now to the essence of Definition 6.1, that lies in the axiom (4) there, compatibility of the gluing of the tangles with the composition of the multilinear maps. We will comment on this later, once we will have some examples of planar algebras. In the meantime, let us mention that it is possible to be more abstract here, by talking about the planar operad, and planar algebras as modules over this operad. But again, we will comment on this later, once we will have some examples of planar algebras.

Finally, let us mention now that Definition 6.1 is something quite simplified. As explained in [42], in order for subfactors to produce planar algebras and vice versa, there are quite a number of supplementary axioms that must be added. More on this later.

But probably too much talking, let us see some illustrations for this. As a first, very basic example of a planar algebra, we have the Temperley-Lieb algebra:

THEOREM 6.2. The Temperley-Lieb algebra TL_N , viewed as graded algebra

$$TL_N = (TL_N(n))_{n \in \mathbb{N}}$$

is a planar algebra, with the corresponding linear maps associated to the planar tangles

$$TL_N(n_1) \otimes \ldots \otimes TL_N(n_k) \to TL_N(n)$$

appearing by putting the various $TL_N(n_i)$ diagrams into the small boxes of the given tangle, which produces a $TL_N(n)$ diagram.

PROOF. This is something trivial, which follows from definitions:

(1) Assume indeed that we are given a planar tangle π , as in Definition 6.1, consisting of a box having 2n marked points on its boundary, and containing k small boxes, having respectively $2n_1, \ldots, 2n_k$ marked points on their boundaries, and then a total of $n + \sum n_i$ noncrossing strings, connecting the various $2n + \sum 2n_i$ marked points.

(2) We want to associate to this tangle π a linear map as follows:

$$T_{\pi}: TL_N(n_1) \otimes \ldots \otimes TL_N(n_k) \to TL_N(n)$$

For this purpose, by linearity, it is enough to construct elements as follows, for any choice of Temperley-Lieb diagrams $\sigma_i \in TL_N(n_i)$, with $i = 1, \ldots, k$:

$$T_{\pi}(\sigma_1 \otimes \ldots \otimes \sigma_k) \in TL_N(n)$$

(3) But constructing such an element is obvious, just by putting the various diagrams $\sigma_i \in TL_N(n_i)$ into the small boxes the given tangle π . Indeed, this procedure produces a certain diagram in $TL_N(n)$, that we can call $T_{\pi}(\sigma_1 \otimes \ldots \otimes \sigma_k)$, as above.

(4) Finally, we have to check that everything is well-defined up to planar isotopy, and that the gluing of tangles corresponds to the composition of linear maps. But both these checks are trivial, coming from the definition of TL_N , and we are done.

As a conclusion to all this, $P = TL_N$ is indeed a planar algebra, but of somewhat "trivial" type, with the triviality coming from the fact that, in this case, the elements of P are planar diagrams themselves, and so the planar structure appears trivially.

The Temperley-Lieb planar algebra TL_N is however an important planar algebra, because it is the "smallest" one, appearing inside the planar algebra of any subfactor. But more on this later, when talking about planar algebras and subfactors.

Moving ahead now, here is our second basic example of a planar algebra, which is also "trivial" in the above sense, with the elements of the planar algebra being planar diagrams themselves, but which appears in a bit more complicated way:

THEOREM 6.3. The Fuss-Catalan algebra $FC_{N,M}$, obtained by coloring the Temperley-Lieb diagrams with black and white colors, clockwise, as follows,

• ● • • • • • • • • ● • •

and keeping those diagrams whose strings connect either $\circ - \circ$ or $\bullet - \bullet$, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

 $FC_{N,M}(n_1) \otimes \ldots \otimes FC_{N,M}(n_k) \to FC_{N,M}(n)$

appearing by putting the various $FC_{N,M}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N,M}(n)$ diagram.

PROOF. The proof here is nearly identical to the proof of Theorem 6.2, with the only change appearing at the level of the colors. To be more precise:

(1) Forgetting about upper and lower sequences of points, which must be joined by strings, a Temperley-Lieb diagram can be thought of as being a collection of strings, say black strings, which compose in the obvious way, with the rule that the value of the circle, which is now a black circle, is N. And it is this obvious composition rule that gives the planar algebra structure, as explained in the proof of Theorem 6.2.

(2) Similarly, forgetting about points, a Fuss-Catalan diagram can be thought of as being a collection of strings, which come now in two colors, black and white. These Fuss-Catalan diagrams compose then in the obvious way, with the rule that the value of the black circle is N, and the value of the white circle is M. And it is this obvious composition rule that gives the planar algebra structure, as before for TL_N .

Even more generally now, we can talk about the multicolored Fuss-Catalan algebra, generalizing both the Temperley-Lieb and Fuss-Catalan algebras, as follows:

THEOREM 6.4. The multicolored Fuss-Catalan algebra $FC_{N_1,...,N_s}$, obtained by coloring the Temperley-Lieb diagrams with s colors, clockwise, as follows,

$$1 \dots ss \dots 11 \dots ss \dots 1 \dots \dots 1 \dots ss \dots 1$$

and keeping those diagrams whose strings connect i - i, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

$$FC_{N_1,\ldots,N_s}(n_1)\otimes\ldots\otimes FC_{N_1,\ldots,N_s}(n_k)\to FC_{N_1,\ldots,N_s}(n)$$

appearing by putting the various $FC_{N_1,...,N_s}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N_1,...,N_s}(n)$ diagram.

PROOF. This is a straightforward remake of Theorems 6.2 and 6.3, which correspond respectively to the cases s = 1, 2, with the only thing that must be added being the fact that the values of the circles of colors $1, \ldots, s$ are respectively the numbers N_1, \ldots, N_s . And with this we are led, as before, to the conclusions in the statement.

Getting back now to generalities, and to Definition 6.1 as stated, that of a general planar algebra, we have so far a few illustrations for it, which, while all important, are all "trivial", with the planar structure simply coming from the fact that, in all the above cases, the elements of the planar algebra are planar diagrams themselves.

In general, the planar algebras can be more complicated than this, and we will see some further examples in a moment. However, the idea is very simple, namely:

PRINCIPLE 6.5. The elements of a planar algebra are not necessarily diagrams, but they behave like diagrams.

And important principle this is. If there is something to be known, in order to understand planar algebras, and the whole quantum algebra theory based on them, it is this principle. But, do we really understand this principle? Not yet, because as already mentioned, our examples so far of planar algebras, namely Temperley-Lieb and Fuss-Catalan, are both "trivial", with the elements of the planar algebra being themselves diagrams.

Nevermind. We will get to understand this principle, via more examples, and via some theory too. Please be sure that once this book read, Principle 6.5 will be understood.

6B. BASIC TANGLES

6b. Basic tangles

What is next? Instead of looking right away for further examples, which can be substantially more complicated than Temperley-Lieb and Fuss-Catalan, let us enjoy what we have. To be more precise, with these two basic examples in hand, Temperley-Lieb and Fuss-Catalan, let us try to say more about the arbitrary planar algebras, as in Definition 6.1, with a bit of inspiration from what happens for these examples.

To start with, we have a number of remarkable planar tangles, whose algebraic action must be well understood, before anything. The first basic tangle is as follows:

EXAMPLE 6.6. The identity tangle is the following tangle, with 2n outer legs,



and this tangle must act via the identity, $T_{\pi}(x) = x$, for any $x \in P_n$.

To be more precise here, consider the tangle in the statement, π . Since applying this tangle obviously does nothing, this tangle must act via the identity map, as stated.

As a more interesting example now, bringing an associative algebra structure to each of the vector spaces P_n that our planar algebra is made of, we have:

EXAMPLE 6.7. The multiplication tangle is as follows, with 2n outer legs,



and this must implement a multiplication map, $T_{\pi}(x \otimes y) = xy$, for any $x, y \in P_n$.

Again, this is something quite self-explanatory, the idea being that the tangle in the statement, or rather its action on P_n , must be an associative multiplication.

Along the same lines, bringing more basic structure to our sequence of vector spaces $P = (P_n)$, which are now a sequence of associative algebras $P = (P_n)$, we have:

EXAMPLE 6.8. The inclusion tangle is as follows, with 2n + 2 outer legs,



and this tangle must act via an inclusion, $T_{\pi}(x) = x$, for any $x \in P_n$.

Again, this is something quite self-explanatory, the idea being that the tangle in the statement, or rather its action $P_n \to P_{n+1}$, must be an inclusion of algebras.

As a conclusion to all this, we already have some interesting structure on our planar algebras, getting well beyond what is totally obvious from Definition 6.1, as follows:

CONCLUSION 6.9. Any planar algebra $P = (P_n)$ is naturally a graded associative algebra over the complex numbers, with multiplication and inclusion maps coming from the action of the multiplication and inclusion tangles, pictured above.

Which looks quite interesting, especially in view of the fact that, due to this coming from the study of some trivial tangles, this can only be the tip of the iceberg. So, let us explore some more what the basic tangles are, and what can be done with them.

Coming first in our second batch of examples, we have:

EXAMPLE 6.10. The expectation tangle is as follows, with 2n outer legs,



and this tangle must act via an expectation, $T_{\pi}: P_{n+1} \to P_n$.

To be more precise, this is something a bit more advanced, the idea here being that the linear map $T_{\pi}: P_{n+1} \to P_n$ associated to the above expectation tangle must be a section, and bimodule map, with respect to the canonical inclusion of algebras $P_n \subset P_{n+1}$, that we constructed before. We will be back to this, with more details, later.

Along the same lines, again at the level of more specialized tangles, we have:

EXAMPLE 6.11. The Jones projection tangle is as follows, with 2n outer legs,



and this tangle corresponds to a rescaled projection $T_{\pi} \in P_n$.

Again, this is something quite self-explanatory, the idea being that, with no inner box present, the Jones projection tangle must simply correspond to a certain element $T_{\pi} \in P_n$. But this element must be an idempotent, up to a N factor, as said above.

Very good all this, so let us upgrade Conclusion 6.9, as follows:

CONCLUSION 6.12 (upgrade). Any planar algebra $P = (P_n)$ is naturally a graded associative algebra, via the action of the multiplication and inclusion tangles, and in addition we have, a bit as for the Temperley-Lieb algebra, expectations and Jones projections.

As already mentioned in the above, in what concerns the last part, regarding the expectations and the Jones projections, this is something a bit more specialized, and definitely in need of more discussion. We will come back to this, a bit later.

Moving ahead, let us discuss now a third batch of basic planar tangles, that we will heavily use as well in what follows. First we have the rotation, which is as follows:

EXAMPLE 6.13. The rotation tangle is as follows, with 2n outer legs,



and this tangle must act via a rotation $T_{\pi}: P_n \to P_n$.

Again, this is something quite self-explanatory, the idea being that the linear map $T_{\pi}: P_n \to P_n$ associated to the above rotation tangle must produce the identity, when raised to the power n, a bit like the usual rotation in the plane, of angle $2\pi/n$, does.

As a last basic tangle, we have the shift, which is constructed as follows:

EXAMPLE 6.14. The shift tangle is as follows, with 2n + 2 outer legs,



and this tangle must act via a shift, $T_{\pi}: P_n \to P_{n+2}$.

6C. TENSOR AND SPIN

Again, this is something quite self-explanatory, and with the remark of course that the shift is not to be confused with the double inclusion map $P_n \to P_{n+2}$. We will get back to this, shift and its properties, with more details, later in this chapter.

As a grand conclusion now to what we did so far, we have:

CONCLUSION 6.15 (final upgrade). Any planar algebra $P = (P_n)$ is naturally a graded associative algebra, and in addition we have, a bit as for the Temperley-Lieb algebra, or for the Fuss-Catalan one, expectations, Jones projections, rotations and shifts.

Which is good knowledge, and we will be back to this, with further details, later on. In any case, we can see here some good evidence for what we said in Principle 6.5, namely that the elements of a planar algebra are not necessarily diagrams, but behave like diagrams. And, more on this on several occassions, in what follows.

Getting back now to theory, we have the following remarkable result, which is something that we will heavily use, in what follows, for all sorts of purposes:

THEOREM 6.16. The following tangles generate the set of all tangles, via gluing:

- (1) Multiplications.
- (2) Inclusions.
- (3) Expectations.
- (4) Jones projections.
- (5) Rotations, or shifts.

PROOF. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list:

(1) To start with, in what regards the list itself, this is the one coming from the above examples, with the identities, which bring nothing to our generation problem, removed.

(2) As a subtlety now, at the end we have a choice, between the rotation and the shift. This is something quite important for the applications, which come in two flavors.

(3) As for the proof, as indicated above, both the results, the one with rotations, and the one with shifts, follow by chopping the tangles, in the obvious way. See [42]. \Box

There are many more things that can be said, along these lines, that is, generalities and basic algebra, in relation with Definition 6.1. We will be back to this later.

6c. Tensor and spin

Let us discuss now some further examples of planar algebras, which are of less trivial nature than TL_N and $FC_{N,M}$, and are of particular interest in relation with algebra and topology. These are the tensor and spin planar algebras $\mathcal{T}_N, \mathcal{S}_N$. Let us start with:

DEFINITION 6.17. The tensor planar algebra \mathcal{T}_N is the sequence of vector spaces $P_k = M_N(\mathbb{C})^{\otimes k}$

with the multilinear maps $T_{\pi}: P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by the formula

$$T_{\pi}(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_{\pi}(i_1, \ldots, i_r : j)e_j$$

with the Kronecker symbols δ_{π} being 1 if the indices fit, and being 0 otherwise.

In other words, we put the indices of the basic tensors on the marked points of the small boxes, in the obvious way, and the coefficients of the output tensor are then given by Kronecker symbols, $\delta_{\pi} \in \{0, 1\}$, which are themselves defined as follows:

 $-\delta_{\pi} = 1$ when all strings join pairs of equal indices.

 $-\delta_{\pi}=0$ otherwise.

The fact that we have indeed a planar algebra is something elementary, and for full details here, we refer to Jones' paper [42]. As illustrations for all this, we have:

EXAMPLE 6.18. Identity.

We recall that the identity 1_k is the (k, k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x, y) = 1$ being the pairs of the form (x, x), this tangle acts as follows:

$$1_k \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix} = \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix}$$

But this action is the identity, as it should.

EXAMPLE 6.19. Multiplication.

The multiplication M_k is the (k, k, k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. This tangle acts in the following way:

$$M_k\left(\begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix} \otimes \begin{pmatrix} l_1 & \cdots & l_k \\ m_1 & \cdots & m_k \end{pmatrix}\right) = \delta_{j_1m_1} \cdots \delta_{j_km_k}\begin{pmatrix} l_1 & \cdots & l_k \\ i_1 & \cdots & i_k \end{pmatrix}$$

Again, this action is the multiplication, as it should.

EXAMPLE 6.20. Inclusion.

The inclusion I_k is the (k, k+1)-tangle which looks like 1_k , but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x, y) = 1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} = \sum_l \begin{pmatrix} j_1 & \dots & j_k & l \\ i_1 & \dots & i_k & l \end{pmatrix}$$

Once again, what we have here is what we can expect from an inclusion.

EXAMPLE 6.21. Expectation.

The expectation U_k is the (k+1, k)-tangle which looks like 1_k , but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k \begin{pmatrix} j_1 & \cdots & j_k & j_{k+1} \\ i_1 & \cdots & i_k & i_{k+1} \end{pmatrix} = \delta_{i_{k+1}j_{k+1}} \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix}$$

This map satisfies then the usual requirements for an expectation.

EXAMPLE 6.22. Jones projection.

The Jones projection E_k is a (0, k+2)-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1) = \sum_{ijl} \begin{pmatrix} i_1 & \dots & i_k & j & j \\ i_1 & \dots & i_k & l & l \end{pmatrix}$$

The elements $e_k = N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra S_N .

EXAMPLE 6.23. Rotation.

The rotation R_k is the (k, k)-tangle which looks like 1_k , but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k = \| \begin{array}{c} \| & | & | & | \\ R_k = \| \\ \| & | & | \\ \| & | & | & | \\ \end{array}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2...i_k}) = e_{i_2...i_ki_1}$$

Thus, what we have indeed is a rotation map.

EXAMPLE 6.24. Shift.

As for the shift S_k , this is the (k, k+2)-tangle which looks like 1_k , but has two more vertical strings, at left of the input box. This tangle acts as follows:

$$S_k \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} = \sum_{lm} \begin{pmatrix} l & m & j_1 & \dots & j_k \\ l & m & i_1 & \dots & i_k \end{pmatrix}$$

Observe that S_k is an inclusion of algebras, which is different from $I_{k+1}I_k$.

Finally, in order for our discussion to be complete, we must talk as well about the *-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix}^* = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$$

As before, we refer to Jones' paper [42] for more on all this.

Let us discuss now a second planar algebra of the same type, which is important as well for various reasons, namely the spin planar algebra \mathcal{S}_N . This planar algebra appears somewhat as a "square root" of the tensor planar algebra \mathcal{T}_N , and its construction is quite similar, but by using this time the algebra \mathbb{C}^N instead of the algebra $M_N(\mathbb{C})$.

There is one subtlety, however, coming from the fact that the general planar algebra formalism, from Definition 6.1, requires the tensors to have even length. Note that this was automatic for the tensor planar \mathcal{T}_N , where the tensors of $M_N(\mathbb{C})$ have length 2. In the case of the spin planar algebra \mathcal{S}_N , we want the vector spaces to be:

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

Thus, we must double the indices of the tensors, in the following way:

DEFINITION 6.25. We write the standard basis of $(\mathbb{C}^N)^{\otimes k}$ in $2 \times k$ matrix form,

$$e_{i_1\dots i_k} = \begin{pmatrix} i_1 & i_1 & i_2 & i_2 & i_3 & \dots & \\ i_k & i_k & i_{k-1} & \dots & \dots & \dots \end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra S_N , also from [42], as follows:

DEFINITION 6.26. The spin planar algebra \mathcal{S}_N is the sequence of vector spaces

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

written as above, with the multiplinear maps $T_{\pi}: P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by

$$T_{\pi}(e_{i_1}\otimes\ldots\otimes e_{i_r})=\sum_j \delta_{\pi}(i_1,\ldots,i_r:j)e_j$$

with the Kronecker symbols δ_{π} being 1 if the indices fit, and being 0 otherwise.

In other words, we are using exactly the same construction as for the tensor planar algebra \mathcal{T}_N , but with $M_N(\mathbb{C})$ replaced by \mathbb{C}^N , with the indices doubled, as in Definition 6.25. As before with the tensor planar algebra \mathcal{T}_N , the fact that the spin planar algebra \mathcal{S}_N is indeed a planar algebra is something rather trivial, coming from definitions.

6C. TENSOR AND SPIN

Observe however that, unlike our previous planar algebras TL_N and $FC_{N,M}$, which were "trivial" planar algebras, their elements being planar diagrams themselves, the planar algebras \mathcal{T}_N and \mathcal{S}_N are not trivial, their elements being not exactly planar diagrams.

Let us also mention that the tensor and spin planar algebras \mathcal{T}_N and \mathcal{S}_N are important for a number of reasons, in the context of group theory, algebra and topology, to be discussed later, at the end of the present chapter, and later on too.

Getting back now to the planar algebra structure of \mathcal{T}_N and \mathcal{S}_N , which is something quite fundamental, worth being well understood, let us have here some more discussion. Generally speaking, the planar calculus for tensors is quite simple, and does not really require diagrams. Indeed, it suffices to imagine that the way various indices appear, travel around and dissapear is by following some obvious strings connecting them.

Here are some illustrations for this general principle, for the spin planar algebra S_N , in relation with the various basic planar tangles, that we know well:

EXAMPLE 6.27. Identity.

The identity 1_k is the (k, k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x, y) = 1$ being the pairs of the form (x, x), this tangle 1_k acts as follows:

$$1_k \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix} = \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix}$$

But this action is the identity, as it should.

EXAMPLE 6.28. Multiplication.

The multiplication M_k is the (k, k, k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. This tangle acts in the following way:

$$M_k\left(\begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix} \otimes \begin{pmatrix} l_1 & \cdots & l_k \\ m_1 & \cdots & m_k \end{pmatrix}\right) = \delta_{j_1m_1} \cdots \delta_{j_km_k}\begin{pmatrix} l_1 & \cdots & l_k \\ i_1 & \cdots & i_k \end{pmatrix}$$

Again, this action is the multiplication, as it should. Observe that, in the present context of the spin planar algebra, this multiplication is commutative.

EXAMPLE 6.29. Inclusion.

The inclusion I_k is the (k, k+1)-tangle which looks like 1_k , but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x, y) = 1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix} = \sum_l \begin{pmatrix} j_1 & \cdots & j_k & l \\ i_1 & \cdots & i_k & l \end{pmatrix}$$

Once again, what we have here is what we can expect from an inclusion.

EXAMPLE 6.30. Expectation.

The expectation U_k is the (k+1, k)-tangle which looks like 1_k , but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k \begin{pmatrix} j_1 & \cdots & j_k & j_{k+1} \\ i_1 & \cdots & i_k & i_{k+1} \end{pmatrix} = \delta_{i_{k+1}j_{k+1}} \begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix}$$

This map satisfies then the usual requirements for an expectation.

EXAMPLE 6.31. Jones projection.

The Jones projection E_k is a (0, k+2)-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1) = \sum_{ijl} \begin{pmatrix} i_1 & \dots & i_k & j & j \\ i_1 & \dots & i_k & l & l \end{pmatrix}$$

The elements $e_k = N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra S_N .

EXAMPLE 6.32. Rotation.

The rotation R_k is the (k, k)-tangle which looks like 1_k , but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k = \left\| \begin{array}{c} & \| & | & | & \| \\ \\ & \| & \| \\ & \| & | & | \\ \end{array} \right.$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2\dots i_k}) = e_{i_2\dots i_ki_1}$$

Thus, what we have indeed is a rotation map.

EXAMPLE 6.33. Shift.

As for the shift S_k , this is the (k, k+2)-tangle which looks like 1_k , but has two more vertical strings, at left of the input box. This tangle acts as follows:

$$S_k \begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix} = \sum_{lm} \begin{pmatrix} l & m & j_1 & \dots & j_k \\ l & m & i_1 & \dots & i_k \end{pmatrix}$$

Observe that S_k is an inclusion of algebras, which is different from $I_{k+1}I_k$.
Finally, in order for our discussion to be complete, we must talk as well about the *-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix}^* = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$$

As before, we refer to Jones' paper [42] for more on all this.

As a perhaps quite obvious question, appearing from the above, we have:

QUESTION 6.34. Is there a general construction of planar algebras, generalizing both the tensor and the spin algebra constructions?

In answer, sure yes, but this will have to wait a bit. The idea indeed is that:

(1) With suitable definitions, the tensor planar algebra appears to be the planar algebra associated to the inclusion $\mathbb{C} \subset M_N(\mathbb{C})$, and the spin planar algebra appears to be the planar algebra associated to the inclusion $\mathbb{C} \subset \mathbb{C}^N$. Thus, as a natural generalization of both these situations, we can look at the planar algebra associated to an inclusion of type $\mathbb{C} \subset B$, with B being an arbitrary finite dimensional algebra.

(2) However, the story is not over here, because for a number of reasons to be become clear later on, basically coming from Jones' notion of basic construction [40], which is the main workhorse when doing quantum algebra, or at least the present type of quantum algebra, it is actually most convenient to introduce directly the planar algebras associated to inclusions of type $A \subset B$, with both A, B being finite dimensional algebras.

(3) But, and here comes the point, while understanding what a finite dimensional algebra A is, in the present complex and involutive algebra context, is not that a big deal, understanding what an inclusion $A \subset B$ of such algebras is is something more complicated, and all in all, the above-mentioned construction of planar algebras associated to such inclusions $A \subset B$ remains something quite complicated, that we will defer for later.

In short, this is the situation, patience and modesty, we are currently learning a bit of this and a bit of that, regarding the planar algebras, because this is how planar algebras are best learned, and once we will learn enough things, in each possible direction, do not worry, we will start something more systematic. Including fully answering Question 6.34, with this being scheduled not very far from now, later in this book.

6. PLANAR ALGEBRAS

6d. Subfactor algebras

The above results raise the question on whether any planar algebra produces a subfactor. The answer here is yes, but with many subtleties, as follows:

THEOREM 6.35. We have the following results:

- (1) Any planar algebra with positivity produces a subfactor.
- (2) In particular, we have TL and FC type subfactors.
- (3) In the amenable case, and with $A_1 = R$, the correspondence is bijective.
- (4) In general, we must take $A_1 = L(F_{\infty})$, and we do not have bijectivity.
- (5) The axiomatization of P, in the case $A_1 = R$, is not known.

PROOF. This is something quite heavy, and for a discussion here, we refer to [42]. We will be back to this, with proofs, on several occasions, in the remainder of this book. \Box

6e. Exercises

Exercises:

EXERCISE 6.36.

Exercise 6.37.

EXERCISE 6.38.

EXERCISE 6.39.

EXERCISE 6.40.

EXERCISE 6.41.

EXERCISE 6.42.

EXERCISE 6.43.

Bonus exercise.

The correspondence

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

EXERCISE 7.2.

Exercise 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

EXERCISE 7.7.

Exercise 7.8.

Universal models

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

Exercise 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

Exercise 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Part III

Finite depth

And Michelle, what will she do Without you, Lady Madeleine And I walk down the avenue And I'm missing you, Lady Madeleine

Commuting squares

9a. Commuting squares

A first question to be discussed in the present chapter, and later on too, is the explicit construction of subfactors by using some suitable combinatorial data, encoded in a structure called "commuting square". Let us start with the following definition:

DEFINITION 9.1. A commuting square in the sense of subfactor theory is a commuting diagram of finite dimensional algebras with traces, as follows,



having the property that the conditional expectations $C_{11} \rightarrow C_{01}$ and $C_{11} \rightarrow C_{10}$ commute, and their product is the conditional expectation $C_{11} \rightarrow C_{00}$.

This notion is in fact something that we already talked about before, when discussing the classification of the finite depth subfactors, following the work of Ocneanu [64], [65] and Popa [69], [70]. To be more precise, it is possible to prove that any finite depth subfactor of R appears from a commuting square, and vice versa. And as a well-known consequence of this, the subfactors of R having index < 4, which are all of finite depth, can be shown to be classified by ADE diagrams. But more on this later.

Getting back now to Definition 9.1 as it is, something quite simple, not obviously subfactor related, there are many examples of such commuting squares, always coming from subtle combinatorial data. In order to discuss this, let us start with:

THEOREM 9.2. Up to a conjugation by a unitary, the pairs of orthogonal MASA in the simplest factor, namely $M_N(\mathbb{C})$, are as follows,

$$A = \Delta$$
 , $B = H\Delta H^*$

with $\Delta \subset M_N(\mathbb{C})$ being the diagonal matrices, and with $H \in M_N(\mathbb{C})$ being Hadamard, in the sense that $|H_{ij}| = 1$ for any i, j, and the rows of H are pairwise orthogonal.

PROOF. Any maximal abelian subalgebra in $M_N(\mathbb{C})$ being conjugated to Δ , we can assume, up to conjugation by a unitary, that we have, with $U \in U_N$:

$$A = \Delta$$
 , $B = U\Delta U^*$

But a straightforward, elementary computation shows that the orthogonality condition reformulates as $|U_{ij}| = 1/\sqrt{N}$, which gives the result.

Along the same lines, we have the following more precise result:

THEOREM 9.3. Given an Hadamard matrix $H \in M_N(\mathbb{C})$, the diagram formed by the associated pair of orthogonal maximal abelian subalgebras of $M_N(\mathbb{C})$,



where $\Delta \subset M_N(\mathbb{C})$ are the diagonal matrices, is a commuting square.

PROOF. The expectation $E_{\Delta} : M_N(\mathbb{C}) \to \Delta$ is the operation $M \to M_{\Delta}$ which consists in keeping the diagonal, and erasing the rest. Consider now the other expectation:

$$E_{H\Delta H^*}: M_N(\mathbb{C}) \to H\Delta H^*$$

It is better to identify this with the following expectation, with $U = H/\sqrt{N}$:

$$E_{U\Delta U^*}: M_N(\mathbb{C}) \to U\Delta U$$

This must be given by a formula of type $M \to UX_{\Delta}U^*$, with X satisfying:

$$\langle M, UDU^* \rangle = \langle UX_{\Delta}U^*, UDU^* \rangle , \quad \forall D \in \Delta$$

The scalar products being given by $\langle a, b \rangle = tr(ab^*)$, this condition reads:

$$tr(MUD^*U^*) = tr(X_{\Delta}D^*) \quad , \quad \forall D \in \Delta$$

Thus $X = U^*MU$, and the formulae of our two expectations are as follows:

$$E_{\Delta}(M) = M_{\Delta}$$
$$E_{U\Delta U^*}(M) = U(U^*MU)_{\Delta}U^*$$

With these formulae in hand, we have the following computation:

$$(E_{\Delta}E_{U\Delta U^*}M)_{ij} = \delta_{ij}(U(U^*MU)_{\Delta}U^*)_{ii}$$

$$= \delta_{ij}\sum_k U_{ik}(U^*MU)_{kk}\bar{U}_{ik}$$

$$= \delta_{ij}\sum_k \frac{1}{N} \cdot (U^*MU)_{kk}$$

$$= \delta_{ij}tr(U^*MU)$$

$$= \delta_{ij}tr(M)$$

$$= (E_{\mathbb{C}}M)_{ij}$$

As for the other composition, the computation here is similar, as follows:

$$(E_{U\Delta U^*}E_{\Delta}M)_{ij} = (U(U^*M_{\Delta}U)_{\Delta}U^*)_{ij}$$

$$= \sum_k U_{ik}(U^*M_{\Delta}U)_{kk}\bar{U}_{jk}$$

$$= \sum_{kl} U_{ik}\bar{U}_{lk}M_{ll}U_{lk}\bar{U}_{jk}$$

$$= \frac{1}{N}\sum_{kl} U_{ik}M_{ll}\bar{U}_{jk}$$

$$= \delta_{ij}tr(M)$$

$$= (E_{\mathbb{C}}M)_{ij}$$

Thus, we have indeed a commuting square, as claimed.

Getting back now to Definition 9.1 as it is, there are many other explicit examples of commuting squares, all coming from subtle combinatorial data, and more on this later. So, leaving aside now examples, let us explain the connection with subfactors. For this purpose, consider an arbitrary commuting square, as in Definition 9.1:



The point is that, under some suitable extra mild assumptions, any such square C produces a subfactor of the hyperfinite II₁ factor R. Indeed, by performing the basic

construction, in finite dimensions, we obtain a whole array, as follows:



To be more precise, by performing the basic construction in both possible directions, namely to the right and upwards, we obtain a whole array of finite dimensional algebras with traces, that we can denote $(C_{ij})_{i,j\geq 0}$, as above. Once this done, we can further consider the von Neumann algebras obtained in the limit, via GNS construction, on each vertical and horizontal line, and denote them A_i, B_j , as above.

With this convention, we have the following result, due to Ocneanu [64], [65]:

THEOREM 9.4. In the context of the above diagram, the limiting von Neumann algebras A_i, B_j are all isomorphic to the hyperfinite II₁ factor R, and:

- (1) $A_0 \subset A_1$ is a subfactor, and $\{A_i\}$ is the Jones tower for it.
- (2) The corresponding planar algebra is given by $A'_0 \cap A_k = C'_{01} \cap C_{k0}$.
- (3) A similar result holds for the "horizontal" subfactor $B_0 \subset B_1$.

PROOF. This is something very standard, with the factoriality of the limiting von Neumann algebras A_i, B_j coming as a consequence of the general commutant computation in (2), which is independent from it, with the hyperfiniteness of the same A_i, B_j algebras being clear by definition, and with the idea for the rest being as follows:

(1) This is somewhat clear from definitions, or rather from a quick verification of the basic construction axioms, as formulated in chapter 3, because the tower of algebras $\{A_i\}$ appears by definition as the $j \to \infty$ limit of the towers of algebras $\{C_{ij}\}$, which are all Jones towers. Thus the limiting tower $\{A_i\}$ is also a Jones tower.

(2) This is the non-trivial result, called Ocneanu compactness theorem, and whose proof is by doing some linear algebra. To be more precise, in one sense the result is clear, because by definition of the algebras $\{A_i\}$, we have inclusions as follows:

$$A'_0 \cap A_k \supset C'_{01} \cap C_{k0}$$

In the other sense things are more tricky, mixing standard linear algebra with some functional analysis too, and we refer here to Ocneanu's lecture notes [64], [65].

(3) This follows from (1,2), by transposing the whole diagram. Indeed, given a commuting square as in Definition 9.1, its transpose is a commuting square as well:



Thus we can apply (1,2) above to this commuting square, and we obtain in this way Jones tower and planar algebra results for the "horizontal" subfactor $B_0 \subset B_1$.

In relation with the examples of commuting squares that we have so far, namely those coming from the Hadamard matrices, from Theorem 9.3, we can upgrade what we have so far into something more conceptual, due to Jones [42], as follows:

THEOREM 9.5. Given a complex Hadamard matrix $H \in M_N(\mathbb{C})$, the diagram formed by the associated pair of orthogonal maximal abelian subalgebras, namely



is a commuting square in the sense of subfactor theory, and the associated planar algebra $P = (P_k)$ is given by the following formula, in terms of H itself,

$$T \in P_k \iff T^{\circ}G^2 = G^{k+2}T^{\circ}$$

where the objects on the right are constructed as follows:

- (1) $T^{\circ} = id \otimes T \otimes id.$ (2) $G_{ia}^{jb} = \sum_{k} H_{ik} \bar{H}_{jk} \bar{H}_{ak} H_{bk}.$ (3) $G_{i_1...i_k,j_1...j_k}^k = G_{i_k i_{k-1}}^{j_k j_{k-1}} \dots G_{i_2 i_1}^{j_2 j_1}$

PROOF. We have several assertions here, the idea being as follows:

(1) The fact that we have indeed a commuting square is something quite elementary, that we already know, from Theorem 9.3.

(2) The computation of the associated planar algebra, directly in terms of H, is something which is definitely possible, thanks to the formula in Theorem 9.4(2).

(3) As for the precise formula of the planar algebra, which emerges by doing the computation, we will be back to it, with full details, later on.

(4) The point indeed is that we want to first develop some better methods in dealing with the Hadamard matrices, and leave the computation of P for later.

9b. Matrix models

Our objective now is to clarify the planar algebra computation for the commuting squares coming from Hadamard matrices, from Theorem 9.5. Our claim is that all this is related, and in a beautiful way, to the quantum permutation groups.

In order to discuss this, and to present as well some generalizations, we will need some preliminaries on the quantum permutation groups, and their matrix models. Let us start with something straightforward, namely:

DEFINITION 9.6. A matrix model for a Woronowicz algebra A = C(G) is a morphism of C^* -algebras of the following type,

$$\pi: C(G) \to M_K(C(T))$$

with T being a compact space, and $K \ge 1$ being an integer.

It is quite clear that assuming that π is faithful leads to the conclusion that C(G) must be a type I algebra, and so that G must be coamenable, and with this being something quite restrictive, excluding for instance all the free quantum groups.

The solution to this problem comes from a weaker notion of faithfulness, called "inner faithfulness", which still allows to recover the combinatorics of G from the combinatorics of the model, but does not potentially exclude any quantum group. We have indeed:

DEFINITION 9.7. Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.

- (1) The Hopf image of π is the smallest quotient Hopf C^* -algebra $C(G) \to C(H)$ producing a factorization of type $\pi : C(G) \to C(H) \to M_K(C(T))$.
- (2) When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.

Here the existence and uniqueness of the Hopf image are standard, coming by dividing the algebra C(G) by a suitable ideal, although we will come in a moment with an explicit Tannakian construction as well. As a basic illustration for these notions, we have two main examples, which are somehow dual to each other, as follows:

(1) In the case where $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation $\rho : \Gamma \to C(T, U_K)$. We conclude that in this case, the minimal factorization constructed in Definition 9.7 is simply the one obtained by taking the image:

$$\rho: \Gamma \to \Lambda \subset C(T, U_K)$$

Thus π is inner faithful when our group satisfies $\Gamma \subset C(T, U_K)$. And we can see here that π , while not being faithful, clearly reminds all of Γ , and so of $G = \widehat{\Gamma}$ too.

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(2) As a second illustration, given a compact group G, and elements $g_1, \ldots, g_K \in G$, we have a representation $\pi : C(G) \to \mathbb{C}^K$, given by $f \to (f(g_1), \ldots, f(g_K))$. The minimal factorization of π is then via C(H), with $H \subset G$ being the following subgroup:

$$H = \overline{\langle g_1, \ldots, g_K \rangle}$$

Thus π is inner faithful precisely when $G = \overline{\langle g_1, \ldots, g_K \rangle}$. Again, we can see here that π , while not being faithful, clearly reminds all of G, and so of $\Gamma = \widehat{G}$ too.

Summarizing, our notion of inner faithfulness does the job, reminding the quantum groups G and $\Gamma = \hat{G}$, and not excluding anything on functional analysis grounds. Which brings us into the question of recapturing the algebraic and analytic properties of G and $\Gamma = \hat{G}$ out the combinatorics of the model. Regarding algebra, we have here:

THEOREM 9.8. Assuming $G \subset U_N^+$, with fundamental corepresentation $u = (u_{ij})$, the Hopf image of $\pi : C(G) \to M_K(C(T))$ comes from the following Tannakian category,

$$C_{kl} = Hom(U^{\otimes k}, U^{\otimes l})$$

where $U_{ij} = \pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

PROOF. This is something standard, which can be actually used as a definition for the Hopf image. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k}, u^{\otimes l}) \subset Hom(U^{\otimes k}, U^{\otimes l})$$

More generally, we have such inclusions when replacing (G, u) with any pair producing a factorization of π . Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions. On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H, v) given by:

$$Hom(v^{\otimes k}, v^{\otimes l}) = Hom(U^{\otimes k}, U^{\otimes l})$$

By the above discussion, C(H) follows to be the Hopf image of π , as claimed.

In what regards now analysis, the result here is as follows:

THEOREM 9.9. Given an inner faithful model $\pi : C(G) \to M_K(C(T))$, we have

$$\int_G = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^k \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$, with $\varphi = tr \otimes \int_T$ being the random matrix trace.

PROOF. Again, this is something very standard. If we denote by \int_G' the limit in the statement, we must prove that this limit converges, and that we have:

$$\int_{G}' = \int_{G}$$

It is enough to check this on the coefficients of corepresentations, and if we let $v = u^{\otimes k}$ be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left(id\otimes\int_{G}'\right)v=\left(id\otimes\int_{G}\right)v$$

We know from chapter 4 that the matrix on the right is the orthogonal projection onto Fix(v). Regarding now the matrix on the left, this is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi \pi)v$. Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have:

$$(id \otimes \varphi \pi)v = (id \otimes \varphi)V$$

Thus, as in chapter 4, we conclude that the 1-eigenspace that we are interested in equals Fix(V). But, according to Theorem 9.8, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired.

9c. Hadamard models

With the above in hand, let us go back now to our von Neumann algebra and subfactor questions. In relation with the complex Hadamard matrices, the connection with the quantum permutations is immediate, coming from the following observation:

PROPOSITION 9.10. If $H \in M_N(\mathbb{C})$ is Hadamard, the rank one projections

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

PROOF. This is clear, the verification for the rows being as follows:

$$\left\langle \frac{H_i}{H_j}, \frac{H_i}{H_k} \right\rangle = \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{kl}}{H_{il}} = \sum_l \frac{H_{kl}}{H_{jl}} = N\delta_{ik}$$

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As for the verification for the columns, this is similar, as follows:

$$\left\langle \frac{H_i}{H_j}, \frac{H_k}{H_j} \right\rangle = \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{jl}}{H_{kl}} = \sum_l \frac{H_{il}}{H_{kl}} = N\delta_{ik}$$

Thus, we have indeed a magic unitary, as claimed.

We are led in this way into the following notion:

DEFINITION 9.11. To any Hadamard matrix $H \in M_N(\mathbb{C})$ we associate the quantum permutation group $G \subset S_N^+$ given by the following Hopf image factorization,



where $\pi(u_{ij}) = Proj(H_i/H_j)$, with $H_1, \ldots, H_N \in \mathbb{T}^N$ being the rows of H.

Our claim now is that this construction $H \to G$ is something really useful, with the quantum group G encoding the combinatorics of H. To be more precise, the idea will be that "H can be thought of as being a kind of Fourier matrix for G". As an illustration for this principle, we first have the following result:

THEOREM 9.12. The construction $H \to G$ has the following properties:

- (1) For a Fourier matrix $H = F_G$ we obtain the group G itself, acting on itself.
- (2) For $H \notin \{F_G\}$, the quantum group G is not classical, nor a group dual.
- (3) For a tensor product $H = H' \otimes H''$ we obtain a product, $G = G' \times G''$.

PROOF. All this material is standard, and elementary, as follows:

(1) Let us first discuss the cyclic group case, $H = F_N$. Here the rows of H are given by $H_i = \rho^i$, where $\rho = (1, w, w^2, \dots, w^{N-1})$. Thus, we have the following formula:

$$\frac{H_i}{H_j} = \rho^{i-j}$$

It follows that the corresponding rank 1 projections $P_{ij} = Proj(H_i/H_j)$ form a circulant matrix, all whose entries commute. Since the entries commute, the corresponding quantum group must satisfy $G \subset S_N$. Now by taking into account the circulant property of $P = (P_{ij})$ as well, we are led to the conclusion that we have $G = \mathbb{Z}_N$, as claimed.

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In the general case now, where $H = F_G$, with G being an arbitrary finite abelian group, the result can be proved either by extending the above proof, of by decomposing $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$ and using (3) below, whose proof is independent from the rest.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative. For details here, along with a number of supplementary facts on the construction $H \to G$, we refer to [6].

(3) Assume that we have a tensor product $H = H' \otimes H''$, and let G, G', G'' be the associated quantum permutation groups. We have then a diagram as follows:



Here all the maps are the canonical ones, with those on the left and on the right coming from N = N'N''. At the level of standard generators, the diagram is as follows:



Now observe that this diagram commutes. We conclude that the representation associated to H factorizes indeed through $C(G') \otimes C(G'')$, and this gives the result.

In order to discuss now the relation with the commuting squares and the subfactors, we can use Theorem 9.8, and we are led to the following result:

THEOREM 9.13. The Tannakian category of the quantum group $G \subset S_N^+$ associated to a complex Hadamard matrix $H \in M_N(\mathbb{C})$ is given by

$$T \in Hom(u^{\otimes k}, u^{\otimes l}) \iff T^{\circ}G^{k+2} = G^{l+2}T^{\circ}$$

where the objects on the right are constructed as follows:

(1) $T^{\circ} = id \otimes T \otimes id.$ (2) $G_{ia}^{jb} = \sum_{k} H_{ik} \bar{H}_{jk} \bar{H}_{ak} H_{bk}.$ (3) $G_{i_1...i_k,j_1...j_k}^k = G_{i_k i_{k-1}}^{j_k j_{k-1}} \dots G_{i_2 i_1}^{j_2 j_1}$

PROOF. According to Theorem 9.8, and with the notations there, we have the following formula for the Tannakian category that we are interested in:

$$Hom(u^{\otimes k}, u^{\otimes l}) = Hom(U^{\otimes k}, U^{\otimes l})$$

The vector space on the right, that we will compute now, consists by definition of the complex $N^l \times N^k$ matrices T satisfying the following relation:

$$TU^{\otimes k} = U^{\otimes l}T$$

If we denote this equality by L = R, the left term L is given by:

$$L_{ij} = (TU^{\otimes k})_{ij}$$

= $\sum_{a} T_{ia} U_{aj}^{\otimes k}$
= $\sum_{a} T_{ia} U_{a_1 j_1} \dots U_{a_k j_k}$

As for the right term R, this is given by a similar formula, as follows:

$$R_{ij} = (U^{\otimes l}T)_{ij}$$

= $\sum_{b} U^{\otimes l}_{ib}T_{bj}$
= $\sum_{b} U_{i_1b_1} \dots U_{i_lb_l}T_{bj}$

Consider now the vectors $\xi_{ij} = H_i/H_j$. Since these vectors span the ambient Hilbert space, the equality L = R is equivalent to the following equality:

$$< L_{ij}\xi_{pq}, \xi_{rs} > = < R_{ij}\xi_{pq}, \xi_{rs} >$$

We use now the following well-known formula, expressing a product of rank one projections P_1, \ldots, P_k in terms of the corresponding image vectors ξ_1, \ldots, ξ_k :

$$< P_1 \dots P_k x, y > = < x, \xi_k > < \xi_k, \xi_{k-1} > \dots < \xi_2, \xi_1 > < \xi_1, y >$$

This gives the following formula for the left term L:

$$< L_{ij}\xi_{pq}, \xi_{rs} > = \sum_{a} T_{ia} < P_{a_{1}j_{1}} \dots P_{a_{k}j_{k}}\xi_{pq}, \xi_{rs} >$$

$$= \sum_{a} T_{ia} < \xi_{pq}, \xi_{a_{k}j_{k}} > \dots < \xi_{a_{1}j_{1}}, \xi_{rs} >$$

$$= \sum_{a} T_{ia} G_{pa_{k}}^{qj_{k}} G_{a_{k}a_{k-1}}^{j_{k}j_{k-1}} \dots G_{a_{2}a_{1}}^{j_{2}j_{1}} G_{a_{1}r}^{j_{1}s}$$

$$= \sum_{a} T_{ia} G_{rap,sjq}^{k+2}$$

$$= (T^{\circ}G^{k+2})_{rip,sjq}$$

As for the right term R, this is given by the following formula:

$$< R_{ij}\xi_{pq}, \xi_{rs} > = \sum_{b} < P_{i_{1}b_{1}} \dots P_{i_{l}b_{l}}\xi_{pq}, \xi_{rs} > T_{bj}$$

$$= \sum_{b} < \xi_{pq}, \xi_{i_{l}b_{l}} > \dots < \xi_{i_{1}b_{1}}, \xi_{rs} > T_{bj}$$

$$= \sum_{b} G_{pi_{l}}^{qb_{l}}G_{i_{l}i_{l-1}}^{b_{l}b_{l-1}} \dots G_{i_{2}i_{1}}^{b_{2}b_{1}}G_{i_{1}r}^{b_{1}s}T_{bj}$$

$$= \sum_{b} G_{rip,sbq}^{l+2}T_{bj}$$

$$= (G^{l+2}T^{\circ})_{rip,sjq}$$

Thus, we obtain the formula in the statement.

The point now is that, with k = 0, we obtain in this way precisely the planar algebra spaces P_l computed by Jones in [42], for the corresponding commuting square, described in Theorem 9.5. Thus, we are led in this way to the following result:

THEOREM 9.14. Let $H \in M_N(\mathbb{C})$ be a complex Hadamard matrix. (1) The planar algebra associated to H is given by the formula

 $P_k = Fix(u^{\otimes k})$

where $G \subset S_N^+$ is the associated quantum permutation group. (2) The Poincaré series $\sum_k \dim(P_k) z^k$ equals the Stieltjes transform

$$f(z) = \int_G \frac{1}{1 - z\chi}$$

of the law of the main character $\chi = \sum_{i} u_{ii}$.

PROOF. This follows as indicated above, by putting together what we have:

(1) As already mentioned above, this simply follows by comparing Theorem 9.13 with the subfactor computation in [42], discussed in Theorem 9.5.

(2) This follows from (1) and from the Peter-Weyl theory, with the statement itself being a nice and concrete application of our main result, (1) above.

Summarizing, in connection with the commuting square problematics from the beginning of this chapter, the conclusion is that for the simplest such commuting squares, namely those coming from Hadamard matrices, the combinatorics ultimately comes from quantum permutation groups. This is something nice, and exploring improvements and generalizations of this will be our main purpose, in the remainder of this chapter.

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9D. FIXED POINTS

9d. Fixed points

We know that the planar algebra associated to an Hadamard matrix $H \in M_N(\mathbb{C})$ appears in fact as the planar algebra associated to a certain related quantum permutation group $G \subset S_N^+$. In view of the various results from chapters 3-4, this suggests that the subfactor itself associated to H should appear as a fixed point subfactor associated to G. We will prove here that this is indeed the case. To be more precise, following [5] and subsequent papers, regarding the subfactor itself, the result here is as follows:

THEOREM 9.15. The subfactor associated to $H \in M_N(\mathbb{C})$ is of the form

$$A^G \subset (\mathbb{C}^N \otimes A)^G$$

with $A = R \rtimes \widehat{G}$, where $G \subset S_N^+$ is the associated quantum permutation group.

PROOF. This is something more technical, the idea being that the basic construction procedure for the commuting squares, explained before Theorem 9.4, can be performed in an "equivariant setting", for commuting squares having components as follows:

$$D \otimes_G E = (D \otimes (E \rtimes \widehat{G}))^G$$

To be more precise, starting with a commuting square formed by such algebras, we obtain by basic construction a whole array of commuting squares as follows, with $\{D_i\}, \{E_i\}$ being by definition Jones towers, and with D_{∞}, E_{∞} being their inductive limits:



The point now is that this quantum group picture works in fact for any commuting square having \mathbb{C} in the lower left corner. In the Hadamard matrix case, that we are

interested in here, the corresponding commuting square is as follows:



Thus, the subfactor obtained by vertical basic construction appears as follows:

$$\mathbb{C}\otimes_G E_\infty \subset \mathbb{C}^N \otimes_G E_\infty$$

But this gives the conclusion in the statement, with the II₁ factor appearing there being by definition $A = E_{\infty} \rtimes \hat{G}$, and with the remark that we have $E_{\infty} \simeq R$.

All the above was of course quite brief, but we will discuss now all this with more details, directly in a more general setting, covering the Hadamard matrix situation. To be more precise, our claim is that the above fixed point subfactor techniques apply, more generally, to the commuting squares having \mathbb{C} in the lower left corner:



In order to discuss this, let us go back to the fixed point subfactors, from chapter 4. In what concerns the fixed point factors, we know from there that we have:

THEOREM 9.16. Consider a Woronowicz algebra $A = (A, \Delta, S)$, and denote by A_{σ} the Woronowicz algebra $(A, \sigma\Delta, S)$, where σ is the flip. Given coactions

$$\beta: B \to B \otimes A$$
$$\pi: P \to P \otimes A_{\sigma}$$

with B being finite dimensional, the following linear map, while not being multiplicative in general, is coassociative with respect to the comultiplication $\sigma\Delta$ of A_{σ} ,

$$\beta \odot \pi : B \otimes P \to B \otimes P \otimes A_{\sigma}$$

$$b \otimes p \to \pi(p)_{23}((id \otimes S)\beta(b))_{13}$$

and its fixed point space, which is by definition the following linear space,

$$(B \otimes P)^{\beta \odot \pi} = \left\{ x \in B \otimes P \, \middle| \, (\beta \odot \pi) x = x \otimes 1 \right\}$$

is then a von Neumann subalgebra of $B \otimes P$. Moreover, such algebras can be used in order to construct the generalized Wassermann subfactors, $(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$.

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PROOF. This is something that we know from chapter 4, and for details, and comments in relation with the non-multiplicativity of $\beta \odot \pi$, we refer to the material there.

Let $\int_A : A \to \mathbb{C}$ be the Haar functional, let $l^2(A)$ be its l^2 -space and let $\widehat{A} \subset B(l^2(A))$ be the dual algebra. If $\alpha : E \to E \otimes \widehat{A}$ is a coaction of \widehat{A} on a finite von Neumann algebra E, the crossed product $E \rtimes_{\alpha} \widehat{A}$ is the von Neumann subalgebra of $E \otimes B(l^2(A))$ generated by $\alpha(E)$ and by $1 \otimes A$. There exists a unique coaction $\widehat{\alpha}$ of A on $E \rtimes_{\alpha} \widehat{A}$ such that $(E \rtimes_{\alpha} \widehat{A})^{\widehat{\alpha}} = \alpha(E)$, and such that the copy $1 \otimes A$ of A is equivariant. With these conventions, again following [5] and subsequent papers, we have the following result:

PROPOSITION 9.17. Let A be a Woronowicz algebra. If $\beta : D \to D \otimes A$ is a coaction on a finite dimensional finite von Neumann algebra and $\alpha : E \to E \otimes \widehat{A}_{\sigma}$ is a coaction on a finite von Neumann algebra then we have the equality

$$(D \otimes (E \rtimes_{\alpha} \widehat{A}_{\sigma}))^{\beta \odot \widehat{\alpha}} = \overline{sp}^{w} \Big\{ \beta(D)_{13} \cdot \alpha(E)_{23} \Big\}$$

as linear subspaces of $D \otimes E \otimes B(l^2(A_{\sigma}))$. Moreover, the following diagram

$$\begin{array}{rcl} \alpha(E)_{23} &\subset & (D \otimes (E \rtimes_{\alpha} \widehat{A}_{\sigma}))^{\beta \odot \widehat{\alpha}} \\ \cup & & \cup \\ \mathbb{C} &\subset & \beta(D)_{13} \end{array}$$

is a non-degenerate commuting square of finite von Neumann algebras.

PROOF. By definition of the crossed product $E \rtimes_{\alpha} \widehat{A}_{\sigma}$, we have the following equalities between subalgebras of $D \otimes E \otimes B(l^2(A_{\sigma}))$:

$$D \otimes (E \rtimes_{\alpha} \widehat{A}_{\sigma}) = D \otimes (\overline{sp}^{w} \{ \alpha(E) \cdot (1 \otimes A_{\sigma}) \})$$
$$= \overline{sp}^{w} \{ (D \otimes A_{\sigma})_{13} \cdot \alpha(E)_{23} \}$$

On the other hand, since the coactions on the finite dimensional algebras are automatically non-degenerate, we have as well the following equality:

$$D \otimes A_{\sigma} = sp\{(1 \otimes A_{\sigma}) \cdot \beta(D)\}$$

Thus, we have the following equality of algebras:

$$D \otimes (E \rtimes_{\alpha} A_{\sigma}) = \overline{sp}^{w} \{ (1 \otimes 1 \otimes A_{\sigma}) \cdot \beta(D)_{13} \cdot \alpha(E)_{23} \}$$

Let us compute now the restriction of the map $\beta \odot \hat{\alpha}$ to the algebra $1 \otimes 1 \otimes A_{\sigma}$, to the algebra $\beta(D)_{13}$, and to the algebra $\alpha(E)_{23}$. This can be done as follows:

(1) The restriction of $\beta \odot \hat{\alpha}$ to the algebra $1 \otimes 1 \otimes A_{\sigma}$ is $1 \otimes 1 \otimes \sigma \Delta$. In particular the map $\beta \odot \hat{\alpha}$ has no fixed points in this algebra $1 \otimes 1 \otimes A_{\sigma}$.

(2) The algebra $\alpha(E)_{23}$ is by definition fixed by $\beta \odot \hat{\alpha}$.

(3) We prove now that the algebra $\beta(D)_{13}$ is also fixed by $\beta \odot \hat{\alpha}$. For this purpose, let $\{u_{ij}\}$ be an orthonormal basis of $l^2(A_{\sigma})$ consisting of coefficients of irreducible corepresentations of A_{σ} . Since we have $\beta(D) \subset D \otimes_{alg} A_{\sigma}$, for any $b \in D$ we can use the notation $\beta(b) = \sum_{uij} b^u_{ij} \otimes u_{ij}$. From the coassociativity of β we obtain:

$$\sum_{uij} \beta(b_{ij}^u) \otimes u_{ij} = \sum_{uijk} b_{ij}^u \otimes u_{kj} \otimes u_{ik}$$

Thus we have $\beta(b_{ik}^u) = \sum_j b_{ij}^u \otimes u_{kj}$ for any u, i, k, and so:

$$(id \otimes S)\beta(b_{ij}^{u}) = (id \otimes S)\left(\sum_{s} b_{is}^{u} \otimes u_{js}\right)$$
$$= \sum_{s} b_{is}^{u} \otimes u_{sj}^{*}$$

Also, we have $\widehat{\alpha}(1 \otimes u_{ij}) = \sum_{k} 1 \otimes u_{ik} \otimes u_{kj}$, and we obtain from this that we have:

$$(\beta \odot \widehat{\alpha})(\beta(b)_{13}) = \sum_{uij} \left(\sum_{k} 1 \otimes 1 \otimes u_{ik} \otimes u_{kj} \right) \left(\sum_{s} b^{u}_{is} \otimes 1 \otimes 1 \otimes u^{*}_{sj} \right)$$
$$= \sum_{uijks} b^{u}_{is} \otimes 1 \otimes u_{ik} \otimes u_{kj} u^{*}_{sj}$$

By summing over j the last term is replaced by $(uu^*)_{ks} = \delta_{k,s}1$. Thus we obtain, as desired, that our algebra consists indeed of fixed points:

$$(\beta \odot \widehat{\alpha})(\beta(b)_{13}) = \sum_{uik} b^u_{ik} \otimes 1 \otimes u_{ik} \otimes 1$$
$$= (\beta(b)_{13}) \otimes 1$$

In order to finish now, observe that (1,2,3) above show that $(D \otimes (E \rtimes_{\alpha} \widehat{A}_{\sigma}))^{\beta \odot \widehat{\alpha}}$, which is the fixed point algebra of $\overline{sp}^{w} \{ (1 \otimes 1 \otimes A_{\sigma}) \cdot \beta(D)_{13} \cdot \alpha(E)_{23} \}$ under the coaction $\beta \odot \widehat{\alpha}$, is equal to $\overline{sp}^{w} \{ \beta(D)_{13} \cdot \alpha(E)_{23} \}$. This finishes the proof of the first assertion, and proves as well the non-degeneracy of the diagram in the statement.

Finally, observe that the diagram in the statement is the dual of the square on the left in the following diagram, where $P = E \rtimes_{\alpha} \widehat{A}_{\sigma}$ and $\pi = \widehat{\alpha}$:

$$\begin{array}{cccc} D &\subset & (D \otimes P)^{\beta \odot \pi} &\subset & D \otimes P \\ \cup & & \cup & & \cup \\ \mathbb{C} &\subset & P^{\pi} &\subset & P \end{array}$$

Since π is dual, the square on the right is a non-degenerate commuting square. We also know that the rectangle is a non-degenerate commuting square. Thus if we denote by $E_X : D \otimes P \to D \otimes P$ the conditional expectation onto X, for any X, then for any $b \in D$ we have $E_{P^{\pi}}(b) = E_P(b) = E_{\mathbb{C}}(b)$, and this proves the commuting square condition. \Box

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Let us denote now by Alg the category having as objects the finite dimensional C^* algebras and having as arrows the inclusions of C^* -algebras which preserve the canonical traces. The above result suggests the following abstract definition:

DEFINITION 9.18. Given objects $(D, \beta) \in A - \mathcal{A}lg$ and $(E, \alpha) \in \widehat{A}_{\sigma} - \mathcal{A}lg$, we let $D\Box_A E = (D \otimes (E \rtimes_{\alpha} \widehat{A}_{\sigma}))^{\beta \odot \widehat{\alpha}}$

be the object in Alg, constructed as in Proposition 9.17.

If $(D', \beta') \subset (D, \beta)$ is an arrow in $A - \mathcal{A}lg$ and $(E', \alpha') \subset (E, \alpha)$ is an arrow in $\widehat{A}_{\sigma} - \mathcal{A}lg$, then we have a canonical embedding, as follows:

$$D'\Box_A E' \subset D\Box_A E$$

Now since both $D'\square_A E'$ and $D\square_A E$ are endowed with their canonical traces, this inclusion is Markov. Thus, we have constructed a bifunctor, as follows:

$$\Box_A: A - \mathcal{A}lg \times \widehat{A}_{\sigma} - \mathcal{A}lg \to \mathcal{A}lg$$

With this convention, we have the following result:

THEOREM 9.19. For any two arrows $D_0 \subset D_1$ in A - Alg and $E_0 \subset E_1$ in $\widehat{A}_{\sigma} - Alg$,

$$\begin{array}{cccc} D_0 \Box_A E_1 & \subset & D_1 \Box_A E_1 \\ \cup & & \cup \\ D_0 \Box_A E_0 & \subset & D_1 \Box_A E_0 \end{array}$$

is a non-degenerate commuting square of finite dimensional von Neumann algebras.

PROOF. This can be proved in several steps, as follows:

Step I. In the simplest case, $D_0 = E_0 = \mathbb{C}$, this follows from the above.

<u>Step II</u>. We prove now the result in the general $E_0 = \mathbb{C}$ case. Indeed, let $E = E_1$, and consider the following diagram:

$$E \subset D_0 \Box_A E \subset D_1 \Box_A E$$
$$\cup \qquad \cup \qquad \cup$$
$$\mathbb{C} \subset D_0 \subset D_1$$

By the result of Step I, both the square on the left and the rectangle are non-degenerate commuting squares. We want to prove that the square on the right is a non-degenerate commuting square. But the non-degeneracy condition follows from:

$$D_1 \Box_A E = sp\{E \cdot D_1\} \subset sp\{D_0 \Box_A E \cdot D_1\}$$

Now let $x \in D_0 \square_A E$ and write $x = \sum_i b_i a_i$ with $b_i \in D_0$ and $a_i \in E$. Then:

$$E_{D_1}(x) = \sum_i b_i E_{D_1}(a_i)$$
$$= \sum_i b_i E_{\mathbb{C}}(a_i)$$
$$= \sum_i b_i E_{D_0}(a_i)$$
$$= E_{D_0}(x)$$

But this proves the commuting square condition, and we are done.

Step III. A similar argument shows that the result holds in the case $D_0 = \mathbb{C}$.

<u>Step IV</u>. General case. We will use many times the following diagram, in which all the rectangles and all the squares, except possibly for the square in the statement, are non-degenerate commuting squares, cf. the conclusions of Steps I, II, III:

The non-degeneracy condition follows from:

$$D_1 \Box_A E_1 = sp\{E_1 \cdot D_1\} \subset sp\{D_0 \Box_A E_1 \cdot D_1 \Box_A E_0\}$$

Now let $x \in D_0 \square_A E_1$ and write $x = \sum_i b_i a_i$ with $b_i \in D_0$ and $a_i \in E_1$. Then:

$$E_{D_1 \square_A E_0}(x) = \sum_i b_i E_{D_1 \square_A E_0}(a_i)$$
$$= \sum_i b_i E_{E_0}(a_i)$$
$$= \sum_i b_i E_{D_0 \square_A E_0}(a_i)$$
$$= E_{D_0 \square_A E_0}(x)$$

But this proves the commuting square condition, and we are done.

We show now that the bifunctor \Box_A behaves well with respect to basic constructions. If A is a Woronowicz algebra, a sequence of two arrows $D_0 \subset D_1 \subset D_2$ in $A - \mathcal{A}lg$ is called a basic construction if $D_0 \subset D_1 \subset D_2$ is a basic construction in $\mathcal{A}lg$ and if its Jones projection $e \in D_2$ is a fixed by the coaction $D_2 \to D_2 \otimes A$. An infinite sequence of basic constructions in $A - \mathcal{A}lg$ is called a Jones tower in $A - \mathcal{A}lg$. We have:

PROPOSITION 9.20. If $D_0 \subset D_1 \subset D_2 \subset D_3 \subset \ldots$ is a Jones tower in A - Alg and $E_0 \subset E_1 \subset E_2 \subset E_3 \subset \ldots$ is a Jones tower in $\widehat{A}_{\sigma} - Alg$ then

is a lattice of basic constructions for non-degenerate commuting squares.

PROOF. We prove only that the rows are Jones towers, the proof for the columns being similar. By restricting the attention to a pair of consecutive inclusions, it is enough to prove that if $D_0 \subset D_1 \subset D_2$ is a basic construction in A - Alg and E is an object of $\widehat{A}_{\sigma} - Alg$ then $D_0 \Box_A E \subset D_1 \Box_A E \subset D_2 \Box_A E$ is a basic construction in Alg.

For this purpose, we will use many times the following diagram, in which all squares and rectangles are non-degenerate commuting squares:

$$E \subset D_0 \Box_A E \subset D_1 \Box_A E \subset D_2 \Box_A E$$
$$\cup \qquad \cup \qquad \cup \qquad \cup$$
$$\mathbb{C} \subset D_0 \subset D_1 \subset D_2$$

We will use the abstract characterization of the basic construction, stating that $N \subset M \subset P$ is a basic construction, with Jones projection $e \in P$, precisely when:

- (1) $P = sp\{M \cdot e \cdot M\}.$
- (2) [e, N] = 0.
- (3) $exe = E_N(x)e$ for any $x \in M$.

(4) $tr(xe) = \lambda tr(x)$ for any $x \in M$, where λ is the inverse of the index of $N \subset M$.

Let $e \in D_2$ be the Jones projection for the basic construction $D_0 \subset D_1 \subset D_2$. With $N = D_0$, $M = D_1$ and $P = D_2$ the verification of (1-4) is as follows:

(1) This follows from the following computation:

$$D_2 \Box_A E = sp\{D_2 \cdot E\}$$

= $sp\{D_1 \cdot e \cdot D_1 \cdot E\}$
= $sp\{D_1 \cdot e \cdot D_1 \Box_A E\}$

(2) This follows from $D_0 \Box_A E = sp\{D_0 \cdot E\}$, from [e, E] = 0 and from $[e, D_0] = 0$.

1

(3) Let
$$x \in D_1 \square_A E$$
, and write $x = \sum_i b_i a_i$ with $b_i \in D_1$ and $a_i \in E$. Then
 $exe = \sum_i eb_i a_i e$
 $= \sum_i eb_i ea_i$
 $= \sum_i E_{D_0}(b_i) ea_i$
 $= \sum_i E_{D_0}(b_i) a_i e$

On the other hand, we have as well the following computation:

$$E_{D_0 \Box_A E}(x)e = \sum_i E_{D_0 \Box_A E}(b_i a_i)e$$
$$= \sum_i E_{D_0 \Box_A E}(b_i)a_i e$$
$$= \sum_i E_{D_0}(b_i)a_i e$$

(4) With the above notations, we have that:

$$E_{D_2}(xe) = \sum_i E_{D_2}(b_i a_i e)$$
$$= \sum_i b_i E_{D_2}(a_i) e$$
$$= \sum_i b_i E_{\mathbb{C}}(a_i) e$$

We also have $b_i E_{\mathbb{C}}(a_i) \in D_1$ for every *i*, and so:

$$tr_{D_2 \Box_A E}(xe) = tr_{D_2}(E_{D_2}(xe))$$
$$= \lambda \sum_i tr_{D_1}(b_i E_{\mathbb{C}}(a_i))$$

On the other hand, we have as well the following computation:

$$tr_{D_{1}\square_{A}E}(x) = tr_{D_{1}}(E_{D_{1}}(x))$$

= $\sum_{i} tr_{D_{1}}(b_{i}E_{D_{1}}(a_{i}))$
= $\sum_{i} tr_{D_{1}}(b_{i}E_{\mathbb{C}}(a_{i}))$

Thus, the fourth condition for a basic construction is verified, as desired.

9D. FIXED POINTS

With standard coaction conventions, from chapter 4, we have:

PROPOSITION 9.21. Given a corepresentation and a representation, as follows,

 $v \in M_n(\mathbb{C}) \otimes A \quad , \quad \pi : A_\sigma \to M_k(\mathbb{C})$

consider, via some standard identifications, the associated objects

$$(M_n(\mathbb{C}), \iota_v) \in A - \mathcal{A}lg \quad , \quad (M_k(\mathbb{C}), \iota_{\check{\pi}}) \in \widehat{A}_{\sigma} - \mathcal{A}lg$$

and form the corresponding algebra $M_n(\mathbb{C}) \square_A M_k(\mathbb{C})$. Then there exists an isomorphism

$$\begin{pmatrix} M_k(\mathbb{C}) &\subset & M_n(\mathbb{C}) \square_A M_k(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} &\subset & M_n(\mathbb{C}) \end{pmatrix} \simeq \begin{pmatrix} \mathbb{C} \otimes M_k(\mathbb{C}) &\subset & M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C} &\subset & u(M_n(\mathbb{C}) \otimes \mathbb{C})u^* \end{pmatrix}$$

sending $z \to 1 \otimes z$ for $z \in M_k(\mathbb{C})$ and $y \to \iota_u(y)$ for $y \in M_n(\mathbb{C})$, where $u = (id \otimes \pi)v$.

PROOF. Consider the following *-morphism of algebras:

$$\Phi: M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes B(l^2(A_\sigma))$$
$$x \to ad(v_{13}\check{\pi}_{23}u_{12}^*)(x \otimes 1)$$

Since both the squares in the statement are non-degenerate commuting squares, all the assertions are consequences of the following formulae, that we will prove now:

 $\Phi(1 \otimes z) = \iota_{\check{\pi}}(z)_{23} \quad , \quad \Phi(\iota_u(y)) = \iota_v(y)_{13}$

The second formula follows from the following computation:

$$\Phi(u(y \otimes 1)u^*) = v_{13}(y \otimes 1 \otimes 1)v_{13}^* = (v(y \otimes 1)v^*)_{13}$$

For the first formula, what we have to prove is that:

$$v_{13}\check{\pi}_{23}u_{12}^*(1\otimes z\otimes 1)u_{12}\check{\pi}_{23}^*v_{13}^* = (\check{\pi}(z\otimes 1)\check{\pi}^*)_{23}$$

By moving the unitaries to the left and to the right we have to prove that:

$$\check{\pi}_{23}^* v_{13} \check{\pi}_{23} u_{12}^* \in (\mathbb{C} \otimes M_k(\mathbb{C}) \otimes \mathbb{C})' = M_n(\mathbb{C}) \otimes \mathbb{C} \otimes B(l^2(H_\sigma))$$

Let us call this unitary U. Since $\check{\pi} = (\pi \otimes id)V'$ we have:

$$U = (id \otimes \pi \otimes id)(V_{23}^*v_{13}V_{23}v_{12}^*)$$

The comultiplication of H_{σ} is given by $\Delta(y) = V^*(1 \otimes y)V$. On the other hand since v^* is a corepresentation of H_{σ} , we have $(id \otimes \Delta)(v^*) = v_{12}^*v_{13}^*$. We get that:

$$V_{23}^* v_{13} V_{23} = (V_{23}^* v_{13}^* V_{23})^*$$

= $((id \otimes \Delta)(v^*))^*$
= $(v_{12}^* v_{13}^*)^*$
= $v_{13} v_{12}$

Thus we have $U = v_{13}$, and we are done.

We are now ready to formulate our main result, as follows:

THEOREM 9.22. Any commuting square having \mathbb{C} in the lower left corner,

$$\begin{array}{rcl} E & \subset & X \\ \cup & & \cup \\ \mathbb{C} & \subset & D \end{array}$$

must appear as follows, for a suitable Woronowicz algebra A, with actions on D, E,

$$\begin{array}{ccc} E & \subset & D \square_A E \\ \cup & & \cup \\ \mathbb{C} & \subset & D \end{array}$$

and the vertical subfactor associated to it is isomorphic to

$$R \subset (D \otimes (R \rtimes_{\gamma_{\pi}} \widehat{A}_{\sigma}))^{\beta_{v} \odot \widehat{\gamma_{\pi}}}$$

which is a fixed point subfactor, in the sense of chapter 4.

PROOF. This is something quite technical, which basically follows by combining the above results, and for full details on this, we refer to $\begin{bmatrix} 6 \end{bmatrix}$ and related papers.

Summarizing, all the commuting squares having \mathbb{C} in the lower left corner are described by quantum groups. This is of course something quite special, and we will study more general commuting squares, not coming from quantum groups, in the next chapters.

9e. Exercises

Exercises: EXERCISE 9.23. EXERCISE 9.24. EXERCISE 9.25. EXERCISE 9.26. EXERCISE 9.27. EXERCISE 9.28. EXERCISE 9.28. EXERCISE 9.29. EXERCISE 9.30. Bonus exercise.

ADE subfactors

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

Exercise 10.8.

Annular structure

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

Exercise 11.2.

Exercise 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Classification results

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

Exercise 12.2.

Exercise 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.
Part IV

Amenability

This is the one Oh, this is the one This is the one She's waited for

Hyperfinite factors

13a. The factor R

Welcome to advanced operator algebra theory, again. What we saw in the previous chapters was in fact just half of the story, and the other half, regarding hyperfiniteness, still remains to be told. The idea indeed is that there has been a considerable amount of work on hyperfiniteness, comparable in size and difficulty with the general classification work for the factors, based on reduction theory, and we will discuss this here.

In order to get started, let us formulate the following definition:

DEFINITION 13.1. A von Neumann algebra $A \subset B(H)$ is called hyperfinite when it appears as the weak closure of an increasing limit of finite dimensional algebras:

$$A = \overline{\bigcup_i A_i}^w$$

When A is a II_1 factor, we call it hyperfinite II_1 factor, and we denote it by R.

As a first observation, there are many hyperfinite von Neumann algebras, for instance because any finite dimensional von Neumann algebra $A = \bigoplus_i M_{n_i}(\mathbb{C})$ is such an algebra, as one can see simply by taking $A_i = A$ for any *i*, in the above definition.

Also, given a measured space X, by using a dense sequence of points inside it, we can write $X = \bigcup_i X_i$ with $X_i \subset X$ being an increasing sequence of finite subspaces, and at the level of the corresponding algebras of functions this gives a decomposition as follows, which shows that the algebra $A = L^{\infty}(X)$ is hyperfinite, in the above sense:

$$L^{\infty}(X) = \overline{\bigcup_{i} L^{\infty}(X_i)}^{c}$$

The interesting point, however, is that when trying to construct II₁ factors which are hyperfinite, all the possible constructions lead in fact to the same factor, denoted R. This is an old theorem of Murray and von Neumann [60], that we will explain now.

In order to get started, we will need a number of technical ingredients. Generally speaking, out main tool will be the expectation $E_i : A \to A_i$ from a hyperfinite von

Neumann algebra A onto its finite dimensional subalgebras $A_i \subset A$, so talking about such conditional expectations will be our first task. Let us start with:

PROPOSITION 13.2. Given an inclusion of finite von Neumann algebras $A \subset B$, there is a unique linear map

$$E: B \to A$$

which is positive, unital, trace-preserving and satisfies the following condition:

$$E(b_1ab_2) = b_1E(a)b_2$$

This map is called conditional expectation from B onto A.

PROOF. We make use of the standard representation of the finite von Neumann algebra B, with respect to its trace $tr: B \to \mathbb{C}$, as constructed in chapter 4:

$$B \subset L^2(B)$$

If we denote by Ω the cyclic and separating vector of $L^2(B)$, we have an identification of vector spaces $A\Omega = L^2(A)$. Consider now the following orthogonal projection:

$$e: L^2(B) \to L^2(A)$$

It follows from definitions that we have an inclusion $e(B\Omega) \subset A\Omega$, and so our projection *e* induces by restriction a certain linear map, as follows:

$$E: B \to A$$

This linear map E and the orthogonal projection e are then related by:

$$exe = E(x)e$$

But this shows that the linear map E satisfies the various conditions in the statement, namely positivity, unitality, trace preservation and bimodule property. As for the uniqueness assertion, this follows by using the same argument, applied backwards, the idea being that a map E as in the statement must come from the projection e.

Following Jones [40], who was a heavy user of such expectations, we will be often interested in what follows in the orthogonal projection $e: L^2(B) \to L^2(A)$ producing the expectation $E: B \to A$, rather than in E itself. So, let us formulate:

DEFINITION 13.3. Associated to any inclusion of finite von Neumann algebras $A \subset B$, as above, is the orthogonal projection

$$e: L^2(B) \to L^2(A)$$

producing the conditional expectation $E: B \to A$ via the following formula:

$$exe = E(x)e$$

This projection is called Jones projection for the inclusion $A \subset B$.

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We will heavily use Jones projections in chapter 16 below, in the context where both the algebras A, B are II₁ factors, when systematically studying the inclusions of such II₁ factors $A \subset B$, called subfactors. In connection with our present hyperfiniteness questions, the idea, already mentioned above, will be that of using the conditional expectation $E_i : A \to A_i$ from a hyperfinite von Neumann algebra A onto its finite dimensional subalgebras $A_i \subset A$, as well as its Jones projection versions $e_i : L^2(A) \to L^2(A_i)$. Let us start with a technical approximation result, as follows:

PROPOSITION 13.4. Assume that a von Neumann algebra $A \subset B(H)$ appears as an increasing limit of von Neumann subalgebras

$$A = \overline{\bigcup_i A_i}^v$$

and denote by $E_i: A \to A_i$ the corresponding conditional expectations.

- (1) We have $||E_i(x) x|| \to 0$, for any $x \in A$.
- (2) If $x_i \in A_i$ is a bounded sequence, satisfying $x_i = E_i(x_{i+1})$ for any *i*, then this sequence has a norm limit $x \in A$, satisfying $x_i = E_i(x)$ for any *i*.

PROOF. Both the assertions are elementary, as follows:

(1) In terms of the Jones projections $e_i : L^2(A) \to L^2(A_i)$ associated to the expectations $E_i : A \to A_i$, the fact that the algebra A appears as the increasing union of its subalgebras A_i translates into the fact that the e_i are increasing, and converging to 1:

 $e_i \nearrow 1$

But this gives $||E_i(x) - x|| \to 0$, for any $x \in A$, as desired.

(2) Let $\{x_i\} \subset A$ be a sequence as in the statement. Since this sequence was assumed to be bounded, we can pick a weak limit $x \in A$ for it, and we have then, for any *i*:

$$E_i(x) = x_i$$

Now by (1) we obtain from this $||x - x_n|| \to 0$, which gives the result.

We have now all the needed ingredients for formulating a first key result, in connection with the hyperfinite II₁ factors, due to Murray-von Neumann [60], as follows:

PROPOSITION 13.5. Given an increasing union on matrix algebras, the following construction produces a hyperfinite II_1 factor

$$R = \overline{\bigcup_{n_i} M_{n_i}(\mathbb{C})}^w$$

called Murray-von Neumann hyperfinite factor.

PROOF. This basically follows from the above, in two steps, as follows:

(1) The von Neumann algebra R constructed in the statement is hyperfinite by definition, with the remark here that the trace on it $tr : R \to \mathbb{C}$ comes as the increasing union of the traces on the matrix components $tr : M_{n_i}(\mathbb{C}) \to \mathbb{C}$, and with all the details here being elementary to check, by using the usual standard form technology.

(2) Thus, it remains to prove that R is a factor. For this purpose, pick an element belonging to its center, $x \in Z(R)$, and consider its expectation on $A_i = M_{n_i}(\mathbb{C})$:

$$x_i = E_i(x)$$

We have then $x_i \in Z(A_i)$, and since the matrix algebra $A_i = M_{n_i}(\mathbb{C})$ is a factor, we deduce from this that this expected value $x_i \in A_i$ is given by:

$$x_i = tr(x_i)\mathbf{1} = tr(x)\mathbf{1}$$

On the other hand, Proposition 13.4 applies, and shows that we have:

$$||x_i - x|| = ||E_i(x) - x|| \to 0$$

Thus our element is a scalar, x = tr(x)1, and so R is a factor, as desired.

Next, we have the following substantial improvement of the above result, also due to Murray-von Neumann [60], which will be our final saying on the subject:

THEOREM 13.6. There is a unique hyperfinite II_1 factor, called Murray-von Neumann hyperfinite factor R, which appears as an increasing union on matrix algebras,

$$R = \overline{\bigcup_{n_i} M_{n_i}(\mathbb{C})}^u$$

with the isomorphism class of this union not depending on the exact sizes of the matrix algebras involved, nor on the particular inclusions between them.

PROOF. We already know from Proposition 13.5 that the union in the statement is a hyperfinite II₁ factor, for any choice of the matrix algebras involved, and of the inclusions between them. Thus, in order to prove the result, it all comes down in proving the uniqueness of the hyperfinite II₁ factor. But this can be proved as follows:

(1) Given a II₁ factor A, a von Neumann subalgebra $B \subset A$, and a subset $S \subset A$, let us write $S \subset_{\varepsilon} B$ when the following condition is satisfied, with $||x||_2 = \sqrt{tr(x^*x)}$:

$$\forall x \in S, \exists y \in B, ||x - y||_2 \le \varepsilon$$

With this convention made, given a II₁ factor A, the fact that this factor is hyperfinite in the sense of Definition 13.1 tells us that for any finite subset $S \subset A$, and any $\varepsilon > 0$, we can find a finite dimensional von Neumann subalgebra $B \subset A$ such that:

$$S \subset_{\varepsilon} B$$

(2) With this observation made, assume that we are given a hyperfinite II₁ factor A. Let us pick a dense sequence $\{x_k\} \subset A$, and let us set:

$$S_k = \{x_1, \dots, x_k\}$$

By choosing $\varepsilon = 1/k$ in the above, we can find, for any $k \in \mathbb{N}$, a finite dimensional von Neumann subalgebra $B_k \subset A$ such that the following condition is satisfied:

$$S_k \subset_{1/k} B_k$$

(3) Our first claim is that, by suitably choosing our subalgebra $B_k \subset A$, we can always assume that this is a matrix algebra, of the following special type:

$$B_k = M_{2^{n_k}}(\mathbb{C})$$

But this is something which is quite routine, which can be proved by starting with a finite dimensional subalgebra $B_k \subset A$ as above, and then perturbing its set of minimal projections $\{e_i\}$ into a set of projections $\{e'_i\}$ which are close in norm, and have as traces multiples of 2^n , with $n \gg 0$. Indeed, the algebra $B'_k \subset A$ having these new projections $\{e'_i\}$ as minimal projections will be then arbitrarily close to the algebra B_k , and so will still contain the subset S_k in the above approximate sense, and due to our trace condition, will be contained in a subalgebra of type $B''_k \simeq M_{2^{n_k}}(\mathbb{C})$, as desired.

(4) Our next claim, whose proof is similar, by using standard perturbation arguments for the corresponding sets of minimal projections, is that in the above the sequence of subalgebras $\{B_k\}$ can be chosen increasing. Thus, up to a rescaling of everything, we can assume that our sequence of subalgebras $\{B_k\}$ is as follows:

$$B_k = M_{2^k}(\mathbb{C})$$

(5) But this finishes the proof. Indeed, according to the above, we have managed to write our arbitrary hyperfinite II₁ factor A as a weak limit of the following type:

$$A = \overline{\bigcup_k M_{2^k}(\mathbb{C})}^{*}$$

Thus we have uniqueness indeed, and our result is proved.

Now back to more concrete things, one question is about how to best think of R, with Theorem 13.6 as stated not providing us with an answer. To be more precise, we would like to know what is the "best model" for R, that is, what exact matrix algebras should we use in practice, and with which inclusions between them. And here, a look at the proof of Theorem 13.6 suggests that the "best writing" of R is as follows:

$$R = \overline{\bigcup_k M_{2^k}(\mathbb{C})}^{*}$$

And we can in fact do even better, by observing that the inclusions between matrix algebras of size 2^k appear via tensor products, and formulating things as follows:

PROPOSITION 13.7. The hyperfinite II_1 factor R appears as

$$R = \overline{\bigotimes_{r \in \mathbb{N}} M_2(\mathbb{C})}^w$$

with the infinite tensor product being defined as an inductive limit, in the obvious way.

PROOF. This follows from the above discussion, and with the remark that there is a binary choice there, of left/right type, to be made when constructing the inductive limit. And we prefer here not to make any choice, and leave things like this, because the best choice here always depends on the precise applications that you have in mind. \Box

Along the same lines, we can ask as well for precise group algebra models for the hyperfinite II₁ factor, $R = L(\Gamma)$, and the canonical choice here is as follows:

PROPOSITION 13.8. The hyperfinite II_1 factor R appears as

$$R = L(S_{\infty})$$

with $S_{\infty} = \bigcup_{r \in \mathbb{N}} S_r$ being the infinite symmetric group.

PROOF. Consider indeed the infinite symmetric group S_{∞} , which is by definition the group of permutations of $\{1, 2, 3, ...\}$ having finite support. Since such an infinite permutation with finite support must appear by extending a certain finite permutation $\sigma \in S_r$, with fixed points outside $\{1, ..., r\}$, we have then, as stated:

$$S_{\infty} = \bigcup_{r \in \mathbb{N}} S_r$$

But this shows that the von Neumann algebra $L(S_{\infty})$ is hyperfinite. On the other hand S_{∞} has the ICC property, and so $L(S_{\infty})$ is a II₁ factor. Thus, $L(S_{\infty}) = R$.

There are of course some more things that can be said here, because other groups of the same type as S_{∞} , namely appearing as increasing limits of finite subgroups, and having the ICC property, will produce as well the hyperfinite factor, $L(\Gamma) = R$, and so there is some group theory to be done here, in order to fully understand such groups. However, we prefer to defer the discussion for later, after learning about amenability, which will lead to a substantial update of our theory, making such things obsolete.

As an interesting consequence of all this, however, let us formulate:

PROPOSITION 13.9. Given two groups Γ, Γ' , each having the ICC property, and each appearing as an increasing union of finite subgroups, we have

$$L(\Gamma) \simeq L(\Gamma')$$

while the corresponding group algebras might not be isomorphic, $\mathbb{C}[\Gamma] \neq \mathbb{C}[\Gamma']$.

13B. AMENABILITY

PROOF. Here the first assertion follows from the above discussion, the von Neumann algebra in question being the hyperfinite II₁ factor R. As for the last assertion, there are countless counterexamples here, all coming from basic group theory.

13b. Amenability

The hyperfinite II₁ factor R, which is a quite fascinating object, was heavily investigated by Murray-von Neumann [60], and then by Connes [16]. There are many things that can be said about it, which all interesting, but are usually quite technical as well.

As a central result here, in what regards advanced hyperfiniteness theory, we have the following theorem of Connes [16], whose proof is something remarkably heavy, and which is arguably the deepest result in operator algebra related functional analysis:

THEOREM 13.10. For a finite von Neumann algebra A, the following are equivalent:

(1) A is hyperfinite in the usual sense, namely it appears as the weak closure of an increasing limit of finite dimensional algebras:

$$A = \overline{\bigcup_i A_i}^i$$

(2) A amenable, in the sense that the standard inclusion $A \subset B(H)$, with $H = L^2(A)$, admits a conditional expectation $E : B(H) \to A$.

PROOF. This result, due to Connes [16], is something fairly heavy, that only a handful of people have really managed to understand, the idea being as follows:

(1) \implies (2) Assuming that the algebra A is hyperfinite, let us write it as the weak closure of an increasing limit of finite dimensional subalgebras:

$$A = \overline{\bigcup_{i} A_{i}}^{u}$$

Consider the inclusion $A \subset B(H)$, with $H = L^2(A)$. In order to construct an expectation $E: B(H) \to A$, let us pick an ultrafilter ω on \mathbb{N} . Given $T \in B(H)$, we can define the following quantity, with μ_i being the Haar measure on the unitary group $U(A_i)$:

$$\psi(T) = \lim_{i \to \omega} \int_{U(A_i)} UTU^* \, d\mu_i(U)$$

With this construction made, by using now the standard involution $J: H \to H$, given by the formula $T \to T^*$, we can further define a map as follows:

$$E: B(H) \to A$$
 , $E(T) = J\psi(T)J$

But this is the expectation that we are looking for, with its left and right invariance properties coming from the left and right invariance of each Haar measure μ_i .

(2) \implies (1) This is something heavy, using lots of advanced functional analysis, and for details here, we refer to Connes' original paper [16].

We should mention that Connes' results in [16], besides proving the above implication $(2) \implies (1)$, provide also a considerable extension of Theorem 13.10, with a number of further equivalent formulations of the notion of amenability, which are a bit more technical, but all good to know. The story here, still a bit simplified, is as follows:

FACT 13.11 (Connes). For a finite von Neumann algebra A, the following conditions are in fact equivalent:

(1) A is hyperfinite, in the sense that it appears as the weak closure of an increasing limit of finite dimensional algebras:

$$A = \overline{\bigcup_i A_i}^u$$

(2) A amenable, in the sense that the standard inclusion $A \subset B(H)$, with $H = L^2(A)$, admits a conditional expectation:

$$E: B(H) \to A$$

(3) There exist unit vectors $\xi_n \in L^2(A) \otimes L^2(A)$ such that, for any $x \in A$:

$$||x\xi_n - \xi_n x||_2 \to 0 \quad , \quad \langle x\xi_n, \xi_n \rangle \to tr(x)$$

(4) For any $x_1, \ldots, x_k \in A$ and $y_1, \ldots, y_k \in A$ we have:

$$\left| tr\left(\sum_{i} x_{i} y_{i}\right) \right| \leq \left\| \sum_{i} x_{i} \otimes y_{i}^{opp} \right\|_{min}$$

Again, this is something technical and advanced, that we won't get into, in this book. Let us mention however that the idea with all this is as follows:

- $(1) \implies (2)$ is elementary, as explained above.
- $(2) \implies (3)$ can be proved by using an inequality due to Powers-Størmer.
- $(3) \implies (4)$ is something quite technical, but doable as well.
- (4) \implies (2) is again something technical, but doable as well.
- $(2) \implies (1)$ is, as before in Theorem 13.10, the difficult implication.

Regarding the difficult implication, $(2) \implies (1)$, the difficulty here comes of course from the fact that, no matter what beautiful abstract functional analysis things you know about A, at some point you will have to get to work, and construct that finite dimensional subalgebras $A_i \subset A$, and it is not even clear where to start from. For a solution to this problem, and for more, we refer to Connes's article [16], and also to his book [17].

13B. AMENABILITY

Getting back now to more everyday mathematics, the above results as stated remain something quite abstract, and advanced, and understanding their concrete implications will be our next task. In the case of the II_1 factors, we have the following result:

THEOREM 13.12. For a II_1 factor R, the following are equivalent:

(1) R amenable, in the sense that we have an expectation, as follows:

 $E: B(L^2(R)) \to R$

(2) R is the Murray-von Neumann hyperfinite II_1 factor.

PROOF. This follows indeed from Theorem 13.10, when coupled with the Murray-von Neumann uniqueness result for the hyperfinite II₁ factor, from Theorem 13.6. \Box

As another application, getting back now to the general case, that of the finite von Neumann algebras, from Theorem 13.10 as stated, a first question is about how all this applies to the group von Neumann algebras, and more generally to the quantum group von Neumann algebras $L(\Gamma)$. In order to discuss this, let us start with the case of the usual discrete groups Γ . We will need the following result, which is standard:

THEOREM 13.13. For a discrete group Γ , the following two conditions are equivalent, and if they are satisfied, we say that Γ is amenable:

- (1) Γ admits an invariant mean $m: l^{\infty}(\Gamma) \to \mathbb{C}$.
- (2) The projection map $C^*(\Gamma) \to C^*_{red}(\Gamma)$ is an isomorphism.

Moreover, the class of amenable groups contains all the finite groups, all the abelian groups, and is stable under taking subgroups, quotients and products.

PROOF. This is something very standard, the idea being as follows:

(1) The equivalence (1) \iff (2) is standard, with the amenability conditions (1,2) being in fact part of a much longer list of amenability conditions, including well-known criteria of Følner, Kesten and others. We will be back to this, with details, in a moment, directly in a more general setting, that of the discrete quantum groups.

(2) As for the last assertion, regarding the finite groups, the abelian groups, and then the stability under taking subgroups, quotients and products, this is something elementary, which follows by using either of the above definitions of the amenability. \Box

Getting back now to operator algebras, we can complement Theorem 13.10 with:

THEOREM 13.14. For a group von Neumann algebra $A = L(\Gamma)$, the following conditions are equivalent:

- (1) A is hyperfinite.
- (2) A amenable.
- (3) Γ is amenable.

PROOF. The group von Neumann algebras $A = L(\Gamma)$ being by definition finite, Theorem 13.10 applies, and gives the equivalence (1) \iff (2). Thus, it remains to prove that we have (2) \iff (3), and we can prove this as follows:

(2) \implies (3) This is something clear, because if we assume that $A = L(\Gamma)$ is amenable, we have by definition a conditional expectation $E : B(L^2(A)) \to A$, and the restriction of this conditional expectation is the desired invariant mean $m : l^{\infty}(\Gamma) \to \mathbb{C}$.

(3) \implies (2) Assume that we are given a discrete amenable group Γ . In view of Theorem 12.13, this means that Γ has an invariant mean, as follows:

$$m: l^{\infty}(\Gamma) \to \mathbb{C}$$

Consider now the Hilbert space $H = l^2(\Gamma)$, and for any operator $T \in B(H)$ consider the following map, which is a bounded sesquilinear form:

$$\varphi_T : H \times H \to \mathbb{C}$$
$$(\xi, \eta) \to m \left[\gamma \to < \rho_\gamma T \rho_\gamma^* \xi, \eta > \right]$$

By using the Riesz representation theorem, we conclude that there exists a certain operator $E(T) \in B(H)$, such that the following holds, for any two vectors ξ, η :

$$\varphi_T(\xi,\eta) = \langle E(T)\xi,\eta \rangle$$

Summarizing, to any operator $T \in B(H)$ we have associated another operator, denoted $E(T) \in B(H)$, such that the following formula holds, for any two vectors ξ, η :

$$\langle E(T)\xi,\eta\rangle = m \left|\gamma \rightarrow \langle \rho_{\gamma}T\rho_{\gamma}^{*}\xi,\eta\rangle\right|$$

In order to prove now that this linear map E is the desired expectation, observe that for any group element $g \in \Gamma$, and any two vectors $\xi, \eta \in H$, we have:

$$<\rho_{g}E(T)\rho_{g}^{*}\xi,\eta> = < E(T)\rho_{g}^{*}\xi,\rho_{g}^{*}\eta>$$

$$= m\left[\gamma \rightarrow <\rho_{\gamma}T\rho_{\gamma}^{*}\rho_{g}^{*}\xi,\rho_{g}^{*}\eta>\right]$$

$$= m\left[\gamma \rightarrow <\rho_{g\gamma}T\rho_{g\gamma}^{*}\xi,\eta>\right]$$

$$= m\left[\gamma \rightarrow <\rho_{\gamma}T\rho_{\gamma}^{*}\xi,\eta>\right]$$

$$= < E(T)\xi,\eta>$$

Since this is valid for any $\xi, \eta \in H$, we conclude that we have, for any $g \in \Gamma$:

$$\rho_g E(T) \rho_g^* = E(T)$$

But this shows that the element $E(T) \in B(H)$ is in the commutant of the right regular representation of Γ , and so belongs to the left regular group algebra of Γ :

$$E(T) \in L(\Gamma)$$

13C. QUANTUM GROUPS

Summarizing, we have constructed a certain linear map $E: B(H) \to L(\Gamma)$. Now by using the above explicit formula of it, in terms of $m: l^{\infty}(\Gamma) \to \mathbb{C}$, which was assumed to be an invariant mean, we conclude that E is indeed an expectation, as desired. \Box

As a very concrete application of all this technology, in relation now with the discrete group algebras which are II_1 factors, the results that we have lead to:

THEOREM 13.15. For a discrete group Γ , the following conditions are equivalent:

- (1) Γ is amenable, and has the ICC property.
- (2) $A = L(\Gamma)$ is the hyperfinite II₁ factor R.

PROOF. This follows indeed from Theorem 13.14, coupled with the standard fact, that we know well from chapter 4, that a group algebra $A = L(\Gamma)$ is a factor, and so a II₁ factor, precisely when the group Γ has the ICC property.

13c. Quantum groups

We would like to discuss now all sorts of questions, for the most open, or at least difficult, in relation with groups and quantum groups, taken finite, discrete or compact, and with more general quantum manifolds and quantum spaces, in connection with the Murray-von Neumann factor R, amenability and hyperfiniteness. As a first such question, in relation with the considerations from chapter 4, we would like to understand which discrete quantum groups Γ produce group algebras as follows:

 $L(\Gamma) \simeq R$

In terms of the compact quantum group duals $G = \widehat{\Gamma}$, the problem is that of understanding which compact quantum groups G produce group algebras as follows:

$$L^{\infty}(G) \simeq R$$

In order to discuss this, we must first talk about amenability. We have here the following result, basically due to Woronowicz [99], and coming from the Peter-Weyl theory, extending to the discrete quantum groups the standard theory for discrete groups:

THEOREM 13.16. Let (A, u) with $u \in M_N(A)$ be a Woronowicz algebra, as axiomatized before. Let A_{full} be the enveloping C^* -algebra of $\mathcal{A} = \langle u_{ij} \rangle$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:

- (1) The Haar functional of A_{full} is faithful.
- (2) The projection map $A_{full} \rightarrow A_{red}$ is an isomorphism.
- (3) The counit map $\varepsilon : A \to \mathbb{C}$ factorizes through A_{red} .
- (4) We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red} .
- (5) $||ax_k \varepsilon(a)x_k|| \to 0$ for any $a \in \mathcal{A}$, for certain norm 1 vectors $x_k \in L^2(\mathcal{A})$.

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. Before starting, we should mention that amenability and the present result are a bit like the spectral theorem, in the sense that knowing that the result formally holds does not help much, and in practice, one needs to remember the proof as well. For this reason, we will work out explicitly all the possible implications between (1-5), whenever possible, adding to the global formal proof, which will be linear, as follows:

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1)$$

In order to prove these implications, and the other ones too, the general idea is that this is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group, and in general, the result follows by adapting the group dual case proof.

(1) \iff (2) This follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red} .

(2) \implies (3) This is trivial, because we have quotient maps $A_{full} \rightarrow A \rightarrow A_{red}$, and so our assumption $A_{full} = A_{red}$ implies that we have $A = A_{red}$.

 $(3) \implies (2)$ Assume indeed that we have a counit map, as follows:

$$\varepsilon: A_{red} \to \mathbb{C}$$

In order to prove $A_{full} = A_{red}$, we can use the right regular corepresentation. Indeed, we can define such a corepresentation by the following formula:

$$W(a \otimes x) = \Delta(a)(1 \otimes x)$$

This corepresentation is unitary, so we can define a morphism as follows:

$$\Delta': A_{red} \to A_{red} \otimes A_{full} \quad , \quad a \to W(a \otimes 1) W^*$$

Now by composing with $\varepsilon \otimes id$, we obtain a morphism as follows:

$$(\varepsilon \otimes id)\Delta' : A_{red} \to A_{full} \quad , \quad u_{ij} \to u_{ij}$$

Thus, we have our inverse for the canonical projection $A_{full} \to A_{red}$, as desired. (3) \implies (4) This implication is clear, because we have:

$$\varepsilon(Re(\chi_u)) = \frac{1}{2} \left(\sum_{i=1}^N \varepsilon(u_{ii}) + \sum_{i=1}^N \varepsilon(u_{ii}^*) \right)$$
$$= \frac{1}{2} (N+N)$$
$$= N$$

Thus the element $N - Re(\chi_u)$ is not invertible in A_{red} , as claimed.

(4) \implies (3) In terms of the corepresentation $v = u + \bar{u}$, whose dimension is 2N and whose character is $2Re(\chi_u)$, our assumption $N \in \sigma(Re(\chi_u))$ reads:

$$\dim v \in \sigma(\chi_v)$$

By functional calculus the same must hold for w = v + 1, and then once again by functional calculus, the same must hold for any tensor power of w:

$$w_k = w^{\otimes k}$$

Now choose for each $k \in \mathbb{N}$ a state $\varepsilon_k \in A^*_{red}$ having the following property:

$$c_k(w_k) = \dim w_k$$

By Peter-Weyl we must have $\varepsilon_k(r) = \dim r$ for any $r \leq w_k$, and since any irreducible corepresentation appears in this way, the sequence ε_k converges to a counit map:

$$\varepsilon: A_{red} \to \mathbb{C}$$

(4) \implies (5) Consider the following elements of A_{red} , which are positive:

$$a_i = 1 - Re(u_{ii})$$

Our assumption $N \in \sigma(Re(\chi_u))$ tells us that $a = \sum a_i$ is not invertible, and so there exists a sequence x_k of norm one vectors in $L^2(A)$ such that:

$$\langle ax_k, x_k \rangle \rightarrow 0$$

Since the summands $\langle a_i x_k, x_k \rangle$ are all positive, we must have, for any *i*:

$$\langle a_i x_k, x_k \rangle \rightarrow 0$$

We can go back to the variables u_{ii} by using the following general formula:

$$||vx - x||^2 = ||vx||^2 + 2 < (1 - Re(v))x, x > -1$$

Indeed, with $v = u_{ii}$ and $x = x_k$ the middle term on the right goes to 0, and so the whole term on the right becomes asymptotically negative, and so we must have:

$$||u_{ii}x_k - x_k|| \to 0$$

Now let $M_n(A_{red})$ act on $\mathbb{C}^n \otimes L^2(A)$. Since u is unitary we have:

$$\sum_{i} ||u_{ij}x_k||^2 = ||u(e_j \otimes x_k)|| = 1$$

From $||u_{ii}x_k|| \to 1$ we obtain $||u_{ij}x_k|| \to 0$ for $i \neq j$. Thus we have, for any i, j:

$$||u_{ij}x_k - \delta_{ij}x_k|| \to 0$$

Now by remembering that we have $\varepsilon(u_{ij}) = \delta_{ij}$, this formula reads:

$$||u_{ij}x_k - \varepsilon(u_{ij})x_k|| \to 0$$

By linearity, multiplicativity and continuity, we must have, for any $a \in \mathcal{A}$, as desired:

$$||ax_k - \varepsilon(a)x_k|| \to 0$$

(5) \implies (1) This is something well-known, which follows via some standard functional analysis arguments, exactly as in the usual group case.

(1) \implies (5) Once again this is something well-known, which follows via some standard functional analysis arguments, exactly as in the usual group case.

Before getting further, with advanced amenability and hyperfiniteness questions, and as a first application of the above, we can now advance on a problem that we left open before, when talking about cocommutative Woronowicz algebras. Indeed, we can now state and prove the following result, which clarifies the situation:

PROPOSITION 13.17. The cocommutative Woronowicz algebras are the intermediate quotients of the following type, with $\Gamma = \langle g_1, \ldots, g_N \rangle$ being a discrete group,

$$C^*(\Gamma) \to C^*_{\pi}(\Gamma) \to C^*_{red}(\Gamma)$$

and with π being a unitary representation of Γ , subject to weak containment conditions of type $\pi \otimes \pi \subset \pi$ and $1 \subset \pi$, which guarantee the existence of Δ, ε .

PROOF. We use the various findings from Theorem 13.16, following Woronowicz, the idea being to proceed in several steps, as follows:

(1) Theorem 13.16 and standard functional analysis arguments show that the cocommutative Woronowicz algebras should appear as intermediate quotients, as follows:

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma)$$

(2) The existence of $\Delta : A \to A \otimes A$ requires our intermediate quotient to appear as follows, with π being a unitary representation of Γ , satisfying the condition $\pi \otimes \pi \subset \pi$, taken in a weak containment sense, and with the tensor product \otimes being taken here to be compatible with our usual maximal tensor product \otimes for the C^* -algebras:

$$C^*(\Gamma) \to C^*_{\pi}(\Gamma) \to C^*_{red}(\Gamma)$$

(3) With this condition imposed, the existence of the antipode $S : A \to A^{opp}$ is then automatic, coming from the group antirepresentation $g \to g^{-1}$.

(4) The existence of the counit $\varepsilon : A \to \mathbb{C}$, however, is something non-trivial, related to amenability, and leading to a condition of type $1 \subset \pi$, as in the statement.

Let us focus now on the Kesten amenability criterion, from Theorem 13.16 (4), which brings connections with interesting mathematics and physics, and which in practice will be our main amenability criterion. In order to discuss this, we will need:

PROPOSITION 13.18. Given a Woronowicz algebra (A, u), with $u \in M_N(A)$, the moments of the main character $\chi = \sum_i u_{ii}$ are given by:

$$\int_G \chi^k = \dim \left(Fix(u^{\otimes k}) \right)$$

In the case $u \sim \bar{u}$ the law of χ is a usual probability measure, supported on [-N, N].

PROOF. The first assertion follows from the Peter-Weyl theory, which tells us that we have the following formula, valid for any corepresentation $v \in M_n(A)$:

$$\int_G \chi_v = \dim(Fix(v))$$

Indeed, with $v = u^{\otimes k}$ we obtain the result. As for the second assertion, if we assume $u \sim \bar{u}$, then we have $\chi = \chi^*$, and so $law(\chi)$ is a real probability measure, supported by the spectrum of χ . But, since the matrix $u \in M_N(A)$ is unitary, we have:

$$uu^* = 1 \implies ||u_{ij}|| \le 1, \forall i, j \implies ||\chi|| \le N$$

Thus the spectrum of the character satisfies $\sigma(\chi) \subset [-N, N]$, as desired.

In relation now with the notion of amenability, we have:

THEOREM 13.19. A Woronowicz algebra (A, u), with $u \in M_N(A)$, is amenable when

$$N \in supp\Big(law(Re(\chi))\Big)$$

and the support on the right depends only on $law(\chi)$.

PROOF. There are two assertions here, the proof being as follows:

(1) According to the Kesten amenability criterion, from Theorem 13.16 (4), the algebra A is amenable when the following condition is satisfied:

$$N \in \sigma(Re(\chi))$$

Now since $Re(\chi)$ is self-adjoint, we know from spectral theory that the support of its spectral measure $law(Re(\chi))$ is precisely its spectrum $\sigma(Re(\chi))$, as desired:

$$supp(law(Re(\chi))) = \sigma(Re(\chi))$$

(2) Regarding the second assertion, once again the variable $Re(\chi)$ being self-adjoint, its law depends only on the moments $\int_G Re(\chi)^p$, with $p \in \mathbb{N}$. But, we have:

$$\int_G Re(\chi)^p = \int_G \left(\frac{\chi + \chi^*}{2}\right)^p = \frac{1}{2^p} \sum_{|k|=p} \int_G \chi^k$$

Thus $law(Re(\chi))$ depends only on $law(\chi)$, and this gives the result.

Let us work out now in detail the group dual case. Here we obtain a very interesting measure, called Kesten measure of the group, as follows:

PROPOSITION 13.20. In the case $A = C^*(\Gamma)$ and $u = diag(g_1, \ldots, g_N)$, and with the normalization $1 \in u = \overline{u}$ made, we have the formula

$$\int_{\widehat{\Gamma}} \chi^p = \# \left\{ i_1, \dots, i_p \middle| g_{i_1} \dots g_{i_p} = 1 \right\}$$

counting the loops based at 1, having length p, on the corresponding Cayley graph.

PROOF. Consider indeed a discrete group $\Gamma = \langle g_1, \ldots, g_N \rangle$. The main character of $A = C^*(\Gamma)$, with fundamental corepresentation $u = diag(g_1, \ldots, g_N)$, is then:

$$\chi = g_1 + \ldots + g_N$$

Given a colored integer $k = e_1 \dots e_p$, the corresponding moment is given by:

$$\int_{\widehat{\Gamma}} \chi^k = \int_{\widehat{\Gamma}} (g_1 + \ldots + g_N)^k = \# \left\{ i_1, \ldots, i_p \middle| g_{i_1}^{e_1} \ldots g_{i_p}^{e_p} = 1 \right\}$$

In the self-adjoint case now, $u \sim \overline{u}$, as in the statement, we are only interested in the moments with respect to usual integers, $p \in \mathbb{N}$, and the above formula becomes:

$$\int_{\widehat{\Gamma}} \chi^p = \# \left\{ i_1, \dots, i_p \middle| g_{i_1} \dots g_{i_p} = 1 \right\}$$

Assume now that we have in addition $1 \in u$, so that the condition $1 \in u = \bar{u}$ in the statement is satisfied. At the level of the generating set $S = \{g_1, \ldots, g_N\}$ this means:

$$1 \in S = S^{-1}$$

Thus the corresponding Cayley graph is well-defined, with the elements of Γ as vertices, and with the edges g - h appearing when the following condition is satisfied:

$$gh^{-1} \in S$$

A loop on this graph based at 1, having length p, is then a sequence as follows:

$$(1) - (g_{i_1}) - (g_{i_1}g_{i_2}) - \ldots - (g_{i_1} \ldots g_{i_{p-1}}) - (g_{i_1} \ldots g_{i_p} = 1)$$

Thus the moments of χ count indeed such loops, as claimed.

In order to generalize the above result to arbitrary Woronowicz algebras, we can use the discrete quantum group philosophy. The fundamental result here is as follows:

THEOREM 13.21. Let (A, u) be a Woronowicz algebra, and assume, by enlarging if necessary u, that we have $1 \in u = \overline{u}$. The following formula

$$d(v,w) = \min\left\{k \in \mathbb{N} \middle| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

defines then a distance on Irr(A), which coincides with the geodesic distance on the associated Cayley graph. In the group dual case we obtain the usual distance.

PROOF. The fact that the lengths are finite follows from Woronowicz's analogue of Peter-Weyl theory, and the other verifications are as follows:

- (1) The symmetry axiom is clear.
- (2) The triangle inequality is elementary to establish as well.
- (3) The Cayley graph assertion is something elementary as well.

(4) Finally, in the group dual case, where our Woronowicz algebra is of the form $A = C^*(\Gamma)$, with $\Gamma = \langle S \rangle$ being a finitely generated discrete group, our normalization condition $1 \in u = \bar{u}$ means that the generating set must satisfy:

$$1 \in S = S^{-1}$$

But this is precisely the normalization condition for the discrete groups, and the fact that we obtain the same metric space is clear. \Box

Summarizing, we have a good understanding of what a discrete quantum group is. We can now formulate a generalization of Proposition 13.20, as follows:

THEOREM 13.22. Let (A, u) be a Woronowicz algebra, with the normalization assumption $1 \in u = \overline{u}$ made. The moments of the main character,

$$\int_G \chi^p = \dim \left(Fix(u^{\otimes p}) \right)$$

count then the loops based at 1, having lenght p, on the corresponding Cayley graph.

PROOF. Here the formula of the moments, with $p \in \mathbb{N}$, is the one coming from Proposition 13.18, and the Cayley graph interpretation comes from Theorem 13.21.

As an application of this, we can introduce the notion of growth, as follows:

DEFINITION 13.23. Given a closed subgroup $G \subset U_N^+$, with $1 \in u = \bar{u}$, consider the series whose coefficients are the ball volumes on the corresponding Cayley graph,

$$f(z) = \sum_{k} b_k z^k$$
, $b_k = \sum_{l(v) \le k} \dim(v)^2$

and call it growth series of the discrete quantum group \widehat{G} . In the group dual case, $G = \widehat{\Gamma}$, we obtain in this way the usual growth series of Γ .

There are many things that can be said about the growth, and we will be back to this. As a first such result, in relation with the notion of amenability, we have:

THEOREM 13.24. Polynomial growth implies amenability.

PROOF. We recall from Theorem 13.21 that the Cayley graph of \widehat{G} has by definition the elements of Irr(G) as vertices, and the distance is as follows:

$$d(v,w) = \min\left\{k \in \mathbb{N} \middle| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

By taking w = 1 and by using Frobenius reciprocity, the lengths are given by:

$$l(v) = \min\left\{k \in \mathbb{N} \middle| v \subset u^{\otimes k}\right\}$$

By Peter-Weyl we have then a decomposition as follows, where B_k is the ball of radius k, and where $m_k(v) \in \mathbb{N}$ are certain multiplicities:

$$u^{\otimes k} = \sum_{v \in B_k} m_k(v) \cdot v$$

By using now Cauchy-Schwarz, we obtain the following inequality:

$$m_{2k}(1)b_k = \sum_{v \in B_k} m_k(v)^2 \sum_{v \in B_k} \dim(v)^2$$
$$\geq \left(\sum_{v \in B_k} m_k(v) \dim(v)\right)^2$$
$$= N^{2k}$$

But shows that if b_k has polynomial growth, then the following happens:

$$\limsup_{k \to \infty} m_{2k}(1)^{1/2k} \ge N$$

Thus, the Kesten type criterion applies, and gives the result.

There are many other things that can be said, as a continuation of the above, notably with explicit computations of growth exponents for all the discrete quantum groups that we know, and with some further generalities too, of functional analytic nature, and in relation with Lie theory and its generalizations too. For more on all this, you can check my quantum group textbook [7], and the quantum group literature cited there.

To summarize now, we have a decent understanding of what a discrete quantum group is, and also of what amenability means, in the discrete quantum group setting. However, all this does not exactly solve the von Neumann algebra questions, and we have:

QUESTION 13.25. Which discrete quantum groups Γ have the property $L(\Gamma) \simeq R$? Equivalently, which compact quantum groups G have the property $L^{\infty}(G) \simeq R$?

Here the equivalence between the above two questions comes from the fact that, with $\Gamma = \hat{G}$, we have $L(\Gamma) = L^{\infty}(G)$. As for the questions themselves, normally the hyperfiniteness part can be dealt with as in the classical group case, by using the amenability theory developed above, and the problem is with the ICC property, guaranteeing factoriality, with no one presently knowing what this "quantum ICC" property is.

As a funny comment here, the equation $L(\Gamma) \simeq R$ is precisely the one Murray and von Neumann were stuck with, in the classical group case, some 90 years ago. Some sort of Connes is needed, coming and solving this problem, with new ideas.

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Finally, let us mention that in connection with amenability and hyperfiniteness, we have as well a series of further questions, in relation with the actions of quantum groups. To be more precise, the problems that we would like to solve are as follows:

(1) We would like to understand, given a compact group or quantum group acting on a von Neumann algebra, $G \curvearrowright P$, when the fixed point algebra P^G is a factor.

(2) More generally, we would like to understand under which assumptions on $G \curvearrowright P$ the fixed point algebra $(B \otimes P)^G$ is a factor, for any finite dimensional algebra B.

(3) In fact, we would like to understand when the fixed point algebra P^G , or more generally all the fixed point algebras $(B \otimes P)^G$, are the hyperfinite II₁ factor R.

These questions are all of interest in subfactor theory, the idea being that a quite standard construction of subfactors is $(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$, coming from a von Neumann algebra P, an inclusion of finite dimensional algebras $B_0 \subset B_1$, and a compact quantum group G acting on everything, provided that the fixed point algebras involved are indeed factors. And then, once such a subfactor constructed and studied, the main problem is that of understanding if this subfactor can be taken to be hyperfinite.

These are quite technical questions, to be discussed in chapter 16 below, when doing subfactors. Let us mention however, coming a bit in advance, that we have:

FACT 13.26. Assuming that $\Gamma = \widehat{G}$ has an outer action on the hyperfinite II₁ factor

 $\Gamma \curvearrowright R$

we can set $P = R \rtimes \Gamma$, and the answer to the above questions is yes.

Which brings us into the very interesting question on whether we have such outer actions $\Gamma \curvearrowright R$, with the status of the subject being as follows:

(1) All this goes back to work in the 80s of Ocneanu, and Wassermann too, with Ocneanu eventually conjecturing that any discrete group Γ , and more generally any discrete quantum group Γ , should have such an action. This question is still open.

(2) In practice, the result is known in the finite case, $|\Gamma| < \infty$, and more generally in the case where $C^*(\Gamma)$ has an inner faithful matrix model, in the sense of chapter 9, with this being worked out in [5] and its follow-ups, and then by Vaes.

(3) And there has been quite some work on this, since then. For the status of the question, and relations with other questions, such as the Connes embedding problem, Voiculescu microstates and more, we refer to Brannan-Chirvasitu-Freslon.

Summarizing, many things going on here, with the philosophy being somehow that, once we want our factors or subfactors to be hyperfinite, isomorphic to R, we are all of

the sudden into all sorts of interesting questions, in relation with advanced mathematics and physics. But more on this later, in chapter 16 below, when doing subfactors.

13d. Hyperfinite factors

Back to general theory, there are many other things that can be said, in relation with hyperfiniteness. We first have a reduction theory result, as follows:

THEOREM 13.27. Any tracial hyperfinite von Neumann algebra appears as

$$A = \int_X A_x \, dx$$

with the factors A_x being either usual matrix algebras, or the factor R.

PROOF. This follows indeed by combining the von Neumann reduction theory from [87] with the theory of R of Murray-von Neumann [60] and Connes [16].

More generally, we have the following result, this time in arbitrary type:

THEOREM 13.28. Given a hyperfinite von Neumann algebra $A \subset B(H)$, write its center as follows, with X being a measured space:

$$Z(A) = L^{\infty}(X)$$

The whole algebra A decomposes then over this measured space X, as follows,

$$A = \int_X A_x \, dx$$

with the fibers A_x being hyperfinite von Neumann factors, which can be of type I, II, III.

PROOF. This is again something heavy, combining the general reduction theory results of von Neumann with the work of Connes in the hyperfinite case. \Box

Which brings us into the question of classifying all hyperfinite factors. The result here, due to Connes [16], with a key contribution by Haagerup [33], is as follows:

THEOREM 13.29. The hyperfinite factors are as follows, with 1 factor in each class

 $I_{\rm N}, I_\infty$

II_1, II_∞

$III_0, III_\lambda, III_1$

and with the type II_1 one R being the most important, basically producing the others too.

PROOF. This is again heavy, based on early work of Murray-von Neumann in type II [60], then on heavy work by Connes in type II and III [15], [16], basically finishing the classification, and with a final contribution by Haagerup in type III₁ [33].

Getting back now to the II_1 factors, and beyond hyperfiniteness, where things are understood, with R being the only example, there is a whole classification program here, by Popa and others, going on. Let us mention that a main open problem is that of deciding whether the free group factors are isomorphic or not:

QUESTION 13.30. Are the von Neumann algebras of free groups isomorphic,

$$L(F_N) \simeq L(F_M)$$

for $N \neq M$, or not?

This question can be of course asked in crossed product form, in the spirit of the various crossed product results evoked above, and of advanced ergodic theory in general, with the space in question, producing the crossed product, being the point:

$$\{.\} \rtimes F_N \simeq^? \{.\} \rtimes F_M$$

This formulation, used by Popa, has the advantage of putting the above problem into a more conceptual framework, with lots of advanced machinery available around. However, it is not clear whether this formulation simplifies or not the original problem.

There are as well a number of alternative approaches to this question, and notably the Voiculescu one, using free probability, which is particularly conceptual and beautiful, the idea being that of recapturing the number $N \in \mathbb{N}$ from the knowledge of the von Neumann algebra $L(F_N)$, via an entropy-type invariant:

$$L(F_N) \rightsquigarrow N$$

This latter program, while not solving the original problem, due to technical difficulties, is however very successful, in the sense that it has led to a lot of interesting results and computations, in relation with a lot of mathematics and physics.

Is the free group factor problem something belonging to logic, as the difficult problems in functional analysis usually end up being? No one really knows the answer here.

Interestingly, the question is difficult to the point where the conjectural answer, yes or no, is not known. And even worse, excluding the many people who have spent considerable time on the matter, years or more, working on yes or no, most people familiar with the question don't even really know what to wish for, yes or no, as an answer.

In what concerns us, we have been quite close in this book to the ideas of Voiculescu, but, as a surprise, these very ideas of Voiculescu lead us into wishing for a yes answer to the above question, which is opposite to his no wish, and work using free entropy. Indeed, to put things in context, let us formulate the question in the following way:

QUESTION 13.31. Is there a factor F, standing as a free counterpart for R?

And wouldn't you wish for a yes answer to this question, with F being of course all the free group factors $L(F_N)$ combined, and probably many more, coming from all sorts of free quantum groups, free homogeneous spaces, or other free manifolds. It would be good to know in free geometry that what we get by default is this factor F.

As a last comment here, later on, when doing subfactors, we will see that the particular factor $F = L(F_{\infty})$ quite does the job there, in subfactors, being more of less the only "free factor" that is needed, for that theory. But this does not really solve Question 13.31 in the context of subfactor theory because, ironically, the main questions there, including the "free" ones, rather concern the subfactors of the good old hyperfinite factor R.

13e. Exercises

Exercises:

EXERCISE 13.32. EXERCISE 13.33. EXERCISE 13.34. EXERCISE 13.35. EXERCISE 13.36. EXERCISE 13.37. EXERCISE 13.38. EXERCISE 13.38. EXERCISE 13.39. Bonus exercise.

Amenable subfactors

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

Exercise 14.2.

Exercise 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.

Hyperfinite subfactors

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

EXERCISE 15.7.

EXERCISE 15.8.

Bonus exercise.

Spectral measures

16a. Small index

We have seen so far the foundations of Jones' subfactor theory, along with results regarding the most basic classes of such subfactors, namely those coming from compact groups, discrete group duals, and more generally compact quantum groups. These subfactors all have integer index, $N \in \mathbb{N}$, and appear as subfactors of the Murray-von Neumann hyperfinite II₁ factor R, either by definition, or by theorem, or by conjecture.

This suggests looking into the classification of subfactors of integer index, or into the classification of the subfactors of R, or into the classification of the subfactors of R having integer index. These are all good questions, that we will discuss here.

Before starting, however, and in order to have an idea on what we want to do, we should discuss the following question: should the index $N \in [1, \infty)$ be small, or big? This is something quite philosophical, and non-trivial, the situation being as follows:

- (1) Mathematics and basic common sense suggest that subfactors should fall into two main classes, "series" and "exceptional". From this perspective, the series, corresponding to uniform values of the index, must be investigated first.
- (2) In practice now, passed a few simple cases, such as the FC or TL subfactors, we cannot hope for the index to take full uniform values. The more reasonable question here is that of looking at the case where $N \in \mathbb{N}$ is uniform.
- (3) The problem now is that, in the lack of theory here, this basically brings us back to groups, group duals, and more generally compact quantum groups, whose combinatorics is notoriously simpler than that of the arbitrary subfactors.
- (4) In short, naivity and pure mathematics tell us to investigate the "big index" case first, but with the remark however that we are missing something, and so that we must do in parallel some study in the "small index" case too.

All this does not look very clear, and so after this discussion, we are basically still in the dark. So, should the answer come then from physics, and applications?

Unfortunately, things here are quite complicated too, basically due to our current poor understanding of quantum mechanics, and of what precisely is to be done, in order to

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have things in physics moving. And in fact, things here are in fact split too, a bit in the same way as above, the situation being basically as follows:

- (1) The very small index range, $N \in [1, 4]$, is subject to the remarkable "quantization" result of Jones, stating that we should have $N = 4\cos^2(\frac{\pi}{n})$, and has strong ties with a number of considerations in conformal field theory.
- (2) In what concerns the other end, N >> 0, this is in relation with statistical mechanics, once again following work of Jones on the subject, and with the index itself corresponding to physicists' famous "big N" variable.

In short, no hope for an answer here. At least with our current knowledge of the subject. Probably most illustrating here is the fact that the main experts, starting with Jones himself, have always being split, working on both small and big index.

Getting away now from these philosophical difficulties, and back to our present book, which is rather elementary and mathematical, in this final chapter we will survey the main structure and classification results available, both in small and big index.

As already mentioned, we will focus on the subfactors of the Murray-von Neumann hyperfinite II₁ factor R, by taking for granted the fact that these subfactors are the most "important", and related to physics. With the side remark, however, that this is actually subject to debate too, with many mathematicians opting for bigger factors like $L(F_{\infty})$, and with some physicists joining them too. But let us not get into this here.

In order to get started now, in order to talk about classification, we need invariants for our subfactors. Which brings us into a third controversy, namely the choice between algebraic and analytic invariants. The situation here is as follows:

DEFINITION 16.1. Associated to any finite index subfactor $A \subset B$, having planar algebra $P = (P_k)$, are the following invariants:

- (1) Its principal graph X, which describes the inclusions $P_0 \subset P_1 \subset P_2 \subset \ldots$, with the reflections coming from basic constructions removed.
- (2) Its fusion algebra F, which describes the fusion rules for the various types of bimodules that can appear, namely A A, A B, B A, B B.
- (3) Its Poincaré series f, which is the generating series of the graded components of the planar algebra, $f(z) = \sum_k \dim(P_k) z^k$.
- (4) Its spectral measure μ , which is the probability measure having as moments the dimensions of the planar algebra components, $\int x^k d\mu(x) = \dim(P_k)$.

This definition is of course something a bit informal, and there is certainly some work to be done, in order to fully define all the above invariants X, F, f, μ , and to work out the precise relation between them. We will be back to this later, but for the moment, let us keep in mind the fact that associated to a given subfactor $A \subset B$ are several combinatorial

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invariants, which are not exactly equivalent, but are definitely versions of the same thing, the "combinatorics of the subfactor", and which come in algebraic or analytic flavors.

So, what to use? As before, in relation with the previous controversies, the main experts, starting with Jones himself, have always being split themselves on this question, working with both algebraic and analytic invariants. Generally speaking, the algebraic invariants, which are (1) and (2) in the above list, tend to be more popular in small index, while the analytic invariants, (3) and (4), are definitely more popular in big index.

In order to get started now, let us first discuss the question of classifying the subfactors of the hyperfinite II₁ factor R, up to isomorphism, having index $N \leq 4$.

This is something quite tricky, and the main idea here will be the fact, coming from the proof of the Jones index restriction theorem, explained in chapter 3, that the index $N \in (1, 4]$ must be the squared norm of a certain graph:

$$N = ||X||^2$$

Now with this observation in hand, the assumption $N \leq 4$ forces X to be one of the Coxeter-Dynkin graphs of type ADE, and then a lot of work, both of classification and exclusion, leads to an ADE classification for the subfactors of R having index $N \leq 4$.

This was for the idea. More in detail now, let us begin by explaining in detail how our subfactor invariant here, which will be the principal graph X, is constructed.

Consider first an arbitrary finite index irreducible subfactor $A_0 \subset A_1$, with associated planar algebra $P_k = A'_0 \cap A_k$, and let us look at the following system of inclusions:

$$P_0 \subset P_1 \subset P_2 \subset \ldots$$

By taking the Bratelli diagram of this system of inclusions, and then deleting the reflections coming from basic constructions, which automatically appear at each step, according to the various results from chapter 3, we obtain a certain graph X, called principal graph of $A_0 \subset A_1$. The main properties of X can be summarized as follows:

THEOREM 16.2. The principal graph X has the following properties:

- (1) The higher relative commutant $P_k = A'_0 \cap A_k$ is isomorphic to the abstract vector space spanned by the 2k-loops on X based at the root.
- (2) In the amenable case, where $A_1 = R$ and when the subfactor is "amenable", the index of $A_0 \subset A_1$ is given by $N = ||X||^2$.

PROOF. This is something standard, the idea being as follows:

(1) The statement here, which explains among others the relation between the principal graph X, and the other subfactor invariants, from Definition 16.1, comes from the definition of the principal graph, as a Bratelli diagram, with the reflections removed.

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(2) This is actually a quite subtle statement, but for our purposes here, we can take the equality $N = ||X||^2$, which reminds a bit the Kesten amenability condition for discrete groups, as a definition for the amenability of the subfactor. With the remark that for the Popa diagonal subfactors what we have here is precisely the Kesten amenability condition for the underlying discrete group Γ , and that, more generally, for the arbitrary generalized Popa or Wassermann subfactors, what we have here is precisely the Kesten type amenability condition for the underlying discrete quantum group Γ .

As a consequence of the above, in relation with classification questions, we have:

THEOREM 16.3. The principal graph of a subfactor having index $N \leq 4$ must be one of the Coxeter-Dynkin graphs of type ADE.

PROOF. This follows indeed from the formula $N = ||X||^2$ from the above result, and from the considerations from the proof of the Jones index restriction theorem, explained in chapter 3. For full details on all this, we refer for instance to [28].

More in detail now, the usual Coxeter-Dynkin graphs are as follows:

Here the graphs A_n with $n \ge 2$ and D_n with $n \ge 3$ have by definition n vertices each, \tilde{A}_{2n} with $n \ge 1$ has 2n vertices, and \tilde{D}_n with $n \ge 4$ has n + 1 vertices. Thus, the first graph in each series is by definition as follows:

$$A_2 = \bullet - \circ \qquad D_3 = \bullet - \circ \qquad \tilde{A}_2 = \bullet \qquad \tilde{D}_4 = \bullet - \circ - \circ$$

There are also a number of exceptional Coxeter-Dynkin graphs. First we have:

$$E_{6} = \bullet - \circ - \circ - \circ - \circ$$

$$E_{7} = \bullet - \circ - \circ - \circ - \circ - \circ - \circ$$

$$E_{8} = \bullet - \circ - \circ - \circ - \circ - \circ - \circ$$

Also, we have as well index 4 versions of the above exceptional graphs, as follows:

Getting back now to Theorem 16.3, with this list in hand, the story is not over here, because we still have to understand which of these graphs can really appear as principal graphs of subfactors. And, for those graphs which can appear, we must understand the structure and classification of the subfactors of R, having them as principal graphs.

In short, there is still a lot of work to be done, as a continuation of Theorem 16.3. The subfactors of index ≤ 4 were intensively studied in the 80s and early 90s, and about 10 years after Jones' foundational paper [40], a complete classification result was found, with contributions by many authors. A simplified form of this result is as follows:

THEOREM 16.4. The principal graphs of subfactors of index ≤ 4 are:

- (1) Index < 4 graphs: A_n , D_{even} , E_6 , E_8 .
- (2) Index 4 finite graphs: \tilde{A}_{2n} , \tilde{D}_n , \tilde{E}_6 , \tilde{E}_7 , \tilde{E}_8 .
- (3) Index 4 infinite graphs: A_{∞} , $A_{-\infty,\infty}$, D_{∞} .

PROOF. As already mentioned, this is something quite heavy, with contributions by many authors, and among the main papers to be read here, let us mention [40], [64], [65], [69]. Observe that the graphs D_{odd} and E_7 don't appear in the above list. This is one of the subtle points of subfactor theory. For a discussion here, see [23].

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There are many other things that can be said about the subfactors of index $N \leq 4$, both at the theoretical level, of the finite depth and more generally of the amenable subfactors, and at the level of the ADE classification, which makes connections with other ADE classifications. We refer here to [23], [28], [64], [65], [69], [70].

Regarding now the subfactors of index $N \in (4, 5]$, and also of small index above 5, these can be classified, but this is a long and complicated story. Let us just record here the result in index 5, which is something quite easy to formulate, as follows:

THEOREM 16.5. The principal graphs of the irreducible index 5 subfactors are:

- (1) A_{∞} , and a non-extremal perturbation of $A_{\infty}^{(1)}$.
- (2) The McKay graphs of $\mathbb{Z}_5, D_5, GA_1(5), A_5, S_5$.
- (3) The twists of the McKay graphs of A_5, S_5 .

PROOF. This is a heavy result, and we refer to [45] for the whole story. The above formulation is the one from [45], with the subgroup subfactors there replaced by fixed point subfactors, and with the cyclic groups denoted as usual by \mathbb{Z}_N .

As a comment here, the above N = 5 result was much harder to obtain than the classification in index N = 4, obtained as a consequence of Theorem 16.4. However, at the level of the explicit construction of such subfactors, things are quite similar at N = 4 and N = 5, with the fixed point subfactors associated to quantum permutation groups $G \subset S_N^+$ providing most of the examples. We refer here to [6] and related papers.

In index N = 6 now, the subfactors cannot be classified, at least in general, due to several uncountable families, coming from groups, group duals, and more generally compact quantum groups. The exact assumption to be added is not known yet.

Summarizing, the current small index classification problem meets considerable difficulties in index N = 6, and right below. In small index N > 6 the situation is largely unexplored. We refer here to [53] and the recent literature on the subject.

16b. Spectral measures

Before getting into the case where the index is big, N >> 0, let us comment on one of the key ingredients for the above classification results, at N < 6. This is the Jones annular theory of subfactors, which is something very beautiful and useful, regarding the case where the index is arbitrary, $N \in [1, \infty)$. The main result is as follows:

THEOREM 16.6. The theta series of a subfactor of index N > 4, which is given by

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

with $f = \sum_k \dim(P_k) z^k$ being the Poincaré series, has positive coefficients.

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PROOF. This is something quite advanced, the idea being that Θ is the generating series of a certain series of multiplicities associated to the subfactor, and more specifically associated to the canonical inclusion $TL_N \subset P$. We refer here to Jones' paper [44].

In relation to this, and to some questions from physics as well, coming from conformal field theory, an interesting question is that of computing the "blowup" of the spectral measure of the subfactor, via the Jones change of variables, namely:

$$z \to \frac{q}{1+q^2}$$

This question makes sense in any index, meaning both $N \in [1, 4]$, where Theorem 16.6 does not apply, and $N \in (4, \infty)$, where Theorem 16.6 does apply. We will discuss in what follows both these questions, by starting with the small index one, $N \in [1, 4]$.

Following [6] and related papers, it is convenient to stay, at least for the beginning, at a purely elementary level, and associate such series to any rooted bipartite graph. Let us start with the following definition, which is something straightforward, inspired by the definition of the Poincaré series of a subfactor, and by Theorem 16.2:

DEFINITION 16.7. The Poincaré series of a rooted bipartite graph X is

$$f(z) = \sum_{k=0}^{\infty} \operatorname{loop}_X(2k) z^k$$

where $loop_X(2k)$ is the number of 2k-loops based at the root.

In the case where X is the principal graph of a subfactor $A_0 \subset A_1$, this series f is the Poincaré series of the subfactor, in the usual sense:

$$f(z) = \sum_{k=0}^{\infty} \dim(A'_0 \cap A_k) z^k$$

In general, the Poincaré series should be thought of as being a basic representation theory invariant of the underlying group-like object. For instance for the Wassermann type subfactor associated to a compact Lie group $G \subset U_N$, the Poincaré series is:

$$f(z) = \int_G \frac{1}{1 - Tr(g)z} \, dg$$

Regarding now the theta series, this can introduced as a version of the Poincaré series, via the change of variables $z^{-1/2} = q^{1/2} + q^{-1/2}$, as follows:

DEFINITION 16.8. The theta series of a rooted bipartite graph X is

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

where f is the Poincaré series.

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The theta series can be written as $\Theta(q) = \sum a_r q^r$, and it follows from the above formula, via some simple manipulations, that its coefficients are integers:

 $a_r \in \mathbb{Z}$

In fact, we have the following explicit formula from Jones' paper [44], relating the coefficients of $\Theta(q) = \sum a_r q^r$ to those of the Poincaré series $f(z) = \sum c_k z^k$:

$$a_r = \sum_{k=0}^{r} (-1)^{r-k} \frac{2r}{r+k} \binom{r+k}{r-k} c_k$$

In the case where X is the principal graph of a subfactor $A_0 \subset A_1$ of index N > 4, it is known from [44] that the numbers a_r are certain multiplicities associated to the planar algebra inclusion $TL_N \subset P$, as explained in Theorem 16.6 and its proof. In particular, the coefficients of the theta series are in this case positive integers:

$$a_r \in \mathbb{N}$$

Before getting into computations, let us discuss as well the measure-theoretic versions of the above invariants. Once again, we start with an arbitrary rooted bipartite graph X. We can first introduce a measure μ , whose Stieltjes transform is f, as follows:

DEFINITION 16.9. The real measure μ of a rooted bipartite graph X is given by

$$f(z) = \int_0^\infty \frac{1}{1 - xz} \, d\mu(x)$$

where f is the Poincaré series.

In the case where X is the principal graph of a subfactor $A_0 \subset A_1$, we recover in this way the spectral measure of the subfactor, as introduced in Definition 16.1, with the remark however that the existence of such a measure μ was not discussed there. In general, and so also in the particular subfactor case, clarifying the things here, the fact that μ as above exists indeed comes from the following simple fact:

PROPOSITION 16.10. The real measure μ of a rooted bipartite graph X is given by the following formula, where $L = MM^t$, with M being the adjacency matrix of the graph,

$$\mu = law(L)$$

and with the probabilistic computation being with respect to the expectation

$$A \rightarrow < A >$$

with $\langle A \rangle$ being the (*, *)-entry of a matrix A, where * is the root.
PROOF. With the conventions in the statement, namely $L = MM^t$, with M being the adjacency matrix, and with $\langle A \rangle$ being the (*, *)-entry of a matrix A, we have:

$$f(z) = \sum_{k=0}^{\infty} \operatorname{loop}_{X}(2k) z^{k}$$
$$= \sum_{k=0}^{\infty} \langle L^{k} \rangle z^{k}$$
$$= \left\langle \frac{1}{1 - Lz} \right\rangle$$

But this shows that we have the formula $\mu = law(L)$, as desired.

In the subfactor case some further interpretations are available as well. For instance in the case of the fixed point subfactors coming from of a compact group $G \subset U_N$, discussed after Definition 16.7 above, μ is the spectral measure of the main character:

$$\mu = law(\chi)$$

In relation now with the theta series, things are more tricky, in order to introduce its measure-theoretic version. Following [6], let us introduce the following notion:

DEFINITION 16.11. The circular measure ε of a rooted bipartite graph X is given by

$$d\varepsilon(q) = d\mu((q+q^{-1})^2)$$

where μ is the associated real measure.

In other words, the circular measure ε is by definition the pullback of the usual real measure μ via the following map, coming from the theory of the theta series in [44]:

$$\mathbb{R} \cup \mathbb{T} \to \mathbb{R}_+$$
$$q \to (q + q^{-1})^2$$

As we will see, all this best works in index $N \in [1, 4]$, with the circular measure ε being here the best-looking invariant, among all subfactor invariants. In index N > 4 things will turn to be quite complicated, but more on this later.

As a basic example for all this, assume that μ is a discrete measure, supported by n positive numbers $x_1 < \ldots < x_n$, with corresponding densities p_1, \ldots, p_n :

$$\mu = \sum_{i=1}^{n} p_i \delta_{x_i}$$

For each $i \in \{1, ..., n\}$ the equation $(q + q^{-1})^2 = x_i$ has four solutions, that we can denote $q_i, q_i^{-1}, -q_i, -q_i^{-1}$. With this notation, we have:

$$\varepsilon = \frac{1}{4} \sum_{i=1}^{n} p_i \left(\delta_{q_i} + \delta_{q_i^{-1}} + \delta_{-q_i} + \delta_{-q_i^{-1}} \right)$$

In general, the basic properties of ε can be summarized as follows:

PROPOSITION 16.12. The circular measure has the following properties:

- (1) ε has equal density at $q, q^{-1}, -q, -q^{-1}$.
- (2) The odd moments of ε are 0.
- (3) The even moments of ε are half-integers.
- (4) When X has norm $\leq 2, \varepsilon$ is supported by the unit circle.
- (5) When X is finite, ε is discrete.
- (6) If K is a solution of $L = (K + K^{-1})^2$, then $\varepsilon = \text{law}(K)$.

PROOF. These results can be deduced from definitions, the idea being that (1-5) are trivial, and that (6) follows from the formula of μ from Proposition 16.10.

In addition to the above, we have the following key formula, which gives the even moments of ε , and makes the connection with the Jones theta series:

THEOREM 16.13. We have the Stieltjes transform type formula

$$2\int \frac{1}{1-qu^2} d\varepsilon(u) = 1 + T(q)(1-q)$$

where the T series of a rooted bipartite graph X is by definition given by

$$T(q) = \frac{\Theta(q) - q}{1 - q}$$

with Θ being the associated theta series.

PROOF. This follows by applying the change of variables $q \to (q + q^{-1})^2$ to the fact that f is the Stieltjes transform of μ . Indeed, we obtain in this way:

$$2\int \frac{1}{1-qu^2} d\varepsilon(u) = 1 + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right) \\ = 1 + \Theta(q) - q \\ = 1 + T(q)(1-q)$$

Thus, we are led to the conclusion in the statement.

As a final theoretical result about all these invariants, which is this time something non-trivial, in the subfactor case, we have the following result, due to Jones [44]:

THEOREM 16.14. In the case where X is the principal graph of an irreducible subfactor of index > 4, the moments of ε are positive numbers.

PROOF. This follows indeed from the result in [44] that the coefficients of Θ are positive numbers, as explained in Theorem 16.6, via the formula in Theorem 16.13.

Summarizing, we have a whole menagery of subfactor, planar algebra and bipartite graph invariants, which come in several flavors, namely series and measures, and which can be linear or circular, and which all appear as versions of the Poincaré series.

Our claim now is that the circular measure ε is the "best" invariant. As a first justification for this claim, let us compute ε for the simplest possible graph in the index range $N \in [1, 4]$, namely the graph \tilde{A}_{2n} . We obtain here something nice, as follows:

THEOREM 16.15. The circular measure of the basic index 4 graph, namely

$$\tilde{A}_{2n} = \begin{vmatrix} \circ - \circ - \circ \cdots \circ - \circ - \circ \\ \bullet & - \circ - \circ - \circ - \circ - \circ \end{vmatrix}$$

is the uniform measure on the 2n-roots of unity.

PROOF. Let us identify the vertices of $X = \tilde{A}_{2n}$ with the group $\{w^k\}$ formed by the 2*n*-th roots of unity in the complex plane, where $w = e^{\pi i/n}$. The adjacency matrix of X acts then on the functions $f \in C(X)$ in the following way:

$$Mf(w^{s}) = f(w^{s-1}) + f(w^{s+1})$$

But this shows that we have $M = K + K^{-1}$, where K is given by:

$$Kf(w^s) = f(w^{s+1})$$

Thus we can use the last assertion in Proposition 16.12, and we get $\varepsilon = \text{law}(K)$, which is the uniform measure on the 2*n*-roots of unity. See [6] for details.

In order to discuss all this more systematically, and for all the ADE graphs, the idea will be that of looking at the combinatorics of the roots of unity. Let us introduce:

DEFINITION 16.16. The series of the form

$$\xi(n_1,\ldots,n_s:m_1,\ldots,m_t) = \frac{(1-q^{n_1})\ldots(1-q^{n_s})}{(1-q^{m_1})\ldots(1-q^{m_t})}$$

with $n_i, m_i \in \mathbb{N}$ are called cyclotomic.

It is technically convenient to allow as well $1 + q^n$ factors, to be designated by n^+ symbols in the above writing. For instance we have, by definition:

$$\xi(2^+:3) = \xi(4:2,3)$$

Also, it is convenient in what follows to use the following notations:

$$\xi' = \frac{\xi}{1-q}$$
 , $\xi'' = \frac{\xi}{1-q^2}$

The Poincaré series of the ADE graphs are given by quite complicated formulae. However, the corresponding T series are all cyclotomic, as follows:

THEOREM 16.17. The T series of the ADE graphs are as follows:

- (1) For A_{n-1} we have $T = \xi(n-1:n)$.
- (2) For D_{n+1} we have $T = \xi(n 1^+ : n^+)$.
- (3) For \tilde{A}_{2n} we have $T = \xi'(n^+ : n)$.
- (4) For \tilde{D}_{n+2} we have $T = \xi''(n+1^+:n)$.
- (5) For E_6 we have $T = \xi(8:3,6^+)$.
- (6) For E_7 we have $T = \xi(12:4,9^+)$.
- (7) For E_8 we have $T = \xi(5^+, 9^+ : 15^+)$.
- (8) For \tilde{E}_6 we have $T = \xi(6^+ : 3, 4)$.
- (9) For E_7 we have $T = \xi(9^+ : 4, 6)$.
- (10) For E_8 we have $T = \xi(15^+ : 6, 10)$.

PROOF. These formulae were obtained in [6], by counting loops, then by making the change of variables $z^{-1/2} = q^{1/2} + q^{-1/2}$, and factorizing the resulting series. An alternative proof for these formulae can be obtained by using planar algebra methods.

Our purpose now will be that of converting the above technical results, regarding the T series, into some final results, regarding the corresponding circular measures ε . For this purpose, we will use the conversion formula in Theorem 16.13.

In order to formulate our results, we will need some more theory. First, we have:

DEFINITION 16.18. A cyclotomic measure is a probability measure ε on the unit circle, having the following properties:

- (1) ε is supported by the 2n-roots of unity, for some $n \in \mathbb{N}$.
- (2) ε has equal density at $q, q^{-1}, -q, -q^{-1}$.

It follows from Theorem 16.17 that the circular measures of the finite ADE graphs are supported by certain roots of unity, hence are cyclotomic. We will be back to this.

At the general level now, let us introduce as well the following notion:

DEFINITION 16.19. The T series of a cyclotomic measure ε is given by:

$$1 + T(q)(1-q) = 2 \int \frac{1}{1-qu^2} d\varepsilon(u)$$

Observe that this formula is nothing but the one in Theorem 16.13, written now in the other sense. In other words, if the cyclotomic measure ε happens to be the circular measure of a rooted bipartite graph, then the T series as defined above coincides with the T series as defined before. This is useful for explicit computations.

We are now ready to discuss the circular measures of the various ADE graphs. The idea is that these measures are all cyclotomic, of level ≤ 3 , and can be expressed in terms of the basic polynomial densities of degree ≤ 6 , namely:

$$\alpha = Re(1 - q^2)$$

$$\beta = Re(1 - q^4)$$

$$\gamma = Re(1 - q^6)$$

To be more precise, we have the following result, with α, β, γ being as above, with d_n being the uniform measure on the 2*n*-th roots of unity, and with $d'_n = 2d_{2n} - d_n$ being the uniform measure on the odd 4n-roots of unity:

THEOREM 16.20. The circular measures of the ADE graphs are given by:

(1)
$$A_{n-1} \to \alpha_n$$
.
(2) $\tilde{A}_{2n} \to d_n$.
(3) $D_{n+1} \to \alpha'_n$.
(4) $\tilde{D}_{n+2} \to (d_n + d'_1)/2$.
(5) $E_6 \to \alpha_{12} + (d_{12} - d_6 - d_4 + d_3)/2$.
(6) $E_7 \to \beta'_9 + (d'_1 - d'_3)/2$.
(7) $E_8 \to \alpha'_{15} + \gamma'_{15} - (d'_5 + d'_3)/2$.
(8) $\tilde{E}_{n+3} \to (d_n + d_3 + d_2 - d_1)/2$.

PROOF. This follows from the T series formulae in Theorem 16.17, via some routine manipulations, based on the general conversion formulae given above. \Box

It is possible to further build along the above lines, with a combinatorial refinement of the formulae in Theorem 16.20, making appear a certain connection with the Deligne work on the exceptional series of Lie groups, which is not understood yet.

16c. Measure blowup

All the above, which was quite nice, was about index $N \in [1, 4]$, where the Jones annular theory result from [44] does not apply. In higher index now, $N \in (4, \infty)$, where the Jones result does apply, the precise correct "blowup" manipulation on the spectral measure is not known yet. The known results here are as follows:

- (1) One one hand, there is as a computation for some basic Hadamard subfactors, with nice blowup, on a certain noncommutative manifold [6].
- (2) On the other hand, there are many computations by Evans-Pugh, with quite technical blowup results, on some suitable real algebraic manifolds [6].

We will discuss in what follows (1), and to be more precise the computation of the spectral measure, and then the blowup problem, for the subfactors coming from the deformed Fourier matrices. Let us start with the following definition:

DEFINITION 16.21. Given two finite abelian groups G, H, we consider the corresponding deformed Fourier matrix, given by the formula

$$(F_G \otimes_Q F_H)_{ia,jb} = Q_{ib}(F_G)_{ij}(F_H)_{ab}$$

and we factorize the associated representation π_Q of the algebra $C(S^+_{G \times H})$,



with $C(G_Q)$ being the Hopf image of this representation π_Q .

Explicitly computing the above quantum permutation group $G_Q \subset S^+_{G \times H}$, as function of the parameter matrix $Q \in M_{G \times H}(\mathbb{T})$, will be our main purpose, in what follows. In order to do so, we first have the following elementary result:

PROPOSITION 16.22. We have a factorization as follows,



given on the standard generators by the formulae

$$U_{ab}^{(i)} = \sum_{j} W_{ia,jb} \quad , \quad V_{ij} = \sum_{a} W_{ia,jb}$$

independently of b, where W is the magic matrix producing π_Q .

PROOF. With $K = F_G$, $L = F_H$ and M = |G|, N = |H|, the formula of the magic matrix $W \in M_{G \times H}(M_{G \times H}(\mathbb{C}))$ associated to $H = K \otimes_Q L$ is as follows:

$$(W_{ia,jb})_{kc,ld} = \frac{1}{MN} \cdot \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot \frac{K_{ik}K_{jl}}{K_{il}K_{jk}} \cdot \frac{L_{ac}L_{bd}}{L_{ad}L_{bc}}$$
$$= \frac{1}{MN} \cdot \frac{Q_{ic}Q_{jd}}{Q_{id}Q_{jc}} \cdot K_{i-j,k-l}L_{a-b,c-d}$$

Our claim now is that the representation π_Q constructed in Definition 16.21 can be factorized in three steps, up to the factorization in the statement, as follows:



Indeed, the construction of the map on the left is standard. Regarding the second factorization, this comes from the fact that since the elements V_{ij} depend on i - j, they satisfy the defining relations for the quotient algebra $C(S_G^+) \to C(G)$. Finally, regarding the third factorization, observe that $W_{ia,jb}$ depends only on i, j and on a - b. By summing over j we obtain that the elements $U_{ab}^{(i)}$ depend only on a - b, and we are done.

We have now all needed ingredients for refining Proposition 16.22, as follows:

PROPOSITION 16.23. We have a factorization as follows,



where the group on the bottom is given by:

$$\Gamma_{G,H} = H^{*G} / \left\langle \left[c_1^{(i_1)} \dots c_s^{(i_s)}, d_1^{(j_1)} \dots d_s^{(j_s)} \right] = 1 \middle| \sum_r c_r = \sum_r d_r = 0 \right\rangle$$

PROOF. Assume that we have a representation, as follows:

$$\pi: C^*(\Gamma) \rtimes C(G) \to M_L(\mathbb{C})$$

Let Λ be a G-stable normal subgroup of Γ , so that G acts on Γ/Λ , and we can form the product $C^*(\Gamma/\Lambda) \rtimes C(G)$, and assume that π is trivial on Λ . Then π factorizes as:



With $\Gamma = H^{*G}$, this gives the result.

We have now all the needed ingredients for proving a main result, as follows: THEOREM 16.24. When Q is generic, the minimal factorization for π_Q is



where on the bottom

$$\Gamma_{G,H} \simeq \mathbb{Z}^{(|G|-1)(|H|-1)} \rtimes H$$

is the discrete group constructed above.

PROOF. Consider the factorization in Proposition 16.23, which is as follows, where L denotes the Hopf image of π_Q :

$$\theta: C^*(\Gamma_{G,H}) \rtimes C(G) \to L$$

To be more precise, this morphism produces the following commutative diagram:



The first observation is that the injectivity assumption on C(G) holds by construction, and that for $f \in C(G)$, the matrix $\pi(f)$ is "block scalar". Now for $r \in \Gamma_{G,H}$ with $\theta(r \otimes 1) = \theta(1 \otimes f)$ for some $f \in C(G)$, we see, using the commutative diagram, that $\pi(r \otimes 1)$ is block scalar. Thus, modulo some standard algebra, we are done.

Summarizing, we have computed the quantum permutation groups associated to the Dită deformations of the tensor products of Fourier matrices, in the case where the deformation matrix Q is generic. For some further computations, in the case where the deformation matrix Q is no longer generic, we refer to the follow-ups of [6].

Let us compute now the Kesten measure $\mu = law(\chi)$, in the case where the deformation matrix is generic, as before. Our results here will be a combinatorial moment formula, a geometric interpretation of it, and an asymptotic result. We first have:

16C. MEASURE BLOWUP

THEOREM 16.25. We have the moment formula

$$\int \chi^p = \frac{1}{|G| \cdot |H|} \# \left\{ \begin{array}{l} i_1, \dots, i_p \in G \\ d_1, \dots, d_p \in H \end{array} \middle| = [(i_1, d_1), (i_2, d_2), \dots, (i_p, d_p)] \\ = [(i_1, d_p), (i_2, d_1), \dots, (i_p, d_{p-1})] \end{array} \right\}$$

where the sets between square brackets are by definition sets with repetition.

PROOF. According to the various formulae above, the factorization found in Theorem 16.24 is, at the level of standard generators, as follows:

$$\begin{array}{cccc} C(S_{G\times H}^+) & \to & C^*(\Gamma_{G,H}) \otimes C(G) & \to & M_{G\times H}(\mathbb{C}) \\ u_{ia,jb} & \to & \frac{1}{|H|} \sum_c F_{b-a,c} c^{(i)} \otimes v_{ij} & \to & W_{ia,jb} \end{array}$$

Thus, the main character of the quantum permutation group that we found in Theorem 16.24 is given by the following formula:

$$\chi = \frac{1}{|H|} \sum_{iac} c^{(i)} \otimes v_{ii}$$
$$= \sum_{ic} c^{(i)} \otimes v_{ii}$$
$$= \left(\sum_{ic} c^{(i)}\right) \otimes \delta_{1}$$

Now since the Haar functional of $C^*(\Gamma) \rtimes C(H)$ is the tensor product of the Haar functionals of $C^*(\Gamma), C(H)$, this gives the following formula, valid for any $p \ge 1$:

$$\int \chi^p = \frac{1}{|G|} \int_{\widehat{\Gamma}_{G,H}} \left(\sum_{ic} c^{(i)} \right)^p$$

Consider the elements $S_i = \sum_c c^{(i)}$. With standard notations, we have:

$$S_i = \sum_c (b_{i0} - b_{ic}, c)$$

Now observe that these elements multiply as follows:

$$S_{i_1} \dots S_{i_p} = \sum_{c_1 \dots c_p} \begin{pmatrix} b_{i_10} - b_{i_1c_1} + b_{i_2c_1} - b_{i_2,c_1+c_2} \\ + b_{i_3,c_1+c_2} - b_{i_3,c_1+c_2+c_3} + \dots \\ \dots + b_{i_p,c_1+\dots+c_{p-1}} - b_{i_p,c_1+\dots+c_p} \end{pmatrix}, \quad c_1 + \dots + c_p \end{pmatrix}$$

In terms of the new indices $d_r = c_1 + \ldots + c_r$, this formula becomes:

$$S_{i_1} \dots S_{i_p} = \sum_{d_1 \dots d_p} \begin{pmatrix} b_{i_10} - b_{i_1d_1} + b_{i_2d_1} - b_{i_2d_2} \\ + b_{i_3d_2} - b_{i_3d_3} + \dots & , d_p \\ \dots + b_{i_pd_{p-1}} - b_{i_pd_p} \end{pmatrix}$$

Now by integrating, we must have $d_p = 0$ on one hand, and on the other hand:

$$[(i_1,0),(i_2,d_1),\ldots,(i_p,d_{p-1})] = [(i_1,d_1),(i_2,d_2),\ldots,(i_p,d_p)]$$

Equivalently, we must have $d_p = 0$ on one hand, and on the other hand:

$$[(i_1, d_p), (i_2, d_1), \dots, (i_p, d_{p-1})] = [(i_1, d_1), (i_2, d_2), \dots, (i_p, d_p)]$$

Thus, by translation invariance with respect to d_p , we obtain:

$$\int_{\widehat{\Gamma}_{G,H}} S_{i_1} \dots S_{i_p} = \frac{1}{|H|} \# \left\{ d_1, \dots, d_p \in H \middle| \begin{bmatrix} (i_1, d_1), (i_2, d_2), \dots, (i_p, d_p) \end{bmatrix} \right\}$$

It follows that we have the following moment formula:

$$\int_{\widehat{\Gamma}_{G,H}} \left(\sum_{i} S_{i} \right)^{p} = \frac{1}{|H|} \# \left\{ \begin{array}{l} i_{1}, \dots, i_{p} \in G \\ d_{1}, \dots, d_{p} \in H \end{array} \middle| \begin{bmatrix} (i_{1}, d_{1}), (i_{2}, d_{2}), \dots, (i_{p}, d_{p}) \end{bmatrix} \right\}$$

Now by dividing by |G|, we obtain the formula in the statement.

The formula in Theorem 16.25 can be further interpreted as follows:

THEOREM 16.26. With M = |G|, N = |H| we have the formula

$$law(\chi) = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}law(A)$$

where the matrix

$$A \in C(\mathbb{T}^{MN}, M_M(\mathbb{C}))$$

is given by A(q) = Gram matrix of the rows of q.

PROOF. According to Theorem 16.25, we have the following formula:

$$\int \chi^{p} = \frac{1}{MN} \sum_{i_{1}...i_{p}} \sum_{d_{1}...d_{p}} \delta_{[i_{1}d_{1},...,i_{p}d_{p}],[i_{1}d_{p},...,i_{p}d_{p-1}]}$$

$$= \frac{1}{MN} \int_{\mathbb{T}^{MN}} \sum_{i_{1}...i_{p}} \sum_{d_{1}...d_{p}} \frac{q_{i_{1}d_{1}} \cdots q_{i_{p}d_{p}}}{q_{i_{1}d_{p}} \cdots q_{i_{p}d_{p-1}}} dq$$

$$= \frac{1}{MN} \int_{\mathbb{T}^{MN}} \sum_{i_{1}...i_{p}} \left(\sum_{d_{1}} \frac{q_{i_{1}d_{1}}}{q_{i_{2}d_{1}}} \right) \left(\sum_{d_{2}} \frac{q_{i_{2}d_{2}}}{q_{i_{3}d_{2}}} \right) \cdots \left(\sum_{d_{p}} \frac{q_{i_{p}d_{p}}}{q_{i_{1}d_{p}}} \right) dq$$

Consider now the Gram matrix in the statement, namely:

$$A(q)_{ij} = < R_i, R_j >$$

Here R_1, \ldots, R_M are the rows of the following matrix:

$$q \in \mathbb{T}^{MN} \simeq M_{M \times N}(\mathbb{T})$$

We have then the following computation:

$$\int \chi^{p} = \frac{1}{MN} \int_{\mathbb{T}^{MN}} \langle R_{i_{1}}, R_{i_{2}} \rangle \langle R_{i_{2}}, R_{i_{3}} \rangle \dots \langle R_{i_{p}}, R_{i_{1}} \rangle$$

$$= \frac{1}{MN} \int_{\mathbb{T}^{MN}} A(q)_{i_{1}i_{2}}A(q)_{i_{2}i_{3}}\dots A(q)_{i_{p}i_{1}}$$

$$= \frac{1}{MN} \int_{\mathbb{T}^{MN}} Tr(A(q)^{p}) dq$$

$$= \frac{1}{N} \int_{\mathbb{T}^{MN}} tr(A(q)^{p}) dq$$

But this gives the formula in the statement, and we are done.

In general, the moments of the Gram matrix A are given by a quite complicated formula, and we cannot expect to have a refinement of Theorem 16.26, with A replaced by a plain, non-matricial random variable, say over a compact abelian group.

However, this kind of simplification does appear at M = 2, and since this phenomenon is quite interesting, we will explain this now. We first have:

PROPOSITION 16.27. For $F_2 \otimes_Q F_H$, with $Q \in M_{2 \times N}(\mathbb{T})$ generic, we have

$$N\int \left(\frac{\chi}{N}\right)^p = \int_{\mathbb{T}^N} \sum_{k\geq 0} \binom{p}{2k} \left|\frac{a_1+\ldots+a_N}{N}\right|^{2k} da$$

where the integral on the right is with respect to the uniform measure on \mathbb{T}^N .

PROOF. In order to prove the result, consider the following quantity, which appeared in the proof of Theorem 16.26:

$$\Phi(q) = \sum_{i_1...i_p} \sum_{d_1...d_p} \frac{q_{i_1d_1}\dots q_{i_pd_p}}{q_{i_1d_p}\dots q_{i_pd_{p-1}}}$$

We can "half-dephase" the matrix $q \in M_{2 \times N}(\mathbb{T})$ if we want to, as follows:

$$q = \begin{pmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_N \end{pmatrix}$$

Let us compute now the above quantity $\Phi(q)$, in terms of the numbers a_1, \ldots, a_N . Our claim is that we have the following formula:

$$\Phi(q) = 2\sum_{k\geq 0} N^{p-2k} \binom{p}{2k} \left| \sum_{i} a_{i} \right|^{2k}$$

Indeed, the idea is that:

(1) The $2N^k$ contribution will come from $i = (1 \dots 1)$ and $i = (2 \dots 2)$.

(2) Then we will have a $p(p-1)N^{k-2}|\sum_i a_i|^2$ contribution coming from indices of type $i = (2 \dots 21 \dots 1)$, up to cyclic permutations.

(3) Then we will have a $2\binom{p}{4}N^{p-4}|\sum_i a_i|^4$ contribution coming from indices of type $i = (2 \dots 21 \dots 12 \dots 21 \dots 1).$

(4) And so on.

In practice now, in order to prove our claim, in order to find the $N^{p-2k}|\sum_i a_i|^{2k}$ contribution, we have to count the circular configurations consisting of p numbers 1, 2, such that the 1 values are arranged into k non-empty intervals, and the 2 values are arranged into k non-empty intervals, and the 2 values are arranged into k non-empty intervals as well. Now by looking at the endpoints of these 2k intervals, we have $2\binom{p}{2k}$ choices, and this gives the above formula.

Now by integrating, this gives the formula in the statement.

Observe now that the integrals in Proposition 16.27 can be computed as follows:

$$\int_{\mathbb{T}^N} |a_1 + \ldots + a_N|^{2k} da = \int_{\mathbb{T}^N} \sum_{i_1 \dots i_k} \sum_{j_1 \dots j_k} \frac{a_{i_1} \dots a_{i_k}}{a_{j_1} \dots a_{j_k}} da$$
$$= \# \left\{ i_1 \dots i_k, j_1 \dots j_k \middle| [i_1, \dots, i_k] = [j_1, \dots, j_k] \right\}$$
$$= \sum_{k=\sum r_i} \binom{k}{r_1, \dots, r_N}^2$$

We obtain in this way the following "blowup" result, for our measure:

PROPOSITION 16.28. For $F_2 \otimes_Q F_H$, with $Q \in M_{2 \times N}(\mathbb{T})$ generic, we have

$$\mu = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{2N}\left(\Psi_*^+\varepsilon + \Psi_*^-\varepsilon\right)$$

where ε is the uniform measure on \mathbb{T}^N , and where the blowup function is:

$$\Psi^{\pm}(a) = N \pm \left| \sum_{i} a_{i} \right|$$

PROOF. We use the formula found in Proposition 16.27, along with the following standard identity, coming from the Taylor formula:

$$\sum_{k \ge 0} \binom{p}{2k} x^{2k} = \frac{(1+x)^p + (1-x)^p}{2}$$

By using this identity, Proposition 16.27 reformulates as follows:

$$N \int \left(\frac{\chi}{N}\right)^p = \frac{1}{2} \int_{\mathbb{T}^N} \left(1 + \left|\frac{\sum_i a_i}{N}\right|\right)^p + \left(1 - \left|\frac{\sum_i a_i}{N}\right|\right)^p da$$

Now by multiplying by N^{p-1} , we obtain the following formula:

$$\int \chi^k = \frac{1}{2N} \int_{\mathbb{T}^N} \left(N + \left| \sum_i a_i \right| \right)^p + \left(N - \left| \sum_i a_i \right| \right)^p da$$

.. ...

But this gives the formula in the statement, and we are done.

We can further improve the above result, by reducing the maps Ψ^{\pm} appearing there to a single one, and we are led to the following statement:

THEOREM 16.29. For $F_2 \otimes_Q F_H$, with $Q \in M_{2 \times N}(\mathbb{T})$ generic, we have

$$\mu = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}\Phi_*\varepsilon$$

where ε is the uniform measure on $\mathbb{Z}_2 \times \mathbb{T}^N$, and where the blowup map is:

$$\Phi(e,a) = N + e \left| \sum_{i} a_i \right|$$

PROOF. This is clear indeed from Proposition 16.28.

As already mentioned, the above results at M = 2 are something quite special. In the general case, $M \in \mathbb{N}$, it is not clear how to construct a nice blowup of the measure.

Asymptotically, things are however quite simple. Let us go back indeed to the general case, where $M, N \in \mathbb{N}$ are both arbitrary. The problem that we would like to solve now is that of finding the good regime, of the following type, where the measure in Theorem 16.25 converges, after some suitable manipulations:

$$M = f(K)$$
 , $N = g(K)$, $K \to \infty$

In order to do so, we have to do some combinatorics. Let NC(p) be the set of noncrossing partitions of $\{1, \ldots, p\}$, and for $\pi \in P(p)$ we denote by $|\pi| \in \{1, \ldots, p\}$ the number of blocks. With these conventions, we have the following result:

PROPOSITION 16.30. With $M = \alpha K, N = \beta K, K \to \infty$ we have:

$$\frac{c_p}{K^{p-1}} \simeq \sum_{r=1}^p \#\left\{\pi \in NC(p) \middle| |\pi| = r\right\} \alpha^{r-1} \beta^{p-r}$$

In particular, with $\alpha = \beta$ we have:

$$c_p \simeq \frac{1}{p+1} \binom{2p}{p} (\alpha K)^{p-1}$$

PROOF. We use the combinatorial formula in Theorem 16.25. Our claim is that, with $\pi = \ker(i_1, \ldots, i_p)$, the corresponding contribution to c_p is:

$$C_{\pi} \simeq \begin{cases} \alpha^{|\pi|-1} \beta^{p-|\pi|} K^{p-1} & \text{if } \pi \in NC(p) \\ O(K^{p-2}) & \text{if } \pi \notin NC(p) \end{cases}$$

As a first observation, the number of choices for a multi-index $(i_1, \ldots, i_p) \in X^p$ satisfying ker $i = \pi$ is:

$$M(M-1)...(M-|\pi|+1) \simeq M^{|\pi|}$$

Thus, we have the following estimate:

$$C_{\pi} \simeq M^{|\pi|-1} N^{-1} \# \left\{ d_1, \dots, d_p \in Y \left| [d_{\alpha} | \alpha \in b] = [d_{\alpha-1} | \alpha \in b], \forall b \in \pi \right\} \right\}$$

Consider now the following partition:

$$\sigma = \ker d$$

The contribution of σ to the above quantity C_{π} is then given by:

$$\Delta(\pi,\sigma)N(N-1)\dots(N-|\sigma|+1)\simeq\Delta(\pi,\sigma)N^{|\sigma|}$$

Here the quantities on the right are as follows:

$$\Delta(\pi, \sigma) = \begin{cases} 1 & \text{if } |b \cap c| = |(b-1) \cap c|, \forall b \in \pi, \forall c \in \sigma \\ 0 & \text{otherwise} \end{cases}$$

We use now the standard fact that for $\pi, \sigma \in P(p)$ satisfying $\Delta(\pi, \sigma) = 1$ we have:

$$|\pi| + |\sigma| \le p + 1$$

In addition, the equality case is well-known to happen when $\pi, \sigma \in NC(p)$ are inverse to each other, via Kreweras complementation. This shows that for $\pi \notin NC(p)$ we have:

$$C_{\pi} = O(K^{p-2})$$

Also, this shows that for $\pi \in NC(p)$ we have:

$$C_{\pi} \simeq M^{|\pi|-1} N^{-1} N^{p-|\pi|-1}$$

= $\alpha^{|\pi|-1} \beta^{p-|\pi|} K^{p-1}$

Thus, we have obtained the result.

We denote by D the dilation operation, given by:

$$D_r(law(X)) = law(rX)$$

With this convention, we have the following result:

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THEOREM 16.31. With $M = \alpha K, N = \beta K, K \to \infty$ we have:

$$\mu = \left(1 - \frac{1}{\alpha\beta K^2}\right)\delta_0 + \frac{1}{\alpha\beta K^2}D_{\frac{1}{\beta K}}(\pi_{\alpha/\beta})$$

In particular with $\alpha = \beta$ we have:

$$\mu = \left(1 - \frac{1}{\alpha^2 K^2}\right)\delta_0 + \frac{1}{\alpha^2 K^2} D_{\frac{1}{\alpha K}}(\pi_1)$$

PROOF. At $\alpha = \beta$, this follows from Proposition 16.30. In general now, we have:

$$\frac{c_p}{K^{p-1}} \simeq \sum_{\pi \in NC(p)} \alpha^{|\pi|-1} \beta^{p-|\pi|}$$
$$= \frac{\beta^p}{\alpha} \sum_{\pi \in NC(p)} \left(\frac{\alpha}{\beta}\right)^{|\pi|}$$
$$= \frac{\beta^p}{\alpha} \int x^p d\pi_{\alpha/\beta}(x)$$

When $\alpha \geq \beta$, where $d\pi_{\alpha/\beta}(x) = \varphi_{\alpha/\beta}(x)dx$ is continuous, we obtain:

$$c_p = \frac{1}{\alpha K} \int (\beta K x)^p \varphi_{\alpha/\beta}(x) dx$$
$$= \frac{1}{\alpha \beta K^2} \int x^p \varphi_{\alpha/\beta}\left(\frac{x}{\beta K}\right) dx$$

But this gives the formula in the statement. When $\alpha \leq \beta$ the computation is similar, with a Dirac mass as 0 disapearing and reappearing, and gives the same result.

We refer to **6** and related papers for more on the above.

16d. Big index

In big index now, the philosophy is that the index of subfactors $N \in [1, \infty)$ should be regarded as being the well-known N variable from physics, which must be big:

$$N \to \infty$$

More precisely, the idea is that the constructions involving groups, group duals, or more generally compact quantum groups, producing subfactors of integer index, $N \in \mathbb{N}$, can be used with "uniform objects" as input, and so produce an asymptotic theory.

The problem however is how to axiomatize the uniformity notion which is needed, in order to have some control on the resulting planar algebra $P = (P_k)$. The answer here comes from the notion of easiness, that we already met before, and its various technical extensions, which are in fact not currently unified, or even fully axiomatized.

The main technical questions here are the classification of the easy quantum groups on one hand, and the axiomatization of the super-quizzy quantum groups on the other hand. We also have the question of better understanding the relation between easiness, subfactors, planar algebras, noncommutative geometry and free probability, and we refer here to [6], [10], [14], [19], [53], [62], [78], [95].

Summarizing, we have many interesting questions, both in small and big index. As a common ground here, both these questions happen inside the Murray-von Neumann factor R, although this is conjectural in big index, related to existence questions for outer actions and matrix models. Thus, as a good problem to finish with, which is from Jones' original subfactor paper [40], and is due to Connes, we have the question of axiomatizing the finite index subfactors of the Murray-von Neumann hyperfinite factor R.

As already mentioned on several occasions, this longstanding question is in need of some new, brave functional analysis input, in relation with the notion of hyperfiniteness, which is probably of quite difficult type, beyond what the current experts can do.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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