Initiation to subfactors

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

2010 *Mathematics Subject Classification.* 46L37

Key words and phrases. Subfactor, Amenability

Abstract. This is an introduction to the theory of subfactors, as developed by Jones and others. A von Neumann algebra is a weakly closed operator $*$-algebra $A \subset B(H)$, a factor is such an algebra having trivial center, $Z(A) = \mathbb{C}$, and a subfactor is an inclusion of such factors, $A \subset B$. We first discuss the basics of the theory, notably with Jones' discovery that any subfactor $A \subset B$ produces a representation of the Temperley-Lieb algebra, and in fact, produces a planar algebra, which can be thought of as being the algebra of invariants of a certain quantum group type object G. Then we discuss more specialized aspects, dealing with the case where G is finite, or more generally amenable. Finally, we provide an introduction to the most interesting case, where both factors A, B are isomorphic to the Murray-von Neumann hyperfinite factor R.

Preface

The algebras of bounded linear operators $A \subset B(H)$ on a complex Hilbert space H, typically taken separable, were studied starting with the work of von Neumann in the 1930s. Von Neumann was interested in quantum mechanics, a new discipline at that time, developed in the 1920s by Heisenberg, Schrödinger, Dirac and others. One of the conclusions of quantum mechanics was the fact that the states of a quantum system are described by the vectors of a complex Hilbert space H, and the observables are described by certain linear operators, which are possibly unbounded, $T : H \to H$. By looking at the commutants of such observables, $A = \{T\}'$, von Neumann was led into the study of the weakly closed operator $*$-algebras $A \subset B(H)$, now called von Neumann algebras.

There has been a lot of work on the von Neumann algebras, with the fundamentals developed by Murray and von Neumann in the 1930s and 1940s, and with some further fundamentals developed by Connes in the early 1970s. The conclusion of this work, which is something non-trivial, is that that the "building blocks" of the whole theory are the von Neumann algebras $A \subset B(H)$ which are infinite dimensional, $\dim(A) = \infty$, have trivial center, $Z(A) = \mathbb{C}$, and have a trace $tr : A \to \mathbb{C}$. These are called II_1 factors.

In view of this finding, an interesting question, which is something very natural, mathematically speaking, and is motivated as well by various questions from quantum mechanics, is to look at the inclusions $A \subset B$ of such II_1 factors, which are called subfactors. This was done by Jones in the late 1970s, and then all over the 1980s and 1990s, with the remarkable conclusion that any such subfactor $A \subset B$ produces a representation of the Temperley-Lieb algebra, and in fact, produces a planar algebra, which can be thought of as being the algebra of invariants of a certain quantum group type object G.

This book is an introduction to this, subfactor theory, as developed by Jones and others, and its ramifications. The book is organized in four parts, as follows:

(1) We first discuss the basics of the theory, notably with explanations on the above-mentioned key findings of Jones, involving diagrams and planar algebras.

(2) We discuss then some more specialized aspects, dealing with the case where the above-mentioned underlying quantum group type object G is finite.

(3) We then go on an even more specialized discussion, dealing with the more general case where the above-mentioned quantum group type object G is amenable.

(4) Finally, we provide an introduction to the most interesting case, where both factors A, B are isomorphic to the Murray-von Neumann hyperfinite factor R.

In the hope that you will find this book useful. The material here will be quite old style, basically stopping around 2000, but do not worry, the old-fashioned questions at the end, regarding the subfactors of R, are still open, and waiting for your input.

Many thanks to everyone, having helped me to progress in my subfactor learning, over the years. Thanks as well to my cats, for some help with functional analysis.

Cergy, March 2025
Teo Banica

Contents

Part I

Subfactors

Farewell, Angelina
The bells of the crown
Are being stolen by bandits
I must follow the sound

CHAPTER 1

Factors

1a. Finite factors

Welcome to subfactors. In this chapter we discuss basic the study of the II_1 factors, following Murray and von Neumann [58], [59], [60], [85], [86], which is the basis for everything advanced and modern, in relation with the operator algebras.

We will only present here the very basic theory of the II_1 factors, and we will come back to them, on a regular basis, later. In fact, as we will soon discover, these II_1 factors are the "building blocks" of the whole von Neumann algebra theory.

Let us first talk about general factors. There are several possible ways of introducing them, and dividing them into several classes, for further study. In what concerns us, we will use a rather intuitive approach. The general idea, which is quite natural, is that among the von Neumann algebras $A \subset B(H)$, of particular interest are the "free" ones, having trivial center, $Z(A) = \mathbb{C}$. These algebras are called factors:

Definition 1.1. *A factor is a von Neumann algebra $A \subset B(H)$ whose center*

$$Z(A) = A \cap A'$$

which is a commutative von Neumann algebra, reduces to the scalars, $Z(A) = \mathbb{C}$.

This notion is something quite subtle, that you probably already met, in your learning of operator algebras, but time now to clarify all this. The idea is that there are two main motivations for the study of factors, with each of them being more than enough, as to serve as a strong motivation. First, at the intuitive level, we have:

Principle 1.2 (Freeness). *The following happen:*

1. *The condition $Z(A) = \mathbb{C}$ defining the factors is, obviously, opposite to the condition $Z(A) = A$ defining the commutative von Neumann algebras.*
2. *Therefore, the factors are the von Neumann algebras which are "free", meaning as far as possible from the commutative ones.*
3. *Equivalently, with $A = L^\infty(X)$, the quantum spaces X coming from factors are those which are "free", meaning as far as possible from the classical spaces.*

So, this was for our first principle, which is something reasonable, intuitive, and self-explanatory, and which can surely serve as a strong motivation for the study of factors.

In fact, all that has being said above comes straight from the structure theorem for the commutative von Neumann algebras, $A = L^\infty(X)$, with X being a measured space, that you probably know well, and the above principle is just a corollary of that theorem.

At a more advanced level, another motivation for the study of factors, which among others justifies the name "factors" for them, comes from the reduction theory of von Neumann [87], which is something non-trivial, that can be summarized as follows:

Principle 1.3 (Reduction theory). *Given a von Neumann algebra $A \subset B(H)$, if we write its center $Z(A) \subset A$, which is a commutative von Neumann algebra, as*

$$Z(A) = L^\infty(X)$$

with X being a measured space, then the whole algebra decomposes as

$$A = \int_X A_x \, dx$$

with the fibers A_x being factors, that is, satisfying $Z(A_x) = \mathbb{C}$.

As a first comment, you have already seen an instance of such decomposition results when learning about finite dimensional algebras. Indeed, such algebras decompose, in agreement with the above, as direct sums of matrix algebras, as follows:

$$A = \bigoplus_x M_{n_x}(\mathbb{C})$$

In general, however, things are more complicated than this, and technically speaking, and as opposed to Principle 1.2, which was more of a triviality, Principle 1.3 is a tough theorem, due to von Neumann [87]. More on this later, on several occasions.

This was for the story, and let us close this philosophical discussion with:

Conclusion 1.4. *Regardless of the approach and technical level, be that beginner or advanced, the von Neumann factors are the algebras that matter.*

Getting to work now, there are many things that can be said about factors.

In order to get started, let us first study their projections. We will see that many interesting things happen, with everything coming from the following technical result:

Proposition 1.5. *Given two projections $p, q \neq 0$ in a factor A, we have*

$$puq \neq 0$$

for a certain unitary $u \in A$.

PROOF. Assume by contradiction $puq = 0$, for any unitary $u \in A$. This gives:

$$u^*puq = 0$$

By using this for all the unitaries $u \in A$, we obtain the following formula:

$$\left(\bigvee_{u \in U_A} u^*pu\right) q = 0$$

On the other hand, from $p \neq 0$ we obtain, by factoriality of A:

$$\bigvee_{u \in U_A} u^*pu = 1$$

Thus, our previous formula is in contradiction with $q \neq 0$, as desired. $\square$

Getteing back now to the order on projections, and to the whole von Neumann projection philosophy, in the case of factors things simplify, as follows:

THEOREM 1.6. *Given two projections $p, q \in A$ in a factor, we have*

$$p \preceq q \quad \text{or} \quad q \preceq p$$

and so $\preceq$ is a total order on the equivalence classes of projections $p \in A$.

PROOF. This basically follows from Proposition 1.5, and from the Zorn lemma, by using some standard functional analysis arguments. To be more precise:

(1) Consider indeed the following set of partial isometries:

$$S = \left\{ u \Big| uu^* \leq p, u^*u \leq q \right\}$$

We can then order this set S by saying that we have $u \leq v$ when $u^*u \leq v^*v$, and when $u = v$ holds on the initial domain u^*uH of u. With this convention made, the Zorn lemma applies, and provides us with a maximal element $u \in S$.

(2) In the case where this maximal element $u \in S$ satisfies $uu^* = p$ or $u^*u = q$, we are led to one of the conditions $p \preceq q$ or $q \preceq p$ in the statement, and we are done.

(3) So, assume that we are in the case left, $uu^* \neq p$ and $u^*u \neq q$. By Proposition 1.5 we obtain a unitary $v \neq 0$ satisfying the following conditions:

$$vv^* \leq p - uu^*$$

$$v^*v \leq q - u^*u$$

But these conditions show that the element $u + v \in S$ is strictly bigger than $u \in S$, which is a contradiction, and we are done. $\square$

Moving ahead now, as explained in the beginning of this book, for a variety of reasons, which can be elementary or advanced, and also mathematical or physical, we are mainly interested in the case where our algebras have traces:

$$tr : A \to \mathbb{C}$$

And in relation with the factors, by leaving aside the rather trivial case of the matrix algebras $A = M_N(\mathbb{C})$, we are led in this way to the following key notion:

DEFINITION 1.7. *A* II_1 *factor is a von Neumann algebra* $A \subset B(H)$ *which:*

(1) *Is infinite dimensional,* $\dim A = \infty$.
(2) *Has trivial center,* $Z(A) = \mathbb{C}$.
(3) *Has a trace* $tr : A \to \mathbb{C}$.

Here the order of the axioms is a bit random, with any of the possible $3! = 6$ choices making sense, and corresponding to a slightly different vision on what the II_1 factors truly are. With the above order, with (1) we are making it clear, right from the beginning, that we are not here for revolutionizing linear algebra. Then with (2) we adhere to Definition 1.1, and to what was said next about it, on freeness and reduction. And finally with (3) we adhere to the above principle, that von Neumann algebras must have traces.

More technically now, and leaving aside anything subjective, the above definition is motivated by the heavy classification work of Murray, von Neumann and Connes [15], [16], [58], [59], [60], [85], [86], [87], whose conclusion is more or less that everything in von Neumann algebras reduces, via some quite complicated procedures, we should mention that, to the study of the II_1 factors. With the mantra here being as follows:

FACT 1.8. *The* II_1 *factors are the building blocks of the whole von Neumann algebra theory.*

To be more precise, this statement, that we will get to understand later, is something widely agreed upon, at least among operator algebra experts who are familiar with von Neumann algebras, and with this agreement being something great. What remains controversial, however, is how to start playing with these Lego bricks that we have:

(1) A first option is that of adding the matrix algebras $M_N(\mathbb{C})$, not to be forgotten, and then stacking together such Lego bricks. According to the von Neumann reduction theory, this leads to the von Neumann algebras having traces, $tr : A \to \mathbb{C}$.

(2) A second option, perhaps even more playful, is that of taking crossed products of such Lego bricks by their automorphisms scaling the trace, or performing more general constructions inspired by advanced ergodic theory. This leads to general factors.

(3) And the third option is that of being a bad kid, or perhaps some kind of nerd, engineer in the becoming, and picking such a Lego brick, or a handful of them, and breaking them, see what's inside. Good option too, and more on this later.

Getting to work now, in practice, and forgetting about reduction theory, which raises the possibility of decomposing any tracial von Neumann algebra into factors, in order to obtain explicit examples of II! factors, it is not even clear that such beasts exist. Fortunately the group von Neumann algebras are there, and we have the following result, which provides us with some examples of II_1 factors, to start with:

THEOREM 1.9. *The center of a group von Neumann algebra $L(\Gamma)$ is*

$$Z(L(\Gamma))=\left\{\sum_g\lambda_g g\Big|\lambda_{gh}=\lambda_{hg}\right\}''$$

and if $\Gamma\neq\{1\}$ has infinite conjugacy classes, in the sense that

$$\left|\{ghg^{-1}|g\in G\}\right|=\infty\quad,\quad\forall h\neq 1$$

with this being called ICC property, the algebra $L(\Gamma)$ is a II_1 factor.

PROOF. There are two assertions here, the idea being as follows:

(1) Consider a linear combination of group elements, which is in the weak closure of $\mathbb{C}[\Gamma]$, and so defines an element of the group von Neumann algebra $L(\Gamma)$:

$$a=\sum_g\lambda_g g$$

By linearity, this element $a\in L(\Gamma)$ belongs to the center of $L(\Gamma)$ precisely when it commutes with all the group elements $h\in\Gamma$, and this gives:

$$\begin{aligned}a\in Z(A)&\iff ah=ha\\&\iff\sum_g\lambda_g gh=\sum_g\lambda_g hg\\&\iff\sum_k\lambda_{kh^{-1}}k=\sum_k\lambda_{h^{-1}k}k\\&\iff\lambda_{kh^{-1}}=\lambda_{h^{-1}k}\end{aligned}$$

Thus, we obtain the formula for $Z(L(\Gamma))$ in the statement.

(2) We have to examine the 3 conditions defining the II_1 factors. We already know from basic algebra that the group algebra $L(G)$ has a trace, given by:

$$tr(g)=\delta_{g,1}$$

Regarding now the center, the condition $\lambda_{gh} = \lambda_{hg}$ that we found is equivalent to the fact that $g \to \lambda_g$ is constant on the conjugacy classes, and we obtain:

$$Z(L(\Gamma)) = \mathbb{C} \iff \Gamma = \mathrm{ICC}$$

Finally, assuming that this ICC condition is satisfied, with $\Gamma \neq \{1\}$, then our group Γ is infinite, and so the algebra $L(\Gamma)$ is infinite dimensional, as desired. □

In order to look now for more examples of II_1 factors, an idea would be that of attempting to decompose into factors the group von Neumann algebras $L(\Gamma)$, but this is something difficult, and in fact we won't really exit the group world in this way. Difficult as well is to investigate the factoriality of the von Neumann algebras of discrete quantum groups $L(\Gamma)$, because the basic computations from the proof of Theorem 1.9 won't extend to this setting, where the group elements $g \in \Gamma$ become corepresentations $g \in M_N(L(\Gamma))$. Despite years of efforts, it is presently not known at all what the "quantum ICC" condition should mean, and the problem comes from this. But more on this later.

In short, we have to stop here the construction of examples, and Theorem 1.9 will be what we have, at least for the moment. With this being actually not a big issue, the group factors $L(\Gamma)$ being known to be quite close to the generic II_1 factors.

1b. Basic results

Getting away now from the above difficulties, let us go back to the abstract II_1 factors, as axiomatized in Definition 1.7. In order to investigate them, the idea will be that of looking at the projections, and their equivalence classes.

In the case of the II_1 factors, as a first interesting remark, the presence of the trace trivializes the proof of the main result that we have about projections, as follows:

Theorem 1.10. *Given two projections $p, q \in A$ in a II_1 factor we have, trivially*

$$p \preceq q \quad \text{or} \quad q \preceq p$$

and so $\preceq$ is a total order on the equivalence classes of projections $p \in A$.

Proof. This is something that we already know, from Theorem 1.6, and which actually holds for any factor, with the non-trivial part being the following implication:

$$p \preceq q,\ q \preceq p \implies p \simeq q$$

But this implication is clear in the present II_1 factor setting, by using the trace. □

The above theorem and its proof, which are remarkable, are the first in a series of mysteries, in what concerns the special case of the II_1 factors. More such mysteries to follow. In order to study now the trace of the II_1 factors, we will need:

PROPOSITION 1.11. *Given a weakly closed left ideal $I \subset A$ in a von Neumann algebra, there exists a unique projection $p \in A$ such that:*

$$I = Ap$$

Moreover, if $I \subset A$ is assumed to be a two-sided ideal, then $p \in Z(A)$.

PROOF. We have several things to be proved, the idea being as follows:

(1) Given an ideal $I \subset A$ as in the statement, consider the following intersection:

$$I \cap I^* \subset A$$

This is a weakly closed non-unital $*$-subalgebra of A, so if we denote by $p \in A$ its largest projection, or unit, then we have an inclusion $Ap \subset I$.

(2) Conversely now, let us pick $x \in I$. By polar decomposition we can write $x = u|x|$, and we have the following implications, which prove the reverse inclusion $I \subset Ap$:

$$\begin{aligned} x \in I &\implies |x| = u^*x \in I \\ &\implies |x| \in I \cap I^* \\ &\implies |x|p = |x| \\ &\implies x = u|x| = u|x|p \in Ap \end{aligned}$$

(3) The uniqueness assertion is clear from the comparison theorem for projections.

(4) Regarding now the last assertion, assume that $I \subset A$ is a two-sided weakly closed ideal. Then for any unitary $u \in A$ we have:

$$\begin{aligned} I = uIu^* &\implies uIu^* = Ap \\ &\implies I = Aupu^* \end{aligned}$$

Thus by uniqueness we obtain $upu^* = p$, and so $p \in Z(A)$, as desired. □

As a first main result now regarding the II_1 factors, following the paper of Murray and von Neumann [60], which by the way is a must-read, we have:

THEOREM 1.12. *Given a II_1 factor A, any weakly continuous positive trace*

$$tr : A \to \mathbb{C}$$

is automatically faithful.

PROOF. Consider the null space of the trace, which is by definition:

$$I = \left\{ x \in A \middle| tr(x^*x) = 0 \right\}$$

We have the following inequality, which shows that I is a left ideal:

$$x^*a^*ax \leq ||a||^2 x^*x$$

Now by using the trace condition $tr(ab) = tr(ba)$, we conclude that I is a two-sided ideal. Also, the Cauchy-Schwarz inequality gives:

$$tr(x^*x) = 0 \iff tr(xy) = 0, \forall y \in A$$

We conclude from this that I is an intersection of kernels of weakly closed functionals, which are weakly closed, and so it is weakly closed. Thus the last assertion in Proposition 1.11 applies, and produces a projection $p \in Z(A)$ such that:

$$I = Ap$$

Now since A was assumed to be a factor, we have $Z(A) = \mathbb{C}$. Thus $p = 0$, and so the null ideal of the trace is $I = \{0\}$, and so our trace tr is faithful, as desired. □

Our goal now will be that of proving that the trace on a II_1 factor is unique, and takes on projections any value in $[0, 1]$. Let us start with a technical result, as follows:

Proposition 1.13. *Given a* II_1 *factor* A*, the traces of the projections*

$$tr(p) \in [0, 1]$$

can take arbitrarily small values.

Proof. Consider the set formed by all values of the trace on the projections:

$$S = \left\{ tr(p) \Big| p^2 = p = p^* \in A \right\}$$

We want to prove that the following number equals 0:

$$c = \inf(S - \{0\})$$

In order to do so, assume by contradiction $c > 0$, pick $\varepsilon > 0$ small, and pick a projection $p \in A$ such that the following condition is satisfied:

$$tr(p) < c + \varepsilon$$

Since we are in a II_1 factor, this projection $p \in A$ cannot be minimal, and so we can find another projection $q \in A$ satisfying $q < p$. Now observe that we have:

$$\begin{aligned} tr(p - q) &= tr(p) - tr(q) \\ &\leq tr(p) - c \\ &\leq \varepsilon \end{aligned}$$

Thus with $\varepsilon < c$ we obtain a contradiction, and so $c = 0$, as desired. □

In order to prove our next main result, we will need as well:

Proposition 1.14. *Given a* II_1 *factor* A *on a Hilbert space* H *and a projection* $p \in A$*, the von Neumann algebra* pAp *is a* II_1 *factor on the Hilbert space* pH*.*

Proof. We have to prove that the von Neumann algebra pAp has a trace, and is infinite dimensional, and these two properties can be proved as follows:

(1) In what regards the trace, we know that the trace $tr : A \to \mathbb{C}$ restricts to a trace $tr : pAp \to \mathbb{C}$, which must be nonzero, as desired.

(2) In what regards the infinite dimensionality, this follows from the fact that a minimal projection in pAp would be minimal in A, which is impossible. $\square$

Still following the fundamental paper of Murray and von Neumann [60], we can now formulate a second main result regarding the II_1 factors, as follows:

Theorem 1.15. *Given a II_1 factor A, the traces of projections*

$$tr(p) \in [0, 1]$$

can take any values in $[0, 1]$.

Proof. Given a number $c \in [0, 1]$, consider the following set:

$$S = \left\{ p^2 = p = p^* \in A \Big| tr(p) \leq c \right\}$$

This set satisfies the assumptions of the Zorn lemma, and so by this lemma we can find a maximal element $p \in S$. Assume by contradiction that we have:

$$tr(p) < c$$

The point now is that by using Proposition 1.13 and Proposition 1.14, we can slightly enlarge the trace of p, and we obtain a contradiction, as desired. $\square$

As a third and last main result regarding the II_1 factors, also from [60], we have:

Theorem 1.16. *The trace of a II_1 factor*

$$tr : A \to \mathbb{C}$$

is unique.

Proof. This can be proved in many ways, a standard one being that of proving that any two traces agree on the projections, as a consequence of the above results:

(1) Assume indeed that we have a second trace $tr' : A \to \mathbb{C}$. Since A is generated by its projections, it is enough to show that we have $tr = tr'$ on projections.

(2) As a first observation, since traces on matrix algebras are unique, we obtain that we have $tr = tr'$ on the projections $p \in A$ having rational trace, $tr(p) \in \mathbb{Q}$.

(3) So, let us pick $p \in A$ having non-rational trace, $tr(p) \notin \mathbb{Q}$, and prove that we have $tr(p) = tr'(p)$. The idea will be that of using the result for the projections having rational traces, applied to an infinite direct sum of projections, converging to p.

(4) To be more precise, assume that we have constructed our sequence $p_i \to p$ up to order $n \in \mathbb{N}$, and let us try to construct p_{n+1}. The idea is to use the following algebra:

$$A_n = (p - p_n)A(p - p_n)$$

(5) Indeed this algebra is a II_1 factor, and we can choose inside it a projection p_{n+1} satisfying $p_n \leq p_{n+1} \leq p$, such that $tr = tr'$ on it, and such that:

$$tr(p - p_{n+1}) \leq \frac{1}{2} \cdot tr(p - p_n)$$

(6) According to our choices for these projections p_n, we have:

$$p = \bigvee_{n=1}^{\infty} p_n$$

Thus when evaluating tr, tr' on p we obtain the same result, as desired. □

In what regards illustrations for all this, as examples of II_1 factors we have so far the group von Neumann algebras $L(\Gamma)$, with Γ being an ICC group. In certain cases, it is possible to say more about all the above, and in particular about the projections, for instance with quite explicit procedures for constructing projections $p \in L(\Gamma)$ having an arbitrary prescribed trace $x \in [0, 1]$. We will be back to this later, when discussing more in detail the group von Neumann algebras $L(\Gamma)$, and their generalizations.

Back to theory, we have seen that the II_1 factors are very interesting objects, naturally lying above the matrix algebras $M_N(\mathbb{C})$, which are type I factors. From this perspective, a II_1 factor $A \subset B(H)$ is not really in need of the ambient Hilbert space H, and the question of "representing" it appears. We will discuss this question, in two steps:

(1) A first question is that of understanding the possible embeddings $A \subset B(H)$, with H being a Hilbert space. The main result here will be the construction of a numeric invariant $\dim_A H$, called coupling constant.

(2) A second question is that of understanding the possible embeddings $A \subset B$, with B being another II_1 factor. By using the coupling constant for both A, B we will construct a numeric invariant $[B : A]$, called index.

We will discuss now (1), and leave (2) for later, towards the end of this chapter. In order to get started, let us formulate the following definition:

Definition 1.17. *Given a von Neumann algebra A with a trace $tr : A \to \mathbb{C}$, the emdedding*

$$A \subset B(L^2(A))$$

obtained by GNS construction is called standard form of A.

Here we use the GNS construction, from functional analysis. As the name indicates, the standard representation is something "standard", to be compared with any other representation $A \subset B(H)$, in order to understand this latter representation.

As known from functional analysis, the GNS construction has a number of unique features, that can be exploited. In the present setting, the main result is as follows:

THEOREM 1.18. *In the context of the standard representation we have*

$$A' = JAJ$$

with $J : L^2(A) \to L^2(A)$ *being the antilinear map given by* $T \to T^*$.

PROOF. Observe first that any $T \in A$ can be regarded as a vector $T \in L^2(A)$, to which we can associate, in an antilinear way, the vector $T^* \in L^2(A)$. Thus we have indeed an antilinear map J as in the statement. In terms of the standard cyclic and separating vector Ω for the GNS representation, the formula of this formula J is:

$$J(x\Omega) = x^*\Omega$$

(1) Our first claim is that we have the following formula:

$$< J\xi, J\eta >=< \xi, \eta >$$

Indeed, with $\xi = x\Omega$ and $\eta = y\Omega$, we have the following computation:

$$\begin{aligned} < J\xi, J\eta > &= < yx^*\Omega, \Omega > \\ &= tr(yx^*) \\ &= < \xi, \eta > \end{aligned}$$

(2) Our second claim is that we have the following formula:

$$JxJ(y\Omega) = yx^*\Omega$$

Indeed, this follows from the following computation:

$$JxJ(y\Omega) = J(xy^*\Omega) = yx^*\Omega$$

(3) Our claim now is that we have an inclusion as follows:

$$JAJ \subset A'$$

Indeed, this follows from the formula obtained in (2).

(4) In order to prove the reverse inclusion, our claim is that for $x \in A'$ we have:

$$Jx\Omega = x^*\Omega$$

Indeed, this follows from the following computation, valid for any $y \in A$:

$$\begin{aligned}
< Jx\Omega, y\Omega > &= < Jy\Omega, x\Omega > \\
&= < y^*\Omega, x\Omega > \\
&= < \Omega, xy\Omega > \\
&= < x^*\Omega, y\Omega >
\end{aligned}$$

(5) Our claim now is that the following formula defines a trace on A':

$$Tr(x) =< x\Omega, \Omega >$$

Indeed, for any two elements $x, y \in A'$ we have:

$$\begin{aligned}
< xy\Omega, \Omega > &= < y\Omega, x^*\Omega > \\
&= < y\Omega, Jx\Omega > \\
&= < x\Omega, Jy\Omega > \\
&= < x\Omega, y^*\Omega > \\
&= < yx\Omega, \Omega >
\end{aligned}$$

(6) We can now finish the proof. Indeed, by using the trace constructed in (5), we can apply our results obtained so far to A', and we obtain $JA'J \subset A$, as desired. □

As a basic illustration for the above result, we have:

THEOREM 1.19. *The commutant of a von Neumann group algebra $L(\Gamma)$, which is obtained by definition by using the left regular representation, is the von Neumann group algebra $R(\Gamma)$, obtained by using the right regular representation.*

PROOF. We recall that the left and the right representations of a discrete group Γ are given by the following formulae, by using the standard identification $\Gamma \subset l^2(\Gamma)$:

$$\lambda_g : h \to gh \quad , \quad \rho_g : h \to hg^{-1}$$

We have $Jg = g^{-1}$ for any group element $g \in \Gamma$, and by using this, we obtain:

$$\begin{aligned}
J\lambda_g Jh &= J\lambda_g h^{-1} \\
&= Jgh^{-1} \\
&= hg^{-1} \\
&= \rho_g h
\end{aligned}$$

Thus, the left and right representations are related by the following formula:

$$J\lambda_g J = \rho_g$$

By using now Theorem 1.18 we can compute commutants, as follows:

$$L(\Gamma)' = JL(\Gamma)J = R(\Gamma)$$

Finally, we have $L(\Gamma) = R(\Gamma)'$ too, by taking the commutant. □

As another application of the standard representation, let us go back to the uniqueness of the trace, that we know from Theorem 1.16. There are several alternative proofs for this fact, which are all instructive. As a first such statement and proof, we have:

THEOREM 1.20. *Given a* II_1 *factor* A*, and an element* $a \in A$*, we have the following Dixmier averaging property:*

$$\overline{span\left\{uau^*\Big|u \in U_A\right\}}^w \cap \mathbb{C}1 \neq \emptyset$$

In particular, the II_1 *factor trace* $tr : A \to \mathbb{C}$ *is unique.*

PROOF. We use the basic theory of the regular representation $A \subset L^2(A)$, with respect to the given trace $tr : A \to \mathbb{C}$, explained above. The proof goes as follows:

(1) Given an element $a \in A$, consider the space in the statement, obtained as the weak closure of the space spanned by the spinned versions of a, namely:

$$K_a = \overline{span\left\{uau^*\Big|u \in U_A\right\}}^w$$

This linear space $K_a \subset A$ is by definition weakly closed, and it follows that the subset $K_a\Omega \subset L^2(A)$, where $\Omega \in L^2(A)$ is the canonical trace vector, is a weakly closed convex subset. In particular, we see that $K_a\Omega \subset L^2(A)$ is a norm closed convex subset.

(2) In view of this, we can consider the unique element $b \in K_a$ having the property that $b\Omega$ has a minimal norm. We have then the following formula, for any unitary $u \in U_A$, where $J : L^2(A) \to L^2(A)$ is the standard antilinear map, given by $T \to T^*$:

$$||uJuJb\Omega|| = ||b\Omega||$$

By uniqueness of b, it follows that for any unitary $u \in U_A$, we have:

$$uJuJb\Omega = b\Omega$$

But this shows that for any unitary $u \in U_A$, we have:

$$ubu^* = b$$

We conclude that we have $b \in \mathbb{C}1$, and this proves the first assertion.

(3) Regarding now the second assertion, consider an arbitrary trace $tr : A \to \mathbb{C}$. By using $tr(uau^*) = tr(a)$, we conclude that this trace is constant on the following set:

$$K_a = \overline{span\left\{uau^*\Big|u \in U_A\right\}}^w$$

Now by using the first assertion, we conclude that we have the following formula:

$$\overline{span\left\{uau^*\Big|u \in U_A\right\}}^w \cap \mathbb{C}1 = \left\{tr(a)1\right\}$$

Summarizing, we have obtained a purely algebraic formula for our trace $tr : A \to \mathbb{C}$, and it follows that this trace is indeed unique, as claimed. □

In relation with the above, let us mention that there is as well a third proof for the uniqueness of the trace, due to Yeadon, based on nothing or almost, meaning the definition of the II_1 factors, along with some abstract functional analysis. For more on all this, basic theory of the II_1 factors, we refer to the standard operator algebra books, with some good choices here being the books of Connes [**17**], Takesaki [**77**] and Blackadar [**12**].

1c. Type II factors

Let us go back now to the general theory of the II_1 factors, with the aim of talking about representations of such II_1 factors, inside the category of the II_1 factors, $A \subset B$. For this purpose we will need a key notion, called coupling constant.

In order to discuss the construction of the coupling constant, we will need some further results on the type II factors, complementing those that we already have. The point indeed is that the class of II factors, to be axiomatized later, and with this being not something urgent, comprises, besides the II_1 factors discussed above, the II_∞ factors as well:

DEFINITION 1.21. *A II_∞ factor is a von Neumann algebra of the form*

$$B = A \otimes B(H)$$

with A being a II_1 factor, and with H being an infinite dimensional Hilbert space.

We should mention that there are several possible ways of defining the II_∞ factors, and the above definition is something rather intuitive, the point being that, once you learn the theory of the II_∞ factors, as we will do here, what you remember at the end of the day is what has been said above, $B = A \otimes B(H)$, with A being a II_1 factor.

Getting started now, as a useful characterization of such factors, we have:

PROPOSITION 1.22. *For an infinite factor B, the following are equivalent:*

(1) *There exists a projection $p \in B$ such that pBp is a II_1 factor.*
(2) *B is a II_∞ factor.*

PROOF. This is something elementary, as follows:

(1) $\implies$ (2) Assume indeed that $p \in B$ is a projection such that pBp is a II_1 factor. We choose a maximal family of pairwise orthogonal projections $\{p_i\} \subset B$ satisfying $p_i \simeq p$, for any i, and we consider the following projection, which satisfies $q \preceq p$:

$$q = 1 - \sum_i p_i$$

Since the indexing set for our set of projections $\{p_i\}$ must be infinite, we can use a strict embedding of this index set into itself, as to write a formula as follows:

$$\begin{aligned} 1 &= q + \sum_i p_i \\ &\preceq p_0 + \sum_{i \neq 0} p_i \\ &\preceq 1 \end{aligned}$$

Thus we have $\sum_i p_i \simeq 1$, and we may further suppose that we have in fact:

$$\sum_i p_i = 1$$

Thus the family $\{p_i\}$ can be used in order to construct a copy $B(H) \subset B$, with $H = l^2(\mathbb{N})$, and we must have $B = A \otimes B(H)$, with A being a II_1 factor, as desired.

(2) $\implies$ (1) This is clear, because when assuming $B = A \otimes B(H)$, as in Definition 1.21, we can take our projection $p \in B$ to be of the form $p = 1 \otimes q$, with $q \in B(H)$ being a rank 1 projection, and we have then $pBp = A$, which is a II_1 factor, as desired. □

Getting back now to the original interpretation of the II_∞ factors, from Definition 1.21, the tensor product writing there $B = A \otimes B(H)$ suggests tensoring the trace of the II_1 factor A with the usual operator trace of $B(H)$. We are led in this way to:

Definition 1.23. *Given a II_∞ factor B, written as $B = A \otimes B(H)$, with A being a II_1 factor and with H being an infinite dimensional Hilbert space, we define a map*

$$tr : B_+ \to [0, \infty] \quad , \quad tr((x_{ij})) = \sum_i tr(x_{ii})$$

where we have chosen a basis of H, as to have $H \simeq l^2(\mathbb{N})$, and so $B(H) \subset M_\infty(\mathbb{C})$.

As an important observation, to start with, unlike in the II_1 factor case, that of the factor A, or in the I_∞ factor case, that of the factor $B(H)$, it is not possible to suitably normalize the trace constructed above. This follows indeed from the results below.

On the positive side now, the above trace has many useful properties, as follows:

Proposition 1.24. *The II_∞ factor trace that we constructed above*

$$tr : B_+ \to [0, \infty]$$

has the following properties:

1. *$tr(x + y) = tr(x) + tr(y)$, and $tr(\lambda x) = \lambda tr(x)$ for $\lambda \geq 0$.*
2. *If $x_i \nearrow x$ then $tr(x_i) \to tr(x)$.*
3. *$tr(xx^*) = tr(x^*x)$.*
4. *$tr(uxu^*) = tr(x)$ for any $u \in U_B$.*

PROOF. All this is elementary, the idea being as follows:

(1) This is clear from definitions.

(2) This is again clear from definitions.

(3) This is something which is elementary as well.

(4) This comes from (3), via the formula $uxu^* = u\sqrt{x} \cdot \sqrt{x}u^*$. □

As a main result now regarding the II_∞ factor trace, we have:

THEOREM 1.25. *The II_∞ factor trace $tr : B_+ \to [0, \infty]$ constructed above, when restricted to the projections*

$$tr : P(B) \to [0, \infty]$$

induces an isomorphism between the totally ordered set of equivalence classes of projections in B and the interval $[0, \infty]$.

PROOF. We have several things to be checked here, as follows:

(1) Our first claim is that a projection $p \in B$ is finite precisely when $tr(p) < \infty$.

– Indeed, in one sense, assume that we have $tr(p) < \infty$. If our projection p was to be infinite, we would have a subprojection $q \leq p$ having the same trace as p, and so $r = p - q$ would be a projection of trace 0, which is impossible. Thus p is indeed finite.

– In the other sense now, assuming $tr(p) = \infty$, we have to prove that p is infinite. For this purpose, let us pick a projection $q \leq p$ having finite trace. Then $r = p - q$ satisfies $tr(r) = \infty$, and so we can iterate the procedure, and we end up with an infinite sequence of pairwise orthogonal projections, which are all smaller than p. But this shows that p dominates an infinite projection, and so that p itself is infinite, as desired.

(2) Our second claim is that if $p, q \in B$ are projections, with p finite, then:

$$p \preceq q \iff tr(p) = tr(q)$$

But this follows exactly as in the II_1 factor case, discussed above.

(3) Our third and final claim, which will finish the proof, is that any infinite projection is equivalent to the identity. For this purpose, assume that $p \in B$ is infinite. By definition, this means that we can find a unitary $u \in B$ such that:

$$uu^* = p \quad , \quad u^*u \leq p \quad , \quad uu^* \neq p$$

But these conditions show that $(u^n)^i u^n$ is a strictly decreasing sequence of equivalent projections, and by using this sequence we conclude that we have $1 \preceq p$, as desired. □

Moving ahead now, in order to further investigate the II_∞ factors, we will need:

Theorem 1.26. *Given a* II_1 *factor* $A \subset B(H)$*, there exists an isometry*

$$u : H \to L^2(A) \otimes l^2(\mathbb{N})$$

such that $ux = (x \otimes 1)u$*, for any* $x \in A$*.*

Proof. We use a standard idea, that we used many times before, namely an amplification trick. Given a II_1 factor $A \subset B(H)$, consider the following Hilbert space:

$$K = H \oplus L^2(A) \otimes l^2(\mathbb{N})$$

Consider, as operators over this space K, the following projections:

$$p = id \oplus 0 \quad , \quad q = 0 \oplus id$$

Both these projections p, q belong then to A', which is a type II_∞ factor. Now since $q \in A'$ is infinite, by Theorem 1.25 we can find a partial isometry $u \in A'$ such that:

$$u^*u = p \quad , \quad uu^* \leq q$$

Now let us represent this partial isometry $u \in B(K)$ as a 2×2 matrix, as follows:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The above conditions $u^*u = p$ and $uu^* \leq q$ reformulate then as follows:

$$b^*b + d^*d = 0 \quad , \quad aa^* + bb^* = 0$$

We conclude that our partial isometry $u \in B(K)$ has the following special form:

$$u = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

But the operator $c : H \to l^2(A) \otimes l^2(\mathbb{N})$ that we found in this way must be an isometry, and from $u \in A'$ we obtain $ux = (x \otimes 1)u$, for any $x \in A$, as desired. □

As a basic consequence of the above result, which is something good to know, and that we will use many times in what follows, we have:

Theorem 1.27. *The commutant of a* II_1 *factor is a* II_1 *factor, or a* II_∞ *factor.*

Proof. This follows indeed from the explicit interpretation of the operator algebra embedding $A \subset B(H)$ of our II_1 factor A, found in Theorem 1.26. □

Summarizing, we have an extension of the general theory of the II_1 factors, developed before, to the general case of the type II factors, which comprises by definition the II_1 factors and the II_∞ factors. All this is of course technically very useful.

1d. Coupling constant

We are now in position of constructing the coupling constant. The idea here, following as usual the key paper of Murray and von Neumann [60], will be that given a representation of a II_1 factor $A\subset B(H)$, we can try to understand how far is this representation from the standard form, where $H=L^2(A)$, from "above" or from "below".

In order to discuss this, which is something quite technical, let us start with:

PROPOSITION 1.28. *Given a II_1 factor $A\subset B(H)$, with its embedding into $B(H)$ being represented as above, in terms of an isometry*

$$u:H\to L^2(A)\otimes l^2(\mathbb N)\quad,\quad ux=(x\otimes1)u$$

the following quantity does not depend on the choice of this isometry u:

$$C=tr(uu^*)$$

Moreover, for the standard form, where $H=L^2(A)$, this constant takes the value 1.

PROOF. Assume indeed that we have an isometry u as in the statement, and that we have as well a second such isometry, of the same type, namely:

$$v:H\to L^2(A)\otimes l^2(\mathbb N)\quad,\quad vx=(x\otimes1)v$$

We have then $uu^*=uv^*vu^*$, and by using this, we obtain:

$$\begin{aligned}C_u&=tr(uu^*)\\&=tr(uv^*vu^*)\\&=tr(vu^*uv^*)\\&=tr(vv^*)\\&=C_v\end{aligned}$$

Thus, we are led to the conclusion in the statement. As for the last assertion, regarding the standard form, this is clear from definitions, because here we can take $u=1$. □

As a conclusion to all this, given a II_1 factor $A\subset B(H)$, we know from Theorem 1.26 that H must appear as an "inflated" version of $L^2(A)$. The corresponding inflation constant is a certain number, that we can call coupling constant, as follows:

DEFINITION 1.29. *Given a representation of a II_1 factor $A\subset B(H)$, we can talk about the corresponding coupling constant, as being the number*

$$\dim_A H\in(0,\infty]$$

constructed as follows, with $u:H\to L^2(A)\otimes l^2(\mathbb N)$ isometry satisfying $ux=(x\otimes1)u$:

$$\dim_A H=tr(uu^*)$$

For the standard form, where $H=L^2(A)$, this coupling constant takes the value 1.

This definition might seem a bit complicated, but things here are quite non-trivial, and there is no way of doing something substantially simpler. Alternatively, we can define the coupling constant via the following formula, after proving first that the number on the right is indeed independent of the choice on a nonzero vector $x \in H$:

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

This latter formula was in fact the original definition of the coupling constant, by Murray and von Neumann [60]. However, technically speaking, things are slightly easier when using the approach in Definition 1.29. We will be back to this key formula of Murray and von Neumann, with full explanations, in a moment.

Let us start our study of the coupling constant with some basic results, coming from definitions and from what we already have, as results, as follows:

PROPOSITION 1.30. *The coupling constant* $\dim_A H \in (0, \infty]$ *associated to a* II_1 *factor representation* $A \subset B(H)$ *has the following properties:*

(1) *For the standard form,* $H = L^2(A)$*, we have* $\dim_A H = 1$.
(2) *For the usual representation on* $H = L^2(A) \otimes l^2(\mathbb{N})$*, we have* $\dim_A H = \infty$.
(3) *We have* $\dim_A H < \infty$ *precisely when* A' *is a* II_1 *factor.*
(4) *We have additivity,* $\dim_A(\oplus_i H_i) = \sum_i \dim_A H_i$.
(5) *We have* $\dim_A(L^2(A)p) = tr(p)$*, for any projection* $p \in A$.
(6) *The coupling constant can take any value in* $(0, \infty]$.

PROOF. All these assertions are elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something that comes from definitions too.

(3) This comes from the general properties of the II_∞ factors, and their traces.

(4) Again, this is clear from the definition of the coupling constant.

(5) This follows by using $u(x) = x \otimes \xi$, with $\xi \in l^2(\mathbb{N})$ being of norm 1.

(6) This follows by starting with (5), and then making direct sums, as in (4). □

At a more advanced level now, in relation with projections and compressions, and getting towards the above-mentioned Murray-von Neumann approach, we have:

PROPOSITION 1.31. *We have the compression formula*

$$\dim_{pAp}(pH) = \frac{\dim_A H}{tr_A(p)}$$

valid for any projection $p \in A$.

Proof. We can prove this result in two steps, as follows:

(1) Assume that H is as follows, with $q \in A$ being a projection satisfying $q \leq p$:

$$H = L^2(A)q$$

We can use the following unitary, intertwining the left and right actions of pAp:

$$L^2(pAp) \to pL^2(A)p \quad , \quad pxp\Omega \to p(x\Omega)p$$

Indeed, we obtain that the following algebras are unitarily equivalent:

$$pAp \subset B(pL^2(A)q) \quad , \quad pAp \subset B(L^2(pAp)q)$$

Thus, by using the formula (5) in Proposition 1.30 we obtain, as desired:

$$\begin{aligned} \dim_{pAp}(pH) &= tr_{pAp}(q) \\ &= \frac{tr_A(q)}{tr_A(p)} \\ &= \frac{\dim_A H}{tr_A(p)} \end{aligned}$$

(2) In the general case now, where H is arbitrary, the result follows from what we proved above, and from the additivity property from Proposition 1.30 (4). □

With all these properties established, we can now recover, as a theorem, the original definition of the coupling constant, due to Murray and von Neumann, as follows:

Theorem 1.32. *Given a II_1 factor $A \subset B(H)$, with the commutant $A' \subset B(H)$ assumed to be finite, the corresponding coupling constant is finite, given by*

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

with the number on the right being independent of the choice on a nonzero vector $x \in H$. In the case where A' is infinite, the corresponding coupling constant is infinite.

Proof. There are several things to be proved here, the idea being as follows:

(1) We know from Proposition 1.30 (3) that we have $\dim_A H < \infty$ precisely when the commutant $A' \subset B(H)$ is finite. Thus, we may assume that we are in this case.

(2) Assuming so, we have the following formula, valid for any projection $p \in A'$, which follows from the basic properties of the coupling constant, established above:

$$\dim_{Ap}(pH) = tr_{A'}(p) \dim_A H$$

(3) Now with this formula in hand, the formula in the statement follows as well, once again by doing a number of standard amplification and compression manipulations. □

As an illustration for all this, given an inclusion of ICC groups $\Lambda \subset \Gamma$, whose group algebras are both II_1 factors, we have the following formula:

$$\dim_{L(\Lambda)} L^2(\Gamma) = [\Gamma : \Lambda]$$

There are many other examples of explicit computations of the coupling constant, all leading into interesting mathematics. We will be back to this.

As a last topic for this chapter, given a II_1 factor A, let us discuss now the representations of type $A \subset B$, with B being another II_1 factor. This is a quite natural notion, perhaps even more natural than the representations $A \subset B(H)$, because we have previously decided that the II_1 factors B, and not the full operator algebras $B(H)$, are the correct infinite dimensional generalization of the usual matrix algebras $M_N(\mathbb{C})$.

This was for the philosophy, and one can of course agree or not with this. Or at least agree or not at the present point of the presentation, because once we will get into the structure of the subfactors $A \subset B$, which is something amazing, there is no way back.

In practice now, given an inclusion of II_1 factors $A \subset B$, a first question is that of defining its index, measuring how big is B compared to A. The first thought here goes into defining the index of $A \subset B$ as being a purely algebraic quantity, as follows:

$$N = \dim_A B$$

However, this is non-trivial, due to the fact that we are in the "continuous dimension" setting, and so our algebraic intuition, where indices are always integers, will not help us much. We will be back to this question later, with a technical solution to it.

In order to solve our index problem, a much better approach is by using the ambient operator algebra $B(H)$, or rather the ambient Hilbert space H, as follows:

THEOREM 1.33. *Given an inclusion of* II_1 *factors* $A \subset B$*, the number*

$$N = \frac{\dim_A H}{\dim_B H}$$

is independent of the ambient Hilbert space H*, and is called index.*

PROOF. The fact that the index of the subfactor $A \subset B$, as defined by the above formula, is indeed independent of the ambient Hilbert space H, comes from the various basic properties of the coupling constant, established above. □

There are many examples of subfactors coming from groups, and every time we obtain the intuitive index. More suprisingly now, Jones proved in [40] that the index, when small, is in fact "quantized", subject to the following unexpected restriction:

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \Big| n \geq 3 \right\} \cup [4, \infty]$$

This is in fact part of a series of non-trivial results about the subfactors, due to Jones, and also Ocneanu, Popa, Wassermann and others, and involving as well the Temperley-Lieb algebra, and many more. We will be back to this, in a moment.

1e. Exercises

Exercises:

EXERCISE 1.34.

EXERCISE 1.35.

EXERCISE 1.36.

EXERCISE 1.37.

EXERCISE 1.38.

EXERCISE 1.39.

EXERCISE 1.40.

EXERCISE 1.41.

Bonus exercise.

CHAPTER 2

Subfactors

2a. Subfactors

We recall that a II_1 factor is a von Neumann algebra $A \subset B(H)$ which has trivial center, $Z(A) = \mathbb{C}$, is infinite dimensional, and has a trace $tr : A \to \mathbb{C}$. For a number of reasons, ranging from simple and intuitive to fairly advanced, explained in chapter 1, such algebras are the core at the whole von Neumann algebra theory.

The world of II_1 factors is a bit similar to the world of the usual matrix algebras $M_N(\mathbb{C})$, which are actually called type I factors, in the sense that it is "self-sufficient", with no need to go further than that. In particular, a nice representation theory for such II_1 factors can be obtained by staying inside the class of II_1 factors, and we have the following definition to start with, which will keep us busy for the rest of this book:

Definition 2.1. *A subfactor is an inclusion of* II_1 *factors* $A \subset B$.

We will see later some examples of such inclusions, along with motivations for their study. In order to get started now, the first thing to be done with such an inclusion is that of defining its index, as a quantity of the following type:

$$[B : A] = \dim_A B$$

Since both A, B are infinite dimensional algebras, this is not exactly obvious. In addition, in view of our previous experience with the II_1 factors, and notably with their "continuous dimension" features, we can only expect the index to range as follows:

$$[B : A] \in [1, \infty]$$

In order to discuss this, let us recall from chapter 1 that given a representation of a II_1 factor $A \subset B(H)$, we can construct a number as follows, called coupling constant, which for the standard form, where $H = L^2(A)$, takes the value 1, and which in general mesures how far is $A \subset B(H)$ from the standard form:

$$\dim_A H \in (0, \infty]$$

Getting now to the subfactors, in the sense of Definition 2.1, we have the following construction, that we know as well from chapter 1:

THEOREM 2.2. *Given a subfactor $A\subset B$, the number*

$$N=\frac{\dim_AH}{\dim_BH}\in[1,\infty]$$

is independent of the ambient Hilbert space H, and is called index.

PROOF. This is something that we know from chapter 1, the idea being that the independence of the index from the choice of the ambient Hilbert space H comes from the various basic properties of the coupling constant. □

There are many examples of subfactors, and we will discuss this gradually, in what follows. Following Jones [40], the most basic examples of subfactors are as follows:

PROPOSITION 2.3. *Assuming that G is a compact group, acting on a II_1 factor P in a minimal way, in the sense that we have*

$$(P^G)'\cap P=\mathbb C$$

and that $H\subset G$ is a closed subgroup of finite index, we have a subfactor

$$P^G\subset P^H$$

having index $N=[G:H]$, called Jones subfactor.

PROOF. This is something standard, the idea being that the factoriality of P^G,P^H comes from the minimality of the action, and that the index formula is clear. We will be back with full details about this in chapter 3, directly in a more general setting. □

In order to study the subfactors, let us start with the following standard result:

PROPOSITION 2.4. *Given a subfactor $A\subset B$, there is a unique linear map*

$$E:B\to A$$

which is positive, unital, trace-preserving and satisfies the following condition:

$$E(b_1ab_2)=b_1E(a)b_2$$

This map is called conditional expectation from B onto A.

PROOF. We make use of the standard representation of the II_1 factor B, with respect to its unique trace $tr:B\to\mathbb C$, as constructed in chapter 1:

$$B\subset L^2(B)$$

If we denote by Ω the standard cyclic and separating vector of $L^2(B)$, we have an identification $A\Omega=L^2(A)$. Consider now the following orthogonal projection:

$$e:L^2(B)\to L^2(A)$$

It follows from definitions that we have an inclusion as follows:

$$e(B\Omega)\subset A\Omega$$

Thus e induces by restriction a certain linear map $E : B \to A$. This linear map E and the orthogonal projection e are then related by:

$$exe = E(x)e$$

But this shows that the linear map E satisfies the various conditions in the statement, namely positivity, unitality, trace preservation and bimodule property. As for the uniqueness assertion, this follows by using the same argument, applied backwards, the idea being that a map E as in the statement must come from the projection e. □

Following Jones [40], we will be interested in what follows in the orthogonal projection $e : L^2(B) \to L^2(A)$ producing the expectation $E : B \to A$, rather than in E itself:

Definition 2.5. *Associated to any subfactor $A \subset B$ is the orthogonal projection*

$$e : L^2(B) \to L^2(A)$$

producing the conditional expectation $E : B \to A$ via the following formula:

$$exe = E(x)e$$

This projection is called Jones projection for the subfactor $A \subset B$.

Quite remarkably, the subfactor $A \subset B$, as well as its commutant, can be recovered from the knowledge of this projection, in the following way:

Proposition 2.6. *Given a subfactor $A \subset B$, with Jones projection e, we have*

$$A = B \cap \{e\}'$$

$$A' = (B' \cap \{e\})''$$

as equalities of von Neumann algebras, acting on the space $L^2(B)$.

Proof. These formulae basically follow from $exe = E(x)e$, as follows:

(1) Let us first prove that we have $A \subset B \cap \{e\}'$. Given $x \in A$, we have:

$$xe = E(x)e = exe$$

$$x^*e = E(x^*)e = ex^*e$$

Thus, we obtain, as desired, that x commutes with e:

$$ex = (x^*e)^* = (ex^*e)^* = exe = xe$$

(2) Let us prove now that $B \cap \{e\}' \subset A$. Assuming $ex = xe$, we have:

$$E(x)e = exe = xe^2 = xe$$

We conclude from this that we have the following equality:

$$(E(x) - x)\Omega = (E(x) - x)e\Omega = 0$$

Now since Ω is separating for B we have, as desired:

$$x = E(x) \in A$$

(3) In order to prove now $A' =< B', e >$, observe that we have:

$$A = B \cap \{e\}' = B'' \cap \{e\}' = (B' \cap \{e\})'$$

Now by taking the commutant, we obtain $A' = (B' \cap \{e\})''$, as desired. □

Still following Jones [40], we are now ready to formulate a key definition:

DEFINITION 2.7. *Associated to any subfactor $A \subset B$ is the basic construction*

$$A \subset_e B \subset C$$

with $C =< B, e >$ being the algebra generated by B and by the Jones projection

$$e : L^2(B) \to L^2(A)$$

acting on the Hilbert space $L^2(B)$.

The idea in what follows will be that $B \subset C$ appears as a kind of "reflection" of $A \subset B$, and also that the basic construction can be iterated, with all this leading to nontrivial results. Let us start by further studying the basic construction:

THEOREM 2.8. *Given a subfactor $A \subset B$ having finite index,*

$$[B : A] < \infty$$

the basic construction $A \subset_e B \subset C$ has the following properties:

(1) $C = JA'J$.
(2) $C = \overline{B + Beb}$.
(3) C *is a* II_1 *factor.*
(4) $[C : B] = [B : A]$.
(5) $eCe = Ae$.
(6) $tr(e) = [B : A]^{-1}$.
(7) $tr(xe) = tr(x)[B : A]^{-1}$, *for any* $x \in B$.

PROOF. All this is standard, the idea being as follows:

(1) We have $JB'J = B$ and $JeJ = e$, which gives:

$$\begin{aligned} JA'J &= J < B', e > J \\ &= < JB'J, JeJ > \\ &= < B, e > \\ &= C \end{aligned}$$

(2) This follows from the fact that the vector space $B + BeB$ is closed under multiplication, and from the fact that we have $exe = E(x)e$.

(3) This follows from the fact, that we know from chapter 1, that our finite index assumption $[B : A] < \infty$ is equivalent to the fact that A' is a factor. But this is in turn equivalent to the fact that $C = JA'J$ is a factor, as desired.

(4) We have indeed the folowing computation:

$$\begin{aligned}
[C:B] &= \frac{\dim_B L^2(B)}{\dim_C L^2(B)}\\
&= \frac{1}{\dim_C L^2(B)}\\
&= \frac{1}{\dim_{JA'J} L^2(B)}\\
&= \frac{1}{\dim_{A'} L^2(B)}\\
&= \dim_A L^2(B)\\
&= [B:A]
\end{aligned}$$

(5) This follows indeed from (2) and from the formula $exe = E(x)e$.

(6) We have the following computation:

$$\begin{aligned}
1 &= \dim_A L^2(A)\\
&= \dim_A(eL^2(B))\\
&= tr_{A'}(e)\dim_A(L^2(B))\\
&= tr_{A'}(a)[B:A]
\end{aligned}$$

Now since $C = JA'J$ and $JeJ = e$, we obtain from this, as desired:

$$tr(e) = tr_{JA'J}(JeJ) = tr_{A'}(e) = [B:A]^{-1}$$

(7) We already know from (6) that the formula in the statement holds for $x = 1$. In order to discuss the general case, observe first that for $x, y \in A$ we have:

$$tr(xye) = tr(yex) = tr(yxe)$$

Thus the linear map $x \to tr(xe)$ is a trace on A, and by uniqueness of the trace on A, we must have, for a certain constant $c > 0$:

$$tr(xe) = c \cdot tr(x)$$

Now by using (6) we obtain $c = [B:A]^{-1}$, so we have proved the formula in the statement for $x \in A$. The passage to the general case $x \in B$ can be done as follows:

$$\begin{aligned}
tr(xe) &= tr(exe)\\
&= tr(E(x)e)\\
&= tr(E(x))c\\
&= tr(x)c
\end{aligned}$$

Thus, we have proved the formula in the statement, in general. □

2b. The Jones tower

The above result is quite interesting, so let us perform now twice the basic construction, and see what we get. The result here, which is more technical, is as follows:

PROPOSITION 2.9. *Associated to $A\subset B$ is the double basic construction*

$$A\subset_e B\subset_f C\subset D$$

with e,f being the following orthogonal projections,

$$e:L^2(B)\to L^2(A)$$

$$f:L^2(C)\to L^2(B)$$

having the following properties:

$$fef=[B:A]^{-1}f$$

$$efe=[B:A]^{-1}e$$

PROOF. We have two formulae to be proved, the idea being as follows:

(1) The first formula is clear, because we have:

$$\begin{aligned}fef&=E(e)f\\&=tr(e)f\\&=[B:A]^{-1}f\end{aligned}$$

(2) Regarding now the second formula, it is enough to check it on the dense subset $(B+BeB)\Omega$. Thus, we must show that for any $x,y,z\in B$, we have:

$$efe(x+yez)\Omega=[B:A]^{-1}e(x+yez)\Omega$$

For this purpose, we will prove that we have, for any $x,y,z\in B$:

$$efex\Omega=[B:A]^{-1}ex\Omega$$

$$efeyez\Omega=[B:A]^{-1}eyez\Omega$$

The first formula can be established as follows:

$$\begin{aligned}efex\Omega&=efexf\Omega\\&=eE(ex)f\Omega\\&=eE(e)xf\Omega\\&=[B:A]^{-1}exf\Omega\\&=[B:A]^{-1}ex\Omega\end{aligned}$$

The second formula can be established as follows:

$$\begin{aligned} efeyez\Omega &= efeyezf\Omega \\ &= eE(eyez)f\Omega \\ &= eE(eye)zf\Omega \\ &= eE(E(y)e)zf\Omega \\ &= eE(y)E(e)zf\Omega \\ &= [B:A]^{-1}eE(y)zf\Omega \\ &= [B:A]^{-1}eyezf\Omega \\ &= [B:A]^{-1}eyez\Omega \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We can in fact perform the basic construction by recurrence, and we obtain:

Theorem 2.10. *Associated to any subfactor $A_0 \subset A_1$ is the Jones tower*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

with the Jones projections having the following properties:

(1) $e_i^2 = e_i = e_i^*$.
(2) $e_i e_j = e_j e_i$ *for* $|i-j| \geq 2$.
(3) $e_i e_{i\pm1} e_i = [B:A]^{-1} e_i$.
(4) $tr(we_{n+1}) = [B:A]^{-1} tr(w)$, *for any word* $w \in < e_1, \ldots, e_n >$.

Proof. This follows from Theorem 2.8 and Proposition 2.9, because the triple basic construction does not need in fact any further study. See Jones [40]. □

2c. Temperley-Lieb

The relations found in Theorem 2.10 are in fact well-known, from the standard theory of the Temperley-Lieb algebra. This algebra, discovered by Temperley and Lieb in the context of statistical mechanics [79], has a very simple definition, as follows:

Definition 2.11. *The Temperley-Lieb algebra of index $N \in [1, \infty)$ is defined as*

$$TL_N(k) = span(NC_2(k,k))$$

with product given by vertical concatenation, with the rule

$$\bigcirc = N$$

for the closed circles that might appear when concatenating.

In other words, the algebra $TL_N(k)$, depending on parameters $k \in \mathbb{N}$ and $N \in [1, \infty)$, is the formal linear span of the pairings $\pi \in NC_2(k, k)$. The product operation is obtained by linearity, for the pairings which span $TL_N(k)$ this being the usual vertical concatenation, with the conventions that things go "from top to bottom", and that each circle that might appear when concatenating is replaced by a scalar factor, equal to N.

In order to make the connection with subfactors, let us start with:

Proposition 2.12. *The Temperley-Lieb algebra $TL_N(k)$ is generated by the diagrams*

$$\varepsilon_1 = \substack{\cup \\ \cap} \quad , \quad \varepsilon_2 = |\substack{\cup \\ \cap} \quad , \quad \varepsilon_3 = ||\substack{\cup \\ \cap} \quad , \quad \ldots$$

which are all multiples of projections, in the sense that their rescaled versions

$$e_i = N^{-1}\varepsilon_i$$

satisfy the abstract projection relations $e_i^2 = e_i = e_i^*$.

Proof. We have two assertions here, the idea being as follows:

(1) The fact that the algebra $TL_N(k)$ is indeed generated by the sequence of diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ follows by drawing pictures, and more specifically by graphically decomposing each basis element $\pi \in NC_2(k, k)$ as a product of such elements ε_i.

(2) Regarding now the projection assertion, when composing ε_i with itself we obtain ε_i itself, times a circle. Thus, according to our multiplication conventions, we have:

$$\varepsilon_i^2 = N\varepsilon_i$$

Also, when turning upside-down ε_i, we obtain ε_i itself. Thus, according to our involution convention for the Temperley-Lieb algebra, we have:

$$\varepsilon_i^* = \varepsilon_i$$

We conclude that the rescalings $e_i = N^{-1}\varepsilon_i$ satisfy $e_i^2 = e_i = e_i^*$, as desired. □

As a second result now, making the link with Theorem 2.10, we have:

Proposition 2.13. *The standard generators $e_i = N^{-1}\varepsilon_i$ of the Temperley-Lieb algebra $TL_N(k)$ have the following properties, where tr is the trace obtained by closing:*

1. *$e_i e_j = e_j e_i$ for $|i - j| \geq 2$.*
2. *$e_i e_{i\pm1} e_i = [B : A]^{-1} e_i$.*
3. *$tr(we_{n+1}) = [B : A]^{-1} tr(w)$, for any word $w \in < e_1, \ldots, e_n >$.*

Proof. This follows indeed by doing some elementary computations with diagrams, in the spirit of those performed in the proof of Proposition 2.12. Indeed:

(1) This is clear from the definition of the diagrams ε_i.

(2) This is clear as well from the definition of the diagrams ε_i.

(3) This is something which is clear too, from the definition of ε_{n+1}. □

With the above results in hand, we can now reformulate our main finding about subfactors, namely Theorem 2.10, into something more conceptual, as follows:

THEOREM 2.14. *Given a finite index subfactor $A_0 \subset A_1$, with Jones tower*

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots\ldots$$

the rescaled sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ produces a representation

$$TL_N \subset B(H)$$

of the Temperley-Lieb algebra of index $N = [A_1 : A_0]$.

PROOF. The idea here is that Theorem 2.10, coming from the study of the basic construction, tells us that the rescaled sequence of projections $e_1, e_2, e_3, \ldots \in B(H)$ behaves algebraically exactly as the sequence of diagrams $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \in TL_N$ given by:

$$\varepsilon_1 = {}^{\cup}_{\cap} \quad , \quad \varepsilon_2 = |^{\cup}_{\cap} \quad , \quad \varepsilon_3 = ||^{\cup}_{\cap} \quad , \quad \ldots$$

But these diagrams generate TL_N, and so we have an embedding $TL_N \subset B(H)$, where H is the Hilbert space where our subfactor $A_0 \subset A_1$ lives, as claimed. □

Before going further, with some examples, more theory, and consequences of Theorem 2.14, let us make the following key observation, also from Jones [40]:

THEOREM 2.15. *Given a finite index subfactor $A_0 \subset A_1$, the graded algebra*

$$P = (P_k)$$

formed by the sequence of higher relative commutants

$$P_k = A_0' \cap A_k$$

contains the copy of the Temperley-Lieb algebra constructed above:

$$TL_N \subset P$$

This graded algebra $P = (P_k)$ is called "planar algebra" of the subfactor.

PROOF. As a first observation, since the Jones projection $e_1 : A_1 \to A_0$ commutes with A_0, as was previously established in the above, we have:

$$e_1 \in P_2'$$

By translation we obtain from this that we have, for any $k \in \mathbb{N}$:

$$e_1, \ldots, e_{k-1} \in P_k$$

Thus we have indeed an inclusion of graded algebras $TL_N \subset P$, as claimed. □

The point with the above result, which explains among others the terminology at the end, is that, in the context of Theorem 2.14, the planar algebra structure of TL_N, obtained by composing diagrams, extends into an abstract planar algebra structure of P. See [42]. We will discuss all this, with full details, in chapter 4 below.

2d. The index theorem

As an interesting consequence of the above results, somehow contradicting the "continuous geometry" philosophy that has being going on so far, in relation with the II_1 factors, we have the following surprising result, also from Jones' original paper [40]:

THEOREM 2.16. *The index of subfactors $A\subset B$ is "quantized" in the $[1,4]$ range,*

$$N\in\left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n\geq 3\right\}\cup[4,\infty]$$

with the obstruction coming from the existence of the representation $TL_N\subset B(H)$.

PROOF. This comes from the basic construction, and more specifically from the combinatorics of the Jones projections $e_1,e_2,e_3,\ldots$, the idea being as folows:

(1) In order to best comment on what happens, when iterating the basic construction, let us record the first few values of the numbers in the statement:

$$4\cos^2\left(\frac{\pi}{3}\right)=1\quad,\quad 4\cos^2\left(\frac{\pi}{4}\right)=2$$

$$4\cos^2\left(\frac{\pi}{5}\right)=\frac{3+\sqrt{5}}{2}\quad,\quad 4\cos^2\left(\frac{\pi}{6}\right)=3$$

$$\ldots$$

(2) When performing a basic construction, we obtain, by trace manipulations on e_1:

$$N\notin(1,2)$$

With a double basic construction, we obtain, by trace manipulations on $<e_1,e_2>$:

$$N\notin\left(2,\frac{3+\sqrt{5}}{2}\right)$$

With a triple basic construction, we obtain, by trace manipulations on $<e_1,e_2,e_3>$:

$$N\notin\left(\frac{3+\sqrt{5}}{2},3\right)$$

Thus, we are led to the conclusion in the statement, by a kind of recurrence, involving a certain family of orthogonal polynomials.

(3) In practice now, the most elegant way of proving the result is by using the fundamental fact, explained in Theorem 2.14, that that sequence of Jones projections $e_1,e_2,e_3,\ldots\subset B(H)$ generate a copy of the Temperley-Lieb algebra of index N:

$$TL_N\subset B(H)$$

With this result in hand, we must prove that such a representation cannot exist in index $N < 4$, unless we are in the following special situation:

$$N = 4\cos^2\left(\frac{\pi}{n}\right)$$

But this can be proved by using some suitable trace and positivity manipulations on TL_N, as in (2) above. For full details here, we refer to [28], [40], [46]. □

The above result raises the question of understanding if there are further restrictions on the index of subfactors $A \subset B$, in the range found there, namely:

$$N \in \left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n \geq 3\right\} \cup [4, \infty]$$

In the simplest formulation of the question, the answer is generally "no", as follows:

THEOREM 2.17. *Consider the Murray-von Neumann hyperfinite* II_1 *factor* R. *Its subfactors* $R_0 \subset R$ *are then as follows:*

(1) *They exist for all admissible index values,* $N \in \left\{4\cos^2\left(\frac{\pi}{n}\right)|n \geq 3\right\} \cup [4, \infty]$.
(2) *In index* $N \leq 4$, *they can be realized as irreducible subfactors,* $R_0' \cap R = \mathbb{C}$.
(3) *In index* $N > 4$, *they can be realized as arbitrary subfactors.*

PROOF. This is something quite tricky, worked out in Jones' original paper [40], and requiring some advanced algebra methods, the idea being as follows:

(1) This basically follows by taking a copy of the Temperley-Lieb algebra TL_N, and then building a subfactor out of it, first by constructing a certain inclusion of inductive limits of finite dimensional algebras, $\mathcal{A} \subset \mathcal{B}$, and then by taking the weak closure, which produces copies of the Murray-von Neumann hyperfinite II_1 factor, $A \simeq B \simeq R$.

(2) This follows by examining and fine-tuning the construction in (1), which can be performed as to have control over the relative commutant.

(3) This follows as well from (1), and with the simplest proof here being in fact quite simple, based on a projection trick. □

As another application now, which is more theoretical, let us go back to the question of defining the index of a subfactor in a purely algebraic manner, which was open since chapter 2. The answer here, due to Pimsner and Popa [67], is as follows:

THEOREM 2.18. *Any finite index subfactor* $A \subset B$ *has an algebraic orthonormal basis, called Pimsner-Popa basis, which is constructed as follows:*

(1) *In integer index,* $N \in \mathbb{N}$, *this is a usual basis, of type* $\{b_1, \ldots, b_N\}$, *whose length is exactly the index.*
(2) *In non-integer index,* $N \notin \mathbb{N}$, *this is something of type* $\{b_1, \ldots, b_n, c\}$, *having length* $n + 1$, *with* $n = [N]$, *and with* $N - n \in (0, 1)$ *being related to* c.

PROOF. This is something quite technical, which follows from the basic theory of the basic construction. We refer here to the paper of Pimsner and Popa [67]. □

2e. Exercises

Exercises:

Exercise 2.19.

Exercise 2.20.

Exercise 2.21.

Exercise 2.22.

Exercise 2.23.

Exercise 2.24.

Exercise 2.25.

Exercise 2.26.

Bonus exercise.

CHAPTER 3

Basic examples

3a. Fixed points

Let us discuss now some basic examples of subfactors, with concrete illustrations for all the above notions, constructions, and general theory. These examples will all come from group actions $G \curvearrowright P$, which are assumed to be minimal, in the sense that:

$$(P^G)' \cap P = \mathbb{C}$$

We will not provide proofs for the next few results to follow, the idea being that these constructions can be unified, and that we would like to keep the proofs for the unifications only. As a starting point, we have the following result, that we already know:

PROPOSITION 3.1. *Assuming that G is a compact group, acting minimally on a II_1 factor P, and that $H \subset G$ is a subgroup of finite index, we have a subfactor*

$$P^G \subset P^H$$

having index $N = [G : H]$, called Jones subfactor.

PROOF. This is something that we know, the idea being that the factoriality of P^G, P^H comes from the minimality of the action, and that the index formula is clear. □

Along the same lines, we have the following result:

PROPOSITION 3.2. *Assuming that G is a finite group, acting minimally on a II_1 factor P, we have a subfactor as follows,*

$$P \subset P \rtimes G$$

having index $N = |G|$, called Ocneanu subfactor.

PROOF. This is standard as well, the idea being that the factoriality of $P \rtimes G$ comes from the minimality of the action, and that the index formula is clear. □

We have as well a third result of the same type, as follows:

PROPOSITION 3.3. *Assuming that G is a compact group, acting minimally on a II_1 factor P, and that $G \to PU_n$ is a projective representation, we have a subfactor*

$$P^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

having index $N = n^2$, called Wassermann subfactor.

Proof. As before, the idea is that the factoriality of $P^G, (M_n(\mathbb{C})\otimes P)^G$ comes from the minimality of the action, and the index formula is clear. □

The above subfactors look quite related, and indeed they are, due to:

Theorem 3.4. *The Jones, Ocneanu and Wassermann subfactors are all of the same nature, and can be written as follows,*

$$\left(P^G\subset P^H\right)\simeq\left((\mathbb{C}\otimes P)^G\subset(l^\infty(G/H)\otimes P)^G\right)$$
$$(P\subset P\rtimes G)\simeq\left((l^\infty(G)\otimes P)^G\subset(\mathcal{L}(l^2(G))\otimes P)^G\right)$$
$$\left(P^G\subset(M_n(\mathbb{C})\otimes P)^G\right)\simeq\left((\mathbb{C}\otimes P)^G\subset(M_n(\mathbb{C})\otimes P)^G\right)$$

with standard identifications for the various tensor products and fixed point algebras.

Proof. This is something very standard, modulo all kinds of standard identifications. We will explain all this more in detail later, after unifying these subfactors. □

In order to unify now the above constructions of subfactors, the idea is quite clear. Given a compact group G, acting minimally on a II_1 factor P, and an inclusion of finite dimensional algebras $B_0\subset B_1$, endowed as well with an action of G, we would like to construct a kind of generalized Wassermann subfactor, as follows:

$$(B_0\otimes P)^G\subset(B_1\otimes P)^G$$

In order to do this, we must talk first about the finite dimensional algebras B, and about inclusions of such algebras $B_0\subset B_1$. Let us start with the following definition:

Definition 3.5. *Associated to any finite dimensional algebra B is its canonical trace, obtained by composing the left regular representation with the trace of $\mathcal{L}(B)$:*

$$tr:B\subset\mathcal{L}(B)\to\mathbb{C}$$

We say that an inclusion of finite dimensional algebras $B_0\subset B_1$ is Markov if it commmutes with the canonical traces of B_0,B_1.

In what regards the first notion, that of the canonical trace, this is something that we know well, from chapter 2. Indeed, as explained there, we can formally write $B=C(X)$, with X being a finite quantum space, and the canonical trace $tr:B\to\mathbb{C}$ is then precisely the integration with respect to the "counting measure" on X.

In what regards the second notion, that of a Markov inclusion, this is something very natural too, and as a first example here, any inclusion of type $\mathbb{C}\subset B$ is Markov. In general, if we write $B_0=C(X_0)$ and $B_1=C(X_1)$, then the inclusion $B_0\subset B_1$ must come from a certain fibration $X_1\to X_0$, and the inclusion $B_0\subset B_1$ is Markov precisely when the fibration $X_1\to X_0$ commutes with the respective counting measures.

We will be back to Markov inclusions and their various properties on several occasions, in what follows. For our next purposes here, we just need the following result:

Proposition 3.6. *Given a Markov inclusion of finite dimensional algebras $B_0 \subset B_1$ we can perform to it the basic construction, as to obtain a Jones tower*

$$B_0 \subset_{e_1} B_1 \subset_{e_2} B_2 \subset_{e_3} B_3 \subset \ldots\ldots$$

exactly as we did in the above for the inclusions of II_1 *factors.*

Proof. This is something quite routine, by following the computations in the above, from the case of the II_1 factors, and with everything extending well. It is of course possible to do something more general here, unifying the constructions for the inclusions of II_1 factors $A_0 \subset A_1$, and for the inclusions of Markov inclusions of finite dimensional algebras $B_0 \subset B_1$, but we will not need this degree of generality, in what follows. □

With these ingredients in hand, getting back now to the Jones, Ocneanu and Wassermann subfactors, from Theorem 3.4, the point is that these constructions can be unified, and then further studied, the final result on the subject being as follows:

Theorem 3.7. *Let G be a compact group, and $G \to Aut(P)$ be a minimal action on a* II_1 *factor. Consider a Markov inclusion of finite dimensional algebras*

$$B_0 \subset B_1$$

and let $G \to Aut(B_1)$ be an action which leaves invariant B_0, and which is such that its restrictions to the centers of B_0 and B_1 are ergodic. We have then a subfactor

$$(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$$

of index $N = [B_1 : B_0]$, called generalized Wassermann subfactor, whose Jones tower is

$$(B_1 \otimes P)^G \subset (B_2 \otimes P)^G \subset (B_3 \otimes P)^G \subset \ldots$$

where $\{B_i\}_{i \geq 1}$ are the algebras in the Jones tower for $B_0 \subset B_1$, with the canonical actions of G coming from the action $G \to Aut(B_1)$, and whose planar algebra is given by:

$$P_k = (B_0' \cap B_k)^G$$

These subfactors generalize the Jones, Ocneanu and Wassermann subfactors.

Proof. There are several things to be proved, the idea being as follows:

(1) As before on various occasions, the idea is that the factoriality of the algebras $(B_i \otimes P)^G$ comes from the minimality of the action $G \to Aut(P)$, and that the index formula is clear as well, from the definition of the coupling constant and of the index.

(2) Regarding the Jones tower assertion, the precise thing to be checked here is that if $A \subset B \subset C$ is a basic construction, then so is the following sequence of inclusions:

$$(A \otimes P)^G \subset (B \otimes P)^G \subset (C \otimes P)^G$$

But this is something standard, which follows by verifying the basic construction conditions. We will be back to this in a moment, directly in a more general setting.

(3) Next, regarding the planar algebra assertion, we have to prove here that for any indices $i \leq j$, we have the following equality between subalgebras of $B_j \otimes P$:

$$((B_i \otimes P)^G)' \cap (B_j \otimes P)^G = (B_i' \cap B_j^G) \otimes 1$$

But this is something which is routine too, following Wassermann [90], and we will be back to this in a moment, with full details, directly in a more general setting.

(4) Finally, the last assertion, regarding the main examples of such subfactors, which are those of Jones, Ocneanu, Wassermann, follows from Theorem 3.4. □

In addition to the Jones, Ocneanu and Wassermann subfactors, discussed and unified in the above, we have the Popa subfactors, which are constructed as follows:

PROPOSITION 3.8. *Given a discrete group $\Gamma =< g_1, \ldots, g_n >$, acting faithfully via outer automorphisms on a II_1 factor Q, we have the following "diagonal" subfactor*

$$\left\{\begin{pmatrix}g_1(q)&&\\&\ddots&\\&&g_n(q)\end{pmatrix}\Big| q \in Q\right\} \subset M_n(Q)$$

having index $N = n^2$, called Popa subfactor.

PROOF. This is something standard, a bit as for the Jones, Ocneanu and Wassermann subfactors, with the result basically coming from the work of Popa, who was the main user of such subfactors. We will come in a moment with a more general result in this direction, involving discrete quantum groups, along with a complete proof. □

In order to unify now Theorem 3.4 and Proposition 3.8, observe that the diagonal subfactors can be written in the following way, by using a group dual:

$$(Q \rtimes \Gamma)^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}}$$

Here the group dual $\widehat{\Gamma}$ acts on $P = Q \rtimes \Gamma$ via the dual of the action $\Gamma \subset Aut(Q)$, and on $M_n(\mathbb{C})$ via the adjoint action of the following representation:

$$\oplus g_i : \widehat{\Gamma} \to \mathbb{C}^n$$

Summarizing, we are led into quantum groups. Our plan in what follows will be that of discussing the quantum extension of Theorem 3.4, covering the diagonal subfactors as well, and this time with full details, and with examples and illustrations as well.

3b. Quantum groups

As a starting point, we have the following key definition, due to Woronowicz [99]:

DEFINITION 3.9. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $v \in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

define morphisms of C^*-algebras $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{C}$, $S : A \to A^{opp}$.

We say that A is cocommutative when $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip. We have the following result, which justifies the terminology and axioms:

PROPOSITION 3.10. *The following are Woronowicz algebras:*

(1) *$C(G)$, with $G \subset U_N$ compact Lie group. Here the structural maps are:*

$$\Delta(\varphi) = [(g,h) \to \varphi(gh)] \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S(\varphi) = [g \to \varphi(g^{-1})]$$

(2) *$C^*(\Gamma)$, with $F_N \to \Gamma$ finitely generated group. Here the structural maps are:*

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

PROOF. In both cases, we have to indicate a certain matrix v. For the first assertion, we can use the matrix $v = (v_{ij})$ formed by matrix coordinates of G, given by:

$$g = \begin{pmatrix} v_{11}(g) & \ldots & v_{1N}(g) \\ \vdots & & \vdots \\ v_{N1}(g) & \ldots & v_{NN}(g) \end{pmatrix}$$

As for the second assertion, we can use here the diagonal matrix formed by generators:

$$v = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Finally, the last assertion follows from the Gelfand theorem, in the commutative case. In the cocommutative case this follows from the Peter-Weyl theory, explained below. □

In view of Proposition 3.10, we can formulate the following definition:

DEFINITION 3.11. *Given a Woronowicz algebra A, we formally write*

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is both commutative and cocommutative, G is a compact abelian group, Γ is a discrete abelian group, and these groups are dual to each other:

$$G = \widehat{\Gamma} \quad , \quad \Gamma = \widehat{G}$$

In general, we still agree to write the formulae $G=\widehat{\Gamma},\Gamma=\widehat{G}$, but in a formal sense. Finally, let us make as well the following convention:

DEFINITION 3.12. *We identify two Woronowicz algebras (A,v) and (B,w), as well as the corresponding quantum groups, when we have an isomorphism of $*$-algebras*

$$< v_{ij} >\simeq< w_{ij} >$$

mapping standard coordinates to standard coordinates.

This convention is here for avoiding amenability issues, as for any compact or discrete quantum group to correspond to a unique Woronowicz algebra. More on this later.

Moving ahead now, let us call corepresentation of A any unitary matrix $u\in M_n(\mathcal{A})$, where $\mathcal{A}=< v_{ij} >$, satisfying the same conditions as those satisfied by u, namely:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad \varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

We have the following key result, due to Woronowicz [99]:

THEOREM 3.13. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $u\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)u=P$$

where P is the orthogonal projection onto $Fix(u)=\{\xi\in\mathbb{C}^n|u\xi=\xi\}$.

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, by linearity we can assume that $a\in A$ is the coefficient of certain corepresentation, $a=(\tau\otimes id)u$. But in this case, an elementary computation gives the following formula, with P_φ being the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)u$:

$$\left(id\otimes\int_\varphi\right)u=P_\varphi$$

(2) Since $u\xi=\xi$ implies $[(id\otimes\varphi)u]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the fixed point space in the statement, namely:

$$Fix(u)=\left\{\xi\in\mathbb C^n\Big|u\xi=\xi\right\}$$

The point now is that when $\varphi\in A^*$ is faithful, by using a standard positivity trick, we can prove that we have $P_\varphi=P$, exactly as in the classical case.

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

We can now develop, again following [99], the Peter-Weyl theory for the corepresentations of A. Consider the dense subalgebra $\mathcal A\subset A$ generated by the coefficients of the fundamental corepresentation v, and endow it with the following scalar product:

$$<a,b>=\int_Gab^*$$

With this convention, we have the following result, from [99]:

Theorem 3.14. *We have the following Peter-Weyl type results:*

1. *Any corepresentation decomposes as a sum of irreducible corepresentations.*
2. *Each irreducible corepresentation appears inside a certain $v^{\otimes k}$.*
3. *$\mathcal A=\bigoplus_{u\in Irr(A)}M_{\dim(u)}(\mathbb C)$, the summands being pairwise orthogonal.*
4. *The characters of irreducible corepresentations form an orthonormal system.*

Proof. All these results are from [99], the idea being as follows:

(1) Given $u\in M_n(A)$, the intertwiner algebra $End(u)=\{T\in M_n(\mathbb C)|Tu=uT\}$ is a finite dimensional C^*-algebra, and so decomposes as $End(u)=M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$. But this gives a decomposition of type $u=u_1+\ldots+u_k$, as desired.

(2) Consider the Peter-Weyl corepresentations, $v^{\otimes k}$ with k colored integer, defined by $v^{\otimes\emptyset}=1$, $v^{\otimes\circ}=v$, $v^{\otimes\bullet}=\bar v$ and multiplicativity. The coefficients of these corepresentations span the dense algebra $\mathcal A$, and by using (1), this gives the result.

(3) Here the direct sum decomposition, which is a $*$-coalgebra isomorphism, follows from (2). As for the second assertion, this follows from the fact that $(id\otimes\int_G)u$ is the orthogonal projection P_u onto the space $Fix(u)$, for any corepresentation u.

(4) Let us define indeed the character of $u\in M_n(A)$ to be the trace, $\chi_u=Tr(u)$. Since this character is a coefficient of u, the orthogonality assertion follows from (3). As for the norm 1 claim, this follows once again from $(id\otimes\int_G)u=P_u$. □

We can now solve a problem that we left open before, namely:

PROPOSITION 3.15. *The cocommutative Woronowicz algebras appear as the quotients*

$$C^*(\Gamma)\to A\to C^*_{red}(\Gamma)$$

given by $A=C^*_\pi(\Gamma)$ with $\pi\otimes\pi\subset\pi$, with Γ being a discrete group.

PROOF. This follows from the Peter-Weyl theory, and clarifies a number of things said before, notably in Proposition 3.10. Indeed, for a cocommutative Woronowicz algebra the irreducible corepresentations are all 1-dimensional, and this gives the results. □

As another consequence of the above results, once again by following Woronowicz [99], we have the following statement, dealing with functional analysis aspects, and extending what we already knew about the C^*-algebras of the usual discrete groups:

THEOREM 3.16. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full}\to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon:A_{full}\to\mathbb{C}$ factorizes through A_{red}.*
4. *We have $N\in\sigma(Re(\chi_v))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A=C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon:A_{red}\to\mathbb{C}$ produces an isomorphism $A_{red}\to A_{full}$, via a formula of type $(\varepsilon\otimes id)\Phi$. See [99].

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N-Re(\chi(v)))=0$, and the converse can be proved by doing some functional analysis. Once again, we refer here to [99]. □

Let us discuss now some new examples of quantum groups, which will play a key role in what follows, in relation with subfactors and planar algebras. Following Wang [89], we first have the following result, which is something quite straightforward:

PROPOSITION 3.17. *The following universal algebras are Woronowicz algebras,*

$$C(O_N^+)=C^*\left((v_{ij})_{i,j=1,\ldots,N}\Big|v=\bar{v},v^t=v^{-1}\right)$$

$$C(U_N^+)=C^*\left((v_{ij})_{i,j=1,\ldots,N}\Big|v^*=v^{-1},v^t=\bar{v}^{-1}\right)$$

so the underlying compact quantum spaces O_N^+,U_N^+ are compact quantum groups.

Proof. This follows from the elementary fact that if a matrix $v = (v_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$v_{ij}^\Delta = \sum_k v_{ik} \otimes v_{kj} \quad , \quad v_{ij}^\varepsilon = \delta_{ij} \quad , \quad v_{ij}^S = v_{ji}^*$$

Thus, we can indeed define morphisms Δ, ε, S as in Definition 3.9, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$, and this gives the result. □

There is a connection here with group duals, coming from:

Proposition 3.18. *Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle v_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda = < g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = v_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, its diagonal entries $g_i = v_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. □

With this notion in hand, we have the following result:

Theorem 3.19. *The diagonal tori of the basic rotation groups are as follows,*

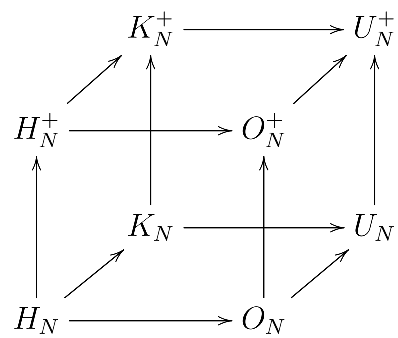
$$\begin{matrix} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{matrix} \quad : \quad \begin{matrix} \mathbb{T}^N & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}} \end{matrix}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

Proof. This is clear indeed from U_N^+, and the other results can be obtained by imposing to the generators of F_N the relations defining the corresponding quantum groups. □

Getting now into more examples, we have the following key result:

Theorem 3.20. *The classical and free, real and complex quantum rotation groups can be complemented with quantum reflection groups, as follows,*



with $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$ being the hyperoctahedral group and the full complex reflection group, and $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$ and $K_N^+ = \mathbb{T} \wr_* S_N^+$ being their free versions.

Proof. This is something quite tricky, the idea being as follows:

(1) The first observation is that S_N, regarded as group of permutations of the N coordinate axes of $\mathbb{R}^N$, is a group of orthogonal matrices, $S_N \subset O_N$. The corresponding coordinate functions $v_{ij} : S_N \to \{0,1\}$ form a matrix $v = (v_{ij})$ which is "magic", in the sense that its entries are projections, summing up to 1 on each row and each column. In fact, by using the Gelfand theorem, we have the following presentation result:

$$C(S_N) = C^*_{comm}\left((v_{ij})_{i,j=1,\ldots,N}\Big| v = \text{magic}\right)$$

(2) Based on the above, and following Wang's paper [89], we can construct the free analogue S_N^+ of the symmetric group S_N via the following formula:

$$C(S_N^+) = C^*\left((v_{ij})_{i,j=1,\ldots,N}\Big| v = \text{magic}\right)$$

Here the fact that we have indeed a Woronowicz algebra is standard, exactly as for the free rotation groups in Proposition 3.17, because if a matrix $v = (v_{ij})$ is magic, then so are the matrices $v^\Delta, v^\varepsilon, v^S$ constructed there, and this gives the existence of Δ, u, S.

(3) Consider now the group $H_N^s \subset U_N$ consisting of permutation-like matrices having as entries the s-th roots of unity. This group decomposes as follows:

$$H_N^s = \mathbb{Z}_s \wr S_N$$

It is straightforward then to construct a free analogue $H_N^{s+} \subset U_N^+$ of this group, for instance by formulating a definition as follows, with $\wr_*$ being a free wreath product:

$$H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

(4) In order to finish, besides the case $s=1$, of particular interest are the cases $s=2,\infty$. Here the corresponding reflection groups are as follows:

$$H_N=\mathbb Z_2\wr S_N\quad,\quad K_N=\mathbb T\wr S_N$$

As for the corresponding quantum groups, these are denoted as follows:

$$H_N^+=\mathbb Z_2\wr_*S_N^+\quad,\quad K_N^+=\mathbb T\wr_*S_N^+$$

Thus, we are led to the conclusions in the statement. □

3c. Diagrams, easiness

Getting now towards easiness, let us start with the following definition:

Definition 3.21. *The Tannakian category associated to a Woronowicz algebra (A,v) is the collection $C_A=(C_A(k,l))$ of vector spaces*

$$C_A(k,l)=Hom(v^{\otimes k},v^{\otimes l})$$

where the corepresentations $v^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer, defined by

$$v^{\otimes\emptyset}=1\quad,\quad v^{\otimes\circ}=v\quad,\quad v^{\otimes\bullet}=\bar v$$

and multiplicativity, $v^{\otimes kl}=v^{\otimes k}\otimes v^{\otimes l}$, are the Peter-Weyl corepresentations.

As a key remark, the fact that $v\in M_N(A)$ is biunitary translates into the following conditions, where $R:\mathbb C\to\mathbb C^N\otimes\mathbb C^N$ is the linear map given by $R(1)=\sum_ie_i\otimes e_i$:

$$R\in Hom(1,v\otimes\bar v)\quad,\quad R\in Hom(1,\bar v\otimes v)$$

$$R^*\in Hom(v\otimes\bar v,1)\quad,\quad R^*\in Hom(\bar v\otimes v,1)$$

We are therefore led to the following abstract definition, summarizing the main properties of the categories appearing from Woronowicz algebras:

Definition 3.22. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C=(C(k,l))$ of subspaces*

$$C(k,l)\subset\mathcal L(H^{\otimes k},H^{\otimes l})$$

satisfying the following conditions:

1. *$S,T\in C$ implies $S\otimes T\in C$.*
2. *If $S,T\in C$ are composable, then $ST\in C$.*
3. *$T\in C$ implies $T^*\in C$.*
4. *Each $C(k,k)$ contains the identity operator.*
5. *$C(\emptyset,\circ\bullet)$ and $C(\emptyset,\bullet\circ)$ contain the operator $R:1\to\sum_ie_i\otimes e_i$.*

The point now is that conversely, we can associate a Woronowicz algebra to any tensor category in the sense of Definition 3.22, in the following way:

PROPOSITION 3.23. *Given a tensor category $C=(C(k,l))$ over $\mathbb C^N$, as above,*

$$A_C=C^*\left((v_{ij})_{i,j=1,\ldots,N}\Big|T\in Hom(v^{\otimes k},v^{\otimes l}),\forall k,l,\forall T\in C(k,l)\right)$$

is a Woronowicz algebra.

PROOF. This is something standard, because the relations $T\in Hom(v^{\otimes k},v^{\otimes l})$ determine a Hopf ideal, so they allow the construction of Δ,ε,S as in Definition 3.9. □

With the above constructions in hand, we have the following result:

THEOREM 3.24. *The Tannakian duality constructions*

$$C\to A_C\quad,\quad A\to C_A$$

are inverse to each other, modulo identifying full and reduced versions.

PROOF. The idea is that we have $C\subset C_{A_C}$, for any algebra A, and so we are left with proving that we have, for any category C, an inclusion as follows:

$$C_{A_C}\subset C$$

But this can be proved indeed, by performing a long series of algebraic manipulations, including a use of von Neumann's bicommutant theorem, and for details we refer to Malacarne [54], and also to Woronowicz [100], where this result was first proved. □

In practice now, all this is quite abstract, and we will rather need Brauer type results, for the specific quantum groups that we are interested in. Let us start with:

DEFINITION 3.25. *Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l. A collection of subsets*

$$D=\bigsqcup_{k,l}D(k,l)$$

with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*

Observe the similarity with the various axioms from Definition 3.22.

In fact, Definition 6.25 is concieved as to be a delinearized version of Definition 3.22, and the relation with the Tannakian categories comes from:

PROPOSITION 3.26. *Given a partition $\pi\in P(k,l)$, consider the linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, where $e_1,\ldots,e_N$ is the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is then categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. The formulae in the statement are all elementary, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}&T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
=&\sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&\sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&\ N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
= & \sum_{i_1\ldots i_p} < T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}), e_{i_1}\otimes\ldots\otimes e_{i_p} > e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix} e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

In relation with quantum groups, we have the following result:

THEOREM 3.27. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the following formula:*

$$Hom(v^{\otimes k},v^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

To be more precise, the spaces on the right form a Tannakian category, and so produce a certain closed subgroup $G_N\subset U_N^+$, via the Tannakian duality correspondence.

PROOF. This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from Malacarne [54], as explained in Theorem 3.24. Indeed, let us set:

$$C(k,l)=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

By using the various axioms in Definition 3.25, and the categorical properties of the operation $\pi\to T_\pi$, from Proposition 3.26, we deduce that $C=(C(k,l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. □

Philosophically speaking, the quantum groups appearing as in Theorem 3.27 are the simplest, from the perspective of Tannakian duality, so let us formulate:

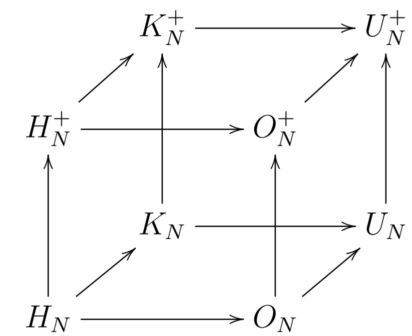
DEFINITION 3.28. *A closed subgroup $G\subset U_N^+$ is called easy when we have*

$$Hom(v^{\otimes k},v^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

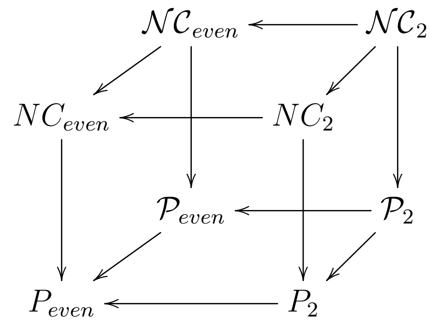
for any colored integers k,l, for a certain category of partitions $D\subset P$.

Getting now to examples, we have the following Brauer type result:

Theorem 3.29. *The basic quantum rotation and reflection groups,*



are all easy, the corresponding categories of partitions being as follows,



with on top, the symbol NC standing everywhere for noncrossing partitions.

Proof. This is something well-known and routine, as follows:

(1) Let us first discuss the easiness property of O_N^+, U_N^+. The quantum group U_N^+ is by definition constructed via the following relations:

$$v^* = v^{-1} \quad , \quad v^t = \bar{v}^{-1}$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \ , \ \pi = \overset{\cap}{\circ\bullet} \quad , \quad T_\pi \ , \ \pi = \overset{\cap}{\bullet\circ}$$

It follows that the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = < \overset{\cap}{\circ\bullet} \ , \ \overset{\cap}{\bullet\circ} > = \mathcal{NC}_2$$

Now by imposing the extra relation $v = \bar{v}$, we obtain the easiness of O_N^+ as well.

(2) In what regards now H_N^+, K_N^+, the first observation is that the magic condition satisfied by v can be reformulated as follows, with $Y \in P(2,1)$ being the fork partition:

$$T_Y \in Hom(v^{\otimes 2}, v)$$

Now by proceeding as in the proof for U_N^+ discussed above, we conclude that the quantum group S_N^+ is indeed easy, the associated category of partitions being:

$$D=< NC_2,Y >= NC$$

With this in hand, we can pass to the quantum groups H_N^+,K_N^+ in a standard way, and we are led to easiness, and the categories in the statement.

(3) Finally, we can pass from the upper face to the lower face of the cube by adding the basic crossing, and this produces the various categories in the statement. □

3d. Actions, invariants

Good news, with the above general quantum group theory in hand, we can now go back to the generalized Wassermann subfactors and the Popa subactors, and unify them. Let us start our discussion with some basic action theory. We first have:

Definition 3.30. *A coaction of a Woronowicz algebra A on a finite von Neumann algebra P is an injective morphism $\Phi:P\to P\otimes A''$ satisfying the following conditions:*

1. *Coassociativity: $(\Phi\otimes id)\Phi=(id\otimes\Delta)\Phi$.*
2. *Trace equivariance: $(tr\otimes id)\Phi=tr(.)1$.*
3. *Smoothness: $\overline{\mathcal P}^w=P$, where $\mathcal P=\Phi^{-1}(P\otimes_{alg}\mathcal A)$.*

The above conditions come from what happens in the commutative case, $A=C(G)$, where they correspond to the usual associativity, trace equivariance and smoothness of the corresponding action $G\curvearrowright P$. Along the same lines, we have as well:

Definition 3.31. *A coaction $\Phi:P\to P\otimes A''$ as above is called:*

1. *Ergodic, if the algebra $P^\Phi=\left\{p\in P\big|\Phi(p)=p\otimes 1\right\}$ reduces to $\mathbb C$.*
2. *Faithful, if the span of $\left\{(f\otimes id)\Phi(P)\big|f\in P_*\right\}$ is dense in A''.*
3. *Minimal, if it is faithful, and satisfies $(P^\Phi)'\cap P=\mathbb C$.*

Observe that the minimality of the action implies in particular that the fixed point algebra P^Φ is a factor. Thus, we are getting here to the case that we are interested in, actions producing factors, via their fixed point algebras. More on this later.

In order to prove our subfactor results, we need of some general theory regarding the minimal actions. Following Wassermann [90], let us start with the following definition:

Definition 3.32. *Let $\Phi:P\to P\otimes A''$ be a coaction. An eigenmatrix for a corepresentation $u\in B(H)\otimes A$ is an element $M\in B(H)\otimes P$ satisfying:*

$$(id\otimes\Phi)M=M_{12}u_{13}$$

A coaction is called semidual if each corepresentation has a unitary eigenmatrix.

As a basic example here, the canonical coaction $\Delta : A \to A \otimes A$ is semidual. We will prove in what follows, following the work of Wassermann in the usual compact group case, that the minimal coactions of Woronowicz algebras are semidual. We first have:

PROPOSITION 3.33. *If $\Phi : P \to P \otimes A''$ is a minimal coaction and $u \in Irr(A)$ is a corepresentation, then u has a unitary eigenmatrix precisely when $P^u \neq \{0\}$.*

PROOF. Given $u \in M_n(A)$, consider the following unitary corepresentation:

$$u^+ = (n \otimes 1) \oplus u = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \in M_2(M_n(\mathbb{C}) \otimes \mathcal{A}) = M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes \mathcal{A}$$

Then, if the following algebra is a factor, u must have a unitary eigenmatrix:

$$X_u = (M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes P)^{\pi_{u^+}}$$

So, let us prove that X_u is a factor. For this purpose, let $x \in Z(X_u)$. We have then $1 \otimes 1 \otimes P^\Phi \subset X_u$, and from the irreducibility of the inclusion $P^\pi \subset P$ we obtain that:

$$x \in M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes 1$$

On the other hand, we have the following formula:

$$X_u \cap M_2(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes 1 = End(u^+) \otimes 1$$

Since our corepresentation u was chosen to be irreducible, it follows that x must be of the following form, with $y \in M_n(\mathbb{C})$, and with $\lambda \in \mathbb{C}$:

$$x = \begin{pmatrix} y & 0 \\ 0 & \lambda I \end{pmatrix} \otimes 1$$

Now let us pick a nonzero element $p \in P^u$, and write:

$$\Phi(p) = \sum_{ij} p_{ij} \otimes u_{ij}$$

Then $\Phi(p_{ij}) = \sum_k p_{kj} \otimes u_{ki}$ for any i, j, and so each column of $(p_{ij})_{ij}$ is a u-eigenvector. Choose such a nonzero column l and let m^i be the matrix having the i-th row equal to l, and being zero elsewhere. Then m_i is a u-eigenmatrix for any i, and this implies that:

$$\begin{pmatrix} 0 & m^i \\ 0 & 0 \end{pmatrix} \in X_u$$

The commutation relation of this matrix with x is as follows:

$$\begin{pmatrix} y & 0 \\ 0 & \lambda I \end{pmatrix} \begin{pmatrix} 0 & m^i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & m^i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & \lambda I \end{pmatrix}$$

But this gives $(y - \lambda I)m^i = 0$. Now by definition of m^i, this shows that the i-th column of $y - \lambda I$ is zero. Thus $y - \lambda I = 0$, and so $x = \lambda 1$, as desired. □

We can now prove a main result about minimal coactions, as follows:

Theorem 3.34. *The minimal coactions are semidual.*

Proof. Let K be the set of finite dimensional unitary corepresentations of A which have unitary eigenmatrices. Then, according to the above, the following happen:

(1) K is stable under taking tensor products, because if M,N are eigenmatrices for u,w, then $M_{13}N_{23}$ is an eigenmatrix for $u\otimes w$. Also, K is stable under taking sums, because if M_i are eigenmatrices for u_i, then $diag(M_i)$ is an eigenmatrix for $\oplus u_i$.

(2) K is stable under substractions. Indeed, if M is an eigenmatrix for $U=\oplus_{i=1}^n u_i$, then the first $\dim(u_1)$ columns of M are formed by elements of P^{u_1}, the next $\dim(u_2)$ columns of M are formed by elements of P^{u_2}, and so on. Now if M is unitary, it is in particular invertible, so all P^{u_i} are different from $\{0\}$, and we may conclude that we can indeed substract corepresentations from U, by using Proposition 3.33.

(3) K is stable under complex conjugation. Indeed, if $u\in Irr(A)$ has a nonzero eigenmatrix M then $\overline{M}$ is an eigenmatrix for $\overline{u}$. By Proposition 3.33 we obtain from this that $P^{\overline{u}}\neq\{0\}$, and we may conclude by using again Proposition 3.33.

Thus, the set K of corepresentations which have unitary eigenmatrices is stable by all the standard operations that can be performed on the finite dimensional unitary corepresentations, and with this in hand, by using Peter-Weyl, we obtain the result. □

Let us construct now the fixed point subfactors. We first have:

Proposition 3.35. *Consider a Woronowicz algebra $A=(A,\Delta,S)$, and denote by A_σ the Woronowicz algebra $(A,\sigma\Delta,S)$, where σ is the flip. Given coactions*

$$\beta:B\to B\otimes A$$

$$\pi:P\to P\otimes A_\sigma$$

with B being finite dimensional, the following linear map, while not being multiplicative in general, is coassociative with respect to the comultiplication $\sigma\Delta$ of A_σ,

$$\beta\odot\pi:B\otimes P\to B\otimes P\otimes A_\sigma$$

$$b\otimes p\to\pi(p)_{23}((id\otimes S)\beta(b))_{13}$$

and its fixed point space, which is by definition the following linear space,

$$(B\otimes P)^{\beta\odot\pi}=\left\{x\in B\otimes P\Big|(\beta\odot\pi)x=x\otimes 1\right\}$$

is then a von Neumann subalgebra of $B\otimes P$.

Proof. This is something standard, which follows from a straightforward algebraic verification, explained in [6]. As mentioned in the statement, to be noted is that the tensor product coaction $\beta\odot\pi$ is not multiplicative in general. See [6]. □

In order to construct now fixed point subfactors, our first task is to investigate the factoriality of such algebras, and we have here the following result:

THEOREM 3.36. *If $\beta : B \to B \otimes A$ is a coaction and $\pi : P \to P \otimes A_\sigma$ is a minimal coaction, then the following conditions are equivalent:*

1. *The von Neumann algebra $(B \otimes P)^{\beta \odot \pi}$ is a factor.*
2. *The coaction β is centrally ergodic, $Z(B) \cap B^\beta = \mathbb{C}$.*

PROOF. The first observation is that, according to Theorem 3.34, the following diagram is a non-degenerate commuting square:

$$\begin{matrix} P & \subset & B \otimes P \\ \cup & & \cup \\ P^\pi & \subset & (B \otimes P)^{\beta \odot \pi} \end{matrix}$$

Thus, it is enough to check the following equality, between subalgebras of $B \otimes P$:

$$Z((B \otimes P)^{\beta \odot \pi}) = (Z(B) \cap B^\beta) \otimes 1$$

So, let x be in the algebra on the left. Then x commutes with $1 \otimes P^\pi$, so it has to be of the form $b \otimes 1$. Thus x commutes with $1 \otimes P$. But x commutes with $(B \otimes P)^{\beta \odot \pi}$, and from the non-degeneracy of the above square, x commutes with $B \otimes P$, and in particular with $B \otimes 1$. Thus we have $b \in Z(B) \cap B^\beta$. As for the other inclusion, this is obvious. □

We will need in what follows the following technical result:

PROPOSITION 3.37. *Consider two commuting squares, as follows:*

$$\begin{matrix} F & \subset & E & \subset & D \\ \cup & & \cup & & \cup \\ A & \subset & B & \subset & C \end{matrix}$$

1. *If the square on the left and the big square are non-degenerate, then so is the square on the right.*
2. *If both squares are non-degenerate, $F \subset E \subset D$ is a basic construction, and the Jones projection $e \in D$ for this basic construction belongs to C, then the square on the right is the basic construction for the square on the left.*

PROOF. The first assertion is clear from the following computation:

$$D = \overline{sp}^w\, CF = \overline{sp}^w\, CBF = \overline{sp}^w\, CE$$

Let $\Psi : D \to C$ be the expectation. By non-degeneracy, we have that:

$$E = \overline{sp}^w\, FB = \overline{sp}^w\, BF$$

We also have $D=\overline{sp}^w\,EeE$ by the basic construction, so we get that:

$$\begin{aligned}C&=\Psi(D)\\&=\Psi(\overline{sp}^w\,EeE)\\&=\Psi(\overline{sp}^w\,BFeFB)\\&=\Psi(\overline{sp}^w\,BeFB)\\&=\overline{sp}^w\,Be\Psi(F)B\\&=\overline{sp}^w\,BeAB\\&=\overline{sp}^w\,BeB\end{aligned}$$

Thus the algebra C is generated by B and e, and this gives the result. □

Next in line, we have the following key technical result:

PROPOSITION 3.38. *If $\beta:B\to B\otimes A$ is a coaction then*

$$\begin{matrix}A&\subset&B\otimes A\\\cup&&\uparrow\beta\\\mathbb{C}&\subset&B\end{matrix}$$

is a non-degenerate commuting square.

PROOF. From the β-equivariance of the trace we get that the inclusion on the left commutes with the traces. Then, from the formula $E_\beta=(id\otimes\int_A)\beta$ we get that the above diagram is a commuting square. Choose now an orthonormal basis $\{b_i\}$ of B, write $\beta:b_i\to\sum_j b_j\otimes u_{ji}$, and consider the corresponding unitary corepresentation:

$$u_\beta=\sum e_{ij}\otimes u_{ij}$$

Then for any k and any $a\in A$ we have the following computation:

$$\sum_i\beta(b_i)(1\otimes u_{ki}^*a)=\sum_{ij}b_j\otimes u_{ji}u_{ki}^*a=b_k\otimes a$$

Thus our commuting square is non-degenerate, as claimed. □

Getting now to the generalized Wassermann subfactors, we first have:

PROPOSITION 3.39. *Given a Markov inclusion of finite dimensional algebras $B_0\subset B_1$, construct its Jones tower, and denote it as follows:*

$$B_0\subset B_1\subset_{e_1}B_2=<B_1,e_1>\subset_{e_2}B_3=<B_2,e_2>\subset_{e_3}\ldots$$

If $\beta_1:B_1\to B_1\otimes A$ is a coaction/anticoaction leaving B_0 invariant then there exists a unique sequence $\{\beta_i\}_{i\geq0}$ of coactions/anticoactions

$$\beta_i:B_i\to B_i\otimes A$$

such that each β_i extends β_{i-1} and leaves invariant the Jones projection e_{i-1}.

Proof. By taking opposite inclusions we see that the assertion for anticoactions is equivalent to the one for coactions, that we will prove now. The uniqueness is clear from $B_i =< B_{i-1}, e_{i-1} >$. For the existence, we can apply Proposition 3.38 to:

$$\begin{matrix} A & \subset & B_0 \otimes A & \subset & B_1 \otimes A \\ \cup & & \uparrow \beta_0 & & \uparrow \beta_1 \\ \mathbb{C} & \subset & B_0 & \subset & B_1 \end{matrix}$$

Indeed, we get in this way that the square on the right is a non-degenerate. Now by performing basic constructions to it, we get a sequence as follows:

$$\begin{matrix} B_0 \otimes A & \subset & B_1 \otimes A & \subset & B_2 \otimes A & \subset & B_3 \otimes A & \subset & \ldots \\ \uparrow \beta_0 & & \uparrow \beta_1 & & \uparrow \beta_2 & & \uparrow \beta_3 & & \\ B_0 & \subset & B_1 & \subset & B_2 & \subset & B_3 & \subset & \ldots \end{matrix}$$

It is easy to see from definitions that the β_i are coactions, that they extend each other, and that they leave invariant the Jones projections. But this gives the result. □

With the above technical results in hand, we can now formulate our main theorem regarding the fixed point subfactors, of the most possible general type, as follows:

Theorem 3.40. *Let G be a compact quantum group, and $G \to Aut(P)$ be a minimal action on a II_1 factor. Consider a Markov inclusion of finite dimensional algebras*

$$B_0 \subset B_1$$

and let $G \to Aut(B_1)$ be an action which leaves invariant B_0 and which is such that its restrictions to the centers of B_0 and B_1 are ergodic. We have then a subfactor

$$(B_0 \otimes P)^G \subset (B_1 \otimes P)^G$$

of index $N = [B_1 : B_0]$, called generalized Wassermann subfactor, whose Jones tower is

$$(B_1 \otimes P)^G \subset (B_2 \otimes P)^G \subset (B_3 \otimes P)^G \subset \ldots$$

where $\{B_i\}_{i \geq 1}$ are the algebras in the Jones tower for $B_0 \subset B_1$, with the canonical actions of G coming from the action $G \to Aut(B_1)$, and whose planar algebra is given by:

$$P_k = (B_0' \cap B_k)^G$$

These subfactors generalize the Jones, Ocneanu, Wassermann and Popa subfactors.

Proof. We have several things to be proved, the idea being as follows:

(1) The first part of the statement, regarding the factoriality, the index and the Jones tower assertions, is something that follows exactly as in the classical group case.

(2) In order to prove now the planar algebra assertion, consider the following diagram, with $i < j$ being arbitrary integers:

$$\begin{array}{ccccc} P & \subset & B_i \otimes P & \subset & B_j \otimes P \\ \cup & & \cup & & \cup \\ P^\pi & \subset & (B_i \otimes P)^{\beta_i \otimes \pi} & \subset & (B_j \otimes P)^{\beta_j \otimes \pi} \end{array}$$

We know from Proposition 3.39 that the big square and the square on the left are both non-degenerate commuting squares. Thus Proposition 3.38 applies, and shows that the square on the right is a non-degenerate commuting square.

(3) Consider now the following sequence of non-degenerate commuting squares:

$$\begin{array}{ccccccc} B_0 \otimes P & \subset & B_1 \otimes P & \subset & B_2 \otimes P & \subset & \ldots \\ \cup & & \cup & & \cup & & \\ (B_0 \otimes P)^{\beta_0 \otimes \pi} & \subset & (B_1 \otimes P)^{\beta_1 \otimes \pi} & \subset & (B_2 \otimes P)^{\beta_2 \otimes \pi} & \subset & \ldots \end{array}$$

Since the Jones projections live in the lower line, Proposition 6.39 applies and shows that this is a sequence of basic constructions for non-degenerate commuting squares. In particular the lower line is a sequence of basic constructions, as desired. □

We discuss now some converses to the above results, which are rather specialized results, of Tannakian nature. Let us start with the following result:

THEOREM 3.41. *Given a quantum permutation group $G \subset S_N^+$, consider the associated coaction map on $C(X)$, where $X = \{1, \ldots, N\}$,*

$$\Phi : C(X) \to C(X) \otimes C(G) \quad , \quad e_j \to \sum_j e_j \otimes u_{ji}$$

and then consider the tensor powers of this coaction, which are the following linear maps:

$$\Phi^k : C(X^k) \to C(X^k) \otimes C(G) \quad , \quad e_{i_1 \ldots i_k} \to \sum_{j_1 \ldots j_k} e_{j_1 \ldots j_k} \otimes u_{j_1 i_1} \ldots u_{j_k i_k}$$

The fixed point spaces of these latter coactions are then given by the formula

$$P_k = Fix(u^{\otimes k})$$

and form a planar subalgebra of the spin planar algebra $\mathcal{S}_N$.

PROOF. This is something which certainly follows from the above results regarding the Wassermann type subfactors, but can be deduced as well directly. □

As a second result now, completing our study, we have:

THEOREM 3.42. *Given a subalgebra $Q \subset \mathcal{S}_N$, there is a unique quantum group*

$$G \subset S_N^+$$

whose associated planar algebra is Q.

Proof. The idea is that this will follow by applying Tannakian duality to the annular category over Q. Let n, m be positive integers. To any element $T_{n+m} \in Q_{n+m}$ we can associate a linear map $L_{nm}(T_{n+m}) : P_n(X) \to P_m(X)$ in the following way:

$$L_{nm}\begin{pmatrix} |\,|\,| \\ T_{n+m} \\ |\,|\,| \end{pmatrix} : \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \to \begin{pmatrix} |\ \ |\ \ \cap \\ \ \ T_{n+m}| \\ ||\ \ |\ | \\ a_n|\ \ |\ | \\ \cup\ \ |\ | \end{pmatrix}$$

That is, we consider the planar $(n, n+m, m)$-tangle having an small input n-box, a big input $n+m$-box and an output m-box, with strings as on the picture of the right. This defines a certain multilinear map, as follows:

$$P_n(X) \otimes P_{n+m}(X) \to P_m(X)$$

Now let us put the element T_{n+m} in the big input box. We obtain in this way a certain linear map $P_n(X) \to P_m(X)$, that we call L_{nm}. Now let us set:

$$Q_{nm} = \left\{ L_{nm}(T_{n+m}) : P_n(X) \to P_m(X) \Big| T_{n+m} \in Q_{n+m} \right\}$$

These spaces form a Tannakian category, and so by [100] we obtain a Woronowicz algebra (A, u), such that the following equalities hold, for any m, n:

$$Hom(u^{\otimes m}, u^{\otimes n}) = Q_{mn}$$

We prove that u is a magic unitary. We have $Hom(1, u^{\otimes 2}) = Q_{02} = Q_2$, so the unit of Q_2 must be a fixed vector of $u^{\otimes 2}$. But $u^{\otimes 2}$ acts on the unit of Q_2 as follows:

$$u^{\otimes 2}(1) = \sum_{kl} \begin{pmatrix} k & k \\ l & l \end{pmatrix} \otimes (uu^t)_{kl}$$

From $u^{\otimes 2}(1) = 1 \otimes 1$ ve get that uu^t is the identity matrix, and together with the unitarity of u, this gives $u^t = u^* = u^{-1}$. Consider now the Jones projection $E_1 \in Q_3$. The linear map $M = L_{21}(E_1)$ is the multiplication $\delta_i \otimes \delta_j \to \delta_{ij}\delta_i$, and we have:

$$(M \otimes id)u^{\otimes 2}\left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) = \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes u_{ki}u_{kj}$$

$$u(M \otimes id)\left(\begin{pmatrix} i & i \\ j & j \end{pmatrix} \otimes 1 \right) = \sum_k \begin{pmatrix} k \\ k \end{pmatrix} \delta_k \otimes \delta_{ij}u_{ki}$$

Thus $u_{ki}u_{kj} = \delta_{ij}u_{ki}$ for any i, j, k, and we deduce from this that u is a magic unitary. Now if P is the planar algebra associated to u, we have $Hom(1, v^{\otimes n}) = P_n = Q_n$, as desired. As for the uniqueness, this is clear from the Peter-Weyl theory from [99]. □

The above results, following old papers from the early 00s, explained in [6], regarding the subgroups $G\subset S_N^+$, have several generalizations, to the subgroups $G\subset O_N^+$ and $G\subset U_N^+$, as well as subfactor versions, going beyond the purely combinatorial level. For the modern story, we refer here to Tarrago-Wahl [78] and related papers.

3e. Exercises

Exercises:

Exercise 3.43.

Exercise 3.44.

Exercise 3.45.

Exercise 3.46.

Exercise 3.47.

Exercise 3.48.

Exercise 3.49.

Exercise 3.50.

Bonus exercise.

CHAPTER 4

Planar algebras

4a. Planar algebras

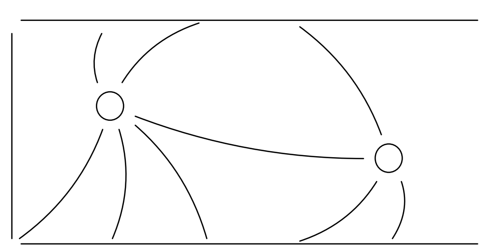
The Temperley-Lieb category that we met in chapter 1 is more than a category, it is a planar algebra. In order to explain this fact, which will be of key importance in what follows, following Jones [42], let us start with the following general definition:

DEFINITION 4.1. *The planar algebras are defined as follows:*

(1) *We consider rectangles in the plane, with the sides parallel to the coordinate axes, and taken up to planar isotopy, and we call such rectangles boxes.*
(2) *A labeled box is a box with $2n$ marked points on its boundary, n on its upper side, and n on its lower side, for some integer $n\in\mathbb N$.*
(3) *A tangle is labeled box, containing a number of labeled boxes, with all marked points, on the big and small boxes, being connected by noncrossing strings.*
(4) *A planar algebra is a sequence of finite dimensional vector spaces $P=(P_n)$, together with linear maps $P_{n_1}\otimes\ldots\otimes P_{n_k}\to P_n$, one for each tangle, such that the gluing of tangles corresponds to the composition of linear maps.*

In this definition we are using rectangles, but everything being up to isotopy, we could have used instead circles with marked points, as in [42]. Our choice for using rectangles comes from the main examples that we have in mind, to be discussed below, where the planar algebra structure is best viewed by using rectangles, as above.

This being said, when convenient, we agree to use circles with marked points for the outer box, or for the inner boxes, or for both, with the convention that the marked point is the lower left corner of the rectangle. Here is a planar tangle, drawn in this way, with the marked points on both circles being by definition those at South-West:



And, exercise for you to see what this tangle becomes, in rectangular notation.

Getting back now to what Definition 4.1 says, in relation with the tangle pictured above, that tangle has two inner boxes, having respectively $2 \times 2 = 4$ and $2 \times 3 = 6$ marked points on their boundaries, and the outer box has $2 \times 4 = 8$ marked points on its boundary. Thus, that tangle π must produce a linear map as follows:

$$T_\pi : P_2 \otimes P_3 \to P_4$$

You get the point, I hope, Definition 4.1 is something very useful in the context of algebra, in order to index various possible operations on a sequence of finite dimensional vector spaces $P = (P_n)$, by diagrams as above. Of course, all this remains very vague for the moment, but we will see many examples and illustrations, in what follows.

Getting now to the essence of Definition 4.1, that lies in the axiom (4) there, compatibility of the gluing of the tangles with the composition of the multilinear maps. We will comment on this later, once we will have some examples of planar algebras. In the meantime, let us mention that it is possible to be more abstract here, by talking about the planar operad, and planar algebras as modules over this operad. But again, we will comment on this later, once we will have some examples of planar algebras.

Finally, let us mention now that Definition 4.1 is something quite simplified. As explained in [**42**], in order for subfactors to produce planar algebras and vice versa, there are quite a number of supplementary axioms that must be added. More on this later.

But probably too much talking, let us see some illustrations for this. As a first, very basic example of a planar algebra, we have the Temperley-Lieb algebra:

Theorem 4.2. *The Temperley-Lieb algebra TL_N, viewed as graded algebra*

$$TL_N = (TL_N(n))_{n\in\mathbb{N}}$$

is a planar algebra, with the corresponding linear maps associated to the planar tangles

$$TL_N(n_1) \otimes \ldots \otimes TL_N(n_k) \to TL_N(n)$$

appearing by putting the various $TL_N(n_i)$ diagrams into the small boxes of the given tangle, which produces a $TL_N(n)$ diagram.

Proof. This is something trivial, which follows from definitions:

(1) Assume indeed that we are given a planar tangle π, as in Definition 4.1, consisting of a box having $2n$ marked points on its boundary, and containing k small boxes, having respectively $2n_1, \ldots, 2n_k$ marked points on their boundaries, and then a total of $n + \Sigma n_i$ noncrossing strings, connecting the various $2n + \Sigma 2n_i$ marked points.

(2) We want to associate to this tangle π a linear map as follows:

$$T_\pi : TL_N(n_1) \otimes \ldots \otimes TL_N(n_k) \to TL_N(n)$$

For this purpose, by linearity, it is enough to construct elements as follows, for any choice of Temperley-Lieb diagrams $\sigma_i \in TL_N(n_i)$, with $i = 1, \ldots, k$:

$$T_\pi(\sigma_1 \otimes \ldots \otimes \sigma_k) \in TL_N(n)$$

(3) But constructing such an element is obvious, just by putting the various diagrams $\sigma_i \in TL_N(n_i)$ into the small boxes the given tangle π. Indeed, this procedure produces a certain diagram in $TL_N(n)$, that we can call $T_\pi(\sigma_1 \otimes \ldots \otimes \sigma_k)$, as above.

(4) Finally, we have to check that everything is well-defined up to planar isotopy, and that the gluing of tangles corresponds to the composition of linear maps. But both these checks are trivial, coming from the definition of TL_N, and we are done. □

As a conclusion to all this, $P = TL_N$ is indeed a planar algebra, but of somewhat "trivial" type, with the triviality coming from the fact that, in this case, the elements of P are planar diagrams themselves, and so the planar structure appears trivially.

The Temperley-Lieb planar algebra TL_N is however an important planar algebra, because it is the "smallest" one, appearing inside the planar algebra of any subfactor. But more on this later, when talking about planar algebras and subfactors.

Moving ahead now, here is our second basic example of a planar algebra, which is also "trivial" in the above sense, with the elements of the planar algebra being planar diagrams themselves, but which appears in a bit more complicated way:

THEOREM 4.3. *The Fuss-Catalan algebra $FC_{N,M}$, obtained by coloring the Temperley-Lieb diagrams with black and white colors, clockwise, as follows,*

$$\circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots\ldots\circ\bullet\bullet\circ$$

and keeping those diagrams whose strings connect either $\circ-\circ$ or $\bullet-\bullet$, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

$$FC_{N,M}(n_1) \otimes \ldots \otimes FC_{N,M}(n_k) \to FC_{N,M}(n)$$

appearing by putting the various $FC_{N,M}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N,M}(n)$ diagram.

PROOF. The proof here is nearly identical to the proof of Theorem 4.2, with the only change appearing at the level of the colors. To be more precise:

(1) Forgetting about upper and lower sequences of points, which must be joined by strings, a Temperley-Lieb diagram can be thought of as being a collection of strings, say black strings, which compose in the obvious way, with the rule that the value of the circle, which is now a black circle, is N. And it is this obvious composition rule that gives the planar algebra structure, as explained in the proof of Theorem 4.2.

(2) Similarly, forgetting about points, a Fuss-Catalan diagram can be thought of as being a collection of strings, which come now in two colors, black and white. These Fuss-Catalan diagrams compose then in the obvious way, with the rule that the value of the black circle is N, and the value of the white circle is M. And it is this obvious composition rule that gives the planar algebra structure, as before for TL_N. □

Even more generally now, we can talk about the multicolored Fuss-Catalan algebra, generalizing both the Temperley-Lieb and Fuss-Catalan algebras, as follows:

Theorem 4.4. *The multicolored Fuss-Catalan algebra $FC_{N_1,\ldots,N_s}$, obtained by coloring the Temperley-Lieb diagrams with s colors, clockwise, as follows,*

$$1\ldots ss\ldots 11\ldots ss\ldots 1\ldots\ldots\ldots 1\ldots ss\ldots 1$$

and keeping those diagrams whose strings connect $i-i$, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

$$FC_{N_1,\ldots,N_s}(n_1)\otimes\ldots\otimes FC_{N_1,\ldots,N_s}(n_k)\to FC_{N_1,\ldots,N_s}(n)$$

appearing by putting the various $FC_{N_1,\ldots,N_s}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N_1,\ldots,N_s}(n)$ diagram.

Proof. This is a straightforward remake of Theorems 4.2 and 4.3, which correspond respectively to the cases $s=1,2$, with the only thing that must be added being the fact that the values of the circles of colors $1,\ldots,s$ are respectively the numbers $N_1,\ldots,N_s$. And with this we are led, as before, to the conclusions in the statement. □

Getting back now to generalities, and to Definition 4.1 as stated, that of a general planar algebra, we have so far a few illustrations for it, which, while all important, are all "trivial", with the planar structure simply coming from the fact that, in all the above cases, the elements of the planar algebra are planar diagrams themselves.

In general, the planar algebras can be more complicated than this, and we will see some further examples in a moment. However, the idea is very simple, namely:

Principle 4.5. *The elements of a planar algebra are not necessarily diagrams, but they behave like diagrams.*

And important principle this is. If there is something to be known, in order to understand planar algebras, and the whole quantum algebra theory based on them, it is this principle. But, do we really understand this principle? Not yet, because as already mentioned, our examples so far of planar algebras, namely Temperley-Lieb and Fuss-Catalan, are both "trivial", with the elements of the planar algebra being themselves diagrams.

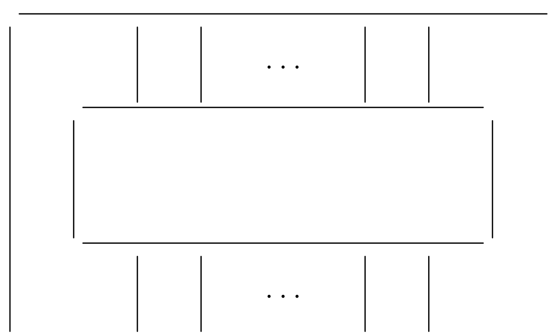
Nevermind. We will get to understand this principle, via more examples, and via some theory too. Please be sure that once this book read, Principle 4.5 will be understood.

4b. Basic tangles

What is next? Instead of looking right away for further examples, which can be substantially more complicated than Temperley-Lieb and Fuss-Catalan, let us enjoy what we have. To be more precise, with these two basic examples in hand, Temperley-Lieb and Fuss-Catalan, let us try to say more about the arbitrary planar algebras, as in Definition 4.1, with a bit of inspiration from what happens for these examples.

To start with, we have a number of remarkable planar tangles, whose algebraic action must be well understood, before anything. The first basic tangle is as follows:

Example 4.6. *The identity tangle is the following tangle, with $2n$ outer legs,*

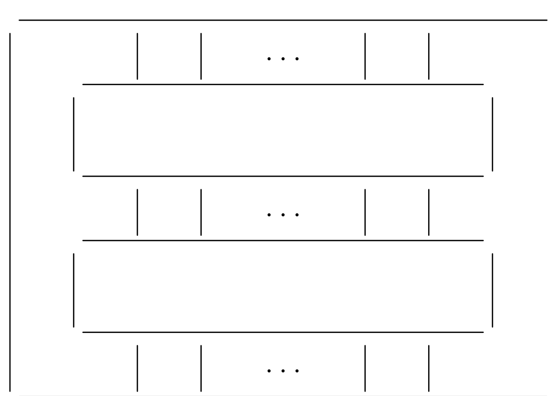


and this tangle must act via the identity, $T_\pi(x) = x$, for any $x \in P_n$.

To be more precise here, consider the tangle in the statement, π. Since applying this tangle obviously does nothing, this tangle must act via the identity map, as stated.

As a more interesting example now, bringing an associative algebra structure to each of the vector spaces P_n that our planar algebra is made of, we have:

Example 4.7. *The multiplication tangle is as follows, with $2n$ outer legs,*

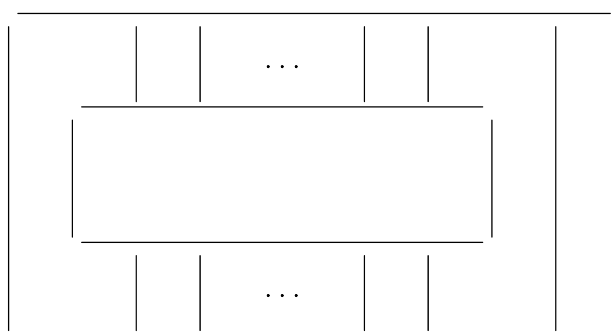


and this must implement a multiplication map, $T_\pi(x \otimes y) = xy$, for any $x, y \in P_n$.

Again, this is something quite self-explanatory, the idea being that the tangle in the statement, or rather its action on P_n, must be an associative multiplication.

Along the same lines, bringing more basic structure to our sequence of vector spaces $P = (P_n)$, which are now a sequence of associative algebras $P = (P_n)$, we have:

Example 4.8. *The inclusion tangle is as follows, with $2n + 2$ outer legs,*



and this tangle must act via an inclusion, $T_\pi(x) = x$, for any $x \in P_n$.

Again, this is something quite self-explanatory, the idea being that the tangle in the statement, or rather its action $P_n \to P_{n+1}$, must be an inclusion of algebras.

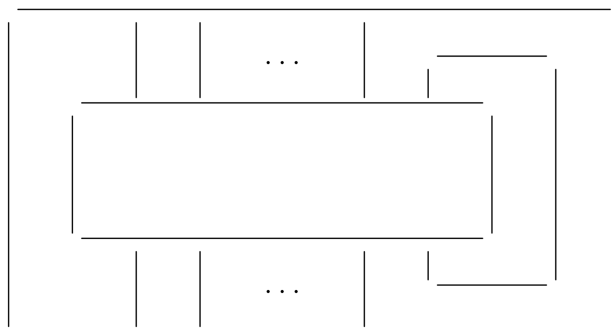
As a conclusion to all this, we already have some interesting structure on our planar algebras, getting well beyond what is totally obvious from Definition 4.1, as follows:

Conclusion 4.9. *Any planar algebra $P = (P_n)$ is naturally a graded associative algebra over the complex numbers, with multiplication and inclusion maps coming from the action of the multiplication and inclusion tangles, pictured above.*

Which looks quite interesting, especially in view of the fact that, due to this coming from the study of some trivial tangles, this can only be the tip of the iceberg. So, let us explore some more what the basic tangles are, and what can be done with them.

Coming first in our second batch of examples, we have:

Example 4.10. *The expectation tangle is as follows, with $2n$ outer legs,*

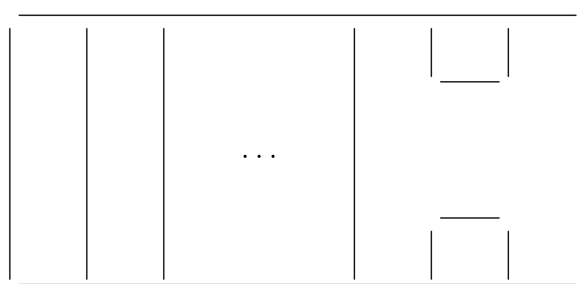


and this tangle must act via an expectation, $T_\pi : P_{n+1} \to P_n$.

To be more precise, this is something a bit more advanced, the idea here being that the linear map $T_\pi : P_{n+1} \to P_n$ associated to the above expectation tangle must be a section, and bimodule map, with respect to the canonical inclusion of algebras $P_n \subset P_{n+1}$, that we constructed before. We will be back to this, with more details, later.

Along the same lines, again at the level of more specialized tangles, we have:

Example 4.11. *The Jones projection tangle is as follows, with $2n$ outer legs,*



and this tangle corresponds to a rescaled projection $T_\pi \in P_n$.

Again, this is something quite self-explanatory, the idea being that, with no inner box present, the Jones projection tangle must simply correspond to a certain element $T_\pi \in P_n$. But this element must be an idempotent, up to a N factor, as said above.

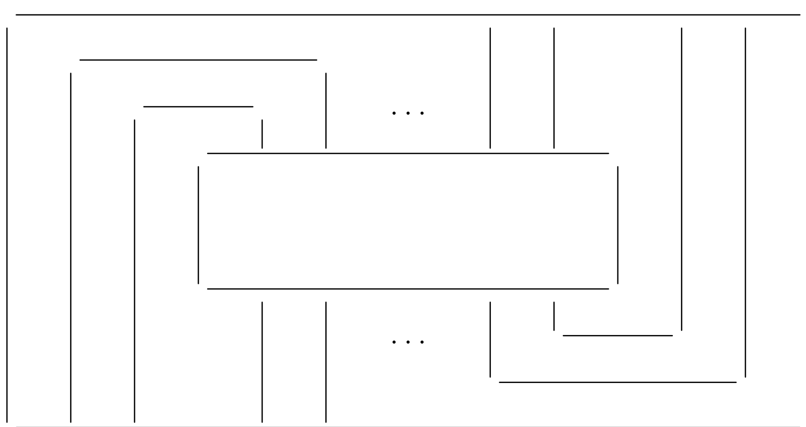
Very good all this, so let us upgrade Conclusion 4.9, as follows:

Conclusion 4.12 (upgrade). *Any planar algebra $P = (P_n)$ is naturally a graded associative algebra, via the action of the multiplication and inclusion tangles, and in addition we have, a bit as for the Temperley-Lieb algebra, expectations and Jones projections.*

As already mentioned in the above, in what concerns the last part, regarding the expectations and the Jones projections, this is something a bit more specialized, and definitely in need of more discussion. We will come back to this, a bit later.

Moving ahead, let us discuss now a third batch of basic planar tangles, that we will heavily use as well in what follows. First we have the rotation, which is as follows:

EXAMPLE 4.13. *The rotation tangle is as follows, with $2n$ outer legs,*

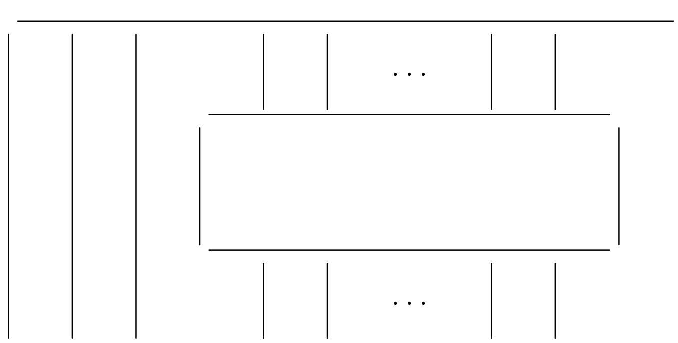


and this tangle must act via a rotation $T_\pi : P_n \to P_n$.

Again, this is something quite self-explanatory, the idea being that the linear map $T_\pi : P_n \to P_n$ associated to the above rotation tangle must produce the identity, when raised to the power n, a bit like the usual rotation in the plane, of angle $2\pi/n$, does.

As a last basic tangle, we have the shift, which is constructed as follows:

EXAMPLE 4.14. *The shift tangle is as follows, with $2n+2$ outer legs,*



and this tangle must act via a shift, $T_\pi : P_n \to P_{n+2}$.

Again, this is something quite self-explanatory, and with the remark of course that the shift is not to be confused with the double inclusion map $P_n \to P_{n+2}$. We will get back to this, shift and its properties, with more details, later in this chapter.

As a grand conclusion now to what we did so far, we have:

CONCLUSION 4.15 (final upgrade). *Any planar algebra $P = (P_n)$ is naturally a graded associative algebra, and in addition we have, a bit as for the Temperley-Lieb algebra, or for the Fuss-Catalan one, expectations, Jones projections, rotations and shifts.*

Which is good knowledge, and we will be back to this, with further details, later on. In any case, we can see here some good evidence for what we said in Principle 4.5, namely that the elements of a planar algebra are not necessarily diagrams, but behave like diagrams. And, more on this on several occassions, in what follows.

Getting back now to theory, we have the following remarkable result, which is something that we will heavily use, in what follows, for all sorts of purposes:

THEOREM 4.16. *The following tangles generate the set of all tangles, via gluing:*

(1) *Multiplications.*
(2) *Inclusions.*
(3) *Expectations.*
(4) *Jones projections.*
(5) *Rotations, or shifts.*

PROOF. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list:

(1) To start with, in what regards the list itself, this is the one coming from the above examples, with the identities, which bring nothing to our generation problem, removed.

(2) As a subtlety now, at the end we have a choice, between the rotation and the shift. This is something quite important for the applications, which come in two flavors.

(3) As for the proof, as indicated above, both the results, the one with rotations, and the one with shifts, follow by chopping the tangles, in the obvious way. See [**42**]. □

There are many more things that can be said, along these lines, that is, generalities and basic algebra, in relation with Definition 4.1. We will be back to this later.

4c. Tensor and spin

Let us discuss now some further examples of planar algebras, which are of less trivial nature than TL_N and $FC_{N,M}$, and are of particular interest in relation with algebra and topology. These are the tensor and spin planar algebras $\mathcal{T}_N, \mathcal{S}_N$. Let us start with:

Definition 4.17. *The tensor planar algebra $\mathcal{T}_N$ is the sequence of vector spaces*

$$P_k = M_N(\mathbb{C})^{\otimes k}$$

with the multilinear maps $T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by the formula

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we put the indices of the basic tensors on the marked points of the small boxes, in the obvious way, and the coefficients of the output tensor are then given by Kronecker symbols, $\delta_\pi \in \{0, 1\}$, which are themselves defined as follows:

– $\delta_\pi = 1$ when all strings join pairs of equal indices.

– $\delta_\pi = 0$ otherwise.

The fact that we have indeed a planar algebra is something elementary, and for full details here, we refer to Jones' paper [42]. As illustrations for all this, we have:

Example 4.18. *Identity.*

We recall that the identity 1_k is the (k, k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x, y) = 1$ being the pairs of the form (x, x), this tangle acts as follows:

$$1_k \begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix} = \begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix}$$

But this action is the identity, as it should.

Example 4.19. *Multiplication.*

The multiplication M_k is the (k, k, k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. This tangle acts in the following way:

$$M_k \left(\begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix} \otimes \begin{pmatrix} l_1 & \ldots & l_k \\ m_1 & \ldots & m_k \end{pmatrix} \right) = \delta_{j_1 m_1} \ldots \delta_{j_k m_k} \begin{pmatrix} l_1 & \ldots & l_k \\ i_1 & \ldots & i_k \end{pmatrix}$$

Again, this action is the multiplication, as it should.

Example 4.20. *Inclusion.*

The inclusion I_k is the $(k, k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x, y) = 1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k \begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix} = \sum_l \begin{pmatrix} j_1 & \ldots & j_k & l \\ i_1 & \ldots & i_k & l \end{pmatrix}$$

Once again, what we have here is what we can expect from an inclusion.

Example 4.21. *Expectation.*

The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

This map satisfies then the usual requirements for an expectation.

Example 4.22. *Jones projection.*

The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1)=\sum_{ijl}\begin{pmatrix}i_1&\ldots&i_k&j&j\\ i_1&\ldots&i_k&l&l\end{pmatrix}$$

The elements $e_k=N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

Example 4.23. *Rotation.*

The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k=\begin{matrix}\Cap\ |\ |\ |\ \|\\ \|\qquad\quad\ \|\\ \|\ |\ |\ |\ \Cup\end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2\ldots i_k})=e_{i_2\ldots i_ki_1}$$

Thus, what we have indeed is a rotation map.

Example 4.24. *Shift.*

As for the shift S_k, this is the $(k,k+2)$-tangle which looks like 1_k, but has two more vertical strings, at left of the input box. This tangle acts as follows:

$$S_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_{lm}\begin{pmatrix}l&m&j_1&\ldots&j_k\\ l&m&i_1&\ldots&i_k\end{pmatrix}$$

Observe that S_k is an inclusion of algebras, which is different from $I_{k+1}I_k$.

Finally, in order for our discussion to be complete, we must talk as well about the $*$-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix} j_1 & \dots & j_k \\ i_1 & \dots & i_k \end{pmatrix}^* = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$$

As before, we refer to Jones' paper [42] for more on all this.

Let us discuss now a second planar algebra of the same type, which is important as well for various reasons, namely the spin planar algebra $\mathcal{S}_N$. This planar algebra appears somewhat as a "square root" of the tensor planar algebra $\mathcal{T}_N$, and its construction is quite similar, but by using this time the algebra $\mathbb{C}^N$ instead of the algebra $M_N(\mathbb{C})$.

There is one subtlety, however, coming from the fact that the general planar algebra formalism, from Definition 4.1, requires the tensors to have even length. Note that this was automatic for the tensor planar $\mathcal{T}_N$, where the tensors of $M_N(\mathbb{C})$ have length 2. In the case of the spin planar algebra $\mathcal{S}_N$, we want the vector spaces to be:

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

Thus, we must double the indices of the tensors, in the following way:

Definition 4.25. *We write the standard basis of $(\mathbb{C}^N)^{\otimes k}$ in $2 \times k$ matrix form,*

$$e_{i_1 \dots i_k} = \begin{pmatrix} i_1 & i_1 & i_2 & i_2 & i_3 & \dots & \dots \\ i_k & i_k & i_{k-1} & \dots & \dots & \dots & \dots \end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra $\mathcal{S}_N$, also from [42], as follows:

Definition 4.26. *The spin planar algebra $\mathcal{S}_N$ is the sequence of vector spaces*

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

written as above, with the multiplinear maps $T_\pi : P_{k_1} \otimes \dots \otimes P_{k_r} \to P_k$ being given by

$$T_\pi(e_{i_1} \otimes \dots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \dots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we are using exactly the same construction as for the tensor planar algebra $\mathcal{T}_N$, but with $M_N(\mathbb{C})$ replaced by $\mathbb{C}^N$, with the indices doubled, as in Definition 4.25. As before with the tensor planar algebra $\mathcal{T}_N$, the fact that the spin planar algebra $\mathcal{S}_N$ is indeed a planar algebra is something rather trivial, coming from definitions.

Observe however that, unlike our previous planar algebras TL_N and $FC_{N,M}$, which were "trivial" planar algebras, their elements being planar diagrams themselves, the planar algebras $\mathcal{T}_N$ and $\mathcal{S}_N$ are not trivial, their elements being not exactly planar diagrams.

Let us also mention that the tensor and spin planar algebras $\mathcal{T}_N$ and $\mathcal{S}_N$ are important for a number of reasons, in the context of group theory, algebra and topology, to be discussed later, at the end of the present chapter, and later on too.

Getting back now to the planar algebra structure of $\mathcal{T}_N$ and $\mathcal{S}_N$, which is something quite fundamental, worth being well understood, let us have here some more discussion. Generally speaking, the planar calculus for tensors is quite simple, and does not really require diagrams. Indeed, it suffices to imagine that the way various indices appear, travel around and dissapear is by following some obvious strings connecting them.

Here are some illustrations for this general principle, for the spin planar algebra $\mathcal{S}_N$, in relation with the various basic planar tangles, that we know well:

Example 4.27. *Identity.*

The identity 1_k is the (k,k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x,y)=1$ being the pairs of the form (x,x), this tangle 1_k acts as follows:

$$1_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

But this action is the identity, as it should.

Example 4.28. *Multiplication.*

The multiplication M_k is the (k,k,k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. This tangle acts in the following way:

$$M_k\left(\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}\otimes\begin{pmatrix}l_1&\ldots&l_k\\ m_1&\ldots&m_k\end{pmatrix}\right)=\delta_{j_1m_1}\ldots\delta_{j_km_k}\begin{pmatrix}l_1&\ldots&l_k\\ i_1&\ldots&i_k\end{pmatrix}$$

Again, this action is the multiplication, as it should. Observe that, in the present context of the spin planar algebra, this multiplication is commutative.

Example 4.29. *Inclusion.*

The inclusion I_k is the $(k,k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x,y)=1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_l\begin{pmatrix}j_1&\ldots&j_k&l\\ i_1&\ldots&i_k&l\end{pmatrix}$$

Once again, what we have here is what we can expect from an inclusion.

Example 4.30. *Expectation.*

The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

This map satisfies then the usual requirements for an expectation.

Example 4.31. *Jones projection.*

The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1)=\sum_{ijl}\begin{pmatrix}i_1&\ldots&i_k&j&j\\ i_1&\ldots&i_k&l&l\end{pmatrix}$$

The elements $e_k=N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

Example 4.32. *Rotation.*

The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k=\begin{matrix}\Cap\,|\,|\,|\,\|\\ \|\qquad\quad\|\\ \|\,|\,|\,|\,\Cup\end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2\ldots i_k})=e_{i_2\ldots i_ki_1}$$

Thus, what we have indeed is a rotation map.

Example 4.33. *Shift.*

As for the shift S_k, this is the $(k,k+2)$-tangle which looks like 1_k, but has two more vertical strings, at left of the input box. This tangle acts as follows:

$$S_k\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}=\sum_{lm}\begin{pmatrix}l&m&j_1&\ldots&j_k\\ l&m&i_1&\ldots&i_k\end{pmatrix}$$

Observe that S_k is an inclusion of algebras, which is different from $I_{k+1}I_k$.

Finally, in order for our discussion to be complete, we must talk as well about the $*$-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}^*=\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}$$

As before, we refer to Jones' paper [42] for more on all this.

As a perhaps quite obvious question, appearing from the above, we have:

QUESTION 4.34. *Is there a general construction of planar algebras, generalizing both the tensor and the spin algebra constructions?*

In answer, sure yes, but this will have to wait a bit. The idea indeed is that:

(1) With suitable definitions, the tensor planar algebra appears to be the planar algebra associated to the inclusion $\mathbb{C}\subset M_N(\mathbb{C})$, and the spin planar algebra appears to be the planar algebra associated to the inclusion $\mathbb{C}\subset\mathbb{C}^N$. Thus, as a natural generalization of both these situations, we can look at the planar algebra associated to an inclusion of type $\mathbb{C}\subset B$, with B being an arbitrary finite dimensional algebra.

(2) However, the story is not over here, because for a number of reasons to be become clear later on, basically coming from Jones' notion of basic construction [40], which is the main workhorse when doing quantum algebra, or at least the present type of quantum algebra, it is actually most convenient to introduce directly the planar algebras associated to inclusions of type $A\subset B$, with both A,B being finite dimensional algebras.

(3) But, and here comes the point, while understanding what a finite dimensional algebra A is, in the present complex and involutive algebra context, is not that a big deal, understanding what an inclusion $A\subset B$ of such algebras is is something more complicated, and all in all, the above-mentioned construction of planar algebras associated to such inclusions $A\subset B$ remains something quite complicated, that we will defer for later.

In short, this is the situation, patience and modesty, we are currently learning a bit of this and a bit of that, regarding the planar algebras, because this is how planar algebras are best learned, and once we will learn enough things, in each possible direction, do not worry, we will start something more systematic. Including fully answering Question 2.34, with this being scheduled not very far from now, later in this book.

4d. Subfactor algebras

The above results raise the question on whether any planar algebra produces a subfactor. The answer here is yes, but with many subtleties, as follows:

THEOREM 4.35. *We have the following results:*

(1) *Any planar algebra with positivity produces a subfactor.*
(2) *In particular, we have TL and FC type subfactors.*
(3) *In the amenable case, and with $A_1 = R$, the correspondence is bijective.*
(4) *In general, we must take $A_1 = L(F_\infty)$, and we do not have bijectivity.*
(5) *The axiomatization of P, in the case $A_1 = R$, is not known.*

PROOF. This is something quite heavy, and for a discussion here, we refer to [42]. We will be back to this, with proofs, on several occasions, in the remainder of this book. □

4e. Exercises

Exercises:

EXERCISE 4.36.

EXERCISE 4.37.

EXERCISE 4.38.

EXERCISE 4.39.

EXERCISE 4.40.

EXERCISE 4.41.

EXERCISE 4.42.

EXERCISE 4.43.

Bonus exercise.

Part II

Finite depth

I was standing by the Nile
When I saw the lady smile
I would take her out for a while
For a while

CHAPTER 5

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Bonus exercise.

CHAPTER 6

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Bonus exercise.

CHAPTER 7

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

EXERCISE 7.1.

EXERCISE 7.2.

EXERCISE 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

EXERCISE 7.7.

EXERCISE 7.8.

Bonus exercise.

CHAPTER 8

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

Exercise 8.1.

Exercise 8.2.

Exercise 8.3.

Exercise 8.4.

Exercise 8.5.

Exercise 8.6.

Exercise 8.7.

Exercise 8.8.

Bonus exercise.

Part III

Amenability

And Michelle, what will she do
Without you, Lady Madeleine
And I walk down the avenue
And I'm missing you, Lady Madeleine

CHAPTER 9

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.

CHAPTER 10

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

EXERCISE 10.8.

Bonus exercise.

CHAPTER 11

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

Exercise 11.1.

Exercise 11.2.

Exercise 11.3.

Exercise 11.4.

Exercise 11.5.

Exercise 11.6.

Exercise 11.7.

Exercise 11.8.

Bonus exercise.

CHAPTER 12

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

Exercise 12.1.

Exercise 12.2.

Exercise 12.3.

Exercise 12.4.

Exercise 12.5.

Exercise 12.6.

Exercise 12.7.

Exercise 12.8.

Bonus exercise.

Part IV

Hyperfiniteness

This is the one
Oh, this is the one
This is the one
She's waited for

CHAPTER 13

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

Bibliography

[1] V.I. Arnold, Mathematical methods of classical mechanics, Springer (1974).

[2] M. Asaeda and U. Haagerup, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$, *Comm. Math. Phys.* **202** (1999), 1–63.

[3] M.F. Atiyah, The geometry and physics of knots, Cambridge Univ. Press (1990).

[4] M.F. Atiyah and I.G. MacDonald, Introduction to commutative algebra, Addison-Wesley (1969).

[5] T. Banica, Calculus and applications (2024).

[6] T. Banica, Principles of operator algebras (2024).

[7] T. Banica, Introduction to modern physics (2025).

[8] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[9] I. Bengtsson and K. Życzkowski, Geometry of quantum states, Cambridge Univ. Press (2006).

[10] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math.* **149** (1999), 1023–1060.

[11] D. Bisch and V.F.R. Jones, Algebras associated to intermediate subfactors, *Invent. Math.* **128** (1997), 89–157.

[12] B. Blackadar, Operator algebras: theory of C^*-algebras and von Neumann algebras, Springer (2006).

[13] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. of Math.* **38** (1937), 857–872.

[14] B. Collins and P. Śniady, Integration with respect to the Haar measure on the unitary, orthogonal and symplectic group, *Comm. Math. Phys.* **264** (2006), 773–795.

[15] A. Connes, Une classification des facteurs de type III, *Ann. Sci. Ec. Norm. Sup.* **6** (1973), 133–252.

[16] A. Connes, Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. of Math.* **104** (1976), 73–115.

[17] A. Connes, Noncommutative geometry, Academic Press (1994).

[18] A. Connes and M. Marcolli, Noncommutative geometry, quantum fields and motives, AMS (2008).

[19] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.

[20] P.A.M. Dirac, Principles of quantum mechanics, Oxford Univ. Press (1930).

[21] J. Dixmier, Von Neumann algebras, Elsevier (1981).

[22] A. Einstein, Relativity: the special and the general theory, Dover (1916).

[23] D.E. Evans and Y. Kawahigashi, Quantum symmetries on operator algebras, Oxford Univ. Press (1998).

[24] E. Fermi, Thermodynamics, Dover (1937).

[25] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics, Caltech (1963).

[26] S.R. Garcia, J. Mashreghi and W.T. Ross, Operator theory by example, Oxford Univ. Press (2023).

[27] M. Goldman, On subfactors of factors of type II_1, *Michigan Math. J.* **7** (1960), 167–172.

[28] F.M. Goodman, P. de la Harpe and V.F.R. Jones, Coxeter graphs and towers of algebras, Springer (1989).

[29] J.M. Gracia-Bondía, J.C. Várilly and H. Figueroa, Elements of noncommutative geometry, Birkhäuser (2001).

[30] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).

[31] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).

[32] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).

[33] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type III_1, *Acta Math.* **158** (1987), 95–148.

[34] U. Haagerup, Principal graphs of subfactors in the index range $4 < [M : N] < 3+\sqrt{2}$, in "Subfactors, Kyuzeso 1993" (1994), 1–38.

[35] J. Harris, Algebraic geometry, Springer (1992).

[36] F. Hiai and D. Petz, The semicircle law, free random variables and entropy, AMS (2000).

[37] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge Univ. Press (1985).

[38] K. Huang, Introduction to statistical physics, CRC Press (2001).

[39] M. Izumi, S. Morrison, D. Penneys, E. Peters and N. Snyder, Subfactors of index exactly 5, *Bull. Lond. Math. Soc.* **47** (2015) 257–269.

[40] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[41] V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math.* **137** (1989), 311–334.

[42] V.F.R. Jones, Planar algebras I (1999).

[43] V.F.R. Jones, The planar algebra of a bipartite graph, in "Knots in Hellas '98" (2000), 94–117.

[44] V.F.R. Jones, The annular structure of subfactors, *Monogr. Enseign. Math.* **38** (2001), 401–463.

[45] V.F.R. Jones, S. Morrison and N. Snyder, The classification of subfactors of index at most 5, *Bull. Amer. Math. Soc.* **51** (2014), 277–327.

[46] V.F.R. Jones and V.S Sunder, Introduction to subfactors, Cambridge Univ. Press (1997).

[47] R.V. Kadison and J.R. Ringrose, Fundamentals of the theory of operator algebras, AMS (1983).

[48] M. Kumar, Quantum: Einstein, Bohr, and the great debate about the nature of reality, Norton (2009).

[49] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).

[50] L.D. Landau and E.M. Lifshitz, Course of theoretical physics, Pergamon Press (1960).

[51] G. Landi, An introduction to noncommutative spaces and their geometry, Springer (1997).

[52] P. Lax, Functional analysis, Wiley (2002).

[53] Z. Liu, S. Morrison and D. Penneys, 1-supertransitive subfactors with index at most $6\frac{1}{5}$, *Comm. Math. Phys.* **334** (2015), 889–922.

[54] S. Malacarne, Woronowicz's Tannaka-Krein duality and free orthogonal quantum groups, *Math. Scand.* **122** (2018), 151–160.

[55] V.A. Marchenko and L.A. Pastur, Distribution of eigenvalues in certain sets of random matrices, *Mat. Sb.* **72** (1967), 507–536.

[56] M.L. Mehta, Random matrices, Elsevier (2004).

[57] G.J. Murphy, C*-algebras and operator theory, Academic Press (1990).

[58] F.J. Murray and J. von Neumann, On rings of operators, *Ann. of Math.* **37** (1936), 116–229.

[59] F.J. Murray and J. von Neumann, On rings of operators. II, *Trans. Amer. math. Soc.* **41** (1937), 208–248.

[60] F.J. Murray and J. von Neumann, On rings of operators. IV, *Ann. of Math.* **44** (1943), 716–808.

[61] J. Nash, The imbedding problem for Riemannian manifolds, *Ann. of Math.* **63** (1956), 20–63.

[62] A. Nica and R. Speicher, Lectures on the combinatorics of free probability, Cambridge Univ. Press (2006).

[63] M.A. Nielsen and I.L. Chuang, Quantum computation and quantum information, Cambridge Univ. Press (2000).

[64] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, *London Math. Soc. Lect. Notes* **136** (1988), 119–172.

[65] A. Ocneanu, Quantum symmetry, differential geometry of finite graphs, and classification of subfactors, Univ. Tokyo Seminar Notes (1990).

[66] G.K. Pedersen, C*-algebras and their automorphism groups, Academic Press (1979).

[67] M. Pimsner and S. Popa, Entropy and index for subfactors, *Ann. Sci. École Norm. Sup.* **19** (1986), 57–106.

[68] S. Popa, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras, *J. Operator Theory* **9** (1983), 253–268.

[69] S. Popa, Classification of subfactors: the reduction to commuting squares, *Invent. Math.* **101** (1990), 19–43.

[70] S. Popa, Classification of amenable subfactors of type II, *Acta Math.* **172** (1994), 163–255.

[71] S. Popa, An axiomatization of the lattice of higher relative commutants of a subfactor, *Invent. Math.* **120** (1995), 427–445.

[72] S. Popa and D. Shlyakhtenko, Universal properties of $L(F_\infty)$ in subfactor theory, *Acta Math.* **191** (2004), 225–257.

[73] W. Rudin, Real and complex analysis, McGraw-Hill (1966).

[74] S. Sakai, C*-algebras and W*-algebras, Springer (1998).

[75] I.R. Shafarevich, Basic algebraic geometry, Springer (1974).

[76] S.V. Strătilă and L. Zsidó, Lectures on von Neumann algebras, Cambridge Univ. Press (1979).

[77] M. Takesaki, Theory of operator algebras, Springer (1979).

[78] P. Tarrago and J. Wahl, Free wreath product quantum groups and standard invariants of subfactors, *Adv. Math.* **331** (2018), 1–57.

[79] N.H. Temperley and E.H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem, *Proc. Roy. Soc. London* **322** (1971), 251–280.

[80] S. Vaes, Strictly outer actions of groups and quantum groups, *J. Reine Angew. Math.* **578** (2005), 147–184.

[81] S. Vaes and R. Vergnioux, The boundary of universal discrete quantum groups, exactness and factoriality, *Duke Math. J.* **140** (2007), 35–84.

[82] D.V. Voiculescu, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66** (1986), 323–346.

[83] D.V. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201–220.

[84] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, AMS (1992).

[85] J. von Neumann, On a certain topology for rings of operators, *Ann. of Math.* **37** (1936), 111–115.

[86] J. von Neumann, On rings of operators. III, *Ann. of Math.* **41** (1940), 94–161.

[87] J. von Neumann, On rings of operators. Reduction theory, *Ann. of Math.* **50** (1949), 401–485.

[88] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).

[89] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[90] A. Wassermann, Coactions and Yang-Baxter equations for ergodic actions and subfactors, *London Math. Soc. Lect. Notes* **136** (1988), 203–236.

[91] J. Watrous, The theory of quantum information, Cambridge Univ. Press (2018).

[92] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).

[93] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).

[94] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *J. Math. Phys.* **19** (1978), 999–1001.

[95] H. Wenzl, C^*-tensor categories from quantum groups, *J. Amer. Math. Soc.* **11** (1998), 261–282.

[96] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).

[97] H. Weyl, The classical groups: their invariants and representations, Princeton Univ. Press (1939).

[98] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* **62** (1955), 548–564.

[99] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[100] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups, *Invent. Math.* **93** (1988), 35–76.

Index