QUANTUM ISOMETRY GROUPS OF DUALS OF FREE POWERS OF CYCLIC GROUPS

TEODOR BANICA AND ADAM SKALSKI

ABSTRACT. We study the quantum isometry groups $G^+(\widehat{\Gamma})$ of the noncommutative Riemannian manifolds associated to discrete group duals $\widehat{\Gamma}$. The basic representation theory problem is to compute the law of the main character $\chi : G^+(\widehat{\Gamma}) \to \mathbb{C}$, and our main result here is as follows: for $\Gamma = \mathbb{Z}_s^{*n}$, with $s \geq 5$ and $n \geq 2$, the variable $\chi/2$ follows the compound free Poisson law $\pi_{\varepsilon/2}$, where ε is the uniform measure on the s-th roots of unity, and $\varepsilon \to \underline{\varepsilon}$ is the canonical projection map from complex to real measures. We discuss as well a number of technical versions of this result, notably with the construction of a new quantum group, which appears as a "representation-theoretic limit", at $s = \infty$.

INTRODUCTION

The notion of compact quantum group was introduced by Woronowicz in [35], [36]. Woronowicz's formalism, which is both quite general, and remarkably easy to handle, allowed Wang to construct in [33], [34] a number of universal quantum groups, namely the free analogues of O_n, U_n, S_n. The next step, developed by Bichon in [16] and by the first-named author in [1], was the construction of quantum automorphism groups of various discrete structures (finite graphs, finite metric spaces). This allowed a considerable extension of Wang's original list of free quantum groups, notably with a free analogue of the hyperoctahedral group H_n [3], and a number of technical versions of it [2], [5], [9].

Another important generalization of Wang's original constructions comes from the work of Goswami [23]. The idea is that Connes' noncommutative geometry theory [20], [21] has shown that a number of important situations, mainly coming from particle physics or number theory, are best described by a certain spectral triple $X = (A, H, D)$. It is therefore natural to ask for the computation of the quantum isometry group $G^+(X)$ of such a spectral triple, constructed in [23]. The theory here has been quickly developed in the last few years, first with a number of foundational papers by Bhowmick and Goswami [11], [12], [13]. The earlier quantum automorphism constructions in [1], [16] were reformulated, unified and generalized in [14], by using the discrete spectral triples constructed

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by Christensen and Ivan in [18]. Another important computation, with several potential applications, is that of the quantum isometry group of Connes' spectral triple of the standard model: this was recently done by Bhowmick, D'Andrea and Dabrowski [10].

One promising direction in view of the general understanding of the quantum isometry groups is that of the explicit computation of $G^+(X)$, in the case where $X = \widehat{\Gamma}$ is a discrete group dual: indeed, a number of very concrete tools, coming from combinatorics, subfactors, or free probability, are available here. The general theory, as well as a number of examples, including an explicit presentation result in the free group case $\Gamma = F_n$, were worked out in [15]. This latter free group case was studied in much detail in [8], where a link with the hyperoctahedral quantum group in [3], with the Fuss-Catalan combinatorics of Bisch and Jones [17], and with the free spheres constructed in [7] was obtained. More generally, it was shown in [8] that the quantum isometry group $G^+(\widehat{F}_n)$ belongs in fact to a general class of "two-parameter quantum symmetry groups", and this makes a link with the recent classification program for the easy quantum groups, started in [5], [9].

The purpose of the present paper is to push one step forward the previous combinatorial and probabilistic considerations in [8], [15]. These considerations basically concern the computation of the law of the main character, and our main result here is as follows.

Theorem. *For $\Gamma = \mathbb{Z}_s^{*n}$ with $s \geq 5$ and $n \geq 2$, the half of the main character of $G^+(\widehat{\Gamma})$ follows the compound free Poisson law $\pi_{\underline{\varepsilon}/2}$, where ε is the uniform measure on the s-roots of unity, and $\varepsilon \to \underline{\varepsilon}$ is the canonical projection map from complex to real measures.*

The proof of this result uses the general diagrammatic methods in [2], [8], Voiculescu's R-transform from [31], [32] and Speicher's notion of free cumulants from [27], [28].

As in the previous paper [2], we end up with a measure which is a compound free Poisson law, in the sense of [24], [29]. However, the situation here is more complicated, combinatorially speaking, because the projection map $\varepsilon \to \underline{\varepsilon}$ appears.

The above theorem holds in fact as well at $s = 3$. However, at $s = 2, 4$ the situation is quite different, and we will present here some results in this direction. The main point here is that the case $s = 4$ is truly special, as already observed in [1], [15].

The limiting case $s \to \infty$ is quite interesting as well, because the quantum groups in the above theorem "converge", in a certain representation theory sense, to a compact quantum group which is different from the quantum isometry group of the dual of the free group $F_n = \mathbb{Z}^{*n}$ studied in [8]. The construction of this new quantum group, for which we refer to section 6 below, will be actually our second main result in this paper.

Finally, let us mention that there are many similarities with the "easy quantum group" results in [2], [5], [6]. One can expect that some further generalizations of the above theorem might provide some answers to the questions raised in [5]. We do not know if it is so, but we will make some comments in this direction, at the end of the paper.

The paper is organized as follows: 1 is a preliminary section, in 2 we discuss the general properties of $G^+(\widehat{\mathbb{Z}_s^{*n}})$, and in 3-5 we compute diagrams, discuss the compound

free Poisson laws, and prove the above theorem. The final sections, 6-7, contain some technical versions of that theorem, and a few concluding remarks.

1. QUANTUM ISOMETRIES

The natural framework for the study of noncommutative objects like $\widehat{\Gamma}$ is Connes' noncommutative geometry [20], [21], where the basic definition is as follows.

Definition 1.1. *A compact spectral triple (A, H, D) consists of the following:*

(1) *a unital C^*-algebra A;*
(2) *a Hilbert space H, on which A acts;*
(3) *an unbounded self-adjoint operator D on H, with compact resolvents, such that $[D, x]$ has a bounded extension, for any x in a dense $*$-subalgebra $\mathcal{A} \subset A$.*

This definition is of course over-simplified, as to best fit with the purposes of the present paper. We refer to [20], [21] for the precise formulation of the axioms.

Now let $\Gamma = < g_1, \ldots, g_m >$ be a discrete group, with the generating set $S = \{g_1, \ldots, g_m\}$ assumed to satisfy $S = S^{-1}$ and $1 \notin S$. The generating set S is of course taken to be a set without repetitions. In most of the examples below, m will be actually minimal.

The length on Γ is given by $l(g) = \min\{r \in \mathbb{N} | \exists h_1 \ldots h_r \in S, g = h_1 \ldots h_r\}$, and the distance on Γ is given by $d(g, h) = l(g^{-1}h)$. Observe that $S = \{g \in G | l(g) = 1\}$.

Consider the group algebra $\mathcal{A} = \mathbb{C}\Gamma$, with involution given by $g^* = g^{-1}$ and antilinearity, and let $A = C^*(\Gamma)$ be the completion of $\mathcal{A}$ with respect to the maximal C^*-norm.

Definition 1.2. *Associated to any discrete group $\Gamma = < g_1, \ldots, g_m >$ as above is a compact spectral triple $\widehat{\Gamma} = (A, H, D)$, as follows:*

(1) $A = C^*(\Gamma)$ *is the full group algebra of Γ.*
(2) $H = l^2(\Gamma)$, *with A acting on it by $g\delta_h = \delta_{gh}$.*
(3) $D(\delta_g) = l(g)\delta_g$, *on the standard basis of H.*

We can think of $\widehat{\Gamma}$ as being a "noncommutative Riemannian manifold", and then consider the quantum isometry group of orientation preserving isometries $G^+(\widehat{\Gamma})$, introduced by Bhowmick and Goswami in Section 2 of [12].

We recall that the standard trace of $\mathbb{C}\Gamma \subset C^*(\Gamma)$ is given by $tr(g) = \delta_{g1}$ and also identify in a natural way $\mathbb{C}\Gamma$ with a vector subspace of $l^2(\Gamma)$.

Proposition 1.3. $G^+(\widehat{\Gamma})$ *is the universal compact quantum group acting on $C^*(\Gamma)$, such that all the eigenspaces of D and the standard trace on $\mathbb{C}\Gamma$ are invariant under this action.*

Proof. This is more or less a consequence of a simpler reformulation of the general definition of the quantum isometry groups of orientation preserving isometries given in [12], in the case where the representation of A on H allows a cyclic and separating vector which spans the kernel of D and satisfies natural conditions with respect to the 'smooth algebra' $\mathcal{A}$ (Corollary 2.27 of [12]). The conditions above are satisfied in the case of manifolds coming from group duals; this, together with the consequences for the computations of $G^+(\widehat{\Gamma})$, is discussed in detail in Section 2 of [15]. Note that the fact that the eigenspaces of D are preserved by the action α implies that $\alpha : \mathbb{C}\Gamma \to \mathbb{C}\Gamma \otimes_{alg} C(G^+(\widehat{\Gamma}))$, so that the trace preservation condition makes sense. $\qquad\square$

Let us look now at the eigenspaces of D. There is one such eigenspace for each positive integer $r \in \mathbb{N}$, namely the span of the set $\Gamma_r = \{g \in \Gamma | l(g) = r\}$ of words of length r.

Proposition 1.4. *For any $r \in \mathbb{N}$, $G^+(\widehat{\Gamma})$ has a unitary representation on $span(\Gamma_r)$, the span of words of Γ of length r. Moreover, the representation at $r = 1$ is faithful.*

Proof. We know from Proposition 1.3 that $G^+(\widehat{\Gamma})$ acts on $span(\Gamma_r)$, and the invariance of the trace shows that the corresponding representation is unitary. The faithfulness property at $r = 1$ follows from the universal property of $G^+(\widehat{\Gamma})$. See [15]. $\qquad\square$

The above two statements make it clear how to explicitly construct $G^+(\widehat{\Gamma})$: since this quantum group has to act faithfully on the span of $\Gamma_1 = S$, we just have to consider the universal quantum group acting on $span(S)$, and then divide by a suitable ideal.

So, let $A_u(m)$ be the universal C^*-algebra generated by the entries of a $m \times m$ matrix u, such that both $u = (u_{ij})$ and $u^t = (u_{ji})$ are unitaries. This is a Hopf C^*-algebra in the sense of Woronowicz [35], with comultiplication $\Delta(u_{ij}) = \Sigma u_{ik} \otimes u_{kj}$, counit $\varepsilon(u_{ij}) = \delta_{ij}$, and antipode $S(u_{ij}) = u_{ji}^*$. Observe that we have $S^2 = id$. See Wang [33].

According to the general results of Woronowicz in [35], we can think of $A_u(m)$ as being the algebra of continuous functions on a certain compact quantum group. We denote this quantum group by U_m^+. That is, U_m^+ is the abstract object given by $A_u(m) = C(U_m^+)$.

Theorem 1.5. $G^+(\widehat{\Gamma})$ *is the subgroup of U_m^+ presented by the following relations:*
  (1) *Those making $\alpha(g_i) = \Sigma g_j \otimes u_{ji}$ a morphism of $*$-algebras.*
  (2) *Those making $span(\Gamma_r)$ invariant under α, for any $r \geq 2$.*

Proof. This result is from [15], where it appears in a slightly different formulation. We present below an explanation of the present formulation, along with a short proof.

We know from Proposition 1.3 and Proposition 1.4 that the quantum $G^+(\widehat{\Gamma})$ acts faithfully on $span(S)$, and that the corresponding representation is unitary. In terms of Hopf algebras, this means that we have a coaction map $\alpha : span(S) \to span(S) \otimes_{alg} C(G^+(\widehat{\Gamma}))$, and that if we denote this map by $\alpha(g_i) = \Sigma g_j \otimes u_{ji}$, then the corepresentation $u = (u_{ij})$ is unitary, and its coefficients u_{ij} generate the C^*-algebra $C(G^+(\widehat{\Gamma}))$.

So, we obtain in this way a surjective morphism of Hopf C^*-algebras $\Phi : A_u(m) \to C(G^+(\widehat{\Gamma}))$, which corresponds to an embedding of compact quantum groups $G^+(\widehat{\Gamma}) \subset U_m^+$.

Let us try to understand now what the kernel of Φ is. According to the universal property of Γ, from Proposition 1.3, and to Woronowicz's Tannakian results in [36], this kernel should be exactly the ideal generated by the following relations:

(1) Those making α a morphism of $*$-algebras. This means that for any $h_i, k_i \in S$ satisfying $h_1 \ldots h_p = k_1 \ldots k_q$ we write the equality $\alpha(h_1 \ldots h_p) = \alpha(k_1 \ldots k_q)$ as a formula of type $\Sigma_{g \in \Gamma} g \otimes E_g = \Sigma_{g \in \Gamma} g \otimes F_g$, and $E_g = F_g$, with $g \in \Gamma$, are our relations.

(2) Those making $span(\Gamma_r)$ invariant under α, for any $r \geq 2$. Once again, an explanation is needed here. The idea is that if we denote by $\overline{\Gamma}_r$ the set of words of length $\leq r$, then $span(\overline{\Gamma}_r)$ is invariant under α. Now since we have $\Gamma_r \subset \overline{\Gamma}_r$, the invariance condition for the subspaces $span(\Gamma_r) \subset span(\overline{\Gamma}_r)$ corresponds indeed to certain relations.

Note that the condition in (2) implies in particular that the action α preserves the trace (as $tr(\Gamma_r) = \{0\}$ for $r \neq 0$), and the proof is completed. $\qquad\square$

Note that in all the cases computed so far in [15] and [8] the relations in (2) actually were a consequence of these imposed by (1) – this will also be the case in this paper. We do not know if this is true for arbitrary Γ.

2. Free products

In this section we discuss the presentation of the quantum group $G^+(\widehat{\Gamma})$, in the case $\Gamma = \mathbb{Z}_s^{*n}$. We will usually assume $s \in \{2, 3, \ldots, \infty\}$, with the convention $\mathbb{Z}_\infty = \mathbb{Z}$.

Let us first recall a basic result from [15], solving the problem at $n = 1$.

Proposition 2.1. *The quantum groups $G^+(\widehat{\mathbb{Z}_s})$ are as follows:*

(1) *At $s = \infty$ we have $\widehat{\mathbb{Z}_s} = \mathbb{T}$, and $G^+(\mathbb{T}) = O_2 = \mathbb{T} \rtimes \mathbb{Z}_2$.*

(2) *More generally, at any $s \neq 4$ we have $G^+(\widehat{\mathbb{Z}_s}) = \mathbb{Z}_s \rtimes \mathbb{Z}_2$.*

(3) *At $s = 4$ the quantum group $G^+(\widehat{\mathbb{Z}_s})$ is non-classical, and infinite.*

Proof. These results can be deduced from Theorem 1.5, the idea being that for $s \neq 4$, the relations there make the coefficients u_{ij} commute. See [15]. $\qquad\square$

In the general case now $n \in \mathbb{N}$, let $g_1, \ldots, g_n$ be the standard generators of $\mathbb{Z}_s^{*n}$, and endow this group with the generating set $S = \{g_{i1}, g_{i2}\}$, where $g_{i1} = g_i, g_{i2} = g_i^{-1}$.

The fundamental coaction is denoted as follows:

$$\alpha(g_{ia}) = \sum_{jb} g_{jb} \otimes u_{jb,ia}$$

The simplest case is when $s = \infty$. Here our group $\Gamma = \mathbb{Z}_s^{*n}$ is the free group F_n, and we have the following result, from [8].

Proposition 2.2. $H_{n0}^+ = G^+(\widehat{F_n})$ *is the subgroup of U_{2n}^+ presented by the following relations, between the generators denoted $u_{ia,jb}$ with $i, j = 1, \ldots, n$ and $a, b = 1, 2$:*

(1) *The entries $u_{ia,jb}$ are partial isometries.*
(2) *We have $u_{ia,jb}^* = u_{i\bar{a},j\bar{b}}$, for any i, j, a, b.*

Proof. This is done in [8], the idea being as follows.

First, we know from Theorem 1.5 that α must preserve the involution, i.e. that we must have $\alpha(g_{ia})^* = \alpha(g_{i\bar{a}})$ for any i, a, and this gives $u_{ia,jb}^* = u_{i\bar{a},j\bar{b}}$, for any i, j, a, b.

Next, the other condition that we get from Theorem 1.5 is:

$$\alpha(g_{ia})\alpha(g_{i\bar{a}}) = 1 \otimes 1 \quad \Longleftrightarrow \quad \sum_{jbkc} g_{jb}g_{kc} \otimes u_{jb,ia}u_{kc,i\bar{a}} = 1 \otimes 1$$

$$\Longleftrightarrow \quad \sum_{jbkc} g_{jb}g_{k\bar{c}} \otimes u_{jb,ia}u_{kc,ia}^* = 1 \otimes 1$$

Thus we must have $u_{jb,ia}u_{kc,ia}^* = 0$ for $(jb) \neq (kc)$, and $\sum_{jb} u_{jb,ia}u_{jb,ia}^* = 1$. By multiplying this latter equality at right by $u_{kc,ia}$, we obtain $u_{kc,ia}u_{kc,ia}^*u_{kc,ia} = u_{kc,ia}$, so $u_{kc,ia}$ must be a partial isometry. This leads to the conclusion in the statement, as the conditions in (2) in Theorem 1.5 are automatically satisfied, see the proof of Theorem 5.1 in [15].   □

In the above statement, the notation H_{n0}^+ comes from the fact that the quantum group under consideration is part of a certain two-parameter series H_{nm}^+, which is such that $H_m^+ = H_{0m}^+$ is the hyperoctahedral quantum group in [3]. See [8].

Let us record as well the following useful reformulation of the above result.

Proposition 2.3. $C(H_{n0}^+)$ *is the universal algebra generated by variables $u_{ia,jb}$ with $i, j = 1, \ldots, n$ and $a, b = 1, 2$, subject to the following relations:*

(1) $u_{ia,jb}$ *are partial isometries, satisfying $u_{ia,jb}^* = u_{i\bar{a},j\bar{b}}$.*
(2) $u_{jb,ia}u_{kc,ia}^* = 0$ *for $(jb) \neq (kc)$, and $\sum_{jb} u_{jb,ia}u_{jb,ia}^* = 1$.*

Proof. This is indeed just a technical reformulation of the above result, the idea being that the relations in (2) correspond to the fact that the matrices u, u^t are unitaries.   □

In the general case now, we have the following statements.

Lemma 2.4. *For $s \geq 5$ and $n \geq 2$, the quantum group $H_{n0}^{s+} = G^+(\widehat{\mathbb{Z}_s^{*n}})$ is the subgroup of H_{n0}^+ presented by the relations coming from $\alpha(g_{ia})^s = 1 \otimes 1$, for any i, a.*

Proof. First, let us mention the fact that the result holds as well at $s = 3$, with the same proof as below, and at $s = \infty$ too, with the convention that the extra relations do not exist in this case. These special cases will be discussed in section 6 below.

Consider the standard coaction, $\alpha(g_{ia}) = \sum_{jb} g_{jb} \otimes u_{jb,ia}$. Our assumption $s \neq 1, 2, 4$ shows that the elements $g_{jb}g_{k\bar{c}}$ are distinct and different from 1 for $(jb) \neq (kc)$, so the computations in the proof of Proposition 2.2 apply, and show that the generators $u_{ia,jb}$ must satisfy the conditions found there. That is, we have indeed $H_{n0}^{s+} \subset H_{n0}^+$.

According now to Theorem 1.5, the conditions which are left are simply those coming from the equalities $\alpha(g_{ia})^s = 1 \otimes 1$, for any i, a, and we are done. The conditions in (2) in

Theorem 1.5 are satisfied due to the fact that we are dealing with a quantum subgroup of H_{n0}^+ and the only extra relations in $\mathbb{Z}_s^{*n}$ are those of the form $g^s = e$ – compare with the proof of Theorem 5.1 in [15]. □

Theorem 2.5. *For $s \geq 5$ and $n \geq 2$, the quantum group $H_{n0}^{s+} = G^+(\widehat{\mathbb{Z}_s^{*n}})$ is the subgroup of H_{n0}^+ presented by the relation $\xi \in Fix(u^{\otimes s})$, where $\xi = \sum_{ia} e_{ia}^{\otimes s}$.*

Proof. According to Lemma 2.4, we just have to understand the meaning of the relations coming from the conditions $\alpha(g_{ia})^s = 1 \otimes 1$, for any i, a. We have:

$$
\begin{aligned}
\alpha(g_{ia})^s &= \sum_{j_1\ldots j_s}\sum_{b_1\ldots b_s} g_{j_1 b_1} \cdots g_{j_s b_s} \otimes u_{j_1 b_1, ia} \cdots u_{j_s b_s, ia} \\
&= \sum_{\gamma} \gamma \otimes \left(\sum_{j_1\ldots j_s}\sum_{b_1\ldots b_s} \delta_{\gamma, g_{j_1 b_1} \cdots g_{j_s b_s}} u_{j_1 b_1, ia} \cdots u_{j_s b_s, ia} \right)
\end{aligned}
$$

We know from Proposition 2.3 that we have $u_{jb,ia}u_{kc,ia} = 0$ for $(jb) \neq (k\bar{c})$, so we can erase from the above formula all the decompositions $\gamma = g_{j_1 b_1} \cdots g_{j_s b_s}$ which are not reduced with respect to the rules $g_{ia}g_{i\bar{a}} = 1$. So, if we denote by W_s the set of reduced words of length s, we have the following formula:

$$
\alpha(g_{ia})^s = 1 \otimes \left(\sum_{jb} u_{jb,ia}^s \right) + \sum_{\gamma \in W_s} \gamma \otimes \left(\sum_{j_1\ldots j_s}\sum_{b_1\ldots b_s} \delta_{\gamma, g_{j_1 b_1} \cdots g_{j_s b_s}} u_{j_1 b_1, ia} \cdots u_{j_s b_s, ia} \right)
$$

Let us look now at an equality of type $\gamma = \gamma'$, with $\gamma, \gamma' \in W_s$. This can happen only in the situation $g_{jb}^p g_{kc}^q = g_{j\bar{b}}^{s-p} g_{k\bar{c}}^{s-q}$, with $j \neq k$ and $p + q = s$. So, if we denote by $W_s' \subset W_s$ the set of words which are not of this type, we have the following formula:

$$
\begin{aligned}
\alpha(g_{ia})^s &= 1 \otimes \left(\sum_{jb} u_{jb,ia}^s \right) + \sum_{j \neq k}\sum_{c}\sum_{s=p+q} g_{j1}^p g_{kc}^q \otimes (u_{j1,ia}^p u_{kc,ia}^q + u_{j2,ia}^{s-p} u_{k\bar{c},ia}^{s-q}) \\
&\quad + \sum_{g_{j_1 b_1} \cdots g_{j_s b_s} \in W_s'} g_{j_1 b_1} \cdots g_{j_s b_s} \otimes u_{j_1 b_1, ia} \cdots u_{j_s b_s, ia}
\end{aligned}
$$

Thus the equality $\alpha(g_{ia})^s = 1 \otimes 1$ is equivalent to the following conditions:

$$
\begin{cases}
\sum_{jb} u_{jb,ia}^s = 1 & \forall\, i, \forall\, a \\
u_{j1,ia}^p u_{kc,ia}^q + u_{j2,ia}^{s-p} u_{k\bar{c},ia}^{s-q} = 0 & \forall\, j \neq k, \forall\, c, \forall\, s = p + q \\
u_{j_1 b_1, ia} \cdots u_{j_s b_s, ia} = 0 & \forall\, g_{j_1 b_1} \cdots g_{j_s b_s} \in W_s'
\end{cases} \qquad (1)
$$

Let us look now at the condition $\xi \in Fix(u^{\otimes s})$. We have:

$$
u^{\otimes s}(\xi \otimes 1) = \left(\sum_{i_r j_r a_r b_r} e_{j_1 b_1, i_1 a_1} \otimes \ldots \otimes e_{j_s b_s, i_s a_s} \otimes u_{j_1 b_1, i_1 a_1} \ldots u_{j_s b_s, i_s a_s} \right) \left(\sum_{ia} e_{ia}^{\otimes s} \otimes 1 \right)
$$

$$
= \sum_{j_1 \ldots j_s \, b_1 \ldots b_s} e_{j_1 b_1} \otimes \ldots e_{j_s b_s} \otimes \left(\sum_{ia} u_{j_1 b_1, ia} \ldots u_{j_s b_s, ia} \right)
$$

By using now the formula $u_{jb,ia} u_{kc,ia} = 0$, valid for any $(jb) \neq (k\bar{c})$, we get:

$$
u^{\otimes s}(\xi \otimes 1) = \sum_{jb} e_{jb}^{\otimes s} \otimes \left(\sum_{ia} u_{jb,ia}^{s} \right)
$$

$$
+ \sum_{g_{j_1 b_1} \ldots g_{j_s b_s} \in W_s} e_{j_1 b_1} \otimes \ldots e_{j_s b_s} \otimes \left(\sum_{ia} u_{j_1 b_1, ia} \ldots u_{j_s b_s, ia} \right)
$$

Thus the condition $\xi \in Fix(u^{\otimes s})$ is equivalent to the following conditions:

$$
\begin{cases} \sum_{jb} u_{jb,ia}^{s} = 1 & \forall i, \forall a \\ \sum_{ia} u_{j_1 b_1, ia} \ldots u_{j_s b_s, ia} = 0 & \forall g_{j_1 b_1} \ldots g_{j_s b_s} \in W_s \end{cases} \quad (2)
$$

Let us prove now that we have $(1) \iff (2)$. We begin with $(1) \implies (2)$. Here all the assertions are clear, except perhaps for the equality $\sum_{ia} u_{j_1 b_1, ia} \ldots u_{j_s b_s, ia} = 0$, with $g_{j_1 b_1} \ldots g_{j_s b_s} \in W_s - W_s'$. Now by recalling the definition of W_s, W_s', our word $g_{j_1 b_1} \ldots g_{j_s b_s}$ must be of the form $g_{jb}^p g_{kc}^q$ with $j \neq k$ and $p + q = s$, and we must prove that we have $\sum_{ia} u_{jb,ia}^p u_{kc,ia}^q = 0$ in this situation. For this purpose, we use the middle condition in (1), namely $u_{j1,ia}^p u_{kc,ia}^q + u_{j2,ia}^{s-p} u_{k\bar{c},ia}^{s-q} = 0$. By multiplying by $u_{j1,ia}^*$ we obtain $u_{j1,ia}^* u_{j1,ia}^p u_{kc,ia}^q = 0$. By multiplying by $u_{j1,ia}$ we obtain $u_{j1,ia} u_{j1,ia}^* u_{j1,ia}^p u_{kc,ia}^q = 0$. But now, since $u_{j1,ia}$ is a partial isometry, and for each partial isometry v we have $vv^*v = v$, we can remove the $u_{j1,ia} u_{j1,ia}^*$ term, and we are left with $u_{j1,ia}^p u_{kc,ia}^q = 0$, and this gives the result, since all terms in the sum $\sum_{ia} u_{jb,ia}^p u_{kc,ia}^q$ are now equal to 0.

In the other direction now, $(2) \implies (1)$, all the assertions are clear as well: indeed, the third relations in (1) can be obtained from the second relations in (2) by multiplying by $u_{jb,ia}$, known to be a partial isometry, and the second relations in (1) can be obtained from the second relations in (2), for the words of type $g_{j_1 b_1} \ldots g_{j_s b_s} \in W_s - W_s'$. $\square$

3. NONCROSSING PARTITIONS

In this section we study the representation theory of $G^+(\widehat{\mathbb{Z}_s^{*n}})$. According to a well-known principle, going back to Woronowicz's work in [36], most of the representation theory invariants are encoded in the structure of the spaces $Hom(u^{\otimes k}, u^{\otimes l})$, which form a tensor category. So, it is this tensor category that we want to compute.

In addition, another well-known principle, once again going back to the considerations in [36], tells us that we should look for an explicit basis for the elements of $Hom(u^{\otimes k}, u^{\otimes l})$, indexed by "diagrams". This latter principle has been proved to be fruitful in a number of situations, see e.g. [5], [8], and we will prove that it applies once again here. Already in [8] we noticed that due to the fact that the quantum isometry groups we study have non-selfadjoint entries in the fundamental representations, it is natural to consider diagrams with vertices of two different colors.

So, let us first construct a candidate for such a basis of $Hom(u^{\otimes k}, u^{\otimes l})$.

Definition 3.1. *We let $D_s(k, l)$ be the set of noncrossing partitions between k upper points and l lower points, with each leg colored black or white, with the following rules:*

(1) *In each block, the signed number of black legs equals the signed number of white legs, modulo s (the sign being $+$ for an upper leg, and $-$ for a lower leg).*
(2) *We agree to identify any colored partition with all the colored partitions obtained by interchanging the two colors, independently in each block.*

In this definition the parameter s ranges in the set $\{1, 2, 3, \ldots, \infty\}$, with the usual convention that $a = b$ modulo ∞ means $a = b$. The condition in (1) means to have the equality $b_{up} - b_{down} = w_{up} - w_{down}$ modulo s, for each block of our colored partition, where $b_{up}, b_{down}, w_{up}, w_{down}$ count the black/white, up/down legs of the block.

Observe that we have inclusions $D_\infty(k, l) \subset D_s(k, l) \subset D_1(k, l)$, for any s. More generally, we have an inclusion $D_s(k, l) \subset D_t(k, l)$, whenever $t | s$.

Here are a few examples of such sets of partitions.

Proposition 3.2. *At $s = 1, 2, \infty$, the sets $D_s(k, l)$ are as follows:*

(1) *$D_1(k, l)$ is the set of noncrossing partitions $NC(k, l)$, with the legs arbitrarily bicolored black or white, taken modulo the block-interchanging of colors.*
(2) *$D_2(k, l)$ is the set of noncrossing partitions with even blocks $NC_{even}(k, l)$, with the legs arbitrarily bicolored black or white, modulo the block-interchanging of colors.*
(3) *$D_\infty(k, l)$ is once again $NC_{even}(k, l)$, this time colored such that the signed numbers of black and white legs are the same, modulo the block-interchanging of colors.*

Proof. This is clear from the fact that the numbers $b_{up}, b_{down}, w_{up}, w_{down}$, which must satisfy $b_{up} - b_{down} = w_{up} - w_{down}$ modulo s, sum up to the size of the block. $\square$

Consider now the Hilbert space $H = \mathbb{C}^{2n}$, with standard basis denoted $\{e_I\} = \{e_{ia}\}$. Associated to any partition $\pi \in D_s(k, l)$ is a linear map $T_\pi : H^{\otimes k} \to H^{\otimes l}$, as follows:

$$T_\pi(e_{I_1} \otimes \ldots \otimes e_{I_k}) = \sum_{J_1 \ldots J_l} \delta_\pi \begin{pmatrix} I_1 & \cdots & I_k \\ J_1 & \cdots & J_l \end{pmatrix} e_{J_1} \otimes \ldots \otimes e_{J_l}$$

Here the symbol $\delta_\pi \in \{0, 1\}$ is defined as follows: we put the indices on the points of π, and we set $\delta_\pi = 1$ when for each block there exists an index $I = ia$ such that all black indices are $I = ia$, and all white indices are $\bar{I} = i\bar{a}$. Otherwise, we set $\delta_\pi = 0$.

Proposition 3.3. *The application $\pi \to T_\pi$ transforms the horizontal concatenation, vertical concatenation and upside-down turning of colored diagrams into the tensor product, composition, and involution of linear maps (with a 2n factor for the circles).*

Proof. All the assertions are clear from definitions, except perhaps for the last assertion, concerning the value of the circles. So, consider the partition $\cap$, with legs colored bw. This partition belongs to $D_s(0,2)$ for any s, and the associated linear map is by definition $T_\cap(1) = \sum_{ia} e_{ia} \otimes e_{i\bar{a}}$. Similarly, the partition $\cup$, with legs colored bw, belongs to $D_s(2,0)$ for any s, and the associated linear map is $T_\cup(e_{ia} \otimes e_{jb}) = \delta_{ia,j\bar{b}}$. Now since we have $T_\cup T_\cap(1) = T_\cup(\sum_{ia} e_{ia} \otimes e_{i\bar{a}}) = \#\{ia\} = 2n$, this gives the result. $\square$

In order to make the link with the quantum groups considered in the previous section, we will need a supplementary category of partitions, constructed in [8]:

Definition 3.4. *We let $\overline{D}_\infty(k,l)$ be the set of partitions in $NC_{even}(k,l)$, with all legs colored black or white, with the following rules:*

(1) *In each block, the top and bottom legs either both follow the pattern bwbw ..., or both follow the pattern wbwb ...*

(2) *We agree to identify any colored partition with all the colored partitions obtained by interchanging the two colors, independently in each block.*

Once again, we can associate linear maps to such partitions, by the same formula as the one given before Proposition 3.3, and an analogue of that proposition holds. See [8].

Observe that for any s we have inclusions as follows:

$$\overline{D}_\infty(k,l) \subset D_\infty(k,l) \subset D_s(k,l) \subset D_1(k,l)$$

Here the first inclusion follows by comparing Definition 3.4 with Proposition 3.2 (3), and the other two inclusions were already noticed, before Proposition 3.2.

As we will see in the proof below, the above inclusions correspond, via Woronowicz's Tannakian duality [36], to inclusions between the quantum groups we are interested in.

Theorem 3.5. *For the quantum group $G^+(\widehat{\mathbb{Z}_s^{*n}})$, with $5 \le s < \infty$ and $n \ge 2$, we have:*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span(T_\pi | \pi \in D_s(k,l))$$

In addition, the linear maps T_π appearing on the right are linearly independent.

Proof. Let us first mention the fact that the result holds as well at $s = 3$. Regarding this case, or more generally all the special cases ($s = 2,3,4,\infty$) we refer to section 6 below.

So, let G be the quantum isometry group in the statement.

Step 1. We first find a purely combinatorial formulation of the problem.

We know from Theorem 2.5 that $G \subset H_{n0}^+$ appears as the subgroup presented by the relations $\xi \in Fix(u^{\otimes k})$, where $\xi = \sum_{ia} e_{ia}^{\otimes s}$. We also know from [8] that the category of

partitions for H_{n0}^+ is the above category $\overline{D}_\infty$. Now since we have $\xi = T_{1_s}$, where 1_s is the 1-block partition having s legs, we conclude that the category of partitions for G is:
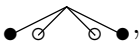
$$D'_s = <\overline{D}_\infty, 1_s>$$

Summarizing, we have to prove that we have $D_s = D'_s$. Since the inclusion $D'_s \subset D_s$ is clear from definitions, we are left with proving the inclusion $D_s \subset D'_s$.

Step 2. We construct some useful elements in $D'_s = <\overline{D}_\infty, 1_s>$.

Consider the 1-block partition $1_s \in NC(0, s)$, with all legs colored black. By capping it at right, from below, with the partition $\vee$, we obtain a 1-block partition in $NC(1, s-1)$, with the upper leg white, and all the lower legs black (in the example $s = 5$):
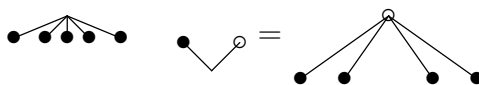
We can repeat this process, and for any $x \in \{1, \ldots, s\}$ we obtain a partition in $NC(x, s-x)$, having all upper legs white, and all the lower legs black. We call these partitions "forks".

Consider now once again the 1-block partition $1_s \in NC(0, s)$, with all legs colored black. By capping it from below in the center with a 'colour inverted' fork in $NC(s-2, 2)$ (so for $s = 5$ it is $\vee\vee$) we obtain the partition $\wedge$, which we call a "comb".

Step 3. We prove first that we have $D_\infty \subset D'_s$.

Since D_∞ is generated by its blocks, it is enough to show that any 1-block partition in D_∞ belongs to D'_s. Also, by Frobenius duality, which diagramatically corresponds to rotating the legs, we can restrict attention to the partitions having no upper legs. So, what we want to prove is that any 1-block partition $\pi \in D_\infty(0, k)$ is in D'_s.

Now recall the definition of D_∞: these are the partitions such that in each block, the signed number of black legs equals the signed number of white legs. Now since our partition π has just one block, and no upper legs, the condition $\pi \in D_\infty(0, k)$ simply tells us that the number of black legs should equal the number of white legs.

We proceed by recurrence on the number of legs (note that if $k = 2$ then $\pi \in \overline{D}_\infty$, so we can start from $k = 4$). First, by using a rotation, we can assume that the last 2 legs at right have different colors, say bw. Depending now on the color of the leg preceding these 2 last legs, we have two cases:

(1) Case $\pi = Rbbw$, with R word in b, w. In this case we put at right of $\pi' = Rb$, supposed by recurrence to be in D'_s, the colour inverted comb $\wedge$, which is as well in D'_s by Step 2, and we cap in the middle with $\vee$ from below. We obtain π, and we are done.

(2) Case $\pi = Rwbw$, with R word in b, w. In this case we put at right of $\pi' = Rw$, supposed by recurrence to be in D'_s, the partition $\wedge$, which is in $\overline{D}_\infty \subset D'_s$, and we cap in the middle with $\vee$ from below. We obtain once again π, and we are done.

Step 4. We prove now that we have indeed $D_s \subset D'_s$.

According to the result in Step 3, it suffices to show that we have $D_s \subset < D_\infty, 1_s >$. In order to do this, we proceed as in Step 3: since D_s is generated by its blocks, and by Frobenius duality, it is enough to check that any 1-block partition $\pi \in D_\infty(0,k)$ is in $< D_\infty, 1_s >$. That is, we have to show that any 1-block partition $\pi \in D_\infty(0,k)$ whose number of black legs equals the number of white legs modulo s is in $< D_\infty, 1_s >$.

Let then $\pi \in D_s$ be a 1-block element, say in $NC(2p + ks)$, where $p, k \in \mathbb{N}$ and p denotes the number of black legs in π. It suffices to observe that it can be always realised as a capping of a 1-block partition $\pi' \in D_\infty \cap NC(2p + 2ks)$, given by π followed by ks black legs, by k copies of our partition 1_s with all black legs from below and on the right. A simple example is given below:
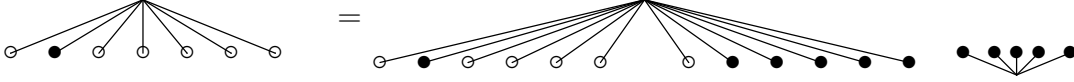
Step 5. It remains to prove that the maps T_π are linearly independent.

The linear independence is a well-known issue, and can be solved by using a standard trick, namely a positivity argument involving a trace, going back to Jones' paper [25]. The idea here is to solve first the problem in the case $k = l$, where the span of the diagrams has a natural structure of C^*-algebra, and then to use Frobenius duality and the natural embeddings $D_s(k,l) \to D_s(k, l + s)$, in order to get the result for any k, l.

However, in our case we can simply use some previously known results, as follows.

Since we have $D_s(k,l) \subset D_1(k,l)$, our problem is actually about the diagrams in D_1, namely the partitions in NC, bicolored, modulo the block-interchanging of colors.

So, let us recall from [8] that if $S_{n0}^+ \subset H_{n0}^+$ denotes the quantum group presented by the relations making all the standard generators $u_{ia,jb}$ projections, then we have $Hom(u^{\otimes k}, u^{\otimes l}) = span(T_\pi | \pi \in D_1(k,l))$. We refer as well to [8] for the precise relation between S_{n0}^+ and the quantum permutation group S_n^+, constructed by Wang in [34].

Let us also recall from [8] that we have an isomorphism $S_{n0}^+ \simeq H_n^+$, where H_n^+ is the hyperoctahedral quantum group constructed in [3], taken with its sudoku representation.

With these results in hand, the linear independence is clear: indeed, the dimension of $Hom(u^{\otimes k}, u^{\otimes l})$ is known from [3], and since a basic diagram count, performed in [8], shows that this is exactly the number of diagrams in $D_1(k,l)$, this gives the result. $\square$

As a first comment, there should be as well a third proof for the linear independence, just by showing that the Gram determinant of the linear maps T_π is nonzero. Indeed, two key "Temperley-Lieb" determinants were computed in [22], [30], and, according to the results in [4], these computations can be usually extended to objects of type D_1.

Note however that the computation of the Gram determinant for D_s with $s \geq 3$ is probably quite a difficult task, because the size of the matrix, which will be shown to be a moment of a certain compound Poisson law, is a quite complicated combinatorial object. So, computing this Gram determinant is a question that we would like to raise here.

Finally, let us mention that these Gram determinant questions are of particular interest in relation with the Weingarten formula [19]. See [6] for some potential applications.

4. Poisson laws

In the remainder of this paper we present a concrete application of Theorem 3.5, namely the computation of the Kesten measure of the discrete quantum group $\Lambda = \widehat{G}$, where $G = G^+(\widehat{\mathbb{Z}_s^{*n}})$. This measure is by definition the spectral measure of the character of the fundamental representation, $\chi = \Sigma u_{ii}$, with respect to the Haar functional.

For the moment, let us start with some probabilistic preliminaries. It is known from [2], [5], [6], [8] that what we can expect to find as Kesten measures for our quantum groups should be some versions of the free Poisson law. So, in this section we will study in detail the free Poisson law, and its "compound" versions, first constructed in [24], [29].

Definition 4.1. *The Poisson law p_t and the free Poisson law π_t are the real probability measures appearing when performing a Poisson limit, respectively a free Poisson limit*

$$p_t = \lim_{n \to \infty} \left(\left(1 - \frac{1}{n} \right) \delta_0 + \frac{1}{n} \delta_t \right)^{*n} \qquad \pi_t = \lim_{n \to \infty} \left(\left(1 - \frac{1}{n} \right) \delta_0 + \frac{1}{n} \delta_t \right)^{\boxplus n}$$

where $*$ is the usual convolution of measures, and $\boxplus$ is Voiculescu's free convolution [31].

A well-known computation shows that at $t = 1$ the Poisson law is $p_1 = \frac{1}{e} \sum_{k=1}^{\infty} \frac{\delta_k}{k!}$, and that the free Poisson law is $\pi_1 = \frac{1}{2\pi} \sqrt{1 - 4x^{-1}} \, dx$. Note that π_1 is the same as the Marchenko-Pastur law, discovered in a slightly different context in [26]. See [27].

More generally, let ρ be a compactly supported positive measure on $\mathbb{R}$. By replacing in the above formulae δ_t by ρ, we obtain measures called compound Poisson/free Poisson laws.

Definition 4.2. *Associated to any compactly supported positive measure ρ on $\mathbb{R}$ are the probability measures*

$$p_\rho = \lim_{n \to \infty} \left(\left(1 - \frac{c}{n} \right) \delta_0 + \frac{1}{n} \rho \right)^{*n} \qquad \pi_\rho = \lim_{n \to \infty} \left(\left(1 - \frac{c}{n} \right) \delta_0 + \frac{1}{n} \rho \right)^{\boxplus n}$$

where $c = mass(\rho)$, called compound Poisson and compound free Poisson laws.

In what follows we will be interested in the case where ρ is discrete, as is for instance the case for $\rho = \delta_t$ with $t > 0$, which produces the Poisson and free Poisson laws.

We recall that the usual convolution operation $*$ is linearized by $\log F$, where F is the Fourier transform. There are several ways of defining and normalizing F, and for the purposes of this paper, we will take as definition $F_{\delta_z}(y) = e^{-iyz}$, on the Dirac masses.

We recall also from Voiculescu [31] that the free convolution operation $\boxplus$ is linearized by the R-transform, constructed as follows: first, we let $f(y) = 1 + m_1 y + m_2 y^2 + \ldots$ be the moment generating function of our measure, so that $G(\xi) = \xi^{-1} f(\xi^{-1})$ is the Cauchy transform; then we set $R(y) = K(y) - y^{-1}$, where $K(y)$ is such that $G(K(y)) = y$.

The following result allows one to detect compound Poisson/free Poisson laws.

Lemma 4.3. *For $\rho = \sum_{i=1}^{s} c_i \delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have*

$$F_{p_\rho}(y) = \exp\left(\sum_{i=1}^{s} c_i(e^{-iyz_i} - 1)\right) \quad R_{\pi_\rho}(y) = \sum_{i=1}^{s} \frac{c_i z_i}{1 - y z_i}$$

where F, R denote respectively the Fourier transform, and Voiculescu's R-transform.

Proof. Let μ_n be the measure appearing in Definition 4.2, under the convolution signs. In the classical case, we have the following well-known computation:

$$F_{\mu_n}(y) = \left(1 - \frac{c}{n}\right) + \frac{1}{n}\sum_{i=1}^{s} c_i e^{-iyz_i} \implies F_{\mu_n^{*n}}(y) = \left(\left(1 - \frac{c}{n}\right) + \frac{1}{n}\sum_{i=1}^{s} c_i e^{-iyz_i}\right)^n$$

$$\implies F_{p_\rho}(y) = \exp\left(\sum_{i=1}^{s} c_i(e^{-iyz_i} - 1)\right)$$

In the free case now, we use a similar method. First, we have:

$$f_{\mu_n}(y) = \left(1 - \frac{c}{n}\right) + \frac{1}{n}\sum_{i=1}^{s} \frac{c_i}{1 - z_i y} \implies G_{\mu_n}(\xi) = \left(1 - \frac{c}{n}\right)\frac{1}{\xi} + \frac{1}{n}\sum_{i=1}^{s} \frac{c_i}{\xi - z_i}$$

$$\implies y = \left(1 - \frac{c}{n}\right)\frac{1}{K_{\mu_n}(y)} + \frac{1}{n}\sum_{i=1}^{s} \frac{c_i}{K_{\mu_n}(y) - z_i}$$

Now since $K_{\mu_n}(y) = y^{-1} + R_{\mu_n}(y) = y^{-1} + R/n$, where $R = R_{\mu_n^{\boxplus n}}(y)$, we get:

$$y = \left(1 - \frac{c}{n}\right)\frac{1}{y^{-1} + R/n} + \frac{1}{n}\sum_{i=1}^{s} \frac{c_i}{y^{-1} + R/n - z_i}$$

$$\implies 1 = \left(1 - \frac{c}{n}\right)\frac{1}{1 + yR/n} + \frac{1}{n}\sum_{i=1}^{s} \frac{c_i}{1 + yR/n - yz_i}$$

Now multiplying by n, rearranging the terms, and letting $n \to \infty$, we get:

$$\frac{c + yR}{1 + yR/n} = \sum_{i=1}^{s} \frac{c_i}{1 + yR/n - yz_i} \implies c + yR_{\pi_\rho}(y) = \sum_{i=1}^{s} \frac{c_i}{1 - yz_i}$$

$$\implies R_{\pi_\rho}(y) = \sum_{i=1}^{s} \frac{c_i z_i}{1 - yz_i}$$

This finishes the proof in the free case, and we are done. $\qquad\square$

We have as well the following result, which provides an alternative to Definition 4.2.

Theorem 4.4. *For $\rho = \sum_{i=1}^{s} c_i \delta_{z_i}$ with $c_i > 0$ and $z_i \in \mathbb{R}$, we have*

$$p_\rho / \pi_\rho = \mathrm{law}\left(\sum_{i=1}^{s} z_i \alpha_i\right)$$

where the variables α_i are Poisson/free Poisson(c_i), independent/free.

Proof. Observe first that the result holds for $\rho = \delta_t$, with $t > 0$. Also, it was shown in [2] that an analogue of this result holds for $\rho =$ uniform measure on the s-th roots of unity (note however that the latter measure is not supported on $\mathbb{R}$).

In the general case, let α be the sum of Poisson/free Poisson variables in the statement. We will show that the Fourier/R-transform of α is given by the formulae in Lemma 4.3.

Indeed, by using some well-known Fourier transform formulae, we have:

$$F_{\alpha_i}(y) = \exp(c_i(e^{-iy} - 1)) \quad \Longrightarrow \quad F_{z_i\alpha_i}(y) = \exp(c_i(e^{-iyz_i} - 1))$$

$$\Longrightarrow \quad F_\alpha(y) = \exp\left(\sum_{i=1}^{s} c_i(e^{-iyz_i} - 1)\right)$$

Also, by using some well-known R-transform formulae, we have:

$$R_{\alpha_i}(y) = \frac{c_i}{1-y} \quad \Longrightarrow \quad R_{z_i\alpha_i}(y) = \frac{c_i z_i}{1 - y z_i}$$

$$\Longrightarrow \quad R_\alpha(y) = \sum_{i=1}^{s} \frac{c_i z_i}{1 - y z_i}$$

Thus we have indeed the same formulae as those in Lemma 4.3, and we are done. $\square$

Remark 4.5. *It is tempting to replace in the above results compactly supported positive measures on $\mathbb{R}$ by their counterparts supported on $\mathbb{C}$. It is easy to see that the classical parts pass through without any modifications; it is the free case that offers a challenge. The free convolution is defined in terms of the distribution of the sum of two free noncommutative random variables. If the variables in question are selfadjoint (which corresponds to the measures being supported on $\mathbb{R}$), then so is their sum. There is however no reason why the sum of normal operators should be normal, so, although the formal computations do not change, it is not clear how they should be interpreted. We thank the referee for pointing this out.*

5. SPECTRAL MEASURES

Let us go back now to our quantum group problematics. We recall from the beginning of section 3 that most of the representation theory invariants of a quantum group (G, u) are known to be encoded in the structure of the linear spaces $Hom(u^{\otimes k}, u^{\otimes l})$.

Here u denotes the fundamental corepresentation of $C(G)$, and we assume that we have $u = \bar{u}$ as corepresentations. Note that this is true for the quantum group H_{n0}^{+} from [8], [15], hence is true as well for the quantum groups investigated in this paper.

The main representation theory invariant is the dimension of the above spaces. By Woronowicz's general results in [35] we have $\dim(Hom(u^{\otimes k}, u^{\otimes l})) = h(\chi^{k+l})$, where $\chi = \Sigma u_{ii}$ is the character of u, and where $h : C(G) \to \mathbb{C}$ is the Haar functional.

So, the main problem is to compute the moments of χ. For several reasons, it is actually more convenient to specify directly the law of χ.

Definition 5.1. *The spectral measure of a compact quantum group (G, u) with $u = \bar{u}$ is the real probability measure μ given by $\int \varphi(x) \, d\mu(x) = h(\varphi(\chi))$, where $\chi = \Sigma u_{ii}$ is the character of the fundamental representation, and $h : C(G) \to \mathbb{C}$ is the Haar functional.*

For a number of other interpretations of this key measure, and a number of motivations for its explicit computation, we refer for instance to [1], [5].

We are now in position to state and prove our main result. Let $\rho \to \underline{\rho}$ the projection map from complex to real measures, given on Dirac masses by $\delta_z \to \delta_{Re(z)}$.

Theorem 5.2. *The spectral measure of $G^+(\widehat{\mathbb{Z}_s^{*n}})$ with $5 \le s < \infty$ and $n \ge 2$ is given by*

$$law(\chi/2) = \pi_{\underline{\varepsilon}/2}$$

where ε is the uniform measure on the s-roots of unity.

Proof. According to the general theory developed by Woronowicz in [35], the moments of the character $\chi = \Sigma u_{ii}$ are given by:

$$\int \chi^r = \dim(Fix(u^{\otimes r}))$$

So, we have to count the diagrams in Theorem 3.5. If we denote by κ_r the number of ways of coloring an r-block, then the number of diagrams in $D_s(0, r)$ is:

$$m_r = \sum_{\pi \in NC(r)} \prod_{b \in \pi} \kappa_{\#b}$$

According now to Speicher's results in [28] (see Proposition 11.4 (3) in [27]), we can conclude that the free cumulants of the measure under consideration are the above numbers κ_r.

We will write $p = q(s)$ to indicate that $p = q$ modulo s. Since for coloring a r-block, as in Definition 3.1, what we have to do is just to pick two numbers p, q satisfying $p + q = r$ and $p = q(s)$, then choose p legs of our block and color them black, and color the remaining q legs white, remembering that the colors can be also exchanged, the numbers κ_r are given by:

$$\kappa_r = \frac{1}{2} \sum_{p+q=r, \ p=q(s)} \binom{r}{p}$$

Consider now the root of unity $w = e^{2\pi i/s}$. Our claim is that we have:

$$\kappa_r = \frac{1}{2s} \sum_{k=1}^{s} (w^k + w^{-k})^r$$

Indeed, the equality between the above two quantities can be checked as follows:

$$\frac{1}{s} \sum_{k=1}^{s} (w^k + w^{-k})^r = \frac{1}{s} \sum_{k=1}^{s} \sum_{p=0}^{r} \binom{r}{p} (w^k)^p (w^{-k})^{r-p}$$

$$= \sum_{p=0}^{r} \binom{r}{p} \left(\frac{1}{s} \sum_{k=1}^{s} (w^{2p-r})^k \right)$$

$$= \sum_{p+q=r,\, p=q(s)} \binom{r}{p}$$

We use now the fact, once again originally proved in Speicher's paper [28], that the free cumulants are the coefficients of the R-transform (actually in [27] the R-transform is first introduced as the power series with the coefficients given by the free cumulants and only later related to the functional equations). So, by using the above formula, we conclude that the R-transform $R_\chi(y) = \kappa_1 + \kappa_2 y + \kappa_3 y^2 + \ldots$ of our measure is given by:

$$R_\chi(y) = \frac{1}{2s} \sum_{k=1}^{s} \frac{w^k + w^{-k}}{1 - (w^k + w^{-k})y}$$

Now by using the well-known dilation formula $R_{a\chi}(y) = aR_\chi(ay)$, we obtain:

$$R_{\chi/2}(y) = \frac{1}{4s} \sum_{k=1}^{s} \frac{w^k + w^{-k}}{1 - (w^k + w^{-k})y/2}$$

Consider now the uniform measure on the s-th roots of unity, $\varepsilon = \frac{1}{s} \sum_{k=1}^{s} \delta_{w^k}$. Its image via the projection map from complex to real measures, $\delta_z \to \delta_{Re(z)}$, is given by:

$$\underline{\varepsilon} = \frac{1}{s} \sum_{k=1}^{s} \delta_{(w^k+w^{-k})/2}$$

Thus the real measure appearing in the statement is:

$$\underline{\varepsilon}/2 = \frac{1}{2s} \sum_{k=1}^{s} \delta_{(w^k+w^{-k})/2}$$

By using now Lemma 4.3, we have:

$$R_{\pi_{\underline{\varepsilon}}/2}(y) = \frac{1}{4s} \sum_{k=1}^{s} \frac{w^k + w^{-k}}{1 - (w^k + w^{-k})y/2}$$

Thus we have $R_{\chi/2}(y) = R_{\pi_{\underline{\varepsilon}}/2}(y)$, and this gives the conclusion in the statement. $\qquad\square$

We can reformulate the above statement in the following way.

Proposition 5.3. *For $\Gamma = \mathbb{Z}_s^{*n}$ with $5 \leq s < \infty$ and $n \geq 2$, the main character of the quantum isometry group $G^+(\widehat{\Gamma})$ decomposes as*

$$\chi = \sum_{k=1}^{s} 2\cos\left(\frac{2k\pi}{s}\right)\alpha_k$$

where $\alpha_1, \ldots, \alpha_s$ are free Poisson variables of parameter $1/(2s)$, free.

Proof. This follows from Theorem 4.4 and from Theorem 5.2. Indeed, if $\alpha_1, \ldots, \alpha_k$ are free Poisson variables as in the statement, we have:

$$law(\chi/2) = \pi_{\underline{\varepsilon}/2} = \pi_{\left\{\frac{1}{2s}\sum_{k=1}^{s}\delta_{(w^k+w^{-k})/2}\right\}} = law\left(\sum_{k=1}^{s}\frac{w^k + w^{-k}}{2}\alpha_k\right)$$

Thus χ has the same law as twice the variable on the right, and we are done. $\qquad\square$

As a first observation, the above results remind those previously found in [2], [5], where the free Poisson law, and its compound versions, also play a central role.

More precisely, for the quantum groups $H_n^s, H_n^{s+}, H_n^{(s)}$ considered in [2], [5], the corresponding asymptotic spectral measures are p_ε, π_ε, $\tilde{p}_\varepsilon$ respectively, where $\varepsilon \to \tilde{\varepsilon}$ is the "squeezing" operation $\delta_z \to \delta_{|z|}$, from the complex to the real probability measures.

So, the quantum group $G^+(\widehat{\mathbb{Z}_s^{*n}})$, while being not exactly "easy" in the sense of [2], [5], seems to belong to the same circle of ideas as the general "hyperoctahedral series" of easy quantum groups, whose existence was conjectured in [5]. This suggests that the hyperoctahedral series, or at least a big part of it, might appear via some simple algebraic manipulations from the quantum isometry groups of type $G^+(\widehat{\Gamma})$, with Γ of the form $\mathbb{Z}_{s_1} * \ldots * \mathbb{Z}_{s_n}$, or perhaps a bit more general. We do not know if it is so.

6. SPECIAL CASES

In this section we discuss the special cases $s = 2, 3, 4, \infty$. We begin with some results at $s = 2, 3, 4$, refining and generalizing those found in [15], at $n = 1$.

Proposition 6.1. $G^+(\widehat{\mathbb{Z}_2^{*n}})$ *is the hyperoctahedral quantum group H_n^+ from [3].*

Proof. Let $g_1, \ldots, g_n$ be the standard generators of $\mathbb{Z}_2^{*n}$. We write the coaction of the quantum isometry group in the statement as $\alpha(g_i) = \Sigma_j g_j \otimes u_{ji}$. Then, according to the general results in section 1, the relations between the generators u_{ij} are as follows:

    (1) First, we have the relations $u_{ij} = u_{ij}^*$ (coming from $g_i = g_i^*$).
    (2) Second, we have $u_{ji}u_{ki} = 0$ for $j \neq k$, and $\Sigma_j u_{ji}^2 = 1$ (coming from $g_i^2 = 1$).

From (2) we get $u_{ki} = \sum_j u_{ji}^2 u_{ki} = u_{ki}^3$, and together with (1), this shows that the elements u_{ij} are partial isometries. With this observation in hand, the above relations are simply the "cubic" relations in [3], for partial isometries, and this gives the result. □

We refer to [3] for a full discussion of the representation theory invariants of H_n^+, and to [2], [6] for some technical versions and generalizations of these computations.

Proposition 6.2. *Theorems 2.5, 3.5, 5.2 hold as well at $s = 3$, and give the presentation, diagrams, and spectral measure for the quantum group $G^+(\widehat{\mathbb{Z}_3^{*n}})$.*

Proof. This is clear from a careful examination of the proofs of the above results: the beginning of the proof of Theorem 2.5 holds as well at $s = 3$, because here we do not have length 2 relations either, and the rest of the proofs simply hold, unchanged. □

At $s = 4$ now, the situation is quite complicated, and we currently do not have general results. Let us just state and prove the following result, slightly improving a result in [15].

Proposition 6.3. *We have a $*$-algebra isomorphism $C(G^+(\widehat{\mathbb{Z}_4})) \simeq C^*(D_\infty \times \mathbb{Z}_2)$.*

Proof. We use the notations in [15]. According to the results there, the fundamental corepresentation of our quantum isometry group must be of the following type:

$$u = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}$$

The generators must satisfy the defining relations for SU_2^{-1}, namely:

$$AA^* + BB^* = 1, \quad AB + BA = 0, \quad AB^* + B^*A = 0$$

In addition, the operator $T = A^2 + B^2$ must satisfy the following relations:

$$T = T^*, \quad T^2 = 1, \quad TA = A^*, \quad TB = B^*$$

See [15]. The point now is that, since T commutes with A, B, A^*, B^*, we can write:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus the quantum isometry algebra in the statement decomposes naturally as a direct sum of two $*$-algebras, and by passing to the generators $A + B, A - B$, we see that each of these $*$-algebras is isomorphic to the group algebra $C^*(D_\infty)$. This gives the result. □

The above result suggests that $G^+(\widehat{\mathbb{Z}_4})$ might be a twist of the dual of $D_\infty \times \mathbb{Z}_2$. This is probably true, but unfortunately we do not see an analogue of this result, for $n > 1$.

Let us discuss now the case $s = \infty$. We have two statements here.

Proposition 6.4. *$G^+(\widehat{F_n})$ is the quantum group H_{n0}^+ from [8].*

Proof. This result, and the precise relation between H_{n0}^+ and H_n^+, are explained in [8]. □

Our second statement concerns the "representation-theoretic" limit, with $s \to \infty$, of the quantum groups $G^+(\widehat{\mathbb{Z}_s^{*n}})$. Quite surprisingly, this limiting quantum group is not the quantum group H_{n0}^+ from [8]. Its main properties can be summarized as follows.

Theorem 6.5. *Let K_n be the quantum subgroup of H_{n0}^+ presented by the relations making all the standard generators u_{ij} normal. Then:*

(1) *$Hom(u^{\otimes k}, u^{\otimes l}) = span(T_\pi | \pi \in D_\infty(k, l))$, for any $k, l \in \mathbb{N}$.*
(2) *$law(\chi/2) = \pi_{\underline{\varepsilon}/2}$, where ε is the uniform measure on the unit circle.*

Proof. We recall from section 3 above that we have inclusions $\overline{D}_\infty(k, l) \subset D_\infty(k, l)$, for any k, l. The idea will be to find first a categorical generation result of type $D_\infty = <\overline{D}_\infty, \pi >$, and then to find the quantum group and probabilistic interpretations of this result.

So, let $\pi \in D_\infty(0, 4)$ be the 1-block partition colored *bbww*. Our claim is that we have $D_\infty = <\overline{D}_\infty, \pi >$. Since the inclusion $\supset$ is clear, we just have to prove the inclusion $\subset$.

In order to do this, let us look at the category $D = <\overline{D}_\infty, \pi >$. By using Frobenius duality, i.e. by using rotations, we can deduce from $\pi \in D$ the fact that D contains all the 1-block partitions having 0 upper legs and 4 lower legs, with the 4 lower legs colored half-black, half-white. So, in other words, we have $D(0, 4) = D_\infty(0, 4)$.

Now by using once again Frobenius duality, i.e. by using rotations, we deduce that we have $D(2, 2) = D_\infty(2, 2)$. In particular, if $\sigma \in D_\infty(2, 2)$ denotes the 1-block partition with the upper legs colored *bw* and the lower legs colored *wb*, then $\sigma \in D(2, 2)$.

With this observation in hand, we can finish the proof of the above claim: indeed, by using σ we can exchange the position of two consecutive black and white colors in any partition in D, and it is easy to see that this operation allows one to construct any element in D_∞ starting from the elements of $\overline{D}_\infty$. So, we have $D_\infty = <\overline{D}_\infty, \pi >$ as claimed.

(1) Let us try to compute the quantum group $K_n' \subset H_{n0}^+$ having as Hom spaces the linear spaces $span(T_\pi | \pi \in D_\infty(k, l))$. Since we have $D_\infty = <\overline{D}_\infty, \pi >$, this quantum group $K_n' \subset H_{n0}^+$ is simply the one presented by the relations coming from $T_\pi \in Fix(u^{\otimes 4})$. Now, according to the definitions in section 3 above, we have:

$$T_\pi = \sum_{ia} e_{ia} \otimes e_{ia} \otimes e_{i\bar{a}} \otimes e_{i\bar{a}}$$

By using the relations in Proposition 2.3, we see that this vector is fixed by $u^{\otimes 4}$ if and only if the following condition is satisfied, where $p_{ia,jb} = u_{ia,jb} u_{ia,jb}^*$, $q_{ia,jb} = u_{ia,jb}^* u_{ia,jb}$:

$$q_{ia,jb} p_{ia,kc} q_{ia,ld} = \begin{cases} q_{ia,jb} & \text{if } (jb) = (kc) = (ld) \\ 0 & \text{otherwise} \end{cases}$$

This condition is easily seen to be equivalent to the "normality" relations $p_{ia,jb} = q_{ia,jb}$, for any i, j, a, b, so the quantum group K_n' that we are currently computing is nothing but the quantum group K_n appearing in the statement, and we are done.

(2) This follows from (1), as in the proof of Theorem 5.2, after of course performing some obvious modifications in the preliminary material in sections 4 and 5. □

One question arising from the above result is that of finding a suitable geometric interpretation of K_n. Since this is a subgroup of $H_{n0}^+ = G^+(\widehat{F}_n)$, we can expect K_n to consist of the "quantum isometries" of $\widehat{F}_n$ preserving not only the length, but also some "extra structure". A careful study here leads to the following informal answer: "under the action of K_n a word in $F_n = \mathbb{Z} * \cdots * \mathbb{Z}$ should get sent to the words of the same length, with the switches between the generators coming from various copies of $\mathbb{Z}$ appearing at the same places". This will be explained in detail in a future paper from the present series.

7. Concluding remarks

We have seen in this paper that the representation theory invariants of $G^+(\widehat{\Gamma})$ can be explicitly computed in the case $\Gamma = \mathbb{Z}_s^{*n}$, by using diagrammatic techniques. The answer that we obtain – an explicit formula for the spectral measure, as a compound free Poisson law $\pi_{\underline{\varepsilon}}$ – appears to be quite interesting from the point of view of free probability.

There are several questions arising from the present work. The main one is probably the computation of the invariants for general free products of cyclic groups, $\Gamma = \mathbb{Z}_{s_1} * \ldots * \mathbb{Z}_{s_n}$. Here we can definitely expect to have some very interesting, new combinatorics, ultimately coming from the arithmetic properties of the sequence of indices $s_1, \ldots, s_n$.

Understanding this next-step combinatorics looks like an important task towards the general understanding of the quantum isometry groups introduced in [23], and of the easy quantum groups introduced in [9]. We intend to come back to this fundamental question, at least with partial results, in some future work.

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T.B.: Department of Mathematics, Cergy-Pontoise University, 95000 Cergy-Pontoise, France. `teodor.banica@u-cergy.fr`

A.S.: Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland. `a.skalski@impan.pl`