Quantum projective manifolds

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

2010 *Mathematics Subject Classification.* 81R50

Key words and phrases. Quantum manifold, Projective manifold

Abstract. This is an introduction to quantum projective manifolds. We first review the general theory of quantum algebraic manifolds, taken in an operator algebra sense, and under a number of supplementary axioms, including the fact that the corresponding integration functional must be a trace. Then we discuss in detail what happens in the projective manifold case, with this assumption bringing many simplifications. We then go for an even more detailed study of the quantum projective manifolds, by working out a number of relevant abstract algebra methods. Finally, we discuss a number of analytic aspects, on one hand in relation with the underlying differential geometry, and on the other hand, in relation with the notion of matrix models for our manifolds.

Preface

What is a quantum manifold? Good question, certainly requiring us to know both some mathematics, and some quantum physics, and unfortunately, with the quantum physics which is required being, for the moment, most likely beyond our reach.

There is of course a long story here, which is something quite subjective, but to put things squarely, in quantum physics the master theorem is the Standard Model for elementary particles, going back to the 1970s. This model has not changed much, despite 50 years of efforts, of both applied mathematicians, and theoretical physicists. The main challenge is to go beyond this model, and the one who will eventually do that, one day, will be most likely entitled to teach us what the quantum manifolds really are.

Nevermind. These things are obviously difficult, but life goes on, mathematics goes on too, with or without knowing what we're doing, and having more classes of quantum manifold wannabees constructed, and abstractly studied, can only be a good thing.

The present book is an introduction to the quantum projective manifolds, basically taken in an algebraic geometry sense, but with a bit of differential geometry flavor too, and with everything being taken in an operator algebra sense. In short, expect some tricky combination of axioms, inspired by some physics that we don't have yet, and then a lot of mathematics, making some sense or not, time will tell, based on these axioms.

The book is organized in four parts, as follows:

(1) We first review the general theory of quantum algebraic manifolds, taken in an operator algebra sense, and under a number of supplementary axioms, including the fact that the corresponding integration functional must be a trace.

(2) Then we discuss in detail what happens in the projective manifold case, with the projectivity assumption bringing many simplifications, somehow in analogy with the same simplification phenomenon, well-known to appear in the classical manifold case.

(3) We then go for an even more detailed study of the quantum projective manifolds, by working out a number of relevant abstract algebra methods, in analogy with the well-known results regarding the correspondence between classical manifolds and ideals.

(4) Finally, we discuss a number of analytic aspects, on one hand in relation with the underlying differential geometry, comprising a Laplace operator, and an integration functional, and on the other hand, in relation with matrix models for our manifolds.

Many thanks to my colleagues and collaborators, for substantial joint work on this, and on related topics. Thanks as well to my cats, for some help with the algebra.

Cergy, March 2025
Teo Banica

Contents

Part I

Quantum manifolds

Bang bang, he shot me down
Bang bang, I hit the ground
Bang bang, that awful sound
Bang bang, my baby shot me down

CHAPTER 1

Free spheres

1a. Free tori

Welcome to noncommutative geometry. Many speculatory things can be said here, so let us start our journey with the following definition, which is something rock-solid:

Definition 1.1. *The free torus $\mathbb{T}_N^+$ is the dual of the free group F_N,*

$$\mathbb{T}_N^+ = \widehat{F_N}$$

in analogy with the fact that the usual torus $\mathbb{T}_N = \mathbb{T}^N$ appears as

$$\mathbb{T}_N = \widehat{\mathbb{Z}^N}$$

with on the right the group $\mathbb{Z}^N$ being the free abelian group.

Before getting into details regarding all this, recall that $\mathbb{R}^N$ is as interesting as $\mathbb{C}^N$. So, let us formulate as well the real version of Definition 1.1, as follows:

Definition 1.2. *The free real torus, or free cube, T_N^+ is the dual*

$$T_N^+ = \widehat{L_N}$$

of the group $L_N = F_N / < g_i^2 = 1 >$, in analogy with the fact that the usual cube is

$$T_N = \widehat{\mathbb{Z}_2^N}$$

with on the right the group $\mathbb{Z}_2^N$ being the free real abelian group.

Here the "real" at the end stands for the fact that the generators must satisfy the real reflection condition $g^2 = 1$. As for the fact that "real torus = cube", as stated, this needs some thinking, and in the hope that, after such thinking, you will agree with me that there is indeed a standard torus inside $\mathbb{R}^N$, and that is the unit cube.

Summarizing, all this sounds good, we have a beginning of free geometry, both real and complex, worth developing, by knowing at least what the torus of each theory is. In practice now, at the level of details, in order to talk about $\mathbb{T}_N^+ = \widehat{F_N}$ and $T_N^+ = \widehat{L_N}$ we need an extension of the usual Pontrjagin duality theory for the abelian groups, and this is best done via operator algebras, and the related notion of compact quantum group.

1b. Quantum spaces

In view of the above, in order to fully understand what happens, let us start with operator algebras. You have probably already heard about infinite matrices, operators and operator algebras, from Heisenberg, Schrödinger, Dirac and others. As a starting point for this, we need a complex Hilbert space H, with the main example in mind being the space $H=L^2(\mathbb{R}^3)$ of the wave functions of the electron. So, let us formulate:

Definition 1.3. *A Hilbert space is a complex vector space H, given with a scalar product $<x,y>$, satisfying the following conditions:*

(1) *$<x,y>$ is linear in x, and antilinear in y.*
(2) *$\overline{<x,y>}=<y,x>$, for any x,y.*
(3) *$<x,x>>0$, for any $x\neq 0$.*
(4) *H is complete with respect to the norm $||x||=\sqrt{<x,x>}$.*

Moving ahead, we need to talk about operators. Again, you might have heard of these from Heisenberg, Schrödinger, Dirac and others, and with the theory being quite complicated to read and digest, because these operators, while fortunately self-adjoint, are unfortunately unbounded. However, for our purposes here, we will only need bounded operators. So, let us formulate, as a first theorem for our book:

Theorem 1.4. *The linear operators $T:H\to H$ which are bounded, meaning that*

$$||T||=\sup_{||x||\leq 1}||Tx||$$

is finite, form a complex algebra $B(H)$, having the following properties:

(1) *$B(H)$ is complete with respect to $||.||$, so we have a Banach algebra.*
(2) *$B(H)$ has an involution $T\to T^*$, given by $<Tx,y>=<x,T^*y>$.*

In addition, the norm and involution are related by the formula $||TT^*||=||T||^2$.

Proof. The fact that we have an algebra is clear, and the completness comes from the fact that, assuming that $\{T_n\}\subset B(H)$ is Cauchy, then $\{T_nx\}$ is Cauchy for any $x\in H$, so we can define the limit $T=\lim_{n\to\infty}T_n$ by setting:

$$Tx=\lim_{n\to\infty}T_nx$$

Regarding $T\to T^*$, this comes from the fact that $\varphi(x)=<Tx,y>$ being a linear form $\varphi:H\to\mathbb{C}$, we must have $\varphi(x)=<x,T^*y>$, for a certain vector $T^*y\in H$. Thus we have a well-defined involution $T\to T^*$, which stays inside $B(H)$, because:

$$\begin{aligned}||T|| &= \sup_{||x||=1}\sup_{||y||=1}<Tx,y>\\ &= \sup_{||y||=1}\sup_{||x||=1}<x,T^*y>\\ &= ||T^*||\end{aligned}$$

Regarding now the last assertion, observe first that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned}||T||^2 &= \sup_{||x||=1} |< Tx, Tx >| \\ &= \sup_{||x||=1} |< x, T^*Tx >| \\ &\leq ||T^*T||\end{aligned}$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, so we are done. □

Observe that when H comes with an orthonormal basis $\{e_i\}_{i\in I}$, the linear map $T \to M$ given by $M_{ij} =< Te_j, e_i >$ produces an embedding as follows:

$$B(H) \subset M_I(\mathbb{C})$$

Moreover, in this picture the operation $T \to T^*$ takes a very simple form, namely:

$$(M^*)_{ij} = \overline{M}_{ji}$$

Moving ahead, the conditions found in Theorem 1.4 suggest formulating:

Definition 1.5. *A C^*-algebra is a complex algebra A, having:*
1. *A norm $a \to ||a||$, making it a Banach algebra.*
2. *An involution $a \to a^*$, satisfying $||aa^*|| = ||a||^2$.*

As basic examples, we have $B(H)$ itself, as well as any norm closed $*$-subalgebra $A \subset B(H)$. It is possible to prove that any C^*-algebra appears in this way, but we will not need in what follows this deep result, called GNS theorem after Gelfand, Naimark, Segal. So, let us simply agree that, by definition, the C^*-algebras A are some sort of "generalized operator algebras", and their elements $a \in A$ can be thought of as being some kind of "generalized operators", on some Hilbert space which is not present.

In practice, this vague idea is all that we need. Indeed, by taking some inspiration from linear algebra, we can emulate spectral theory in our setting, as follows:

Theorem 1.6. *Given $a \in A$, define its spectrum as being the set*

$$\sigma(a) = \left\{\lambda \in \mathbb{C}\Big| a - \lambda \notin A^{-1}\right\}$$

and its spectral radius $\rho(a)$ as the radius of the smallest centered disk containing $\sigma(a)$.
1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

Proof. Our first claim is that for any polynomial $f\in\mathbb C[X]$, and more generally for any rational function $f\in\mathbb C(X)$ having poles outside $\sigma(a)$, we have:

$$\sigma(f(a))=f(\sigma(a))$$

This indeed something well-known for the usual matrices. In the general case, assume first that we have a polynomial, $f\in\mathbb C[X]$. If we pick an arbitrary number $\lambda\in\mathbb C$, and write $f(X)-\lambda=c(X-r_1)\ldots(X-r_k)$, we have then, as desired:

$$\begin{aligned}\lambda\notin\sigma(f(a))&\iff f(a)-\lambda\in A^{-1}\\&\iff c(a-r_1)\ldots(a-r_k)\in A^{-1}\\&\iff a-r_1,\ldots,a-r_k\in A^{-1}\\&\iff r_1,\ldots,r_k\notin\sigma(a)\\&\iff \lambda\notin f(\sigma(a))\end{aligned}$$

Assume now that we are in the general case, $f\in\mathbb C(X)$. We pick $\lambda\in\mathbb C$, we write $f=P/Q$, and we set $F=P-\lambda Q$. By using the above finding, we obtain, as desired:

$$\begin{aligned}\lambda\in\sigma(f(a))&\iff F(a)\notin A^{-1}\\&\iff 0\in\sigma(F(a))\\&\iff 0\in F(\sigma(a))\\&\iff \exists\mu\in\sigma(a),F(\mu)=0\\&\iff \lambda\in f(\sigma(a))\end{aligned}$$

Regarding now the assertions in the statement, these basically follows from this:

(1) This comes from the following formula, valid when $||a||<1$:

$$\frac{1}{1-a}=1+a+a^2+\ldots$$

(2) Assuming $a^*=a^{-1}$, if we denote by D the unit disk, we have, by using (1):

$$||a||=1\implies\sigma(a)\subset D$$

$$||a^{-1}||=1\implies\sigma(a^{-1})\subset D$$

On the other hand, by using the rational function $f(z)=z^{-1}$, we have:

$$\sigma(a^{-1})\subset D\implies\sigma(a)\subset D^{-1}$$

Now by putting everything together we obtain, as desired:

$$\sigma(a)\subset D\cap D^{-1}=\mathbb T$$

(3) This follows from (2), by using the rational function $f(z)=(z+it)/(z-it)$. Indeed, for $t>>0$ we have the following computation:

$$\left(\frac{a+it}{a-it}\right)^*=\frac{a-it}{a+it}=\left(\frac{a+it}{a-it}\right)^{-1}$$

Thus the element $f(a)$ is a unitary, and by using (2) its spectrum is contained in $\mathbb{T}$. We conclude from this that we have, as desired, $f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$.

(4) We already know that we have $\rho(a) \leq ||a||$, for any $a \in A$. For the reverse inequality, when a is normal, we fix a number $\rho > \rho(a)$. We have then:

$$\begin{aligned}\int_{|z|=\rho} \frac{z^n}{z-a}\,dz &= \int_{|z|=\rho} \sum_{k=0}^{\infty} z^{n-k-1} a^k\,dz\\ &= \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz\right) a^k\\ &= a^{n-1}\end{aligned}$$

By applying the norm and taking n-th roots we obtain from this formula:

$$\rho \geq \lim_{n\to\infty} ||a^n||^{1/n}$$

When $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of type $n = 2^k$, by using the C^*-algebra condition $||aa^*|| = ||a||^2$, and by taking n-th roots we get, as desired:

$$\rho(a) \geq ||a||$$

In the general normal case now, $aa^* = a^*a$, we have $a^n(a^n)^* = (aa^*)^n$, and by using this, along with the result for self-adjoints, applied to aa^*, we obtain:

$$\begin{aligned}\rho(a) &\geq \lim_{n\to\infty} ||a^n||^{1/n}\\ &= \sqrt{\lim_{n\to\infty} ||a^n(a^n)^*||^{1/n}}\\ &= \sqrt{\lim_{n\to\infty} ||(aa^*)^n||^{1/n}}\\ &= \sqrt{\rho(aa^*)}\\ &= \sqrt{||a||^2}\\ &= ||a||\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Generally speaking, Theorem 1.6 is all that you need to know, for doing further operator algebras, only military grade weapons there. As a main application, we have:

THEOREM 1.7 (Gelfand). *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a commutative C^*-algebra, with structure as follows:*

1. *The norm is the usual sup norm, $||f|| = \sup_{x\in X} |f(x)|$.*
2. *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

Conversely, any commutative C^*-algebra is of the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.

PROOF. Given a commutative C^*-algebra A, we can define indeed X to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras:

$$ev : A \to C(X)$$

We first prove that ev is involutive. We use the following formula:

$$a = \frac{a + a^*}{2} - i \cdot \frac{i(a - a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$. So, claim proved. Also, since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The Gelfand theorem suggests formulating the following definition:

DEFINITION 1.8. *Given a C^*-algebra A, not necessarily commutative, we write*

$$A = C(X)$$

and call the abstract object X a "compact quantum space".

This might look quite revolutionary, but in practice, this definition changes nothing to what we have been doing so far, namely studying the C^*-algebras. So, we will keep studying the C^*-algebras, but by using the above fancy quantum space terminology. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

Let us discuss now the other basic result regarding the C^*-algebras, namely the GNS representation theorem. We will need some more spectral theory, as follows:

PROPOSITION 1.9. *For a normal element $a \in A$, the following are equivalent:*

(1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
(2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
(3) *$a = cc^*$, for some $c \in A$.*

PROOF. This is something very standard, as follows:

(1) $\implies$ (2) Since our element a is normal the algebra $< a >$ that is generates is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can set $c = b$.

(3) $\implies$ (1) We can proceed here by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$. But this contradicts the elementary fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, which can be checked by explicit inversion. □

Here is now the GNS representation theorem, along with the idea of the proof:

THEOREM 1.10 (GNS theorem). *Let A be a C^*-algebra.*

(1) *A appears as a closed $*$-subalgebra $A \subset B(H)$, for some Hilbert space H.*
(2) *When A is separable (usually the case), H can be chosen to be separable.*
(3) *When A is finite dimensional, H can be chosen to be finite dimensional.*

PROOF. Let us first discuss the commutative case, $A = C(X)$. Our claim here is that if we pick a probability measure on X, we have an embedding as follows:

$$C(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

Indeed, given a function $f \in C(X)$, consider the operator $T_f(g) = fg$, acting on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 dx} \leq ||f||_\infty ||g||_2$$

Thus, $f \to T_f$ provides us with a C^*-algebra embedding $C(X) \subset B(H)$, as claimed. In general now, assuming that a linear form $\varphi : A \to \mathbb{C}$ has some suitable positivity properties, making it analogous to the integration functionals $\int_X : A \to \mathbb{C}$ from the commutative case, we can define a scalar product on A, by the following formula:

$$< a, b >= \varphi(ab^*)$$

By completing we obtain a Hilbert space H, and we have an embedding as follows:

$$A \subset B(H) \quad , \quad a \to (b \to ab)$$

Thus we obtain the assertion (1), and a careful examination of the construction $A \to H$, outlined above, shows that the assertions (2,3) are in fact proved as well. □

Good time now to get back towards Definitions 1.1 and 1.2. We will need:

THEOREM 1.11. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^* = g^{-1}$.*

(1) *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
(2) *When Γ is abelian, we have an isomorphism $C^*(\Gamma) \simeq C(G)$, where $G = \widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.*

Proof. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma]\subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H=l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi:\mathbb{C}[\Gamma]\subset B(H)\quad,\quad \pi(g)(h)=gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h)=gh$ that $g\to\pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g\in\Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A=C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A=C(X)$, with:

$$X=Spec(A)$$

But the spectrum $X=Spec(A)$, consisting of the characters $\chi:C^*(\Gamma)\to\mathbb{C}$, can be identified with the Pontrjagin dual $G=\widehat{\Gamma}$, and this gives the result. $\square$

The above result suggests the following definition:

Definition 1.12. *Given a discrete group Γ, the compact quantum space G given by*

$$C(G)=C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G=\widehat{\Gamma}$.

Good news, this definition is exactly what we need, in order to understand the meaning of Definitions 1.1 and 1.2. To be more precise, we have the following result:

Theorem 1.13. *The basic tori are all group duals, as follows,*

$$\begin{array}{ccc}T_N^+&\longrightarrow&\mathbb{T}_N^+\\ \uparrow&&\uparrow\\ T_N&\longrightarrow&\mathbb{T}_N\end{array}\qquad=\qquad\begin{array}{ccc}\widehat{L_N}&\longrightarrow&\widehat{F_N}\\ \uparrow&&\uparrow\\ \mathbb{Z}_2^N&\longrightarrow&\mathbb{T}^N\end{array}$$

where $F_N=\mathbb{Z}^{*N}$ is the free group on N generators, and $L_N=\mathbb{Z}_2^{*N}$ is its real version.

Proof. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 1.11 (2), this gives the result. $\square$

Moving ahead, now that we have our formalism, we can start developing free geometry. As a first objective, we would like to better understand the relation between the classical and free tori. In order to discuss this, let us introduce the following notion:

Definition 1.14. *Given a compact quantum space X, its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I = < [a,b] >$$

In this situation, we also say that X appears as a "liberation" of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space.

In relation now with our tori, we have the following result:

Theorem 1.15. *We have inclusions between the various tori, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and the free tori on top appear as liberations of the tori on the bottom.

Proof. This is indeed clear from definitions, because commutativity of a group algebra means precisely that the group in question is abelian. □

1c. Free spheres

In order to extend now the free geometries that we have, real and complex, let us begin with the spheres. Following [13], we have the following notions:

Definition 1.16. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1,\ldots,x_N\Big|x_i = x_i^*,\sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

Here the fact that these algebras are indeed well-defined comes from the following estimate, which shows that the biggest C^*-norms on these $*$-algebras are bounded:

$$||x_i||^2 = ||x_ix_i^*|| \leq \left|\left|\sum_i x_ix_i^*\right|\right| = 1$$

As a first result now, regarding the above free spheres, we have:

Theorem 1.17. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{array}{ccc} C(S^{N-1}_{\mathbb{R},+}) & \longleftarrow & C(S^{N-1}_{\mathbb{C},+}) \\ \downarrow & & \downarrow \\ C(S^{N-1}_{\mathbb{R}}) & \longleftarrow & C(S^{N-1}_{\mathbb{C}}) \end{array}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right)$$

It is enough to establish the second isomorphism. So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_{\mathbb{C}}$ satisfy the defining relations for A, we have a quotient map of as follows:

$$A \to C(S^{N-1}_{\mathbb{C}})$$

Conversely, let us write $A = C(S)$, by using the Gelfand theorem. The variables $x_1,\ldots,x_N$ become in this way true coordinates, providing us with an embedding $S \subset \mathbb{C}^N$. Also, the quadratic relations become $\sum_i |x_i|^2 = 1$, so we have $S \subset S^{N-1}_{\mathbb{C}}$. Thus, we have a quotient map $C(S^{N-1}_{\mathbb{C}}) \to A$, as desired, and this gives all the results. □

1d. Algebraic manifolds

By using the free spheres constructed above, we can now formulate:

DEFINITION 1.18. *A real algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i \in \mathbb{C} < x_1, \ldots, x_N >$$

We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinates $x_1, \ldots, x_N$.

As a basic example here, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_{\mathbb{C}}, S^{N-1}_{\mathbb{R}}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem:

THEOREM 1.19. *If $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class} = \left\{ x \in S^{N-1}_{\mathbb{C}} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class}.

PROOF. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_{\mathbb{C}}$. Now since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

DEFINITION 1.20. *We agree to identify two real algebraic submanifolds $X, Y \subset S^{N-1}_{\mathbb{C},+}$ when we have a $*$-algebra isomorphism between $*$-algebras of coordinates*

$$f : \mathcal{C}(Y) \to \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

We will see later the reasons for making this convention, coming from amenability. Now back to the tori, as constructed before, we can see that these are examples of algebraic manifolds, in the sense of Definition 1.18. In fact, we have the following result:

Theorem 1.21. *The four main quantum spheres produce the main quantum tori*

$$\begin{matrix} S^{N-1}_{\mathbb{R},+} & \to & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \to & S^{N-1}_{\mathbb{C}} \end{matrix} \qquad \to \qquad \begin{matrix} T_N^+ & \to & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \to & \mathbb{T}_N \end{matrix}$$

via the formula $T = S \cap \mathbb{T}_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb{C},+}$.

Proof. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb{T}_N^+ = S^{N-1}_{\mathbb{C},+} \cap \mathbb{T}_N^+$. But this is something trivial, because we have $\mathbb{T}_N^+ \subset S^{N-1}_{\mathbb{C},+}$.

(2) Free real case. Here the formula in the statement reads $T_N^+ = S^{N-1}_{\mathbb{R},+} \cap \mathbb{T}_N^+$. But this is clear as well, the real version of $\mathbb{T}_N^+$ being T_N^+.

(3) Classical complex case. Here the formula in the statement reads $\mathbb{T}_N = S^{N-1}_{\mathbb{C}} \cap \mathbb{T}_N^+$. But this is clear as well, the classical version of $\mathbb{T}_N^+$ being $\mathbb{T}_N$.

(4) Classical real case. Here the formula in the statement reads $T_N = S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). $\square$

1e. Exercises

Exercises:

Exercise 1.22.

Exercise 1.23.

Exercise 1.24.

Exercise 1.25.

Exercise 1.26.

Exercise 1.27.

Exercise 1.28.

Exercise 1.29.

Bonus exercise.

CHAPTER 2

Free rotations

2a. Quantum groups

In order to better understand the structure of $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}$, we need to talk about free rotations. Following Woronowicz [99], let us start with:

Definition 2.1. *A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate A, such that the formulae*

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

define morphisms of C^*-algebras as follows,

$$\Delta:A\to A\otimes A\quad,\quad\varepsilon:A\to\mathbb{C}\quad,\quad S:A\to A^{opp}$$

called comultiplication, counit and antipode.

Obviously, this is something tricky, and we will see details in a moment, the idea being that these are the axioms which best fit with what we want to do, in this book. Let us also mention, technically, that $\otimes$ in the above can be any topological tensor product, and with the choice of $\otimes$ being irrelevant, but more on this later. Also, A^{opp} is the opposite algebra, with multiplication $a\cdot b=ba$, and more on this later too.

We say that A is cocommutative when $\Sigma\Delta=\Delta$, where $\Sigma(a\otimes b)=b\otimes a$ is the flip. With this convention, we have the following key result, from Woronowicz [99]:

Proposition 2.2. *The following are Woronowicz algebras:*

1. *$C(G)$, with $G\subset U_N$ compact Lie group. Here the structural maps are:*
$$\Delta(\varphi)=(g,h)\to\varphi(gh)\quad,\quad\varepsilon(\varphi)=\varphi(1)\quad,\quad S(\varphi)=g\to\varphi(g^{-1})$$
2. *$C^*(\Gamma)$, with $F_N\to\Gamma$ finitely generated group. Here the structural maps are:*
$$\Delta(g)=g\otimes g\quad,\quad\varepsilon(g)=1\quad,\quad S(g)=g^{-1}$$

Moreover, we obtain in this way all the commutative/cocommutative algebras.

Proof. This is something very standard, the idea being as follows:

(1) Given $G \subset U_N$, we can set $A = C(G)$, which is a Woronowicz algebra, together with the matrix $u = (u_{ij})$ formed by coordinates of G, given by:

$$g = \begin{pmatrix} u_{11}(g) & \ldots & u_{1N}(g) \\ \vdots & & \vdots \\ u_{N1}(g) & \ldots & u_{NN}(g) \end{pmatrix}$$

Conversely, if (A, u) is a commutative Woronowicz algebra, by using the Gelfand theorem we can write $A = C(X)$, with X being a certain compact space. The coordinates u_{ij} give then an embedding $X \subset M_N(\mathbb{C})$, and since the matrix $u = (u_{ij})$ is unitary we actually obtain an embedding $X \subset U_N$, and finally by using the maps Δ, ε, S we conclude that our compact subspace $X \subset U_N$ is in fact a compact Lie group, as desired.

(2) Consider a finitely generated group $F_N \to \Gamma$. We can set $A = C^*(\Gamma)$, which is by definition the completion of the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by $g^* = g^{-1}$, for any $g \in \Gamma$, with respect to the biggest C^*-norm, and we obtain a Woronowicz algebra, together with the diagonal matrix formed by the generators of Γ:

$$u = \begin{pmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_N \end{pmatrix}$$

Conversely, if (A, u) is a cocommutative Woronowicz algebra, the Peter-Weyl theory of Woronowicz, to be explained below, shows that the irreducible corepresentations of A are all 1-dimensional, and form a group Γ, and so we have $A = C^*(\Gamma)$, as desired. Thus, theorem proved, modulo a representation theory discussion, to come soon. □

In general now, the structural maps Δ, ε, S have the following properties:

Proposition 2.3. *Let (A, u) be a Woronowicz algebra.*

(1) *Δ, ε satisfy the usual axioms for a comultiplication and a counit, namely:*

$$\begin{aligned} (\Delta \otimes id)\Delta &= (id \otimes \Delta)\Delta \\ (\varepsilon \otimes id)\Delta &= (id \otimes \varepsilon)\Delta = id \end{aligned}$$

(2) *S satisfies the antipode axiom, on the $*$-subalgebra generated by entries of u:*

$$m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(.)1$$

(3) *In addition, the square of the antipode is the identity, $S^2 = id$.*

Proof. Observe first that the result holds in the case where A is commutative. Indeed, by using Proposition 2.2 (1) we can write:

$$\Delta = m^t \quad , \quad \varepsilon = u^t \quad , \quad S = i^t$$

The 3 conditions in the statement come then by transposition from the basic 3 group theory conditions satisfied by m,u,i, which are as follows, with $\delta(g)=(g,g)$:

$$m(m\times id)=m(id\times m)$$
$$m(id\times u)=m(u\times id)=id$$
$$m(id\times i)\delta=m(i\times id)\delta=1$$

Observe also that the result holds as well in the case where A is cocommutative, by using Proposition 2.2 (1). In the general case now, the proof goes as follows:

(1) We have the following computation:

$$(\Delta\otimes id)\Delta(u_{ij})=\sum_l\Delta(u_{il})\otimes u_{lj}=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

We have as well the following computation, which gives the first formula:

$$(id\otimes\Delta)\Delta(u_{ij})=\sum_k u_{ik}\otimes\Delta(u_{kj})=\sum_{kl}u_{ik}\otimes u_{kl}\otimes u_{lj}$$

On the other hand, we have the following computation:

$$(id\otimes\varepsilon)\Delta(u_{ij})=\sum_k u_{ik}\otimes\varepsilon(u_{kj})=u_{ij}$$

We have as well the following computation, which gives the second formula:

$$(\varepsilon\otimes id)\Delta(u_{ij})=\sum_k\varepsilon(u_{ik})\otimes u_{kj}=u_{ij}$$

(2) By using the fact that the matrix $u=(u_{ij})$ is unitary, we obtain:

$$\begin{aligned}m(id\otimes S)\Delta(u_{ij}) &= \sum_k u_{ik}S(u_{kj})\\ &= \sum_k u_{ik}u_{jk}^*\\ &= (uu^*)_{ij}\\ &= \delta_{ij}\end{aligned}$$

We have as well the following computation, which gives the result:

$$\begin{aligned}m(S\otimes id)\Delta(u_{ij}) &= \sum_k S(u_{ik})u_{kj}\\ &= \sum_k u_{ki}^*u_{kj}\\ &= (u^*u)_{ij}\\ &= \delta_{ij}\end{aligned}$$

(3) Finally, the formula $S^2=id$ holds as well on generators, and so in general too. $\square$

Let us record as well the following technical result:

PROPOSITION 2.4. *Given a Woronowicz algebra (A, u), we have $u^t = \bar{u}^{-1}$, so u is biunitary, in the sense that it is unitary, with unitary transpose.*

PROOF. We have the following computation, based on the fact that u is unitary:

$$\begin{aligned}(uu^*)_{ij} = \delta_{ij} \quad &\implies \quad \sum_k S(u_{ik}u_{jk}^*) = \delta_{ij}\\ &\implies \quad \sum_k u_{kj}u_{ki}^* = \delta_{ij}\\ &\implies \quad (u^t\bar{u})_{ji} = \delta_{ij}\end{aligned}$$

Similarly, we have the following computation, once agan using the unitarity of u:

$$\begin{aligned}(u^*u)_{ij} = \delta_{ij} \quad &\implies \quad \sum_k S(u_{ki}^*u_{kj}) = \delta_{ij}\\ &\implies \quad \sum_k u_{jk}^*u_{ik} = \delta_{ij}\\ &\implies \quad (\bar{u}u^t)_{ji} = \delta_{ij}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, the Woronowicz algebras appear to have nice properties. In view of Proposition 2.2 and Proposition 2.3, we can formulate the following definition:

DEFINITION 2.5. *Given a Woronowicz algebra A, we formally write*

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

When A is commutative and cocommutative, G and Γ are usual abelian groups, dual to each other. In general, we still agree to write $G = \widehat{\Gamma}, \Gamma = \widehat{G}$, but in a formal sense. As a final piece of general theory now, let us complement Definition 2.1 with:

DEFINITION 2.6. *Given two Woronowicz algebras (A, u) and (B, v), we write*

$$A \simeq B$$

and identify the corresponding quantum groups, when we have an isomorphism

$$< u_{ij} > \simeq < v_{ij} >$$

of $*$-algebras, mapping standard coordinates to standard coordinates.

With this convention, which is in tune with our conventions for algebraic manifolds from chapter 1, and more on this later, any compact or discrete quantum group corresponds to a unique Woronowicz algebra, up to equivalence. Also, we can see now why in

Definition 2.1 the choice of the exact topological tensor product $\otimes$ is irrelevant. Indeed, no matter what tensor product $\otimes$ we use there, we end up with the same Woronowicz algebra, and the same compact and discrete quantum groups, up to equivalence.

In practice, we will use in what follows the simplest such tensor product $\otimes$, which is the maximal one, obtained as completion of the usual algebraic tensor product with respect to the biggest C^*-norm. With the remark that this product is something rather abstract, and so can be treated, in practice, as a usual algebraic tensor product.

Moving ahead now, let us call corepresentation of A any unitary matrix $v\in M_n(\mathcal{A})$, where $\mathcal{A}=<u_{ij}>$, satisfying the same conditions are those satisfied by u, namely:

$$\Delta(v_{ij})=\sum_k v_{ik}\otimes v_{kj}\quad,\quad\varepsilon(v_{ij})=\delta_{ij}\quad,\quad S(v_{ij})=v_{ji}^*$$

These corepresentations can be then thought of as corresponding to the finite dimensional unitary smooth representations of the underlying compact quantum group G. Following Woronowicz [99], we have the following key result:

THEOREM 2.7. *Any Woronowicz algebra has a unique Haar integration functional,*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

which can be constructed by starting with any faithful positive form $\varphi\in A^*$, and setting

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v\in M_n(\mathbb{C})\otimes A$ we have

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.

PROOF. Following [99], this can be done in 3 steps, as follows:

(1) Given $\varphi\in A^*$, our claim is that the following limit converges, for any $a\in A$:

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

Indeed, we can assume, by linearity, that a is the coefficient of a corepresentation:

$$a=(\tau\otimes id)v$$

But in this case, an elementary computation shows that we have the following formula, where P_φ is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$:

$$\left(id\otimes\int_\varphi\right)v=P_\varphi$$

(2) Since $v\xi=\xi$ implies $[(id\otimes\varphi)v]\xi=\xi$, we have $P_\varphi\geq P$, where P is the orthogonal projection onto the following fixed point space:

$$Fix(v)=\left\{\xi\in\mathbb{C}^n\Big|v\xi=\xi\right\}$$

The point now is that when $\varphi\in A^*$ is faithful, by using a standard positivity trick, one can prove that we have $P_\varphi=P$. Assume indeed $P_\varphi\xi=\xi$, and let us set:

$$a=\sum_i\left(\sum_j v_{ij}\xi_j-\xi_i\right)\left(\sum_k v_{ik}\xi_k-\xi_i\right)^*$$

We must prove that we have $a=0$. Since v is biunitary, we have:

$$\begin{aligned}
a &= \sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar{\xi}_k-\frac{1}{N}\bar{\xi}_i\right)\right)\\
&= \sum_{ijk}v_{ij}v_{ik}^*\xi_j\bar{\xi}_k-\frac{1}{N}v_{ij}\xi_j\bar{\xi}_i-\frac{1}{N}v_{ik}^*\xi_i\bar{\xi}_k+\frac{1}{N^2}\xi_i\bar{\xi}_i\\
&= \sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar{\xi}_i-\sum_{ik}v_{ik}^*\xi_i\bar{\xi}_k+\sum_i|\xi_i|^2\\
&= ||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&= 2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $P_\varphi\xi=\xi$, we obtain from this:

$$\begin{aligned}
\varphi(a) &= 2\varphi(||\xi||^2-Re(<v\xi,\xi>))\\
&= 2(||\xi||^2-Re(<P_\varphi\xi,\xi>))\\
&= 2(||\xi||^2-||\xi||^2)\\
&= 0
\end{aligned}$$

Now since φ is faithful, this gives $a=0$, and so $v\xi=\xi$. Thus $\int_\varphi$ is independent of φ, and is given on coefficients $a=(\tau\otimes id)v$ by the following formula:

$$\left(id\otimes\int_\varphi\right)v=P$$

(3) With the above formula in hand, the left and right invariance of $\int_G=\int_\varphi$ is clear on coefficients, and so in general, and this gives all the assertions. See [99]. □

Consider the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the coefficients of the fundamental corepresentation u, and endow it with the following scalar product:

$$< a, b >= \int_G ab^*$$

We have then the following result, also due to Woronowicz [99]:

THEOREM 2.8. *We have the following Peter-Weyl type results:*
(1) *Any corepresentation decomposes as a sum of irreducible corepresentations.*
(2) *Each irreducible corepresentation appears inside a certain $u^{\otimes k}$.*
(3) *$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$, the summands being pairwise orthogonal.*
(4) *The characters of irreducible corepresentations form an orthonormal system.*

PROOF. All these results are from [99], the idea being as follows:

(1) Given a corepresentation $v \in M_n(A)$, consider its interwiner algebra:

$$End(v) = \left\{ T \in M_n(\mathbb{C}) \Big| Tv = vT \right\}$$

It is elementary to see that this is a finite dimensional C^*-algebra, and we conclude from this that we have a decomposition as follows:

$$End(v) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

To be more precise, such a decomposition appears by writing the unit of our algebra as a sum of minimal projections, as follows, and then working out the details:

$$1 = p_1 + \ldots + p_k$$

But this decomposition allows us to define subcorepresentations $v_i \subset v$, which are irreducible, so we obtain, as desired, a decomposition $v = v_1 + \ldots + v_k$.

(2) To any corepresentation $v \in M_n(A)$ we associate its space of coefficients, given by $C(v) = span(v_{ij})$. The construction $v \to C(v)$ is then functorial, in the sense that it maps subcorepresentations into subspaces. Observe also that we have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

Now given an arbitrary corepresentation $v \in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v) \subset \mathcal{A}$, and so we must have, for certain positive integers $k_1, \ldots, k_p$, an inclusion of vector spaces, as follows:

$$C(v) \subset C(u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v \subset u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}$$

Thus, by using (1), we are led to the conclusion in the statement.

(3) By using (1) and (2), we obtain a linear space decomposition as follows:

$$\mathcal{A}=\sum_{v\in Irr(A)}C(v)=\sum_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$$

In order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C(v)\perp C(w)$$

As a first observation, which follows from an elementary computation, for any two corepresentations v,w we have a Frobenius type isomorphism, as follows:

$$Hom(v,w)\simeq Fix(\bar{v}\otimes w)$$

Now let us set $P_{ia,jb}=\int_G v_{ij}w_{ab}^*$. According to Theorem 2.7, the matrix P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(\bar{v},\bar{w})=\{0\}$$

Thus we have $P=0$, and so $C(v)\perp C(w)$, which gives the result.

(4) The algebra $\mathcal{A}_{central}$ contains indeed all the characters, because we have:

$$\Sigma\Delta(\chi_v)=\sum_{ij}v_{ji}\otimes v_{ij}=\Delta(\chi_v)$$

The fact that the characters span $\mathcal{A}_{central}$, and form an orthogonal basis of it, follow from (3). Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl}=\int_G v_{ij}v_{kl}^*$$

We know from Theorem 2.7 that these integrals form the orthogonal projection onto $Fix(v\otimes\bar{v})\simeq End(\bar{v})=\mathbb{C}1$. By using this fact, we obtain the following formula:

$$\int_G\chi_v\chi_v^*=\sum_{ij}\int_G v_{ii}v_{jj}^*=\sum_i\frac{1}{N}=1$$

Thus the characters have indeed norm 1, and we are done. □

We refer to Woronowicz [99] for full details on all the above, and for some applications as well. Let us just record here the fact that in the cocommutative case, we obtain from (4) that the irreducible corepresentations must be all 1-dimensional, and so that we must have $A=C^*(\Gamma)$ for some discrete group Γ, as mentioned in Proposition 2.2.

At a more technical level now, we have a number of more advanced results, from Woronowicz [99], [100] and other papers, that must be known as well. We will present them quickly, and for details you check my book [9]. First we have:

THEOREM 2.9. *Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of A_{full} is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.*

If this is the case, we say that the underlying discrete quantum group Γ is amenable.

PROOF. This is well-known in the group dual case, $A = C^*(\Gamma)$, with Γ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\iff$ (2) This simply follows from the fact that the GNS construction for the algebra A_{full} with respect to the Haar functional produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial, and conversely, a counit map $\varepsilon : A_{red} \to \mathbb{C}$ produces an isomorphism $A_{red} \to A_{full}$, via a formula of type $(\varepsilon \otimes id)\Phi$.

(3) $\iff$ (4) Here $\implies$ is clear, coming from $\varepsilon(N - Re(\chi(u))) = 0$, and the converse can be proved by doing some standard functional analysis. □

Yet another important result is Tannakian duality, as follows:

THEOREM 2.10. *The following operations are inverse to each other:*

1. *The construction $A \to C$, which associates to any Woronowicz algebra A the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
2. *The construction $C \to A$, which associates to a tensor category C the Woronowicz algebra A presented by the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. This is something quite deep, the idea being as follows:

(1) We have indeed a construction $A \to C$ as above, whose output is a tensor C^*-subcategory with duals of the tensor C^*-category of Hilbert spaces.

(2) We have as well a construction $C \to A$ as above, simply by dividing the free $*$-algebra on N^2 variables by the relations in the statement.

Regarding now the bijection claim, after some elementary algebra we are left with proving $C_{A_C} \subset C$. But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. See [100]. □

2b. Free rotations

Good news, with the above general theory in hand, we can go back now to our free geometry program, as developed in chapter 1, and substantially build on that. Indeed, the point is that we can talk now about free rotations. Following Wang [89], we have:

Theorem 2.11. *The following constructions produce compact quantum groups,*

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)$$

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

which appear respectively as liberations of the groups O_N and U_N.

Proof. This first assertion follows from the elementary fact that if a matrix $u=(u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u^\Delta_{ij}=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u^\varepsilon_{ij}=\delta_{ij}\quad,\quad u^S_{ij}=u^*_{ji}$$

Indeed, the biunitarity of u^Δ can be checked by a direct computation. Regarding now the matrix $u^\varepsilon=1_N$, this is clearly biunitary. Also, regarding the matrix u^S, there is nothing to prove here either, because its unitarity its clear too. And finally, observe that if u has self-adjoint entries, then so do the above matrices $u^\Delta,u^\varepsilon,u^S$.

Thus our claim is proved, and we can define morphisms Δ,ε,S as in Definition 2.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$. As for the second assertion, this follows exactly as for the free spheres, by adapting the sphere proof from chapter 1. □

The basic properties of O_N^+,U_N^+ can be summarized as follows:

Theorem 2.12. *The quantum groups O_N^+,U_N^+ have the following properties:*

1. *The closed subgroups $G\subset U_N^+$ are exactly the $N\times N$ compact quantum groups. As for the closed subgroups $G\subset O_N^+$, these are those satisfying $u=\bar{u}$.*
2. *We have liberation embeddings $O_N\subset O_N^+$ and $U_N\subset U_N^+$, obtained by dividing the algebras $C(O_N^+),C(U_N^+)$ by their respective commutator ideals.*
3. *We have as well embeddings $\widehat{L}_N\subset O_N^+$ and $\widehat{F}_N\subset U_N^+$, where L_N is the free product of N copies of $\mathbb{Z}_2$, and where F_N is the free group on N generators.*

Proof. All these assertions are elementary, as follows:

(1) This is clear from definitions, with the remark that, in the context of Definition 2.1, the formula $S(u_{ij})=u^*_{ji}$ shows that the matrix $\bar{u}$ must be unitary too.

(2) This follows from the Gelfand theorem. To be more precise, this shows that we have presentation results for $C(O_N),C(U_N)$, similar to those in Theorem 2.11, but with the commutativity between the standard coordinates and their adjoints added:

$$C(O_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},u^t=u^{-1}\right)$$

$$C(U_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1},u^t=\bar{u}^{-1}\right)$$

Thus, we are led to the conclusion in the statement.

(3) This follows indeed from (1) and from Proposition 2.2, with the remark that with $u = diag(g_1, \ldots, g_N)$, the condition $u = \bar{u}$ is equivalent to $g_i^2 = 1$, for any i. $\square$

The last assertion in Theorem 2.12 suggests the following construction:

PROPOSITION 2.13. *Given a closed subgroup $G \subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T \subset G$ constructed as follows:*

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \middle| \forall i \neq j \right\rangle$$

This torus is then a group dual, $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by the elements $g_i = u_{ii}$, which are unitaries inside $C(T)$.

PROOF. Since u is unitary, its diagonal entries $g_i = u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

It follows that we have $C(T) = C^*(\Lambda)$, modulo identifying as usual the C^*-completions of the various group algebras, and so that we have $T = \widehat{\Lambda}$, as claimed. $\square$

With this notion in hand, Theorem 2.12 (3) reformulates as follows:

THEOREM 2.14. *The diagonal tori of the basic unitary groups are the basic tori:*

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \qquad \to \qquad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

In particular, the basic unitary groups are all distinct.

PROOF. This is something clear and well-known in the classical case, and in the free case, this is a reformulation of Theorem 2.12 (3), which tells us that the diagonal tori of O_N^+, U_N^+, in the sense of Proposition 2.13, are the group duals $\widehat{L}_N, \widehat{F}_N$. $\square$

There is an obvious relation here with the considerations from chapter 1, that we will analyse later on. As a second result now regarding our free quantum groups, relating them this time to the free spheres constructed in chapter 1, we have:

Theorem 2.15. *We have embeddings of algebraic manifolds as follows, obtained in double indices by rescaling the coordinates, $x_{ij}=u_{ij}/\sqrt{N}$:*

$$\begin{matrix}O_N^+&\to&U_N^+\\ \uparrow&&\uparrow\\ O_N&\to&U_N\end{matrix}\qquad\to\qquad\begin{matrix}S^{N^2-1}_{\mathbb R,+}&\to&S^{N^2-1}_{\mathbb C,+}\\ \uparrow&&\uparrow\\ S^{N^2-1}_{\mathbb R}&\to&S^{N^2-1}_{\mathbb C}\end{matrix}$$

Moreover, the quantum groups appear from the quantum spheres via

$$G=S\cap U_N^+$$

with the intersection being computed inside the free sphere $S^{N^2-1}_{\mathbb C,+}$.

Proof. As explained in Theorem 2.12, the biunitarity of the matrix $u=(u_{ij})$ gives an embedding of algebraic manifolds, as follows:

$$U_N^+\subset S^{N^2-1}_{\mathbb C,+}$$

Now since the relations defining $O_N,O_N^+,U_N\subset U_N^+$ are the same as those defining $S^{N^2-1}_{\mathbb R},S^{N^2-1}_{\mathbb R,+},S^{N^2-1}_{\mathbb C}\subset S^{N^2-1}_{\mathbb C,+}$, this gives the result. □

Summarizing, we have now up and working some free rotation groups, which are closely related to the free spheres and tori constructed in chapter 1.

2c. Quantum isometries

In order to further discuss now the relation with the spheres, which can only come via some sort of "isometric actions", let us start with the following standard fact:

Proposition 2.16. *Given a closed subset $X\subset S^{N-1}_{\mathbb C}$, the formula*

$$G(X)=\left\{U\in U_N\Big|U(X)=X\right\}$$

defines a compact group of unitary matrices, or isometries, called affine isometry group of X. For the spheres $S^{N-1}_{\mathbb R},S^{N-1}_{\mathbb C}$ we obtain in this way the groups O_N,U_N.

Proof. The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X\subset\mathbb C^N$, but we are not interested here in such general spaces. □

Observe that in the case of the real and complex spheres, the affine isometry group $G(X)$ leaves invariant the Riemannian metric, because this metric is equivalent to the one inherited from $\mathbb C^N$, which is preserved by our isometries $U\in U_N$.

Thus, we could have constructed as well $G(X)$ as being the group of metric isometries of X, with of course some extra care in relation with the complex structure, as for the complex sphere $X=S^{N-1}_\mathbb{C}$ to produce $G(X)=U_N$ instead of $G(X)=O_{2N}$. But, such things won't really work for the free spheres, and so are to be avoided.

The point now is that we have the following quantum analogue of Proposition 2.16, which is a perfect analogue, save for the fact that X is now assumed to be algebraic, for some technical reasons, which allows us to talk about quantum isometry groups:

Theorem 2.17. *Given an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G\subset U_N^+$ acting affinely on X, in the sense that the formula*

$$\Phi(x_i)=\sum_j x_j\otimes u_{ji}$$

defines a morphism of C^*-algebras $\Phi:C(X)\to C(X)\otimes C(G)$, has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of X.

Proof. Assume indeed that our manifold $X\subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_\alpha(x_1,\ldots,x_N)=0\Big\rangle$$

In order to prove the result, consider the following variables:

$$X_i=\sum_j x_j\otimes u_{ji}\in C(X)\otimes C(U_N^+)$$

Our claim is that the quantum group in the statement $G=G^+(X)$ appears as:

$$C(G)=C(U_N^+)\Big/\Big\langle f_\alpha(X_1,\ldots,X_N)=0\Big\rangle$$

In order to prove this, pick one of the defining polynomials, and write it as follows:

$$f_\alpha(x_1,\ldots,x_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\cdot x_{i_1^r}\ldots x_{i_{s_r}^r}$$

With $X_i=\sum_j x_j\otimes u_{ji}$ as above, we have the following formula:

$$f_\alpha(X_1,\ldots,X_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\sum_{j_1^r\ldots j_{s_r}^r}x_{j_1^r}\ldots x_{j_{s_r}^r}\otimes u_{j_1^r i_1^r}\ldots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f_\alpha(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables u_{ij}. Thus, we have indeed a closed subspace $G\subset U_N^+$, with a universal map, as follows:

$$\Phi:C(X)\to C(X)\otimes C(G)$$

In order to show now that G is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider as well the following elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f_\alpha(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f_\alpha(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f_\alpha(X_1,\ldots,X_N)=0$$

Thus we can map $u_{ij}\to u_{ij}^\gamma$ for any $\gamma\in\{\Delta,\varepsilon,S\}$, and we are done. $\square$

We can now formulate a result about spheres and rotations, as follows:

Theorem 2.18. *The quantum isometry groups of the basic spheres are*

$$\begin{matrix}
S^{N-1}_{\mathbb{R},+}&\longrightarrow&S^{N-1}_{\mathbb{C},+}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb{R}}&\longrightarrow&S^{N-1}_{\mathbb{C}}
\end{matrix}
\qquad\to\qquad
\begin{matrix}
O_N^+&\longrightarrow&U_N^+\\
\uparrow&&\uparrow\\
O_N&\longrightarrow&U_N
\end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. We have 4 results to be proved, the idea being as follows:

$\underline{S^{N-1}_{\mathbb{C},+}}$. Let us first construct an action $U_N^+\curvearrowright S^{N-1}_{\mathbb{C},+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

By using the biunitarity of u, we have the following computation:

$$\sum_i X_iX_i^*=\sum_{ijk}x_jx_k^*\otimes u_{ji}u_{ki}^*=\sum_j x_jx_j^*\otimes 1=1\otimes 1$$

Once again by using the biunitarity of u, we have as well:

$$\sum_i X_i^*X_i=\sum_{ijk}x_j^*x_k\otimes u_{ji}^*u_{ki}=\sum_j x_j^*x_j\otimes 1=1\otimes 1$$

Thus we have an action $U_N^+\curvearrowright S^{N-1}_{\mathbb{C},+}$, which gives $G^+(S^{N-1}_{\mathbb{C},+})=U_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R},+}}$. Let us first construct an action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$. We already know that the variables $X_i = \sum_j x_j \otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, so we just have to check that these variables are self-adjoint. But this is clear from $u = \bar{u}$, as follows:

$$X_i^* = \sum_j x_j^* \otimes u_{ji}^* = \sum_j x_j \otimes u_{ji} = X_i$$

Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{R},+}$, with $G \subset U_N^+$. The variables $X_i = \sum_j x_j \otimes u_{ji}$ must be then self-adjoint, and the above computation shows that we must have $u = \bar{u}$. Thus our quantum group must satisfy $G \subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{C}}}$. The fact that we have an action $U_N \curvearrowright S^{N-1}_{\mathbb{C}}$ is clear. Conversely, assume that we have an action $G \curvearrowright S^{N-1}_{\mathbb{C}}$, with $G \subset U_N^+$. We must prove that this implies $G \subset U_N$, and we will use a standard trick of Bhowmick-Goswami [13]. We have:

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

By multiplying this formula with itself we obtain:

$$\Phi(x_i x_k) = \sum_{jl} x_j x_l \otimes u_{ji} u_{lk}$$

$$\Phi(x_k x_i) = \sum_{jl} x_l x_j \otimes u_{lk} u_{ji}$$

Since the variables x_i commute, these formulae can be written as:

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{ji} u_{lk} + u_{li} u_{jk}) + \sum_j x_j^2 \otimes u_{ji} u_{jk}$$

$$\Phi(x_i x_k) = \sum_{j<l} x_j x_l \otimes (u_{lk} u_{ji} + u_{jk} u_{li}) + \sum_j x_j^2 \otimes u_{jk} u_{ji}$$

Since the tensors at left are linearly independent, we must have:

$$u_{ji} u_{lk} + u_{li} u_{jk} = u_{lk} u_{ji} + u_{jk} u_{li}$$

By applying the antipode to this formula, then applying the involution, and then relabelling the indices, we succesively obtain:

$$u_{kl}^* u_{ij}^* + u_{kj}^* u_{il}^* = u_{ij}^* u_{kl}^* + u_{il}^* u_{kj}^*$$

$$u_{ij} u_{kl} + u_{il} u_{kj} = u_{kl} u_{ij} + u_{kj} u_{il}$$

$$u_{ji} u_{lk} + u_{jk} u_{li} = u_{lk} u_{ji} + u_{li} u_{jk}$$

Now by comparing with the original formula, we obtain from this:

$$u_{li} u_{jk} = u_{jk} u_{li}$$

In order to finish, it remains to prove that the coordinates u_{ij} commute as well with their adjoints. For this purpose, we use a similar method. We have:

$$\Phi(x_ix_k^*)=\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*$$

$$\Phi(x_k^*x_i)=\sum_{jl}x_l^*x_j\otimes u_{lk}^*u_{ji}$$

Since the variables on the left are equal, we deduce from this that we have:

$$\sum_{jl}x_jx_l^*\otimes u_{ji}u_{lk}^*=\sum_{jl}x_jx_l^*\otimes u_{lk}^*u_{ji}$$

Thus we have $u_{ji}u_{lk}^*=u_{lk}^*u_{ji}$, and so $G\subset U_N$, as claimed.

$\underline{S^{N-1}_\mathbb{R}}$. The fact that we have an action $O_N\curvearrowright S^{N-1}_\mathbb{R}$ is clear. In what regards the converse, this follows by combining the results that we already have, as follows:

$$\begin{aligned}G\curvearrowright S^{N-1}_\mathbb{R}\quad&\implies\quad G\curvearrowright S^{N-1}_{\mathbb{R},+},S^{N-1}_\mathbb{C}\\&\implies\quad G\subset O_N^+,U_N\\&\implies\quad G\subset O_N^+\cap U_N=O_N\end{aligned}$$

Thus, we conclude that we have $G^+(S^{N-1}_\mathbb{R})=O_N$, as desired. □

2d. Haar integration

Let us discuss now the correspondence $U\to S$. In the classical case the situation is very simple, because the sphere $S=S^{N-1}$ appears by rotating the point $x=(1,0,\ldots,0)$ by the isometries in $U=U_N$. Moreover, the stabilizer of this action is the subgroup $U_{N-1}\subset U_N$ acting on the last $N-1$ coordinates, and so the sphere $S=S^{N-1}$ appears from the corresponding rotation group $U=U_N$ as an homogeneous space, as follows:

$$S^{N-1}=U_N/U_{N-1}$$

In functional analytic terms, all this becomes even simpler, the correspondence $U\to S$ being obtained, at the level of algebras of functions, as follows:

$$C(S^{N-1})\subset C(U_N)\quad,\quad x_i\to u_{1i}$$

In general now, the straightforward homogeneous space interpretation of S as above fails. However, we can have some theory going by using the functional analytic viewpoint, with an embedding $x_i\to u_{1i}$ as above. Let us start with the following result:

Theorem 2.19. *For the basic spheres, we have a diagram as follows,*

$$\begin{array}{ccc}
C(S) & \xrightarrow{\ \ \Phi\ \ } & C(S)\otimes C(U)\\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha\otimes id}\\
C(U) & \xrightarrow{\ \ \Delta\ \ } & C(U)\otimes C(U)
\end{array}$$

where on top $\Phi(x_i)=\sum_j x_j\otimes u_{ji}$, *and on the left* $\alpha(x_i)=u_{1i}$.

Proof. The diagram in the statement commutes indeed on the standard coordinates, the corresponding arrows being as follows, on these coordinates:

$$\begin{array}{ccc}
x_i & \longrightarrow & \sum_j x_j\otimes u_{ji}\\
\downarrow & & \downarrow\\
u_{1i} & \longrightarrow & \sum_j u_{1j}\otimes u_{ji}
\end{array}$$

Thus by linearity and multiplicativity, the whole the diagram commutes. □

The point now is that, by further building on the above result, we obtain the desired correspondence $U\to S$, and some useful integration results as well.

At the level of the fine structure of the free spheres $S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}$ now, we have some obvious formal eigenspaces for the Laplace operator, and a Weingarten integration formula as well, both coming from the representation theory of O_N^+,U_N^+. Moreover, it is possible to get beyond this, with a full construction of a Laplace operator.

Regarding other possible invariants, orientability does not work, the Dirac operator does not exist, smoothness does not work either, and in what regards K-theory, with our free objects we are a bit too far away from the traditional "reasonable" range of K-theory, usually requiring amenability, or at least some form of K-amenability.

However, after some thinking, maybe including some physical thoughts too, in connection with what is smoothness and is that wished or not, in the present situation, all this is normal. So, no worries, and as we will soon discover, we will get away with the tools that we have, namely Laplace operator and the Weingarten formula, which are not that bad, technically speaking, for all the problems that we will choose to solve.

2e. Exercises

Exercises:

EXERCISE 2.20.

EXERCISE 2.21.

EXERCISE 2.22.

EXERCISE 2.23.

EXERCISE 2.24.

EXERCISE 2.25.

EXERCISE 2.26.

EXERCISE 2.27.

Bonus exercise.

CHAPTER 3

Fine structure

3a. Diagrams, easiness

We have so far a beginning of free geometry, in the real case with a triple of basic objects $(S^{N-1}_{\mathbb{R},+},O_N^+,T_N^+)$, and in the complex case with objects $(S^{N-1}_{\mathbb{C},+},U_N^+,\mathbb{T}_N^+)$. This is not bad, and our purpose in what follows will be that of expanding these two collections of objects, from 3 items each, to 10, 100, 1000, or as many as we can, and the more the merrier, in the name of pure mathematics, where new objects are always welcome.

This being said, what to start with? Leaving aside the tori, which are just duals of discrete groups, and as old as modern mathematics, we face a choice between spheres S, and rotation groups U. As a first observation, these two types of objects are closely related, because in the classical case, given a sphere S, we can recover U as being its isometry group, and conversely, given a group U, we can recover S just by rotating a point. And, as seen in chapter 2, the situation is quite similar in the free case.

This being said, spheres S are not the same thing as rotation groups U, and we will have to make a choice. Normally spheres S look a bit more important, but on the other hand physics, or even mathematics, tell us that no matter what we want to do, of advanced type, about either S or U, we will always end up in struggling with U.

So, we will go for U, and our goal in this chapter will be that of better understanding O_N^+,U_N^+, and also look for more free quantum groups, as many as we can find. And regarding spheres S and other such manifolds, we will leave this for later. Sounds good, doesn't it? Before getting into this, however, let us check with physics and cat:

Cat 3.1. *Gauge invariance gives you everything. But don't forget to do some manifolds too, all our kittens learn that, and it's good learning.*

Thanks cat, this is a pleasure to hear, and in tune with my mathematical intuition. Getting started now, we would like to have a better understanding of the liberation operations that we have, $O_N\to O_N^+$ and $U_N\to U_N^+$, and also have more examples of liberation operations of the same type, $G_N\to G_N^+$. And then, once we will have enough theory and examples, look for classification results for the free quantum groups $\{G_N^+\}$.

Let us start with the construction of more examples, which is certainly a very exciting business, and leave the abstractions for later. Following Wang [89], we first have:

Proposition 3.2. *Consider the symmetric group S_N, viewed as permutation group of the N coordinate axes of $\mathbb{R}^N$. The coordinate functions on $S_N \subset O_N$ are given by*

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

and the matrix $u = (u_{ij})$ that these functions form is magic, in the sense that its entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column.

Proof. The action of S_N on the standard basis $e_1, \ldots, e_N \in \mathbb{R}^N$ being given by $\sigma : e_j \to e_{\sigma(j)}$, this gives the formula of u_{ij} in the statement. As for the fact that the matrix $u = (u_{ij})$ that these functions form is magic, this is clear. □

With a bit more effort, we obtain the following nice characterization of S_N:

Proposition 3.3. *The algebra of functions on S_N has the following presentation,*

$$C(S_N) = C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = \text{magic}\right)$$

and the multiplication, unit and inversion map of S_N appear from the maps

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

defined at the algebraic level, of functions on S_N, by transposing.

Proof. The universal algebra A in the statement being commutative, by the Gelfand theorem it must be of the form $A = C(X)$, with X being a certain compact space. Now since we have coordinates $u_{ij} : X \to \mathbb{R}$, we have an embedding $X \subset M_N(\mathbb{R})$. Also, since we know that these coordinates form a magic matrix, the elements $g \in X$ must be 0-1 matrices, having exactly one 1 entry on each row and each column, and so $X = S_N$. Thus we have proved the first assertion, and the second assertion is clear as well. □

Still following Wang [89], we can now liberate S_N, as follows:

Theorem 3.4. *The following universal C^*-algebra, with magic meaning as usual formed by projections ($p^2 = p^* = p$), summing up to 1 on each row and each column,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \middle| u = \text{magic}\right)$$

is a Woronowicz algebra, with comultiplication, counit and antipode given by:

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

Thus the space S_N^+ is a compact quantum group, called quantum permutation group.

PROOF. As a first observation, the universal C^*-algebra in the statement is indeed well-defined, because the conditions $p^2 = p^* = p$ satisfied by the coordinates give:

$$||u_{ij}|| \leq 1$$

In order to prove now that we have a Woronowicz algebra, we must construct maps Δ, ε, S given by the formulae in the statement. Consider the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}$$

Our claim is that, since u is magic, so are these three matrices. Indeed, regarding u^Δ, its entries are idempotents, as shown by the following computation:

$$(u_{ij}^\Delta)^2 = \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj} = \sum_{kl} \delta_{kl}u_{ik} \otimes \delta_{kl}u_{kj} = u_{ij}^\Delta$$

These elements are self-adjoint as well, as shown by the following computation:

$$(u_{ij}^\Delta)^* = \sum_k u_{ik}^* \otimes u_{kj}^* = \sum_k u_{ik} \otimes u_{kj} = u_{ij}^\Delta$$

The row and column sums for the matrix u^Δ can be computed as follows:

$$\sum_j u_{ij}^\Delta = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1$$

$$\sum_i u_{ij}^\Delta = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1$$

Thus, u^Δ is magic. Regarding now u^ε, u^S, these matrices are magic too, and this for obvious reasons. Thus, all our three matrices $u^\Delta, u^\varepsilon, u^S$ are magic, so we can define Δ, ε, S by the formulae in the statement, by using the universality property of $C(S_N^+)$. □

Our first task now is to make sure that Theorem 3.4 produces indeed a new quantum group, which does not collapse to S_N. Still following Wang [89], we have:

THEOREM 3.5. *We have an embedding $S_N \subset S_N^+$, given at the algebra level by:*

$$u_{ij} \to \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

This is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

PROOF. The fact that we have indeed an embedding as above follows from Proposition 3.3. Observe that in fact more is true, because our results above give:

$$C(S_N) = C(S_N^+) \Big/ \Big\langle ab = ba \Big\rangle$$

Thus, the inclusion $S_N \subset S_N^+$ is a "liberation", in the sense that S_N is the classical version of S_N^+. We will often use this basic fact, in what follows. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N=2$. The fact that S_2^+ is indeed classical, and hence collapses to S_2, is trivial, because the 2×2 magic matrices are as follows, with p being a projection:

$$U=\begin{pmatrix}p&1-p\\1-p&p\end{pmatrix}$$

Indeed, this shows that the entries of U commute. Thus $C(S_2^+)$ is commutative, and so equals its biggest commutative quotient, which is $C(S_2)$. Thus, $S_2^+=S_2$.

Case $N=3$. By using the same argument as in the $N=2$ case, and the symmetries of the problem, it is enough to check that u_{11},u_{22} commute. But this follows from:

$$\begin{aligned}u_{11}u_{22}&=u_{11}u_{22}(u_{11}+u_{12}+u_{13})\\&=u_{11}u_{22}u_{11}+u_{11}u_{22}u_{13}\\&=u_{11}u_{22}u_{11}+u_{11}(1-u_{21}-u_{23})u_{13}\\&=u_{11}u_{22}u_{11}\end{aligned}$$

Indeed, by applying the involution to this formula, we obtain that we have as well $u_{22}u_{11}=u_{11}u_{22}u_{11}$. Thus, we obtain $u_{11}u_{22}=u_{22}u_{11}$, as desired.

Case $N=4$. Consider the following matrix, with p,q being projections:

$$U=\begin{pmatrix}p&1-p&0&0\\1-p&p&0&0\\0&0&q&1-q\\0&0&1-q&q\end{pmatrix}$$

This matrix is magic, and we can choose $p,q\in B(H)$ as for the algebra $<p,q>$ to be noncommutative and infinite dimensional. We conclude that $C(S_4^+)$ is noncommutative and infinite dimensional as well, and so S_4^+ is non-classical and infinite, as claimed.

Case $N\geq 5$. Here we can use the standard embedding $S_4^+\subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u\to\begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this in hand, the fact that S_4^+ is a non-classical, infinite compact quantum group implies that S_N^+ with $N\geq 5$ has these two properties as well. □

With the above results in hand, we can introduce as well quantum reflections:

THEOREM 3.6. *The following constructions produce compact quantum groups,*

$$\begin{aligned}C(H_N^+)&=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u_{ij}=u_{ij}^*,\ (u_{ij}^2)=\text{magic}\right)\\C(K_N^+)&=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|[u_{ij},u_{ij}^*]=0,\ (u_{ij}u_{ij}^*)=\text{magic}\right)\end{aligned}$$

which appear as liberations of the reflection groups $H_N=\mathbb{Z}_2\wr S_N$ and $K_N=\mathbb{T}\wr S_N$.

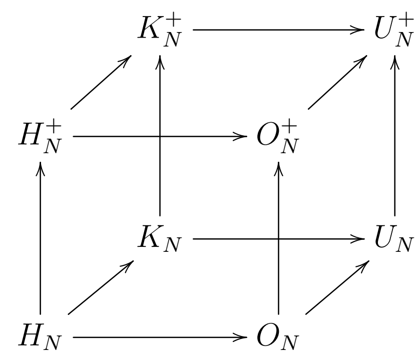
PROOF. This can be proved in the usual way, with the first assertion coming from the fact that if u satisfies the relations in the statement, then so do the matrices $u^\Delta, u^\varepsilon, u^S$, and with the second assertion being trivial. Let us also mention that, in analogy with $H_N = \mathbb{Z}_2 \wr S_N$ and $K_N = \mathbb{T} \wr S_N$, we have decomposition results as follows:

$$H_N^+ = \mathbb{Z}_2 \wr_* S_N^+ \quad , \quad K_N^+ = \mathbb{T} \wr_* S_N^+$$

To be more precise, here $\wr_*$ is a free wreath product, and these formulae can be established a bit as in the classical case. For more on all this, we refer to [10]. □

All the above is very nice, and as a conclusion to all this, let us record the following result, which collects and refines the various liberation statements formulated above:

THEOREM 3.7. *The quantum unitary and reflection groups are as follows,*



and in this diagram, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

PROOF. The fact that we have inclusions as in the statement follows from the definition of the various quantum groups involved. As for the various intersection claims, these follow as well from definitions. For some further details on all this, we refer to [10]. □

As a comment here, observe that the symmetric group S_N and its free analogue S_N^+, while certainly being very interesting objects, had not made the cut for appearing in the above almighty cube, called "standard cube" in quantum algebra. However, this is something quite natural, because S_N and S_N^+ are objects on their own, neither real or complex, and for practical purposes, like ours with our cube, these quantum groups must be replaced with H_N, H_N^+ in the real case, and with K_N, K_N^+ in the free case.

Actually I'm not quite sure about this, time to ask the cat. Who says:

CAT 3.8. *Do not worry, the high speed world is projective anyway, and it is better to use reflections instead of permutations.*

Thanks cat, not that I really understand what you say, but it fits with my purposes and cube, which looks really cool. But I will keep this in mind, and discuss later the relation between affine and projective geometry, in the free setting, that is promised.

With this done, let us get now into the second question that we were having, namely the conceptual understanding of the various liberation operations $G_N \to G_N^+$. In order to discuss this, we will need Tannakian duality, and Brauer type theorems. Let us start with Tannakian duality. This is a rather abstract statement, as follows:

THEOREM 3.9. *The following operations are inverse to each other:*

(1) *The construction $G \to C$, which associates to a closed subgroup $G \subset_u U_N^+$ the tensor category formed by the intertwiner spaces $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.*
(2) *The construction $C \to G$, associating to a tensor category C the closed subgroup $G \subset_u U_N^+$ coming from the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$, with $T \in C_{kl}$.*

PROOF. We have indeed a construction $G \to C_G$, whose output is a subcategory of the tensor C^*-category of finite dimensional Hilbert spaces, as follows:

$$(C_G)_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$$

We have as well a construction $C \to G_C$, obtained by setting:

$$C(G_C) = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C_{kl} \right\rangle$$

Regarding now the bijection claim, some elementary algebra shows that $C = C_{G_C}$ implies $G = G_{C_G}$, and that $C \subset C_{G_C}$ is automatic. Thus we are left with proving:

$$C_{G_C} \subset C$$

But this latter inclusion can be proved indeed, by doing some algebra, and using von Neumann's bicommutant theorem, in finite dimensions. □

The above result is something quite abstract, yet powerful. We will see applications of it in a moment, in the form of Brauer theorems for S_N, O_N, U_N and S_N^+, O_N^+, U_N^+, and other quantum groups. In order to formulate these Brauer theorems, let us start with:

DEFINITION 3.10. *Let $P(k,l)$ be the set of partitions between an upper row of k points, and a lower row of l points. A collection of sets*

$$D = \bigsqcup_{k,l} D(k,l)$$

with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:

(1) *Stability under the horizontal concatenation, $(\pi, \sigma) \to [\pi\sigma]$.*
(2) *Stability under the vertical concatenation, $(\pi, \sigma) \to [^\sigma_\pi]$.*
(3) *Stability under the upside-down turning, $\pi \to \pi^*$.*
(4) *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
(5) *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

As a basic example, we have the category of all partitions P itself. Other basic examples are the category of pairings P_2, and the categories NC, NC_2 of noncrossing partitions, and pairings. We have as well the category $\mathcal{P}_2$ of pairings which are "matching", in the sense that they connect $\circ-\circ$, $\bullet-\bullet$ on the vertical, and $\circ-\bullet$ on the horizontal, and its subcategory $\mathcal{NC}_2\subset\mathcal{P}_2$ consisting of the noncrossing matching pairings.

There are many other examples, and we will be back to this, gradually, in what follows. Regarding now the relation with the Tannakian categories, this comes from:

PROPOSITION 3.11. *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}\quad,\quad T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

PROOF. The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

As for the composition and involution axioms, their proof is similar. □

In relation now with quantum groups, we have the following result:

THEOREM 3.12. *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

which produces a Tannakian category, and so a closed subgroup $G_N\subset_u U_N^+$.

Proof. Let call C_{kl} the spaces on the right. By using the axioms in Definition 3.10, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 3.11, we see that $C = (C_{kl})$ is a Tannakian category. Thus Theorem 3.9 applies, and gives the result. □

We can now formulate a key definition, as follows:

Definition 3.13. *A compact quantum group G_N is called easy when we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi\Big|\pi \in D(k,l)\right)$$

for any colored integers k, l, for a certain category of partitions $D \subset P$.

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. The terminology is quite natural, because Tannakian duality is basically our only serious tool. In relation now with the orthogonal, unitary and symmetric quantum groups, here is the result:

Theorem 3.14. *The basic quantum permutation and rotation groups,*

$$\begin{array}{ccccc} S_N^+ & \longrightarrow & O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ S_N & \longrightarrow & O_N & \longrightarrow & U_N \end{array}$$

are all easy, the corresponding categories of partitions being as follows,

$$\begin{array}{ccccc} NC & \longleftarrow & NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P & \longleftarrow & P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

with 2 standing for pairings, NC for noncrossing, and calligraphic for matching.

Proof. This is something quite fundamental, the proof being as follows:

(1) The quantum group U_N^+ is defined via the following relations:

$$u^* = u^{-1} \quad , \quad u^t = \bar{u}^{-1}$$

But, by doing some elementary computations, these relations tell us precisely that the following two operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi = \underset{\circ\bullet}{\cap} \ , \ \underset{\bullet\circ}{\cap}$$

Thus, the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D = < \underset{\circ\bullet}{\cap} \ , \ \underset{\bullet\circ}{\cap} > = \mathcal{N}C_2$$

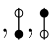
(2) The subgroup $O_N^+ \subset U_N^+$ is defined by imposing the following relations:

$$u_{ij} = \bar{u}_{ij}$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi =$$ 

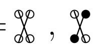
We conclude that the Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = < \mathcal{NC}_2,$$  $$>= NC_2$$

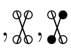
(3) The subgroup $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0 \quad , \quad [u_{ij}, \bar{u}_{kl}] = 0$$

Thus, the following operators must be in the associated Tannakian category C:

$$T_\pi \quad : \quad \pi =$$ 

Thus the associated Tannakian category is $C = span(T_\pi | \pi \in D)$, with:

$$D = < \mathcal{NC}_2,$$  $$>= \mathcal{P}_2$$

(4) In order to deal now with O_N, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us that O_N is indeed easy, coming from:

$$D = < NC_2, \mathcal{P}_2 >= P_2$$

(5) We know that the subgroup $S_N^+ \subset O_N^+$ appears as follows:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle u = \text{magic} \Big\rangle$$

In order to interpret the magic condition, consider the fork partition:

$$Y \in P(2,1)$$

Given a corepresentation u, we have the following formulae:

$$(T_Y u^{\otimes 2})_{i,jk} = \sum_{lm} (T_Y)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

$$(u T_Y)_{i,jk} = \sum_l u_{il} (T_Y)_{l,jk} = \delta_{jk} u_{ij}$$

We conclude that we have the following equivalence:

$$T_Y \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+) \Big/ \Big\langle T_Y \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Thus S_N^+ is indeed easy, the corresponding category of partitions being:

$$D = < Y > = NC$$

(6) Finally, in order to deal with S_N, we can use the following formula:

$$S_N = S_N^+ \cap O_N$$

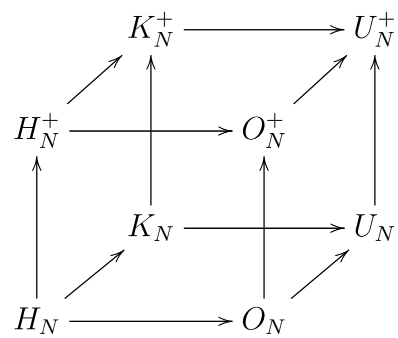
At the categorical level, this tells us that S_N is indeed easy, coming from:

$$D = < NC, P_2 > = P$$

Thus, we are led to the conclusions in the statement. □

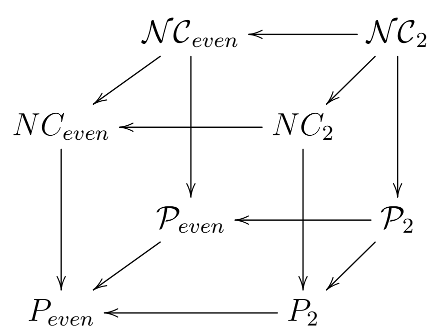
Moving ahead, we can upgrade what we have into a cube result, as follows:

THEOREM 3.15. *The basic quantum unitary and reflection groups,*



are all easy, and the corresponding categories of partitions form an intersection diagram.

PROOF. The precise claim is that the categories are as follows, with P_{even} being the category of partitions having even blocks, and with $\mathcal{P}_{even}(k,l) \subset P_{even}(k,l)$ consisting of the partitions satisfying $\# \circ = \# \bullet$ in each block, when flattening the partition:



But this is something that we already know for the right face, from Theorem 3.14, and in what regards the left face, the proof here is similar, by using the results for S_N, S_N^+ from that same Theorem 3.14. As for the last assertion, this is something trivial. □

The above results are something quite deep, and we will see in what follows countless applications of them. As a first such application, rather philosophical, we have:

THEOREM 3.16. *The constructions $G_N \to G_N^+$ with $G = O, U, S, H, K$ are easy quantum group liberations, in the sense that they come from the construction*

$$D \to D \cap NC$$

at the level of the associated categories of partitions.

PROOF. This is clear indeed from Theorem 3.14 and Theorem 3.15, and from the following trivial equalities, connecting the categories found there:

$$NC_2 = P_2 \cap NC \quad , \quad \mathcal{NC}_2 = \mathcal{P}_2 \cap NC$$

$$NC = P \cap NC$$

$$NC_{even} = P_{even} \cap NC \quad , \quad \mathcal{NC}_{even} = \mathcal{P}_{even} \cap NC$$

Thus, we are led to the conclusion in the statement. □

The above result is quite nice, because the various constructions $G_N \to G_N^+$ that we made so far, although natural, were something quite ad-hoc. Now all this is no longer ad-hoc, and the next time that we will have to liberate a subgroup $G_N \subset U_N$, we know what the recipe is, namely check if G_N is easy, and if so, simply define $G_N^+ \subset U_N^+$ as being the easy quantum group coming from the category $D = D_G \cap NC$.

3b. Uniformity, characters

In general, the study of the free quantum groups, in the "easy" sense explained above, is something quite complex. In order to cut a bit from complexity, we will use:

PROPOSITION 3.17. *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D \subset P$, the following conditions are equivalent:*

1. *$G_{N-1} = G_N \cap U_{N-1}^+$, via the embedding $U_{N-1}^+ \subset U_N^+$ given by $u \to diag(u, 1)$.*
2. *$G_{N-1} = G_N \cap U_{N-1}^+$, via the N possible diagonal embeddings $U_{N-1}^+ \subset U_N^+$.*
3. *D is stable under the operation which consists in removing blocks.*

PROOF. We use the general easiness theory, as explained above:

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N \subset G_N$, which makes everything S_N-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given a subgroup $K \subset U_{N-1}^+$, with fundamental corepresentation u, consider the $N \times N$ matrix $v = diag(u, 1)$. Our claim is that for any $\pi \in P(k)$ we have:

$$\xi_\pi \in Fix(v^{\otimes k}) \iff \xi_{\pi'} \in Fix(v^{\otimes k'}), \forall \pi' \in P(k'), \pi' \subset \pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}\xi_\pi\in Fix(v^{\otimes k}) &\iff (v^{\otimes k}\xi_\pi)_{i_1\ldots i_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\ &\iff \sum_j(v^{\otimes k})_{i_1\ldots i_k,j_1\ldots j_k}(\xi_\pi)_{j_1\ldots j_k}=(\xi_\pi)_{i_1\ldots i_k},\forall i\\ &\iff \sum_j\delta_\pi(j_1,\ldots,j_k)v_{i_1j_1}\ldots v_{i_kj_k}=\delta_\pi(i_1,\ldots,i_k),\forall i\end{aligned}$$

Now let us recall that our corepresentation has the special form $v=diag(u,1)$. We conclude from this that for any index $a\in\{1,\ldots,k\}$, we must have:

$$i_a=N\implies j_a=N$$

With this observation in hand, if we denote by i',j' the multi-indices obtained from i,j obtained by erasing all the above $i_a=j_a=N$ values, and by $k'\leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'}\delta_\pi(j_1,\ldots,j_k)(v^{\otimes k'})_{i'j'}=\delta_\pi(i_1,\ldots,i_k),\forall i$$

Here the index j is by definition obtained from j' by filling with N values. In order to finish now, we have two cases, depending on i, as follows:

Case 1. Assume that the index set $\{a|i_a=N\}$ corresponds to a certain subpartition $\pi'\subset\pi$. In this case, the N values will not matter, and our formula becomes:

$$\sum_{j'}\delta_\pi(j'_1,\ldots,j'_{k'})(v^{\otimes k'})_{i'j'}=\delta_\pi(i'_1,\ldots,i'_{k'})$$

Case 2. Assume now the opposite, namely that the set $\{a|i_a=N\}$ does not correspond to a subpartition $\pi'\subset\pi$. In this case the indices mix, and our formula reads:

$$0=0$$

Thus, we are led to $\xi_{\pi'}\in Fix(v^{\otimes k'})$, for any subpartition $\pi'\subset\pi$, as claimed. Thus our claim is proved, and with this in hand, the result follows from Tannakian duality. □

Based on the above result, let us formulate the following definition:

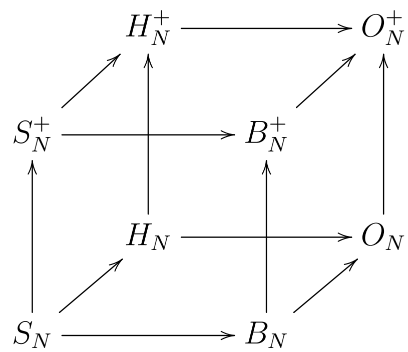
DEFINITION 3.18. *An easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, is called uniform when we have, for any $N\in\mathbb N$:*

$$G_{N-1}=G_N\cap U_{N-1}^+$$

Equivalently, D must be stable under the operation which consists in removing blocks.

For classification purposes the uniformity axiom is something very natural and useful, substantially cutting from complexity, and we have the following result:

Theorem 3.19. *The classical and free uniform orthogonal easy quantum groups are*



with B_N, B_N^+ being the classical and quantum bistochastic groups.

Proof. There are several things to be proved, the idea being as follows:

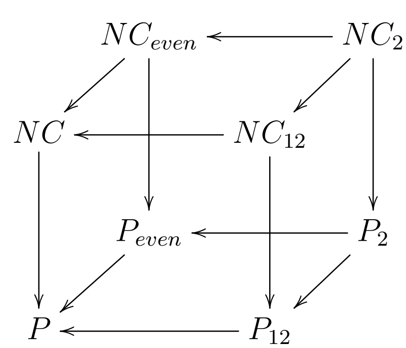
(1) We first recall that the bistochastic group $B_N \subset O_N$ consists of the orthogonal matrices whose entries sum up to 1 on each row, or equivalently, sum up to 1 on each column. Thus, if we denote by $\xi \in \mathbb{C}^N$ the all-one vector, we have:

$$B_N = \left\{U \in O_N \Big| U\xi = \xi\right\}$$

Based on this, we can construct a free analogue of B_N as follows, and the fact that we obtain indeed a quantum group follows exactly as for O_N^+, U_N^+:

$$C(B_N^+) = C(O_N^+) \Big/ \left\langle u\xi = \xi \right\rangle$$

(2) Since the relation $u\xi = \xi$ reads $T_| \in Fix(u)$, with $| \in P(0,1)$ being the singleton partition, we conclude that B_N, B_N^+ are easy, coming from the categories P_{12}, NC_{12} of singletons and pairings, and noncrossing singletons and pairings. Thus, all the quantum groups in the statement are easy, the corresponding categories of partitions being:



(3) Regarding now the classification, consider an easy quantum group $S_N \subset G_N \subset O_N$. This must come from a category $P_2 \subset D \subset P$, and if we assume $G = (G_N)$ to be uniform,

then D is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in D. Our claim is that the admissible sets are as follows:

– $L = \{2\}$, producing O_N.

– $L = \{1, 2\}$, producing B_N.

– $L = \{2, 4, 6, \ldots\}$, producing H_N.

– $L = \{1, 2, 3, \ldots\}$, producing S_N.

(4) Indeed, in one sense, this follows from our easiness results for O_N, B_N, H_N, S_N. In the other sense now, assume that $L \subset \mathbb{N}$ is such that the set P_L consisting of partitions whose sizes of the blocks belong to L is a category of partitions. We know from the axioms of the categories of partitions that the semicircle $\cap$ must be in the category, so we have $2 \in L$. We claim that the following conditions must be satisfied as well:

$$k, l \in L,\ k > l \implies k - l \in L$$

$$k \in L,\ k \geq 2 \implies 2k - 2 \in L$$

(5) Indeed, we will prove that both conditions follow from the axioms of the categories of partitions. Let us denote by $b_k \in P(0, k)$ the one-block partition:

$$b_k = \left\{\begin{matrix} \sqcap\!\sqcap & \ldots & \sqcap \\ 1\ 2 & \ldots & k \end{matrix}\right\}$$

For $k > l$, we can write b_{k-l} in the following way:

$$b_{k-l} = \left\{\begin{matrix} \sqcap\!\sqcap & \ldots & \ldots & \ldots & \ldots & \sqcap \\ 1\ 2 & \ldots & l & l+1 & \ldots & k \\ \sqcup\!\sqcup & \ldots & \sqcup & | & \ldots & | \\ & & & 1 & \ldots & k-l \end{matrix}\right\}$$

In other words, we have the following formula:

$$b_{k-l} = (b_l^* \otimes |^{\otimes k-l}) b_k$$

Since all the terms of this composition are in P_L, we have $b_{k-l} \in P_L$, and this proves our first claim. As for the second claim, this can be proved in a similar way, by capping two adjacent k-blocks with a 2-block, in the middle.

(6) With these conditions in hand, we can conclude in the following way:

Case 1. Assume $1 \in L$. By using the first condition with $l = 1$ we get:

$$k \in L \implies k - 1 \in L$$

This condition shows that we must have $L = \{1, 2, \ldots, m\}$, for a certain number $m \in \{1, 2, \ldots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} m \in L &\implies 2m - 2 \in L \\ &\implies 2m - 2 \leq m \\ &\implies m \in \{1, 2, \infty\} \end{aligned}$$

The case $m = 1$ being excluded by the condition $2 \in L$, we reach to one of the two sets producing the groups S_N, B_N.

Case 2. Assume $1 \notin L$. By using the first condition with $l = 2$ we get:

$$k \in L \implies k - 2 \in L$$

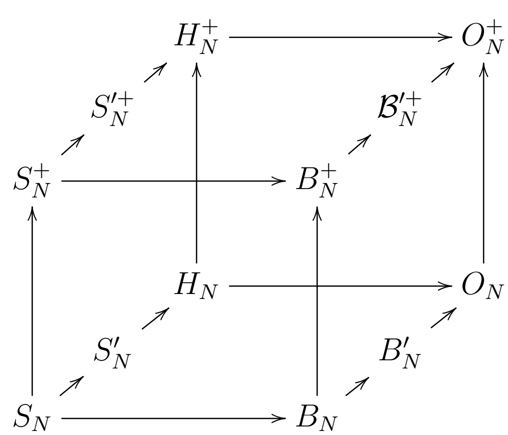
This condition shows that we must have $L = \{2, 4, \ldots, 2p\}$, for a certain number $p \in \{1, 2, \ldots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned} 2p \in L &\implies 4p - 2 \in L \\ &\implies 4p - 2 \leq 2p \\ &\implies p \in \{1, \infty\} \end{aligned}$$

Thus L must be one of the two sets producing O_N, H_N, and we are done. In the free case, $S_N^+ \subset G_N \subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time $O_N^+, B_N^+, H_N^+, S_N^+$. □

When removing the uniformity axiom things become more complicated, as follows:

THEOREM 3.20. *The classical and free orthogonal easy quantum groups are*



with $S_N' = S_N \times \mathbb{Z}_2$, $B_N' = B_N \times \mathbb{Z}_2$, and with $S_N'^+, \mathcal{B}_N'^+$ being their liberations, where $\mathcal{B}_N'^+$ stands for the two possible such liberations, $B_N'^+ \subset B_N''^+$.

PROOF. The idea here is that of jointly classifying the "classical" categories of partitions $P_2 \subset D \subset P$, and the "free" ones $NC_2 \subset D \subset NC$:

(1) At the classical level this leads, via a study which is quite similar to that from the proof of Theorem 3.19, to 2 more groups, namely S_N', B_N'.

(2) At the free level we obtain 3 more quantum groups, $S_N'^+, B_N'^+, B_N''^+$, with the inclusion $B_N'^+ \subset B_N''^+$, which is something a bit surprising, being best thought of as coming from an inclusion $B_N' \subset B_N''$, which happens to be an isomorphism. $\square$

It is possible to obtain similar results in the general unitary case, first with a quite simple statement, regarding the uniform case, and then with something more complicated, regarding the non-uniform case. We refer here to the paper of Tarrago-Weber [81].

Importantly, the uniformity assumption has some interesting analytic consequences, making the link with the Bercovici-Pata bijection [19]. In order to discuss this, we first need to know how to integrate on the easy quantum groups, and we have here:

THEOREM 3.21. *Assuming that a closed subgroup $G \subset U_N^+$ is easy, coming from a category of partitions $D \subset P$, we have the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k} = \sum_{\pi,\sigma\in D(k)} \delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where $\delta \in \{0,1\}$ are the usual Kronecker type symbols, and where the Weingarten matrix $W_{kN} = G_{kN}^{-1}$ is the inverse of the Gram matrix $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$.

PROOF. We know from the general theory in chapter 1 that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(u^{\otimes k}) = span\left(\xi_\pi\Big|\pi \in D(k)\right)$$

In order to prove the result, consider the following linear map:

$$E(x) = \sum_{\pi\in D(k)} < x, \xi_\pi > \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse on $Fix(u^{\otimes k})$ of the restriction of E. But this restriction is the linear map given by G_{kN}, and so W is the linear map given by W_{kN}, and this gives the result. $\square$

In relation now with characters, we have the following moment formula:

PROPOSITION 3.22. *The moments of truncated characters are given by the formula*

$$\int_G (u_{11} + \ldots + u_{ss})^k = Tr(W_{kN}G_{ks})$$

where G_{kN} and $W_{kN} = G_{kN}^{-1}$ are the associated Gram and Weingarten matrices.

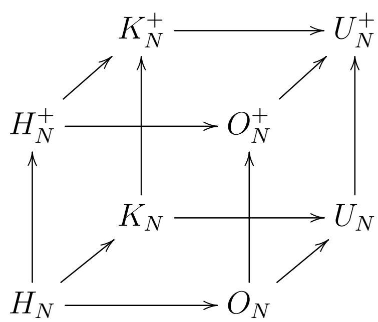
PROOF. We have indeed the following computation:

$$\begin{aligned}\int_G (u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s \ldots \sum_{i_k=1}^s \int u_{i_1 i_1}\ldots u_{i_k i_k}\\ &= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)\sum_{i_1=1}^s \ldots \sum_{i_k=1}^s \delta_\pi(i)\delta_\sigma(i)\\ &= \sum_{\pi,\sigma\in D(k)} W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\ &= Tr(W_{kN}G_{ks})\end{aligned}$$

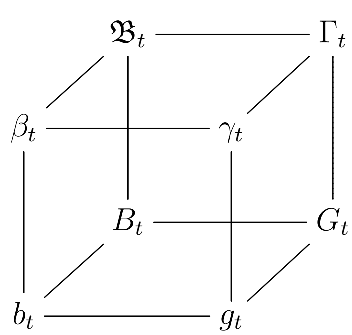
Thus, we have obtained the formula in the statement. □

With the above general theory in hand, we can now formulate our character results for the main examples of uniform easy quantum groups, as follows:

THEOREM 3.23. *For the main quantum rotation and reflection groups,*

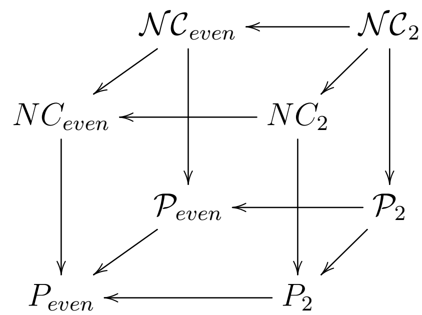


the corresponding truncated characters follow with $N\to\infty$ the laws



which are the main limiting laws in classical and free probability.

Proof. We know from Theorem 3.15 that the above quantum groups are all easy, coming from the following categories of partitions:



Now by using Proposition 3.22, we obtain the following formula:

$$\lim_{N\to\infty}\int_{G_N}\chi_t^k=\sum_{\pi\in D(k)}t^{|\pi|}$$

But this gives the laws in the statement, via some standard calculus. □

3c. Temperley-Lieb

All the above is sweet, and there are many other things that can be said, along the same lines, about the liberation operations $G_N\to G_N^+$, using easiness and partitions. This being said, we are rather interested in free quantum groups, so we do not need partitions with crossings, and this leads us to a quite puzzling question, as follows:

Question 3.24. *Among the many objects which are in bijection with the noncrossing partitions, which are the most adapted to the study of the free quantum groups?*

To be more precise here, in order to give you a taste on what this question is about, you have surely heard for instance about the Catalan numbers:

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

These Catalan numbers count the partitions in $NC(k)$, but they count as well a zillion other interesting things, just ask and any expert in combinatorics will probably get you stuck for 1 hour in the coffee room, in explaining you all this, and our problem is, among these zillion things, what are the best for the study of free quantum groups.

This does not look obvious, and so time to ask the cat. And cat says:

Cat 3.25. *You're getting old, double the strings as to have Temperley-Lieb diagrams, as in the heyday of free quantum group theory.*

Thanks cat, and yes indeed, age does not help much with knowledge and memory, in fact Question 3.24 is something that I already thought about, some 30 years ago, when developing the basic theory of free quantum groups. Following Temperley-Lieb, who by the way were first-class physicists, and then Jones, who was a first-class physicist too, and many others, including myself when younger, not to forget cat of course, we will of course go for this, doubling strings and using Temperley-Lieb diagrams.

Let us start with the following result, which is well-known:

PROPOSITION 3.26. *We have a bijection $NC(k) \simeq NC_2(2k)$, as follows:*

(1) *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
(2) *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

PROOF. The fact that the above two operations are indeed inverse to each other is clear, by drawing pictures, and computing the corresponding compositions. □

With the above result in hand, we can axiomatize the free quantum groups, in terms of Temperley-Lieb diagrams NC_2, and say many interesting things about them, based on the work of Jones and others on subfactor theory and planar algebras [67].

We can compute representations and their fusion rules, Cayley graphs, growth exponents, laws of characters and more, by using diagrams, and more specifically Temperley-Lieb diagrams NC_2, which are quite often the most adapted, to our questions.

As a basic example for what can be done here, regarding O_N^+, we have:

THEOREM 3.27. *The irreducible representations of O_N^+ with $N \geq 2$ can be labelled by positive integers, r_k with $k \in \mathbb{N}$, the fusion rules for these representations are*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

and the dimensions are $\dim r_k = (q^{k+1} - q^{-k-1})/(q - q^{-1})$, *with $q + q^{-1} = N$.*

PROOF. The idea is to skilfully recycle the well-known proof for SU_2. Our claim is that we can construct, by recurrence on $k \in \mathbb{N}$, a sequence $r_0, r_1, r_2, \ldots$ of irreducible, self-adjoint and distinct representations of O_N^+, satisfying:

$$r_0 = 1 \quad , \quad r_1 = u \quad , \quad r_{k-1} \otimes r_1 = r_{k-2} + r_k$$

In order to do so, we can use the formula $r_{k-2} \otimes r_1 = r_{k-3} + r_{k-1}$ and Frobenius duality, and we conclude there exists a certain representation r_k such that:

$$r_{k-1} \otimes r_1 = r_{k-2} + r_k$$

As a first observation, r_k is self-adjoint, because its character is a certain polynomial with integer coefficients in χ, which is self-adjoint. In order to prove now that r_k is irreducible, and non-equivalent to $r_0,\ldots,r_{k-1}$, let us split as before $u^{\otimes k}$, as follows:

$$u^{\otimes k}=c_kr_k+c_{k-2}r_{k-2}+c_{k-4}r_{k-4}+\ldots$$

The point now is that we have the following equalities and inequalities:

$$C_k=\sum_i c_i^2\leq\dim(End(u^{\otimes k}))\leq|NC_2(k,k)|=C_k$$

Indeed, the equality at left is clear as before, then comes a standard inequality, then an inequality coming from easiness, then a standard equality. Thus, we have equality, so r_k is irreducible, and non-equivalent to $r_{k-2},r_{k-4},\ldots$ Moreover, r_k is not equivalent to $r_{k-1},r_{k-3},\ldots$ either, by using the same argument as for SU_2, and the end of the proof is exactly as for SU_2. As for dimensions, by recurrence we obtain, with $q+q^{-1}=N$:

$$\dim r_k=q^k+q^{k-2}+\ldots+q^{-k+2}+q^{-k}$$

But this gives the dimension formula in the statement, and we are done. □

It is possible to use similar methods for the other main examples of free quantum groups, and do many other things, in relation with the Temperley-Lieb algebra.

3d. Meander determinants

We discuss now, following Di Francesco [38] and others, the computation of the Gram determinants for the free quantum groups, which is a very interesting question, related to many things. But let us start with S_N and other classical groups. We will need:

DEFINITION 3.28. *The Möbius function of any lattice, and so of P, is given by*

$$\mu(\pi,\sigma)=\begin{cases}1&\text{if }\pi=\sigma\\ -\sum_{\pi\leq\tau<\sigma}\mu(\pi,\tau)&\text{if }\pi<\sigma\\ 0&\text{if }\pi\not\leq\sigma\end{cases}$$

with the construction being performed by recurrence.

As an illustration here, for $P(2)=\{||,\sqcap\}$, we have by definition:

$$\mu(||,||)=\mu(\sqcap,\sqcap)=1$$

Also, $||<\sqcap$, with no intermediate partition in between, so we obtain:

$$\mu(||,\sqcap)=-\mu(||,||)=-1$$

Finally, we have $\sqcap\not\leq||$, and so we have as well the following formula:

$$\mu(\sqcap,||)=0$$

We will need the Möbius inversion formula, which can be formulated as follows:

THEOREM 3.29. *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1 & \text{if } \pi\leq\sigma\\ 0 & \text{if } \pi\not\leq\sigma\end{cases}$$

is the Möbius matrix of P, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.

PROOF. This is well-known, coming from the fact that A_k is upper triangular. Indeed, when inverting, we are led into the recurrence for μ, from Definition 3.28. □

As an illustration, for $P(2)$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following result:

PROPOSITION 3.30. *The Gram matrix of the vectors ξ_π with $\pi\in P(k)$,*

$$G_{\pi\sigma}=N^{|\pi\vee\sigma|}$$

decomposes as a product of upper/lower triangular matrices, $G_k=A_kL_k$, where

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1) & \text{if } \sigma\leq\pi\\ 0 & \text{otherwise}\end{cases}$$

and where A_k is the adjacency matrix of $P(k)$.

PROOF. We have indeed the following computation:

$$\begin{aligned}G_k(\pi,\sigma)&=N^{|\pi\vee\sigma|}\\&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According now to the definition of A_k,L_k, this formula reads:

$$\begin{aligned}G_k(\pi,\sigma)&=\sum_{\tau\geq\pi}L_k(\tau,\sigma)\\&=\sum_\tau A_k(\pi,\tau)L_k(\tau,\sigma)\\&=(A_kL_k)(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix}N^2&N\\ N&N\end{pmatrix}=\begin{pmatrix}1&1\\ 0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\ N&N\end{pmatrix}$$

We are led in this way to the following formula, due to Lindstöm:

THEOREM 3.31. *The determinant of the Gram matrix G_k is given by*

$$\det(G_k)=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

with the convention that in the case $N<k$ we obtain 0.

PROOF. If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, A_k is upper triangular, and L_k is lower triangular. Thus, we have:

$$\begin{aligned}\det(G_k)&=\det(A_k)\det(L_k)\\ &=\det(L_k)\\ &=\prod_\pi L_k(\pi,\pi)\\ &=\prod_\pi N(N-1)\ldots(N-|\pi|+1)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Let us discuss as well the case of the orthogonal group O_N. Here the combinatorics is that of the Young diagrams. We denote by $|.|$ the number of boxes, and we use quantity f^λ, which gives the number of standard Young tableaux of shape λ. We have then:

THEOREM 3.32. *The determinant of the Gram matrix of O_N is given by*

$$\det(G_{kN})=\prod_{|\lambda|=k/2}f_N(\lambda)^{f^{2\lambda}}$$

where the quantities on the right are $f_N(\lambda)=\prod_{(i,j)\in\lambda}(N+2j-i-1)$.

PROOF. For the group O_N the Gram matrix is diagonalizable, as follows:

$$G_{kN}=\sum_{|\lambda|=k/2}f_N(\lambda)P_{2\lambda}$$

Here $1=\sum P_{2\lambda}$ is the standard partition of unity associated to the Young diagrams having $k/2$ boxes, and the coefficients $f_N(\lambda)$ are those in the statement. Now since we have $Tr(P_{2\lambda})=f^{2\lambda}$, this gives the formula in the statement. □

In order to deal now with O_N^+,S_N^+, we will need the following fact:

PROPOSITION 3.33. *The Gram matrices of $NC_2(2k) \simeq NC(k)$ are related by*

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi',\sigma')$$

where $\pi \to \pi'$ is the shrinking operation, and Δ_{kn} is the diagonal of G_{kn}.

PROOF. In the context of the bijection from Proposition 3.26, we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we are led to the formula in the statement. □

Now back to O_N^+, S_N^+, let us begin with some examples. We first have:

PROPOSITION 3.34. *The first Gram matrices and determinants for O_N^+ are*

$$\det\begin{pmatrix} N^2 & N \\ N & N^2 \end{pmatrix} = N^2(N^2-1)$$

$$\det\begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^3 & N & N & N^2 \\ N^2 & N & N^3 & N & N^2 \\ N^2 & N & N & N^3 & N^2 \\ N & N^2 & N^2 & N^2 & N^3 \end{pmatrix} = N^5(N^2-1)^4(N^2-2)$$

with the matrices being written by using the lexicographic order on $NC_2(2k)$.

PROOF. The formula at $k = 2$, where $NC_2(4) = \{\sqcap\sqcap, \bigcap\!\!\!\!\cap\}$, is clear from definitions. At $k = 3$ however, things are tricky. The partitions here are as follows:

$$NC(3) = \{|||, \sqcap|, \sqcap\!\!\!|\,, |\sqcap, \sqcap\!\sqcap\}$$

The Gram matrix and its determinant are, according to Theorem 3.31:

$$\det\begin{pmatrix} N^3 & N^2 & N^2 & N^2 & N \\ N^2 & N^2 & N & N & N \\ N^2 & N & N^2 & N & N \\ N^2 & N & N & N^2 & N \\ N & N & N & N & N \end{pmatrix} = N^5(N-1)^4(N-2)$$

By using Proposition 3.33, the Gram determinant of $NC_2(6)$ is given by:

$$\begin{aligned} \det(G_{6N}) &= \frac{1}{N^2\sqrt{N}} \times N^{10}(N^2-1)^4(N^2-2) \times \frac{1}{N^2\sqrt{N}} \\ &= N^5(N^2-1)^4(N^2-2) \end{aligned}$$

Thus, we have obtained the formula in the statement. □

In general, such tricks won't work, because $NC(k)$ is strictly smaller than $P(k)$ at $k\geq 4$. However, following Di Francesco [38], we have the following result:

THEOREM 3.35. *The determinant of the Gram matrix for O_N^+ is given by*

$$\det(G_{kN})=\prod_{r=1}^{[k/2]}P_r(N)^{d_{k/2,r}}$$

where P_r are the Chebycheff polynomials, given by

$$P_0=1\quad,\quad P_1=X\quad,\quad P_{r+1}=XP_r-P_{r-1}$$

and $d_{kr}=f_{kr}-f_{k,r+1}$, with f_{kr} being the following numbers, depending on $k,r\in\mathbb Z$,

$$f_{kr}=\binom{2k}{k-r}-\binom{2k}{k-r-1}$$

with the convention $f_{kr}=0$ for $k\notin\mathbb Z$.

PROOF. This is something quite technical, obtained by using a decomposition as follows of the Gram matrix G_{kN}, with the matrix T_{kN} being lower triangular:

$$G_{kN}=T_{kN}T_{kN}^t$$

Thus, a bit as in the proof of the Lindstöm formula, we obtain the result, but the problem lies however in the construction of T_{kN}, which is non-trivial. See [38]. □

With this in hand, we have as well a similar formula for S_N^+, obtained from Theorem 3.35 via Proposition 3.33. For the other free quantum groups, the computations can be done as well. For more on all this, we refer to [38] and related papers.

3e. Exercises

Exercises:

EXERCISE 3.36.

EXERCISE 3.37.

EXERCISE 3.38.

EXERCISE 3.39.

EXERCISE 3.40.

EXERCISE 3.41.

EXERCISE 3.42.

EXERCISE 3.43.

Bonus exercise.

CHAPTER 4

Free manifolds

4a. Quotient spaces

Let us begin with some generalities regarding the quotient spaces, and more general homogeneous spaces. Regarding the quotients, we have the following construction:

Proposition 4.1. *Given a quantum subgroup $H\subset G$, with associated quotient map $\rho:C(G)\to C(H)$, if we define the quotient space $X=G/H$ by setting*

$$C(X)=\left\{f\in C(G)\Big|(\rho\otimes id)\Delta f=1\otimes f\right\}$$

then we have a coaction map as follows,

$$\Phi:C(X)\to C(X)\otimes C(G)$$

obtained as the restriction of the comultiplication of $C(G)$. In the classical case, we obtain in this way the usual quotient space $X=G/H$.

Proof. Observe that the linear subspace $C(X)\subset C(G)$ defined in the statement is indeed a subalgebra, because it is defined via a relation of type $\varphi(f)=\psi(f)$, with both φ,ψ being morphisms of algebras. Observe also that in the classical case we obtain the algebra of continuous functions on the quotient space $X=G/H$, because:

$$\begin{aligned}(\rho\otimes id)\Delta f=1\otimes f &\iff (\rho\otimes id)\Delta f(h,g)=(1\otimes f)(h,g),\forall h\in H,\forall g\in G\\ &\iff f(hg)=f(g),\forall h\in H,\forall g\in G\\ &\iff f(hg)=f(kg),\forall h,k\in H,\forall g\in G\end{aligned}$$

Regarding now the construction of Φ, observe that for $f\in C(X)$ we have:

$$\begin{aligned}(\rho\otimes id\otimes id)(\Delta\otimes id)\Delta f &= (\rho\otimes id\otimes id)(id\otimes\Delta)\Delta f\\ &= (id\otimes\Delta)(\rho\otimes id)\Delta f\\ &= (id\otimes\Delta)(1\otimes f)\\ &= 1\otimes\Delta f\end{aligned}$$

Thus the condition $f\in C(X)$ implies $\Delta f\in C(X)\otimes C(G)$, and this gives the existence of Φ. Finally, the other assertions are all clear. □

As an illustration, in the group dual case we have:

Proposition 4.2. *Assume that $G=\widehat{\Gamma}$ is a discrete group dual.*

(1) *The quantum subgroups of G are $H=\widehat{\Lambda}$, with $\Gamma\to\Lambda$ being a quotient group.*
(2) *For such a quantum subgroup $\widehat{\Lambda}\subset\widehat{\Gamma}$, we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, where:*

$$\Theta=\ker(\Gamma\to\Lambda)$$

Proof. This is well-known, the idea being as follows:

(1) In one sense, this is clear. Conversely, since the algebra $C(G)=C^*(\Gamma)$ is cocommutative, so are all its quotients, and this gives the result.

(2) Consider a quotient map $r:\Gamma\to\Lambda$, and denote by $\rho:C^*(\Gamma)\to C^*(\Lambda)$ its extension. Consider a group algebra element, written as follows:

$$f=\sum_{g\in\Gamma}\lambda_g\cdot g\in C^*(\Gamma)$$

We have then the following computation:

$$\begin{aligned}
f\in C(\widehat{\Gamma}/\widehat{\Lambda})\quad&\iff\quad(\rho\otimes id)\Delta(f)=1\otimes f\\
&\iff\quad\sum_{g\in\Gamma}\lambda_g\cdot r(g)\otimes g=\sum_{g\in\Gamma}\lambda_g\cdot 1\otimes g\\
&\iff\quad\lambda_g\cdot r(g)=\lambda_g\cdot 1,\forall g\in\Gamma\\
&\iff\quad supp(f)\subset\ker(r)
\end{aligned}$$

But this means that we have $\widehat{\Gamma}/\widehat{\Lambda}=\widehat{\Theta}$, with $\Theta=\ker(\Gamma\to\Lambda)$, as claimed. □

Given two compact quantum spaces X,Y, we say that X is a quotient space of Y when we have an embedding of C^*-algebras $\alpha:C(X)\subset C(Y)$. We have:

Definition 4.3. *We call a quotient space $G\to X$ homogeneous when*

$$\Delta(C(X))\subset C(X)\otimes C(G)$$

where $\Delta:C(G)\to C(G)\otimes C(G)$ is the comultiplication map.

In other words, an homogeneous quotient space $G\to X$ is a quantum space coming from a subalgebra $C(X)\subset C(G)$, which is stable under the comultiplication. The relation with the quotient spaces from Proposition 4.1 is as follows:

Theorem 4.4. *The following results hold:*

(1) *The quotient spaces $X=G/H$ are homogeneous.*
(2) *In the classical case, any homogeneous space is of type G/H.*
(3) *In general, there are homogeneous spaces which are not of type G/H.*

PROOF. Once again these results are well-known, the proof being as follows:

(1) This is clear from Proposition 4.1.

(2) Consider a quotient map $p : G \to X$. The invariance condition in the statement tells us that we must have an action $G \curvearrowright X$, given by:

$$g(p(g')) = p(gg')$$

Thus, we have the following implication:

$$p(g') = p(g'') \implies p(gg') = p(gg''),\ \forall g \in G$$

Now observe that the following subset $H \subset G$ is a subgroup:

$$H = \left\{ g \in G \middle| p(g) = p(1) \right\}$$

Indeed, $g, h \in H$ implies that we have:

$$p(gh) = p(g) = p(1)$$

Thus we have $gh \in H$, and the other axioms are satisfied as well. Our claim now is that we have an identification $X = G/H$, obtained as follows:

$$p(g) \to Hg$$

Indeed, the map $p(g) \to Hg$ is well-defined and bijective, because $p(g) = p(g')$ is equivalent to $p(g^{-1}g') = p(1)$, and so to $Hg = Hg'$, as desired.

(3) Given a discrete group Γ and an arbitrary subgroup $\Theta \subset \Gamma$, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is homogeneous. Now by using Proposition 4.2, we can see that if $\Theta \subset \Gamma$ is not normal, the quotient space $\widehat{\Gamma} \to \widehat{\Theta}$ is not of the form G/H. □

With the above formalism in hand, let us try now to understand the general properties of the homogeneous spaces $G \to X$, in the sense of Theorem 4.4. We have:

PROPOSITION 4.5. *Assume that a quotient space $G \to X$ is homogeneous.*

(1) *We have a coaction map as follows, obtained as restriction of Δ:*

$$\Phi : C(X) \to C(X) \otimes C(G)$$

(2) *We have the following implication:*

$$\Phi(f) = f \otimes 1 \implies f \in \mathbb{C}1$$

(3) *We have as well the following formula:*

$$\left(id \otimes \int_G \right) \Phi f = \int_G f$$

(4) *The restriction of $\int_G$ is the unique unital form satisfying:*

$$(\tau \otimes id)\Phi = \tau(.)1$$

PROOF. These results are all elementary, the proof being as follows:

(1) This is clear from definitions, because Δ itself is a coaction.

(2) Assume that $f \in C(G)$ satisfies $\Delta(f) = f \otimes 1$. By applying the counit we obtain:

$$(\varepsilon \otimes id)\Delta f = (\varepsilon \otimes id)(f \otimes 1)$$

We conclude from this that we have $f = \varepsilon(f)1$, as desired.

(3) The formula in the statement, $(id \otimes \int_G)\Phi f = \int_G f$, follows indeed from the left invariance property of the Haar functional of $C(G)$, namely:

$$\left(id \otimes \int_G\right)\Delta f = \int_G f$$

(4) We use here the right invariance of the Haar functional of $C(G)$, namely:

$$\left(\int_G \otimes id\right)\Delta f = \int_G f$$

Indeed, we obtain from this that $tr = (\int_G)_{|C(X)}$ is G-invariant, in the sense that:

$$(tr \otimes id)\Phi f = tr(f)1$$

Conversely, assuming that $\tau : C(X) \to \mathbb{C}$ satisfies $(\tau \otimes id)\Phi f = \tau(f)1$, we have:

$$\begin{aligned}
\left(\tau \otimes \int_G\right)\Phi(f) &= \int_G (\tau \otimes id)\Phi(f)\\
&= \int_G (\tau(f)1)\\
&= \tau(f)
\end{aligned}$$

On the other hand, we can compute the same quantity as follows:

$$\begin{aligned}
\left(\tau \otimes \int_G\right)\Phi(f) &= \tau\left(id \otimes \int_G\right)\Phi(f)\\
&= \tau(tr(f)1)\\
&= tr(f)
\end{aligned}$$

Thus we have $\tau(f) = tr(f)$ for any $f \in C(X)$, and this finishes the proof. □

Summarizing, we have a notion of noncommutative homogeneous space, which perfectly covers the classical case. In general, however, the group dual case shows that our formalism is more general than that of the quotient spaces G/H.

We discuss now an extra issue, of analytic nature. The point indeed is that for one of the most basic examples of actions, namely $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, the associated morphism $\alpha : C(X) \to C(G)$ is not injective. The same is true for other basic actions, in the free setting. In order to include such examples, we must relax our axioms:

Definition 4.6. *An extended homogeneous space over a compact quantum group G consists of a morphism of C^*-algebras, and a coaction map, as follows,*

$$\alpha : C(X) \to C(G)$$

$$\Phi : C(X) \to C(X) \otimes C(G)$$

such that the following diagram commutes

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \ \Phi\ \ } & C(X) \otimes C(G) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha \otimes id} \\ C(G) & \xrightarrow{\ \ \Delta\ \ } & C(G) \otimes C(G) \end{array}$$

and such that the following diagram commutes as well,

$$\begin{array}{ccc} C(X) & \xrightarrow{\ \ \Phi\ \ } & C(X) \otimes C(G) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle id \otimes \int} \\ C(G) & \xrightarrow{\ \ \int(.)1\ \ } & C(X) \end{array}$$

where $\int$ is the Haar integration over G. We write then $G \to X$.

As a first observation, when the morphism α is injective we obtain an homogeneous space in the previous sense. The examples with α not injective, which motivate the above formalism, include the standard action $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, and the standard action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$. Here are a few general remarks on the above axioms:

Proposition 4.7. *Assume that we have morphisms of C^*-algebras*

$$\alpha : C(X) \to C(G)$$

$$\Phi : C(X) \to C(X) \otimes C(G)$$

satisfying the coassociativity condition $(\alpha \otimes id)\Phi = \Delta\alpha$.

1. *If α is injective on a dense $*$-subalgebra $A \subset C(X)$, and $\Phi(A) \subset A \otimes C(G)$, then Φ is automatically a coaction map, and is unique.*
2. *The ergodicity type condition $(id \otimes \int)\Phi = \int \alpha(.)1$ is equivalent to the existence of a linear form $\lambda : C(X) \to \mathbb{C}$ such that $(id \otimes \int)\Phi = \lambda(.)1$.*

Proof. This is something elementary, the idea being as follows:

(1) Assuming that we have a dense $*$-subalgebra $A \subset C(X)$ as in the statement, satisying $\Phi(A) \subset A \otimes C(G)$, the restriction $\Phi_{|A}$ is given by:

$$\Phi_{|A} = (\alpha_{|A} \otimes id)^{-1} \Delta \alpha_{|A}$$

This restriction and is therefore coassociative, and unique. By continuity, the morphism Φ itself follows to be coassociative and unique, as desired.

(2) Assuming $(id\otimes\int)\Phi=\lambda(.)1$, we have:

$$\left(\alpha\otimes\int\right)\Phi=\lambda(.)1$$

On the other hand, we have as well the following formula:

$$\left(\alpha\otimes\int\right)\Phi=\left(id\otimes\int\right)\Delta\alpha=\int\alpha(.)1$$

Thus we obtain $\lambda=\int\alpha$, as claimed. ☐

Given an extended homogeneous space $G\to X$ in our sense, with associated map $\alpha:C(X)\to C(G)$, we can consider the image of this latter map:

$$\alpha:C(X)\to C(Y)\subset C(G)$$

Equivalently, at the level of the associated noncommutative spaces, we can factorize the corresponding quotient map $G\to Y\subset X$. With these conventions, we have:

Proposition 4.8. *Consider an extended homogeneous space $G\to X$.*

(1) $\Phi(f)=f\otimes 1\implies f\in\mathbb{C}1$.
(2) *$tr=\int\alpha$ is the unique unital G-invariant form on $C(X)$.*
(3) *The image space obtained by factorizing, $G\to Y$, is homogeneous.*

Proof. We have several assertions to be proved, the idea being as follows:

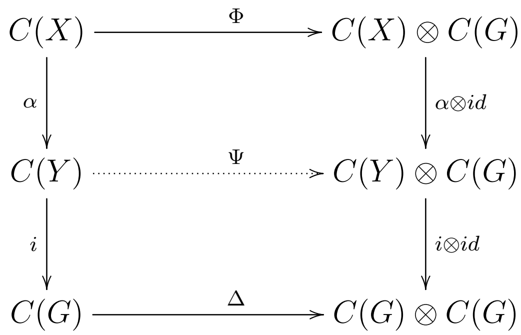
(1) This follows indeed from $(id\otimes\int)\Phi(f)=\int\alpha(f)1$, which gives $f=\int\alpha(f)1$.

(2) The fact that $tr=\int\alpha$ is indeed G-invariant can be checked as follows:

$$\begin{aligned}(tr\otimes id)\Phi f&=(\textstyle\int\alpha\otimes id)\Phi f\\&=(\textstyle\int\otimes id)\Delta\alpha f\\&=\textstyle\int\alpha(f)1\\&=tr(f)1\end{aligned}$$

As for the uniqueness assertion, this follows as before.

(3) The condition $(\alpha \otimes id)\Phi = \Delta\alpha$, together with the fact that i is injective, allows us to factorize Δ into a morphism Ψ, as follows:

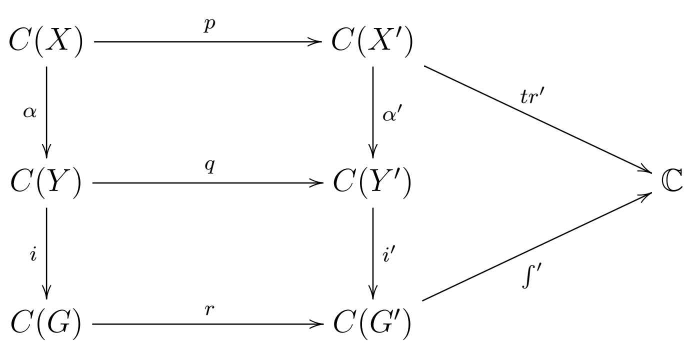


Thus the image space $G \to Y$ is indeed homogeneous, and we are done. $\square$

Finally, we have the following result:

THEOREM 4.9. *Let $G \to X$ be an extended homogeneous space, and construct quotients $X \to X'$, $G \to G'$ by performing the GNS construction with respect to $\int\alpha, \int$. Then α factorizes into an inclusion $\alpha' : C(X') \to C(G')$, and we have an homogeneous space.*

PROOF. We factorize $G \to Y \subset X$ as above. By performing the GNS construction with respect to $\int i\alpha, \int i, \int$, we obtain a diagram as follows:



Indeed, with $tr = \int\alpha$, the GNS quotient maps p, q, r are defined respectively by:

$$\begin{aligned}
\ker p &= \left\{ f \in C(X) \Big| tr(f^*f) = 0 \right\} \\
\ker q &= \left\{ f \in C(Y) \Big| \int(f^*f) = 0 \right\} \\
\ker r &= \left\{ f \in C(G) \Big| \int(f^*f) = 0 \right\}
\end{aligned}$$

Next, we can define factorizations i', α' as above. Observe that i' is injective, and that α' is surjective. Our claim now is that α' is injective as well. Indeed:

$$\begin{aligned}\alpha' p(f) = 0 &\implies q\alpha(f) = 0\\ &\implies \int \alpha(f^*f) = 0\\ &\implies tr(f^*f) = 0\\ &\implies p(f) = 0\end{aligned}$$

We conclude that we have $X' = Y'$, and this gives the result. □

4b. Partial isometries

Our task now will be that of finding a suitable collection of "free homogeneous spaces", generalizing at the same time the free spheres S, and the free unitary groups U. This can be done at several levels of generality, and central here is the construction of the free spaces of partial isometries, which can be done in fact for any easy quantum group. In order to explain this, let us start with the classical case. We have here:

Definition 4.10. *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{T : E \to F \text{ isometry}\Big| E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L\right\}$$

$$U_{MN}^L = \left\{T : E \to F \text{ isometry}\Big| E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L\right\}$$

where the notion of isometry is with respect to the usual real/complex scalar products.

As a first observation, at $L = M = N$ we obtain the groups O_N, U_N:

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of O_{1N}^1 are the isometries $T : E \to \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$, and so our results here are:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1} \quad , \quad U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of O_{1N}^1 are the isometries $T : \mathbb{R} \to F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely determined by $T(1) \in \mathbb{R}^M$, which must belong to $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$, and so our results here are:

$$O_{M1}^1 = S_{\mathbb{R}}^{M-1} \quad , \quad U_{M1}^1 = S_{\mathbb{C}}^{M-1}$$

In general, the most convenient is to view the elements of O_{MN}^L, U_{MN}^L as rectangular matrices, and to use matrix calculus for their study. We have indeed:

Proposition 4.11. *We have identifications of compact spaces*

$$O_{MN}^L\simeq\left\{U\in M_{M\times N}(\mathbb R)\Big|UU^t=\text{projection of trace }L\right\}$$

$$U_{MN}^L\simeq\left\{U\in M_{M\times N}(\mathbb C)\Big|UU^*=\text{projection of trace }L\right\}$$

with each partial isometry being identified with the corresponding rectangular matrix.

Proof. We can indeed identify the partial isometries $T:E\to F$ with their corresponding extensions $U:\mathbb R^N\to\mathbb R^M$, $U:\mathbb C^N\to\mathbb C^M$, obtained by setting $U_{E^\perp}=0$. Then, we can identify these latter maps U with the corresponding rectangular matrices. $\square$

As an illustration, at $L=M=N$ we recover in this way the usual matrix description of O_N,U_N. Also, at $L=M=1$ we obtain the usual description of $S^{N-1}_\mathbb R,S^{N-1}_\mathbb C$, as row spaces over the corresponding groups O_N,U_N. Finally, at $L=N=1$ we obtain the usual description of $S^{N-1}_\mathbb R,S^{N-1}_\mathbb C$, as column spaces over the corresponding groups O_N,U_N.

Now back to the general case, observe that the isometries $T:E\to F$, or rather their extensions $U:\mathbb K^N\to\mathbb K^M$, with $\mathbb K=\mathbb R,\mathbb C$, obtained by setting $U_{E^\perp}=0$, can be composed with the isometries of $\mathbb K^M,\mathbb K^N$, according to the following scheme:

$$\begin{array}{ccccccc}
\mathbb K^N & \xrightarrow{B^*} & \mathbb K^N & \overset{U}{\dashrightarrow} & \mathbb K^M & \xrightarrow{A} & \mathbb K^M\\
\uparrow & & \uparrow & & \uparrow & & \uparrow\\
B(E) & \dashrightarrow & E & \xrightarrow{T} & F & \dashrightarrow & A(F)
\end{array}$$

With the identifications in Proposition 4.11 made, the precise statement here is:

Proposition 4.12. *We have action maps as follows, which are both transitive,*

$$O_M\times O_N\curvearrowright O_{MN}^L\quad,\quad (A,B)U=AUB^t$$

$$U_M\times U_N\curvearrowright U_{MN}^L\quad,\quad (A,B)U=AUB^*$$

whose stabilizers are respectively $O_L\times O_{M-L}\times O_{N-L}$ *and* $U_L\times U_{M-L}\times U_{N-L}$.

Proof. We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer G of the following point:

$$U=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

Since $(A,B)\in G$ satisfy $AU=UB$, their components must be of the following form:

$$A=\begin{pmatrix}x&*\\0&a\end{pmatrix}\quad,\quad B=\begin{pmatrix}x&0\\ *&b\end{pmatrix}$$

Now since A,B are unitaries, these matrices follow to be block-diagonal, and so:

$$G=\left\{(A,B)\Big|A=\begin{pmatrix}x&0\\0&a\end{pmatrix},B=\begin{pmatrix}x&0\\0&b\end{pmatrix}\right\}$$

The stabilizer of U is parametrized by triples (x,a,b) belonging to $O_L\times O_{M-L}\times O_{N-L}$ and $U_L\times U_{M-L}\times U_{N-L}$, and we are led to the conclusion in the statement. □

Finally, let us work out the quotient space description of O_{MN}^L,U_{MN}^L. We have here:

THEOREM 4.13. *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned}O_{MN}^L&=(O_M\times O_N)/(O_L\times O_{M-L}\times O_{N-L})\\U_{MN}^L&=(U_M\times U_N)/(U_L\times U_{M-L}\times U_{N-L})\end{aligned}$$

with the quotient maps being given by $(A,B)\to AUB^*$, *where* $U=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

PROOF. This is just a reformulation of Proposition 4.12, by taking into account the fact that the fixed point used in the proof there was $U=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. □

Once again, the basic examples here come from the cases $L=M=N$ and $L=M=1$. At $L=M=N$ the quotient spaces at right are respectively:

$$O_N\quad,\quad U_N$$

At $L=M=1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1}\quad,\quad U_N/U_{N-1}$$

In fact, in the general $L=M$ case we obtain the following spaces:

$$O_{MN}^M=O_N/O_{N-M}\quad,\quad U_{MN}^M=U_N/U_{N-M}$$

Similarly, the examples coming from the cases $L=M=N$ and $L=N=1$ are particular cases of the general $L=N$ case, where we obtain the following spaces:

$$O_{MN}^N=O_N/O_{M-N}\quad,\quad U_{MN}^N=U_N/U_{M-N}$$

Summarizing, we have here some basic homogeneous spaces, unifying the spheres with the rotation groups. The point now is that we can liberate these spaces, as follows:

DEFINITION 4.14. *Associated to any integers* $L\leq M,N$ *are the algebras*

$$\begin{aligned}C(O_{MN}^{L+})&=C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace }L\right)\\C(U_{MN}^{L+})&=C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace }L\right)\end{aligned}$$

with the trace being by definition the sum of the diagonal entries.

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij}u_{ij}u_{ij}^*=\sum_{ij}u_{ij}^*u_{ij}=L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc}
O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+}\\
\uparrow & & \uparrow\\
O_{MN}^L & \longrightarrow & U_{MN}^L
\end{array}$$

At the level of basic examples now, at $L=M=1$ and at $L=N=1$ we obtain the following diagrams, showing that our formalism covers indeed the free spheres:

$$\begin{array}{ccc}
S^{N-1}_{\mathbb R,+} & \longrightarrow & S^{N-1}_{\mathbb C,+}\\
\uparrow & & \uparrow\\
S^{N-1}_{\mathbb R} & \longrightarrow & S^{N-1}_{\mathbb C}
\end{array}\qquad\qquad
\begin{array}{ccc}
S^{M-1}_{\mathbb R,+} & \longrightarrow & S^{M-1}_{\mathbb C,+}\\
\uparrow & & \uparrow\\
S^{M-1}_{\mathbb R} & \longrightarrow & S^{M-1}_{\mathbb C}
\end{array}$$

We have as well the following result, in relation with the free rotation groups:

Proposition 4.15. *At $L=M=N$ we obtain the diagram*

$$\begin{array}{ccc}
O_N^+ & \longrightarrow & U_N^+\\
\uparrow & & \uparrow\\
O_N & \longrightarrow & U_N
\end{array}$$

consisting of the groups O_N,U_N, and their liberations.

Proof. We recall that the various quantum groups in the statement are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$\begin{aligned}
C(O_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=u^tu=1\right)\\
C(U_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1,\bar{u}u^t=u^t\bar{u}=1\right)
\end{aligned}$$

On the other hand, according to Proposition 4.11 and to Definition 4.14, we have the following presentation results:

$$\begin{aligned}C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u}, uu^t=\text{projection of trace } N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace } N\right)\end{aligned}$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. We use as well the following formulae:

$$Tr(uu^*)=Tr(u^t\bar{u})\quad,\quad Tr(\bar{u}u^t)=Tr(u^*u)$$

We therefore obtain the following formulae:

$$\begin{aligned}C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t,u^tu=\text{projections of trace } N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,u^*u,\bar{u}u^t,u^t\bar{u}=\text{projections of trace } N\right)\end{aligned}$$

Now observe that, in tensor product notation, the conditions at right are all of the form $(tr\otimes id)p=1$. Thus, p must be follows, for the above conditions:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

We therefore obtain that, for any faithful state φ, we have $(tr\otimes\varphi)(1-p)=0$. It follows from this that the following projections must be all equal to the identity:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

But this leads to the conclusion in the statement. □

Regarding now the homogeneous space structure of $O_{MN}^{L\times},U_{MN}^{L\times}$, the situation here is a bit more complicated in the free case than in the classical case, due to a number of algebraic and analytic issues. We first have the following result:

PROPOSITION 4.16. *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times\times U_N^\times\curvearrowright U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$.*
(2) *We have a map $U_M^\times\times U_N^\times\to U_{MN}^{L\times}$, given by $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$.*

Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.

PROOF. In the classical case, consider the following action and quotient maps:

$$U_M\times U_N\curvearrowright U_{MN}^L\quad,\quad U_M\times U_N\to U_{MN}^L$$

The transposes of these two maps are as follows, where $J=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi &\to ((U,A,B)\to\varphi(AUB^*))\\ \varphi &\to ((A,B)\to\varphi(AJB^*))\end{aligned}$$

But with $\varphi=u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u = u$, let us set:

$$U_{ij} = \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^*$$

We have then the following computation:

$$\begin{aligned}
(UU^*U)_{ij} &= \sum_{pq} \sum_{klmnst} u_{kl}u_{mn}^*u_{st} \otimes a_{ki}a_{mq}^*a_{sq} \otimes b_{lp}^*b_{np}b_{tj}^* \\
&= \sum_{klmt} u_{kl}u_{ml}^*u_{mt} \otimes a_{ki} \otimes b_{tj}^* \\
&= \sum_{kt} u_{kt} \otimes a_{ki} \otimes b_{tj}^* \\
&= U_{ij}
\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}
\sum_{ij} U_{ij}U_{ij}^* &= \sum_{ij} \sum_{klst} u_{kl}u_{st}^* \otimes a_{ki}a_{si}^* \otimes b_{lj}^*b_{tj} \\
&= \sum_{kl} u_{kl}u_{kl}^* \otimes 1 \otimes 1 \\
&= L
\end{aligned}$$

(2) Assuming $uu^*u = u$, let us set:

$$V_{ij} = \sum_{r \leq L} a_{ri} \otimes b_{rj}^*$$

We have then the following computation:

$$\begin{aligned}
(VV^*V)_{ij} &= \sum_{pq} \sum_{x,y,z \leq L} a_{xi}a_{yq}^*a_{zq} \otimes b_{xp}^*b_{yp}b_{zj}^* \\
&= \sum_{x \leq L} a_{xi} \otimes b_{xj}^* \\
&= V_{ij}
\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}
\sum_{ij} V_{ij}V_{ij}^* &= \sum_{ij} \sum_{r,s \leq L} a_{ri}a_{si}^* \otimes b_{rj}^*b_{sj} \\
&= \sum_{l \leq L} 1 \\
&= L
\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. $\square$

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X = G/H$, the associated action and quotient maps are given by:

$$\begin{cases} a : X \times G \to X & : \quad (Hg, h) \to Hgh \\ p : G \to X & : \quad g \to Hg \end{cases}$$

Thus we have $a(p(g), h) = p(gh)$. In our context, a similar result holds:

THEOREM 4.17. *With $G = G_M \times G_N$ and $X = G_{MN}^L$, where $G_N = O_N^\times, U_N^\times$, we have*

$$\begin{array}{ccc} G \times G & \xrightarrow{\quad m \quad} & G \\ \Big\downarrow{\scriptstyle p\times id} & & \Big\downarrow{\scriptstyle p} \\ X \times G & \xrightarrow{\quad a \quad} & X \end{array}$$

where a, p are the action map and the map constructed in Proposition 4.16.

PROOF. At the level of the associated algebras of functions, we must prove that the following diagram commutes, where Φ, α are morphisms of algebras induced by a, p:

$$\begin{array}{ccc} C(X) & \xrightarrow{\quad \Phi \quad} & C(X \times G) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha\otimes id} \\ C(G) & \xrightarrow{\quad \Delta \quad} & C(G \times G) \end{array}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned} (\alpha \otimes id)\Phi(u_{ij}) &= (\alpha \otimes id) \sum_{kl} u_{kl} \otimes a_{ki} \otimes b_{lj}^* \\ &= \sum_{kl} \sum_{r \leq L} a_{rk} \otimes b_{rl}^* \otimes a_{ki} \otimes b_{lj}^* \end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where F_{23} is the flip between the second and the third components:

$$\begin{aligned} \Delta\pi(u_{ij}) &= F_{23}(\Delta \otimes \Delta) \sum_{r \leq L} a_{ri} \otimes b_{rj}^* \\ &= F_{23} \left(\sum_{r \leq L} \sum_{kl} a_{rk} \otimes a_{ki} \otimes b_{rl}^* \otimes b_{lj}^* \right) \end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. □

4c. Partial permutations

Let us discuss now some discrete extensions of the above constructions. We have:

DEFINITION 4.18. *Associated to a partial permutation, $\sigma : I \simeq J$ with $I \subset \{1, \ldots, N\}$ and $J \subset \{1, \ldots, M\}$, is the real/complex partial isometry*

$$T_\sigma : span\left(e_i\Big|i \in I\right) \to span\left(e_j\Big|j \in J\right)$$

given on the standard basis elements by $T_\sigma(e_i) = e_{\sigma(i)}$.

Let S_{MN}^L be the set of partial permutations $\sigma : I \simeq J$ as above, with range $I \subset \{1, \ldots, N\}$ and target $J \subset \{1, \ldots, M\}$, and with $L = |I| = |J|$. We have:

PROPOSITION 4.19. *The space of partial permutations signed by elements of $\mathbb{Z}_s$,*

$$H_{MN}^{sL} = \left\{T(e_i) = w_i e_{\sigma(i)}\Big|\sigma \in S_{MN}^L, w_i \in \mathbb{Z}_s\right\}$$

is isomorphic to the quotient space

$$(H_M^s \times H_N^s)/(H_L^s \times H_{M-L}^s \times H_{N-L}^s)$$

via a standard isomorphism.

PROOF. This follows by adapting the computations in the proof of Proposition 4.12 and Theorem 4.13. Indeed, we have an action map as follows, which is transitive:

$$H_M^s \times H_N^s \to H_{MN}^{sL} \quad , \quad (A, B)U = AUB^*$$

Consider now the following point:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The stabilizer of this point follows to be the following group:

$$H_L^s \times H_{M-L}^s \times H_{N-L}^s$$

To be more precise, this group is embedded via:

$$(x, a, b) \to \left[\begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & b \end{pmatrix}\right]$$

But this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$. The result here is as follows:

PROPOSITION 4.20. *The compact quantum space H_{MN}^{sL+} associated to the algebra*

$$C(H_{MN}^{sL+}) = C(U_{MN}^{L+}) \Big/ \left\langle u_{ij} u_{ij}^* = u_{ij}^* u_{ij} = p_{ij} = \text{projections}, u_{ij}^s = p_{ij} \right\rangle$$

has an action map, and is the target of a quotient map, as in Theorem 4.17.

PROOF. We must show that if the variables u_{ij} satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\quad,\quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We use the fact that the standard coordinates a_{ij},b_{ij} on the quantum groups H_M^{s+},H_N^{s+} satisfy the following relations, for any $x\neq y$ on the same row or column of a,b:

$$xy=xy^*=0$$

We obtain, by using these relations, the following formula:

$$U_{ij}U_{ij}^*=\sum_{klmn}u_{kl}u_{mn}^*\otimes a_{ki}a_{mi}^*\otimes b_{lj}^*b_{mj}=\sum_{kl}u_{kl}u_{kl}^*\otimes a_{ki}a_{ki}^*\otimes b_{lj}^*b_{lj}$$

On the other hand, we have as well the following formula:

$$V_{ij}V_{ij}^*=\sum_{r,t\leq L}a_{ri}a_{ti}^*\otimes b_{rj}^*b_{tj}=\sum_{r\leq L}a_{ri}a_{ri}^*\otimes b_{rj}^*b_{rj}$$

In terms of the projections $x_{ij}=a_{ij}a_{ij}^*$, $y_{ij}=b_{ij}b_{ij}^*$, $p_{ij}=u_{ij}u_{ij}^*$, we have:

$$U_{ij}U_{ij}^*=\sum_{kl}p_{kl}\otimes x_{ki}\otimes y_{lj}\quad,\quad V_{ij}V_{ij}^*=\sum_{r\leq L}x_{ri}\otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij},V_{ij}^*V_{ij}$ are given by the same formulae. Finally, once again by using the relations of type $xy=xy^*=0$, we have:

$$U_{ij}^s=\sum_{k_rl_r}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i}\ldots a_{k_si}\otimes b_{l_1j}^*\ldots b_{l_sj}^*=\sum_{kl}u_{kl}^s\otimes a_{ki}^s\otimes(b_{lj}^*)^s$$

On the other hand, we have as well the following formula:

$$V_{ij}^s=\sum_{r_l\leq L}a_{r_1i}\ldots a_{r_si}\otimes b_{r_1j}^*\ldots b_{r_sj}^*=\sum_{r\leq L}a_{ri}^s\otimes(b_{rj}^*)^s$$

Thus the conditions of type $u_{ij}^s=p_{ij}$ are satisfied as well, and we are done. □

Let us discuss now the general case. We have the following result:

PROPOSITION 4.21. *The various spaces G_{MN}^L constructed so far appear by imposing to the standard coordinates of U_{MN}^{L+} the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with $s=(e_1,\ldots,e_s)$ ranging over all the colored integers, and with $\pi,\sigma\in D(0,s)$.

PROOF. According to the various constructions above, the relations defining the quantum space G_{MN}^L can be written as follows, with σ ranging over a family of generators, with no upper legs, of the corresponding category of partitions D:

$$\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=\delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned}\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s} &= \sum_{i_1\ldots i_s}\delta_\pi(i)\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}\\ &= \sum_{i_1\ldots i_s}\delta_\pi(i)\delta_\sigma(i)\\ &= L^{|\pi\vee\sigma|}\end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping π fixed, and by making σ vary over all the partitions in the category. □

In the general case now, where $G=(G_N)$ is an arbitary uniform easy quantum group, we can construct spaces G_{MN}^L by using the above relations, and we have:

THEOREM 4.22. *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

with π,σ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 4.17.

PROOF. We proceed as in the proof of Proposition 4.20. We must prove that, if the variables u_{ij} satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\quad,\quad V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Regarding the variables U_{ij}, the computation here goes as follows:

$$\begin{aligned}&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\ =&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\otimes\delta_\pi(i)\delta_\sigma(j)a_{k_1i_1}^{e_1}\ldots a_{k_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\ =&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\\ =&\ L^{|\pi\vee\sigma|}\end{aligned}$$

For the variables V_{ij} the proof is similar, as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{l_1i_1}^{e_1}\ldots a_{l_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)\\
=&\ L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 4.20, and the commutation of the diagram in Theorem 4.17 is then trivial. □

4d. Integration results

Let us discuss now the integration over the various noncommutative spaces constructed so far, and notably over the spaces G_{MN}^L, which are quite general. We first have:

Definition 4.23. *The integration functional of G_{MN}^L is the composition*

$$\int_{G_{MN}^L}:C(G_{MN}^L)\to C(G_M\times G_N)\to\mathbb{C}$$

of the representation $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$ with the Haar functional of $G_M\times G_N$.

Observe that in the case $L=M=N$ we obtain the integration over G_N. Also, at $L=M=1$, or at $L=N=1$, we obtain the integration over the sphere.

In the general case now, we first have the following result:

Proposition 4.24. *The integration functional of G_{MN}^L has the invariance property*

$$\left(\int_{G_{MN}^L}\otimes\, id\right)\Phi(x)=\int_{G_{MN}^L}x$$

with respect to the coaction map, namely:

$$\Phi(u_{ij})=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

Proof. We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L}\otimes\, id\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned}
X &= \left(\int_{G_{MN}^L} \otimes\, id\right) \sum_{k_x l_x} u_{k_1 l_1} \ldots u_{k_s l_s} \otimes a_{k_1 i_1} \ldots a_{k_s i_s} \otimes b^*_{l_1 j_1} \ldots b^*_{l_s j_s} \\
&= \sum_{k_x l_x} \sum_{r_x \leq L} a_{k_1 i_1} \ldots a_{k_s i_s} \otimes b^*_{l_1 j_1} \ldots b^*_{l_s j_s} \int_{G_M} a_{r_1 k_1} \ldots a_{r_s k_s} \int_{G_N} b^*_{r_1 l_1} \ldots b^*_{r_s l_s} \\
&= \sum_{r_x \leq L} \sum_{k_x} a_{k_1 i_1} \ldots a_{k_s i_s} \int_{G_M} a_{r_1 k_1} \ldots a_{r_s k_s} \otimes \sum_{l_x} b^*_{l_1 j_1} \ldots b^*_{l_s j_s} \int_{G_N} b^*_{r_1 l_1} \ldots b^*_{r_s l_s}
\end{aligned}$$

By using now the invariance property of the Haar functionals of G_M, G_N, we obtain:

$$\begin{aligned}
X &= \sum_{r_x \leq L} \left(\int_{G_M} \otimes\, id\right) \Delta(a_{r_1 i_1} \ldots a_{r_s i_s}) \otimes \left(\int_{G_N} \otimes\, id\right) \Delta(b^*_{r_1 j_1} \ldots b^*_{r_s j_s}) \\
&= \sum_{r_x \leq L} \int_{G_M} a_{r_1 i_1} \ldots a_{r_s i_s} \int_{G_N} b^*_{r_1 j_1} \ldots b^*_{r_s j_s} \\
&= \left(\int_{G_M} \otimes \int_{G_N}\right) \sum_{r_x \leq L} a_{r_1 i_1} \ldots a_{r_s i_s} \otimes b^*_{r_1 j_1} \ldots b^*_{r_s j_s}
\end{aligned}$$

But this gives the formula in the statement, and we are done. □

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

Theorem 4.25. *We have the Weingarten type formula*

$$\int_{G_{MN}^L} u_{i_1 j_1} \ldots u_{i_s j_s} = \sum_{\pi\sigma\tau\nu} L^{|\pi \vee \tau|} \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)$$

where the matrices on the right are given by $W_{sM} = G_{sM}^{-1}$*, with* $G_{sM}(\pi, \sigma) = M^{|\pi \vee \sigma|}$.

Proof. We make use of the usual quantum group Weingarten formula, that we know from chapters 2-3. By using this formula for G_M, G_N, we obtain:

$$\begin{aligned}
\int_{G_{MN}^L} u_{i_1 j_1} \ldots u_{i_s j_s} &= \sum_{l_1 \ldots l_s \leq L} \int_{G_M} a_{l_1 i_1} \ldots a_{l_s i_s} \int_{G_N} b^*_{l_1 j_1} \ldots b^*_{l_s j_s} \\
&= \sum_{l_1 \ldots l_s \leq L} \sum_{\pi\sigma} \delta_\pi(l) \delta_\sigma(i) W_{sM}(\pi, \sigma) \sum_{\tau\nu} \delta_\tau(l) \delta_\nu(j) W_{sN}(\tau, \nu) \\
&= \sum_{\pi\sigma\tau\nu} \left(\sum_{l_1 \ldots l_s \leq L} \delta_\pi(l) \delta_\tau(l)\right) \delta_\sigma(i) \delta_\nu(j) W_{sM}(\pi, \sigma) W_{sN}(\tau, \nu)
\end{aligned}$$

The coefficient being $L^{|\pi \vee \tau|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of the integration, as follows:

THEOREM 4.26. *The integration of G_{MN}^L is the unique positive unital trace*

$$C(G_{MN}^L)\to\mathbb{C}$$

which is invariant under the action of the quantum group $G_M\times G_N$.

PROOF. We use a standard method, from [13], the point being to show that we have the following ergodicity formula:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)=\int_{G_{MN}^L}x$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:

$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

By using the Weingarten formula, the left term can be written as follows:

$$\begin{aligned}X&=\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\int_{G_M}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\&=\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\sum_{\pi\sigma}\delta_\pi(k)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\&=\sum_{\pi\sigma\tau\nu}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\tau(l)u_{k_1l_1}\ldots u_{k_sl_s}\end{aligned}$$

By using now the summation formula in Theorem 4.25, we obtain:

$$X=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

Now by comparing with the Weingarten formula for G_{MN}^L, this proves our claim. Assume now that $\tau:C(G_{MN}^L)\to\mathbb{C}$ satisfies the invariance condition. We have:

$$\begin{aligned}\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)&=\left(\tau\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)\\&=\left(\int_{G_M}\otimes\int_{G_N}\right)(\tau\otimes id)\Phi(x)\\&=\left(\int_{G_M}\otimes\int_{G_N}\right)(\tau(x)1)\\&=\tau(x)\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\begin{aligned}\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \tau(tr(x)1)\\ &= tr(x)\end{aligned}$$

Thus we obtain $\tau = tr$, and this finishes the proof. □

As a main application of the above results, we have:

PROPOSITION 4.27. *For a sum of coordinates of the following type,*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

with the coordinates not overlapping on rows and columns, we have

$$\int_{G_{MN}^L}\chi_E^s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

where $K=|E|$ is the cardinality of the indexing set.

PROOF. With $K=|E|$, we can write $E=\{(\alpha(i),\beta(i))\}$, for certain embeddings:

$$\alpha:\{1,\ldots,K\}\subset\{1,\ldots,M\}$$

$$\beta:\{1,\ldots,K\}\subset\{1,\ldots,N\}$$

In terms of these maps α,β, the moment in the statement is given by:

$$M_s=\int_{G_{MN}^L}\left(\sum_{i\leq K}u_{\alpha(i)\beta(i)}\right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned}&M_s\\ =\;&\int_{G_{MN}^L}\sum_{i_1\ldots i_s\leq K}u_{\alpha(i_1)\beta(i_1)}\ldots u_{\alpha(i_s)\beta(i_s)}\\ =\;&\sum_{i_1\ldots i_s\leq K}\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(\alpha(i_1),\ldots,\alpha(i_s))\delta_\tau(\beta(i_1),\ldots,\beta(i_s))W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\\ =\;&\sum_{\pi\sigma\tau\nu}\left(\sum_{i_1\ldots i_s\leq K}\delta_\pi(i)\delta_\tau(i)\right)L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\end{aligned}$$

But, as explained before, in the proof of Theorem 4.25, the coefficient on the left in the last formula is $C=K^{|\pi\vee\tau|}$. We therefore obtain the formula in the statement. □

At a more concrete level now, we have the following conceptual result, making the link with the Bercovici-Pata bijection [19]:

THEOREM 4.28. *In the context of the liberation operations*

$$O^L_{MN}\to O^{L+}_{MN}\quad,\quad U^L_{MN}\to U^{L+}_{MN}\quad,\quad H^{sL}_{MN}\to H^{sL+}_{MN}$$

the laws of the sums of non-overlapping coordinates,

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

are in Bercovici-Pata bijection, in the

$$|E|=\kappa N, L=\lambda N, M=\mu N$$

regime and $N\to\infty$ limit.

PROOF. This follows indeed from the formula in Proposition 4.27. □

4e. Exercises

Exercises:

EXERCISE 4.29.

EXERCISE 4.30.

EXERCISE 4.31.

EXERCISE 4.32.

EXERCISE 4.33.

EXERCISE 4.34.

EXERCISE 4.35.

EXERCISE 4.36.

Bonus exercise.

Part II

Projective manifolds

Homely girl
You used to be so lonely
You're a beautiful woman
Oh, homely girl

CHAPTER 5

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

EXERCISE 5.7.

EXERCISE 5.8.

Bonus exercise.

CHAPTER 6

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

EXERCISE 6.6.

EXERCISE 6.7.

EXERCISE 6.8.

Bonus exercise.

CHAPTER 7

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.

Part III

Abstract algebra

I don't want to hear sad songs anymore
I only want to hear love songs
I found my heart up
In this place tonight

CHAPTER 9

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.

CHAPTER 10

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

EXERCISE 10.7.

EXERCISE 10.8.

Bonus exercise.

CHAPTER 11

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.

CHAPTER 12

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Analytic aspects

As soon as you're born they make you feel small
By giving you no time instead of it all
Till the pain is so big you feel nothing at all
A working class hero is something to be

CHAPTER 13

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Bonus exercise.

CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.

CHAPTER 15

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

Bibliography

[1] V.I. Arnold, Ordinary differential equations, Springer (1973).

[2] V.I. Arnold, Mathematical methods of classical mechanics, Springer (1974).

[3] V.I. Arnold, Catastrophe theory, Springer (1974).

[4] V.I. Arnold, Lectures on partial differential equations, Springer (1997).

[5] V.I. Arnold and B.A. Khesin, Topological methods in hydrodynamics, Springer (1998).

[6] M.F. Atiyah, The geometry and physics of knots, Cambridge Univ. Press (1990).

[7] M.F. Atiyah, The geometry and physics of knots, Cambridge Univ. Press (1990).

[8] M.F. Atiyah and I.G. MacDonald, Introduction to commutative algebra, Addison-Wesley (1969).

[9] T. Banica, Principles of operator algebras (2024).

[10] T. Banica, Affine noncommutative geometry (2024).

[11] T. Banica, Introduction to modern physics (2025).

[12] T. Banica and J. Bichon, Matrix models for noncommutative algebraic manifolds, *J. Lond. Math. Soc.* **95** (2017), 519–540.

[13] T. Banica and D. Goswami, Quantum isometries and noncommutative spheres, *Comm. Math. Phys.* **298** (2010), 343–356.

[14] T. Banica and S. Mészáros, Uniqueness results for noncommutative spheres and projective spaces, *Illinois J. Math.* **59** (2015), 219–233.

[15] T. Banica, A. Skalski and P.M. Sołtan, Noncommutative homogeneous spaces: the matrix case, *J. Geom. Phys.* **62** (2012), 1451–1466.

[16] G.K. Batchelor, An introduction to fluid dynamics, Cambridge Univ. Press (1967).

[17] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[18] E.J. Beggs and S. Majid, Quantum Riemannian geometry, Springer (2020).

[19] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math.* **149** (1999), 1023–1060.

[20] J. Bhowmick, F. D'Andrea and L. Dabrowski, Quantum isometries of the finite noncommutative geometry of the standard model, *Comm. Math. Phys.* **307** (2011), 101–131.

[21] J. Bhowmick, F. D'Andrea, B. Das and L. Dabrowski, Quantum gauge symmetries in noncommutative geometry, *J. Noncommut. Geom.* **8** (2014), 433–471.

[22] J. Bichon, Half-liberated real spheres and their subspaces, *Colloq. Math.* **144** (2016), 273–287.

[23] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. of Math.* **38** (1937), 857–872.

[24] S.M. Carroll, Spacetime and geometry, Cambridge Univ. Press (2004).

[25] A.H. Chamseddine and A. Connes, The spectral action principle, *Comm. Math. Phys.* **186** (1997), 731–750.

[26] A.H. Chamseddine and A. Connes, Why the standard model, *J. Geom. Phys.* **58** (2008), 38–47.

[27] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic groups, *Comm. Math. Phys.* **264** (2006), 773–795.

[28] A. Connes, Noncommutative geometry, Academic Press (1994).

[29] A. Connes, On the spectral characterization of manifolds, *J. Noncommut. Geom.* **7** (2013), 1–82.

[30] A. Connes and M. Dubois-Violette, Moduli space and structure of noncommutative 3-spheres, *Lett. Math. Phys.* **66** (2003), 91–121.

[31] A. Connes and G. Landi, Noncommutative manifolds, the instanton algebra and isospectral deformations, *Comm. Math. Phys.* **221** (2001), 141–160.

[32] A. Connes and M. Marcolli, Noncommutative geometry, quantum fields and motives, AMS (2008).

[33] W.N. Cottingham and D.A. Greenwood, An introduction to the standard model of particle physics, Cambridge Univ. Press (2012).

[34] F. D'Andrea, L. Dabrowski and G. Landi, The noncommutative geometry of the quantum projective plane, *Rev. Math. Phys.* **20** (2008), 979–1006.

[35] B. Das, U. Franz and X. Wang, Invariant Markov semigroups on quantum homogeneous spaces, *J. Noncommut. Geom.* **15** (2021), 531–580.

[36] B. Das and D. Goswami, Quantum Brownian motion on noncommutative manifolds: construction, deformation and exit times, *Comm. Math. Phys.* **309** (2012), 193–228.

[37] P.A. Davidson, Introduction to magnetohydrodynamics, Cambridge Univ. Press (2001).

[38] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.

[39] M.P. do Carmo, Differential geometry of curves and surfaces, Dover (1976).

[40] M.P. do Carmo, Riemannian geometry, Birkhäuser (1992).

[41] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).

[42] A. Einstein, Relativity: the special and the general theory, Dover (1916).

[43] L.C. Evans, Partial differential equations, AMS (1998).

[44] S.J. Farlow, Partial differential equations for scientists and engineers, Dover (1982).

[45] W. Feller, An introduction to probability theory and its applications, Wiley (1950).

[46] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics I: mainly mechanics, radiation and heat, Caltech (1963).

[47] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics II: mainly electromagnetism and matter, Caltech (1964).

[48] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics III: quantum mechanics, Caltech (1966).

[49] W. Fulton, Algebraic topology, Springer (1995).

[50] W. Fulton and J. Harris, Representation theory, Springer (1991).

[51] H. Goldstein, C. Safko and J. Poole, Classical mechanics, Addison-Wesley (1980).

[52] D. Goswami, Quantum group of isometries in classical and noncommutative geometry, *Comm. Math. Phys.* **285** (2009), 141–160.

[53] D. Goswami and J. Bhowmick, Quantum isometry groups, Springer (2016).

[54] J.M. Gracia-Bondía, J.C. Várilly and H. Figueroa, Elements of noncommutative geometry, Birkhäuser (2001).

[55] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).

[56] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).

[57] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).

[58] P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley (1994).

[59] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique, IHES (1967).

[60] A. Grothendieck et al., Séminaire de géométrie algébrique, IHES (1972).

[61] J. Harris, Algebraic geometry, Springer (1992).

[62] R. Hartshorne, Algebraic geometry, Springer (1977).

[63] A. Hatcher, Algebraic topology, Cambridge Univ. Press (2002).

[64] J.E. Humphreys, Introduction to Lie algebras and representation theory, Springer (1972).

[65] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.

[66] V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. Math.* **137** (1989), 311–334.

[67] V.F.R. Jones, Planar algebras I (1999).

[68] T. Kibble and F.H. Berkshire, Classical mechanics, Imperial College Press (1966).

[69] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).

[70] L.D. Landau and E.M. Lifshitz, Mechanics, Pergamon Press (1960).

[71] L.D. Landau and E.M. Lifshitz, The classical theory of fields, Addison-Wesley (1951).

[72] L.D. Landau and E.M. Lifshitz, Quantum mechanics: non-relativistic theory, Pergamon Press (1959).

[73] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, Quantum electrodynamics, Butterworth-Heinemann (1982).

[74] G. Landi, An introduction to noncommutative spaces and their geometry, Springer (1997).

[75] Y.I. Manin, Quantum groups and noncommutative geometry, Springer (2018).

[76] V.A. Marchenko and L.A. Pastur, Distribution of eigenvalues in certain sets of random matrices, *Mat. Sb.* **72** (1967), 507–536.

[77] M. Marcolli, Noncommutative cosmology, World Scientific (2018).

[78] M.L. Mehta, Random matrices, Elsevier (2004).

[79] J. Nash, The imbedding problem for Riemannian manifolds, *Ann. of Math.* **63** (1956), 20–63.

[80] I.R. Shafarevich, Basic algebraic geometry, Springer (1974).

[81] P. Tarrago and M. Weber, Unitary easy quantum groups: the free case and the group case, *Int. Math. Res. Not.* **18** (2017), 5710–5750.

[82] J.R. Taylor, Classical mechanics, Univ. Science Books (2003).

[83] D.V. Voiculescu, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66** (1986), 323–346.

[84] D.V. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201–220.

[85] D.V. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, *Comm. Math. Phys.* **155** (1993), 71–92.

[86] D.V. Voiculescu, A hydrodynamic exercise in free probability: setting up free Euler equations, *Exposition. Math.* **38** (2020), 271–283.

[87] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, AMS (1992).

[88] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).

[89] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[90] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).

[91] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).

[92] S. Weinberg, Lectures on astrophysics, Cambridge Univ. Press (2019).

[93] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *J. Math. Phys.* **19** (1978), 999–1001.

[94] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).

[95] H. Weyl, The classical groups: their invariants and representations, Princeton Univ. Press (1939).

[96] H. Weyl, Space, time, matter, Princeton Univ. Press (1918).

[97] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* **62** (1955), 548–564.

[98] E. Witten, Quantum field theory and the Jones polynomial, *Comm. Math. Phys.* **121** (1989), 351–399.

[99] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[100] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups, *Invent. Math.* **93** (1988), 35–76.