Complex matrices and diagonalization

Teo Banica

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Complex numbers 1/3

The complex numbers are $z = a + ib$, with $i^2 = -1$.

They can be represented in the plane, with z being $\binom{a}{b}$.
We have $z = re^{it}$, with $r = \sqrt{a^2 + b^2}$, and $\tan t = b/a$.

The equation $x^2 = -1$ has two solutions, $x = \pm i$.
In fact, the equation $P(x) = 0$ has $N = \deg P$ solutions.

Also, complex numbers are important in quantum physics.

Complex numbers 2/3

Consider the rotation of angle $t \in \mathbb{R}$:

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

This rotation has 2 complex eigenvectors (!), because:

$$R_t \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos t - i \sin t \\ \sin t + i \cos t \end{pmatrix} = e^{-it} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$R_t \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos t + i \sin t \\ \sin t - i \cos t \end{pmatrix} = e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Thus, good news, R_t is diagonalizable over $\mathbb{C}$.

Complex numbers 3/3

More magics. When identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, the rotation of angle $t \in \mathbb{R}$ becomes a 1×1 matrix (!):

$$R_t = (e^{it})$$

Thus, with complex numbers, this rotation R_t of angle $t \in \mathbb{R}$ in the plane is something completely trivial. Very nice.

Theory 1/4

The theory from the real case extends to this setting:

Theorem. Any linear map $f:\mathbb{C}^N\to\mathbb{C}^N$ is of the form $f(v)=Av$, with $A\in M_N(\mathbb{C})$.

Theorem. More generally, any linear map $f:\mathbb{C}^N\to\mathbb{C}^M$ is of the form $f(v)=Av$, with $A\in M_{M\times N}(\mathbb{C})$.

Theorem. With $f_A(v)=Av$, we have $f_{AB}=f_Af_B$. In particular f_A is invertible when A is invertible, and $f_A^{-1}=f_{A^{-1}}$.

Theory 2/4

The theory of the determinant extends as well:

Definition. The determinant of a matrix $A \in M_N(\mathbb{C})$ is

$$\det A = \sum_{\sigma \in S_N} \varepsilon(\sigma) A_{1\sigma(1)} \dots A_{N\sigma(N)}$$

where $\varepsilon(\sigma) = (-1)^c$, c being the number of inversions.

Theorem. The determinant is subject to the following rules:

(1) $\det(\lambda u, \{w_i\}) = \lambda \det(u, \{w_i\})$.

(2) $\det(u, v, \{w_i\}) = \det(u - v, v, \{w_i\})$.

Also, we have $\det(AB) = \det A \cdot \det B$, $\det(A^t) = \det A$.

Theory 3/4

The theory of the eigenvalues extends as well:

<u>Definition</u>. Given $A \in M_N(\mathbb{C})$, if $v \in \mathbb{C}^N$ and $\lambda \in \mathbb{C}$ satisfy

$$Av = \lambda v$$

we say that v is an eigenvector of A, with eigenvalue λ.

<u>Theorem</u>. The eigenvalues are the roots of the polynomial

$$P(\lambda) = \det(A - \lambda 1_N)$$

called characteristic polynomial of the matrix.

Theory 4/4

Theorem. Consider a 2×2 real or complex matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(1) The characteristic polynomial is $P(\lambda) = \lambda^2 - S\lambda + P$, with:

$$S = a + d \quad , \quad P = ad - bc$$

(2) We have two complex eigenvalues, given by:

$$\lambda_1 + \lambda_2 = S \quad , \quad \lambda_1 \lambda_2 = P$$

(3) Equivalently, we have the following formula:

$$\lambda_{1,2} = \frac{S \pm \sqrt{S^2 - 4P}}{2}$$

Diagonalization 1/4

Theorem. Given $A \in M_N(\mathbb{C})$, consider its characteristic polynomial $P(x) = \det(A - x1_N)$, and decompose it into factors:

$$P(x) = (-1)^N (x - \lambda_1) \ldots (x - \lambda_N)$$

For $\lambda \in \{\lambda_1, \ldots, \lambda_N\}$ consider the corresponding eigenspace:

$$E_\lambda = \left\{ v \in \mathbb{C}^N \Big| Av = \lambda v \right\}$$

We have then dimension inequalities as follows, for any λ,

$$1 \leq \dim(E_\lambda) \leq \#(\lambda \in \{\lambda_1, \ldots, \lambda_N\})$$

and A is diagonalizable precisely when we have equalities at right.

Diagonalization 2/4

In practice, the above result can be used as follows:

(1) Compute the characteristic polynomial $P(x) = \det(A - x1_N)$, and factorize it as $P(x) = (-1)^N(x - \lambda_1)\ldots(x - \lambda_N)$.

(2) Remark: if λ_i are distinct, A is certainly diagonalizable. Also, if $\lambda_i \notin \mathbb{R}$ for some i, A is not diagonalizable over $\mathbb{R}$.

(3) Solve $Av = \lambda_i v$ for any i. If a space of solutions E_{λ_i} satisfies $\dim(E_{\lambda_i}) < \#(\lambda \in \{\lambda_1, \ldots, \lambda_N\})$, A is not diagonalizable.

(4) Otherwise, find a basis of each of these spaces E_{λ_i}, and put all eigenvectors found into a matrix P (the "passage matrix").

(5) Put as well all eigenvalues found on the diagonal of a matrix D. Compute P^{-1}. We have then $A = PDP^{-1}$.

Diagonalization 3/4

Some tricks and tips:

(1) In 2 dimensions, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the eigenvalues are best computed by using $x + y = a + d$, $xy = ad - bc$.

(2) In fact, in N dimensions, it is known that the eigenvalues satisfy $\lambda_1 + \ldots + \lambda_N = Tr(A)$ and $\lambda_1 \ldots \lambda_N = \det A$.

(3) If P has integer coefficients, $P \in \mathbb{Z}[X]$, look first for integer roots, $\lambda \in \mathbb{Z}$. These must divide the coefficient of X^0.

Diagonalization 4/4

More tricks and tips:

(1) When computing eigenspaces E_{λ_i}, start with eigenvalues having big multiplicity, because the computation here might lead to the conclusion that A is not diagonalizable, and so you're done.

(2) Always check and doublecheck your computations. If your matrix depends on a parameter t, plug in $t = 0$ or so from time to time, in order to doublecheck. Good luck!

Advanced 1/4

Theorem. With respect to $< x, y >= \sum_i x_i \bar{y}_i$ we have

$$< Ax, y >=< x, A^* y >$$

with A^* being the adjoint matrix, given by $(A^*)_{ij} = \bar{A}_{ji}$.

Theorem. For a matrix $U \in M_N(A)$, the following are equivalent:

(1) U is a unitary, $< Ux, Uy >=< x, y >$.

(2) U satisfies the equation $U^* = U^{-1}$.

Proof. We have indeed $< Ux, Uy >=< x, U^* Uy >$, as desired.

Advanced 2/4

Theorem. The matrices which are normal, in the sense that

$$AA^* = A^*A$$

are diagonalizable.

Theorem. The matrices which are self-adjoint, in the sense that

$$A = A^*$$

are diagonalizable. Moreover, their eigenvalues are real.

Theorem. The matrices which are unitary, in the sense that

$$U^* = U^{-1}$$

are diagonalizable. Their eigenvalues are on the unit circle.

Advanced 3/4

<u>Theorem</u>. The following happen, inside $M_N(\mathbb{C})$:

(1) The matrices having distinct eigenvalues are dense.

(2) The diagonalizable matrices are dense.

<u>Proof</u>. Here (1) follows by using the resultant $R(P, P')$, because the equation $R = 0$ defines a hypersurface in $M_N(\mathbb{C})$, having dense complement. As for (2), this follows from (1).

<u>Comment</u>. This is interesting, because it tells us that "any formula which is true for diagonalizable matrices is true in general".

Advanced 4/4

<u>Theorem</u>. Any matrix $A \in M_N(\mathbb{C})$ can be put in Jordan form

$$A \sim \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$$

with each Jordan block being of the following type,

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

with the numbers λ ranging over the eigenvalues of A.