Linear algebra and calculus questions

Teo Banica

"Introduction to linear algebra", 4/6

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Systems 1/3

Theorem. Any linear system of equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = v_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = v_2 \\ \quad\vdots \\ a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = v_N \end{cases}$$

can be written in matrix form, as follows,

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & & & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}$$

and when A is invertible, its solution is given by $x = A^{-1}v$.

Systems 2/3

Theorem. Any linear recurrence system

$$\begin{cases} x_{k+1} = a_{11}x_k + a_{12}y_k + a_{13}z_k + \ldots \\ y_{k+1} = a_{21}x_k + a_{22}y_k + a_{23}z_k + \ldots \\ z_{k+1} = a_{31}x_k + a_{32}y_k + a_{33}z_k + \ldots \\ \qquad\qquad\vdots \end{cases}$$

can be written in matrix form, as follows,

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_k \\ y_k \\ z_k \\ \vdots \end{pmatrix}$$

and the solution is obtained by applying A^k to the inital data.

Systems 3/3

In order to compute A^k, we must diagonalize the matrix,

$$A = PDP^{-1}$$

and then the powers are given by the following formula:

$$A^k = PD^k P^{-1}$$

This formula holds in fact for any $k \in \mathbb{Z}$, or even $k \in \mathbb{R}$.

Calculus 1/4

Theorem. Any function can be locally approximated as

$$f(x+t) \simeq f(x) + at$$

where $a = f'(x)$ is the derivative of f at the point x.

Proof. Let us recall indeed the definition of the derivative:

$$f'(x) = \lim_{t \to 0} \frac{f(x+t) - f(x)}{t}$$

But this gives the formula in the statement.

Calculus 2/4

Theorem. Any function of several variables, written as

$$f = (f_1, \ldots, f_N)$$

can be locally approximated as follows,

$$f(x+t) \simeq f(x) + At$$

with A being the matrix of partial derivatives at x,

$$A = \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{ij}$$

acting on the vectors t by usual multiplication.

Calculus 3/4

Theorem. We have the change of variable formula

$$\int_a^b f(x)dx = \int_c^d f(\varphi(t))\varphi'(t)dt$$

where $c = \varphi^{-1}(a)$ and $d = \varphi^{-1}(b)$.

Proof. This follows with $f = F'$ from the rule

$$(F\varphi)'(t) = F'(\varphi(t))\varphi'(t)$$

by integrating between c and d.

Calculus 4/4

Theorem. Given a transformation in several variables,

$$\varphi = (\varphi_1, \ldots, \varphi_N)$$

we have the following change of variable formula,

$$\int_E f(x)dx = \int_{\varphi^{-1}(E)} f(\varphi(t))J_\varphi(t)dt$$

with the J_φ quantity, called Jacobian, being given by:

$$J_\varphi(t) = \det\left[\left(\frac{\partial \varphi_i}{\partial x_j}(x)\right)_{ij}\right]$$

Polar coordinates 1/4

Theorem. We have polar coordinates in 2 dimensions,

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$$

and the corresponding Jacobian is $J(r,t) = r$.

Proof. The Jacobian is by definition given by:

$$\begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} = r$$

Thus, we have indeed the formula in the statement.

Polar coordinates 2/4

$$\int_{\mathbb{R}} e^{-x^2} dx = ?$$

Polar coordinates 3/4

Theorem. We have the following formula:

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

Proof. The square of the integral is given by:

$$\begin{aligned} I^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dxdy \\ &= \int_0^{2\pi} \int_0^{\infty} re^{-r^2} drdt \\ &= \int_0^{2\pi} \left[-\frac{e^{-r^2}}{2} \right]_0^{\infty} dt \end{aligned}$$

We obtain $I^2 = (2\pi) \times \frac{1}{2} = \pi$, and so $I = \sqrt{\pi}$.

Polar coordinates 4/4

Definition. The normal law of parameter 1 is:

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

Remark. The Gauss formula gives with $x = y/\sqrt{2t}$

$$\int_{\mathbb{R}} e^{-y^2/2t} dy = \sqrt{2\pi t}$$

so these laws have indeed mass 1.

Spheres 1/4

Theorem. We have spherical coordinates in 3 dimensions,

$$\begin{cases} x = r\cos s \\ y = r\sin s\cos t \\ z = r\sin s\sin t \end{cases}$$

and the corresponding Jacobian is $J(r,s,t) = r^2\sin s$.

Proof. The Jacobian is given by:

$$\begin{vmatrix} \cos s & -r\sin s & 0 \\ \sin s\cos t & r\cos s\cos t & -r\sin s\sin t \\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t \end{vmatrix} = r^2\sin s$$

Thus, we have indeed the formula in the statement.

Spheres 2/4

Theorem. We have spherical coordinates in N dimensions,

$$\begin{cases} x_1 &= r\cos t_1 \\ x_2 &= r\sin t_1 \cos t_2 \\ \vdots \\ x_{N-1} &= r\sin t_1 \ldots \sin t_{N-2} \cos t_{N-1} \\ x_N &= r\sin t_1 \ldots \sin t_{N-2} \sin t_{N-1} \end{cases}$$

and the corresponding Jacobian is:

$$J(r,t) = r^{N-1} \sin^{N-2} t_1 \sin^{N-3} t_2 \ldots \sin^2 t_{N-3} \sin t_{N-2}$$

Remark. This generalizes the previous coordinates at $N = 2, 3$.

Spheres 3/4

<u>Theorem</u>. The volume of the sphere in $\mathbb{R}^N$ is given by

$$\frac{V}{2^N} = \left(\frac{\pi}{2}\right)^{[N/2]} \frac{1}{(N+1)!!}$$

with $N!! = (N-1)(N-3)(N-5)\ldots$, stopping at 1 or 2.

(1) At $N = 1$ we obtain $V/2 = 1$, so $V = 2$.

(2) At $N = 2$ we obtain $V/4 = \pi/2 \cdot 1/2$, so $V = \pi$.

(3) At $N = 3$ we obtain $V/8 = \pi/2 \cdot 1/3$, so $V = 4\pi/3$.

(4) At $N = 4$ we obtain $V/16 = \pi^2/4 \cdot 1/8$, so $V = \pi^2/2$.

Spheres 4/4

Proof. By using spherical coordinates, and Fubini, we are left with computing integrals over the circle. But these are given by

$$\frac{2}{\pi}\int_0^{\pi/2} \cos^p t \sin^q t \, dt = \left(\frac{2}{\pi}\right)^{\delta(p,q)} \frac{p!!q!!}{(p+q+1)!!}$$

where $\delta(a,b) = 0$ if both a, b are even, and $\delta(a,b) = 1$ otherwise, and by plugging in these quantities, we obtain the result.