Real matrices and their properties

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"Introduction to linear algebra", 1/6

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Rotations 1/3

Problem: what's the formula of the rotation of angle t?

Rotations 2/3

The points in the plane $\mathbb{R}^2$ can be represented as vectors $\binom{x}{y}$. The 2×2 matrices "act" on such vectors, as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Many simple transformations (symmetries, projections..) can be written in this form. What about the rotation of angle t?

Rotations 3/3

A quick picture shows that we must have:

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Also, by paying attention to positives and negatives:

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

Thus, the matrix of our rotation can only be:

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

By "linear algebra”, this is the correct answer.

Linear maps 1/4

Theorem. The maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ which are linear, in the sense that they map lines through 0 to lines through 0, are:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Remark. If we make the multiplication convention

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

the theorem says $f(v) = Av$, with A being a 2×2 matrix.

Linear maps 2/4

Examples. The identity and null maps are given by:

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}\quad,\quad\begin{pmatrix}0&0\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

The projections on the horizontal and vertical axes are given by:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\0\end{pmatrix}\quad,\quad\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\y\end{pmatrix}$$

The symmetry with respect to the $x=y$ diagonal is given by:

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}y\\x\end{pmatrix}$$

We have as well the rotation of angle $t\in\mathbb{R}$, studied before.

Linear maps 3/4

Theorem. The maps $f : \mathbb{R}^N \to \mathbb{R}^N$ which are linear, in the sense that they map lines through 0 to lines through 0, are:

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1N}x_N \\ \vdots \\ a_{N1}x_1 + \ldots + a_{NN}x_N \end{pmatrix}$$

Remark. With the matrix multiplication convention

$$\begin{pmatrix} a_{11} & \ldots & a_{1N} \\ \vdots & & \vdots \\ a_{N1} & \ldots & a_{NN} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1N}x_N \\ \vdots \\ a_{N1}x_1 + \ldots + a_{NN}x_N \end{pmatrix}$$

the theorem says $f(v) = Av$, with A being a $N \times N$ matrix.

Linear maps 4/4

Example. Consider the all-1 matrix. This acts as follows:

$$\begin{pmatrix}1&\ldots&1\\ \vdots&\ddots&\vdots\\ 1&\ldots&1\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}=\begin{pmatrix}x_1+\ldots+x_N\\ \vdots\\ x_1+\ldots+x_N\end{pmatrix}$$

But this formula can be written as follows:

$$\frac{1}{N}\begin{pmatrix}1&\ldots&1\\ \vdots&\ddots&\vdots\\ 1&\ldots&1\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_N\end{pmatrix}=\frac{x_1+\ldots+x_N}{N}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

And this latter map is the projection on the all-1 vector.

Theory 1/4

Definition. We can multiply $M \times N$ matrices with $N \times K$ matrices,

$$\begin{pmatrix} a_{11} & \ldots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & \ldots & a_{MN} \end{pmatrix} \begin{pmatrix} b_{11} & \ldots & b_{1K} \\ \vdots & & \vdots \\ b_{N1} & \ldots & b_{NK} \end{pmatrix}$$

the product being a $M \times K$ matrix, given by the formula

$$\begin{pmatrix} a_{11}b_{11} + \ldots + a_{1N}b_{N1} & \ldots\ldots & a_{11}b_{1K} + \ldots + a_{1N}b_{NK} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{M1}b_{11} + \ldots + a_{MN}b_{N1} & \ldots\ldots & a_{M1}b_{1K} + \ldots + a_{MN}b_{NK} \end{pmatrix}$$

obtained via the rule "multiply rows by columns".

Theory 2/4

Better definition. The matrix multiplication is given by

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

with A_{ij} being the entry on the i-th row and j-th column.

Theorem. The linear maps $f : \mathbb{R}^N \to \mathbb{R}^M$ are those of the form

$$f_A(v) = Av$$

with A being a $N \times M$ matrix.

Remark. Size check $(N \times 1) = (N \times M)(M \times 1)$, ok.

Theory 3/4

Theorem. With the above convention $f_A(v) = Av$, we have

$$f_A f_B = f_{AB}$$

"the product of matrices corresponds to the composition of maps".

Theorem. A linear map $f : \mathbb{R}^N \to \mathbb{R}^N$ is invertible when the matrix $A \in M_N(\mathbb{R})$ which produces it is invertible, and we have:

$$(f_A)^{-1} = f_{A^{-1}}$$

Theory 4/4

Theorem. The inverses of the 2×2 matrices are given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof. When $ad = bc$ the columns are proportional, so the matrix cannot be invertible. When $ad - bc \neq 0$, let us solve:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

We must solve the following equations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

But this leads to the formula in the statement.

Eigenvectors 1/4

Definition. Let $A \in M_N(\mathbb{R})$ be a square matrix, and assume that A multiplies by $\lambda \in \mathbb{R}$ in the direction of a vector $v \in \mathbb{R}^N$:

$$Av = \lambda v$$

In this case, we say that:

(1) $v \in \mathbb{R}^N$ is an eigenvector of A.

(2) $\lambda \in \mathbb{R}$ is the corresponding eigenvalue.

Eigenvectors 2/4

Examples. The identity has all vectors as eigenvectors, with $\lambda = 1$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The same goes for the null matrix, with $\lambda = 0$ this time:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the projection on the horizontal axis, $P\binom{x}{y} = \binom{x}{0}$, we have:

$$Pv = \lambda v \quad \Longleftrightarrow \quad v = \begin{pmatrix} 0 \\ y \end{pmatrix}, \lambda = 0 \quad \text{or} \quad v = \begin{pmatrix} x \\ 0 \end{pmatrix}, \lambda = 1$$

A similar result holds for the projection on the vertical axis.

Eigenvectors 3/4

More examples. For the symmetry $S\binom{x}{y} = \binom{y}{x}$, we have:

$$Sv = \lambda v \iff v = \begin{pmatrix} x \\ x \end{pmatrix}, \lambda = 1 \text{ or } v = \begin{pmatrix} x \\ -x \end{pmatrix}, \lambda = -1$$

For the transformation $T\binom{x}{y} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\binom{x}{y} = \binom{y}{0}$ we have:

$$Tv = \lambda v \quad \iff \quad v = \begin{pmatrix} x \\ 0 \end{pmatrix}, \lambda = 0$$

For the rotation of angle $t \neq 0$, we must have $v = 0, \lambda = 0$.

Eigenvectors 4/4

Definition. We say that a matrix $A \in M_N(\mathbb{R})$ is diagonalizable if it has N eigenvectors $v_1, \ldots, v_N$ which form a basis of $\mathbb{R}^N$.

Remark. When A is diagonalizable, in that basis we can write:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

This means that we have $A = PDP^{-1}$, with D diagonal.

Problems. Which matrices are diagonalizable? And, how to diagonalize them?