Infinite matrices and spectral theory

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"Introduction to linear algebra", 5/6

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Linear spaces 1/3

Definition. A complex vector space is a set V with operations

$$(u, v) \to u + v \quad , \quad (\lambda, u) \to \lambda u$$

having the following properties:

(1) $u + v = v + u$.

(2) $(u + v) + w = u + (v + w)$.

(3) $(\lambda + \mu)u = \lambda u + \mu u$.

(4) $(\lambda\mu)u = \lambda(\mu u)$.

(5) $\lambda(u + v) = \lambda u + \lambda v$.

Examples. $\mathbb{C}^N$, $\mathbb{C}^\infty$, $M_N(\mathbb{C})$, $C[0,1]$ and many other.

Linear spaces 2/3

Definition. A map $f : V \to W$ is called linear when:

(1) $f(u+v) = f(u) + f(v)$.

(2) $f(\lambda u) = \lambda f(u)$.

Theorem. Let $f : V \to W$ be a linear map.

(1) $\ker f = \{v \in V | f(v) = 0\}$ is a linear space.

(2) $\mathrm{Im}\, f = \{f(v) | v \in V\}$ is a linear space.

(3) $\dim \ker f + \dim \mathrm{Im}\, f = \dim V$.

Linear spaces 3/3

Theorem. In finite dimensions, any vector space V has a basis $\{e_i\}$, which is such that any $v \in V$ can be written, uniquely, as:

$$v = v_1 e_1 + \ldots + v_N e_N$$

Thus we have $V = \mathbb{C}^N$, the identification being given by:

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$$

As a consequence, any linear map $f : V \to W$ between finite dimensional vector spaces corresponds to a matrix.

Hilbert spaces 1/4

Definition. A scalar product on a complex vector space H is an operation $(x, y) \to < x, y >$, satisfying:

(1) $< x, y >$ is linear in x, and antilinear in y.

(2) $\overline{< x, y >} = < y, x >$, for any x, y.

(3) $< x, x > > 0$, for any $x \neq 0$.

Theorem. If we set $||x|| = \sqrt{< x, x >}$ then:

(1) $| < x, y > | \leq ||x|| \cdot ||y||$.

(2) $||x + y|| \leq ||x|| + ||y||$.

(3) $d(x, y) = ||x - y||$ is a distance.

Proof. (1) follows from the fact that the degree 2 polynomial $f(t) = ||tx + y||^2$ is positive, and (1) $\implies$ (2) $\implies$ (3).

Hilbert spaces 2/4

Definition. A Hilbert space is a complex vector space H with a scalar product $< x, y >$, which is complete with respect to

$$||x|| = \sqrt{< x, x >}$$

in the sense that the Cauchy sequences with respect to the associated distance $d(x, y) = ||x - y||$ converge.

Examples.

(1) $H = \mathbb{C}^N$, with $< x, y >= \sum_i x_i \bar{y}_i$.
(2) $H = l^2(\mathbb{N})$, with $< x, y >= \sum_i x_i \bar{y}_i$.
(3) $H = L^2(X)$, with $< f, g >= \int_X f(x)\overline{g(x)}dx$.

Hilbert spaces 3/4

Theorem. Any Hilbert space H has an orthonormal basis $\{e_i\}_{i\in I}$, and so we have an indentification $H = l^2(I)$.

Proof. The basis can be constructed by starting with an "algebraic" basis, and using the Gram-Schmidt method.

Warning. For spaces like $H = L^2[0,1]$, this is something not trivial.

Theorem. Let H be a Hilbert space, with basis $\{e_i\}_{i\in I}$. We have

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

with $T : H \to H$ linear corresponding to the following matrix:

$$M_{ij} =< Te_j, e_i >$$

In particular, when $\dim(H) = N < \infty$, we obtain $\mathcal{L}(H) \simeq M_N(\mathbb{C})$.

Hilbert spaces 4/4

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra with unit $B(H)$, which:

(1) is complete with respect to $||.||$ (Banach algebra).

(2) has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Proof. Here "complex algebra" is elementary, (1) follows by setting $Tx = \lim_{n\to\infty} T_n x$, (2) comes from the fact that $\varphi(x) =< Tx, y >$ is linear, and (3) can be proved by double inequality.

Spectral theory 1/4

Definition. A C^*-algebra is a complex algebra with unit A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

Definition. The spectrum of an element $a \in A$ is the set:

$$\sigma(a) = \left\{\lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1}\right\}$$

Theorem. $\sigma(ab) = \sigma(ba)$ outside $\{0\}$.

Proof. Indeed, $c = (1 - ab)^{-1} \implies 1 + cba = (1 - ba)^{-1}$.

Remark. In infinite dimensions, $S^*S = 1, SS^* \neq 1$ (shift).

Spectral theory 2/4

Theorem. We have the following formula, for any rational function $f \in \mathbb{C}(X)$ having its poles outside $\sigma(a)$:

$$\sigma(f(a)) = f(\sigma(a))$$

Proof. In the polynomial case, $f \in \mathbb{C}[X]$, we can factorize,

$$f(X) - \lambda = c(X - r_1) \ldots (X - r_n)$$

and the result can be proved as follows:

$$\begin{aligned} \lambda \notin \sigma(f(a)) &\iff a - r_1, \ldots, a - r_n \in A^{-1} \\ &\iff r_1, \ldots, r_n \notin \sigma(a) \\ &\iff \lambda \notin f(\sigma(a)) \end{aligned}$$

In the general case, $f = P/Q$, we can use $F = P - \lambda Q$.

Spectral theory 3/4

Definition. Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. Let A be a C^*-algebra.

(1) The spectrum of a norm 1 element is in the unit disk.

(2) The spectrum of a unitary ($a^* = a^{-1}$) is on the unit circle.

(3) The spectrum of a self-adjoint element ($a = a^*$) is real.

(4) ρ of a normal element ($aa^* = a^*a$) equals its norm.

Spectral theory 4/4

(1) Clear from $(1-a)^{-1} = 1 + a + a^2 + \ldots$ for $||a|| < 1$.

(2) Follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

(3) Follows from (2), by using $f(z) = (z + it)/(z - it)$.

(4) By (1) we have $\rho(a) \leq ||a||$. Given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

If $a = a^*$ we are done. In general, we can use $||aa^*|| = ||a||^2$.

Advanced 1/4

Theorem. Given a compact space X, the complex algebra

$$C(X) = \{f : X \to \mathbb{C} \text{ continuous}\}$$

is a C^*-algebra, with norm and involution given by:

$$||f|| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

This algebra is commutative, in the sense that $fg = gf$.

Proof. It is well-known that $C(X)$ is complete with respect to the sup norm, and the other conditions are trivially satisfied.

Advanced 2/4

<u>Theorem</u>. Any commutative C^*-algebra is the form $C(X)$, with its "spectrum" $X = Spec(A)$ consisting of the characters:

$$\chi : A \to \mathbb{C}$$

<u>Proof</u>. Set $X = Spec(A)$, with topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

Advanced 3/4

Theorem. Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.

(1) We can define $f(a) \in A$, with $f \to f(a)$ being a morphism.

(2) We have the formula $\sigma(f(a)) = f(\sigma(a))$.

Proof. Since a is normal, $B =< a >$ is commutative, and the Gelfand theorem gives $B = C(X)$, with $X = Spec(B)$.

The map $X \to \sigma(a)$ given by evaluation at a being bijective, we have $X = \sigma(a)$. Thus $B = C(\sigma(a))$, and we are done.

Advanced 4/4

Definition. Given an arbitrary C^*-algebra A, we can write

$$A = C(X)$$

and call X a "noncommutative compact space".