Life and economy

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

2010 *Mathematics Subject Classification.* 91B08

Key words and phrases. Life models, Global economy

Abstract. This is a mathematical introduction to life models and global economy questions, starting from the physics and chemistry basics, and ending with politics.

Preface

This is a mathematical introduction to life models and global economy questions, starting from the physics and chemistry basics, and ending with politics.

Many thanks go to my colleagues, for various useful discussions. Thanks as well to my cats, who seem to effortlessly get away without politics or democracy, or even economy and money. But the truth is, not many mice caught, and I'm paying for the food.

Contents

Part I

Molecules, life

Don't say it in Russian
Don't say it in German
Say it in broken English
Say it in broken English

CHAPTER 1

Atoms, molecules

1a. Atoms, molecules

Welcome to social science. The first thing to be known here is that humans are not machines, and you cannot apply to them the same methods as in hard sciences likes physics, chemistry or engineering, that is, all sorts of experiments, then quick, cold modelling based on these experiments, and finally calculus for solving your models.

What does work, although some exceptions to this rule exist, is the study of History. That is, History, or parts of it which are revelant to your social science work, is the "experiment", coming of course with a huge amount of data, and for free. And then, after studying History, you can come as usual with conclusions, models, and mathematics.

In what concerns us, with our economy motivations, we will be interested in pretty much everything. That is, pretty much everything that happened in the history of mankind, and even before, teaches us interesting things about what is life, who we are, life models, organization of society, money and economy. So, coming as first news, with respect to what has been said above about social sciences in general, economy turns to be a tricky business, requiring as preliminary a critical look at the whole History.

Generally speaking, the credit for such ideas, that understanding economy requires as preliminary a critical look at the whole History, goes to Karl Marx. In addition, the main opus of Marx, which is his 3-volume Capital [72], [73], [74], based on this idea, is quite a masterpiece, which remains hard to beat, even nowadays. To be more precise, the masterpiece in all this are the first two volumes [72], [73], which look in a first-class, objective scientific way, with economy motivations in mind, at the history of mankind. This study was brilliant when it came out, and as already said, is hard to beat, even nowadays. As for the third volume [74], this rather draws conclusions, and no surprise here, that conclusions are rather what we call today "marxism".

In short, what we were planning do to here, in this book, is more or less what Marx did in the first two volumes of Capital [72], [73], and so, no surprise, we will mostly rely on him, for our presentation. However, there will be a few differences.

In the hope that you will find this exciting. Finally, as a disclaimer here, I should mention that I am not a social scientist, but just a modest quantum physicist, and this will be I guess the average way quantum physics sees social science, at large.

Getting to work now, as already said above, as a first goal, we would like to understand what life is, and more specifically, we have the following question to be solved:

QUESTION 1.1. *What is life? How does life organize? Is there some sort of money and economy present, perhaps in some hidden form?*

In order to answer this question, we need a lot of science. On the menu, physics of all types, culminating with quantum mechanics, and the atomic theory, then chemistry and molecules, then organic molecules and life, and then a lot of biology, culminating with Darwin's findings, which will provide us with an answer to Question 1.1.

Getting started now, we need a crash course in general physics, going up to the atomic theory. This is something quite ambitious, our plan being as follows:

1. Classical mechanics.
2. Electrostatics.
3. Electrodynamics.
4. Relativity theory.
5. Quantum mechanics.
6. Atomic theory.

Generally speaking, all this can be learned from many places, with the classic being the books of Feynman [35], [36], [37], [38]. Alternatively, you can learn this from the equally lovely books of Griffiths [46], [47], [48], [49], or of Huang [52], [53], [54], [55], or at least these are, along with Feynman, my personal favorites. In case you already know some physics, and want to learn more, go with Weinberg [96], [97], [98], [99].

As another comment on physics books, all the above authors, Feynman, Griffiths, Huang and Weinberg, were mostly interested themselves in quantum mechanics, and certain less fashionable aspects of modern physics, such as classical mechanics, suffer a bit from this. So, for classical mechanics better go with Kibble [59] or Taylor [90] for learning the basics, and with Goldstein [44] or Landau-Lifshitz [63] for more advanced theory. And if you happen to be a mathematician, not afraid of difficult mathematics, go with Arnold [1], [2], [3], [4], who was the one on this planet best knowing classical mechanics, and who in addition knew how to write lovely mathematics books.

As yet another comment about learning physics, observe that thermodynamics and statistical mechanics, which are certainly some of the most exciting disciplines in physics, and in fact, in the context of modern physics, lie on par with quantum mechanics at spot #1, among physics trends on the internet, are missing from our above list (1-6). This is because we will not need these right now, for understanding the atoms and molecules. However, thermodynamics will come into play later in this book, when discussing economy, and some reading here in advance would be useful. Besides Feynman, Griffiths, Huang and Weinberg, you have here Fermi [**34**], and also Blundell and Blundell [**12**], Kadanoff [**57**], Pathria and Beale [**79**], Schroeder [**85**] and Steane [**87**], all good books.

Finally, if you are a social scientist not willing to read any technical books, sure yes, but check the popular physics books of Weinberg, he wrote quite a few of them, which are all very good, coming from someone who really knew well physics. Also, still speaking popular books, and as an advice for everyone now, be them mathematicians, physicists, other scientists or social scientists, get a copy of the book of Kumar [**61**], that is about quantum mechanics, very readable, and is my own favorite popular physics book.

Back to our business, looks like we forgot our to-do list (1-6), with all this discussion. Typical social science phenomenon, I guess. That list was as follows:

1. Classical mechanics.
2. Electrostatics.
3. Electrodynamics.
4. Relativity theory.
5. Quantum mechanics.
6. Atomic theory.

So, getting started for good now, at the beginnings of physics was classical mechanics, whose main findings can be summarized as follows:

Fact 1.2 (Classical mechanics). *The force of attraction between two bodies of masses m_1, m_2, having distance $d > 0$ between them, is given by*

$$||F|| = G \cdot \frac{m_1 m_2}{d^2}$$

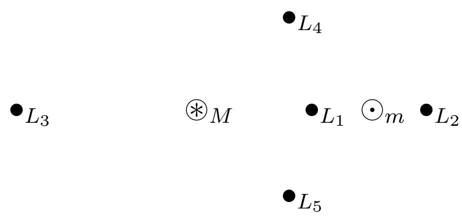
where $G = 6.674 \times 10^{-11}$ is a constant. This force alters the trajectory of one body with respect to the other according to the following formula, a being the acceleration:

$$F = ma$$

This trajectory is a curve of degree 2, called conic, which can be an ellipsis, parabola, or hyperbola. However, for 3 or more objects, all this can lead to order, or chaos.

In what regards the last sentence, that is a very short summary of what happens for the N-body problem with $N \geq 3$, and as a piece of advice here, have a look at Earth scientific satellites and Lagrange points, in relation with $N = 3$, and also go on internet for more about $N = 3$, including weird solutions, all this is very interesting.

As a piece of advertisement here, the interesting problem at $N = 3$ is how to position a specialized scientific satellite, deep in space, and away from the dust and radiation of the usual orbits around the Earth, as to stay there, under the joint influence of the gravity of the Sun M and of the Earth m. And there are 5 possible solutions here, called Lagrange points L1-L5, whose positions with respect to M, m are as follows:



Moreover, and here comes another interesting point, L4, L5 are stable, in the sense that a satellite installed there will really stay there, regardless of the various tiny little things that might happen, like an asteroid passing by, while L1, L2, L3 are unstable, in the sense that a satellite installed there will need constant tiny adjustments, in order to really stay there. So, which one would you choose for installing your satellite?

You would probably say L4, L5, but this is precisely the wrong answer, because due to their stability, these points attract a lot of asteroids and space garbage, and our satellite will certainly not perform well there, in that crowd. So, with L4, L5 ruled out, and with L3 ruled out too, being too far, the correct choices are L1, L2. But here, you still need to learn a lot more mechanics, for understanding how to do this, in practice.

In what regards now electrostatics, this is again something very fundamental, that you know well, the basics here being summarized as follows:

Fact 1.3 (Electrostatics). *Ordinary matter is made of electrons* $-$, *protons* $+$ *and neutrons* 0, *with the number of* $+, -$ *being roughly equal. We set*

$$q = \#\{+\} - \#\{-\}$$

and if $q \neq 0$, *we call this a charge. Any pair of charges* $q_1, q_2 \in \mathbb{R}$ *is then subject to a force as follows, which is attractive if* $q_1 q_2 < 0$ *and repulsive if* $q_1 q_2 > 0$,

$$||F|| = K \cdot \frac{|q_1 q_2|}{d^2}$$

where $K = 8.988 \times 10^9$. *However, unlike in classical mechanics,* $q_1 < 0$ *will not spin around* $q_2 > 0$ *on an ellipsis, due to magnetism, relativity, and quantum mechanics.*

Here you are certainly familiar with the Coulomb law formula in the statement, which is very similar to the Newton law formula from Fact 1.2. This normally suggests that when the force is attractive, $q_1q_2 < 0$, the negative charge, say an electron $-$, will spin around the positive charge, say a proton $+$, on an ellipsis. But this is well-known to be wrong, with the solution of this 2-body problem, which corresponds to the hydrogen atom, being far more complicated, due to the numerous reasons mentioned in the statement.

Regarding now electrodynamics, this comes as a continuation of electrostatics, with the aim of fixing some of the obvious bugs there, the basics being as follows:

Fact 1.4 (Electrodynamics). *Moving charges produce magnetic fields, and the dynamics of the electric fields E and magnetic fields B is governed by the formulae*

$$< \nabla, E >= \frac{\rho}{\varepsilon_0}$$

$$< \nabla, B >= 0$$

$$\nabla \times E = -\dot{B}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \dot{E}$$

called Maxwell equations. Also, accelerating or decelerating charges produce electromagnetic radiation, of various wavelengths, called light, of various colors.

Obviously, we are now into serious science here, with the Maxwell equations being something quite complicated, and the pride of 19th century physics, and still the nightmare of everyone using them. To start with, electrodynamics is the science of moving electrical charges. And the problem is that, unlike in classical mechanics, where the Newton law is good for both the static and the dynamic setting, the Coulomb law, which is actually very similar to the Newton law, does the job when the charges are static, but no longer describes well the situation when the charges are moving.

The problem comes from the fact that moving charges produce magnetism, and with this being visible when putting together two electric wires, which will attract or repel, depending on orientation. Thus, in contrast with classical mechanics, where static or dynamic problems are described by a unique field, the gravitational one, in electrodynamics we have two fields, namely the electric field E, and the magnetic field B.

Fortunately, there is a full set of equations relating the electric field E and the magnetic field B, those found by Maxwell and others, given above. Regarding the math, $<,>$ and

$\times$ are the usual scalar and vector products on $\mathbb{R}^3$, the dots denote derivatives with respect to time, and ∇ is the gradient operator, or space derivative, given by:

$$\nabla = \begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \\ \frac{d}{dz} \end{pmatrix}$$

As for the physics, the first formula is the Gauss law, ρ being the charge, and ε_0 being a constant, and with this Gauss law more or less replacing the Coulomb law from electrostatics. The second formula is something basic, and anonymous. The third formula is the Faraday law. As for the fourth formula, this is the Ampère law, as modified by Maxwell, with J being the volume current density, and μ_0 being a constant.

Importantly, in addition to what is said in Fact 1.4, it is also known that the constants there μ_0, ε_0, which are electrodynamic quantities, are subject to the following magic formula, due to Biot-Savart, with $c = 299\,792\,458$ m/s being the speed of light:

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

In what regards now the last sentence, this is something fundamental too, putting an end to centuries or even millenia of discussions, regarding the nature of light. Speaking light, here is the table coming from Fact 1.4, which is a must-know:

Frequency	Type	Wavelength
	$-$	
$10^{18} - 10^{20}$	γ rays	$10^{-12} - 10^{-10}$
$10^{16} - 10^{18}$	X $-$ rays	$10^{-10} - 10^{-8}$
$10^{15} - 10^{16}$	UV	$10^{-8} - 10^{-7}$
	$-$	
$10^{14} - 10^{15}$	blue	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	yellow	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	red	$10^{-7} - 10^{-6}$
	$-$	
$10^{11} - 10^{14}$	IR	$10^{-6} - 10^{-3}$
$10^{9} - 10^{11}$	microwave	$10^{-3} - 10^{-1}$
$1 - 10^{9}$	radio	$10^{-1} - 10^{8}$

Observe the tiny space occupied by the visible light, all colors there, and the many more missing, being squeezed under the $10^{14} - 10^{15}$ frequency banner. Here is a zoom on that part, with of course the remark that all this, colors, is something subjective:

Frequency THz $= 10^{12}$ Hz	Color	Wavelength nm $= 10^{-9}$ m
$670 - 790$	violet	$380 - 450$
$620 - 670$	blue	$450 - 485$
$600 - 620$	cyan	$485 - 500$
$530 - 600$	green	$500 - 565$
$510 - 530$	yellow	$565 - 590$
$480 - 510$	orange	$590 - 625$
$400 - 480$	red	$625 - 750$

Hang on, we are not done yet with the Maxwell equations, and their consequences. Yet another feature of these equations is that these can be regarded as well as a precursor of Einstein's relativity theory, which can be summarized as follows:

FACT 1.5 (Relativity theory). *The speeds are bounded, $v < c$, by the speed of light in vaccuum, which is the same for all intertial observers, given by:*

$$c = 299\,792\,458 \text{ m/s}$$

In view of this, classical mechanics must be fixed, and the correct formula for the addition of speeds, guaranteeing $v < c$ for the sum, is Einstein's formula

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB}v_{BC}/c^2}$$

which at small speeds reduces to the usual Galileo formula $v_{AC} = v_{AB} + v_{BC}$. Moreover, the improved theory is invariant under the space-time Lorentz transformation

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^2)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, exactly as the Maxwell equations. Gravity can be added, too.

Obviously, many deep things going on here, and many other things can be said, for instance $E = mc^2$ comes from this too. This being said, the idea of Einstein is very simple, based only on $v < c$. Indeed, by rescaling things as to have $c = 1$, we are looking for a speed addition formula $(u, v) \to u +_e v$ satisfying the following condition:

$$u, v \leq 1 \implies u +_e v \leq 1$$

But here, thinking at the math, not many choices, with the obvious choice being:

$$u +_e v = \frac{u + v}{1 + uv}$$

And the miracle is that this formula, which is the one in the statement after rescaling by c, is indeed the correct one. With everything coming afterwards, namely Lorentz transformation, and gravity added, being more or less straightforward mathematics.

Finally, no discussion of relativity would be complete without a proof of $E = mc^2$. The idea here is that the relativistic energy of an object of rest mass $m > 0$ is as follows, making it clear that at speed $v = 0$, the energy should be $E = mc^2$:

$$\begin{aligned}\mathcal{E} &= \frac{mc^2}{\sqrt{1 - v^2/c^2}}\\ &= mc^2\left(1 + \frac{v^2}{2c^2} + \ldots\right)\\ &= mc^2 + \frac{mv^2}{2} + \ldots\end{aligned}$$

Now still speaking deep things, and going back to the Maxwell equations from Fact 1.4, although almighty, and compatible with relativity too, via the mathematics of the Lorentz transformation, these still do not solve the 2-body problem in electrodynamics, which is the functioning problem for the hydrogen atom. The problem comes from quantum mechanics, whose basic philosophy can be summarized as follows:

FACT 1.6 (Quantum mechanics). *Small particles like electrons and protons do not have clear positions and speeds. This is how things are, at that scale, and it is all about the probability of finding the particle here or there, and with this or that speed.*

This might seem overly vague, but sometimes a totally new and weird thought, of course in the hands of someone having the technical know-how, is enough to make science advance. Besides the above fact, which is something mathematical and theoretical, of key importance was the discovery, by Balmer, Rydberg and others, of the mechanism of the spectral lines of hydrogen H. These lines, depending on integer parameters $n_1 < n_2$, are given by the Rydberg formula, which is as follows, with $R = 1.096\ 775\ 83 \times 10^7$:

$$\frac{1}{\lambda_{n_1n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

Interestingly, and perhaps reminding a bit speed addition in relativity, these spectral lines combine according to the Ritz-Rydberg principle, which is as follows:

$$\frac{1}{\lambda_{n_1n_2}} + \frac{1}{\lambda_{n_2n_3}} = \frac{1}{\lambda_{n_1n_3}}$$

In practice, all these lines came from the efforts of several people, namely Balmer in 1885, in the visible range, then Lyman in 1906 in UV, Paschen in 1908 in IR, and later

Brackett in 1922, Pfund in 1924, Humphreys in 1953, and others aftwerwards, with all the extra lines being in far IR. The simplified complete table is as follows:

n_1	n_2	Series name	Wavelength $n_2=\infty$	Color $n_2=\infty$
		–	–	
1	$2-\infty$	Lyman	91.13 nm	UV
2	$3-\infty$	Balmer	364.51 nm	UV
3	$4-\infty$	Paschen	820.14 nm	IR
		–	–	
4	$5-\infty$	Brackett	1458.03 nm	far IR
5	$6-\infty$	Pfund	2278.17 nm	far IR
6	$7-\infty$	Humphreys	3280.56 nm	far IR
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now back to the Ritz-Rydberg principle, which is the main theoretical result in all this, this reminds the following multiplication formula for the usual matrix units $e_{ij}:e_j\to e_i$, perhaps taken in infinite dimensions, as to allow infinite-ranging indices:

$$e_{n_1n_2}e_{n_2n_3}=e_{n_1n_3}$$

But this is very interesting, suggesting that the observables of the hydrogen atom should be some sort of infinite matrices, making the link with Fact 1.6.

Obviously, what we have here is a first-class scientific puzzle. Based on all this, and on some earlier predictions of Bohr, who was the initiator of the whole program, Heisenberg and Schrödinger, and then De Broglie, Dirac, Pauli and others were able to solve this puzzle, and develop a quantum mechanics theory starting from Fact 1.6, with the main applications, to the functioning of hydrogen and of other atoms, being as follows:

FACT 1.7 (Atomic theory). *The atoms are formed by a core of protons and neutrons, surrounded by a cloud of electrons, basically obeying to a modified version of electromagnetism. And with a fine mechanism involved, as follows:*

(1) *The electrons are free to move only on certain specified elliptic orbits, labelled* $1,2,3,\ldots$, *situated at certain specific heights.*
(2) *The electrons can jump or fall between orbits* $n_1<n_2$, *absorbing or emitting light and heat, that is, electromagnetic waves, as accelerating charges.*
(3) *The energy of such a wave, coming from* $n_1\to n_2$ *or* $n_2\to n_1$, *is given, via the Planck viewpoint, by the Rydberg formula, applied with* $n_1<n_2$.
(4) *The simplest such jumps are those observed by Lyman, Balmer, Paschen. And multiple jumps explain the Ritz-Rydberg formula.*

Still with me, I hope? We are certainly now into complicated physics, and even seem to be somewhere towards the end of science, as understandable by humans. But, thinking well, we are in fact only at the beginning, because Fact 1.7 is not that useful as such, for the simple reason that atoms usually don't come alone, but rather tend to attach to each other, and form molecules. So, with physics understood, welcome to chemistry.

Getting into chemistry now, we first need a better understanding of the atoms, further building on Fact 1.7. The basics of chemistry can be summarized as follows:

FACT 1.8 (Basic chemistry). *Atoms can be labeled according to their atomic number, which is the number of protons in their nucleus, in practice*

$$Z = 1, \ldots, 118$$

and tend to attach to each other, and form molecules, with the electron distribution on the orbitals being responsible for this mechanism.

All this is very interesting, and truly corresponding to what happens in the real life, meaning at our scale, our usual temperature, our usual pressure, and so on. More precisely now, there are two assertions here. First is a continuation of the atomic physics from chapter 1, which leads to the conclusion that the known atoms, also called chemical elements, basically depend only on their atomic number $Z = 1, \ldots, 118$. These chemical elements can be arranged in a table, called periodic table, as follows:

	1	2		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	H 1																		He 2
2	Li 3	Be 4												B 5	C 6	N 7	O 8	F 9	Ne 10
3	Na 11	Mg 12												Al 13	Si 14	P 15	S 16	Cl 17	Ar 18
4	K 19	Ca 20		Sc 21	Ti 22	V 23	Cr 24	Mn 25	Fe 26	Co 27	Ni 28	Cu 29	Zn 30	Ga 31	Ge 32	As 33	Se 34	Br 35	Kr 36
5	Rb 37	Sr 38		Y 39	Zr 40	Nb 41	Mo 42	Tc 43	Ru 44	Rh 45	Pd 46	Ag 47	Cd 48	In 49	Sn 50	Sb 51	Te 52	I 53	Xe 54
6	Cs 55	Ba 56	l	Lu 71	Hf 72	Ta 73	W 74	Re 75	Os 76	Ir 77	Pt 78	Au 79	Hg 80	Tl 81	Pb 82	Bi 83	Po 84	At 85	Rn 86
7	Fr 87	Ra 88	a	Lr 103	Rf 104	Db 105	Sg 106	Bh 107	Hs 108	Mt 109	Ds 110	Rg 111	Cn 112	Nh 113	Fl 114	Mc 115	Lv 116	Ts 117	Og 118
			l:	La 57	Ce 58	Pr 59	Nd 60	Pm 61	Sm 62	Eu 63	Gd 64	Tb 65	Dy 66	Ho 67	Er 68	Tm 69	Yb 70		
			a:	Ac 89	Th 90	Pa 91	U 92	Np 93	Pu 94	Am 95	Cm 96	Bk 97	Cf 98	Es 99	Fm 100	Md 101	No 102		

Here the horizontal parameter $1, \ldots, 18$ is called the group, and the vertical parameter $1, \ldots, 7$ is called the period. The two rows on the bottom consist of lanthanum ${}_{57}$La and its followers, called lanthanides, and of actinium ${}_{89}$Ac and its followers, called actinides. These are to be inserted in the main table, where indicated, lanthanides between barium ${}_{56}$Ba and lutetium ${}_{71}$Lu, and actinides between radium ${}_{88}$Ra and lawrencium ${}_{103}$Lr.

Thus, the periodic table, when correctly drawn, but no one does that because of obvious typographical reasons, is in fact a 7×32 table. Note here that, according to our 7×18 convention, which is the standard one, lanthanides and actinides don't have a group number $1, \ldots, 18$. Their group is by definition "lanthanides" and "actinides".

In order to go now towards chemistry, as a first requirement, you cannot call yourself a chemist if not knowing all the elements up to krypton ${}_{36}$Kr, which are absolutely needed for everything, at least a little bit. The names of these elements are as follows:

(1) Hydrogen ${}_{1}$H, helium ${}_{2}$He.

(2) Lithium ${}_{3}$Li, berylium ${}_{4}$Be, boron ${}_{5}$B, carbon ${}_{6}$C, nitrogen ${}_{7}$N, oxygen ${}_{8}$O, fluorine ${}_{9}$F, neon ${}_{10}$Ne.

(3) Sodium ${}_{11}$Na, magnesium ${}_{12}$Mg, aluminium ${}_{13}$Al, silicon ${}_{14}$Si, phosphorus ${}_{15}$P, sulfur ${}_{16}$S, chlorine ${}_{17}$Cl, argon ${}_{18}$Ar.

(4) Potassium ${}_{19}$K, calcium ${}_{20}$Ca, scandium ${}_{21}$Sc, titanium ${}_{22}$Ti, vanadium ${}_{23}$V, and chromium ${}_{24}$Cr, manganese ${}_{25}$Mn, iron ${}_{26}$Fe, cobalt ${}_{27}$Co.

(5) Nickel ${}_{28}$Ni, copper ${}_{29}$Cu, zinc ${}_{30}$Zn, gallium ${}_{31}$Ga, germanium ${}_{32}$Ge, arsenic ${}_{33}$As, selenium ${}_{34}$Se, bromine ${}_{35}$Br, krypton ${}_{36}$Kr.

Observe that all names fit with the abbreviations, expect for sodium ${}_{11}$Na, coming from the Latin natrium, potassium ${}_{19}$K, coming from the Latin kalium, iron ${}_{26}$Fe coming from the Latin ferrum, and also copper ${}_{29}$Cu, coming from the Latin cuprum.

In what regards the elements heavier than krypton ${}_{36}$Kr, it was heartbreaking to sort them out, I just love them all, but as a useful complement to the above list, we can at least list some remarkable elements, for various reasons, among them. These include:

(6) Noble gases: xenon ${}_{54}$Xe, radon ${}_{86}$Rn.

(7) Noble metals: silver ${}_{47}$Ag, iridium ${}_{77}$Ir, platinum ${}_{78}$Pt, gold ${}_{47}$Au.

(8) Heavy metals: mercury ${}_{80}$Hg, lead ${}_{82}$Pb.

(9) Radioactive: polonium ${}_{84}$Po, radium ${}_{88}$Ra, uranium ${}_{92}$U, plutonium ${}_{94}$Pu.

(10) Miscellaneous: rubidium $_{37}$Rb, strontium $_{38}$Sr, molybdenum $_{42}$Mo, technetium $_{43}$Tc, cadmium $_{48}$Cd, tin $_{50}$Sn, iodine $_{53}$I, caesium $_{55}$Cs, tungsten $_{74}$Tu, bismuth $_{83}$Bi, francium $_{87}$Fr, americium $_{95}$Am.

Here the abbreviations not fitting with English names come from the Latin or sometimes Greek argentum $_{47}$Ag, aurum $_{47}$Au, hydrargyrum $_{80}$Hg, plumbum $_{82}$Pb and stannum $_{50}$Sn. The noble gases in (1) normally include oganesson $_{118}$Og as well. The noble metals in (2) are something subjective. There are of course plenty of other heavy metals (3), or radioactive elements (4). As for the list in (5), this is something subjective, basically a mixture of well-known metals used in engineering, and some well-known bad guys in the context of nuclear fallout. Technetium $_{43}$Tc is a bizarre element, human-made.

Regarding now the second assertion in Fact 1.8, regarding the formation of molecules, this again comes from Fact 1.7, but via a more complicated mechanism. The idea here is that given two or several atoms, which can have the same atomic number Z or not, what happens is that, depending on their respective Z, these atoms might choose to share some electrons, with this coming somehow from less energy needed for functioning, in this new configuration. And so, we are led to clusters of atoms, called molecules.

As an example here, or rather counterexample, let us look at the group 18 elements, helium $_2$He, neon $_{10}$Ne, argon $_{18}$Ar, krypton $_{36}$Kr, xenon $_{54}$Xe and radon $_{86}$Rn. These are called noble gases, and are allergic to chemistry, because the group 18 elements are precisely those with all possible electron positions fully occupied, up to a certain $n \in \mathbb{N}$, which makes them very unfriendly to any chemistry proposition from the outside.

1b.

1c.

1d.

1e. Exercises

Exercises:

Exercise 1.9.

Exercise 1.10.

Exercise 1.11.

Exercise 1.12.

Exercise 1.13.

Exercise 1.14.

Bonus exercise.

CHAPTER 2

Organic molecules

2a. Organic molecules

We discuss here organic chemistry. The basics here can be summarized as follows:

FACT 2.1 (Organic chemistry). *Advanced molecules, called organic, are typically long and contain lots of carbon* $_6$C *and hydrogen* $_1$H*, which tend to team together.*

Here the formation mechanism is something quite complicated, relying on the remarkable properties of the carbon $_6$C and hydrogen $_1$H elements, which tend indeed to team together, and form all sorts of amazing molecules, which are typically very long.

2b.

2c.

2d.

2e. Exercises

Exercises:

EXERCISE 2.2.

EXERCISE 2.3.

EXERCISE 2.4.

EXERCISE 2.5.

EXERCISE 2.6.

EXERCISE 2.7.

Bonus exercise.

CHAPTER 3

Cells and life

3a. Cells and life

Getting now to biology, the basics here can be summarized as follows:

FACT 3.1 (Basic biology). *Organic molecules tend to team together, as to form cells, which themselves tend to team together too, as to form advanced forms of life.*

Finally, as a last piece of theory, we need to talk about mutations, from a chemical viewpoint, with these being responsible for the evolution of the various forms of life.

3b.

3c.

3d.

3e. Exercises

Exercises:

EXERCISE 3.2.

EXERCISE 3.3.

EXERCISE 3.4.

EXERCISE 3.5.

EXERCISE 3.6.

EXERCISE 3.7.

Bonus exercise.

CHAPTER 4

Mutations, Darwin

4a.

4b.

4c.

4d.

4e. Exercises

Exercises:

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Bonus exercise.

Part II

Computers, brain

At the age of thirty-seven
She realised she'd never ride
Through Paris in a sports car
With the warm wind in her hair

CHAPTER 5

Computer science

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

Bonus exercise.

CHAPTER 6

Neurons, brain

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Bonus exercise.

CHAPTER 7

Basic models

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

EXERCISE 7.1.

EXERCISE 7.2.

EXERCISE 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

Bonus exercise.

CHAPTER 8

Artificial intelligence

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

Exercise 8.1.

Exercise 8.2.

Exercise 8.3.

Exercise 8.4.

Exercise 8.5.

Exercise 8.6.

Bonus exercise.

Part III

Economy, games

And Michelle, what will she do
Without you, Lady Madeleine
And I walk down the avenue
And I'm missing you, Lady Madeleine

CHAPTER 9

Games and money

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

Bonus exercise.

CHAPTER 10

Bible and Quran

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

Bonus exercise.

CHAPTER 11

Marx and Freud

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

Bonus exercise.

CHAPTER 12

Industrial society

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

Bonus exercise.

Part IV

Into politics

There's room at the top they are telling you still
But first you must learn how to smile as you kill
If you want to live like the folks on the hill
A working class hero is something to be

CHAPTER 13

Politics, elections

13a. Winning elections

Welcome to politics. Generally speaking, politics is all about adapting the views of your party, by staying of course a bit yourself, to the changes in the distribution of the citizens over the political spectrum, as to win the elections, and govern afterwards. And there is a lot of interesting mathematics here, depending on how exactly you model the political spectrum, with things here ranging from simple to very advanced, how the elections are organized, and so on, that we intend to talk about, in this book.

But let us start with simple things. The simplest possible question concerns the situation where there are two parties, say hearts ♡ and spades ♠, and the citizens have already made their choices. With the question being, of course, who wins?

Wins the party with the most votes, you would say, and sure, so that is. But, for complicating a bit things, and with this being not that far from what happens in the real life, assume for instance that spades ♠ are in power, and they did some bad things and lost votes, to the point that hearts ♡ have now the majority in the polls, and they are facing reelection. Can they win, by suitably reshaping the electoral districts?

And the answer here can be yes. Check this out, with the spades ♠ winning 66% of the parliament, while only having $4/9 = 44\%$ of the popular vote:

$$\begin{matrix} \spadesuit & \spadesuit & \heartsuit & \longrightarrow & \spadesuit \\ \spadesuit & \spadesuit & \heartsuit & \longrightarrow & \spadesuit \\ \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit \end{matrix}$$

This looks quite interesting, with surely some mathematics involved, that we can try to work out. So, as a first question that we would like to solve, we have:

Question 13.1. *Assuming that spades ♠ are in power, and can reshape at will the electoral districts, how low can they afford to get in the polls, as to still win?*

This question does not look very difficult, and in order to study it, let us go back to the above 4/9 configuration. Obviously, there is some loss there for spades ♠, because they win 66% of the parliament, instead of the required 51%. And also, each of their parliament representatives gets elected with 66% of the voices, instead of the required 51%. Thus, there is certainly room for improvement, here.

In order to get a better idea of what is going on, let us fine-tune our electoral picture, by using 100 voting citizens, instead of just 9. And here, we can see that spades $\spadesuit$ can win 60% of the parliament, while only having 36% of the popular vote:

$$\begin{array}{cccccccccccc}
\spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\
\spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit
\end{array}$$

Now for doing even better, let us assume that we have 10,000 voting citizens, organized in 100 voting districts, as bit as above. Then, by using the same strategy, we can see that spades $\spadesuit$ can still win, provided that their popular vote is higher than:

$$\frac{51^2}{10,000} = 26\%$$

In case we have 1,000,000 voting citizens, we get an even better figure, namely:

$$\frac{501^2}{1,000,000} = 25.1\%$$

Obviously, what we have here is a mathematical limit, which equals 25%. So, we have now an answer to our question, as follows:

Answer 13.2. *Spades $\spadesuit$ can win, provided that they have $1/4$ of the votes.*

This is of course something a bit simplified, because for being truly correct, mathematically, we should rather say that they need $1/4+\varepsilon$ of the votes, with $\varepsilon > 0$ depending on the number n and the size N of the electoral districts. But, unless your state is really very small, we can assume in practice that these numbers are very big, $n, N >> 0$, which in practice makes $\varepsilon = 0$, leading to the above answer, as formulated.

This latter convention is actually very useful for doing the math, without bothering with all sorts of small numbers and limits, so let us enshrine it, as follows:

Convention 13.3. *We can assume for computations that in the case of a tie the spades $\spadesuit$ win, with this being easy to implement, via $N >> 0$ leading to $\varepsilon = 0$ arguments.*

With this convention made, time to draw a new picture, say with 16 voting citizens, and with the spades $\spadesuit$ winning now 50% of the parliament, and so the majority, according to our convention, while only having 25% of the popular vote:

$$\begin{matrix} \spadesuit & \spadesuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\ \spadesuit & \spadesuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit \end{matrix}$$

Now, story over with this? Not yet, because once the parliament elected, a similar phenomenon can happen there, with a 1/4 minority imposing their will. Or, equivalently, we can investigate the case of 2-stage elections, where voters elect grand electors, who afterwards elect the parliament representatives. In any case, assuming that the process happens twice, one way or another, it is quite clear that the votes needed for a final win are just a small $(1/2)^3 = 1/8$, with the simplified picture being as follows:

$$\begin{matrix} \spadesuit & \spadesuit & \heartsuit & \heartsuit & \longrightarrow & \spadesuit & & \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit & \longrightarrow & \spadesuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit & \longrightarrow & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit & \longrightarrow & \heartsuit & & \end{matrix}$$

To be more precise, we have assumed here, as per Convention 13.3, that in the case of a tie the spades $\spadesuit$ win. And, with this convention, we can see that what we have in the end is a win for the spades $\spadesuit$, despite having only $2/16 = 1/8$ of the popular vote.

Let us record this finding, along with a bit more, as follows:

Conclusion 13.4. *Assuming that spades $\spadesuit$ are in power, and they can modify the electoral districts, at all levels, at will, in order to win they need*

$$v = \left(\frac{1}{2}\right)^{k+1}$$

of the popular vote, with k being the number of stages of the electoral process. In particular, with a 5-stage electoral process, they just need $1/64 \simeq 1\%$ of the votes.

Here the first assertion comes from the above discussion, iterated k times, with each iteration leading to $v \to v/2$. As for the numerics, with $k = 5$ we have, as claimed:

$$v = \left(\frac{1}{2}\right)^6 = \frac{1}{64} \simeq 1\%$$

In case you still don't believe, here is the picture, making it clear that spades ♠ can win with 1/64 of the popular vote, with the 1st, 2nd, 3rd level electoral districts being pictured as 1×2 rectangles, and with the 4th, 5th level ones, as 2×1 rectangles:

♠ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♠ ♡ ♡ ♡ → ♠ ♡ → ♠

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡ → ♠

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡ → ♡ → ♠

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡ → ♡ → ♡

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡ → ♡

♡ ♡ ♡ ♡ ♡ ♡ ♡ ♡ → ♡ ♡ ♡ ♡ → ♡ ♡ → ♡

As a funny consequence of this, assuming that the spades ♠ are the party of the richest 1%, in the context of a 5-stage electoral process as above, or rather 6-stage process, as to have $v = 1/128 < 1\%$ neatly, they can forever get reelected, and this without even doing any political campaign, or showing any sort of respect for the remaining 99% of the citizens. What they have to do is simply vote for themselves, and it's a win.

With this being a social science book, we will not hesitate to record the outcome of these latter speculations, in the form of a grand philosophical conclusion, as follows:

Grand conclusion 13.5. *The more complicated the electoral process is, and the bureaucracy in general is, the more that favors the bad-intentioned.*

And we will see of course some other illustrations for this general principle, duly accompanied by rigorous thinking and mathematical proofs, later in this book.

13b. Electoral maps

The story is not over with the above, because we have neglected, in our modeling, the precise 2D shape of the electorate distribution, and of the electoral map. Assume for instance that we have 16 voters, with the spades $\spadesuit$ clumped in the upper left corner:

$$\begin{matrix}\spadesuit & \spadesuit & \heartsuit & \heartsuit\\ \spadesuit & \spadesuit & \heartsuit & \heartsuit\\ \heartsuit & \heartsuit & \heartsuit & \heartsuit\\ \heartsuit & \heartsuit & \heartsuit & \heartsuit\end{matrix}$$

Then, for the spades $\spadesuit$ to win, as usual as per our asymptotic Convention 13.3, what they have to do is to design the electoral map in the form of 4 rectangles, as follows:

$$\begin{matrix}1&1&1&1\\2&2&2&2\\3&3&3&3\\4&4&4&4\end{matrix}$$

However, in case the law of the land requires the voting districts to have no more than 3 citizens, this 4 rectangle configuration would be illegal, and the spades $\spadesuit$ would have to use small 2-voter districts, geographically organized in a funny way, such as:

$$\begin{matrix}1&2&2&5\\3&4&4&5\\3&1&6&6\\7&7&8&8\end{matrix}$$

To be more precise, this configuration certainly leads to a win for the spades $\spadesuit$, winning in the districts $1,2,3,4$, and losing in the districts $5,6,7,8$, as so winning overall, according to our Convention 13.3. However, geographically, this does not look very good, with the district 1 being not connected, and with the problem here coming from the spade voter up left, surrounded by its spade mates, who therefore needs a non-connected voter district, in order to "fool" a heart voter $\heartsuit$, that is, win over that voter with 50%.

Hope you get my point, there is a topology problem here, with each clump of spade $\spadesuit$ voters requiring the spade party to design funny, non-connected voting districts. As an even more striking example here, assume that the country map looks like a lollipop, with the spade voters being all amassed at the sweet region, as follows:

$$\begin{matrix}&\spadesuit\\ \spadesuit&\spadesuit&\spadesuit&\heartsuit&\heartsuit&\heartsuit&\ldots\ldots&\heartsuit&\heartsuit&\heartsuit\\ &\spadesuit\end{matrix}$$

In this case, it is clear that only the spade voter right can afford voting in a normal-looking, connected district, with all the others requiring non-connected districts, as to reach to the various heart voters on the right, and win against them with 50%.

Summarizing, we have a new problem here, coming from the two-dimensionality of the land, and from connectedness issues for electoral districts. But, can we formulate a precise mathematical question, that we can study afterwards, out of this observation?

Well, this does not look easy, due to a variety of reasons, that will become clear in a moment. So, instead of torturing our brains with formulating a precise question, let us do some mathematics instead, in connection with the above, and in the hope that a good question will emerge from all this. As a first result on the subject, we have:

THEOREM 13.6. *Assuming that the country is connected, and that the electoral process and map take place in the continuum, as opposed to the above discrete models, the spades* $\spadesuit$ *can still win with* $1/4$ *of the votes, and with the districts being connected.*

PROOF. This is something purely mathematical, the idea being as follows:

(1) In math terms, what we have to prove is that given a connected set $X \subset \mathbb{R}^2$, representing the country, with $1/4$ of it colored black, and $3/4$ of it colored white, with these colors representing the $\spadesuit$ and $\heartsuit$ voters, and given $n \in \mathbb{N}$, the required number of electoral districts, we can always decompose X as a disjoint union as follows, with each set X_i, corresponding to a wanted electoral district, having area precisely $1/n$ of the total area of X, and being colored either $50-50$ black and white, or entirely white:

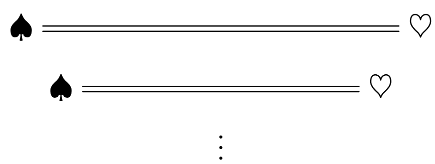
$$X = X_1 \sqcup \ldots \sqcup X_n$$

(2) But this is quite clear, because as explained before, the problem can only come from clumps of $\spadesuit$ voters, and point now is that, since we assumed that we are dealing with the continuum, each such $\spadesuit$ voter, even if isolated inside a clump, can send a "tentacle" towards the $\heartsuit$ voters, with of course the width of this tentacle being chosen to be $\varepsilon \simeq 0$, inside the $\spadesuit$ clump. Thus, we are led to the conclusion in the statement.

(3) As an illustration for this, let us discuss the example of the lollipop country mentioned before, with the spade voters being all amassed at the sweet region:

$$\begin{matrix} & \spadesuit & & & & & & & & \\ \spadesuit & \spadesuit & \spadesuit & \heartsuit & \heartsuit & \heartsuit & \ldots\ldots & \heartsuit & \heartsuit & \heartsuit \\ & \spadesuit & & & & & & & & \end{matrix}$$

For this particular country, every voter $\spadesuit$, or group of voters $\spadesuit$, depending on the total number n of required electoral districts, must send a tentacle towards the $\heartsuit$ voters, and the electoral districts will look as follows, to be stacked together, with the length of the various tentacles being adjusted, as for the conquered ♡ regions not to overlap:



(4) You get my point, what we say in the statement is definitely possible for the lollipop country, and for any other connected country as well, by using the same idea, namely send tentacles, and so on. Of course, if you are a mathematics nerd, exercise for you to write down a formal proof of this, after of course fine-tuning a bit the mathematical problem raised in (1), as to avoid various measure-theoretic related pathologies. □

In view of the above, in order to have some questions going on, in relation with the connectedness of the electoral districts, we must go back to our discrete grid models. Or perhaps keep dealing with the continuum, but asking this time for a substantial minimal width $w > 0$ for the electoral districts. But with this latter question looking far too complicated, we will go back to grid models, that is the reasonable thing to do.

As a first observation, when talking connectedness of various subsets of a grid, we have in fact two possible notions here, that we can use, as follows:

Definition 13.7. *A subset X of the integer grid $\mathbb{Z} \times \mathbb{Z}$ is called:*

1. *Truly connected, if each pair of squares $i, j \in X$ can be joined by a path of squares inside X, with at each step, the squares sharing a common edge.*
2. *Formally connected, if each pair of squares $i, j \in X$ can be joined by a path of squares inside X, with at each step, the squares sharing a common vertex.*

In other words, we have a difference coming from the truly connected spaces, looking a bit like tetris bits, and the connected spaces in the sense of formal mathematics. In what follows we will rather use the truly connected spaces, first because this is more realistic, say if you want to build roads of width $w > 0$ on your electoral districts, and also because this helps with mathematics. Indeed, by translating everything by 0.5 both on the horizontal and vertical, we can model our sets X by the midpoints $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ of their corresponding square blocks, and in this way, the notion of edge adjacency needed for talking about true connectedness reformulates in a very simple way, as follows:

$$(i, j) \sim (i \pm 1, j) \quad \text{and} \quad (i, j) \sim (i, j \pm 1)$$

Getting now to our goals, which remain that of formulating some sort of useful and doable mathematical question, in relation with all this, things remain complicated. However, after some thinking, the difficulty comes from the fact that we are trying to deal with two questions at the same time, one being that of evaluating the number and size of ♠ clumps, and the other one being that of how to deal with these clumps, as to win.

In view of this, the best is to split the problem in half, as follows:

QUESTIONS 13.8. *Given a set $Y \subset \{1,\ldots,N\} \times \{1,\ldots,N\}$, indexed by a binary matrix $A \in M_N(0,1)$, corresponding to the ♠ voters in a square country:*

(1) *How to read, on the matrix A, the distribution of the connected components of Y, that is, of the ♠ clumps, meaning number, size, and location on the map?*
(2) *And then, how to compute, again in terms of A, the maximal number of votes the ♠ party can get, via a suitable design of connected electoral districts?*

We will see in what follows that these are both good questions, that can be both investigated with tools from graph theory, and from probability.

13c. Graphs, topology

Graphs, topology.

13d. Probability methods

Probability methods.

13e. Exercises

Exercises:

EXERCISE 13.9.

EXERCISE 13.10.

EXERCISE 13.11.

EXERCISE 13.12.

EXERCISE 13.13.

EXERCISE 13.14.

Bonus exercise.

CHAPTER 14

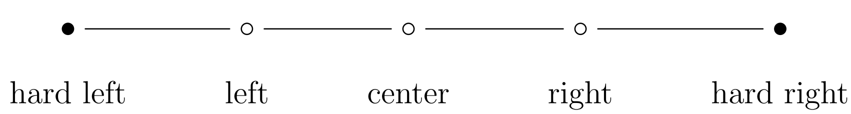
Left and right

14a. Left and right

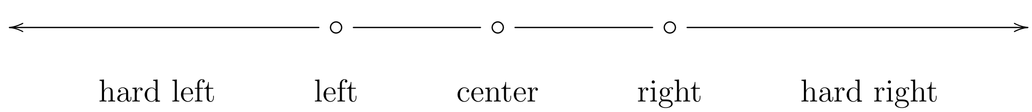
You certainly know about left and right, and about the wide implications our left-right system has, to questions regarding politics, economy, and society in general. Passed this, there are personal matters involved as well, who has not had heated political arguments with family, friends, colleagues, or even people in the street. And also, at the other side of what can happen or not happen, because of left and right, and politics in general, we have as well very serious consequences, such as poverty, famine, disease and war.

But, what exactly are left and right? Skipping what the left and right say about themselves, which can vary a lot, in time and in space, and we will not be here for commenting much on this, left and right are just a convention. They are a matter of dividing the individuals in two camps, as to have democracy and elections working.

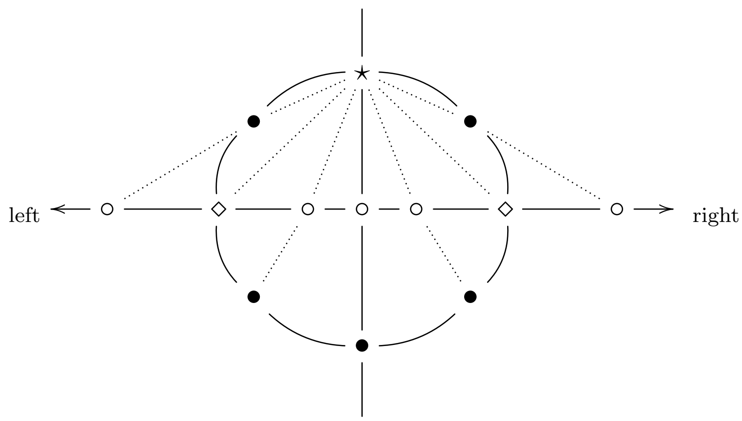
In practice, things are a bit more complicated than this, because when you have two precise camps, these camps will end up fighting, with the bigger one winning. Due to this, politics requires in fact a continuum of opinions, from hard left to center to hard right, for various parties to have some breathing space, and place for continuously adjusting their positions, with the basic picture of the political spectrum looking as follows:



However, this is just a basic picture. A version of it, which can sometimes explain many things, is by allowing both ends of the interval to be loose, as follows:

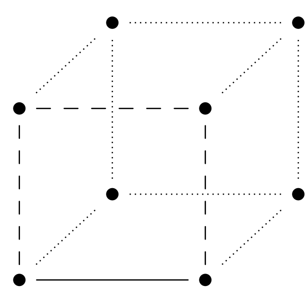


Among others, this latter model allows, say by wrapping everything into a circle, for the hard left and hard right to join each other, which is something that often happens in the real life. And with this, we are certainly into some mathematics, with the picture here, that of the circular model and of its usual linear projection, being as follows:



But the story is far from being over here, because all this remains 1D. Imagine for instance that you receive a questionnaire, with 10 political questions. This means that there might be in fact 10 political dimensions, and with the end of the questionnaire being that of determining, out of your $2^{10} = 1024$ answers, where exactly you are in the 1D model, or in the higher dimensional model used by those who wrote the questionnaire.

Thus, we can have more accurate "flat" models of the political spectrum in 2D, 3D and so on, by replacing the original interval by a square, cube and so on:



Moreover, we can further wrap these higher dimensional models into spheres, tori, and all sorts of other geometric objects, a bit as we did above, with the 1D interval.

The remainder of this book is an introduction to this, politics from a topological viewpoint, and its applications to economy questions. We will mostly insist on the 3D models, cubic as above, or with loose ends, or curved in various mathematical ways.

We will need the formula of the stereographic projection, which is as follows:

THEOREM 14.1. *The map wrapping $[-1,1]$ into the unit circle, and then stereographically projecting on $\mathbb{R}$ is given by the formula*

$$\varphi(u)=\tan\left(\frac{\pi u}{2}\right)$$

with the convention that our wrapping is the most straightforward one, making correspond $\pm 1\to i$, with negatives on the left, and positives on the right.

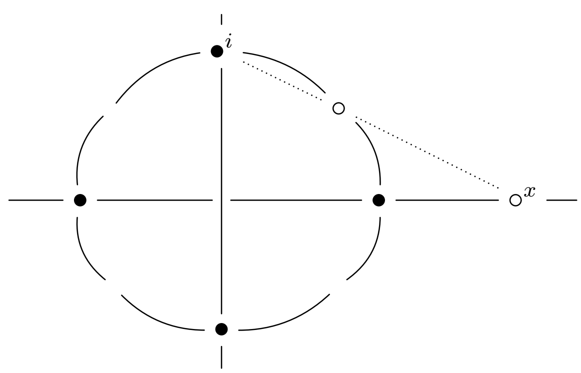
PROOF. Regarding the wrapping, as indicated, this is given by:

$$u\to e^{it}\quad,\quad t=\pi u-\frac{\pi}{2}$$

Indeed, this correspondence wraps $[-1,1]$ as above, the basic instances of our correspondence being as follows, and with everything being fine modulo 2π:

$$-1\to\frac{\pi}{2}\quad,\quad -\frac{1}{2}\to-\pi\quad,\quad 0\to-\frac{\pi}{2}\quad,\quad \frac{1}{2}\to 0\quad,\quad 1\to\frac{\pi}{2}$$

Regarding now the stereographic projection, the picture here is as follows:



Thus, by Thales, the formula of the stereographic projection is as follows:

$$\frac{\cos t}{x}=\frac{1-\sin t}{1}\implies x=\frac{\cos t}{1-\sin t}$$

Now if we compose our wrapping operation above with the stereographic projection, what we get is, via the above Thales formula, and some trigonometry:

$$\begin{aligned} x &= \frac{\cos t}{1-\sin t} \\ &= \frac{\cos\left(\pi u - \frac{\pi}{2}\right)}{1-\sin\left(\pi u - \frac{\pi}{2}\right)} \\ &= \frac{\cos\left(\frac{\pi}{2} - \pi u\right)}{1+\sin\left(\frac{\pi}{2} - \pi u\right)} \\ &= \frac{\sin(\pi u)}{1+\cos(\pi u)} \\ &= \frac{2\sin\left(\frac{\pi u}{2}\right)\cos\left(\frac{\pi u}{2}\right)}{2\cos^2\left(\frac{\pi u}{2}\right)} \\ &= \tan\left(\frac{\pi u}{2}\right) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Bonus exercise.

CHAPTER 15

Two dimensions

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

EXERCISE 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

Bonus exercise.

CHAPTER 16

Three dimensions

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

Bibliography

[1] V.I. Arnold, Ordinary differential equations, Springer (1973).

[2] V.I. Arnold, Mathematical methods of classical mechanics, Springer (1974).

[3] V.I. Arnold, Lectures on partial differential equations, Springer (1997).

[4] V.I. Arnold, Catastrophe theory, Springer (1974).

[5] J. Baik, P. Deift and T. Suidan, Combinatorics and random matrix theory, AMS (2016).

[6] T. Banica, Calculus and applications (2024).

[7] T. Banica, Measure and integration (2024).

[8] T. Banica, Principles of thermodynamics (2024).

[9] G.K. Batchelor, An introduction to fluid dynamics, Cambridge Univ. Press (1967).

[10] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[11] I. Bengtsson and K. Życzkowski, Geometry of quantum states, Cambridge Univ. Press (2006).

[12] S.J. Blundell and K.M. Blundell, Concepts in thermal physics, Oxford Univ. Press (2006).

[13] B. Bollobás, Modern graph theory, Springer (1998).

[14] B. Bollobás, Random graphs, Cambridge Univ. Press (1985).

[15] A. Borodin, I. Corwin and A. Guionnet, eds., Random matrices, AMS (2019).

[16] A. Bose and M. Bhattacharjee, Large covariance and autocovariance matrices, CRC Press (2018).

[17] E.A. Coddington, An introduction to ordinary differential equations, Dover (1961).

[18] J.B. Conway, A course in functional analysis, Springer (1985).

[19] C. Darwin, On the origin of species (1859).

[20] W. de Launey and D. Flannery, Algebraic design theory, AMS (2011).

[21] M.P. do Carmo, Differential geometry of curves and surfaces, Dover (1976).

[22] M.P. do Carmo, Riemannian geometry, Birkhäuser (1992).

[23] S.T. Dougherty, Combinatorics and finite geometry, Springer (2020).

[24] M. Dresher, The mathematics of games of strategy, Dover (1981).

[25] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).

[26] R. Durrett, Random graph dynamics, Cambridge Univ. Press (2006).

[27] A. Einstein, Relativity: the special and the general theory, Dover (1916).

[28] M. Eliade, A history of religious ideas, I: from the Stone Age to the Eleusinian Mysteries (1978).

[29] M. Eliade, A history of religious ideas, II: from Gautama Buddha to the triumph of Christianity (1982).

[30] M. Eliade, A history of religious ideas, III: from Muhammad to the Age of the Reforms (1985).

[31] L.C. Evans, Partial differential equations, AMS (1998).

[32] S.J. Farlow, Partial differential equations for scientists and engineers, Dover (1982).

[33] W. Feller, An introduction to probability theory and its applications, Wiley (1950).

[34] E. Fermi, Thermodynamics, Dover (1937).

[35] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics I: mainly mechanics, radiation and heat, Caltech (1963).

[36] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics II: mainly electromagnetism and matter, Caltech (1964).

[37] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman lectures on physics III: quantum mechanics, Caltech (1966).

[38] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals, Dover (1965).

[39] M. Foucault, Madness and civilization: a history of insanity in the age of reason (1961).

[40] S. Freud, The interpretation of dreams (1899).

[41] S. Freud, Three essays on the theory of sexuality (1905).

[42] S. Freud, The future of an illusion (1927).

[43] C. Godsil and G. Royle, Algebraic graph theory, Springer (2001).

[44] H. Goldstein, C. Safko and J. Poole, Classical mechanics, Addison-Wesley (1980).

[45] E.H. Gombrich, The story of art (1950).

[46] D.J. Griffiths, Introduction to electrodynamics, Cambridge Univ. Press (2017).

[47] D.J. Griffiths and D.F. Schroeter, Introduction to quantum mechanics, Cambridge Univ. Press (2018).

[48] D.J. Griffiths, Introduction to elementary particles, Wiley (2020).

[49] D.J. Griffiths, Revolutions in twentieth-century physics, Cambridge Univ. Press (2012).

[50] R.W. Hamming, Numerical methods for scientists and engineers, Dover (1962).

[51] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge Univ. Press (1985).

[52] K. Huang, Introduction to statistical physics, CRC Press (2001).

[53] K. Huang, Quantum field theory, Wiley (1998).

[54] K. Huang, Quarks, leptons and gauge fields, World Scientific (1982).

[55] K. Huang, Fundamental forces of nature, World Scientific (2007).

[56] T.J. Kaczynski, Industrial society and its future (1995).

[57] L.P. Kadanoff, Statistical physics: statics, dynamics and renormalization, World Scientific (2000).

[58] I. Kant, Critique of pure reason (1781).

[59] T. Kibble and F.H. Berkshire, Classical mechanics, Imperial College Press (1966).

[60] D.E. Knuth, The art of computer programming, Addison-Wesley (1968).

[61] M. Kumar, Quantum: Einstein, Bohr, and the great debate about the nature of reality, Norton (2009).

[62] T. Lancaster and K.M. Blundell, Quantum field theory for the gifted amateur, Oxford Univ. Press (2014).

[63] L.D. Landau and E.M. Lifshitz, Course of theoretical physics, Pergamon Press (1960).

[64] S. Lang, Algebra, Addison-Wesley (1993).

[65] P. Lax, Linear algebra and its applications, Wiley (2007).

[66] P. Lax, Functional analysis, Wiley (2002).

[67] P. Lax and M.S. Terrell, Calculus with applications, Springer (2013).

[68] P. Lax and M.S. Terrell, Multivariable calculus with applications, Springer (2018).

[69] T.M. Liggett, Interacting particle systems, Springer (1985).

[70] S. Ling and C. Xing, Coding theory: a first course, Cambridge Univ. Press (2004).

[71] G. Livan, M. Novaes and P. Vivo, Introduction to random matrices: theory and practice, Springer (2018).

[72] K. Marx, Capital, I (1867).

[73] K. Marx, Capital, II (1885).

[74] K. Marx, Capital, III (1894).

[75] K. Marx and F. Engels, The communist manifesto (1848).

[76] M.L. Mehta, Random matrices, Elsevier (1967).

[77] M.A. Nielsen and I.L. Chuang, Quantum computation and quantum information, Cambridge Univ. Press (2000).

[78] F.W. Nietzsche, Thus spoke Zarathustra (1883).

[79] R.K. Pathria and and P.D. Beale, Statistical mechanics, Elsevier (1972).

[80] M. Potters and J.P. Bouchaud, A first course in random matrix theory, Cambridge Univ. Press (2020).

[81] J. Preskill, Quantum information and computation, Caltech (1998).

[82] W. Rudin, Principles of mathematical analysis, McGraw-Hill (1964).

[83] W. Rudin, Real and complex analysis, McGraw-Hill (1966).

[84] W. Rudin, Functional analysis, McGraw-Hill (1973).

[85] D.V. Schroeder, An introduction to thermal physics, Oxford Univ. Press (1999).

[86] N.J.A. Sloane and S. Plouffe, Encyclopedia of integer sequences, Academic Press (1995).

[87] A.M. Steane, Thermodynamics, Oxford Univ. Press (2016).

[88] S. Sternberg, Dynamical systems, Dover (2010).

[89] D.R. Stinson, Combinatorial designs: constructions and analysis, Springer (2006).

[90] J.R. Taylor, Classical mechanics, Univ. Science Books (2003).

[91] M. Tenenbaum and H. Pollard, Ordinary differential equations, Dover (1963).

[92] R.J. Trudeau, Introduction to graph theory, Dover (1993).

[93] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).

[94] J. von Neumann and O. Morgenstern, Theory of games and economic behavior, Princeton Univ. Press (1944).

[95] J. Watrous, The theory of quantum information, Cambridge Univ. Press (2018).

[96] S. Weinberg, Foundations of modern physics, Cambridge Univ. Press (2011).

[97] S. Weinberg, Lectures on quantum mechanics, Cambridge Univ. Press (2012).

[98] S. Weinberg, Lectures on astrophysics, Cambridge Univ. Press (2019).

[99] S. Weinberg, Cosmology, Oxford Univ. Press (2008).

[100] H. Weyl, Space, time, matter, Princeton Univ. Press (1918).