Principles of lattice models

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Abstract. A lattice model is a discretization of a usual, continuous problem taking place in $\mathbb{R}^3$, over the lattice $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, with $\varepsilon \simeq 0$. Examples of such models abound, with the general idea of discretization being in fact the main idea of statistical mechanics, and with a main example being the Ising model for ferromagnetism. We discuss here such questions, with an introduction to the subject, and with an introduction to more advanced aspects as well, using methods from probability and quantum algebra.

Preface

A lattice model is a discretization of a usual, continuous problem taking place in $\mathbb{R}^3$, over the lattice $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, with $\varepsilon \simeq 0$. That is, our modelling method consists in replacing the space $\mathbb{R}^3$ with the discrete lattice $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, chosen to be reasonably fine, $\varepsilon \simeq 0$, then reformulating our question in terms of this lattice $\varepsilon\mathbb{Z}^3$, in the hope that the mathematics will get simpler, and that with $\varepsilon \to 0$ we will get to interesting results.

All this is extremely vague and general, and actually reminds computers, and you would probably say "yes, sounds like a lazy scientist who wants the computer to do all the work". Which is a good point, but it is not about numeric methods that we want to talk about in this book. Our claim is that discretization can be a powerful method even in the context of pure mathematics and physics, with no computers allowed.

Here are a few examples of all this, and judge for yourself:

(1) Waves. Not that the wave equation $\ddot{\varphi} = v^2\Delta\varphi$ really needs help from discretization, but one thing that you probably know, and which is really beautiful, is that this wave equation can be deduced by approximating $\mathbb{R}^3$ with a lattice $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, with $\varepsilon \to 0$, consisting of balls connected by springs. So, forgetting now about the wave equation itself, we have an answer to an interesting philosophical question, regarding the very nature of our universe. To be more precise, based on this, we can declare that our universe is simply something made of tiny little balls, connected by tiny little springs.

(2) Along the same lines, by making use of some imagination, we can discretize if we want many other things, with every time interesting philosophical conclusions. Want to talk about the propagation of heat, $\dot{\varphi} = \alpha\Delta\varphi$? There surely should be a discrete model for that. What about fluid mechanics, and the various equations there? Just discretize the fluids, as being formed of tiny little cubes, and here's your model. And so on.

(3) We can in fact even discretize gravity itself, if we really want to, by saying that if we place a mass $M > 0$ at the origin, the "news" about this mass $M > 0$ will spread quickly among the points of $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, from neighbor to neighbor, with everyone saying around something of type "hey buddies, there's a mass $M > 0$ somewhere towards the origin, please comply and get attracted by it, and pass the word to your neighbors". So that in the end everyone gets aware of this mass, and so gravity works.

(4) And with the remark that, with this gravity example, we might be well now into serious physics. Indeed, one of the main findings of modern physics is that there is a “Planck scale” $\varepsilon > 0$ where the discretization of physics might actually be something exact, and understanding gravity at the Planck scale is a big open problem. And with the side remark here that having such a thing done would eventually put an end to the “physics of the void” that we are so used to, for centuries. Probably a good thing.

(5) Less speculatively now, examples of discretization abound in classical statistical mechanics, with this meaning in phenomena appearing from thermodynamics. In fact, the general idea of discretization is the main idea of statistical mechanics.

(6) And even less speculatively now, examples of lattices $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$ are naturaly met in the context of materials like metals, where electrons can be thought of as being arranged on such a lattice. And, with this in mind, a main example of a lattice model, having countless concrete applications, is the Ising model for ferromagnetism.

(7) And for ending with more speculation, quantum mechanics in its advanced form used nowadays, which is Feynman’s quantum electrodynamics (QED), is by definition something of a lattice model too, with a particle, in Feynman’s vision, hesitating a billion times per second on where to go, left, right, up, down and so on, a bit like being totally lost inside a certain lattice $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, with ε being very small.

So, this was for the general idea of lattice models, and as you can see, this idea is as wide as physics itself. And by throwing now into the picture computers and numeric methods, that we have not talked about in the above, the idea here being that computers understand much better discrete problems, formulated over a lattice $\varepsilon\mathbb{Z}^3 \subset \mathbb{R}^3$, than continuous problems, formulated over $\mathbb{R}^3$ itself, we are led to the conclusion that, in practice, lattice models might well account for more than half of physics.

The present book is an introduction to all this. Our goal with be that of discussing such lattice models, from a theoretical and abstract viewpoint, with as many interesting examples as possible, and by being reasonably strong and modern on mathematical aspects, with a slight preference for algebraic and probabilistic methods.

We will mostly insist on (5,6), statistical mechanics and the Ising model, which are the topics where there are many things can be said, and by using for our study quantum groups, along the lines of the Yang-Baxter equation, and of the work of Vaughan Jones. And we will also talk a bit about more speculative aspects, such as (1,2,3,4,7).

Part of this book uses some quantum group mathematics that I developed since the mid 90s, and notably the notion of matrix model for quantum groups, and I would like to thank my coworkers. Many thanks go as well to my cats, for some help with computations. The mouse discretization result from the end of chapter 16 is due to them.

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