

The physics of low temperatures

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ABSTRACT. This is an introduction to the physics of low temperatures. We discuss both the physics and chemistry around absolute zero, and in the interstellar space.

Preface

Afraid of zero Kelvin? So am I, that is -273.15 C, most likely freezing cold, far more severe than in the Dracula mountains where I grew up. Certainly no room there for vegetation, animals, love and everything, as we know it, from our planet Earth.

This being said, zero Kelvin is a fascinating concept, physically speaking. You might know it from relativity, speeds are bounded from above by the speed of light, $v < c$, and the existence of absolute zero is the same thing as saying that all temperatures are bounded from below by this zero, $T > 0$, and so, a similar phenomenon. Want it or not, the world that we live in is doubly curved, by both Einstein and Kelvin.

From a less abstract perspective, but still remaining into rather theoretical physics, temperatures around zero Kelvin are actually something extremely natural. Forgetting about our small life here on Earth, and forgetting even about our beloved Sun, the average temperature in the interstellar space is $T = 2.73$ K. So, from this cosmic perspective, $T < 2.73$ K means cold, and $T > 2.73$ K means warm. As simple as that.

Quite interestingly, if you are familiar with some calculus, $2.73 \simeq 2.71 = e$, our favorite constant in mathematics, and with this being related to the various things said above, both about curvature, and about the interstellar space. Indeed, when writing the Kelvin temperature as $T = e^L$, with L standing for logarithmic Kelvin degrees, the range of these latter degrees is $L \in (-\infty, \infty)$, so the curvature disappears, and also from the cosmic perspective, and up to an epsilon, $L < 0$ means cold, and $L > 0$ means warm. Which is of course very nice, and many further speculations can be made here.

But are we here for talking philosophy, or science and engineering. Very low temperatures, close to zero Kelvin, can be obtained in labs here on Earth, via a variety of methods, and the physics and chemistry there are quite fascinating. Liquid helium and its properties, superconductivity, Bose-Einstein condensates, and many more. All this looks like a wonderland for physics and engineering, and is certainly to be explored.

The present book is an introduction to this, the physics of low temperatures, by exploring both the philosophical and the more concrete aspects of what happens there, and by mixing things that we can make happen here on Earth, in our labs, and things that happen naturally in the outer space. A bit in the spirit of the above discussion.

The book has 4 parts, with Part I dealing with the math, physics and philosophy of zero Kelvin, then Part II and Part III dealing with the physics at small temperatures, and with Part IV going back to philosophy, with a trip to the outer space. Sort of a sandwich organization, and with the whole thing being served, of course, refreshingly cold.

Many thanks go to my probability and physics friends and colleagues, for various discussions about this, over the time. Thanks as well to my cats, freeze then attack, that is a strategy that I learned from them, to be widely used in this book.

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Part I

Absolute zero

*Back with another one
Of those block rockin' beats
We're about ready
To rock steady*

CHAPTER 1

Thermodynamics

1a. Thermodynamics

We must first talk about gases. Let us start with the following basic fact, which was the beginning of everything, going back to work of Boyle, Charles, Avogadro, Gay-Lussac, Clapeyron and others from the 17th, 18th and 19th centuries, and with final touches from Maxwell, Boltzmann, Gibbs and others, in the late 19th and early 20th centuries:

FACT 1.1. *The ideal gases satisfy the equation $PV = kT$, where:*

- (1) *V is the volume of the gas, independently of the shape of the container used.*
- (2) *P is the pressure of the gas, measured with a manometer.*
- (3) *T is the temperature of the gas, measured with a thermometer.*
- (4) *k is a constant, depending on the gas.*

That is, $PV = kT$ basically tells us that “pressure and temperature are the same thing”.

At the first glance, for instance if you are a mathematician not used to this, this looks more like a joke. Why not defining then $P = T$ or vice-versa, you would say, and what is the point with that long list of distinguished gentlemen having worked hard on this.

Error. The point indeed comes from the following:

EXPLANATION 1.2. *In the equation of state $PV = kT$, as formulated above, the pressure P and the temperature T appear more precisely as follows,*

- (1) *The manometer read comes from the gas molecules pushing a piston, so P is a statistical quantity, coming from the statistics of the molecular speeds,*
- (2) *The thermometer read is something even more complicated, and T is as well a statistical quantity, coming from the statistics of the molecular speeds,*

so $PV = kT$ is something non-trivial, telling us that the mathematical machinery producing P, T , via manometer and thermometer, out of the molecular speeds, is the same.

Hope you got my point, and getting back now to historical details, Boyle, Charles, Avogadro, Gay-Lussac, Clapeyron, joined by Clausius, Carnot, Joule, Lord Kelvin and others, first observed $PV = kT$, and then reached to a good understanding of what this means, via an axiomatization of P and T . Later Maxwell started to look into the molecular speeds and their statistics, then Boltzmann came with a tough mathematical

computation, proving $PV = kT$, and then, even later, Gibbs and others further built on all this, by formalizing modern thermodynamics, in the form that is still used today.

But probably too much talking, let us get to work. As a first result now, dealing with pressure only, and for the gases without collisions between molecules, we have:

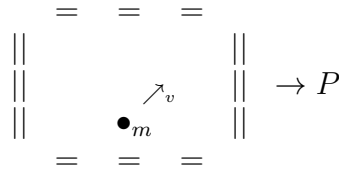
THEOREM 1.3. *The pressure P , volume V and total kinetic energy K of a gas, having point molecules, with no collisions between them, satisfy*

$$PV = \frac{2K}{d}$$

where $d = 1, 2, 3$ is the dimensionality of the gas, $d = 3$ for usual 3D gases.

PROOF. We can do this in several steps, as follows:

(1) Let us first assume that the gas is enclosed in a cubic volume, $V = L^3$. We want to compute the pressure P on the right wall. Since there are no collisions, we can assume by linearity that our gas has 1 molecule, having mass m and travelling at speed v . We must compute the pressure P exerted by this molecule on the right wall:



(2) We first look at a 1D gas. Our molecule hits the right wall at every $\Delta t = 2L/v$ interval, with its change of momentum being $\Delta p = 2mv$. We obtain, as desired:

$$P = \frac{F}{L^2} = \frac{\Delta p}{L^2 \Delta t} = \frac{2mv}{L^2 \cdot 2L/v} = \frac{mv^2}{L^3} = \frac{2K}{V}$$

(3) In the case of a d -dimensional gas, exactly the same computation takes place, but this time with v being replaced by its horizontal component v_1 . Thus, we have:

$$P = \frac{mv_1^2}{V}$$

But, we have the following formula, with the equality on the right being understood in a statistical sense, our molecule being assumed to follow a random direction:

$$\|v\|^2 = v_1^2 + \dots + v_d^2 = dv_1^2$$

Thus, the pressure in this case is given by the following formula, as desired:

$$P = \frac{m\|v\|^2}{dV} = \frac{2K}{dV}$$

(4) It remains to extend our result to arbitrary volume shapes. For this purpose, let us first redo the above computations for a parallelepiped, $V = L_1L_2L_3$. Here the above 1D gas computation carries on, and gives the same result, as follows:

$$P = \frac{F}{L_2L_3} = \frac{\Delta p}{L_2L_3\Delta t} = \frac{2mv}{L_2L_3 \cdot 2L_1/v} = \frac{mv^2}{L_1L_2L_3} = \frac{2K}{V}$$

Thus the d -dimensional computation carries on too, and gives the result.

(5) In order now to reach to arbitrary shapes, the idea will be that of stacking thin parallelepipeds, best approximating the shape that we have in mind, as follows:



(6) But for this purpose it is better to drop our assumption that the gas has 1 molecule, and use N molecules instead. With $\rho = N/V$ being the molecular density, and K_0 being the kinetic energy of a single molecule, our computation in (4) for the parallelepiped, with now N molecules instead of 1, reformulates as follows:

$$P = \frac{2K}{dV} = \frac{2NK_0}{dV} = \frac{2\rho VK_0}{dV} = \frac{2\rho K_0}{d}$$

(7) But this latter formula shows that the pressure has nothing to do with the precise volume V , but just with the molecular density $\rho = N/V$. Thus, we can stack indeed parallelepipeds, with of course the assumption that ρ is constant over these parallelepipeds, and we obtain that the above formula holds for an arbitrary volume shape V :

$$P = \frac{2\rho K_0}{d}$$

Now by getting back to the volume V , we obtain the following formula:

$$P = \frac{2\rho K_0}{d} = \frac{2NK_0}{dV} = \frac{2K}{dV}$$

Thus, we are led to the conclusion in the statement. \square

Observe that, when carefully looking at the above proof, and more specifically at step (3), we can see that pressure P is not something instantaneous, but rather something statistical. Moreover, we can also see that the pressure reading needs some time $t > 0$.

At a more advanced level now, dealing with the internal mechanism of the $PV = kT$ formula, we have the following famous result, due to Maxwell and Boltzmann:

THEOREM 1.4 (Maxwell, Boltzmann). *The molecular speeds $v \in \mathbb{R}^3$ of a gas in thermal equilibrium are subject to the Maxwell-Boltzmann distribution formula*

$$P(v) = \left(\frac{m}{2\pi bT}\right)^{3/2} \exp\left(-\frac{m\|v\|^2}{2bT}\right)$$

with m being the mass of the molecules, and b being the Boltzmann constant.

PROOF. As before with other things, this is something in between fact and theorem. Maxwell came upon it as a fact, or perhaps as a sort of pseudo-theorem, and a bit later Boltzmann came with a proof. In what follows we will present the original argument of Maxwell, then briefly discuss Boltzmann's proof. Here is Maxwell's argument:

(1) We are looking for the precise probability distribution P of the molecular speeds $v = (v_1, v_2, v_3)$ which makes the mechanics of gases work. Intuition tells us that P has no correlations between the x, y, z directions of space, and so we must have:

$$P(v) = f(v_1)g(v_2)h(v_3)$$

Moreover, by rotational symmetry the functions f, g, h must coincide, and so:

$$P(v) = f(v_1)f(v_2)f(v_3)$$

(2) Further thinking, again invoking rotational symmetry, leads to the conclusion that $P(v)$ must depend only on the magnitude $\|v\|$ of the velocity $v \in \mathbb{R}^3$, and not on the direction. Thus, we must have as well a formula of the following type:

$$P(v) = \varphi(\|v\|^2)$$

(3) Now by comparing the requirements in (1) and (2), we are led via some math to the conclusion that φ must be an exponential, which amounts in saying that:

$$P(v) = \lambda \exp(-C\|v\|^2)$$

(4) Obviously we must have $C > 0$, for things to be bounded, and then by integrating we can obtain λ as function of C , and our formula becomes:

$$P(v) = \left(\frac{C}{\pi}\right)^{3/2} \exp(-C\|v\|^2)$$

(5) It remains to find the value of $C > 0$. But for this purpose, observe that, now that we have our distribution, be that still depending on $C > 0$, we can compute everything that we want to, just by integrating. In particular, we find that on average:

$$v_1^2 = v_2^2 = v_3^2 = \frac{1}{2C}$$

Thus the average magnitude of the molecular speed is given by:

$$\|v\| = \frac{3}{2C}$$

It follows that the average kinetic energy of the molecules is:

$$K_0 = \frac{m\langle |v|^2 \rangle}{2} = \frac{3m}{4C}$$

(6) On the other hand, recall from our discussions above that one of the many equivalent formulations of $PV = kT$, using $PV = 2K/3$, was as follows:

$$\frac{2K_0}{3} = bT$$

Thus we obtain $m/(2C) = bT$, and so $C = m/(2bT)$, as desired.

(7) Observe that the above proof has in fact little physical content, with the whole thing being basically obtained by using $PV = 2K/3$, which is a mathematical theorem, then $PV = kT$, which is a crucial physics fact, and finally by invoking several times a number of clever symmetry arguments, and doing some calculus.

(8) However, Boltzmann came later with a more rigorous argument, fully establishing the above formula, via detailed computations in relation with the internal collisions. \square

At a more advanced level, still regarding the gases, the following key result:

THEOREM 1.5. *Beyond the ideal gas setting, stating that we should have*

$$PV = kT$$

the gases are subject to the Van der Waals equation

$$\left(P + \frac{\alpha}{V^2}\right)(V - \beta) = kT$$

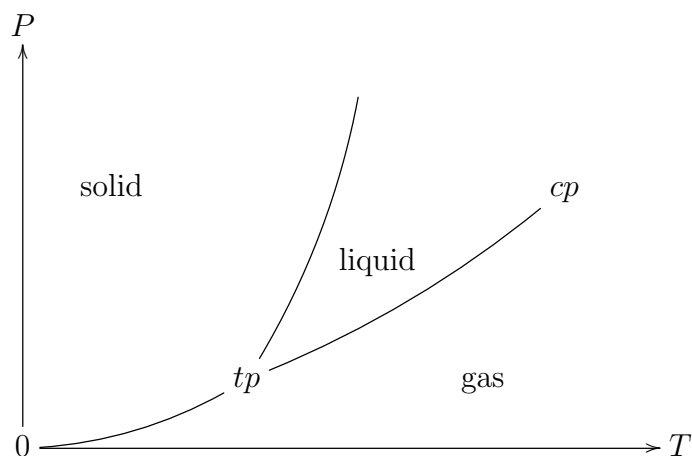
depending on two parameters $\alpha, \beta > 0$.

PROOF. This is something quite tricky, with the correction parameters $\alpha, \beta > 0$ appearing from a detailed study of the gas, from a kinetic viewpoint. \square

The above result is of key importance, and takes us into rethinking everything that we know about the ideal gases, which must be replaced with Van der Waals gases, at the advanced level. Among the main consequences of this replacement, the isobars, isochores, isothermals and adiabatics of the ideal gases, given by simple formulae, must be replaced by isobars, isochores, isothermals and adiabatics for the Van der Waals gases, which are no longer something trivial, with some interesting math being now involved.

Among others, the Van der Waals gas study makes appear some interesting points on the isothermals, called triple and critical points of the gas. But this makes the connection with the general theory of matter, which can be summarized as follows:

FACT 1.6. *Ordinary matter appears in 3 forms, namely solid, liquid and gaseous, roughly appearing according to the following generic diagram*



with tp, cp standing for the triple and critical points. Also, at low or high temperatures we have interesting phenomena like Bose-Einstein condensation, and plasma.

Passed the gases, that we are quite familiar with, the problem is with the liquids and solids, plus of course with all sorts of exotic matter that can appear at the extremes.

1b.

1c.

1d.

1e. Exercises

Exercises:

EXERCISE 1.7.

EXERCISE 1.8.

EXERCISE 1.9.

EXERCISE 1.10.

EXERCISE 1.11.

EXERCISE 1.12.

Bonus exercise.

CHAPTER 2

Absolute zero

2a. Absolute zero

Regarding absolute zero in thermodynamics, here is the precise claim:

CLAIM 2.1. *The temperature as measured before, with an arbitrarily calibrated thermometer, can be linearly rescaled into an absolute temperature, which is positive,*

$$T > 0$$

and is given by the following equation, valid for any $T_1 < T_2$,

$$\frac{T_1}{T_2} = 1 - \eta$$

where η is the efficiency of a reversible engine operating in the regime $[T_1, T_2]$.

In order to prove this claim, we will use engine techniques from chapter 1. Let us first review the material there, with abstract applications in mind. First, we have:

DEFINITION 2.2. *The efficiency of an arbitrary cyclic engine functioning between temperatures $T_1 < T_2$ is the quantity $\eta \in (0, 1)$ given by*

$$\eta = \frac{W}{Q_2} = \frac{Q_2 - Q_1}{Q_2} = 1 - \frac{Q_1}{Q_2}$$

where Q_2 is the heat absorbed at T_2 , from the heat source of the engine, W is the work done, and Q_1 is the loss in the process $Q_2 \rightarrow W$, appearing as heat expelled at T_1 .

We refer to chapter 1 for more on this, explanations, theory and examples, with a summary of what we know so far about this being as follows:

(1) Many useful heat engines, with variable efficiencies, can be designed, for practical use in the real life, well-known being the Otto, Diesel and Rankine ones.

(2) When it comes to efficiency, as being the quantity defined above, this is the same, and is maximal, for the reversible engines running at $[T_1, T_2]$.

(3) These reversible engines, which are something that are usually not implemented in the real life, due to a number of reasons, including the Carnot ones.

All this might seem a bit away from the questions that we want to investigate here, but with Claim 2.1 in mind, the notion of efficiency, and the findings in (2) and (3), can be useful for us. The precise result here, that we know from chapter 1, is as follows:

THEOREM 2.3. *Among the engines functioning in a given $[T_1, T_2]$ regime, the maximum efficiency*

$$\eta = 1 - \frac{Q_1}{Q_2}$$

is achieved by the reversible engines, including the Carnot ones, and with all these reversible engines having the same efficiency.

PROOF. This is something that we know from chapter 1, but since we will be heavily using all this, we will review the proof, with more details. This goes as follows:

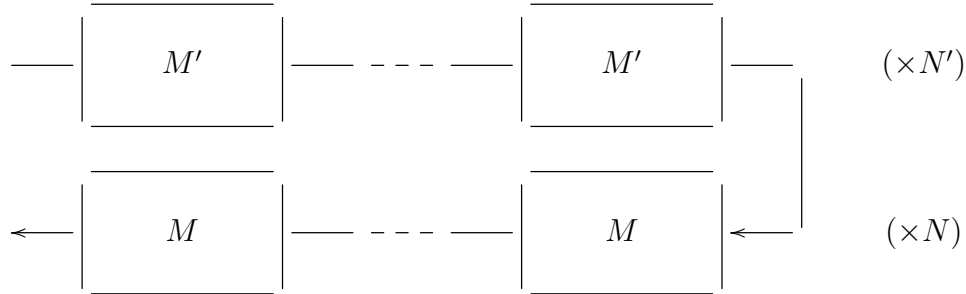
(1) It is enough to prove the following inequality, for any two engines M, M' , with the first engine M being assumed to be reversible:

$$\frac{Q_1}{Q_2} \leq \frac{Q'_1}{Q'_2}$$

So, assume that this is wrong, and pick a rational number in between:

$$\frac{Q_1}{Q'_1} > \frac{N'}{N} > \frac{Q_2}{Q'_2}$$

(2) In order to come up with a contradiction, based on this, we use a trick. Consider the engine \mathcal{M} consisting of N' cycles of M' , and N reverse cycles of M :



(3) The data for this new, complex engine \mathcal{M} is then as follows:

$$Q_1 = N'Q'_1 - NQ_1$$

$$Q_2 = N'Q'_2 - NQ_2$$

$$\mathcal{W} = N'W' - NW$$

(4) We have then $\mathcal{W} = Q_2 - Q_1$. On the other hand, the inequalities in (1) read:

$$Q_1 < 0 \quad , \quad Q_2 > 0$$

We conclude from this that the total work of our complex engine \mathcal{M} is positive:

$$\mathcal{W} > 0$$

(5) Thus, what we have here a true, usual engine, producing work. But the point now is that, as explained in chapter 1, this would imply via the second law that we have $\mathcal{Q}_2 > \mathcal{Q}_1 > 0$. Thus, we have our contradiction, and we are done. \square

As a consequence of Theorem 2.3, we can now fine-tune our definition of efficiency, for the reversible engines, that we will use exclusively in this chapter, as follows:

DEFINITION 2.4. *The efficiency of a reversible engine operating in the regime $[T_1, T_2]$ is the number $\eta = \eta(T_1, T_2)$ defined as follows,*

$$\eta(T_1, T_2) = 1 - \frac{Q_1}{Q_2}$$

known to depend only on the temperatures T_1, T_2 .

To be more precise here, the formula for η is the previous one, from Definition 2.2, and the fact that this number η depends only on T_1, T_2 comes from Theorem 2.3.

We have now all the ingredients for stating our main result, as follows:

THEOREM 2.5. *The temperature as measured before, with an arbitrarily calibrated thermometer, can be linearly rescaled into an absolute temperature, which is positive,*

$$T > 0$$

and is given by the following equation, valid for any $T_1 < T_2$,

$$\frac{T_1}{T_2} = 1 - \eta$$

where η is the efficiency of a reversible engine operating in the regime $[T_1, T_2]$.

PROOF. This follows from the theory developed above, and from some supplementary tricks, of the same type, combining engines, the idea being as follows:

(1) Consider the following function, defined for any temperatures $T_1 < T_2$:

$$f(T_1, T_2) = \frac{1}{1 - \eta(T_1, T_2)} = \frac{Q_2}{Q_1}$$

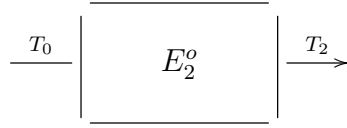
Our claim, which will basically lead to the result, via some elementary mathematics, is that we have the following formula, valid for any temperatures $T_0 < T_1 < T_2$:

$$f(T_0, T_2) = f(T_0, T_1)f(T_1, T_2)$$

(2) In order to prove this claim, consider a reversible engine E_1 running in the regime $[T_0, T_1]$, and a reversible engine E_2 running in the regime $[T_0, T_2]$:



Since E_2 is reversible, we can run it in reverse sense as well, as follows:



(3) We denote by Q_0, Q_1 the amount of heat expelled/absorbed by E_1 at T_0/T_1 , and, after suitably normalizing E_2 , we denote by Q_0, Q_2 the amount of heat expelled/absorbed by E_2 at T_0/T_2 . According to our definition of efficiency, we have the following formulae:

$$f(T_0, T_1) = \frac{Q_1}{Q_0} \quad , \quad f(T_0, T_2) = \frac{Q_2}{Q_0}$$

(4) Consider now an engine E , obtained by running a cycle of E_1 , and then a reverse cycle of E_2 , denoted as above by E_2^o , with the picture being as follows:



(5) Let us analyze how this complex engine E works at the temperatures T_0, T_1, T_2 . According to our assumptions, the situation here is as follows:

- At T_0 , no heat is exchanged, Q_0 being absorbed by the reverse E_2 .
- At T_1 , what happens is that an amount of Q_1 is expelled by E_1 .
- At T_2 , what happens is that an amount of Q_2 is absorbed by E_2 .

(6) We conclude from this that E is a reversible engine running in the regime $[T_1, T_2]$, and so, by our definition of efficiency, we have the following formula:

$$f(T_1, T_2) = \frac{Q_2}{Q_1}$$

Now by comparing with the two formulae obtained at (3), we get, as claimed:

$$f(T_0, T_2) = f(T_0, T_1)f(T_1, T_2)$$

(7) But this tells us that we can find a certain function $\theta = \theta(T)$ such that:

$$f(T_1, T_2) = \frac{\theta(T_2)}{\theta(T_1)}$$

Moreover, this function $\theta = \theta(T)$ must be strictly positive, and increasing.

(8) Getting back now to efficiencies, we have the following formula:

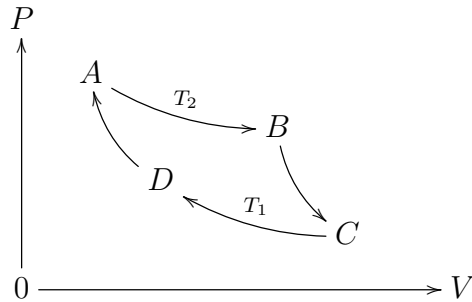
$$\eta(T_1, T_2) = 1 - \frac{1}{f(T_1, T_2)} = 1 - \frac{\theta(T_1)}{\theta(T_2)}$$

The point now is that, for computing θ , we can use this formula, and any reversible engine, of our choice. And by choosing here the Carnot engine operating on $[T_1, T_2]$, and doing the computation, based on the adiabatics for the ideal gases, we obtain:

$$\eta(T_1, T_2) = 1 - \frac{T_1}{T_2}$$

Thus the absolute temperature $\theta = \theta(T) \in (0, \infty)$ constructed in (7) is given by $\theta(T) = \lambda T$, for some $\lambda \in \mathbb{R}$, and we are led to the conclusion in the statement.

(9) To be more precise here, and getting to details, let us first recall from chapter 1 the general functioning scheme of the Carnot engines. This is as follows, with the horizontals AB, CD being the isothermals, and the verticals BC, DA being the adiabatics:



(10) Assuming now that our Carnot engine runs with an ideal gas, and by using the fact, that we know from chapter 1, that the energy of such an ideal gas depends only on the temperature, we have, by looking what happens on the above AB isothermal:

$$Q_2 = L_{AB} = RT_2 \log \frac{V_B}{V_A}$$

Similarly, by looking what happens on the CD isothermal, we have:

$$Q_1 = L_{CD} = RT_1 \log \frac{V_C}{V_D}$$

(11) As a second ingredient now, also as a consequence of the results from chapter 1, since the points A, D lie on an adiabat, we have the following formula:

$$T_1 V_D^{k-1} = T_2 V_A^{k-1}$$

Similarly, since the points B, C lie on an adiabat, we have:

$$T_1 V_C^{k-1} = T_2 V_B^{k-1}$$

Now by taking the quotients of these two formulae, we obtain:

$$\frac{V_B}{V_A} = \frac{V_C}{V_D}$$

(12) But with this in hand, we can go back to the two formulae obtained in (10). Since the logarithms on the right are the same, by dividing these formulae, we obtain:

$$\frac{Q_2}{Q_1} = \frac{T_2}{T_1}$$

Thus, we are led to the conclusions in the statement. \square

So, that was it, and truly amazing, isn't it. Obviously, having $T > 0$ is something of similar impact to Einstein's $v < c$. And, not only this discovery came before Einstein's, but its proof, explained above, was via a sort of enormous Gedankenexperiment, far more technically involved than the trains and clocks needed for special relativity.

2b.

2c.

2d.

2e. Exercises

Exercises:

EXERCISE 2.6.

EXERCISE 2.7.

EXERCISE 2.8.

EXERCISE 2.9.

EXERCISE 2.10.

EXERCISE 2.11.

Bonus exercise.

CHAPTER 3

Scales, resolution

3a.

3b.

3c.

3d.

3e. Exercises

Exercises:

EXERCISE 3.1.

EXERCISE 3.2.

EXERCISE 3.3.

EXERCISE 3.4.

EXERCISE 3.5.

EXERCISE 3.6.

Bonus exercise.

CHAPTER 4

Some cosmology

4a.

4b.

4c.

4d.

4e. Exercises

Exercises:

EXERCISE 4.1.

EXERCISE 4.2.

EXERCISE 4.3.

EXERCISE 4.4.

EXERCISE 4.5.

EXERCISE 4.6.

Bonus exercise.

Part II

Low temperatures

*Four to the floor, I was sure
Never seeing clear
I could have it all
Whenever you are near*

CHAPTER 5

Liquid helium

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

Bonus exercise.

CHAPTER 6

Superconductivity

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

EXERCISE 6.1.

EXERCISE 6.2.

EXERCISE 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

EXERCISE 6.6.

Bonus exercise.

CHAPTER 7

Bose-Einstein

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

EXERCISE 7.1.

EXERCISE 7.2.

EXERCISE 7.3.

EXERCISE 7.4.

EXERCISE 7.5.

EXERCISE 7.6.

Bonus exercise.

CHAPTER 8

Chemistry, tables

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

Bonus exercise.

Part III

Cooling methods

Cause you ride on time
Cause you ride on time
Ride on time
Cause you ride on time

CHAPTER 9

Refrigerators

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

Bonus exercise.

CHAPTER 10

Vacuum pumps

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

EXERCISE 10.1.

EXERCISE 10.2.

EXERCISE 10.3.

EXERCISE 10.4.

EXERCISE 10.5.

EXERCISE 10.6.

Bonus exercise.

CHAPTER 11

Laser cooling

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

Bonus exercise.

CHAPTER 12

Outer space

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

Bonus exercise.

Part IV

Interstellar space

*Heaven holds a sense of wonder
And I wanted to believe
That I'd get caught up
When the rage in me subsides*

CHAPTER 13

Cold planets

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

Bonus exercise.

CHAPTER 14

Interstellar space

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

Bonus exercise.

CHAPTER 15

Maxwell demon

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

EXERCISE 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

Bonus exercise.

CHAPTER 16

Into darkness

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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