Complex reflections and Bessel laws

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"Introduction to matrix groups", 4/6

08/20

Reflection groups 1/3

Definition. The reflection group H_N^s, depending on parameters

$$N\in\mathbb N \quad , \quad s\in\mathbb N\cup\{\infty\}$$

is the group of $N\times N$ matrices with entries in

$$\mathbb Z_s\cup\{0\}$$

having one nonzero entry on each row and each column.

Examples. At $s=1$ we have the symmetric group $S_N\subset O_N$.
At $s=2$ we have the hyperoctahedral group $H_N\subset O_N$.
At $s=3,4,\ldots$ we have a certain finite subgroup $H_N^s\subset U_N$.
At $s=\infty$ we have a certain infinite subgroup $K_N\subset U_N$.

Reflection groups 2/3

Theorem. We have $H_N^s = \mathbb{Z}_s \wr S_N$, wreath product decomposition.

Proof. This basically says that the elements $g \in H_N^s$ appear as permutations $\sigma \in S_N$ "decorated" with signs $\varepsilon \in \mathbb{Z}_s^N$, which is something that we already know, from the matrix picture.

Theorem. The irreducible complex reflection groups are

$$H_N^{sd} = \{U \in H_N^s \,|\, \det U \in \mathbb{Z}_d\}$$

along with 34 exceptional examples.

Proof. This is something complicated, due to Shephard and Todd.

Reflection groups 3/3

Theorem. The groups H_N^s are easy, in the sense that

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

are Brauer type algebras, spanned by diagrams.

Proof. This holds indeed, with $D_{kl} \subset \mathcal{P}(k,l)$ being defined by the condition $\#\circ = \#\bullet\,(s)$, weighted count, in each block.

Problem. What is the law of the main character χ for H_N^s? And, what about the laws of truncated characters χ_t?

Comment. At $s = 1$, where the group is S_N, we have $\chi \sim p_1$, and more generally $\chi_t \sim p_t$, Poisson laws, with $N \to \infty$.

Real reflections 1/4

Definition. The hyperoctahedral group $H_N \subset O_N$ is:

(1) The symmetry group of the unit hypercube $\square_N \subset \mathbb{R}^N$.

(2) The group of symmetries of the N coordinate axes of $\mathbb{R}^N$.

(3) The group of permutation-like matrices with ± 1 entries.

Theory. We have $H_N = \mathbb{Z}_2 \wr S_N$, the reflection subgroups reduce to $SH_N = H_N \cap SO_N$, and we have easiness, with $D = P_{even}$.

Real reflections 2/4

Theorem. The laws of truncated characters for H_N are

$$\mathrm{law}(\chi_t) \simeq e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

for any $t \in (0,1]$, in the $N \to \infty$ limit.

Proof. Inclusion-exclusion principle, exactly as for S_N, but this time with the permutations $\sigma \in S_N$ being decorated by signs $\varepsilon \in \mathbb{Z}_2^N$.

Real reflections 3/4

Remark. The limiting truncated character law for H_N is

$$b_t = e^{-t} \sum_{k \in \mathbb{Z}} \delta_k \, f_k(t/2)$$

where f_k is the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

Due to this fact, we call b_t Bessel law, of parameter t.

Real reflections 4/4

Theorem. The Bessel laws b_t have the semigroup property

$$b_s * b_t = b_{s+t}$$

with respect to the usual convolution of real measures.

Theorem. The Bessel laws are compound Poisson laws,

$$b_t = \mathrm{law}(a - b)$$

with a, b being independent, both following the Poisson law p_t.

Proofs. Similar to the proofs for S_N, using the Fourier transform.

Bessel laws 1/4

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$p_\nu = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{1}{n}\nu\right)^{*n}$$

and the measure p_ν is called compound Poisson law. For

$$\nu = \sum_{i=1}^s t_i \delta_{z_i}$$

with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$p_\nu = \mathrm{law}\left(\sum_{i=1}^s z_i \alpha_i\right)$$

whenever the variables α_i are Poisson (t_i), independent.

Bessel laws 2/4

Definition. The higher Bessel laws are the compound Poisson laws

$$b_t^s = p_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s-roots of unity.

Comments. By the above, this means that we have:

$$b_t^s = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\varepsilon_s\right)^{*n}$$

Equivalently, we have the following formula,

$$b_t^s = \mathrm{law}\left(\sum_{r=1}^{s} w^r\alpha_i\right)$$

where $w = e^{2\pi i/s}$, and where $\alpha_i \sim p_t$, independent.

Bessel laws 3/4

Examples.

(1) At $s = 1$ we obtain the Poisson laws p_t.

(2) At $s = 2$ we obtain the Bessel laws b_t.

(3) At $s = 3, 4, \ldots$ we obtain certain discrete complex measures.

(4) At $s = \infty$ we obtain certain complex measures B_t.

Bessel laws 4/4

Theorem. The Fourier transform of b_t^s is given by:

$$F_{b_t^s}(y) = \exp\left(t\sum_{r=1}^s (e^{iw^r y} - 1)\right)$$

Theorem. The Bessel laws for a convolution semigroup:

$$b_t^s * b_{t'}^s = b_{t+t'}^s$$

Proofs. The first formula is clear from the $b_t^s = \mathrm{law}\left(\sum_{r=1}^s w^r \alpha_i\right)$ interpretation, and the second formula follows from it.

Complex reflections 1/4

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Complex reflections 2/4

Theorem. The laws of truncated characters for H_N^s are

$$\mathrm{law}(\chi_t) \simeq b_t^s$$

for any $t \in (0,1]$, in the $N \to \infty$ limit.

Proof. Inclusion-exclusion principle, exactly as for S_N, but this time with the permutations $\sigma \in S_N$ being decorated by signs $\varepsilon \in \mathbb{Z}_s^N$.

Remark. This extends and unifies all our previous results.

Complex reflections 3/4

In the order to further extend all this, a first idea would be to look at the general series of complex reflection groups:

$$H_N^{sd} = \left\{ U \in H_N^s \middle| \det U \in \mathbb{Z}_d \right\}$$

However, this does not seem to bring new laws, at least at order 0. The study of the fluctuations is an interesting problem.

Complex reflections 4/4

Another type of extension comes by staying with H_N^s, but looking at the fluctuations of the characters

$$g\to Tr(g)$$

or of the truncated characters

$$g\to\sum_{i=1}^{[tN]}g_{ii}\quad,\quad t\in(0,1]$$

or of the Diaconis-Shahshahani variables

$$g\to Tr(g^k)\quad,\quad k\in\mathbb N$$

and so on. Things here are quite well understood at $s=1,2$.