A MAXIMALITY RESULT FOR ORTHOGONAL QUANTUM GROUPS

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Abstract. We prove that the quantum group inclusion $O_n \subset O_n^*$ is "maximal", where O_n is the usual orthogonal group and O_n^* is the half-liberated orthogonal quantum group, in the sense that there is no intermediate compact quantum group $O_n \subset G \subset O_n^*$. In order to prove this result, we use: (1) the isomorphism of projective versions $PO_n^* \simeq PU_n$, (2) some maximality results for classical groups, obtained by using Lie algebras and some matrix tricks, and (3) a short five lemma for cosemisimple Hopf algebras.

Introduction

Quantum groups were introduced by Drinfeld [13] and Jimbo [15] in order to study "non-classical" symmetries of complex systems. This was followed by the fundamental work of Woronowicz [21], [22] on compact quantum groups. The key examples which were constructed by Drinfeld and Jimbo, and further analyzed by Woronowicz, were q-deformations G_q of classical Lie groups G. The idea is as follows: consider the commutative algebra $A = C(G)$. For a suitable choice of generating "coordinates" of this algebra, replace commutativity by the q-commutation relations $ab = qba$, where $q > 0$ is a parameter. In this way one obtains an algebra $A_q = C(G_q)$, where G_q is a quantum group. When $q = 1$ one then recovers the classical group G.

For $G = O_n, U_n, S_n$ it was later discovered by Wang [19], [20] that one can also obtain compact quantum groups by "removing" the commutation relations entirely. In this way one obtains "free" versions O_n^+, U_n^+, S_n^+ of these classical groups. This construction has been axiomatized in [11] in terms of the "easiness" condition for compact quantum groups, and has led to several applications in probability. See [9], [10].

It is clear from the construction that one has $G \subset G^+$ for $G = O_n, U_n, S_n$. Since G^+ can be viewed a "liberation" of G, it is natural to wonder whether there are any intermediate quantum groups $G \subset G' \subset G^+$, which could be seen as "partial liberations" of G. For O_n, S_n this problem has been solved in the case of "easy" intermediate quantum groups [12], [8]. For S_n there are no intermediate easy quantum groups $S_n \subset G' \subset S_n^+$. However for O_n there is exactly one intermediate easy quantum group $O_n \subset O_n^* \subset O_n^+$, called the "half-liberated" orthogonal group, which was constructed in [11]. At the level of relations

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among coordinates, this is constructed by replacing the commutation relations $ab = ba$ with the half-commutation relations $abc = cba$.

In the larger category of compact quantum groups it is an open problem whether there are intermediate quantum groups $S_n \subset G \subset S_n^+$, or $O_n \subset G \subset O_n^+$ with $G \neq O_n^*$. This is an important question for better understanding the "liberation" procedure of [11]. At $n = 4$ (the smallest value at which $S_n \neq S_n^+$), it follows from the results in [5] that the inclusion $S_n \subset S_n^+$ is indeed maximal, and it was conjectured in [6] that this is the case, for any $n \in \mathbb{N}$. Likewise the inclusion $O_n \subset O_n^* \subset O_n^+$ is known to be maximal at $n = 2$, thanks to the results of Podleś in [16]. In general it is likely that these two problems are related to each other via combinatorial invariants [12] or cocycle twists [7].

In this paper we make some progress towards solving this problem in the orthogonal case, by showing that the inclusion $O_n \subset O_n^*$ is maximal. A key tool in our analysis will be the fact the "projective version" of O_n^* is the same as that of the classical unitary group U_n. By using a version of the five lemma for cosemisimple Hopf algebras (following ideas from [1], [3]), we are thus able to reduce the problem to showing that the inclusion of groups $PO_n \subset PU_n$ is maximal. We then solve this problem by using some Lie algebra techniques inspired from [4], [14].

The paper is organized as follows: Section 1 contains background and preliminaries. In Section 2 we prove that $PO_n \subset PU_n$ is maximal. In Section 3 we prove a short five lemma for cosemisimple Hopf algebras, which may be of independent interest. We then use this in Section 4 to prove our main result, namely that $O_n \subset O_n^*$ is maximal.

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1. Orthogonal quantum groups

In this section we briefly recall the free and half-liberated orthogonal quantum groups from [19], [11], and the notion of "projective version" for a unitary compact quantum group. We will work at the level of Hopf $*$-algebras of representative functions.

First we have the following fundamental definition, arising from Woronowicz' work [21].

Definition 1.1. *A unitary Hopf algebra is a $*$-algebra A which is generated by elements $\{u_{ij} | 1 \leq i, j \leq n\}$ such that $u = (u_{ij})$ and $\overline{u} = (u_{ij}^*)$ are unitaries, and such that:*

1. *There is a $*$-algebra map $\Delta : A \to A \otimes A$ such that $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$.*
2. *There is a $*$-algebra map $\varepsilon : A \to \mathbb{C}$ such that $\varepsilon(u_{ij}) = \delta_{ij}$.*
3. *There is a $*$-algebra map $S : A \to A^{op}$ such that $S(u_{ij}) = u_{ji}^*$.*

If $u_{ij} = u_{ij}^*$ *for* $1 \leq i, j \leq n$, *we say that* A *is an orthogonal Hopf algebra.*

It follows that Δ, ε, S satisfy the usual Hopf algebra axioms. The motivating examples of unitary (resp. orthogonal) Hopf algebra is $A = \mathcal{R}(G)$, the algebra of representative function of a compact subgroup $G \subset U_n$ (resp. $G \subset O_n$). Here the standard generators u_{ij} are the coordinate functions which take a matrix to its (i,j)-entry.

In fact every commutative unitary Hopf algebra is of the form $\mathcal{R}(G)$ for some compact group $G \subset U_n$. In general we use the suggestive notation "$A = \mathcal{R}(G)$" for any unitary (resp. orthogonal) Hopf algebra, where G is a *unitary (resp. orthogonal) compact quantum group*. Of course any group-theoretic statements about G must be interpreted in terms of the Hopf algebra A.

It can be shown that shown that a unitary Hopf algebra has an enveloping C^*-algebra, satisfying Woronowicz' axioms in [21]. In general there are several ways to complete a unitary Hopf algebra into a C^*-algebra, but in this paper we will ignore this problem and work at the level of unitary Hopf algebras.

The following examples of Wang [19] are fundamental to our considerations.

Definition 1.2. *The universal unitary Hopf algebra* $A_u(n)$ *is the universal* $*$*-algebra generated by elements* $\{u_{ij} | 1 \leq i, j \leq n\}$ *such that the matrices* $u = (u_{ij})$ *and* $\overline{u} = (u_{ij}^*)$ *in* $M_n(A_u(n))$ *are unitaries.*

The universal orthogonal Hopf algebra $A_o(n)$ *is the universal* $*$*-algebra generated by self-adjoint elements* $\{u_{ij} | 1 \leq i, j \leq n\}$ *such that the matrix* $u = (u_{ij})_{1 \leq i,j \leq n}$ *in* $M_n(A_o(n))$ *is orthogonal.*

The existence of the Hopf algebra structural morphisms follows from the universal properties of $A_u(n)$ and $A_o(n)$. As discussed above, we use the notations $A_u(n) = \mathcal{R}(U_n^+)$ and $A_o(n) = \mathcal{R}(O_n^+)$, where U_n^+ is the *free unitary quantum group* and O_n^+ is the *free orthogonal quantum group*.

Note that we have $\mathcal{R}(O_n^+) \twoheadrightarrow \mathcal{R}(O_n)$, in fact $\mathcal{R}(O_n)$ is the quotient of $\mathcal{R}(O_n^+)$ by the relations that the coordinates u_{ij} commute. At the level of quantum groups, this means that we have an inclusion $O_n \subset O_n^+$.

In other words, $\mathcal{R}(O_n^+)$ is obtained from $\mathcal{R}(O_n)$ by "removing commutativity" among the coordinates u_{ij}. It was discovered in [11] that one can obtain a natural orthogonal quantum group by requiring instead that the coordinates "half-commute".

Definition 1.3. *The half-liberated othogonal Hopf algebra* $A_o^*(n)$ *is the universal* $*$*-algebra generated by self-adjoint elements* $\{u_{ij} | 1 \leq i, j \leq n\}$ *which half-commute in the sense that* $abc = cba$ *for any* $a, b, c \in \{u_{ij}\}$, *and such that the matrix* $u = (u_{ij})_{1 \leq i,j \leq n}$ *in* $M_n(A_o^*(n))$ *is orthogonal.*

The existence of the Hopf algebra structural morphisms again follows from the universal properties of $A_o^*(n)$. We use the notation $A_o^*(n) = \mathcal{R}(O_n^*)$, where O_n^* is the *half-liberated orthogonal quantum group*. Note that we have $\mathcal{R}(O_n^+) \twoheadrightarrow \mathcal{R}(O_n^*) \twoheadrightarrow \mathcal{R}(O_n)$, i.e. $O_n \subset$

$O_n^* \subset O_n^+$. As discussed in the introduction, our aim in this paper is to show that the inclusion $O_n \subset O_n^*$ is maximal. A key tool in our analysis will be the projective version of a unitary quantum group, which we now recall.

Definition 1.4. *The projective version of a unitary compact quantum group $G \subset U_n^+$ is the quantum group $PG \subset U_{n^2}^+$, having as basic coordinates the elements $v_{ij,kl} = u_{ik}u_{jl}^*$.*

In other words, $P\mathcal{R}(G) = \mathcal{R}(PG) \subset \mathcal{R}(G)$ is the subalgebra generated by the elements $v_{ij,kl} = u_{ik}u_{jl}^*$. It is clearly a Hopf $*$-subalgebra of $\mathcal{R}(G)$. In the case where $G \subset U_n$ is classical we recover of course the well-known formula $PG = G/(G \cap \mathbb{T})$, where $\mathbb{T} \subset U_n$ is the group of norm one multiples of the identity.

The following key result was proved in [12].

Theorem 1.5. *We have an isomorphism $PO_n^* \simeq PU_n$.*

Proof. First, thanks to the half-commutation relations between the standard coordinates on O_n^*, for any $a, b, c, d \in \{u_{ij}\}$ we have $abcd = cbad = cdab$. Thus the standard coordinates on the quantum group PO_n^* commute ($ab \cdot cd = cd \cdot ab$), so this quantum group is actually a classical group. A representation theoretic study, based on the diagrammatic results in [11], allows then to show this classical group is actually PU_n. See [12]. $\square$

Note that in fact the techniques developed in the present paper enable us to give a very simple proof of this theorem, avoiding the diagramatic techniques from [11], [12]. See the last remark in Section 4.

2. Classical group results

In this section we prove that the inclusion $PO_n \subset PU_n$ is maximal in the category of compact groups (we assume throughout the paper that $n \geq 2$, otherwise there is nothing to prove). We will see later on, in Sections 3 and 4 below, that this result can be "twisted", in order to reach to the maximality of the inclusion $O_n \subset O_n^*$.

Let $\tilde{O}_n$ be the group generated by O_n and $\mathbb{T} \cdot I_n$ (the group of multiples of identity of norm one). That is, $\tilde{O}_n$ is the preimage of PO_n under the quotient map $U_n \twoheadrightarrow PU_n$. Let $\widetilde{SO}_n \subset \tilde{O}_n$ be the group generated by SO_n and $\mathbb{T} \cdot I_n$. Note that $\tilde{O}_n = \widetilde{SO}_n$ if n is odd, and if n is even then $\tilde{O}_n$ has two connected components and $\widetilde{SO}_n$ is the component containing the identity.

It is a classical fact that a compact matrix group is a Lie group, so $\widetilde{SO}_n$ is a Lie group. Let $\mathfrak{so}_n$ (resp. $\mathfrak{u}_n$) be the real Lie algebras of SO_n (resp. U_n). It is known that $\mathfrak{u}_n$ consists of the matrices $M \in M_n(\mathbb{C})$ satisfying $M^* = -M$, and $\mathfrak{so}_n = \mathfrak{u}_n \cap M_n(\mathbb{R})$. It is easy to see that the Lie algebra of $\widetilde{SO}_n$ is $\mathfrak{so}_n \oplus i\mathbb{R}$.

First we need the following lemma:

Lemma 2.1. *If $n \geq 2$, the adjoint representation of SO_n on the space of real symmetric matrices of trace zero is irreducible.*

Proof. Let $X \in M_n(\mathbb{R})$ be symmetric with trace zero, and let V be the span of $\{UXU^t : U \in SO_n\}$. We must show that V is the space of all real symmetric matrices of trace zero.

First we claim that V contains all diagonal matrices of trace zero. Indeed, since we may diagonalize X by conjugating with an element of SO_n, V contains some non-zero diagonal matrix of trace zero. Now if $D = diag(d_1, d_2, \ldots, d_n)$ is a diagonal matrix in V, then by conjugating D by

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \in SO_n$$

we have that V also contains $diag(d_2, d_1, d_3, \ldots, d_n)$. By a similar argument we see that for any $1 \leq i, j \leq n$ the diagonal matrix obtained from D by interchanging d_i and d_j lies in V. Since S_n is generated by transpositions, it follows that V contains any diagonal matrix obtained by permuting the entries of D. But it is well-known that this representation of S_n on diagonal matrices of trace zero is irreducible, and hence V contains all such diagonal matrices as claimed.

Now if Y is any real symmetric matrix of trace zero, we can find a U in SO_n such that UYU^t is a diagonal matrix of trace zero. But we then have $UYU^t \in V$, and hence also $Y \in V$ as desired. □

Proposition 2.2. *The inclusion $\widetilde{SO}_n \subset U_n$ is maximal in the category of connected compact groups.*

Proof. Let G be a connected compact group satisfying $\widetilde{SO}_n \subset G \subset U_n$. Then G is a Lie group, let $\mathfrak{g}$ denote its Lie algebra, which satisfies $\mathfrak{so}_n \oplus i\mathbb{R} \subset \mathfrak{g} \subset \mathfrak{u}_n$.

Let ad_G be the action of G on $\mathfrak{g}$ obtained by differentiating the adjoint action of G on itself. This action turns $\mathfrak{g}$ into a G-module. Since $SO_n \subset G$, $\mathfrak{g}$ is also an SO_n-module.

Now if $G \neq \widetilde{SO}_n$, then since G is connected we must have $\mathfrak{so}_n \oplus i\mathbb{R} \neq \mathfrak{g}$. It follows from the real vector space structure of the Lie algebras $\mathfrak{u}_n$ and $\mathfrak{so}_n$ that there exists a non-zero symmetric real matrix of trace zero X such that $iX \in \mathfrak{g}$.

But by Lemma 2.1 the space of symmetric real matrices of trace zero is an irreducible representation of SO_n under the adjoint action. So $\mathfrak{g}$ must contain all such X, and hence $\mathfrak{g} = \mathfrak{u}_n$. But since U_n is connected, it follows that $G = U_n$. □

Our aim is to extend this result to the category of compact groups. To do this we need to compute the *normalizer* of $\widetilde{SO}_n$ in U_n, i.e. the subgroup of U_n consisting of unitary U for which $U^{-1}XU \in \widetilde{SO}_n$ for all $X \in \widetilde{SO}_n$. For this we need two lemmas.

Lemma 2.3. *The commutant of SO_n in $M_n(\mathbb{C})$, denoted SO_n', is as follows:*

(1) $SO_2' = \{\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\ \alpha, \beta \in \mathbb{C}\}$.

(2) *If* $n \geq 3$, $SO_n' = \{\alpha I_n, \alpha \in \mathbb{C}\}$.

Proof. At $n = 2$ this is a direct computation. For $n \geq 3$, an element in $X \in SO'_n$ commutes with any diagonal matrix having exactly $n-2$ entries equal to 1 and two entries equal to -1. Hence X is a diagonal matrix. Now since X commutes with any even permutation matrix and $n \geq 3$, it commutes in particular with the permutation matrix associated with the cycle (i, j, k) for any $1 < i < j < k$, and hence all the entries of X are the same: we conclude that X is a scalar matrix. □

Lemma 2.4. *The set of matrices with non-zero trace is dense in SO_n.*

Proof. At $n = 2$ this is clear since the set of elements in SO_2 having a given trace is finite. Assume that $n > 2$ and let $T \in SO_n \simeq SO(\mathbb{R}^n)$ with $Tr(T) = 0$. Let $E \subset \mathbb{R}^n$ be a 2-dimensional subspace preserved by T and such that $T_{|E} \in SO(E)$. Let $\epsilon > 0$ and let $S_\epsilon \in SO(E)$ with $||T_{|E} - S_\epsilon|| < \epsilon$ and $Tr(T_{|E}) \neq Tr(S_\epsilon)$ ($n = 2$ case). Now define $T_\epsilon \in SO(\mathbb{R}^n) = SO_n$ by $T_{\varepsilon|E} = S_\epsilon$ and $T_{\epsilon|E^\perp} = T_{|E^\perp}$. It is clear that $||T - T_\epsilon|| \leq ||T_{|E} - S_\epsilon|| < \epsilon$ and that $Tr(T_\epsilon) = Tr(S_\epsilon) + Tr(T_{|E^\perp}) \neq 0$. □

Proposition 2.5. *$\tilde{O}_n$ is the normalizer of $\widetilde{SO}_n$ in U_n.*

Proof. It is clear that $\tilde{O}_n$ normalizes $\widetilde{SO}_n$, so we must show that if $U \in U_n$ normalizes $\widetilde{SO}_n$ then $U \in \tilde{O}_n$. First note that U normalizes SO_n. Indeed if $X \in SO_n$ then $U^{-1}XU \in \widetilde{SO}_n$, so $U^{-1}XU = \lambda Y$ for $\lambda \in \mathbb{T}$ and $Y \in SO_n$. If $Tr(X) \neq 0$, we have $\lambda \in \mathbb{R}$ and hence $\lambda Y = U^{-1}XU \in SO_n$. The set of matrices having non-zero trace is dense in SO_n by Lemma 2.4, so since SO_n is closed and the matrix operations are continous, we conclude that $U^{-1}XU \in SO_n$ for all $X \in SO_n$.

Thus for any $X \in SO_n$, we have $(UXU^{-1})^t(UXU^{-1}) = I_n$ and hence $X^tU^tUX = U^tU$. This means that $U^tU \in SO'_n$. Hence if $n \geq 3$, we have $U^tU = \alpha I_n$ by Lemma 2.3, with $\alpha \in \mathbb{T}$ since U is unitary. Hence we have $U = \alpha^{1/2}(\alpha^{-1/2}U)$ with $\alpha^{-1/2}U \in O_n$, and $U \in \widetilde{O_n}$. If $n = 2$, Lemma 2.3 combined with the fact that $(U^tU)^t = U^tU$ gives again that $U^tU = \alpha I_2$, and we conclude as in the previous case. □

We can now extend Proposition 2.2 as follows.

Proposition 2.6. *The inclusion $\tilde{O}_n \subset U_n$ is maximal in the category of compact groups.*

Proof. Suppose that $\tilde{O}_n \subset G \subset U_n$ is a compact group such that $G \neq U_n$. It is a well known fact that the connected component of the identity in G is a normal subgroup, denoted G_0. Since we have $\widetilde{SO}_n \subset G_0 \subset U_n$, by Proposition 2.2 we must have $G_0 = \widetilde{SO}_n$. But since G_0 is normal in G, G normalizes $\widetilde{SO}_n$ and hence $G \subset \tilde{O}_n$ by Proposition 2.5. □

We are now ready to state and prove the main result in this section.

Theorem 2.7. *The inclusion $PO_n \subset PU_n$ is maximal in the category of compact groups.*

Proof. It follows directly from the observation that the maximality of $\tilde{O}_n$ in U_n implies the maximality of PO_n in PU_n. Indeed, if $PO_n \subset G \subset PU_n$ were an intermediate subgroup, then its preimage under the quotient map $U_n \twoheadrightarrow PU_n$ would be an intermediate subgroup of $\tilde{O}_n \subset U_n$, contradicting Proposition 2.6. □

3. A short five lemma

In this section we prove a short five lemma for cosemisimple Hopf algebras (Theorem 3.4 below), which is a result having its own interest, to be used in Section 4 below.

Definition 3.1. *A sequence of Hopf algebra maps*

$$\mathbb{C} \to B \xrightarrow{i} A \xrightarrow{p} L \to \mathbb{C}$$

is called pre-exact if i is injective, p is surjective and $i(B) = A^{cop}$, where:

$$A^{cop} = \{a \in A | (id \otimes p)\Delta(a) = a \otimes 1\}$$

The example that we are interested in is as follows.

Proposition 3.2. *Let A be an orthogonal Hopf algebra with generators u_{ij}. Assume that we have surjective Hopf algebra map $p : A \to \mathbb{C}\mathbb{Z}_2$, $u_{ij} \to \delta_{ij} g$, where $< g >= \mathbb{Z}_2$. Let PA be the projective version of A, i.e. the subalgebra generated by the elements $u_{ij}u_{kl}$ with the inclusion $i : PA \subset A$. Then the sequence*

$$\mathbb{C} \to PA \xrightarrow{i} A \xrightarrow{p} \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$

is pre-exact.

Proof. We have:

$$(id \otimes p)\Delta(u_{i_1j_1} \ldots u_{i_mj_m}) = \begin{cases} u_{i_1j_1} \ldots u_{i_mj_m} \otimes 1 & \text{if } m \text{ is even} \\ u_{i_1j_1} \ldots u_{i_mj_m} \otimes g & \text{if } m \text{ is odd} \end{cases}$$

Thus A^{cop} is the span of monomials of even length, which is clearly PA. □

A pre-exact sequence as in Definition 3.1 is said to be exact [2] if in addition we have $i(B)^+A = \ker(\pi) = Ai(B)^+$, where $i(B)^+ = i(B) \cap \ker(\varepsilon)$. The pre-exact sequence in Proposition 3.2 is actually exact, but we only need its pre-exactness in what follows.

In order to prove the short five lemma, we use the following well-known result. We give a proof for the sake of completeness.

Lemma 3.3. *Let $\theta : A \to A'$ be a Hopf algebra morphism with A, A' cosemisimple and let $h_A, h_{A'}$ be the respective Haar integrals of A, A'. Then θ is injective iff $h_{A'}\theta = h_A$.*

Proof. For $a \in A$, we have:

$$\theta(h_{A'}(\theta(a_1))a_2) = h_{A'}(\theta(a)_1)\theta(a)_2 = \theta(h_{A'}\theta(a)1)$$

Thus if θ is injective then $h_{A'}\theta$ is a Haar integral on A, and the result follows from the uniqueness of the Haar integral.

Conversely, assume that $h_A = h_{A'}\theta$. Then for all $a, b \in A$, we have $h_A(ab) = h_{A'}(\theta(a)\theta(b))$, so if $\theta(a) = 0$, we have $h_A(ab) = 0$ for all $b \in A$. It follows from the orthogonality relations that $a = 0$, and hence θ is injective. $\square$

Theorem 3.4. *Consider a commutative diagram of cosemisimple Hopf algebras*

$$\begin{tikzcd} k \arrow[r] & B \arrow[r, "i"] \arrow[d, equal] & A \arrow[r, "\pi"] \arrow[d, "\theta"] & L \arrow[r] \arrow[d, equal] & k \\ k \arrow[r] & B \arrow[r, "i'"] & A' \arrow[r, "\pi'"] & L \arrow[r] & k \end{tikzcd}$$

where the rows are pre-exact. Then θ is injective.

Proof. We have to show that $h_A = h_{A'}\theta$, where $h_A, h_{A'}$ are the respective Haar integrals of A, A'. Let Λ be the set of isomorphism classes of simple L-comodules and consider the Peter-Weyl decomposition of L:

$$L = \bigoplus_{\lambda\in\Lambda} L(\lambda)$$

We view A as a right L-comodule via $(id \otimes \pi)\Delta$. Then A has a decomposition into isotypic components as follows, where $A_\lambda = \{a \in A \mid (id \otimes \pi) \circ \Delta(a) \in A \otimes L(\lambda)\}$:

$$A = \bigoplus_{\lambda\in\Lambda} A_\lambda$$

It is clear that $A_1 = A^{co\pi}$. Then if $\lambda \neq 1$, we have $h_A(A_\lambda) = 0$. Indeed for $a \in A_\lambda$, we have:

$$a_1 \otimes \pi(a_2) \in A \otimes L(\lambda) \implies h_A(a)1 = \pi(h_A(a_1)a_2) \in L(\lambda) \implies h_A(a) = 0$$

Since $\pi'\theta = \pi$, it is easy to see that $\theta(A_\lambda) \subset A'_\lambda$ and hence for $\lambda \neq 1$, $h_{A'|A'_\lambda} = h_{A'}\theta_{|A_\lambda} = 0 = h_{A|A_\lambda}$. For $\lambda = 1$, we have $i(B) = A_1$ and θ is injective on $i(B)$ since $\theta i = i'$. Hence by Lemma 3.3 we have $h_{A'}\theta_{|A_1} = h_{A_1} = h_{A|A_1}$. Since $A = \oplus_{\lambda\in\Lambda} A_\lambda$ we conclude $h_A = h_{A'}\theta$ and by Lemma 3.3 we get that θ is injective. $\square$

It follows from discussions with Alexandru Chirvasitu that the theorem can be improved by showing that θ is an isomorphism. Indeed, since L is assumed to be cosemisimple, A is automatically faithfully coflat as a left L-comodule, and hence by Theorem 1.4 in [18] $i : B \to A$ and $i' : B \to A'$ are L-Galois extensions with A and A' faitfully flat as left B-modules. Since $\theta : A \to A'$ is an L-colinear algebra map, it follows from Remark 3.11 in [17] that θ is an isomorphism.

4. The main result

We have now all the ingredients for stating and proving our main result in this paper.

Theorem 4.1. *The inclusion $O_n \subset O_n^*$ is maximal in the category of compact quantum groups.*

Proof. Consider a sequence of surjective Hopf $*$-algebra maps as follows, whose composition is the canonical surjection:

$$A_o^*(n) \xrightarrow{f} A \xrightarrow{g} \mathcal{R}(O_n)$$

By Proposition 3.2 we get a commutative diagram of Hopf algebra maps with pre-exact rows:

$$\begin{array}{ccccccccc}
\mathbb{C} & \longrightarrow & PA_o^*(n) & \xrightarrow{i_1} & A_o^*(n) & \xrightarrow{p_1} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C} \\
 & & \downarrow{\scriptstyle f_|} & & \downarrow{\scriptstyle f} & & \| & & \\
\mathbb{C} & \longrightarrow & PA & \xrightarrow{i_2} & A & \xrightarrow{p_2} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C} \\
 & & \downarrow{\scriptstyle g_|} & & \downarrow{\scriptstyle g} & & \| & & \\
\mathbb{C} & \longrightarrow & P\mathcal{R}(O_n) & \xrightarrow{i_3} & \mathcal{R}(O_n) & \xrightarrow{p_3} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C}
\end{array}$$

Consider now the following composition, with the isomorphism on the left coming from Theorem 1.5:

$$\mathcal{R}(PU_n) \simeq PA_o^*(n) \xrightarrow{f_|} PA \xrightarrow{g_|} P\mathcal{R}(O_n) \simeq \mathcal{R}(PO_n)$$

This induces, at the group level, the embedding $PO_n \subset PU_n$. By Theorem 2.7 $f_|$ or $g_|$ is an isomorphism. If $f_|$ is an isomorphism we get a commutative diagram of Hopf algebra morphisms with pre-exact rows:

$$\begin{array}{ccccccccc}
\mathbb{C} & \longrightarrow & PA_o^*(n) & \xrightarrow{i_1} & A_o^*(n) & \xrightarrow{p_1} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C} \\
 & & \| & & \downarrow{\scriptstyle f} & & \| & & \\
\mathbb{C} & \longrightarrow & PA_o^*(n) & \xrightarrow{i_2 \circ f_|} & A & \xrightarrow{p_2} & \mathbb{C}\mathbb{Z}_2 & \longrightarrow & \mathbb{C}
\end{array}$$

Then f is an isomorphism by Theorem 3.4. Similarly if $g_|$ is an isomorphism, then g is an isomorphism. $\square$

Observe that the technique in the proof of Theorem 4.1 also enables us to prove that $PO_n^* \simeq PU_n$ independently from [12]. Indeed, since $PA_o^*(n)$ is commutative, there exists a compact group G with $PA_o^*(n) \simeq \mathcal{R}(G)$ and $PO_n \subset G \subset PU_n$. Then Theorem 2.7 gives $G = PO_n$ or $G = PU_n$. If $G = PO_n$, then as in the proof of Theorem 4.1, Theorem 3.4 gives that $A_o^*(n) \twoheadrightarrow \mathcal{R}(O_n)$ is an isomorphism, which is false since $A_o^*(n)$ is a not commutative if $n \geq 2$. Hence $G = PU_n$.

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