Symmetric groups and Poisson laws

Teo Banica

"Introduction to matrix groups", 3/6

08/20

Characters 1/3

Definition. A representation of a finite group G is a morphism

$$\pi : G \to U_N$$

and the character of this representation is the map

$$\chi : G \to \mathbb{C}$$

obtained by taking the trace of the images of group elements:

$$\chi(g) = Tr(\pi(g))$$

When G comes as $G \subset_\pi U_N$, we call χ the "main character".

Characters 2/3

Remark. The characters are central functions on the group, in the sense that they satisfy the following condition:

$$\chi(gh) = \chi(hg)$$

We will see later that any central function on the group is a linear combination of characters. This is something non-trivial.

Remark. We can talk, more generally, about representations and characters of compact groups, with the representations

$$\pi : G \to U_N$$

being assumed to be continuous. We will do this later on.

Characters 3/3

Problem. Given $\pi : G \to U_N$, we want to compute the law of:

$$\chi = Tr \circ \pi : G \to \mathbb{C}$$

That is, we would like to compute the following probabilities,

$$P(\chi = k) \in [0, 1] \quad , \quad k \in \mathbb{C}$$

and then the complex discrete measure encoding them:

$$\mu = \sum_{k \in \mathbb{C}} P(\chi = k)\delta_k$$

There are many motivations for this question. Details later.

Fixed points 1/4

<u>Theorem</u>. For the symmetric group S_N, regarded as subgroup

$$S_N \subset O_N$$

permuting the coordinate axes of $\mathbb{R}^N$, the main character is

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

and its law is a discrete probability measure, supported by $\mathbb{N}$.

<u>Proof</u>. Each $\sigma \in S_N \subset O_N$ is a 0-1 matrix, whose trace $Tr(\chi)$ counts the 1 diagonal entries, corresponding to fixed points.

Fixed points 2/4

Theorem. The probability for a permutation $\sigma \in S_N$ to be a derangement is, in the $N \to \infty$ limit:

$$P_0 \simeq \frac{1}{e}$$

Proof. We must be outside the union $F = \bigcup_i F_i$, where:

$$F_i = \left\{ \sigma \in S_N \middle| \sigma(i) = i \right\}$$

The inclusion-exclusion principle gives:

$$F^c = N! - \sum_i |F_i| + \sum_{i<j} |F_i \cap F_j| - \sum_{i<j<k} |F_i \cap F_j \cap F_k| + \ldots$$

We obtain $P_0 = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots \simeq \frac{1}{e}$, as claimed.

Fixed points 3/4

Theorem. The probability for a permutation $\sigma \in S_N$ to have exactly $k \in \mathbb{N}$ fixed points is

$$P_k \simeq \frac{1}{e} \cdot \frac{1}{k!}$$

once again in the $N \to \infty$ limit.

Proof. We already know that the result holds at $k = 0$. In general the proof is similar, by using the inclusion-exclusion principle.

Fixed points 4/4

Theorem. The character of the standard representation

$$S_N \subset O_N$$

obtained by permuting the coordinate axes of $\mathbb{R}^N$ is given by

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

and follows with $N \to \infty$ the following law:

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

Proof. This follows by putting together the above results.

Poisson laws 1/4

Definition. The Poisson law of parameter 1 is:

$$p_1 = \frac{1}{e}\sum_k \frac{\delta_k}{k!}$$

More generally, the Poisson law of parameter $t > 0$ is:

$$p_t = e^{-t}\sum_k \frac{t^k}{k!}\,\delta_k$$

Remark. These laws have indeed mass 1.

Poisson laws 2/4

Theorem. We have the following formula, for any $s, t > 0$:

$$p_s * p_t = p_{s+t}$$

Proof. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula:

$$\begin{aligned} p_s * p_t &= e^{-s} \sum_k \frac{s^k}{k!} \delta_k * e^{-t} \sum_l \frac{t^l}{l!} \delta_l \\ &= e^{-s-t} \sum_n \delta_n \sum_{k+l=n} \frac{s^k t^l}{k! l!} \\ &= e^{-s-t} \sum_n \frac{(s+t)^n}{n!} \delta_n \end{aligned}$$

Thus, we obtain the Poisson law p_{s+t}, as claimed.

Poisson laws 3/4

Theorem. The Fourier transform of p_t is given by:

$$F_{p_t}(x) = \exp\left((e^{ix} - 1)t\right)$$

Proof. By using $F_f(x) = \mathbb{E}(e^{ixf})$, we obtain:

$$\begin{aligned} F_{p_t}(x) &= e^{-t} \sum_k \frac{t^k}{k!} e^{ikx} \\ &= e^{-t} \sum_k \frac{(e^{ix} t)^k}{k!} \\ &= \exp(-t) \exp(e^{ix} t) \end{aligned}$$

Thus, we obtain the formula in the statement.

Poisson laws 4/4

Theorem. We have the following convergence, in moments:

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

Proof. We have the following computation:

$$\begin{aligned}
F_{\delta_t}(x)=e^{itx} \quad&\implies\quad F_{\mu_n}(x)=\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\\
&\implies\quad F_{\mu_n^{*n}}(x)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n\\
&\implies\quad F_{\mu_n^{*n}}(x)=\left(1+\frac{(e^{ix}-1)t}{n}\right)^n\\
&\implies\quad F(x)=\exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Thus, we obtain the Fourier transform of p_t.

Truncation 1/4

Problem. We know that for $S_N \subset O_N$ with $N \to \infty$, the main character follows the Poisson law p_1.

What about the general Poisson law p_t, of parameter $t > 0$? Can we obtain this law in the representation theory context?

Truncation 2/4

<u>Definition</u>. Given a group representation $\pi : G \to U_N$, its truncated character with respect to a parameter $t \in (0,1]$,

$$\chi_t : G \to \mathbb{C}$$

is the map given by the following formula:

$$\chi_t(g) = \sum_{i=1}^{[tN]} \pi(g)_{ii}$$

When G comes as a group of matrices, $G \subset_\pi U_N$, we call this map χ_t the "main truncated character" of the group.

Truncation 3/4

Theorem. The main truncated character of the symmetric group

$$S_N \subset O_N$$

which permutes the coordinate axes of $\mathbb{R}^N$, is given by

$$\chi_t(\sigma) = \#\left\{i \in \{1, \ldots, [tN]\}\middle|\sigma(i) = i\right\}$$

and follows with $N \to \infty$ the Poisson law of parameter t,

$$p_t = e^{-t} \sum_k \frac{t^k}{k!}\,\delta_k$$

for any value of the parameter $t \in (0, 1]$.

Truncation 4/4

Proof. We already know that the formula holds at $t = 1$. The same method, inclusion-exclusion, gives, more generally:

$$\lim_{N \to \infty} P(\chi = k) = \frac{1}{e^t} \cdot \frac{t^k}{k!}$$

Thus, we obtain with $N \to \infty$ the Poisson law p_t, as claimed.

Comment. We will see later extensions and interpretations of all this, in the advanced representation theory context.