

MODELLING QUESTIONS FOR QUANTUM PERMUTATIONS

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ABSTRACT. Given a quantum permutation group $G \subset S_N^+$, with orbits having the same size K , we construct a universal matrix model $\pi : C(G) \rightarrow M_K(C(X))$, having the property that the images of the standard coordinates $u_{ij} \in C(G)$ are projections of rank ≤ 1 . Our conjecture is that this model is inner faithful under suitable algebraic assumptions, and is in addition stationary under suitable analytic assumptions. We fully discuss this conjecture for classical groups and prove it for several families of duals of discrete groups.

INTRODUCTION

The present paper is motivated by some questions in quantum algebra. Wang introduced in [18] the free analogue S_N^+ of the quantum permutation group S_N . While many things are known about S_N^+ , the analytic structure of the algebra $C(S_N^+)$ is still subject to investigation. One key open problem, slightly stronger than the Connes embedding property, is whether $C(S_N^+)$ has an inner faithful matrix model. See [5], [8], [10].

A Grassmannian model approach to this question was proposed in [6]. The idea is that the magic bases of \mathbb{C}^N form a real algebraic manifold X_N , and the problem is whether the corresponding representation $\pi_N : C(S_N^+) \rightarrow M_N(C(X_N))$ is inner faithful or not. In order to solve this question, some methods are available from [5], [19], but their application would require the construction of a measure on X_N . An idea here, proposed in [6], is that of using the push-forward of the Haar measure on U_N^N , via a Sinkhorn type algorithm [15]. But the convergence of the algorithm is not established yet.

A perhaps simpler question, with many degrees of freedom, is that of looking first at the various quantum subgroups $G \subset S_N^+$. As explained in [1], the matrix model construction is available in this setting, with the model space $X_G \subset X_N$ being obtained by imposing the abstract Tannakian conditions which define G . However, in the non-transitive case the model space collapses to the null space. We will fix here this issue with a new construction, the idea being to allow 0 entries in our magic basis when the orbits of G are non-trivial. To be more precise, we will assume that G is quasi-transitive, in the sense that its orbits have the same size $K \in \mathbb{N}$, with $K|N$, and we will construct a universal matrix model $\pi : C(G) \rightarrow M_K(C(X))$, having the property that the images of the standard coordinates $u_{ij} \in C(G)$ are projections of rank ≤ 1 .

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One important source of examples when trying to understand properties of compact quantum groups are duals of discrete groups. This is where our construction is interesting. Indeed, the only transitive group duals are cyclic groups, while there are plenty of quasi-transitive examples coming from free products of cyclic group. We can therefore do computations and give explicit examples of inner faithful matrix models in this enlarged setting.

Our conjecture is that the quasi-flat model is inner faithful under suitable uniformity assumptions on G , and is in addition stationary under suitable analytic assumptions on G . We will discuss this conjecture for the classical groups $G \subset S_N$, and we will investigate it as well for the group duals $\widehat{\Gamma} \subset S_N^+$. The general case, including that of $G = S_N^+$ itself, and of other transitive subgroups $G \subset S_N^+$, remains an open problem.

The paper is organized as follows: 1-2 contain preliminaries on quasi-transitive quantum groups, in 3-4 we construct the universal models and we formulate the conjectures, in 5-6 we perform some basic work on these conjectures, in the classical group and in the group dual cases, and in 7-8 we discuss in detail the group dual case.

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1. QUANTUM PERMUTATIONS

We are interested in the quantum analogues of the permutation groups $G \subset S_N$ and we will therefore use the setting of compact quantum groups from [21]. We denote by \otimes the minimal tensor product of C^* -algebras and, for a quantum group G , we write $C(G)$ for its maximal C^* -algebra. In order to introduce quantum permutations, let us recall that a magic unitary is a square matrix over a C^* -algebra, $u \in M_N(A)$, whose entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column. The following key definition is due to Wang [18]:

Definition 1.1. $C(S_N^+)$ is the universal C^* -algebra generated by the entries of a $N \times N$ magic unitary matrix $u = (u_{ij})$, with the morphisms given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

as comultiplication, counit and antipode.

This algebra satisfies Woronowicz' axioms in [21] and the underlying space S_N^+ is therefore a compact quantum group, called quantum permutation group. Observe that any magic unitary $v \in M_N(A)$ produces a representation $\pi : C(S_N^+) \rightarrow A$, given by $\pi(u_{ij}) = v_{ij}$. In particular, we have a representation as follows:

$$\pi : C(S_N^+) \rightarrow C(S_N) \quad : \quad u_{ij} \rightarrow \chi(\sigma \in S_N \mid \sigma(j) = i)$$

The corresponding embedding $S_N \subset S_N^+$ is an isomorphism at $N = 2, 3$, but not at $N \geq 4$, where S_N^+ is infinite. Moreover, it is known that we have $S_4^+ \simeq SO_3^{-1}$, and that any S_N^+ with $N \geq 4$ has the same fusion semiring as SO_3 . See [4], [18].

Let us say that H is a quantum subgroup of G and write $H \subset G$ if there is a surjective $*$ -homomorphism $C(G) \rightarrow C(H)$ intertwining the coproducts. Quantum permutations are given by quantum subgroups of S_N^+ so that they act on $\{1, \dots, N\}$. The orbit decomposition theory for these quantum groups was developed in [7] but we present here an alternative approach, based on the following simple fact:

Proposition 1.2. *Given a quantum group $G \subset S_N^+$, with standard coordinates denoted $u_{ij} \in C(G)$, the following defines an equivalence relation on $\{1, \dots, N\}$:*

$$i \sim j \text{ when } u_{ij} \neq 0$$

In the classical case, $G \subset S_N$, this is the orbit equivalence coming from the action of G .

Proof. We first check the fact that we have indeed an equivalence relation:

- (1) $i \sim i$ follows from $\varepsilon(u_{ij}) = \delta_{ij}$, which gives $\varepsilon(u_{ii}) = 1$, and so $u_{ii} \neq 0$, for any i .
- (2) $i \sim j \implies j \sim i$ follows from $S(u_{ij}) = u_{ji}$, which gives $u_{ij} \neq 0 \implies u_{ji} \neq 0$.
- (3) $i \sim j, j \sim k \implies i \sim k$ follows from $\Delta(u_{ik}) = \sum_j u_{ij} \otimes u_{jk}$. Indeed, in this formula, the right-hand side is a sum of projections, so assuming $u_{ij} \neq 0, u_{jk} \neq 0$ for a certain index j , we have $u_{ij} \otimes u_{jk} > 0$, and so $\Delta(u_{ik}) > 0$, which gives $u_{ik} \neq 0$, as desired.

In the classical case now, $G \subset S_N$, the standard coordinates are the characteristic functions $u_{ij} = \chi(\sigma \in G | \sigma(j) = i)$. Thus the condition $u_{ij} \neq 0$ is equivalent to the existence of an element $\sigma \in G$ such that $\sigma(j) = i$, and this means precisely that i and j must be in the same orbit under the action of G , as claimed. \square

Summarizing, we have a quantum analogue of the orbit decomposition from the classical case. We will always assume from now on that the orbits of $G \subset S_N^+$ come in increasing order, in the sense that the equivalence classes of \sim are

$$\pi = \{1, \dots, K_1\}, \dots, \{K_1 + \dots + K_{M-1} + 1, \dots, K_1 + \dots + K_M\}$$

Indeed, at least for the questions that we are interested in here, we can always assume that it is so, simply by conjugating everything by a suitable permutation $\sigma \in S_N$. In analogy with the classical case, we have as well the following notion:

Definition 1.3. *We call $G \subset S_N^+$ transitive when $u_{ij} \neq 0$ for any i, j , i.e. there is only one orbit.*

Let us discuss now the quantum analogue of the fact that given a subgroup $G \subset S_N$, with orbits of lengths K_1, \dots, K_M , we have an inclusion as follows:

$$G \subset S_{K_1} \times \dots \times S_{K_M}$$

Given two quantum permutation groups $G \subset S_K^+, H \subset S_L^+$ with magic corepresentations denoted by u and v , we can consider the free product C^* -algebra $A = C(G) * C(H)$,

together with the magic matrix $w = \text{diag}(u, v)$. The pair (A, w) satisfies Woronowicz's axioms, and we therefore obtain a quantum permutation group denoted by $G \hat{*} H \subset S_{K+L}^+$, see [17]. With this notion in hand, we have the following result:

Proposition 1.4. *Given a quantum group $G \subset S_N^+$, with orbits of size K_1, \dots, K_M , we have an inclusion*

$$G \subset S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+$$

In the classical case, $G \subset S_N$, we obtain in this way the usual inclusion $G \subset S_{K_1} \times \dots \times S_{K_M}$.

Proof. Since the standard coordinates $u_{ij} \in C(G)$ satisfy $u_{ij} = 0$ for $i \not\sim j$, the algebra $C(G)$ appears as quotient of the following algebra:

$$C(S_N^+) / \langle u_{ij} = 0, \forall i \not\sim j \rangle = C(S_{K_1}^+) * \dots * C(S_{K_M}^+) = C(S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+)$$

Thus, we have an inclusion of quantum groups, as in the statement. Moreover, the classical version of the quantum group $S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+$ is

$$(S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+)_{\text{class}} = S_{K_1} \times \dots \times S_{K_M}$$

Thus in the classical case we obtain $G \subset S_{K_1} \times \dots \times S_{K_M}$, as claimed. \square

Let us discuss now what happens in the group dual case, where the situation is non-trivial. Following the work of Bichon in [7], we have the following result:

Proposition 1.5. *Given a decomposition $N = K_1 + \dots + K_M$, and a quotient group $\mathbb{Z}_{K_1} * \dots * \mathbb{Z}_{K_M} \rightarrow \Gamma$, we have an embedding*

$$\widehat{\Gamma} \subset \mathbb{Z}_{K_1} \hat{*} \dots \hat{*} \mathbb{Z}_{K_M} \subset S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+ \subset S_N^+$$

Moreover, modulo the action of $S_N \times S_N$ on the magic unitaries, obtained by permuting the rows and columns, we obtain in this way all the group dual subgroups $\widehat{\Gamma} \subset S_N^+$.

For more on the group duals $\widehat{\Gamma} \subset S_N^+$, we refer to [7]. We will back later on to these quantum groups, under the extra assumption $K_1 = \dots = K_M$.

2. QUASI-TRANSITIVITY

We discuss now an extension of the notion of transitivity, that we call quasi-transitivity. We will see later on that the universal flat matrix model construction from [1], [6], which works well in the transitive case, adapts to the quasi-transitive case.

Definition 2.1. *A quantum permutation group $G \subset S_N^+$ is called quasi-transitive when all its orbits have the same size, i.e. \sim has equivalence classes of same size.*

Of course, if G is transitive then it is quasi-transitive. In general, if we denote by $K \in \mathbb{N}$ the common size of the blocks, and by $M \in \mathbb{N}$ their multiplicity, then we have $N = KM$. Moreover, it follows from Proposition 1.4 above that

$$G \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$$

Let us now discuss examples. Assume that $G \subset S_K^+ \hat{*} \dots \hat{*} S_K^+$. If u, v are the fundamental corepresentations of $C(S_N^+), C(S_K^+)$, consider the quotient map $\pi_i : C(S_N^+) \rightarrow C(S_K^+)$ constructed as follows:

$$u \mapsto \text{diag}(1_K, \dots, 1_K, \underbrace{v}_{i\text{-th term}}, 1_K, \dots, 1_K)$$

and set $C(G_i) = \pi_i(C(G))$. This should be thought of as the intersection of G with the i -th component. Assume that G_i is transitive for all i . Because

$$G_1 \times \dots \times G_M \subset G \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}},$$

the size of any orbit of G is at least K (it contains $G_1 \times \dots \times G_M$) and at most K (it is contained in $S_K^+ \hat{*} \dots \hat{*} S_K^+$). Thus, G is quasi-transitive.

We call the quasi-transitive subgroups appearing as above “of product type”. Observe that there are quasi-transitive groups which are not of product type, as for instance the group $G = S_2 \subset S_2 \times S_2 \subset S_4$ obtained by using the embedding $\sigma \rightarrow (\sigma, \sigma)$. Indeed, the quasi-transitivity is clear, say by letting G act on the vertices of a square. On the other hand, since we have $G_1 = G_2 = \{1\}$, this group is not of product type.

In general, we can construct examples by using various product operations. Besides the usual product \times and the dual free product $\hat{*}$, we can use for instance:

Definition 2.2. *The half-classical product operation \circ for algebras is given by*

$$A_1 \circ \dots \circ A_M = A_1 * \dots * A_M / \left\langle abc = cba \mid \forall a \in A_i, b \in A_j, c \in A_k, i, j, k \text{ distinct} \right\rangle$$

and we denote by $\hat{\circ}$ the induced operation, at the quantum permutation group level.

In other words, given quantum permutation groups $G_i \subset S_{K_i}^+$, we can construct a quantum permutation group $G_1 \hat{\circ} \dots \hat{\circ} G_M \subset S_{K_1}^+ \hat{*} \dots \hat{*} S_{K_M}^+$ as follows:

$$C(G_1 \hat{\circ} \dots \hat{\circ} G_M) = C(G_1) \circ \dots \circ C(G_M)$$

The fact that we indeed obtain a quantum group comes from the fact that the half-commutation relations $abc = cba$ are of “easy” type, see e.g. [14]. With this notion in hand, we have the following examples given transitive subgroups $G_1, \dots, G_M \subset S_K^+$:

- The direct product $G = G_1 \times \dots \times G_M$.
- The half-classical product $G = G_1 \hat{\circ} \dots \hat{\circ} G_M$.
- The dual free product $G = G_1 \hat{*} \dots \hat{*} G_M$.

In the group dual case, we have the following result:

Proposition 2.3. *The group duals $\widehat{\Gamma} \subset \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$ which are of product type are precisely those appearing from intermediate groups of the following type:*

$$\underbrace{\mathbb{Z}_K * \dots * \mathbb{Z}_K}_{M \text{ terms}} \rightarrow \Gamma \rightarrow \underbrace{\mathbb{Z}_K \times \dots \times \mathbb{Z}_K}_{M \text{ terms}}$$

Proof. It is clear that any intermediate quotient Γ as in the statement produces a quantum permutation group $\widehat{\Gamma} \subset S_N^+$ which is of product type. Conversely, given a group dual $\widehat{\Gamma} \subset S_N^+$ coming from a quotient group $\mathbb{Z}_K^{*M} \rightarrow \Gamma$, the subgroups $G_i \subset \widehat{\Gamma}$ constructed above must be group duals as well, $G_i = \widehat{\Gamma}_i$, for certain quotient groups $\Gamma \rightarrow \Gamma_i$. Now if $\widehat{\Gamma}$ is of product type, $\widehat{\Gamma}_i \subset S_K^+$ must be transitive hence equal to $\widehat{\mathbb{Z}}_K$. We then conclude that we have $\widehat{\mathbb{Z}}_K^M \subset \widehat{\Gamma}$ and so $\Gamma \rightarrow \mathbb{Z}_K^M$. \square

To give now examples which are not of product type, we will use the notion of normality for compact quantum groups from [11], [19].

Definition 2.4. *Given a quantum subgroup $H \subset G$, coming from a quotient map $\pi : C(G) \rightarrow C(H)$, the following are equivalent:*

- (1) $A = \{a \in C(G) \mid (id \otimes \pi)\Delta(a) = a \otimes 1\}$ satisfies $\Delta(A) \subset A \otimes A$.
- (2) $B = \{a \in C(G) \mid (\pi \otimes id)\Delta(a) = 1 \otimes a\}$ satisfies $\Delta(B) \subset B \otimes B$.
- (3) We have $A = B$, as subalgebras of $C(G)$.

If these conditions are satisfied, we say that $H \subset G$ is a normal subgroup.

As explained in [11], in the classical case we obtain the usual notion of normal subgroup. Also, in the group dual case the normality of any subgroup, which must be a group dual subgroup, is automatic. Now with this notion in hand, we have:

Theorem 2.5. *If $G \subset S_N^+$ is transitive and $H \subset G$ is normal, then $H \subset S_N^+$ is quasi-transitive.*

Proof. Consider the quotient map $\pi : C(G) \rightarrow C(H)$, as in Definition 2.7, given at the level of standard coordinates by $u_{ij} \mapsto v_{ij}$. Consider two orbits O_1, O_2 of H and set:

$$x_i = \sum_{j \in O_1} u_{ij} \quad , \quad y_i = \sum_{j \in O_2} u_{ij}$$

These two elements are orthogonal projections in $C(G)$ and they are nonzero, because they are sums of nonzero projections by transitivity of G . We have:

$$(id \otimes \pi)\Delta(x_i) = \sum_k \sum_{j \in O_1} u_{ik} \otimes v_{kj} = \sum_{k \in O_1} \sum_{j \in O_1} u_{ik} \otimes v_{kj} = \sum_{k \in O_1} u_{ik} \otimes 1 = x_i \otimes 1$$

Thus by normality of H we have $(\pi \otimes id)\Delta(x_i) = 1 \otimes x_i$. On the other hand, assuming that we have $i \in O_2$, we obtain:

$$(\pi \otimes id)\Delta(x_i) = \sum_k \sum_{j \in O_1} v_{ik} \otimes u_{kj} = \sum_{k \in O_2} v_{ik} \otimes x_k$$

Multiplying this by $v_{ik} \otimes 1$ with $k \in O_2$ yields $v_{ik} \otimes x_k = v_{ik} \otimes x_i$, that is to say $x_k = x_i$. In other words, x_i only depends on the orbit of i . The same is of course true for y_i . By using this observation, we can compute the following element:

$$z = \sum_{k \in O_2} \sum_{j \in O_1} u_{kj} = \sum_{k \in O_2} x_k = |O_2|x_i$$

On the other hand, by applying the antipode, we have as well:

$$S(z) = \sum_{k \in O_2} \sum_{j \in O_1} u_{jk} = \sum_{j \in O_1} y_j = |O_1|y_j$$

We therefore obtain the following formula : $S(x_i) = \frac{|O_1|}{|O_2|}y_j$. Since both x_i and y_j have norm one, we conclude that the two orbits have the same size, and this finishes the proof. \square

If G_{class} is the Gelfand spectrum of the maximal abelian quotient of $C(G)$, it is conjectured that $G \subset S_N^+$ is transitive if and only if its classical version $G_{class} \subset S_N$ is, with the remark that “ \Leftarrow ” is of course trivial. In the quasi-transitive case a similar statement makes sense, but with “ \Leftarrow ” being this time no longer clear.

3. MATRIX MODELS

Given a quantum permutation group $G \subset S_N^+$, we will be interested in what follows in the matrix models of type $\pi : C(G) \rightarrow M_K(C(X))$, with X being a compact space. There are many examples of such models, and the “simplest” ones are as follows:

Definition 3.1. *Given a subgroup $G \subset S_N^+$, a random matrix model of type*

$$\pi : C(G) \rightarrow M_K(C(X))$$

is called quasi-flat when the fibers $P_{ij}^x = \pi(u_{ij})(x)$ all have rank ≤ 1 .

As a first observation, the functions $x \mapsto r_{ij}^x = \text{rank}(P_{ij}^x)$ are locally constant over X , so they are constant over the connected components of X . Thus, when X is connected, our assumption is that we have $r_{ij}^x = r_{ij} \in \{0, 1\}$, for any $x \in X$, and any i, j . Observe that in the case $K = N$ these questions disappear, because we must have $r_{ij}^x = 1$ for any i, j , and any $x \in X$. In that case the model is called flat, see [6].

It was observed in [6] that the algebra $C(S_N^+)$ has a universal flat model. If we identify the rank one projections in $M_N(\mathbb{C})$ with the elements of the complex projective space $P_{\mathbb{C}}^{N-1}$, it is given by

$$\pi : C(S_N^+) \rightarrow M_N(C(X_N)) \quad , \quad \pi_N(u_{ij}) = [P \mapsto P_{ij}]$$

where X_N is the set of matrices $P \in M_N(P_{\mathbb{C}}^{N-1})$ which are bistochastic with sums 1.

In order to construct now universal quasi-flat models, we use a reformulation of Woronowicz's Tannakian duality [22], [1]. Given an inclusion $G \subset S_N^+$, with the corresponding fundamental corepresentations denoted by $w \mapsto u$, we have the following formula:

$$C(G) = C(S_N^+) / \left(T \in \text{Hom}(w^{\otimes k}, w^{\otimes l}), \forall k, l \in \mathbb{N}, \forall T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \right)$$

with the Hom-spaces at left being taken in a formal sense. With this in hand we have the following result :

Theorem 3.2. *Given a quasi-transitive subgroup $G \subset S_N^+$, with orbits of size K , we have a universal quasi-flat model $\pi : C(G) \rightarrow M_K(C(X))$, constructed as follows:*

- (1) For $G = \underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$ with $N = KM$, the model space is $X_{N,K} = \underbrace{X_K \times \dots \times X_K}_{M \text{ terms}}$, and with $u = \text{diag}(u^1, \dots, u^M)$ the map is $\pi_{N,K}(u_{ij}^r) = [(P^1, \dots, P^M) \mapsto P_{ij}^r]$.
- (2) In general, the model space is the submanifold $X_G \subset X_{N,K}$ obtained via the Tannakian relations defining G .

Proof. This result is known since [1], [6] in the flat case, the idea being to use the Tannakian description of the quotient.

(1) This follows from the description of the universal flat model for S_N^+ since the 0 entries of the model must appear exactly where $u = (u_{ij})$ has 0 entries.

(2) Assume that $G \subset S_N^+$ is quasi-transitive, with orbits of size K . We then have an inclusion $G \subset S_K^+ \hat{*} \dots \hat{*} S_K^+$ with M terms, and in order to construct the universal quasi-flat model for $C(G)$ we need a universal solution to the following factorization problem:

$$\begin{array}{ccc} C(\underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}) & \rightarrow & M_K(C(X_{N,K})) \\ \downarrow & & \downarrow \\ C(G) & \rightarrow & M_K(C(X_G)) \end{array}$$

The solution to this latter question is given by the following construction, with the Hom-spaces at left being taken as usual in a formal sense:

$$C(X_G) = C(X_{N,K}) / \left(T \in \text{Hom}(P^{\otimes k}, P^{\otimes l}), \forall k, l \in \mathbb{N}, \forall T \in \text{Hom}(u^{\otimes k}, u^{\otimes l}) \right)$$

With this result in hand, the Gelfand spectrum of the algebra on the left is then an algebraic submanifold $X_G \subset X_{N,K}$, having the desired universality property. \square

Observe that talking about quasi-flat models for quantum groups which are not necessarily quasi-transitive perfectly makes sense. The universal model spaces can be constructed as above, and this was in fact already discussed in [1], but nothing guarantees

that in the non-quasi-flat case, the model spaces are non-empty. So, we prefer to restrict the attention to the quasi-transitive case, and formulate Theorem 3.2 as it is.

4. INNER FAITHFULNESS

We formulate in what follows two conjectures concerning quasi-flat models. We first review the notions of inner faithfulness and stationarity, from [1], [2], [5]. Following [2], we first have:

Definition 4.1. *Let $\pi : C(G) \rightarrow M_K(C(X))$ be a matrix model.*

- (1) *The Hopf image of π is the smallest quotient Hopf C^* -algebra $C(G) \rightarrow C(H)$ producing a factorization of type $\pi : C(G) \rightarrow C(H) \rightarrow M_K(C(X))$.*
- (2) *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

Observe that when $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation $\rho : \Gamma \rightarrow C(X, U_K)$, and the above factorization is the one obtained by taking the image, $\rho : \Gamma \rightarrow \Gamma' \subset C(X, U_K)$. Thus π is inner faithful when ρ is faithful. Also, given a compact group G , and elements $g_1, \dots, g_K \in G$, we have a representation $\pi : C(G) \rightarrow \mathbb{C}^K$, given by $f \rightarrow (f(g_1), \dots, f(g_K))$. The minimal factorization of π is then via $C(G')$, with $G' = \overline{\langle g_1, \dots, g_K \rangle}$, and π is inner faithful when $G = G'$.

In practice, X is often a compact Lie group, or a compact homogeneous space, or a more general compact probability space. In that case, inner faithfulness was characterized in [20].

Proposition 4.2. *A model $\pi : C(G) \rightarrow M_K(C(X))$, with X being assumed to be a compact probability space, is inner faithful if and only if*

$$\int_G = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{r=1}^k \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$ and $\varphi = \text{tr} \otimes \int_X$ is the random matrix trace with tr the normalized trace on $M_N(\mathbb{C})$.

Following [1], we call a matrix model stationary when the Cesàro limiting convergence in Proposition 4.2 is stationary. In other words, we have the following definition:

Definition 4.3. *A matrix model $\pi : C(G) \rightarrow M_K(C(X))$, with X assumed to be a compact probability space, is called stationary when:*

$$\int_G = \left(\text{tr} \otimes \int_X \right) \pi$$

In the general case, where X is only assumed to be a compact space, π will be called stationary if it is stationary with respect to some probability measure on X .

There are many interesting examples of such models, see [1]. However, the stationarity condition is a very strong assumption, and we have the following result, from [1]:

Proposition 4.4. *Let $\pi : C(G) \rightarrow M_K(C(X))$ be a stationary model. Then,*

- (1) π is faithful.
- (2) $C(G)$ is a type I C^* -algebra, hence the discrete dual $\Gamma = \widehat{G}$ is amenable.

For group duals, the existence of a stationary model can be completely characterized, thanks to a result of Thoma [16]. Recall from [21] that any finitely generated group $\Gamma = \langle g_1, \dots, g_N \rangle$ produces a closed subgroup $\widehat{\Gamma} \subset U_N^+$, with fundamental corepresentation $u_{ij} = \delta_{ij}g_i$, and that the group dual subgroups $\widehat{\Gamma} \subset U_N^+$ all appear in this way, modulo a conjugation of the fundamental corepresentation $u = (u_{ij})$ by a unitary $U \in U_N$.

Theorem 4.5 (Thoma). *Given a group dual $G = \widehat{\Gamma} \subset U_N^+$, the following are equivalent:*

- (1) $C(G)$ is of type I.
- (2) $C(G)$ has a stationary model.
- (3) $C(G)$ has a stationary model, over an homogeneous space.
- (4) Γ is virtually abelian.

Proof. (1) \Rightarrow (4) is Thoma's theorem [16] and (3) \implies (2) \implies (1) are trivial implications. We therefore only have to prove (4) \implies (3). Let $\Lambda < \Gamma$ be an abelian subgroup of finite index and let $K = [\Lambda : \Gamma]$. We define a matrix model $\pi : C^*(\Gamma) \rightarrow M_K(C(\widehat{\Lambda}))$ by

$$\pi(\gamma)(\chi) = \text{Ind}_{\Lambda}^{\Gamma}(\chi)(\gamma)$$

To see that this model is faithful, take $\gamma \in \Gamma$ and recall that the character ψ of $\text{Ind}_{\Lambda}^{\Gamma}(\chi)$ is given by

$$\psi(\gamma) = \text{Tr}(\text{Ind}_{\Lambda}^{\Gamma}(\chi)) = \sum_{x \in \Gamma/\Lambda} \delta_{x^{-1}\gamma x \in \Lambda} \chi(x^{-1}\gamma x)$$

where Tr is the usual (non-normalized) trace on $M_K(\mathbb{C})$. Thus,

$$\left(\text{tr} \otimes \int_{\widehat{\Lambda}} \right) \pi(\gamma) = \frac{1}{K} \sum_{x \in \Gamma/\Lambda} \delta_{x^{-1}\gamma x \in \Lambda} \int_{\widehat{\Lambda}} \chi(x^{-1}\gamma x) d\chi$$

Since the integral over all characters is the indicator function of the trivial element, the expression above equals $\delta_{\gamma, e}$ and the model is stationary. \square

Let us formulate now our two main conjectures regarding the universal quasi-flat models for the quantum permutation groups. We first have:

Conjecture 4.6. *Assuming that $G \subset S_N^+$ is quasi-transitive, with orbits of size K , and that $\Gamma = \widehat{G}$ satisfies a suitable ‘‘virtual abelianity’’ condition, we have:*

- (1) *The universal quasi-flat model space X_G is an homogeneous space.*
- (2) *The corresponding model $\pi : C(G) \rightarrow M_K(C(X_G))$ is stationary.*

The evidence here comes from Thoma’s theorem, in its conjectural stronger form presented above, as well from a number of explicit verifications, to be performed below, and notably from a verification in the case where G is classical. We do not know yet what the “virtual abelianity” condition should mean. When Γ is classical, as in Thoma’s theorem, this condition states that we must have an abelian subgroup $\Lambda < \Gamma$ such that $F = \Gamma/\Lambda$ is finite. Regarding now the notion of inner faithfulness, we have here:

Conjecture 4.7. *Assuming that $G \subset S_N^+$ is quasi-transitive, and satisfies in addition a suitable “uniformity” condition, the universal quasi-flat model is inner faithful.*

We do not know yet what the precise “uniformity” condition should be. We believe that all this is related to the notion of easiness [12], [14], and this will be confirmed by some of the verifications performed below, but in general, we have no results.

Finally, let us mention that in the transitive case, the very first question here concerns $G = S_N^+$ itself, and the problem is difficult and open. Indeed, assuming that the conjecture holds, S_N^+ would follow to be inner linear (in a parametric sense) and we would therefore obtain that $L^\infty(S_N^+)$ has the Connes embedding property. Thus, we would have here a solution to an old open problem.

5. THE CLASSICAL CASE

In this section we discuss the classical case, $G \subset S_N$. First note that the identity map $C(G) \rightarrow C(G) = M_1(C(G))$ is always a stationary matrix model. Our question here is the following one: assume that G is quasi-transitive with orbits of size K , under which condition is there a stationary model $\pi : C(G) \rightarrow M_K(C(X))$? We will use the following notion:

Definition 5.1. *A “sparse Latin square” is a square matrix $L \in M_N(*, 1, \dots, K)$ having the property that each of its rows and columns consists of a permutation of the numbers $1, \dots, K$, completed with $*$ entries.*

In the case $K = N$, where there are no $*$ symbols, we recover the usual Latin squares. In general, however, the combinatorics of these matrices seems to be more complicated than that of the usual Latin squares. Here are a few examples of such matrices:

$$\begin{pmatrix} 1 & 2 & * \\ 2 & * & 1 \\ * & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * & * \\ 2 & * & 1 & * \\ * & 1 & * & 2 \\ * & * & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * & * \\ 2 & 1 & * & * \\ * & * & 1 & 2 \\ * & * & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & * & * & * \\ 2 & * & 1 & * & * \\ * & 1 & 2 & * & * \\ * & * & * & 1 & 2 \\ * & * & * & 2 & 1 \end{pmatrix}$$

With this notion in hand, the result that we need is as follows:

Proposition 5.2. *The quasi-flat representations $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$ appear as*

$$u_{ij} \mapsto P_{L_{ij}}$$

where $P_1, \dots, P_K \in M_K(\mathbb{C})$ are rank one projections, summing up to 1, and where $L \in M_N(*, 1, \dots, K)$ is a sparse Latin square, with the convention $P_* = 0$.

Proof. Assuming that $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$ is quasi-flat, the elements $P_{ij} = \pi(u_{ij})$ are projections of rank at most one which pairwise commute and form a magic unitary. Let us denote by $P_1, \dots, P_K \in M_K(\mathbb{C})$ the rank one projections appearing in the first row of $P = (P_{ij})$. Since these projections form a partition of unity with rank one projections, any rank one projection $Q \in M_K(\mathbb{C})$ commuting with all of them satisfies $Q \in \{P_1, \dots, P_K\}$. In particular we have $P_{ij} \in \{P_1, \dots, P_K\}$ for any i, j such that $P_{ij} \neq 0$. Thus we can write $u_{ij} \mapsto P_{L_{ij}}$, for a certain matrix $L \in M_N(*, 1, \dots, K)$, with the convention $P_* = 0$.

In order to conclude, note that $u_{ij} \mapsto P_{L_{ij}}$ defines a representation $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$ precisely when the matrix $P = (P_{L_{ij}})_{ij}$ is magic. But this condition tells us precisely that L must be a sparse Latin square. \square

Our task now is to compute the associated Hopf image.

Proposition 5.3. *Given a sparse Latin square $L \in M_N(*, 1, \dots, K)$, consider the permutations $\sigma_1, \dots, \sigma_K \in S_N$ given by:*

$$\sigma_x(j) = i \iff L_{ij} = x$$

The Hopf image associated to a representation $\pi : C(S_N) \rightarrow M_K(\mathbb{C})$, $u_{ij} \mapsto P_{L_{ij}}$ as above is then the algebra $C(G_L)$, where $G_L = \langle \sigma_1, \dots, \sigma_K \rangle \subset S_N$.

Proof. We use a method from [3]. The image of π being generated by P_1, \dots, P_K , we have an isomorphism of algebras $\alpha : \text{Im}(\pi) \simeq C(1, \dots, K)$ given by $P_i \mapsto \delta_i$. Consider the following diagram:

$$\begin{array}{ccccc} C(S_N) & \xrightarrow{\pi} & \text{Im}(\pi) & \longrightarrow & M_K(\mathbb{C}) \\ & \searrow \varphi & \downarrow \alpha & & \\ & & C(1, \dots, K) & & \end{array}$$

where the map on the right is the canonical inclusion and $\varphi = \alpha\pi$. Since the Hopf image of π coincides with the one of φ , it is enough to compute the latter. We know that φ is given by $\varphi(u_{ij}) = \delta_{L_{ij}}$, with the convention $\delta_* = 0$. By Gelfand duality, φ must come from a certain map $\sigma : \{1, \dots, K\} \rightarrow S_N$, via the transposition formula $\varphi(f)(x) = f(\sigma_x)$. With the choice $f = u_{ij}$, we obtain:

$$\delta_{L_{ij}}(x) = u_{ij}(\sigma_x)$$

Now observe that these two quantities are by definition given by:

$$\delta_{L_{ij}}(x) = \begin{cases} 1 & \text{if } L_{ij} = x \\ 0 & \text{otherwise} \end{cases}, \quad u_{ij}(\sigma_x) = \begin{cases} 1 & \text{if } \sigma_x(j) = i \\ 0 & \text{otherwise} \end{cases}$$

We conclude that σ_x is the permutation in the statement. Summarizing, we have shown that φ comes by transposing the map $x \rightarrow \sigma_x$, with σ_x being as in the statement. By using now the general theory in [2], the Hopf image of φ is the algebra $C(G_L)$, with $G_L = \langle \sigma_1, \dots, \sigma_K \rangle$, and this finishes the proof. \square

Let us discuss the construction of the universal model space, and then Conjecture 4.6, in the classical case. We agree to identify the rank one projections in $M_K(\mathbb{C})$ with the corresponding elements of the projective space $P_{\mathbb{C}}^{K-1}$. Combining Proposition 5.2 and Proposition 5.3 above, we have

Corollary 5.4. *Assuming that $G \subset S_N$ is quasi-transitive, with orbits of size K , the universal quasi-flat model space for G is given by $X_G = E_K \times L_{N,K}^G$, where:*

$$E_K = \left\{ (P_1, \dots, P_K) \in (P_{\mathbb{C}}^{K-1})^K \mid P_i \perp P_j, \forall i \neq j \right\}$$

$$L_{N,K}^G = \left\{ L \in M_N(*, 1, \dots, K) \text{ sparse Latin square} \mid G_L \subset G \right\}$$

In particular, X_G has a canonical probability measure, obtained as the homogeneous space measure on E_K times the normalized counting measure on $L_{N,K}^G$.

Note that this model above is empty if there is no sparse Latin square L such that $G_L \subset G$. The existence of such a sparse Latin square is a strong condition on G , here is an intrinsic characterization:

Proposition 5.5. *Let $G \subset S_N$ be a classical group. The following are equivalent:*

- (1) $L_{N,K}^G \neq \emptyset$.
- (2) *There exists K elements of G , denoted by $\sigma_1, \dots, \sigma_K$, such that $\sigma_1(i), \dots, \sigma_K(i)$ are pairwise distinct for all $1 \leq i \leq N$.*

Proof. (1) \Rightarrow (2) : If $\sigma_1, \dots, \sigma_K$ are the permutations associated to L , then $\sigma_x(i) = \sigma_y(i)$ means by definition that $x = L_{i\sigma_x(i)} = L_{i\sigma_y(i)} = y$.

(2) \Rightarrow (1) : consider such permutations $\sigma_1, \dots, \sigma_K$. If i is fixed then for each j there is at most one index x such that $\sigma_x(i) = j$. We set $L_{ij} = x$ in that case and $L_{ij} = *$ otherwise. Then, L is a sparse Latin square and the associated permutations are $\sigma_1^{-1}, \dots, \sigma_K^{-1}$, which belong to G . \square

It turns out that as soon as $L_{N,K}^G \neq \emptyset$, the universal quasi-flat model is stationary. To prove this, let us first prove an elementary fact.

Lemma 5.6. *Given $G \subset S_N$, we have an action $G \curvearrowright L_{N,K}^G$, given by*

$$(L^\tau)_{ij} = L_{\tau^{-1}(i)j}$$

Proof. Given a sparse Latin square $L \in M_N(*, 1, \dots, K)$, the matrix $L^\tau \in M_N(*, 1, \dots, K)$ constructed in the statement is a sparse Latin square too and $\tau \rightarrow (L \mapsto L^\tau)$ is a group

morphism, so it remains to check that $G_L \subset G$ implies $G_{L^\tau} \subset G$. To do this, let us write $G_L = \langle \sigma_1, \dots, \sigma_K \rangle$ and $G_{L^\tau} = \langle \sigma'_1, \dots, \sigma'_K \rangle$. Then,

$$\begin{aligned} \sigma'_x(j) = i &\iff (L^\tau)_{ij} = x \iff L_{\tau^{-1}(i)j} = x \\ &\iff \sigma_x(j) = \tau^{-1}(i) \iff \tau\sigma_x(j) = i \end{aligned}$$

Thus, $\sigma'_x = \tau\sigma_x$ and $G_{L^\tau} \subset G$. \square

We can now prove Conjecture 4.6 in the classical group case:

Theorem 5.7. *If $L_{N,K}^G$ is not empty, then the universal flat model for a quasi-transitive subgroup $G \subset S_N$,*

$$\pi : C(G) \rightarrow M_K(C(X_G))$$

is stationary with respect to $\nu \otimes m$ where m is the normalized counting measure on $L_{N,K}^G$ and ν is any probability measure on E_K .

Proof. Write $M_K(C(X_G)) = M_K(\mathbb{C}) \otimes C(E_K) \otimes C(L_{N,K}^G)$ and consider, for $P \in E_K$, the map

$$\pi^P = (id \otimes ev_P \otimes id) : C(G) \rightarrow M_K(C(L_{N,K}^G))$$

Recall that for $\tau \in G$ and $f \in C(G)$, $\tau.f$ denotes the map $h \mapsto f(\tau^{-1}h)$. This corresponds to the regular action of G on itself. Moreover,

$$\pi^P(u_{ij})(\tau^{-1}L) = \pi^P(u_{\tau(i)j}) = \pi^P(\tau.u_{ij})$$

Since the normalized counting measure m on $L_{N,K}^G$ is G -invariant, it follows (writing again m for the integration with respect to m on $C(L_{N,K}^G)$) that $(tr \otimes m)\pi^P$ is a G -invariant state on $C(G)$, i.e. it is \int_G . Summarizing, we have proved that $(tr \otimes id \otimes m)\pi$ is the constant function equal to \int_G and the result follows. \square

As pointed to us by the referee, by composing with the coefficient maps on $M_K(\mathbb{C})$ we obtain a stronger result : the map $(id \otimes id \otimes m)$ is constant on E_K with value $\int_G .1_{M_K(\mathbb{C})}$.

6. GROUP DUALS

In what follows we discuss universal quasi-flat models in the group dual case. Let us start with $\Gamma = \mathbb{Z}_N$, where the class of 1 will be denoted by g , and called the canonical generator.

Lemma 6.1. *We have an isomorphism of Hopf algebras*

$$C^*(\mathbb{Z}_N) = C(S_N^+) / \left\langle u_{ij} = u_{kl} \mid j - i = l - k \pmod{N} \right\rangle$$

which is such that $g = \sum_{j=1}^N w^{j-i} u_{ij}$, for any i , where $w = e^{2\pi i/N}$.

Proof. The quotient algebra A in the statement being generated by the entries of the first row of $u = (u_{ij})$, it is commutative. If we identify the elements of $\widehat{\mathbb{Z}}_N$ with the powers of w , and the elements of \mathbb{Z}_N with the functions $w \mapsto w^k$, then $\pi(u_{ij}) = \delta_{w^{j-i}}$ defines an isomorphism between A and the algebra $C^*(\mathbb{Z}_N) = C(\widehat{\mathbb{Z}}_N)$. Moreover,

$$(\pi \otimes \pi)\Delta(u_{ij}) = \sum_k \delta_{w^{k-i}} \otimes \delta_{w^{j-k}} = \Delta_{\widehat{\mathbb{Z}}_K}(\delta_{w^{j-i}})$$

so that we have an isomorphism of Hopf algebras, as stated. The formula for g in the statement then follows from the formula for π . \square

This translates into a convenient description of the universal flat matrix model.

Proposition 6.2. *The universal flat model space for $C^*(\mathbb{Z}_N)$ is the space*

$$E_N = \left\{ (P_1, \dots, P_N) \in (P_{\mathbb{C}}^{N-1})^N \mid P_i \perp P_j, \forall i \neq j \right\}$$

with the model map given by $\pi(g)(P_1, \dots, P_N) = \sum_{j=1}^N w^j P_j$.

Proof. According to Proposition 6.1 above, $\pi(u_{ij})(P_1, \dots, P_N) = P_{j-i}$ is a flat matrix model map, which gives the formula in the statement for $\pi(g)$.

Let X be a flat matrix model space for $C^*(\mathbb{Z}_N)$, with model map π_X . For each $x \in X$ the matrix $\pi_X(g)(x)$ is circulant, so this matrix is completely determined by its first row, which is an element of E_N . Let us set $P_j^x = P_{Nj}^x$, and let P^x be the circulant matrix with first row (P_1^x, \dots, P_N^x) . The map $\Phi : X \rightarrow E_N$ given by $\Phi(x) = P^x$ being a continuous embedding, we get a surjective $*$ -homomorphism $\Psi : M_N(C(E_N)) \rightarrow M_N(C(X))$ through which π_X factors by construction. Thus, we obtain the universality of E_N , as claimed. \square

Regarding now the general group dual case, we have here the following result:

Theorem 6.3. *Given a quotient group $\mathbb{Z}_K^{*M} \rightarrow \Gamma$, the universal quasi-flat model $\pi : C^*(\Gamma) \rightarrow M_K(C(Y_\Gamma))$ appears as follows:*

- (1) *For $\Gamma = \mathbb{Z}_K^{*M}$, the model space is $Y_\Gamma = Y_{N,K} = \underbrace{E_K \times \dots \times E_K}_{M \text{ terms}}$, and the model map is given by the formula in Proposition 6.2, on each of the components.*
- (2) *In general, the model space is the subspace $Y_\Gamma \subset Y_{N,K}$ consisting of the elements x such that $\pi(\gamma)(x) = Id$ for any $\gamma \in \ker(\mathbb{Z}_K^{*M} \rightarrow \Gamma)$.*

Proof. The first assertion is clear from Proposition 6.2. Regarding now the second assertion, let us go back to Theorem 3.5 (2) above. The statement there tells us that the model space for $\widehat{\Gamma}$ appears from the model space for $\underbrace{S_K^+ \hat{*} \dots \hat{*} S_K^+}_{M \text{ terms}}$ via the Tannakian conditions

defining $\widehat{\Gamma}$. But these Tannakian conditions, when expressed in terms of the generators of \mathbb{Z}_K^{*M} are precisely the relations $\gamma = 1$ with $\gamma \in \ker(\mathbb{Z}_K^{*M} \rightarrow \Gamma)$, as stated. \square

7. MAXIMAL TORI

We will now prove Conjecture 4.7 for several families of duals of discrete groups. The proofs will be more convenient to write by using a lift of the quasi-flat models in the following sense :

Definition 7.1. *The affine lift of the universal quasi-flat model for $C^*(\mathbb{Z}_K^{*M})$ is $\pi : C^*(\mathbb{Z}_K^{*M}) \rightarrow M_K(C(U_K^M))$ given on the canonical generator g_i of the i -th factor by*

$$\pi(g_i)(U^1, \dots, U^M) = \sum_j w^j P_{U_j^i}$$

where U_j^i is the j -th column of U^i and P_ξ denotes the orthogonal projection onto $\mathbb{C}\xi$.

According to the results of [7] explained in section 1 above, the maximal group dual subgroups $\widehat{\Gamma} \subset S_N^+$, which can be regarded as being “maximal tori”, are the free products of type $\mathbb{Z}_{K_1} * \dots * \mathbb{Z}_{K_M}$ with $K_1 + \dots + K_M = N$. In the quasi-transitive case, where $K_1 = \dots = K_M = K$ with $K|N$, we have the following result:

Theorem 7.2. *The universal quasi-flat model for \mathbb{Z}_K^{*M} is inner faithful.*

Proof. It is enough to prove that the affine lift of the model is inner faithful. Let us consider a reduced word $\gamma \in \mathbb{Z}_K^{*M}$, and write it as $\gamma = g_{i_1}^{k_1} \dots g_{i_n}^{k_n}$, with $i_t \neq i_{t+1}$, and with $1 \leq k_t \leq K - 1$. Then:

$$\begin{aligned} \pi(\gamma)(U^1, \dots, U^M) &= \sum_{j_1 \dots j_n=1}^K w^{k_1 j_1} P_{U_{j_1}^{i_1}} \dots w^{k_n j_n} P_{U_{j_n}^{i_n}} \\ &= \sum_{j_1 \dots j_n=1}^K w^{\langle k, j \rangle} P_{U_{j_1}^{i_1}} \dots P_{U_{j_n}^{i_n}} \end{aligned}$$

Our aim is to prove that there is at least one tuple (U^1, \dots, U^M) for which the matrix above is not the identity. Recall the following formula valid for any vectors in a Hilbert space $\xi_1, \dots, \xi_l \in H$, with the scalar product being linear on the left:

$$P_{\xi_1} \dots P_{\xi_l}(x) = \langle x, \xi_l \rangle \langle \xi_l, \xi_{l-1} \rangle \dots \langle \xi_2, \xi_1 \rangle \xi_1$$

To compute the trace of this operator, one can consider any orthonormal basis containing ξ_l , yielding $\langle \xi_1, \xi_l \rangle \langle \xi_l, \xi_{l-1} \rangle \dots \langle \xi_2, \xi_1 \rangle$. Applying this to $\pi(\gamma)$ and using the equality $\langle U_i, V_j \rangle = \sum_l U_{ki} \bar{V}_{kj} = (V^*U)_{ji}$, we get:

$$\begin{aligned} \text{tr} \circ \pi(\gamma) &= \frac{1}{K} \sum_{j_1 \dots j_n=1}^K w^{\langle k, j \rangle} \langle U_{j_1}^{i_1}, U_{j_n}^{i_n} \rangle \langle U_{j_n}^{i_n}, U_{j_{n-1}}^{i_{n-1}} \rangle \dots \langle U_{j_2}^{i_2}, U_{j_1}^{i_1} \rangle \\ &= \frac{1}{K} \sum_{j_1 \dots j_n=1}^K w^{\langle k, j \rangle} (U^{i_n} * U^{i_1})_{j_n j_1} (U^{i_{n-1}} * U^{i_n})_{j_{n-1} j_n} \dots (U^{i_1} * U^{i_2})_{j_1 j_2} \end{aligned}$$

Denoting by W the diagonal matrix given by $W_{ij} = \delta_{ij}w^i$, we have:

$$\sum_{j_1} w^{k_1 j_1} U_{j_n j_1} U_{j_1 j_2}^* = \sum_{j_1 l} U_{j_n j_1} W_{j_1 l}^{k_1} U_{l j_2}^* = (UW^{k_1}U^*)_{j_n j_2}$$

Applying this n times in the above formula for $tr \circ \pi(\gamma)$ yields:

$$\begin{aligned} tr \circ \pi(\gamma) &= tr (U^{i_n} U^{i_1} W^{k_1} U^{i_1} U^{i_2} W^{k_2} \dots W^{k_{n-1}} U^{i_{n-1}} U^{i_n} W^{k_n}) \\ &= tr (U^{i_1} W^{k_1} U^{i_1} \dots U^{i_n} W^{k_n} U^{i_n}) \end{aligned}$$

Assume now that $\pi(\gamma)(U^1, \dots, U^M) = Id$ for all tuples of unitary matrices. The trace of a unitary matrix can only be equal to 1 if it is the identity, hence

$$\prod_{p=1}^n U^{i_p} W^{k_p} U^{i_p} = Id$$

for all (U^1, \dots, U^M) . In other words, the noncommutative polynomial

$$P = \prod_{p=1}^n X^{i_p} W^{k_p} X^{i_p} - 1$$

vanishes on all of U_K^M . But this is impossible if $k_t \neq 0 \pmod K$ for all t , hence $\pi(\gamma)$ is not always the identity and π is inner faithful. \square

The above result can be extended by allowing arbitrary direct products as components of the free product. More precisely, assume that $M = M_1 + \dots + M_n$. Then, the free product $\mathbb{Z}_K^{M_1} * \dots * \mathbb{Z}_K^{M_n}$ is a quotient of \mathbb{Z}_K^M which is still quasi-transitive with orbits of size K and we prove that its universal quasi-flat matrix model is still inner faithful.

Proposition 7.3. *The universal quasi-flat model for $\Gamma = \mathbb{Z}_K^{M_1} * \dots * \mathbb{Z}_K^{M_n}$ is inner faithful.*

Proof. We first need a picture of the affine lift of the model for the direct product \mathbb{Z}_K^M . Note that if two complete families of rank one orthogonal projections commute, then they are a permutation of one another. We may therefore consider the space $U_K \times S_K^M$ as the affine lift of our model. If g_1, \dots, g_M are the canonical generators of the direct product, their action is then given by:

$$\pi(g_i)(U, \sigma_1, \dots, \sigma_M) = \sum_{j=1}^K w^j P_{U_{\sigma_i^{-1}(j)}} = \sum_{j=1}^K w^{\sigma_i(j)} P_{U_j}$$

This yields the following formula for a general element:

$$\pi(g_1^{k_1} \dots g_M^{k_M})(U, \sigma_1, \dots, \sigma_M) = \sum_{j=1}^K w^{k_1 \sigma_1(j) + \dots + k_M \sigma_M(j)} P_{U_j}$$

Let $g_1(i), \dots, g_{M_i}(i)$ be the generators of $\mathbb{Z}_K^{M_i}$, and let consider a reduced word:

$$\gamma = \left(g_1(i_1)^{k_1(1)} \dots g_{M_{i_1}}(i_1)^{k_{M_{i_1}}(1)} \right) \dots \left(g_1(i_n)^{k_1(n)} \dots g_{M_{i_n}}(i_n)^{k_{M_{i_n}}(n)} \right)$$

The computation of $tr \circ \pi(\gamma)$ is similar to the one in the proof of Theorem 7.2 until the introduction of the matrices W . Here we have to replace W^{k_t} by $\prod_{s=1}^{M_t} W_{\sigma_s^t}^{k_s(t)}$, where $(U^t, \sigma_1^t, \dots, \sigma_{M_t}^t)_{1 \leq t \leq n}$ is the element to which we are applying $\pi(\gamma)$ and $(W_\sigma)_{ij} = \delta_{ij} w^{\sigma(i)}$.

Assuming that $\pi(\gamma) = 1$, we can apply the same strategy as before: we have a polynomial which must vanish on all tuples of unitary matrices and this is impossible unless all the matrices appearing in the polynomial are the identity. We therefore get the condition $\prod_{s=1}^{M_t} W_{\sigma_s^t}^{k_s(t)} = Id$ for all t , which translates into

$$\prod_{s=1}^{M_t} w^{\sigma_s^t(i)k_s(t)} = 1$$

for all $1 \leq i \leq N$ and all $(\sigma_1^t, \dots, \sigma_{M_t}^t)$. To derive a contradiction, we will simply sum the above equation over all permutations, getting

$$\frac{1}{(K!)^M} \sum_{\sigma_1, \dots, \sigma_{M_t} \in S_K} w^{k_1(t)\sigma_1(i) + \dots + k_{M_t}(t)\sigma_{M_t}(i)} = \frac{1}{(K!)^M} \prod_{s=1}^{M_t} \left(\sum_{\sigma_s \in S_K} w^{k_s(t)\sigma_s(i)} \right)$$

For any i' , there are $(K-1)!$ permutations σ such that $\sigma_s(i) = i'$. This leads to:

$$\begin{aligned} \sum_{\sigma_s \in S_K} w^{k_s(t)\sigma_s(i)} &= \sum_{i'=1}^K (K-1)! w^{k_s(t)i'} \\ &= K! \delta_{k_s(t), 0} \end{aligned}$$

Gathering everything yields

$$\prod_{s=1}^{M_t} \delta_{k_s(t), 0} = 1$$

so that $k_s(t) = 0$ for all t and all s , a contradiction. \square

8. FURTHER RESULTS

In this section we give two other extensions of Theorem 7.2. Before that, let us give an example showing that the universal model is not always inner faithful for duals of discrete groups. Consider the group $\Gamma = (\mathbb{Z}_K * \mathbb{Z}_K) \times \mathbb{Z}_K$ and let

$$\pi : C^*(\Gamma) \rightarrow M_K(C(X))$$

be a matrix model. For all x , $\pi(g_1)(x)$ and $\pi(g_3)(x)$ are commuting diagonalizable operators with all eigenvalues of multiplicity one. The same holds for $\pi(g_2)(x)$ and $\pi(g_3)(x)$

so that $\pi(g_1)(x)$ and $\pi(g_2)(x)$ must commute. Since this holds for any $x \in X$, we conclude that π factors through the quotient \mathbb{Z}_K^3 , so that the model is not inner faithful. The ‘‘uniformity assumption’’ of Conjecture 4.7 should enable to distinguish between the groups in Proposition 7.3 and the example above. It is not clear however what the precise statement should be.

We now extend Theorem 7.2 by allowing the free product to be amalgamated over a subgroup isomorphic to \mathbb{Z}_L . Assume that we have $N = KM$ as before, and assume in addition that we have $K = LR$. We can then consider the following quotient:

$$\Gamma = \mathbb{Z}_K^{*M} / \langle g_i^R = g_j^R \mid \forall i, j \rangle$$

It is enough for our purpose to provide an explicit model which is inner faithful and we define it through its affine lift. Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of \mathbb{C}^K and consider for $1 \leq t \leq L$ the subspace $V_t \subset \mathbb{C}^K$ spanned by the vectors e_i with $(t-1)R+1 \leq i \leq tR$, so that

$$\mathbb{C}^K = V_1 \oplus \dots \oplus V_L$$

This gives a block-diagonal embedding $U_R^L \subset U_K$, which is the affine lift of the model space. However, for computations it is simpler to use a permutation of this model:

Definition 8.1. *We denote by $X_{K,L}$ the set of unitary matrices $U \subset U_K$ such that $(U_t, U_{t+L}, \dots, U_{t+(R-1)L})$ is a basis of V_t for all $1 \leq t \leq L$.*

Defining $\pi(g_i)$ by the usual formula, we see that the following element does not depend on i :

$$\pi(g_i)^R(U^1, \dots, U^M) = \sum_{j=1}^K w^{jR} P_{U_j^i} = \sum_{t=1}^L w^{tR} \left(\sum_{s=0}^{R-1} P_{U_{t+sL}^i} \right) = \sum_{t=1}^L w^{tR} P_{V_t}$$

Thus, we get a representation $\pi : C^*(\Gamma) \rightarrow M_N(C(X_{K,L}^M))$. We have:

Theorem 8.2. *The quasi-flat model $\pi : C^*(\Gamma) \rightarrow M_K(C(X_{K,L}^M))$ for the amalgamated free product $\Gamma = \mathbb{Z}_K^{*M} / \langle g_i^R = g_j^R \mid \forall i, j \rangle$ is inner faithful.*

Proof. As usual, we consider an arbitrary element $\gamma \in \Gamma$. Writing $h = g_i^R$, we can assume that $\gamma = h^l g_{i_1}^{k_1} \dots g_{i_n}^{k_n}$ with $0 < l < L$ and $0 < k_i < R$ for all i . Thus:

$$\begin{aligned} \pi(\gamma)(U^1, \dots, U^M) &= \pi(h^l)(U^1, \dots, U^M) \left(\sum_{j_1 \dots j_n} w^{\langle k, j \rangle} P_{U_{j_1}^{i_1}} \dots P_{U_{j_n}^{i_n}} \right) \\ &= \left(\sum_{t=1}^L w^{tR} P_{V_t} \right) \left(\sum_{j_1 \dots j_n} w^{\langle k, j \rangle} P_{U_{j_1}^{i_1}} \dots P_{U_{j_n}^{i_n}} \right) \\ &= \sum_{t=1}^L \sum_{j_1 \dots j_n} w^{tR} w^{\langle k, j \rangle} P_{V_t} P_{U_{j_1}^{i_1}} \dots P_{U_{j_n}^{i_n}} \end{aligned}$$

Let us consider the term corresponding to a fixed t . The product of projections vanishes unless $U_{j_1} \in V_t$. This forces $U_{i_2} \in V_t$ and by induction all the terms must be in V_t . This means that there are s_1, \dots, s_n such that $j_m = t + s_m L$ for all $1 \leq m \leq n$. Thus:

$$\pi(\gamma)(U^1, \dots, U^M) = \sum_{t=1}^L w^{tR} w^{t(k_1 + \dots + k_n)} \sum_{s_1, \dots, s_n=0}^{R-1} w^{L\langle k, s \rangle} P_{U_{t+s_1 L}^{i_1}} \dots P_{U_{t+s_n L}^{i_n}}$$

Let us denote by $U^i(t)$ the unitary operator on V_t obtained from the appropriate columns of U^i . The trace of $\pi(\gamma)$ can be expressed using these operators. To do this, simply note that the normalized trace tr can be written as $L^{-1}tr_R$, where tr_R is the normalized trace on U_R . A computation similar to that of Theorem 7.2 then yields:

$$tr \circ \pi(\gamma) = \frac{1}{L} \sum_{t=1}^L w^{t(lR+k_1+\dots+k_n)} tr_R \left(\prod_{p=1}^n (U^{i_p}(t) W^{Lk_p} U^{i_p^*}(t)) \right)$$

Assuming that $\pi(\gamma) = Id$, we can now derive a contradiction. Indeed, this forces

$$\frac{1}{L} \sum_{t=1}^L w^{t(lR+k_1+\dots+k_n)} \prod_{p=1}^n (U^{i_p}(t) W^{Lk_p} U^{i_p}(t)) = Id$$

Because $U^i(t)$ only acts on V_t and the decomposition is orthogonal, this equation is equivalent to the system formed by the equations for each fixed t . As before, this system cannot always be satisfied unless $k_p = 0$ for all p . In that case, we are left with $w^{tR} = 1$ for all t , implying $l = 0$ and the proof is complete. \square

We end with another construction. This time, we do not identify the copies of \mathbb{Z}_L but simply make them commute. More precisely, we set

$$\Gamma = \mathbb{Z}_K^{*M} / \langle g_i^R g_j^R = g_j^R g_i^R \mid \forall i, j \rangle$$

This is in a sense a mix between the free product of direct products and the amalgamated free products. Thus, the model space is $X_\Gamma = X_{K,L}^M \times S_L^M$ understood as tuples $(U^1, \dots, U^M, \sigma_1, \dots, \sigma_M)$ where U^i satisfies that $(U_t^i, U_{t+L}^i, \dots, U_{t+(r-1)L}^i)$ is a basis of $V_{\sigma_i^{-1}(t)}$ for all t . Still using the usual formula for the canonical generators, we see that

$$\pi(g_i)^R(U^1, \dots, U^M, \sigma_1, \dots, \sigma_M) = \sum_{t=1}^L w^{\sigma_i(t)R} P_{V_t}$$

commutes with $\pi(g_j)^R$ so that we indeed have a representation $\pi : C^*(\Gamma) \rightarrow M_K(C(X_\Gamma))$.

Proposition 8.3. *The quasi-flat model $\pi : C^*(\Gamma) \rightarrow M_K(C(X_\Gamma))$ for the group $\Gamma = \mathbb{Z}_K^{*M} / \langle g_i^R g_j^R = g_j^R g_i^R \mid \forall i, j \rangle$ is inner faithful.*

Proof. As usual, we consider a non-trivial word $\gamma = g_{i_1}^{k_1} \dots g_{i_n}^{k_n}$ in a particular form. For each i , let $k_i = k'_i + a_i R$ be the euclidean division of k_i by R . Then:

$$\begin{aligned} \pi(\gamma)(U^1, \dots, U^M, \sigma_1, \dots, \sigma_M) &= \prod_{s=1}^n \left(\sum_{t_s=1}^L w^{\sigma_{i_s}(t_s) a_s R} P_{V_{t_s}} \right) \left(\sum_{j_s=1}^K w^{k'_s j_s} P_{U_{j_s}^{t_s}} \right) \\ &= \sum_{t_1, \dots, t_n=1}^L \sum_{j_1, \dots, j_n=1}^K w^{R \langle \sigma(t), a \rangle} w^{\langle k', j \rangle} P_{V_{t_1}} P_{U_{j_1}^{t_1}} \dots P_{V_{t_n}} P_{U_{j_n}^{t_n}} \end{aligned}$$

For the product to be nonzero we need all the V_t 's to be the same so that the first sum reduces to only one index. Moreover, this forces as before $j_m = t + s_m L$, so that we get:

$$\sum_{t=1}^L \sum_{s_1, \dots, s_n=0}^{R-1} w^{R(a_1 \sigma_{i_1}(t) + \dots + a_n \sigma_{i_n}(t))} w^{t(k'_1 + \dots + k'_n)} w^{L \langle k', s \rangle} P_{U_{t+s_1 L}^{i_1}} \dots P_{U_{t+s_n L}^{i_n}}$$

Using the same argument as in the proof of Theorem 8.1, we conclude that $k'_p = 0$ for all p and that

$$w^{R(a_1 \sigma_{i_1}(t) + \dots + a_n \sigma_{i_n}(t))} = 1$$

for all t . Summing over S_L^M then yields $\prod \delta_{a_i, 0} = 1$ and the proof is complete. \square

It is very likely that similar computations yield inner faithfulness results for free products of amalgamated free products or similar constructions. One may even expect a general stability result under free products. As for the uniformity assumption in Conjecture 4.7, it is still unclear but must include strong conditions on commuting elements in order to avoid the counter-example given at the beginning of this section.

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