Lectures on measure theory

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2010 *Mathematics Subject Classification.* 28A20

Key words and phrases. Measure theory, Function space

Abstract. This is an introduction to measure theory and function spaces, with all the needed preliminaries included, and with some applications included as well. We first discuss some motivations, coming from basic probability theory, and from logic. Then we go ahead with measure theory and function spaces, developed in a standard way. Finally, we come back to logic and probability, with a discussion regarding ∞, and its connections with various subtle questions from theoretical physics.

Preface

You probably know some calculus, in one and several variables, along with some basic algebra, geometry and probability. For most questions in science and engineering, that knowledge is enough, although learning new methods and tricks, from time to time, say from older colleagues that you will work with, will always improve your science.

This being said, why not learning some new methods and tricks right away, now that you're still studying. You would perhaps say, why bothering with this, I already had enough troubles with math, and learned enough of it, as to get started in life. But let me ask you a few questions, about your current knowledge, and judge yourself:

(1) The probability for a randomly picked $x \in \mathbb{R}$ to be $x = 1$ is, and you surely know that, $P = 0$. But what does this exactly mean, mathematically speaking?

(2) Along the same lines, but more concerning, everything that happens in the real life, happens with probability $P = 0$. And that is something that we must clarify.

(3) Even worse, in certain physics disciplines like quantum mechanics, and even statistical mechanics, P is the only tool available, and $P = 0$, your daily nightmare.

(4) Another interesting thing that you surely know is that the reals are uncountable, $|\mathbb{R}| > |\mathbb{N}| = \infty$. So, how many possible ∞ beasts are there, in mathematics?

(5) What about arbitrarily small infinitesimals, how many $1/\infty$ beasts are there? And with the comment that these can be, technically speaking, very useful.

(6) Regarding these latter questions, big and small infinities appear in a subtle way in advanced physics, namely relativity, particle physics, and the Big Bang.

(7) Less philosophically now, you are surely a heavy user of $dxdy = dydx$, called Fubini theorem, for integrating 2-variable functions. But, does Fubini always work?

(8) Also, when using polar coordinates, the formula is $dxdy = Jdrdt$, with $J = r$. But no one ever fully proves this in class, so is this formula really correct, and why?

(9) You surely know about the Dirac δ_x function from electrostatics, or other physics. But, what is the precise mathematics behind this delta function?

(10) In fact, even mathematicians can pull out tricks based on δ_x. For instance they have a theory where the basic step function is differentiable, with derivative δ_0.

(11) Still talking classical analysis, you might know from Fourier transform theory that functions can be of type L^1, L^2, L^∞ and so on, and clarifying all this matters.

(12) As even worse classical analysis, arbitrary functions can be non-integrable at all, at least in theory. And with these being non-negligible in quantum mechanics.

(13) In fact, talking worse things that can happen, these are wrong formulae, obtained via correct proofs. So, make sure that you know as much theory as possible.

(14) How to construct random numbers? With this being certainly a question of interest if you are an engineer, doing computer simulations for everything.

(15) Also at the concrete level, what about multivariate logic? We all function on a "yes, no, maybe" system, can we have some mathematics going, out of that?

(16) And with the remark that, in relation with what has been said before in relation with ∞, these latter questions can be serious business, in advanced physics.

In the hope that I convinced you that learning some more analysis would be a great investment, and the present book will be here for that. The above questions all fall under the banner "measure theory", and we will discuss here measure theory, with an introduction to the subject, and answers, ranging from partial to full, to the above questions.

This book was concieved as a complement to my calculus book [7], or to any other modern calculus book, typically weak on measure theory. Of course, you can learn too calculus directly from here, but that would be quite formal, quick, and painful. And finally, if you're brave enough for learning full calculus directly, the old way, with measure theory included, go with some Cold War books, such as those of Rudin [74], [75].

Many thanks go to my students, who had to endure several beta versions of the material presented here, once for instance, long ago, with a measure theory class featuring a full proof for the Zorn Lemma. Thanks as well go to my cats, who seem to deal with everything, successfully, just by using fast binary logic. But we are not all that fast.

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No never no more

CHAPTER 1

Probability zero

1a. Probability zero

Welcome to measure theory. What we will be doing in this book, namely measure theory and function spaces, is standard advanced undergraduate mathematics, sometimes erring on the graduate side, and can be best described as being a mixture of "advanced calculus" and "rigorous calculus". That is, you are supposed to know reasonably well basic calculus, in one and several variables, learned from one of the many modern calculus books available, such as those of Lax and Terrell [65], [66], or why not mine [7], and looking for a continuation of that, more advanced of course, but more rigorous too.

I can hear you thinking "why bothering with more rigorous calculus, what I want to learn is more advanced calculus, that is what looks the most useful". Good point, and if you are really hardline about this, I can only recommend you, as a continuation of your calculus experience, to start reading some physics books. You have here the famous course of Feynman [32], or the lovely books of Griffiths [42], [43], [44], or those of Weinberg [93], [94], if you already know some physics. So go with them, for advanced calculus, physicists know best, that is their job, to come up via advanced calculus to all sorts of amazing formulae, and you will certainly learn many interesting things from them.

This being said, even if you are genuinely interested in physics, a bit of rigor, that you will not necessarily learn from Feynman, Griffiths or Weinberg, can help. To make my point, let me formulate the following question, which is something very natural:

QUESTION 1.1. *The probability for a randomly picked $x\in\mathbb R$ to be $x=1$ is obviously $P=0$. But what does this mean, mathematically speaking?*

And I hope that you agree with me, this is a very good question, for anyone having a bit of calculus background, and looking to learn more, regardless of precise goals. For instance, don't expect to understand any sort of truly advanced physics, which quite often relies on probabilistic techniques, without having such things perfectly understood.

1b. Normal laws

So, let us examine our question. And surprise here, this is something quite tricky, even if you have some reasonable probability background. Indeed, the uniform measure on $\mathbb{R}$, that you first think of, when it comes to randomly pick numbers $x \in \mathbb{R}$, has mass ∞, and so is not a probability measure. Damn. So our formula in Question 1.1, namely $P(x = 1) = 0$, while being something undoubtedly true, and it would be foolish to deny this, and think otherwise, looks more like a "social science fact", coming from our human methods of picking random numbers $x \in \mathbb{R}$, than something mathematical.

So, in order to answer our question, here we are into some deep thinking, both in social science, and in abstract mathematics. And the situation here is as follows:

(1) Humanities first. If you ask people on the street for random numbers $x \in \mathbb{R}$, it is most likely that most of them will choose small numbers, say $x \in [-1000, 1000]$, and basically no one will think at choices of type $x = 872, 563, 267, 127, 167, 176$.

(2) Of course, there is a similar phenomenon involving decimals too, but let us ignore that. Our finding is that P should come from some sort of bell-shaped curve around the origin 0, and that can only be, either by decree, or by the CLT, a normal law.

(3) Regarding now the mathematics, this is quite clear. Once we have our normal law, the probability $P(x = 1)$ that we want to compute is obtained by integrating the density of this law between 1 and 1. And we obtain of course 0, no matter what $t > 0$ is.

As a conclusion to all this, mystery solved, our answer to Question 1.1 being:

ANSWER 1.2. *The probability for a randomly picked $x \in \mathbb{R}$ to be $x = 1$ is*

$$P = \frac{1}{\sqrt{2\pi t}} \int_1^1 e^{-x^2/2t} dx$$

which means, by computing the integral, $P = 0$.

Well done, but that wasn't trivial, wasn't it. Which leads us now into a bit of a philosophy scare, so if dealing with such a simple question like Question 1.1 was non-trivial, what about more complicated questions of the same type, based on the obvious fact that "everything that happens in the real life, happens with probability $P = 0$".

Very concerning all this, hope you agree with me. In short, we have to review now all our basic probability knowledge, by putting that on a fully rigorous basis, in order to never ever get annoyed by such $P = 0$ things, and have these fully understood, in a fully rigorous way. And good news, this is what we will be doing in this book, by systematically developing measure theory. So, let us record this finding, as follows:

FACT 1.3. *Everything that happens in the real life, happens with probability $P = 0$, and in order to clarify this, we must develop measure theory.*

Which is very nice, at least we know one thing, and have now a plan.

1c. Poisson laws

Before developing measure theory, however, let us further explore Question 1.1, with the aim of bringing even more mess to what we have, namely Answer 1.2. Indeed, there are in fact some bugs with that Answer 1.2, our rationale being as follows:

(1) If you ask people on the street for random numbers $x \in \mathbb{R}$, frankly, you will certainly have some answering $x = 1$. Thus, we have in fact $P(x = 1) > 0$.

(2) Looking back at our reasoning leading to Answer 1.2, the modelling error there was by ignoring what happens to the decimals. People don't like them, for sure.

(3) Thus, by simplifying, we must rather understand what happens when picking random integers $x \in \mathbb{Z}$. Or even better, by picking numbers $x \in \mathbb{N}$.

In view of this, let us formulate a more realistic version of Question 1.1, as follows:

QUESTION 1.4. *The probability for a randomly picked $x \in \mathbb{N}$ to be $x = 1$ is obviously $P > 0$. But what does this mean, mathematically speaking, and what exactly is P?*

So, here we go again with modelling questions, belonging to both social science, and to mathematics. To be more precise, from a pure mathematics viewpoint, the first measure that comes to mind, namely the uniform measure on $\mathbb{N}$, has mass ∞, and so is not a probability measure. Thus, wrong way, and we must throw in a bit of social science.

But social science tells us that, exactly as before in the continuous setting, people prefer small numbers, say $x \in [0, 1000]$, to beasts such as $x = 872, 563, 267, 127, 167, 176$. Thus, we can only expect to have a density decreasing from 0 to ∞, and a bit of probabilistic thinking, say based on the PLT, tells us that what we need is a Poisson law.

Time now to answer Question 1.4. Based on the above, we can formulate:

ANSWER 1.5. *The probability for a randomly picked $x \in \mathbb{N}$ to be $x = 1$ is*

$$P = \frac{1}{e^t} \int_1^1 \sum_{k=0}^{\infty} \frac{t^k \delta_k}{k!}$$

which means, by computing the integral, $P = t/e^t$.

Of course, this was something a bit mathematical, with the $P > 0$ phenomenon being definitely explained, but with the actual figure $P = t/e^t$ still depending on a parameter $t > 0$, whose realistic value remains to be computed, based on social science considerations. But we will not get here into this latter question, which looks quite difficult.

As a conclusion to this, let us complement Fact 1.3 with:

FACT 1.6. *Some things happen in the real life with probability $P > 0$, but even these need, mathematically speaking, measure theory, for dealing with the Dirac masses.*

Long story short, the conclusion is clear, we must develop measure theory.

1d. Conclusions

We have seen in this chapter that, no matter what we want to do advanced probability, or even basic probability, to be fully honest, we must develop measure theory first. Before that, however, it is better to talk a bit about the foundations of mathematics itself, and about phenomena like $0 \times \infty = 1$ which can appear, and are behind measures and probability, but in the simpler context of sets and logic. We will do this in the remainder of Part I, and then afterwards in Part II we will go for measure theory.

1e. Exercises

Exercises:

EXERCISE 1.7.

EXERCISE 1.8.

EXERCISE 1.9.

EXERCISE 1.10.

EXERCISE 1.11.

EXERCISE 1.12.

Bonus exercise.

CHAPTER 2

Numbers, sets

2a. Real numbers

As explained in the previous chapter, in order to reach to some form of truly advanced calculus, we must first develop measure theory. But this requires first rethinking various mathematical foundations, so let us start with the basics, namely numbers and sets.

To start with, the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ was invented by God. Or at least this is the commonly accepted view on this. Let us mention too, as alternative opinions, that according to certain mathematicians, God only invented the empty set $\emptyset$, and everything including $\mathbb{N}$ naturally came afterwards. Also, according to certain physicists, God only invented the Big Bang, and everything including $\mathbb{N}$ naturally came afterwards. But shall we trust all this modern science, better stick with good old traditional religion.

Once you have $\mathbb{N}$, solving $a + b = c$ naturally leads you to the set of all integers $\mathbb{Z}$. Then, once you have $\mathbb{Z}$, solving $ab = c$ naturally leads you to the set of all rationals $\mathbb{Q}$. So, this will be our starting point, the set of rationals $\mathbb{Q}$, defined as follows:

Definition 2.1. *The rational numbers are the quotients $r = a/b$, with $a, b \in \mathbb{Z}$ and $b \neq 0$, identified according to the usual rule for quotients, namely:*

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

These quotients add, multiply and invert according to the following formulae:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad , \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad , \quad \left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

We denote the set of rational numbers by $\mathbb{Q}$, standing for "quotients".

In more advanced mathematical terms, the above operations, namely sum, product and inversion, tells us that $\mathbb{Q}$ is a field. Thus, what we did so far, with our philosophical discussion, was to construct the simplest possible field. Let us record this as follows:

Fact 2.2. *$\mathbb{Q}$ is the smallest field, containing $\{0, 1\}$.*

Observe that we recorded this as a fact, rather than a theorem, and also that we did not include in our field axioms the conditions $0, 1 \in F$. This is for full rigor in our approach, in order to deal, whenever needed, say for some future considerations regarding the Big Bang, with the fields present before the first God intervention, such as $\emptyset$.

Many things can be done with $\mathbb{Q}$, but one thing that fails is solving $x^2 = 2$. Indeed, assuming that $r = a/b$ with $a, b \in \mathbb{N}$ prime to each other satisfies $r^2 = 2$, we have $a^2 = 2b^2$, so $a \in 2\mathbb{N}$. But by using again $a^2 = 2b^2$ we obtain $b \in 2\mathbb{N}$, contradiction.

In short, in order to advance with our mathematics, we are now in need to introduce the field of real numbers $\mathbb{R}$. You would probably say that this is very easy, via decimal writing, like everyone does, but before doing that, let me ask you a few questions:

(1) Honestly, do you really like the addition of real numbers, using the decimal form? Let us take, as example, the following computation:

$$12.456,783,872$$

$$+\ 27.536,678,377$$

This computation can surely be done, but, annoyingly, it must be done from right to left, instead of left to right, as we would prefer. I mean, personally I would be most interested in knowing first what happens at left, if the integer part is 39 or 40, but go do all the computation, starting from the right, in order to figure out that. In short, my feeling is that this addition algorithm, while certainly good, is a bit deceiving.

(2) What about multiplication. Here things become even more complicated, imagine for instance that Mars attacks, with δ-rays, which are something unknown to us, and 100,000 stronger than γ-rays, and which have paralyzed all our electronics, and that in order to protect Planet Earth, you must do the following multiplication by hand:

$$12.456,783,872$$

$$\times\ 27.536,678,377$$

That does not look very inviting, doesn't it. In short, as before with the addition, there is a bit of a bug with all this, the algorithm being too complicated.

(3) Getting now to the problem that we were interested in, what about extracting square roots, and more specifically the square root of 2. So, problem for us:

$$\sqrt{2} = ?$$

Normally there should be here an algorithm too, and I even used to know this when younger, but I forgot. So, again we are running here into troubles, it is not even clear that $\sqrt{2}$ exists, in decimal form, and don't count on such things, if Mars attacks.

Quite concerning all this. Let us record these findings as follows:

FACT 2.3. *The real numbers $x \in \mathbb{R}$ can be certainly introduced via their decimal form, but with this, the field structure of $\mathbb{R}$ remains something quite unclear.*

It looks like we are a bit stuck. However, there is a clever solution to this, invented by Dedekind. His definition for the real numbers is as follows:

DEFINITION 2.4. *The real numbers $x \in \mathbb{R}$ are formal cuts in the set of rationals,*

$$\mathbb{Q} = \mathbb{Q}_{\leq x} \sqcup \mathbb{Q}_{>x}$$

with such a cut being by definition subject to the following condition:

$$p \in \mathbb{Q}_{\leq x}\ ,\ q \in \mathbb{Q}_{>x} \implies p < q$$

These numbers add and multiply by adding and multiplying the corresponding cuts.

This might look quite original, but believe me, there is some genius behind this definition. As a first observation, we have an inclusion $\mathbb{Q} \subset \mathbb{R}$, obtained by identifying each rational number $r \in \mathbb{Q}$ with the obvious cut that it produces, namely:

$$\mathbb{Q}_{\leq r} = \left\{p \in \mathbb{Q}\Big| p \leq r\right\} \quad , \quad \mathbb{Q}_{>r} = \left\{q \in \mathbb{Q}\Big| q > r\right\}$$

As a second observation, the addition and multiplication of real numbers, obtained by adding and multiplying the corresponding cuts, in the obvious way, is something very simple. To be more precise, in what regards the addition, the formula is as follows:

$$\mathbb{Q}_{\leq x+y} = \mathbb{Q}_{\leq x} + \mathbb{Q}_{\leq y}$$

As for the multiplication, the formula here is similar, namely $\mathbb{Q}_{\leq xy} = \mathbb{Q}_{\leq x}\mathbb{Q}_{\leq y}$, up to some mess with positives and negatives, which is quite easy to untangle, and with this being a good exercise. We can also talk about order between real numbers, as follows:

$$x \leq y \iff \mathbb{Q}_{\leq x} \subset \mathbb{Q}_{\leq y}$$

But let us perhaps leave more abstractions for later, and go back to more concrete things. As a first success of our theory, we can formulate the following theorem:

THEOREM 2.5. *The equation $x^2 = 2$ has two solutions over the real numbers, namely the positive solution, denoted $\sqrt{2}$, and its negative counterpart, which is $-\sqrt{2}$.*

PROOF. By using $x \to -x$, it is enough to prove that $x^2 = 2$ has exactly one positive solution $\sqrt{2}$. But this is clear, because $\sqrt{2}$ can only come from the following cut:

$$\mathbb{Q}_{\leq\sqrt{2}} = \mathbb{Q}_- \bigsqcup \left\{p \in \mathbb{Q}_+\Big| p^2 \leq 2\right\} \quad , \quad \mathbb{Q}_{>\sqrt{2}} = \left\{q \in \mathbb{Q}_+\Big| q^2 > 2\right\}$$

Thus, we are led to the conclusion in the statement. □

More generally, the same method works in order to extract the square root $\sqrt{r}$ of any number $r \in \mathbb{Q}_+$, or even of any number $r \in \mathbb{R}_+$, and we have the following result:

THEOREM 2.6. *The solutions of* $ax^2+bx+c=0$ *with* $a,b,c\in\mathbb{R}$ *are*

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

provided that $b^2-4ac\geq 0$. *In the case* $b^2-4ac<0$, *there are no solutions.*

PROOF. We can write our equation in the following way:

$$\begin{aligned}
ax^2+bx+c=0 &\iff x^2+\frac{b}{a}x+\frac{c}{a}=0\\
&\iff \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0\\
&\iff \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}\\
&\iff x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Summarizing, we have a nice abstract definition for the real numbers, that we can certainly do some math with. As a first result, which is something very useful, and puts as back into real life, and science and engineering technology, we have:

THEOREM 2.7. *The real numbers* $x\in\mathbb{R}$ *can be written in decimal form,*

$$x=\pm a_1\ldots a_n.b_1b_2b_3\ldots\ldots$$

with $a_i,b_i\in\{0,1,\ldots,9\}$, *with the convention* $\ldots b999\ldots=\ldots(b+1)000\ldots$

PROOF. This is something non-trivial, even for the rationals $x\in\mathbb{Q}$ themselves, which require some work in order to be put in decimal form, the idea being as follows:

(1) First of all, our precise claim is that any $x\in\mathbb{R}$ can be written in the form in the statement, with the integer $\pm a_1\ldots a_n$ and then each of the digits $b_1,b_2,b_3,\ldots$ providing the best approximation of x, at that stage of the approximation.

(2) Moreover, we have a second claim as well, namely that any expression of type $x=\pm a_1\ldots a_n.b_1b_2b_3\ldots\ldots$ corresponds to a real number $x\in\mathbb{R}$, and that with the convention $\ldots b999\ldots=\ldots(b+1)000\ldots$, the correspondence is bijective.

(3) In order to prove now these two assertions, our first claim is that we can restrict the attention to the case $x\in[0,1)$, and with this meaning of course $0\leq x<1$, with respect to the order relation for the reals discussed in the above.

(4) Getting started now, let $x\in\mathbb{R}$, coming from a cut $\mathbb{Q}=\mathbb{Q}_{\leq x}\sqcup\mathbb{Q}_{>x}$. Since the set $\mathbb{Q}_{\leq x}\cap\mathbb{Z}$ consists of integers, and is bounded from above by any element $q\in\mathbb{Q}_{>x}$ of your

choice, this set has a maximal element, that we can denote $[x]$:

$$[x] = \max\left(\mathbb{Q}_{\leq x} \cap \mathbb{Z}\right)$$

It follows from definitions that $[x]$ has the usual properties of the integer part, namely:

$$[x] \leq x < [x] + 1$$

Thus we have $x = [x] + y$ with $[x] \in \mathbb{Z}$ and $y \in [0,1)$, and getting back now to what we want to prove, namely (1,2) above, it is clear that it is enough to prove these assertions for the remainder $y \in [0,1)$. Thus, we have proved (3), and we can assume $x \in [0,1)$.

(5) So, assume $x \in [0,1)$. We are first looking for a best approximation from below of type $0.b_1$, with $b_1 \in \{0,\ldots,9\}$, and it is clear that such an approximation exists, simply by comparing x with the numbers $0.0, 0.1, \ldots, 0.9$. Thus, we have our first digit b_1, and then we can construct the second digit b_2 as well, by comparing x with the numbers $0.b_10, 0.b_11, \ldots, 0.b_19$. And so on, which finishes the proof of our claim (1).

(6) In order to prove now the remaining claim (2), let us restrict again the attention, as explained in (4), to the case $x \in [0,1)$. First, it is clear that any expression of type $x = 0.b_1b_2b_3\ldots$ defines a real number $x \in [0,1]$, simply by declaring that the corresponding cut $\mathbb{Q} = \mathbb{Q}_{\leq x} \sqcup \mathbb{Q}_{>x}$ comes from the following set, and its complement:

$$\mathbb{Q}_{\leq x} = \bigcup_{n \geq 1} \left\{ p \in \mathbb{Q} \Big| p \leq 0.b_1 \ldots b_n \right\}$$

(7) Thus, we have our correspondence between real numbers as cuts, and real numbers as decimal expressions, and we are left with the question of investigating the bijectivity of this correspondence. But here, the only bug that happens is that numbers of type $x = \ldots b999\ldots$, which produce reals $x \in \mathbb{R}$ via (6), do not come from reals $x \in \mathbb{R}$ via (5). So, in order to finish our proof, we must investigate such numbers.

(8) So, consider an expression of type $\ldots b999\ldots$ Going back to the construction in (6), we are led to the conclusion that we have the following equality:

$$\mathbb{Q}_{\leq \ldots b999\ldots} = \mathbb{Q}_{\leq \ldots (b+1)000\ldots}$$

Thus, at the level of the real numbers defined as cuts, we have:

$$\ldots b999\ldots = \ldots (b+1)000\ldots$$

But this solves our problem, because by identifying $\ldots b999\ldots = \ldots (b+1)000\ldots$ the bijectivity issue of our correspondence is fixed, and we are done. □

The above theorem was of course quite difficult, but this is how things are. Alternatively, we have the following third possible definition for the real numbers:

FACT 2.8. *The field of real numbers $\mathbb{R}$ can be defined as well as the analytic completion of $\mathbb{Q}$ with respect to the usual distance on the rationals, namely*

$$d\left(\frac{a}{b},\frac{c}{d}\right)=\left|\frac{a}{b}-\frac{c}{d}\right|$$

and with the operations on $\mathbb{R}$ coming from those on $\mathbb{Q}$, via Cauchy sequences. This is compatible with both the Dedekind cuts, and the decimal writing.

Here we assume of course some knowledge of abstract analysis, with δ and ε, and exercise for you to clarify all this, and decide which definition of $\mathbb{R}$ you prefer.

2b. Equations, roots

Motivated by measure theory and ∞, let us see now what the passage $\mathbb{Q}\to\mathbb{R}$ teaches us. And here, surprise, we run right away into a very difficult question, namely:

QUESTION 2.9. *The reals are uncountable, so that we have*

$$|\mathbb{R}|>|\mathbb{Q}|=\infty$$

and is there anything in between.

As aready mentioned, this is a very difficult question in set theory, and we will comment more on this in chapter 3 below, when talking logic. However, before that, we can still have some fun with numbers, by looking for intermediate fields, as follows:

$$\mathbb{Q}\subset F\subset\mathbb{R}$$

Here we can talk for instance about fields like $\mathbb{Q}[\sqrt{2}]$, but all this is not very appropriate, right now, because the first question that comes to mind, in relation with such things, is solving $x^2=-1$. So, let us introduce the complex numbers $\mathbb{C}$. We have:

THEOREM 2.10. *Any polynomial $P\in\mathbb{C}[X]$ decomposes as*

$$P=c(X-a_1)\ldots(X-a_N)$$

with $c\in\mathbb{C}$ and with $a_1,\ldots,a_N\in\mathbb{C}$.

PROOF. The problem is that of proving that our polynomial has at least one root, because afterwards we can proceed by recurrence. We prove this by contradiction. So, assume that P has no roots, and pick a number $z\in\mathbb{C}$ where $|P|$ attains its minimum:

$$|P(z)|=\min_{x\in\mathbb{C}}|P(x)|>0$$

Since $Q(t)=P(z+t)-P(z)$ is a polynomial which vanishes at $t=0$, this polynomial must be of the form ct^k + higher terms, with $c\neq 0$, and with $k\geq 1$ being an integer. We obtain from this that, with $t\in\mathbb{C}$ small, we have the following estimate:

$$P(z+t)\simeq P(z)+ct^k$$

Now let us write $t = rw$, with $r > 0$ small, and with $|w| = 1$. Our estimate becomes:

$$P(z + rw) \simeq P(z) + cr^k w^k$$

Now recall that we have assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$\begin{aligned} |P(z + rw)| &\simeq |P(z) + cr^k w^k| \\ &= |P(z)|(1 - |c|r^k) \end{aligned}$$

Now by choosing $r > 0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(z + rw)| < |P(z)|$$

But this contradicts our definition of $z \in \mathbb{C}$, as a point where $|P|$ attains its minimum. Thus P has a root, and by recurrence it has N roots, as stated. □

As a concrete question, let us try to solve a degree 3 equation, $aX^3 + bX^2 + cX + d = 0$. By linear transformations we can always assume $a = 1, b = 0$, and then it is convenient to write $c = 3p, d = 2q$. Thus, our equation becomes $x^3 + 3px + 2q = 0$, and regarding such equations, we have the following famous result, due to Cardano:

Theorem 2.11. *For a normalized degree 3 equation, namely*

$$x^3 + 3px + 2q = 0$$

the discriminant is $\Delta = -108(p^3 + q^2)$. Assuming $p, q \in \mathbb{R}$ and $\Delta < 0$, the number

$$x = \sqrt[3]{-q + \sqrt{p^3 + q^2}} + \sqrt[3]{-q - \sqrt{p^3 + q^2}}$$

is a real solution of our equation.

Proof. The formula of Δ is clear from definitions, and with $108 = 4 \times 27$. Now with x as in the statement, by using $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$, we have:

$$\begin{aligned} x^3 &= \left(\sqrt[3]{-q + \sqrt{p^3 + q^2}} + \sqrt[3]{-q - \sqrt{p^3 + q^2}}\right)^3 \\ &= -2q + 3\sqrt[3]{-q + \sqrt{p^3 + q^2}} \cdot \sqrt[3]{-q - \sqrt{p^3 + q^2}} \cdot x \\ &= -2q + 3\sqrt[3]{q^2 - p^3 - q^2} \cdot x \\ &= -2q - 3px \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

There are many more things that can be said about degree 3 equations, along these lines, and we will certainly have an exercise about this, at the end of this chapter.

2c. Galois theory

With the above discussed, let us go back now to the question that we raised in the beginning of the previous section, but we forgot afterwards about it, namely finding the intermediate fields $\mathbb{Q} \subset F \subset \mathbb{R}$. Obviously, the good question is as follows:

QUESTION 2.12. *What are the intermediate fields* $\mathbb{Q} \subset F \subset \mathbb{C}$*?*

There is a lot of interesting theory that can be developed here, following Galois and others. Among others, we reach in this way to some answers regarding the equations of degree 4, and more. For more on all this, we recommend a number theory book.

Still speaking fields, it is quite remarkable that both $\mathbb{R}$ and $\mathbb{R}^2$ have field structures. This fails for $\mathbb{R}^3$, but then for $\mathbb{R}^4$ something, of rather physics flavor, can be done.

2d. Prime numbers

Getting back to the passage $\mathbb{Q} \to \mathbb{R}$, via norms and Cauchy sequences, what about other norms on $\mathbb{Q}$? This leads us into p-adic norms and numbers, which are quite fascinating objects, and are useful for our purposes here, improving our knowledge of ∞.

Finally, no discussion about the prime numbers would be complete without a word on the Riemann zeta function. And there is a lot of interesting analysis and theoretical physics behind, that we can try to get a bit, into. We will be back to all this later.

2e. Exercises

Exercises:

EXERCISE 2.13 (Erdős). *If a subset* $P \subset \mathbb{N}$ *satisfies* $\sum_{p \in P} \frac{1}{p} = \infty$*, we have arbitrarily long arithmetic progressions, inside* P*.*

EXERCISE 2.14.

EXERCISE 2.15.

EXERCISE 2.16.

EXERCISE 2.17.

EXERCISE 2.18.

Bonus exercise.

CHAPTER 3

Logic, machines

3a. Logic, ordinals

We learned many interesting things about numbers, in the previous chapter, certainly improving our knowledge of ∞, and all this new knowledge is golden, that will certainly help us later, when we will be lost into the mysteries of measure theory.

This being said, all this number theory material, mostly concerning fields F, does not help us in relation with one interesting problem that we met, namely the continuum hypothesis. Indeed, all our fields F are, by either definition, or by a trivial argument based on definitions, either isomorphic to $\mathbb{Q}$, or to $\mathbb{R}$, when viewed as sets.

So, bad news, time to leave number theory, known as Queen of mathematics, and focus on sets instead. To start with, we have the following simple fact:

PROPOSITION 3.1. *Given a set S, the set $P(S)$ of its parts $P \subset S$ is strictly bigger,*

$$|P(S)| > |S|$$

with this meaning that we have an embedding $\supset$, but not an embedding $\subset$.

PROOF. As a first observation, this is clear for the finite sets, $|S| = N < \infty$, because here we have $|P(S)| = 2^N > N$. Also, for $S = \mathbb{N}$ this is standard, say by using a diagonal procedure for the non-embedding, and in general, the proof is similar. □

By using the above result, we can construct beasts called ordinals, as follows:

DEFINITION 3.2. *We can construct big numbers called ordinals, which are all infinite, by starting with the smallest possible infinity, namely $\infty_1 = |\mathbb{N}|$, and setting*

$$\infty_k = 2^{\infty_{k-1}}$$

with the operation $N \to 2^N$ corresponding to $S \to P(S)$, at the set level.

Many interesting things can be said about ordinals. An interesting question is that of understanding their reciprocals $1/\infty_k$, which can be very useful in the context of advanced calculus. We will be back to this later, in Part IV of the present book.

3b. Zorn lemma

Getting now to more applied logic, one very useful tool is the Zorn lemma, allowing us to do transfinite recursion, for a variety of real-life situations. The statement and proof of this lemma, which are something non-trivial, are as follows:

LEMMA 3.3. *Zorn lemma, allowing us to do transfinite recursion.*

PROOF. This is something quite technical, the idea being as follows:

(1) As a first comment on this, I've been personally writing various math books since ages, on all disciplines, and never had any Lemma in any of my books, with this being the first one. And, ain't no joke with these mathematicians' lemmas.

(2) In practice, all this requires developing a bit more logic, and with all this being, after all, a quite beautiful and refreshing business. And, needless to say, having this proof fully worked out and understood is useful, with respect to our ∞ experience. □

We will see many applications of the Zorn lemma in this book, when doing measure theory and function spaces. As an advertisement, however, for its power, we have:

THEOREM 3.4. *Let H be a Hilbert space.*

(1) *Any algebraic basis of this space $\{f_i\}_{i\in I}$ can be turned into an orthonormal basis $\{e_i\}_{i\in I}$, by using the Gram-Schmidt procedure.*
(2) *Thus, H has an orthonormal basis, and so we have $H \simeq l^2(I)$, with I being the indexing set for this orthonormal basis.*

PROOF. All this is standard by Gram-Schmidt, the idea being as follows:

(1) First of all, in finite dimensions an orthonormal basis $\{e_i\}_{i\in I}$ is by definition a usual algebraic basis, satisfying $< e_i, e_j >= \delta_{ij}$. But the existence of such a basis follows by applying the Gram-Schmidt procedure to any algebraic basis $\{f_i\}_{i\in I}$, as claimed.

(2) In infinite dimensions, a first issue comes from the fact that the standard basis $\{\delta_i\}_{i\in\mathbb{N}}$ of the space $l^2(\mathbb{N})$ is not an algebraic basis in the usual sense, with the finite linear combinations of the functions δ_i producing only a dense subspace of $l^2(\mathbb{N})$, that of the functions having finite support. Thus, we must fine-tune our definition of "basis".

(3) But this can be done in two ways, by saying that $\{f_i\}_{i\in I}$ is a basis of H when the functions f_i are linearly independent, and when either the finite linear combinations of these functions f_i form a dense subspace of H, or the linear combinations with $l^2(I)$ coefficients of these functions f_i form the whole H. For orthogonal bases $\{e_i\}_{i\in I}$ these definitions are equivalent, and in any case, our statement makes now sense.

(4) Regarding now the proof of our result, this time in infinite dimensions, this follows again from Gram-Schmidt, exactly as in the finite dimensional case, but by using this time the Zorn lemma, in order to correctly do the recurrence. □

The above result is something quite subtle, even after forgetting its proof, and as a first application of it, we can now formulate the following key definition:

DEFINITION 3.5. *A Hilbert space H is called separable when the following equivalent conditions are satisfied:*

(1) *H has a countable algebraic basis $\{f_i\}_{i\in\mathbb{N}}$.*
(2) *H has a countable orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$.*
(3) *We have $H\simeq l^2(\mathbb{N})$, isomorphism of Hilbert spaces.*

In what follows we will be mainly interested in the separable Hilbert spaces, where most of the questions coming from quantum physics take place. In view of the above, the following philosophical question appears: why not simply talking about $l^2(\mathbb{N})$?

In answer to this, we cannot really do so, because many of the separable spaces that we are interested in appear as spaces of functions, and such spaces do not necessarily have a very simple or explicit orthonormal basis, as shown by the following result:

THEOREM 3.6. *The Hilbert space $H=L^2[0,1]$ is separable, having as orthonormal basis the orthonormalized version of the algebraic basis $f_n=x^n$ with $n\in\mathbb{N}$.*

PROOF. This follows from the Weierstrass theorem, which provides us with the basis $f_n=x^n$, which can be orthogonalized by using the Gram-Schmidt procedure, as explained in Theorem 3.4. Working out the details here is actually an excellent exercise. □

As a conclusion to all this, we are interested in 1 space, namely the unique separable Hilbert space H, but due to various technical reasons, it is often better to forget that we have $H=l^2(\mathbb{N})$, and say instead that we have $H=L^2(X)$, with X being a separable measured space, or simply say that H is an abstract separable Hilbert space.

There is a relation here with physics as well, in relation with the Schrödinger space $H=L^2(\mathbb{R}^3)$ of wave functions of the electron, whose separability is something that you have to struggle with. More on this later, when talking quantum mechanics.

3c. Life and games

Life and games, in relation with logic. An interesting example here is Conway's Game of Life. There are also many games in physics, in relation with lattice models.

3d. Intelligence, machines

Intelligence, machines. Turing and others. We will be back to this.

3e. Exercises

Exercises:

Exercise 3.7.

Exercise 3.8.

Exercise 3.9.

Exercise 3.10.

Exercise 3.11.

Exercise 3.12.

Bonus exercise.

CHAPTER 4

Random numbers

4a. Monte Carlo

We have built so far a decent knowledge of ∞, and all this will be useful later, when developing measure theory, and then talking about function spaces. Before that, however, one last thing. All that we have been talking about so far, while certainly beautiful and potentially useful, was a bit abstract. So, what about true applications, say for the scientist or engineer using a computer for everything heavy mathematics?

So, let us talk now about integrating functions, which is a favorite pastime of our friends, the computers. As definition for the integral, which is something simple and straightforward, that our computer friend will certainly appreciate, we have:

Theorem 4.1. *We have the Riemann integration formula,*

$$\int_a^b f(x)dx=(b-a)\times\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N f\left(a+\frac{b-a}{N}\cdot k\right)$$

which can serve as a definition for the integral.

Proof. This is standard, by drawing rectangles. We have indeed the following formula, which can stand as a definition for the signed area below the graph of f:

$$\int_a^b f(x)dx=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N\frac{b-a}{N}\cdot f\left(a+\frac{b-a}{N}\cdot k\right)$$

Thus, we are led to the formula in the statement. □

Observe that the above formula suggests that $\int_a^b f(x)dx$ is the length of the interval $[a,b]$, namely $b-a$, times the average of f on the interval $[a,b]$. Thinking a bit, this is indeed something true, with no need for Riemann sums, coming directly from definitions, because area means side times average height. Thus, we can also formulate:

Proposition 4.2. *The integral of a function $f:[a,b]\to\mathbb R$ is given by*

$$\int_a^b f(x)dx=(b-a)\times A(f)$$

where $A(f)$ is the average of f over the interval $[a,b]$.

PROOF. As explained above, this is clear from definitions, via some geometric thinking. Alternatively, this is something which certainly comes from Theorem 4.1. □

Going ahead with more interpretations of the integral, we have:

THEOREM 4.3. *We have the Monte Carlo integration formula,*

$$\int_a^b f(x)dx = (b-a) \times \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N f(x_i)$$

with $x_1, \ldots, x_N \in [a,b]$ being random.

PROOF. We recall from Theorem 4.1 that the idea is to use a formula as follows, with the points $x_1, \ldots, x_N \in [a,b]$ being uniformly distributed:

$$\int_a^b f(x)dx = (b-a) \times \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^N f(x_i)$$

But this works as well when the points $x_1, \ldots, x_N \in [a,b]$ are randomly distributed, for somewhat obvious reasons, and this gives the result. □

Observe that the Monte Carlo integration works better than Riemann integration, for instance when trying to improve the estimate, via $N \to N+1$. Indeed, in the context of Riemann integration, assume that we managed to find an estimate as follows, which in practice requires computing N values of our function f, and making their average:

$$\int_a^b f(x)dx \simeq \frac{b-a}{N} \sum_{k=1}^N f\left(a + \frac{b-a}{N} \cdot k\right)$$

In order to improve this estimate, any extra computed value of our function $f(y)$ will be unuseful. For improving our formula, what we need are N extra values of our function, $f(y_1), \ldots, f(y_N)$, with the points $y_1, \ldots, y_N$ being precisely the midpoints of the previous division of $[a,b]$, so that we can write an improvement of our formula, as follows:

$$\int_a^b f(x)dx \simeq \frac{b-a}{2N} \sum_{k=1}^{2N} f\left(a + \frac{b-a}{2N} \cdot k\right)$$

With Monte Carlo, things are far more flexible. Assume indeed that we managed to find an estimate as follows, which again requires computing N values of our function:

$$\int_a^b f(x)dx \simeq \frac{b-a}{N} \sum_{k=1}^N f(x_i)$$

Now if we want to improve this, any extra computed value of our function $f(y)$ will be helpful, because we can set $x_{n+1}=y$, and improve our estimate as follows:

$$\int_a^b f(x)dx\simeq\frac{b-a}{N+1}\sum_{k=1}^{N+1}f(x_i)$$

And isn't this potentially useful, and powerful, when thinking at practically computing integrals, either by hand, or by using a computer. Let us record this finding as follows:

CONCLUSION 4.4. *Monte Carlo integration works better than Riemann integration, when it comes to computing as usual, by estimating, and refining the estimate.*

As another interesting feature of Monte Carlo integration, this works much better than Riemann integration, for functions having various symmetries, because Riemann integration can get "fooled" by these symmetries, while Monte Carlo remains strong.

As an example for this phenomeon, chosen to be quite drastic, let us attempt to integrate, via both Riemann and Monte Carlo, the following function $f:[0,\pi]\to\mathbb R$:

$$f(x)=\Big|\sin(120x)\Big|$$

The first few Riemann sums for this function are then as follows:

$$I_2(f)=\frac{\pi}{2}(|\sin 0|+|\sin 60\pi|)=0$$

$$I_3(f)=\frac{\pi}{3}(|\sin 0|+|\sin 40\pi|+|\sin 80\pi|)=0$$

$$I_4(f)=\frac{\pi}{4}(|\sin 0|+|\sin 30\pi|+|\sin 60\pi|+|\sin 90\pi|)=0$$

$$I_5(f)=\frac{\pi}{5}(|\sin 0|+|\sin 24\pi|+|\sin 48\pi|+|\sin 72\pi|+|\sin 96\pi|)=0$$

$$I_6(f)=\frac{\pi}{6}(|\sin 0|+|\sin 20\pi|+|\sin 40\pi|+|\sin 60\pi|+|\sin 80\pi|+|\sin 100\pi|)=0$$

$$\vdots$$

Based on this evidence, we will conclude, obviously, that we have:

$$\int_0^\pi f(x)dx=0$$

With Monte Carlo, however, such things cannot happen. Indeed, since there are finitely many points $x\in[0,\pi]$ having the property $\sin(120x)=0$, a random point $x\in[0,\pi]$ will have the property $|\sin(120x)|>0$, so Monte Carlo will give, at any $N\in\mathbb N$:

$$\int_0^\pi f(x)dx\simeq\frac{b-a}{N}\sum_{k=1}^{N}f(x_i)>0$$

Again, this is something interesting, when practically computing integrals, either by hand, or by using a computer. So, let us record, as a complement to Conclusion 4.4:

CONCLUSION 4.5. *Monte Carlo integration is smarter than Riemann integration, because the symmetries of the function can fool Riemann, but not Monte Carlo.*

All this is good to know, and we will be back to this later, when knowing more about random numbers, whose production and manipulation is in fact something quite tricky. To be more precise, the problem usually comes from the fact that the various algorithms producing random numbers have, as any piece of mathematics has, some "symmetries", that you don't want to see on your output numbers. But more on this later.

4b. Random numbers

Random numbers, needed for Monte Carlo, as explained in the above. All this is in fact quite subtle, and we get in this way back into philosophical questions from chapter 1, number theory questions from chapter 2, and logic questions from chapter 3.

4c. Coding, cryptography

Coding and cryptography, now that we talked about big and random numbers.

4d. Economy and war

Economy, and subsequent war, with proof for that, to end this chapter in beauty.

4e. Exercises

Exercises:

EXERCISE 4.6.

EXERCISE 4.7.

EXERCISE 4.8.

EXERCISE 4.9.

EXERCISE 4.10.

EXERCISE 4.11.

Bonus exercise.

Part II

Measure theory

A license to love
Insurance to hold
Melts all your memories
And change into gold

CHAPTER 5

Measure theory

5a. Spaces, topology

Welcome to measure theory, take two. We have seen in Part I a lot of interesting mathematics, coming as a complement to the basic calculus that you surely know, making us more familiar with ∞, and often pushing us into developing measure theory.

So, this is what we will do here, in this Part II, developing mesure theory, in a very standard, rigorous and mathematical way. Then in Part III we will talk about function spaces, by using measure theory tools, again in a very standard way, and with this being traditionally part of "measure theory" too. In short, expect now 200 pages of standard mathematics, which will be something useful, improving your knowledge of calculus.

Before starting, two comments, depending on what exact reader you might be:

(1) In case you are here strictly for abstract analysis, namely measure theory and function spaces, this whole middle of the book, Parts II-III, is for you. Formally speaking, we will not really need Part I, except for the Zorn lemma proved in chapter 3, so you can start your reading here, and things will be fine. This being said, we will be a bit abstract sometimes, and never ask me why I'm doing this, or that, all sorts of abstract things: the answer to these questions often comes from what we did in Part I.

(2) On the opposite, if you read and liked Part I, and sort of got addicted to that, study of ∞ in all its mathematical forms, you might say, that was nice, so bad it is over, and am I ready now for 200 pages of subsequent abstractions. Well, what we will be doing here will be very useful, both for calculus, and for the philosophical study of ∞. And in what regards ∞, do not worry, we will be back to it in Part IV, that time with even more exciting considerations, mixing measure theory, math and physics.

In short, you got it, this book is concieved like a sandwich, with philosophical buns and meat content, so time now to get into the meat. Getting started now, before talking measures, we need spaces. We will use here a very general definition, as folllows:

DEFINITION 5.1. *A topological space is a set X, along with a collection of subsets $U \subset X$ called open sets, satisfying what we can expect from the open sets.*

In practice, for analysis, we will mostly need the case where X is a metric space, with the remark however that abstract topological spaces as above are something quite interesting, for instance in relation with the number theory considerations from chapter 2. So, getting now to metric spaces, the definition that we will need here is as follows:

Definition 5.2. *Let X be a metric space.*

(1) *The open balls are the sets $B_x(r) = \{y \in X | d(x,y) < r\}$.*
(2) *The closed balls are the sets $\bar{B}_x(r) = \{y \in X | d(x,y) \leq r\}$.*
(3) *$E \subset X$ is called open if for any $x \in E$ we have a ball $B_x(r) \subset E$.*
(4) *$E \subset X$ is called closed if its complement $E^c \subset X$ is open.*

At the level of examples, you can quickly convince yourself, by working out a few of them, that our notions above coincide with the usual ones, that we know well, in the case $X = \mathbb{R}, \mathbb{C}$. We will be back to this later, with some general results in this sense, confirming all this. But for the moment, let us work out the basics. We first have the following result, clarifying some terminology issues from Definition 5.2:

Proposition 5.3. *The open balls are open, and the closed balls are closed.*

Proof. This might sound a bit as a joke, but it is not one, because this is the kind of thing that we have to check. Fortunately, all this is elementary, as follows:

(1) Given an open ball $B_x(r)$ and a point $y \in B_x(r)$, by using the triangle inequality we have $B_y(r') \subset B_x(r)$, with $r' = r - d(x,y)$. Thus, $B_x(r)$ is indeed open.

(2) Given a closed ball $\bar{B}_x(r)$ and a point $y \in B_x(r)^c$, by using the triangle inequality we have $B_y(r') \subset B_x(r)^c$, with $r' = d(x,y) - r$. Thus, $\bar{B}_x(r)$ is indeed closed. □

Here is now something more interesting, making the link with our intuitive understanding of the notion of closedness, coming from our experience so far with analysis:

Proposition 5.4. *For a set $E \in X$, the following are equivalent:*

(1) *E is closed in our sense, meaning that E^c is open.*
(2) *We have $x_n \to x, x_n \in E \implies x \in E$.*

Proof. We can prove this by double implication, as follows:

(1) $\implies$ (2) Assume by contradiction $x_n \to x, x_n \in E$ with $x \notin E$. Since we have $x \in E^c$, which is open, we can pick a ball $B_x(r) \subset E^c$. But this contradicts our convergence assumption $x_n \to x$, so we are done with this implication.

(2) $\implies$ (1) Assume by contradiction that E is not closed in our sense, meaning that E^c is not open. Thus, we can find $x \in E^c$ such that there is no ball $B_x(r) \subset E^c$. But with $r = 1/n$ this provides us with a point $x_n \in B_x(1/n) \cap E$, and since we have $x_n \to x$, this contradicts our assumption (2). Thus, we are done with this implication too. □

Here is a now key result, making the link with the axioms in Definition 5.1:

THEOREM 5.5. *Let X be a metric space.*

(1) *If E_i are open, then $\cup_i E_i$ is open.*
(2) *If F_i are closed, then $\cap_i F_i$ is closed.*
(3) *If $E_1, \ldots, E_n$ are open, then $\cap_i E_i$ is open.*
(4) *If $F_1, \ldots, F_n$ are closed, then $\cup_i F_i$ is closed.*

Moreover, both (3) and (4) can fail for infinite intersections and unions.

PROOF. We have several things to be proved, the idea being as follows:

(1) This is clear from definitions, because any point $x \in \cup_i E_i$ must satisfy $x \in E_i$ for some i, and so has a ball around it belonging to E_i, and so to $\cup_i E_i$.

(2) This follows from (1), by using the following well-known set theory formula:

$$\left(\bigcup_i E_i\right)^c = \bigcap_i E_i^c$$

(3) Given an arbitrary point $x \in \cap_i E_i$, we have $x \in E_i$ for any i, and so we have a ball $B_x(r_i) \subset E_i$ for any i. Now with this in hand, let us set:

$$B = B_x(r_1) \cap \ldots \cap B_x(r_n)$$

As a first observation, this is a ball around x, $B = B_x(r)$, of radius given by:

$$r = \min(r_1, \ldots, r_n)$$

But this ball belongs to all the E_i, and so belongs to their intersection $\cap_i E_i$. We conclude that the intersection $\cap_i E_i$ is open, as desired.

(4) This follows from (3), by using the following well-known set theory formula:

$$\left(\bigcap_i E_i\right)^c = \bigcup_i E_i^c$$

(5) Finally, in what regards the counterexamples at the end, we will leave their construction, which is something very elementary, as an instructive exercise. □

Finally, still in relation with open and closed sets, we have as well:

DEFINITION 5.6. *Let X be a metric space, and $E \subset X$ be a subset.*

(1) *The interior $E^\circ \subset E$ is the set of points $x \in E$ which admit around them open balls $B_x(r) \subset E$.*
(2) *The closure $E \subset \bar{E}$ is the set of points $x \in X$ which appear as limits of sequences $x_n \to x$, with $x \in E$.*

These notions are quite interesting, because they make sense for any set E. That is, when E is open, that is open and end of the story, and when E is closed, that is closed and end of the story. In general, however, a set $E \subset X$ is not open or closed, and what we can best do to it, in order to study with our tools, is to "squeeze" it, as follows:

$$E^\circ \subset E \subset \bar{E}$$

In practice now, in order to use the above notions, we need to know a number of things, including that fact that E open implies $E^\circ = E$, the fact that E closed implies $\bar{E} = E$, and many more such results, not to forget the fact that the closures of the open balls $B_r(x)$ are the closed balls $\bar{B}_x(r)$, clarifying an obvious notational issue which appears with respect to Definition 5.2. But all this can be done, and the useful statement here, summarizing all that we need to know about interiors and closures, is as follows:

THEOREM 5.7. *Let X be a metric space, and $E \subset X$ be a subset.*

1. *The interior $E^\circ \subset E$ is the biggest open set contained in E.*
2. *The closure $E \subset \bar{E}$ is the smallest closed set containing E.*

PROOF. We have several things to be proved, the idea being as follows:

(1) Let us first prove that the interior E° is open. For this purpose, pick $x \in E^\circ$. We know that we have a ball $B_x(r) \subset E$, and since this ball is open, it follows that we have $B_x(r) \subset E^\circ$. Thus, the interior E° is open, as claimed.

(2) Let us prove now that the closure $\bar{E}$ is closed. For this purpose, we will prove that the complement $\bar{E}^c$ is open. So, pick $x \in \bar{E}^c$. Then x cannot appear as a limit of a sequence $x_n \to x$ with $x_n \in E$, so we have a ball $B_x(r) \subset \bar{E}^c$, as desired.

(3) Finally, the maximality and minimality assertions regarding E° and $\bar{E}$ are both routine too, coming from definitions, and we will leave them as exercises. ☐

As an application of the theory developed above, and more specifically of the notion of closure from Definition 5.6, we can talk as well about density, as follows:

DEFINITION 5.8. *We say that a subset $E \subset X$ is dense when:*

$$\bar{E} = X$$

That is, any point of X must appear as a limit of points of E.

Obviously, this is something which is in tune with what we know so far from this book, and with the intuitive notion of density. As a basic example, we have $\bar{\mathbb{Q}} = \mathbb{R}$, that we know well from the beginning of this book, and more specifically, from chapter 2.

Moving ahead now, at a more subtle level, again in analogy with what we know about $X = \mathbb{R}, \mathbb{C}$, we can talk about compact sets, and about connected sets. Again things here are quite tricky, in the general metric space framework, actually substantially deviating from what we know, and we will do this in detail. Let us start with:

DEFINITION 5.9. *A set $K \subset X$ is called compact if any cover with open sets*

$$K \subset \bigcup_i E_i$$

has a finite subcover, $K \subset (E_{i_1} \cup \ldots \cup E_{i_n})$.

This definition might seem overly abstract, and perhaps even sound like a joke, but our claim is that this is the correct definition, and that there is no way of doing otherwise. The point indeed is that we have the following counterexample:

PROPOSITION 5.10. *Given an infinite set X with the discrete distance on it, namely $d(p,q) = 1 - \delta_{pq}$, which can be modelled as the basis of a suitable Hilbert space,*

$$X = \{e_x\}_{x \in X} \subset l^2(X)$$

this set is closed and bounded, but not compact.

PROOF. Here the first part, regarding the modelling of X, that we will actually not really need, is something that we already know. Regarding now the second part:

(1) X being the total space, it is by definition closed. As a remark here, that we will need later, since the points of X are obviously open, any subset $E \subset X$ is open, and by taking complements, any set $E \subset X$ is closed as well.

(2) X is also bounded, because all distances are smaller than 1.

(3) However, our set X is not compact, because its points being open, as noted above, $X = \cup_{x \in X}\{x\}$ is an open cover, having no finite subcover. □

Let us develop now the theory of compact sets, as axiomatized above, and see what we get. We first have the following result, confirming that we are on the good track:

PROPOSITION 5.11. *The following hold:*

(1) *Compact implies closed.*
(2) *Closed inside compact is compact.*
(3) *Compact intersected with closed is compact.*

PROOF. These assertions are all clear from definitions, as follows:

(1) Assume that $K \subset X$ is compact, and let us prove that K is closed. For this purpose, we will prove that K^c is open. So, pick $p \in K^c$. For any $q \in K$ we set $r = d(p,q)/3$, and we consider the following balls, separating p and q:

$$U_q = B_p(r) \quad , \quad V_q = B_q(r)$$

We have then $K \subset \cup_{q \in K} V_q$, so we can pick a finite subcover, as follows:

$$K \subset (V_{q_1} \cup \ldots \cup V_{q_n})$$

With this done, consider the following intersection:

$$U = U_{q_1} \cap \ldots \cap U_{q_n}$$

This intersection is then a ball around p, and since this ball avoids $V_{q_1}, \ldots, V_{q_n}$, it avoids the whole K. Thus, we have proved that K^c is open at p, as desired.

(2) Assume that $F \subset K$ is closed, with $K \subset X$ being compact. For proving our result, we can assume, by replacing X with K, that we have $X = K$. In order to prove now that F is compact, consider an open cover of it, as follows:

$$F \subset \bigcup_i E_i$$

By adding the set F^c, which is open, to this cover, we obtain a cover of K. Now since K is compact, we can extract from this a finite subcover Ω, and there are two cases:

– If $F^c \in \Omega$, by removing F^c from Ω we obtain a finite cover of F, as desired.

– If $F^c \notin \Omega$, we are done too, because in this case Ω is a finite cover of F.

(3) This follows from (1) and (2), because if $K \subset X$ is compact, and $F \subset X$ is closed, then $K \cap F \subset K$ is closed inside a compact, so it is compact. □

As a second batch of results, which are useful as well, we have:

Proposition 5.12. *The following hold:*

1. *If $K_i \subset X$ are compact, satisfying $K_{i_1} \cap \ldots \cap K_{i_n} \neq \emptyset$, then $\cap_i K_i \neq \emptyset$.*
2. *If $K_1 \supset K_2 \supset K_3 \supset \ldots$ are non-empty compacts, then $\cap_i K_i \neq \emptyset$.*
3. *If K is compact, and $E \subset K$ is infinite, then E has a limit point in K.*
4. *If K is compact, any sequence $\{x_n\} \subset K$ has a limit point in K.*
5. *If K is compact, any $\{x_n\} \subset K$ has a subsequence which converges in K.*

Proof. Again, these are elementary results, which can be proved as follows:

(1) Assume by contradiction $\cap_i K_i = \emptyset$, and let us pick $K_1 \in \{K_i\}$. Since any $x \in K_1$ is not in $\cap_i K_i$, there is an index i such that $x \in K_i^c$, and we conclude that we have:

$$K_1 \subset \bigcup_{i \neq 1} K_i^c$$

But this can be regarded as being an open cover of K_1, that we know to be compact, so we can extract from it a finite subcover, as follows:

$$K_1 \subset \left(K_{i_1}^c \cup \ldots \cup K_{i_n}^c\right)$$

Now observe that this latter subcover tells us that we have:

$$K_1 \cap K_{i_1} \cap \ldots \cap K_{i_n} = \emptyset$$

But this contradicts our intersection assumption in the statement, and we are done.

(2) This is a particular case of (1), proved above.

(3) We prove this by contradiction. So, assume that E has no limit point in K. This means that any $p\in K$ can be isolated from the rest of E by a certain open ball $V_p=B_p(r)$, and in both the cases that can appear, $p\in E$ or $p\notin E$, we have:

$$|V_p\cap E|=0,1$$

Now observe that these sets V_p form an open cover of K, and so of E. But due to $|V_p\cap E|=0,1$ and to $|E|=\infty$, this open cover of E has no finite subcover. Thus the same cover, regarded now as cover of K, has no finite subcover either, contradiction.

(4) This follows from (3) that we just proved, with $E=\{x_n\}$.

(5) This is a reformulation of (4), that we just proved. □

Getting now to some more exciting theory, here is a key result about compactness, which is less trivial, and that we will need on a regular basis, in what follows:

THEOREM 5.13. *For a subset $K\subset\mathbb R^N$, the following are equivalent:*
(1) *K is closed and bounded.*
(2) *K is compact.*
(3) *Any infinite subset $E\subset K$ has a limiting point in K.*

PROOF. This is something quite tricky, the idea being as follows:

(1) $\implies$ (2) As a first task, in order to establish this implication, let us prove that any product of closed intervals, as follows, is indeed compact:

$$J=\prod_{i=1}^N[a_i,b_i]\subset\mathbb R^N$$

We can assume by linearity that we are dealing with the unit cube:

$$C_1=\prod_{i=1}^N[0,1]\subset\mathbb R^N$$

In order to prove that C_1 is compact, we proceed by contradiction. So, assume that we have an open cover as follows, having no finite subcover:

$$C_1\subset\bigcup_iE_i$$

Now let us cut C_1 into 2^N small cubes, in the obvious way, over the N coordinate axes. Then at least one of these small cubes, which are all covered by $\cup_iE_i$ too, has no finite subcover. So, let us call $C_2\subset C_1$ one of these small cubes, having no finite subcover:

$$C_2\subset\bigcup_iE_i$$

We can then cut C_2 into 2^N small cubes, and by the same reasoning, we obtain a smaller cube $C_3 \subset C_2$ having no finite subcover. And so on by recurrence, and we end up with a decreasing sequence of cubes, as follows, having no finite subcover:

$$C_1 \supset C_2 \supset C_3 \supset \ldots$$

Now since these decreasing cubes have edge size $1, 1/2, 1/4, \ldots$, their intersection must be a point. So, let us call p this point, defined by the following formula:

$$\{p\} = \bigcap_k C_k$$

But this point p must be covered by $\cup_i E_i$, so we can find an index i such that:

$$p \in E_i$$

Now observe that E_i must contain a whole ball around p, and so starting from a certain $K \in \mathbb{N}$, all the cubes C_k will be contained in this ball, and so in E_i:

$$C_k \subset E_i \quad , \quad \forall k \geq K$$

But this is a contradiction, because C_K, and in fact the smaller cubes C_k with $k > K$ as well, were assumed to have no finite subcover. Thus, we have proved our claim.

(1) $\implies$ (2), continuation. But with this claim in hand, the result is now clear. Indeed, assume that $K \subset \mathbb{R}^N$ is closed and bounded. Then, since K is bounded, we can view it as a subset as a suitable big cube, of the following form:

$$K \subset \prod_{i=1}^N [-M, M] \subset \mathbb{R}^N$$

But, what we have here is a closed subset inside a compact set, that follows to be compact, as desired.

(2) $\implies$ (3) This is something that we already know, not needing $K \subset \mathbb{R}^N$.

(3) $\implies$ (1) We have to prove that K as in the statement is both closed and bounded, and we will do both these things by contradiction, as follows:

– Assume first that K is not closed. But this means that we can find a point $x \notin K$ which is a limiting point of K. Now let us pick $x_n \in K$, with $x_n \to x$, and consider the set $E = \{x_n\}$. According to our assumption, E must have a limiting point in K. But this limiting point can only be x, which is not in K, contradiction.

– Assume now that K is not bounded. But this means that we can find points $x_n \in K$ satisfying $||x_n|| \to \infty$, and if we consider the set $E = \{x_n\}$, then again this set must have a limiting point in K, which is impossible, so we have our contradiction, as desired. □

So long for compactness. As a last piece of general topology, in our metric space framework, we can talk as well about connectedness, as follows:

Definition 5.14. *We can talk about connected sets $E \subset X$, as follows:*

(1) *We say that E is connected if it cannot be separated as $E = E_1 \cup E_2$, with the components E_1, E_2 satisfying $E_1 \cap \bar{E}_2 = \bar{E}_1 \cap E_2 = \emptyset$.*
(2) *We say that E is path connected if any two points $p, q \in E$ can be joined by a path, meaning a continuous $f : [0.1] \to X$, with $f(0) = p$, $f(1) = q$.*

All this looks a bit technical, and indeed it is. To start with, (1) is something quite natural, but the separation condition there $E_1 \cap \bar{E}_2 = \bar{E}_1 \cap E_2 = \emptyset$ can be weakened into $E_1 \cap E_2 = \emptyset$, or strengthened into $\bar{E}_1 \cap \bar{E}_2 = \emptyset$, depeding on purposes, and with our (1) as formulated being the good compromise, for most purposes. As for (2), this condition is obviously something stronger, and we have in fact the following implications:

$$\text{convex} \implies \text{path connected} \implies \text{connected}$$

The problem, however, is that connected does not imply path connected, and there are as well various counterexamples in relation with the various versions of (1) that can be formulated, as explained above. Anyway, leaving aside the discussion here, which is something quite technical, once all these questions clarified, the idea is that any set E can be written as a disjoint union of connected components, as follows:

$$E = \bigsqcup_i E_i$$

Getting back now to more concrete things, remember that we are here in this book for studying functions, and doing calculus. And, regarding functions, we have:

Theorem 5.15. *Assuming that $f : X \to Y$ is continuous, the following happen:*

(1) *If O is open, then $f^{-1}(O)$ is open.*
(2) *If C is closed, then $f^{-1}(C)$ is closed.*
(3) *If K is compact, then $f(K)$ is compact.*
(4) *If E is connected, then $f(E)$ is connected.*

Proof. This is something fundamental, which can be proved as follows:

(1) This is clear from the definition of continuity, written with ε, δ. In fact, the converse holds too, in the sense that if $f^{-1}(\text{open}) = \text{open}$, then f must be continuous.

(2) This follows from (1), by taking complements. And again, the converse holds too, in the sense that if $f^{-1}(\text{closed}) = \text{closed}$, then f must be continuous.

(3) Given an open cover $f(K) \subset \cup_i E_i$, we have by using (1) an open cover $K \subset \cup_i f^{-1}(E_i)$, and so by compactness of K, a finite subcover $K \subset f^{-1}(E_{i_1}) \cup \ldots \cup f^{-1}(E_{i_n})$, and so finally a finite subcover $f(K) \subset E_{i_1} \cup \ldots \cup E_{i_n}$, as desired.

(4) This can be proved via the same trick as for (3). Indeed, any separation of $f(E)$ into two parts can be returned via f^{-1} into a separation of E into two parts, contradiction. □

As a comment here, Theorem 5.15 generalizes, and in a clever way, many things that we know from one-variable calculus. Of particular interest is (3), which shows in particular that any continuous function on a compact space $f : X \to \mathbb{R}$ attains its minimum and its maximum, and then (4), which can be regarded as being a general mean value theorem. As for (1) and (2), these are useful in everyday life, and we will see examples of this.

5b. Borel sets, axioms

We are now armed to attack measure theory. We first need measurable sets, and on the bottom line, we would like our open and closed sets, and their "combinations", to be measurable. But these "combinations" can be understood and axiomatized, and are called Borel sets. More generally, we can talk about abstract families of "measurable sets", satisfying certain axioms, which generalize the basic properties of Borel sets.

5c. Measure theory

With this done, we can then come with measures, for measuring our measurable sets. All this is quite standard material, and we will explain this here.

5d. Radon-Nikodym

How many measures are there? For this, we need to compare measures, and the main theorem here is a very useful decomposition result, due to Radon-Nikodym. In the case of $\mathbb{R}^N$, the ultimate conclusion is that any measure has a continuous part $f(x)dx$, whose density is a usual function $f : \mathbb{R}^N \to \mathbb{C}$, and a discrete part, formed by Dirac masses.

5e. Exercises

Exercises:

Exercise 5.16. *Clarify the topological aspects of the p-adic numbers.*

Exercise 5.17.

Exercise 5.18.

Exercise 5.19.

Exercise 5.20.

Exercise 5.21.

Bonus exercise.

CHAPTER 6

Integration theory

6a. Basic integration

We can integrate functions $f : X \to \mathbb{C}$ with respect to measures $\mu \in M(X)$, in the obvious way, by doing Riemann sums. The general results here are very similar to those that you already know, from one or several real variables. However, a number of things can be quite tricky, especially in respect with the precise assumptions to be satisfied, in order for our computations to really converge, and we will explain all this here.

6b. Theorems, Fubini

Goind ahead with more theory, let us discuss the Fubini theorem. You might already know from calculus that Fubini does not always work, and we will clarify here all this, with theorems and counterexamples, directly in our general measure theory setting.

6c. Change of variables

Let us discuss now the change of variables, which is the last theoretical result that we will need. Things here are quite tricky, the theorem and proof being as follows:

THEOREM 6.1. *Given a transformation $\varphi = (\varphi_1, \ldots, \varphi_N)$, we have*

$$\int_E f(x)dx = \int_{\varphi^{-1}(E)} f(\varphi(t))|J_\varphi(t)|dt$$

with the J_φ quantity, called Jacobian, being given by

$$J_\varphi(t) = \det\left[\left(\frac{d\varphi_i}{dx_j}(x)\right)_{ij}\right]$$

and with this generalizing the usual formula from one variable calculus.

PROOF. This is something quite tricky, the idea being as follows:

(1) Observe first that this generalizes indeed the change of variable formula in 1 dimension, that you know well, the point here being that the absolute value on the derivative appears as to compensate for the lack of explicit bounds for the integral.

(2) In general now, we can first argue that, the formula in the statement being linear in f, we can assume $f=1$. Thus we want to prove $vol(E)=\int_{\varphi^{-1}(E)}|J_\varphi(t)|dt$, and with $D=\varphi^{-1}(E)$, this amounts in proving $vol(\varphi(D))=\int_D|J_\varphi(t)|dt$.

(3) Now since this latter formula is additive with respect to D, it is enough to prove that $vol(\varphi(D))=\int_D J_\varphi(t)dt$, for small cubes D, and assuming $J_\varphi>0$. But this basically follows by using the definition of the determinant of real matrices, as a volume.

(4) To be more precise, we first have to review the definition of the determinant of real matrices, have it done old style, as a signed volume. Unfortunately references here are scarce, due to a French colleague of mine, N.B., who found some time ago an alternative approach to this, and to mathematics in general, and then organized satanic sessions of book burning, in linear algebra and mathematics, in France and worldwide.

(5) Adding to the jokes, the story in fact is that the USSR resisted this, and this had dramatic consequences after the fell of the Iron Curtain, with Soviet and Western mathematicians starting to meet, work together and so on, with the former knowing what the determinant is, and the latter, not really. By the way this did not apply to many satellite USSR countries, like my native Romania, during the Cold War we learned there mathematics Western style. Ironically, I relearned them Soviet style later on, after the fell of the Iron Curtain, from fellow Russian mathematicians met here in the West.

(6) So, good luck in finding a good reference for this. Generally speaking, linear algebra books written by people knowing what they're doing, such as Lax [**63**] or Petersen [**73**], are usually quite honest, or perhaps revolutionary, or counter-revolutionary, depending on the viewpoint, and do talk about this, but not with full details. The same goes for my calculus book [**7**], you can find about 10 pages there about this, and for more, about 20 pages, you can go with my own linear algebra book, that you can find on the internet, along by the way with all my other books. As for the full theory, probably covering 50 pages or so, or even more, I don't really have a reference here myself.

(7) With this done, things are unfortunately not over at all, because what we said in (3) above is still very loose, even cheating a bit, and needs about 4-5 pages of tough computations, and new ideas, in order to be clarified. We will explain here all this.

(8) Let us also mention that there are some interesting, alternative proofs to all this, with none of them being of course trivial. A standard proof for instance, that most people like me, from the Generation X, are quite used to, having learned that as students, is the one in the book of Rudin [**74**]. So, a good idea would be that of reading that too.

(9) Finally, let us mention that our present theorem is something quite fascinating, with the challenge being that of coming up with a truly simple proof for it. In fact, every now and then, professional mathematicians have a try, with new ideas on this old subject. In waiting forward for some new ideas and research, from you too, later on. □

Summarizing, quite philosophical, all this. And for adding more philosophy, in relation with my colleague N.B. mentioned in the above proof, once you know well mathematics, as to be black belt level, and have some spare time, have a look at his books. These are really gems, and masterpieces of mathematical extremism, and have been a source of inspiration for many extreme things in mathematics, including my own books.

Ironically, and talking now revolution and counter-revolution, the books of N.B. were originally motivated by marxist ideas and philosophy. But so are my books, including the present one, the idea behind them being that true mathematical marxism exists, and is exactly the opposite of what N.B. was saying. Funny you would say, but turn on the TV and watch the news, quite often all those guerrilla groups fighting each other are in fact all marxists, just having a small argument about what true marxism really is.

Summarizing, life and math are about fighting, and no matter of what camp you choose, you will be useful to mankind. As long as you fight for something, of course.

6d. Geometry, spheres

We have learned so far the basics of measure theory and integration, done in a general, abstract and rigorous way. Time now to get into some applications.

Generally speaking, and this regardless if you are a mathematician or a physicist, the main problem is that of understanding how the geometry, integration and calculus are changing, when passing from $\mathbb{R}^N$ to the unit sphere $S^{N-1}_\mathbb{R} \subset \mathbb{R}^N$, and vice versa:

(1) From a physical viewpoint, all sorts of waves that you will have to deal with, on a daily basis, be them mechanical, or electromagnetic, or even quantum mechanical, propagate from a "source", and so, you need spherical coordinates for their study.

(2) As for the mathematical viewpoint, no surprise here, this is nearly identical to the physics one. Unless you are deep into discrete mathematics, curvature will be your daily joy and nightmare, and the simplest curved object is the unit sphere $S^{N-1}_\mathbb{R} \subset \mathbb{R}^N$.

So, let us go ahead with the study of the spheres, and more specifically, with integrating over them, or integrating on the whole space, using them. In 2 dimensions, we have:

Theorem 6.2. *We have polar coordinates in 2 dimensions,*

$$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$$

the corresponding Jacobian being $J = r$.

PROOF. This is elementary, the Jacobian being as follows:

$$\begin{aligned} J &= \begin{vmatrix} \frac{d(r\cos t)}{dr} & \frac{d(r\cos t)}{dt} \\ \\ \frac{d(r\sin t)}{dr} & \frac{d(r\sin t)}{dt} \end{vmatrix} \\ &= \begin{vmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{vmatrix} \\ &= r\cos^2 t + r\sin^2 t \\ &= r \end{aligned}$$

Thus, we have indeed the formula in the statement. □

We can now compute the Gauss integral, which is the best calculus formula ever:

THEOREM 6.3. *We have the following formula,*

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

called Gauss integral formula.

PROOF. Let I be the above integral. By using polar coordinates, we obtain:

$$\begin{aligned} I^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2} dxdy \\ &= \int_0^{2\pi}\int_0^\infty e^{-r^2} r dr dt \\ &= 2\pi \int_0^\infty \left(-\frac{e^{-r^2}}{2}\right)' dr \\ &= 2\pi \left[0 - \left(-\frac{1}{2}\right)\right] \\ &= \pi \end{aligned}$$

Thus, we are led to the formula in the statement. □

Moving now to 3 dimensions, we have here the following result:

THEOREM 6.4. *We have spherical coordinates in 3 dimensions,*

$$\begin{cases} x = r\cos s \\ y = r\sin s\cos t \\ z = r\sin s\sin t \end{cases}$$

the corresponding Jacobian being $J(r,s,t) = r^2 \sin s$.

PROOF. The fact that we have indeed spherical coordinates is clear. Regarding now the Jacobian, this is given by the following formula:

$$\begin{aligned}
&J(r,s,t)\\
= &\begin{vmatrix}\cos s & -r\sin s & 0\\ \sin s\cos t & r\cos s\cos t & -r\sin s\sin t\\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t\end{vmatrix}\\
= &\ r^2\sin s\sin t\begin{vmatrix}\cos s & -r\sin s\\ \sin s\sin t & r\cos s\sin t\end{vmatrix}+r\sin s\cos t\begin{vmatrix}\cos s & -r\sin s\\ \sin s\cos t & r\cos s\cos t\end{vmatrix}\\
= &\ r\sin s\sin^2 t\begin{vmatrix}\cos s & -r\sin s\\ \sin s & r\cos s\end{vmatrix}+r\sin s\cos^2 t\begin{vmatrix}\cos s & -r\sin s\\ \sin s & r\cos s\end{vmatrix}\\
= &\ r\sin s(\sin^2 t+\cos^2 t)\begin{vmatrix}\cos s & -r\sin s\\ \sin s & r\cos s\end{vmatrix}\\
= &\ r\sin s\times 1\times r\\
= &\ r^2\sin s
\end{aligned}$$

Thus, we have indeed the formula in the statement. □

Let us work out now the general spherical coordinate formula, in arbitrary N dimensions. The formula here, which generalizes those at $N=2,3$, is as follows:

THEOREM 6.5. *We have spherical coordinates in N dimensions,*

$$\begin{cases}x_1 &= r\cos t_1\\ x_2 &= r\sin t_1\cos t_2\\ \vdots\\ x_{N-1} &= r\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1}\\ x_N &= r\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1}\end{cases}$$

the corresponding Jacobian being given by the following formula,

$$J(r,t)=r^{N-1}\sin^{N-2}t_1\sin^{N-3}t_2\,\ldots\,\sin^2 t_{N-3}\sin t_{N-2}$$

and with this generalizing the known formulae at $N=2,3$.

PROOF. As before, the fact that we have spherical coordinates is clear. Regarding now the Jacobian, also as before, by developing over the last column, we have:

$$\begin{aligned}
J_N &= r\sin t_1\ldots\sin t_{N-2}\sin t_{N-1}\times\sin t_{N-1}J_{N-1}\\
&+ r\sin t_1\ldots\sin t_{N-2}\cos t_{N-1}\times\cos t_{N-1}J_{N-1}\\
&= r\sin t_1\ldots\sin t_{N-2}(\sin^2 t_{N-1}+\cos^2 t_{N-1})J_{N-1}\\
&= r\sin t_1\ldots\sin t_{N-2}J_{N-1}
\end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence. □

Let us talk as well about the stereographic projection, which was something dear to Riemann, the founding father of modern geometry. We have here:

THEOREM 6.6. *The stereographic projection is given by inverse maps*

$$\Phi:\mathbb{R}^N\to S^N_\mathbb{R}-\{\infty\}\quad,\quad\Psi:S^N_\mathbb{R}-\{\infty\}\to\mathbb{R}^N$$

given by the following formulae,

$$\Phi(v)=(1,0)+\frac{2}{1+||v||^2}(-1,v)\quad,\quad\Psi(c,x)=\frac{x}{1-c}$$

with the convention $\mathbb{R}^{N+1}=\mathbb{R}\times\mathbb{R}^N$*, and with the coordinate of* $\mathbb{R}$ *denoted* x_0*, and with the coordinates of* $\mathbb{R}^N$ *denoted* $x_1,\ldots,x_N$*.*

PROOF. We are looking for the formulae of the isomorphism $\mathbb{R}^N\simeq S^N_\mathbb{R}-\{\infty\}$, obtained by identifying $\mathbb{R}^N=\mathbb{R}^N\times\{0\}\subset\mathbb{R}^{N+1}$ with the unit sphere $S^N_\mathbb{R}\subset\mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty=(1,0,\ldots,0)$, via the stereographic projection. That is, we need the precise formulae of two inverse maps, as follows:

$$\Phi:\mathbb{R}^N\to S^N_\mathbb{R}-\{\infty\}\quad,\quad\Psi:S^N_\mathbb{R}-\{\infty\}\to\mathbb{R}^N$$

In one sense, according to our conventions above, we must have a formula as follows for our map Φ, with the parameter $t\in(0,1)$ being such that $||\Phi(v)||=1$:

$$\Phi(v)=t(0,v)+(1-t)(1,0)$$

The equation for the parameter $t\in(0,1)$ can be solved as follows:

$$\begin{aligned}(1-t)^2+t^2||v||^2=1\quad&\iff\quad t^2(1+||v||^2)=2t\\&\iff\quad t=\frac{2}{1+||v||^2}\end{aligned}$$

We conclude that the formula of the map Φ is as follows:

$$\Phi(v)=(1,0)+\frac{2}{1+||v||^2}(-1,v)$$

In the other sense now we must have, for a certain $\alpha\in\mathbb{R}$:

$$(0,\Psi(c,x))=\alpha(c,x)+(1-\alpha)(1,0)$$

But from $\alpha c+1-\alpha=0$ we get the following formula for the parameter α:

$$\alpha=\frac{1}{1-c}$$

We conclude that the formula of the map Ψ is as follows:

$$\Psi(c,x)=\frac{x}{1-c}$$

Here, as before, we use the convention in the statement, namely $\mathbb{R}^{N+1}=\mathbb{R}\times\mathbb{R}^N$, with the coordinate of $\mathbb{R}$ denoted x_0, and with the coordinates of $\mathbb{R}^N$ denoted $x_1,\ldots,x_N$. □

There are of course many other possible parametrizations of the sphere, such as the one using cylindrical coordinates, or the one peeling the sphere as an orange, and so on. All this is quite interesting, so have a look at cartography, many things to be learned.

Finally, let us do something tough, with our spheres. We have certainly heard about the heat equation $\dot{f} = \alpha\Delta f$, or about the wave equation $\ddot{f} = v^2\Delta f$, both involving the Laplace operator for the functions $f : \mathbb{R}^N \to \mathbb{C}$, given by the following formula:

$$\Delta f = \sum_{i=1}^N \frac{d^2 f}{dx_i^2}$$

Thinking a bit, it is pretty much clear that, once we will get into serious physics, be that heat or waves or other, we will need the formula of Δ in spherical coordinates. So, let us do this, as a quick calculus exercise. The result, and its proof, are as follows:

THEOREM 6.7. *The Laplace operator in spherical coordinates is*

$$\Delta = \frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{d}{dr}\right) + \frac{1}{r^2\sin s}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d}{ds}\right) + \frac{1}{r^2\sin^2 s}\cdot\frac{d^2}{dt^2}$$

with our standard conventions for these coordinates, in 3D.

PROOF. There are several proofs here, a short, elementary one being as follows:

(1) Let us first see how Δ behaves under a change of coordinates $\{x_i\} \to \{y_i\}$, in arbitrary N dimensions. Our starting point is the chain rule for derivatives:

$$\frac{d}{dx_i} = \sum_j \frac{d}{dy_j}\cdot\frac{dy_j}{dx_i}$$

By using this rule, then Leibnitz for products, then again this rule, we obtain:

$$\begin{aligned}
\frac{d^2 f}{dx_i^2} &= \sum_j \frac{d}{dx_i}\left(\frac{df}{dy_j}\cdot\frac{dy_j}{dx_i}\right)\\
&= \sum_j \frac{d}{dx_i}\left(\frac{df}{dy_j}\right)\cdot\frac{dy_j}{dx_i} + \frac{df}{dy_j}\cdot\frac{d}{dx_i}\left(\frac{dy_j}{dx_i}\right)\\
&= \sum_j \left(\sum_k \frac{d}{dy_k}\cdot\frac{dy_k}{dx_i}\right)\left(\frac{df}{dy_j}\right)\cdot\frac{dy_j}{dx_i} + \frac{df}{dy_j}\cdot\frac{d^2 y_j}{dx_i^2}\\
&= \sum_{jk} \frac{d^2 f}{dy_k dy_j}\cdot\frac{dy_k}{dx_i}\cdot\frac{dy_j}{dx_i} + \sum_j \frac{df}{dy_j}\cdot\frac{d^2 y_j}{dx_i^2}
\end{aligned}$$

(2) Now by summing over i, we obtain the following formula, with A being the derivative of $x \to y$, that is to say, the matrix of partial derivatives dy_i/dx_j:

$$\begin{aligned}\Delta f &= \sum_{ijk} \frac{d^2 f}{dy_k dy_j} \cdot \frac{dy_k}{dx_i} \cdot \frac{dy_j}{dx_i} + \sum_{ij} \frac{df}{dy_j} \cdot \frac{d^2 y_j}{dx_i^2} \\ &= \sum_{ijk} A_{ki} A_{ji} \frac{d^2 f}{dy_k dy_j} + \sum_{ij} \frac{d^2 y_j}{dx_i^2} \cdot \frac{df}{dy_j} \\ &= \sum_{jk} (AA^t)_{jk} \frac{d^2 f}{dy_k dy_j} + \sum_j \Delta(y_j) \frac{df}{dy_j}\end{aligned}$$

(3) So, this will be the formula that we will need. Observe that this formula can be further compacted as follows, with all the notations being self-explanatory:

$$\Delta f = Tr(AA^t H_y(f)) + < \Delta(y), \nabla_y(f) >$$

(4) Getting now to spherical coordinates, $(x, y, z) \to (r, s, t)$, the derivative of the inverse, obtained by differentiating x, y, z with respect to r, s, t, is given by:

$$A^{-1} = \begin{pmatrix} \cos s & -r \sin s & 0 \\ \sin s \cos t & r \cos s \cos t & -r \sin s \sin t \\ \sin s \sin t & r \cos s \sin t & r \sin s \cos t \end{pmatrix}$$

The product $(A^{-1})^t A^{-1}$ of the transpose of this matrix with itself is then:

$$\begin{pmatrix} \cos s & \sin s \cos t & \sin s \sin t \\ -r \sin s & r \cos s \cos t & r \cos s \sin t \\ 0 & -r \sin s \sin t & r \sin s \cos t \end{pmatrix} \begin{pmatrix} \cos s & -r \sin s & 0 \\ \sin s \cos t & r \cos s \cos t & -r \sin s \sin t \\ \sin s \sin t & r \cos s \sin t & r \sin s \cos t \end{pmatrix}$$

But everything simplifies here, and we have the following remarkable formula, which by the way is something very useful, worth to be memorized:

$$(A^{-1})^t A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 s \end{pmatrix}$$

Now by inverting, we obtain the following formula, in relation with the above:

$$AA^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2 \sin^2 s) \end{pmatrix}$$

(5) Let us compute now the Laplacian of r,s,t. We first have the following formula, that we will use many times in what follows, and is worth to be memorized:

$$\begin{aligned}\frac{dr}{dx} &= \frac{d}{dx}\sqrt{x^2+y^2+z^2}\\ &= \frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+y^2+z^2}}\\ &= \frac{x}{r}\end{aligned}$$

Of course the same computation works for y,z too, and we therefore have:

$$\frac{dr}{dx}=\frac{x}{r}\quad,\quad\frac{dr}{dy}=\frac{y}{r}\quad,\quad\frac{dr}{dz}=\frac{z}{r}$$

(6) By using the above formulae, twice, we can compute the Laplacian of r:

$$\begin{aligned}\Delta(r) &= \Delta\left(\sqrt{x^2+y^2+z^2}\right)\\ &= \frac{d}{dx}\left(\frac{x}{r}\right)+\frac{d}{dy}\left(\frac{y}{r}\right)+\frac{d}{dz}\left(\frac{z}{r}\right)\\ &= \frac{r^2-x^2}{r^3}+\frac{r^2-y^2}{r^3}+\frac{r^2-z^2}{r^3}\\ &= \frac{2}{r}\end{aligned}$$

(7) In what regards now s, the computation here goes as follows:

$$\begin{aligned}\Delta(s) &= \Delta\left(\arccos\left(\frac{x}{r}\right)\right)\\ &= \frac{d}{dx}\left(-\frac{\sqrt{r^2-x^2}}{r^2}\right)+\frac{d}{dy}\left(\frac{xy}{r^2\sqrt{r^2-x^2}}\right)+\frac{d}{dz}\left(\frac{xz}{r^2\sqrt{r^2-x^2}}\right)\\ &= \frac{2x\sqrt{r^2-x^2}}{r^4}+\frac{r^2(z^2-2y^2)+2x^2y^2}{r^4\sqrt{r^2-x^2}}+\frac{r^2(y^2-2z^2)+2x^2z^2}{r^4\sqrt{r^2-x^2}}\\ &= \frac{2x\sqrt{r^2-x^2}}{r^4}+\frac{x(2x^2-r^2)}{r^4\sqrt{r^2-x^2}}\\ &= \frac{x}{r^2\sqrt{r^2-x^2}}\\ &= \frac{\cos s}{r^2\sin s}\end{aligned}$$

(8) Finally, in what regards t, the computation here goes as follows:

$$\begin{aligned}\Delta(t) &= \Delta\left(\arctan\left(\frac{z}{y}\right)\right)\\ &= \frac{d}{dx}(0)+\frac{d}{dy}\left(-\frac{z}{y^2+z^2}\right)+\frac{d}{dz}\left(\frac{y}{y^2+z^2}\right)\\ &= 0-\frac{2yz}{(y^2+z^2)^2}+\frac{2yz}{(y^2+z^2)^2}\\ &= 0\end{aligned}$$

(9) We can now plug the data from (4) and (6,7,8) in the general formula that we found in (2) above, and we obtain in this way:

$$\begin{aligned}\Delta f &= \frac{d^2f}{dr^2}+\frac{1}{r^2}\cdot\frac{d^2f}{ds^2}+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2f}{dt^2}+\frac{2}{r}\cdot\frac{df}{dr}+\frac{\cos s}{r^2\sin s}\cdot\frac{df}{ds}\\ &= \frac{2}{r}\cdot\frac{df}{dr}+\frac{d^2f}{dr^2}+\frac{\cos s}{r^2\sin s}\cdot\frac{df}{ds}+\frac{1}{r^2}\cdot\frac{d^2f}{ds^2}+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2f}{dt^2}\\ &= \frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{df}{dr}\right)+\frac{1}{r^2\sin s}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{df}{ds}\right)+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2f}{dt^2}\end{aligned}$$

Thus, we are led to the formula in the statement. □

Still with me, I hope. We will get back later to physics, using our formula above.

6e. Exercises

Exercises:

Exercise 6.8.

Exercise 6.9.

Exercise 6.10.

Exercise 6.11.

Exercise 6.12.

Exercise 6.13.

Bonus exercise.

CHAPTER 7

Probability theory

7a. Probability, revised

Good news, with measure theory developed, we have now all tools in our bag for rewriting probability theory, in a rigorous way. However, as more mixed news perhaps, we will take advantage of this opportunity for rewriting the underlying calculus too, in a more advanced way, by using the moment method, which is something heavily used in quantum physics, and that we will need, later in this book, when talking about ∞.

In short, don't expect the present chapter to be your standard advanced probability text, rigorous and everything, but rather to be something more advanced, based on that. Of course, if just looking for rigorous standard probability, without much moments and stuff, there are some good texts here, that you can now read, with your measure theory knowledge. But better stay with us, we have interesting things to say.

With such ideas in mind, our starting point will be as follows:

Definition 7.1. *Let X be a probability space, that is, a space with a probability measure, and with the corresponding integration denoted $\mathbb{E}$, and called expectation.*

1. *The random variables are the real functions $f \in L^\infty(X)$.*
2. *The moments of such a variable are the numbers $M_k(f) = \mathbb{E}(f^k)$.*
3. *The law of such a variable is the measure given by $M_k(f) = \int_\mathbb{R} x^k d\mu_f(x)$.*

Here the fact that the law μ_f as above exists indeed, as a real probability measure, is not exactly trivial. But we can do this by looking at formulae of the following type:

$$\mathbb{E}(\varphi(f)) = \int_\mathbb{R} \varphi(x) d\mu_f(x)$$

Indeed, having this for monomials $\varphi(x) = x^n$, as in the statement, is the same as having it for polynomials $\varphi \in \mathbb{R}[X]$, which in turn is the same as having it for characteristic functions $\varphi = \chi_I$ of measurable sets $I \subset \mathbb{R}$. Thus, in the end, what we need is:

$$\mathbb{P}(f \in I) = \mu_f(I)$$

But this can serve as a definition for μ_f, and done. Alternatively, with our measure theory knowledge, μ_f is simply the push-forward of the Lebesgue measure, by f.

Regarding now independence, we can formulate here things as follows:

Definition 7.2. *Two variables $f,g\in L^\infty(X)$ are called independent when*

$$\mathbb E(f^kg^l)=\mathbb E(f^k)\,\mathbb E(g^l)$$

happens, for any $k,l\in\mathbb N$.

Again, this definition hides some non-trivial things, the idea being a bit as before, namely that of looking at formulae of the following type:

$$\mathbb E[\varphi(f)\psi(g)]=\mathbb E[\varphi(f)]\,\mathbb E[\psi(g)]$$

To be more precise, passing as before from monomials to polynomials, then to characteristic functions, we are led to the usual definition of independence, namely:

$$\mathbb P(f\in I,g\in J)=\mathbb P(f\in I)\,\mathbb P(g\in J)$$

As a first result now, which is something fundamental, we have:

Theorem 7.3. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$\mu_{f+g}=\mu_f*\mu_g$$

where $*$ is the convolution of real probability measures.

Proof. We have the following computation, using the independence of f,g:

$$\int_\mathbb R x^kd\mu_{f+g}(x)=\mathbb E((f+g)^k)=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)$$

On the other hand, by using the Fubini theorem, we have as well:

$$\begin{aligned}\int_\mathbb R x^kd(\mu_f*\mu_g)(x)&=\int_{\mathbb R\times\mathbb R}(x+y)^kd\mu_f(x)d\mu_g(y)\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

Thus μ_{f+g} and $\mu_f*\mu_g$ have the same moments, so they coincide, as desired. □

As a second result on independence, which is more advanced, we have:

Theorem 7.4. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$F_{f+g}=F_fF_g$$

where $F_f(x)=\mathbb E(e^{ixf})$ is the Fourier transform.

PROOF. This is something very standard, coming from:

$$\begin{aligned}F_{f+g}(x) &= \int_{\mathbb{R}} e^{ixz} d(\mu_f * \mu_g)(z)\\ &= \int_{\mathbb{R}\times\mathbb{R}} e^{ix(z+t)} d\mu_f(z) d\mu_g(t)\\ &= \int_{\mathbb{R}} e^{ixz} d\mu_f(z) \int_{\mathbb{R}} e^{ixt} d\mu_g(t)\\ &= F_f(x)F_g(x)\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Still staying general, we will need as well, from time to time, some key complex analysis results, in relation with measures and their moments. First, we have:

THEOREM 7.5. *The density of a real probability measure μ can be recaptured from the sequence of moments $\{M_k\}_{k\geq 0}$ via the Stieltjes inversion formula*

$$d\mu(x) = \lim_{t\searrow 0} -\frac{1}{\pi} Im\left(G(x+it)\right) \cdot dx$$

where the function on the right, given in terms of moments by

$$G(\xi) = \xi^{-1} + M_1\xi^{-2} + M_2\xi^{-3} + \ldots$$

is the Cauchy transform of the measure μ.

PROOF. This is something quite subtle and heavy, the idea for the proof, along with some basic applications of the formula in the statement, being as follows:

(1) Regarding the proof, the Cauchy transform of our measure μ is given by:

$$G(\xi) = \xi^{-1}\sum_{k=0}^{\infty} M_k\xi^{-k} = \int_{\mathbb{R}} \frac{1}{\xi - y}\, d\mu(y)$$

Now with $\xi = x + it$, we obtain from this the following formula:

$$\begin{aligned}Im(G(x+it)) &= \int_{\mathbb{R}} Im\left(\frac{1}{x-y+it}\right) d\mu(y)\\ &= -\int_{\mathbb{R}} \frac{t}{(x-y)^2+t^2}\, d\mu(y)\end{aligned}$$

By integrating over $[a,b]$ we obtain, with the change of variables $x = y + tz$:

$$\begin{aligned}\int_a^b Im(G(x+it))dx &= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{t}{(tz)^2+t^2}\,t\,dz\,d\mu(y)\\ &= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{1}{1+z^2}\,dz\,d\mu(y)\\ &= -\int_{\mathbb{R}}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)d\mu(y)\end{aligned}$$

(2) The point now is that with $t \searrow 0$ we have the following estimates:

$$\lim_{t\searrow 0}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)=\begin{cases}\frac{\pi}{2}-\frac{\pi}{2}=0 & (y<a)\\ \frac{\pi}{2}-0=\frac{\pi}{2} & (y=a)\\ \frac{\pi}{2}-(-\frac{\pi}{2})=\pi & (a<y<b)\\ 0-(-\frac{\pi}{2})=\frac{\pi}{2} & (y=b)\\ -\frac{\pi}{2}-(-\frac{\pi}{2})=0 & (y>b)\end{cases}$$

We therefore obtain the following formula, which proves our result:

$$\lim_{t\searrow 0}\int_a^b Im(G(x+it))dx = -\pi\left(\mu(a,b)+\frac{\mu(a)+\mu(b)}{2}\right)$$

(3) As applications, the first thing that we can do is to find the measure having as even moments the Catalan numbers, $C_k = \frac{1}{k+1}\binom{2k}{k}$, and having all odd moments 0. But this gives the following measure, called Wigner semicircle law on $[-2,2]$:

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

(4) We can also compute the measure having as plain moments the Catalan numbers, $M_k = C_k$. This gives the following measure, called Marchenko-Pastur law on $[0,4]$:

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

(5) Next, in what regards the measure having as moments the central binomial coefficients, $D_k = \binom{2k}{k}$, we obtain following measure, called arcsine law on $[0,4]$:

$$\alpha_1 = \frac{1}{\pi\sqrt{x(4-x)}}\,dx$$

(6) Finally, for the middle binomial coefficients, $E_k = \binom{k}{[k/2]}$, the computation gives the following law on $[-2,2]$, called "square root" of the arcsine law on $[0,4]$:

$$\sigma_1 = \frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\,dx$$

(7) All this is very nice, but I can hear you screaming, what is the point with all these fringe measures. In answer, these measures are not fringe at all, quite the opposite, they are the main laws in Random Matrix Theory (RMT). More on them later. □

We have as well the following result, at the general level, which can be useful too:

THEOREM 7.6. *A sequence of numbers $M_0, M_1, M_2, M_3, \ldots \in \mathbb{R}$, with $M_0 = 1$, is the series of moments of a real probability measure μ precisely when:*

$$\begin{vmatrix}M_0\end{vmatrix} \geq 0 \quad , \quad \begin{vmatrix}M_0 & M_1\\ M_1 & M_2\end{vmatrix} \geq 0 \quad , \quad \begin{vmatrix}M_0 & M_1 & M_2\\ M_1 & M_2 & M_3\\ M_2 & M_3 & M_4\end{vmatrix} \geq 0 \quad , \quad \ldots$$

That is, the associated Hankel determinants must be all positive.

PROOF. This is something a bit heavier, and as a first observation, the positivity conditions in the statement tell us that the following linear forms must be positive:

$$\sum_{i,j=1}^n c_i\bar{c}_j M_{i+j} \geq 0$$

But this is something very classical, in one sense the result being elementary, coming from the following computation, which shows that we have positivity indeed:

$$\int_\mathbb{R}\left|\sum_{i=1}^n c_i x^i\right|^2 d\mu(x) = \int_\mathbb{R}\sum_{i,j=1}^n c_i\bar{c}_j x^{i+j} d\mu(x) = \sum_{i,j=1}^n c_i\bar{c}_j M_{i+j}$$

As for the other sense, here the result comes once again from the above formula, this time via some standard study, using positivity, and a bit of basic functional analysis. □

7b. Limiting theorems

What really matters in probability, allowing you to do some interesting mathematics, are the limiting theorems. Let us first talk about discrete probability. We have:

DEFINITION 7.7. *The Poisson law of parameter $t > 0$ is the measure*

$$p_t = e^{-t}\sum_{k\in\mathbb{N}}\frac{t^k}{k!}\,\delta_k$$

with the letter "p" standing for Poisson.

We have already met these laws, since chapter 1. So, let us quickly develop now their general theory. Going directly for the kill, Fourier transform computation, we have:

PROPOSITION 7.8. *The Fourier transform of p_t is given by:*

$$F_{p_t}(y) = \exp\left((e^{iy} - 1)t\right)$$

In particular we have $p_s * p_t = p_{s+t}$, called convolution semigroup property.

PROOF. We have indeed the following computation:

$$\begin{aligned}
F_{p_t}(y) &= e^{-t}\sum_k \frac{t^k}{k!}F_{\delta_k}(y)\\
&= e^{-t}\sum_k \frac{t^k}{k!}\,e^{iky}\\
&= e^{-t}\sum_k \frac{(e^{iy}t)^k}{k!}\\
&= \exp\left((e^{iy}-1)t\right)
\end{aligned}$$

As for the second assertion, this follows from the fact that $\log F_{p_t}$ is linear in t, via the linearization property for the convolution from Theorem 7.4. □

The above result suggests that the laws p_t should appear as some sort of exponentials with respect to convolution, and yes indeed, this is the case, as shown by:

PROPOSITION 7.9. *The Poisson laws appear as formal exponentials*

$$p_t = \sum_k \frac{t^k(\delta_1-\delta_0)^{*k}}{k!}$$

with respect to the convolution of measures $*$.

PROOF. By using the binomial formula, the measure on the right is:

$$\begin{aligned}
\mu &= \sum_k \frac{t^k}{k!}\sum_{r+s=k}(-1)^s\frac{k!}{r!s!}\delta_r\\
&= \sum_r \frac{t^r\delta_r}{r!}\sum_s \frac{(-1)^s}{s!}\\
&= \frac{1}{e}\sum_r \frac{t^r\delta_r}{r!}\\
&= p_t
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a main result now, we have the Poisson Limit Theorem, as follows:

THEOREM 7.10 (PLT). *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

for any $t>0$.

PROOF. Indeed, if we denote by ν_n the measure under the convolution sign, we have the following computation, for the Fourier transform of the limit:

$$\begin{aligned}
F_{\delta_r}(y)=e^{iry} &\implies F_{\nu_n}(y)=\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{iy}\\
&\implies F_{\nu_n^{*n}}(y)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{iy}\right)^n\\
&\implies F_{\nu_n^{*n}}(y)=\left(1+\frac{(e^{iy}-1)t}{n}\right)^n\\
&\implies F(y)=\exp\left((e^{iy}-1)t\right)
\end{aligned}$$

Thus, we obtain indeed the Fourier transform of p_t, as desired. □

Finally, regarding the moments, we have here the following interesting result:

THEOREM 7.11. *The moments of p_1 are the Bell numbers,*

$$M_k(p_1)=|P(k)|$$

where $P(k)$ is the set of partitions of $\{1,\ldots,k\}$. More generally, we have

$$M_k(p_t)=\sum_{\pi\in P(k)}t^{|\pi|}$$

for any $t>0$, where $|.|$ is the number of blocks.

PROOF. We know that the moments of p_1 are given by the following formula:

$$M_k=\frac{1}{e}\sum_r\frac{r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned}
M_{k+1} &= \frac{1}{e}\sum_r\frac{r^k}{r!}\left(1+\frac{1}{r}\right)^k\\
&= \frac{1}{e}\sum_r\frac{r^k}{r!}\sum_s\binom{k}{s}r^{-s}\\
&= \sum_s\binom{k}{s}M_{k-s}
\end{aligned}$$

But the Bell numbers $B_k=|P(k)|$ satisfy the same recurrence, so we have $M_k=B_k$, as claimed. Next, we know that the moments of p_t with $t>0$ are given by:

$$N_k=e^{-t}\sum_r\frac{t^rr^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned} N_{k+1} &= e^{-t}\sum_r \frac{t^{r+1}r^k}{r!}\left(1+\frac{1}{r}\right)^k \\ &= e^{-t}\sum_r \frac{t^{r+1}r^k}{r!}\sum_s \binom{k}{s} r^{-s} \\ &= t\sum_s \binom{k}{s} N_{k-s} \end{aligned}$$

But the numbers $S_k = \sum_{\pi\in P(k)} t^{|\pi|}$ are easily seen to satisfy the same recurrence, with the same initial values, namely t and $t+t^2$, so we have $N_k = S_k$, as claimed. □

In the continuous setting now, let us formulate:

DEFINITION 7.12. *The normal law of parameter* 1 *is the following measure:*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ *is the following measure:*

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

These are also called Gaussian distributions, with "g" standing for Gauss.

Observe that the above laws have indeed mass 1, as they should. This follows indeed from the Gauss formula, which gives, with $x = \sqrt{2t}\, y$:

$$\begin{aligned} \int_{\mathbb{R}} e^{-x^2/2t} dx &= \int_{\mathbb{R}} e^{-y^2}\sqrt{2t}\, dy \\ &= \sqrt{2t}\int_{\mathbb{R}} e^{-y^2} dy \\ &= \sqrt{2t}\times\sqrt{\pi} \\ &= \sqrt{2\pi t} \end{aligned}$$

Generally speaking, the normal laws appear as bit everywhere, in real life. The reasons behind this phenomenon come from the Central Limit Theorem (CLT), that we will explain in a moment, after developing some general theory. As a first result, we have:

PROPOSITION 7.13. *We have the variance formula*

$$V(g_t) = t$$

valid for any $t > 0$.

PROOF. The first moment is 0, because our normal law g_t is centered. As for the second moment, this can be computed as follows:

$$\begin{aligned} M_2 &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^2 e^{-x^2/2t}dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} (tx)\left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} te^{-x^2/2t}dx \\ &= t \end{aligned}$$

We conclude from this that the variance is $V = M_2 = t$. □

Here is another result, which is the key one for the study of the normal laws:

THEOREM 7.14. *We have the following formula, valid for any $t > 0$:*

$$F_{g_t}(x) = e^{-tx^2/2}$$

In particular, the normal laws satisfy $g_s * g_t = g_{s+t}$, for any $s, t > 0$.

PROOF. The Fourier transform formula can be established as follows:

$$\begin{aligned} F_{g_t}(x) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-y^2/2t+ixy}dy \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-(y/\sqrt{2t}-\sqrt{t/2}ix)^2-tx^2/2}dy \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-z^2-tx^2/2}\sqrt{2t}dz \\ &= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\int_{\mathbb{R}} e^{-z^2}dz \\ &= \frac{1}{\sqrt{\pi}}e^{-tx^2/2}\cdot\sqrt{\pi} \\ &= e^{-tx^2/2} \end{aligned}$$

As for the last assertion, this follows from the fact that $\log F_{g_t}$ is linear in t. □

We are now ready to state and prove the CLT, as follows:

THEOREM 7.15 (CLT). *Given random variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i \sim g_t$$

where g_t is the Gaussian law of parameter t, having as density $\frac{1}{\sqrt{2\pi t}}e^{-y^2/2t}dy$.

PROOF. With $F_f(x)=\mathbb{E}(e^{ixf})$ as usual, in terms of moments, we have:

$$\begin{aligned}F_f(x)&=\mathbb{E}\left(\sum_{k=0}^\infty\frac{(ixf)^k}{k!}\right)\\&=\sum_{k=0}^\infty\frac{(ix)^k\mathbb{E}(f^k)}{k!}\\&=\sum_{k=0}^\infty\frac{i^kM_k(f)}{k!}\,x^k\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}F(x)&=\left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n\\&=\left[1-\frac{tx^2}{2n}+O(n^{-2})\right]^n\\&\simeq\left[1-\frac{tx^2}{2n}\right]^n\\&\simeq e^{-tx^2/2}\end{aligned}$$

But this latter function being the Fourier transform of g_t, we obtain the result. □

Let us discuss now some further properties of the normal law. We first have:

PROPOSITION 7.16. *The even moments of the normal law are the numbers*

$$M_k(g_t)=t^{k/2}\times k!!$$

where $k!!=(k-1)(k-3)(k-5)\ldots$, and the odd moments vanish.

PROOF. We have the following computation, valid for any integer $k\in\mathbb{N}$:

$$\begin{aligned}M_k&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}y^ke^{-y^2/2t}dy\\&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}(ty^{k-1})\left(-e^{-y^2/2t}\right)'dy\\&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}t(k-1)y^{k-2}e^{-y^2/2t}dy\\&=t(k-1)\times\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}y^{k-2}e^{-y^2/2t}dy\\&=t(k-1)M_{k-2}\end{aligned}$$

Now recall from the proof of Proposition 7.13 that we have $M_0=1$, $M_1=0$. Thus by recurrence, we are led to the formula in the statement. □

We have the following alternative formulation of the above result:

PROPOSITION 7.17. *The moments of the normal law are the numbers*

$$M_k(g_t)=t^{k/2}|P_2(k)|$$

where $P_2(k)$ is the set of pairings of $\{1,\ldots,k\}$.

PROOF. Let us count the pairings of $\{1,\ldots,k\}$. In order to have such a pairing, we must pair 1 with one of the numbers $2,\ldots,k$, and then use a pairing of the remaining $k-2$ numbers. Thus, we have the following recurrence formula:

$$|P_2(k)|=(k-1)|P_2(k-2)|$$

As for the initial data, this is $P_1=0$, $P_2=1$. Thus, we are led to the result. □

We are not done yet, and here is one more improvement of the above:

THEOREM 7.18. *The moments of the normal law are the numbers*

$$M_k(g_t)=\sum_{\pi\in P_2(k)}t^{|\pi|}$$

where $P_2(k)$ is the set of pairings of $\{1,\ldots,k\}$, and $|.|$ is the number of blocks.

PROOF. This follows indeed from Proposition 7.17, because the number of blocks of a pairing of $\{1,\ldots,k\}$ is trivially $k/2$, independently of the pairing. □

Let us discuss now the complex analogues of all this, with a notion of complex normal, or Gaussian law. To start with, we have the following definition:

DEFINITION 7.19. *The complex normal, or Gaussian law of parameter $t>0$ is*

$$G_t=law\left(\frac{1}{\sqrt{2}}(a+ib)\right)$$

where a,b are independent, each following the law g_t.

In short, the complex normal laws appear as natural complexifications of the real normal laws. As in the real case, these measures form convolution semigroups:

PROPOSITION 7.20. *The complex Gaussian laws have the property*

$$G_s*G_t=G_{s+t}$$

for any $s,t>0$, and so they form a convolution semigroup.

PROOF. This follows indeed from the real result, namely $g_s*g_t=g_{s+t}$, established in Theorem 7.14, simply by taking real and imaginary parts. □

We have as well the following complex analogue of the CLT:

THEOREM 7.21 (CCLT). *Given complex variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with common variance $t > 0$, we have*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i \sim G_t$$

with $n \to \infty$, in moments.

PROOF. This follows indeed from the real CLT, established in Theorem 7.15, simply by taking the real and imaginary parts of all the variables involved. □

Finally, it is possible to talk as well about moments. We will be back to this.

7c. Infinite divisibility

According to our various computations above, all the main limiting laws in probability are infinitely divisible, and with this coming in fact as well from the limiting theorems themselves, somehow by definition. In general, the abstract study of the infinitely divisible measures in an interesting question, related to the moment method, and with a lot of subtle complex analysis involved. And, needless to say, having applications too.

7d. Random matrices

We have already met the Wigner, Marchenko-Pastur, arcsine and square root of arcsine laws in the above, somehow by chance, in relation with the moment method, and more specifically as what comes out of Stieltjes inversion, with the "simplest" data for our sequence of moments. Time now to discuss what is really behind these laws. This is an exciting theory, called Random Matrix Theory (RMT), that we will discuss now.

7e. Exercises

Exercises:

EXERCISE 7.22.

EXERCISE 7.23.

EXERCISE 7.24.

EXERCISE 7.25.

EXERCISE 7.26.

EXERCISE 7.27.

Bonus exercise.

CHAPTER 8

Haar integration

8a. Locally compact groups

We have seen that measure theory can be rigorously developed, along with some basic applications, both mathematically as measure theory, or also mathematically I guess, as probability theory. However, there is a bug with all this, at the level of examples and further applications. Indeed, passed a few tricks that we can do with Dirac masses, our measures were all of the following form, with $g:\mathbb{R}^N\to\mathbb{C}$ being a certain density:

$$d\mu(x)=g(x)dx$$

Thus, when it comes to integrating functions $f:\mathbb{R}^N\to\mathbb{C}$ with respect to such measures μ, we are in fact back to basic integration in N dimensions, as we knew it from standard multivariable calculus, with this coming from the following formula:

$$\int_{\mathbb{R}^N}f(x)d\mu(x)=\int_{\mathbb{R}^N}f(x)g(x)dx$$

Of course, you can argue that we can do more complicated things, such as integrating on curves and surfaces and so on, but the truth is that, thinking well, the dx will always be there, and we will be integrating which respect to it, as in standard calculus.

So, it is about dx and its generalizations that we must talk about, if we want our measure theory and probability theory to really take of. And the answer here comes from the locally compact groups, generalizing $\mathbb{R}^N$, with each having its own dx.

Let us begin with the compact group case, which is elementary. We will need:

Definition 8.1. *A unitary representation of a compact group G is a continuous group morphism into a unitary group*

$$v:G\to U_N\quad,\quad g\to v_g$$

which can be faithful or not. The character of such a representation is the function

$$\chi:G\to\mathbb{C}\quad,\quad g\to Tr(v_g)$$

where Tr is the usual, unnormalized trace of the $N\times N$ matrices.

At the level of examples, most of the compact groups that we met so far, finite or continuous, naturally appear as closed subgroups $G\subset U_N$. In this case, the embedding $G\subset U_N$ is of course a representation, called fundamental representation. In general now, let us first discuss the various operations on the representations. We have here:

Proposition 8.2. *The representations of a compact group G are subject to:*

1. *Making sums. Given representations v,w, of dimensions N,M, their sum is the $N+M$-dimensional representation $v+w=diag(v,w)$.*
2. *Making products. Given representations v,w, of dimensions N,M, their product is the NM-dimensional representation $(v\otimes w)_{ia,jb}=v_{ij}w_{ab}$.*
3. *Taking conjugates. Given a N-dimensional representation v, its conjugate is the N-dimensional representation $(\bar{v})_{ij}=\bar{v}_{ij}$.*
4. *Spinning by unitaries. Given a N-dimensional representation v, and a unitary $U\in U_N$, we can spin v by this unitary, $v\to UvU^*$.*

Proof. The fact that the operations in the statement are indeed well-defined, among morphisms from G to unitary groups, is indeed clear from definitions. □

In relation now with characters, we have the following result:

Proposition 8.3. *We have the following formulae, regarding characters*

$$\chi_{v+w}=\chi_v+\chi_w\quad,\quad\chi_{v\otimes w}=\chi_v\chi_w\quad,\quad\chi_{\bar{v}}=\bar{\chi}_v\quad,\quad\chi_{UvU^*}=\chi_v$$

in relation with the basic operations for the representations.

Proof. All these assertions are elementary, by using the following well-known trace formulae, valid for any square matrices V,W, and any unitary U:

$$Tr(diag(V,W))=Tr(V)+Tr(W)\quad,\quad Tr(V\otimes W)=Tr(V)Tr(W)$$

$$Tr(\bar{V})=\overline{Tr(V)}\quad,\quad Tr(UVU^*)=Tr(V)$$

Thus, we are led to the formulae in the statement. □

Assume now that we are given a closed subgroup $G\subset U_N$. By using the above operations, we can construct a whole family of representations of G, as follows:

Definition 8.4. *Given a closed subgroup $G\subset U_N$, its Peter-Weyl representations are the various tensor products between the fundamental representation and its conjugate:*

$$v:G\subset U_N\quad,\quad\bar{v}:G\subset U_N$$

We denote these tensor products $v^{\otimes k}$, with $k=\circ\bullet\bullet\circ\ldots$ being a colored integer, with the colored tensor powers being defined according to the rules

$$v^{\otimes\circ}=v\quad,\quad v^{\otimes\bullet}=\bar{v}\quad,\quad v^{\otimes kl}=v^{\otimes k}\otimes v^{\otimes l}$$

and with the convention that $v^{\otimes\emptyset}$ is the trivial representation $1:G\to U_1$.

Here are a few examples of such representations, namely those coming from the colored integers of length 2, which will often appear in what follows:

$$v^{\otimes\circ\circ}=v\otimes v\quad,\quad v^{\otimes\circ\bullet}=v\otimes\bar{v}$$

$$v^{\otimes\bullet\circ}=\bar{v}\otimes v\quad,\quad v^{\otimes\bullet\bullet}=\bar{v}\otimes\bar{v}$$

In relation now with characters, we have the following result:

Proposition 8.5. *The characters of the Peter-Weyl representations are given by*

$$\chi_{v^{\otimes k}}=(\chi_v)^k$$

with the colored powers being given by $\chi^\circ=\chi$, $\chi^\bullet=\bar{\chi}$ and multiplicativity.

Proof. This follows indeed from the additivity, multiplicativity and conjugation formulae from Proposition 8.3, via the conventions in Definition 8.4. □

Getting back now to our motivations, we can see the interest in the above constructions. Indeed, the joint moments of the main character $\chi=\chi_v$ and its adjoint $\bar{\chi}=\chi_{\bar{v}}$ are the expectations of the characters of various Peter-Weyl representations:

$$\int_G\chi^k=\int_G\chi_{v^{\otimes k}}$$

In order to advance, we must develop some general theory. Let us start with:

Definition 8.6. *Given a compact group G, and two of its representations,*

$$v:G\to U_N\quad,\quad w:G\to U_M$$

we define the space of intertwiners between these representations as being

$$Hom(v,w)=\left\{T\in M_{M\times N}(\mathbb{C})\Big|Tv_g=w_gT,\forall g\in G\right\}$$

and we use the following conventions:

1. *We use the notations $Fix(v)=Hom(1,v)$, and $End(v)=Hom(v,v)$.*
2. *We write $v\sim w$ when $Hom(v,w)$ contains an invertible element.*
3. *We say that v is irreducible, and write $v\in Irr(G)$, when $End(v)=\mathbb{C}1$.*

The terminology here is standard, with Fix, Hom, End standing for fixed points, homomorphisms and endomorphisms. We will see later that irreducible means indecomposable, in a suitable sense. Here are now a few basic results, regarding these spaces:

Proposition 8.7. *The spaces of intertwiners have the following properties:*

1. *$T\in Hom(v,w),S\in Hom(w,z)\implies ST\in Hom(v,z)$.*
2. *$S\in Hom(v,w),T\in Hom(z,t)\implies S\otimes T\in Hom(v\otimes z,w\otimes t)$.*
3. *$T\in Hom(v,w)\implies T^*\in Hom(w,v)$.*

In abstract terms, we say that the Hom spaces form a tensor $*$-category.

PROOF. All the formulae in the statement are clear from definitions, via elementary computations. As for the last assertion, this is something coming from (1,2,3). We will be back to tensor categories later on, with more details on this latter fact. □

As a main consequence of the above result, we have:

PROPOSITION 8.8. *Given a representation $v : G \to U_N$, the linear space*

$$End(v) \subset M_N(\mathbb{C})$$

is a $*$-algebra, with respect to the usual involution of the matrices.

PROOF. By definition, $End(v)$ is a linear subspace of $M_N(\mathbb{C})$. We know from Proposition 8.7 (1) that this subspace $End(v)$ is a subalgebra of $M_N(\mathbb{C})$, and then we know as well from Proposition 8.7 (3) that this subalgebra is stable under the involution $*$. Thus, what we have here is a $*$-subalgebra of $M_N(\mathbb{C})$, as claimed. □

In order to exploit the above fact, we will need a basic result from linear algebra, stating that any $*$-algebra $A \subset M_N(\mathbb{C})$ decomposes as a direct sum, as follows:

$$A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

Indeed, let us write the unit $1 \in A$ as $1 = p_1 + \ldots + p_k$, with $p_i \in A$ being central minimal projections. Then each of the spaces $A_i = p_i A p_i$ is a subalgebra of A, and we have a decomposition $A = A_1 \oplus \ldots \oplus A_k$. But since each central projection $p_i \in A$ was chosen minimal, we have $A_i \simeq M_{N_i}(\mathbb{C})$, with $N_i = rank(p_i)$, as desired.

We can now formulate our first Peter-Weyl type theorem, as follows:

THEOREM 8.9 (PW1). *Let $v : G \to U_N$ be a representation, consider the algebra $A = End(v)$, and write its unit $1 = p_1 + \ldots + p_k$ as above. We have then*

$$v = v_1 + \ldots + v_k$$

with each v_i being an irreducible representation, obtained by restricting v to $Im(p_i)$.

PROOF. This basically follows from Proposition 8.8, as follows:

(1) We first associate to our representation $v : G \to U_N$ the corresponding action map on $\mathbb{C}^N$. If a linear subspace $W \subset \mathbb{C}^N$ is invariant, the restriction of the action map to W is an action map too, which must come from a subrepresentation $w \subset v$.

(2) Consider now a projection $p \in End(v)$. From $pv = vp$ we obtain that the linear space $W = Im(p)$ is invariant under v, and so this space must come from a subrepresentation $w \subset v$. It is routine to check that the operation $p \to w$ maps subprojections to subrepresentations, and minimal projections to irreducible representations.

(3) With these preliminaries in hand, let us decompose the algebra $End(v)$ as above, by using the decomposition $1 = p_1 + \ldots + p_k$ into central minimal projections. If we

denote by $v_i \subset v$ the subrepresentation coming from the vector space $V_i = Im(p_i)$, then we obtain in this way a decomposition $v = v_1 + \ldots + v_k$, as in the statement. □

Here is now our second Peter-Weyl theorem, complementing Theorem 8.9:

THEOREM 8.10 (PW2). *Given a closed subgroup $G \subset_v U_N$, any of its irreducible smooth representations*

$$w : G \to U_M$$

appears inside a tensor product of the fundamental representation v and its adjoint $\bar{v}$.

PROOF. Given a representation $w : G \to U_M$, we define the space of coefficients $C_w \subset C(G)$ of this representation as being the following linear space:

$$C_w = span\Big[g \to w(g)_{ij}\Big]$$

With this notion in hand, the result can be deduced as follows:

(1) The construction $w \to C_w$ is functorial, in the sense that it maps subrepresentations into linear subspaces. This is indeed something which is routine to check.

(2) A closed subgroup $G \subset_v U_N$ is a Lie group, and a representation $w : G \to U_M$ is smooth when we have an inclusion $C_w \subset < C_v >$. This is indeed well-known.

(3) By definition of the Peter-Weyl representations, as arbitrary tensor products between the fundamental representation v and its conjugate $\bar{v}$, we have:

$$< C_v > = \sum_k C_{v^{\otimes k}}$$

(4) Now by putting together the above observations (2,3) we conclude that we must have an inclusion as follows, for certain exponents $k_1, \ldots, k_p$:

$$C_w \subset C_{v^{\otimes k_1} \oplus \ldots \oplus v^{\otimes k_p}}$$

(5) By using now (1), we deduce that we have an inclusion $w \subset v^{\otimes k_1} \oplus \ldots \oplus v^{\otimes k_p}$, and by applying Theorem 8.9, this leads to the conclusion in the statement. □

With this is hand, we can now talk about integration over G. This is something quite technical, the idea being that the uniform measure μ over G can be constructed by starting with an arbitrary probability measure η, and setting:

$$\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \eta^{*k}$$

Thus, our next task will be that of proving this result. It is convenient, for this purpose, to work with the integration functionals with respect to the various measures on G, instead of the measures themselves. Let us begin with the following key result:

Proposition 8.11. *Given a unital positive linear form $\psi : C(G) \to \mathbb{C}$, the limit*

$$\int_\psi f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f)$$

exists, and for a coefficient of a representation $f = (\tau \otimes id)w$ we have

$$\int_\psi f = \tau(P)$$

where P is the orthogonal projection onto the 1-eigenspace of $(id \otimes \psi)w$.

Proof. By linearity it is enough to prove the second assertion. More precisely, we can have the whole result proved if we can establish the following formula:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f) = \tau(P)$$

In order to prove this latter formula, observe that we have:

$$\psi^{*k}(f) = (\tau \otimes \psi^{*k})w = \tau((id \otimes \psi^{*k})w)$$

Consider the matrix $M = (id \otimes \psi)w$. In terms of this matrix, we have:

$$((id \otimes \psi^{*k})w)_{i_0 i_{k+1}} = \sum_{i_1 \ldots i_k} M_{i_0 i_1} \ldots M_{i_k i_{k+1}} = (M^k)_{i_0 i_{k+1}}$$

Thus we have the following formula, valid for any integer $k \in \mathbb{N}$:

$$(id \otimes \psi^{*k})w = M^k$$

It follows that our Cesàro limit is given by the following formula:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \tau(M^k) = \tau\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n M^k\right)$$

Now since w is unitary we have $||w|| = 1$, and we obtain from this that we have:

$$||M|| \leq 1$$

Thus, in the above expression, the Cesàro limit on the right converges, and equals the orthogonal projection onto the 1-eigenspace of M:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n M^k = P$$

Thus our initial Cesàro limit converges as well, to $\tau(P)$, as desired. □

When the linear ψ is chosen faithful, we have the following finer result:

PROPOSITION 8.12. *Given a faithful unital linear form $\psi \in C(G)^*$, the limit*

$$\int_\psi f = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \psi^{*k}(f)$$

exists, and is independent of ψ, given on coefficients of representations by

$$\left(id \otimes \int_\psi\right) w = P$$

where P is the orthogonal projection onto $Fix(w) = \left\{\xi \in \mathbb{C}^n \middle| w\xi = \xi\right\}$.

PROOF. In view of Proposition 8.11, it remains to prove that when ψ is faithful, the 1-eigenspace of $M = (id \otimes \psi)w$ equals the space $Fix(w)$.

"$\supset$" This inclusion is clear, and for any ψ, because $w\xi = \xi \implies M\xi = \xi$.

"$\subset$" Here we must prove that, if ψ is faithful, we have $M\xi = \xi \implies w\xi = \xi$. For this purpose, assume that we have $M\xi = \xi$, and consider the following function:

$$f = \sum_i \left(\sum_j w_{ij}\xi_j - \xi_i\right)\left(\sum_k w_{ik}\xi_k - \xi_i\right)^*$$

We must prove that we have $f = 0$. Since w is unitary, we have:

$$\begin{aligned}
f &= \sum_i \left(\sum_j \left(w_{ij}\xi_j - \frac{1}{N}\xi_i\right)\right)\left(\sum_k \left(w_{ik}^*\bar{\xi}_k - \frac{1}{N}\bar{\xi}_i\right)\right) \\
&= \sum_{ijk} w_{ij}w_{ik}^*\xi_j\bar{\xi}_k - \frac{1}{N}w_{ij}\xi_j\bar{\xi}_i - \frac{1}{N}w_{ik}^*\xi_i\bar{\xi}_k + \frac{1}{N^2}\xi_i\bar{\xi}_i \\
&= \sum_j |\xi_j|^2 - \sum_{ij} w_{ij}\xi_j\bar{\xi}_i - \sum_{ik} w_{ik}^*\xi_i\bar{\xi}_k + \sum_i |\xi_i|^2 \\
&= ||\xi||^2 - < w\xi, \xi > - \overline{< w\xi, \xi >} + ||\xi||^2 \\
&= 2(||\xi||^2 - Re(< w\xi, \xi >))
\end{aligned}$$

By using now our assumption $M\xi = \xi$, we obtain from this that we have:

$$\begin{aligned}
\psi(f) &= 2\psi(||\xi||^2 - Re(< w\xi, \xi >)) \\
&= 2(||\xi||^2 - Re(< M\xi, \xi >)) \\
&= 2(||\xi||^2 - ||\xi||^2) \\
&= 0
\end{aligned}$$

Now since ψ is faithful, this gives $f = 0$, and so $w\xi = \xi$, as claimed. □

We can now formulate a main result, as follows:

THEOREM 8.13. *Any compact group G has a unique Haar integration, which can be constructed by starting with any faithful positive unital form $\psi\in C(G)^*$, and setting:*

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\psi^{*k}$$

Moreover, for any representation w we have the formula

$$\left(id\otimes\int_G\right)w=P$$

where P is the orthogonal projection onto $Fix(w)=\left\{\xi\in\mathbb C^n\middle|w\xi=\xi\right\}$.

PROOF. Let us first go back to the general context of Proposition 8.11. Since convolving one more time with ψ will not change the Cesàro limit appearing there, the functional $\int_\psi\in C(G)^*$ constructed there has the following invariance property:

$$\int_\psi*\psi=\psi*\int_\psi=\int_\psi$$

In the case where ψ is assumed to be faithful, as in Proposition 8.12, our claim is that we have the following formula, valid this time for any $\varphi\in C(G)^*$:

$$\int_\psi*\varphi=\varphi*\int_\psi=\varphi(1)\int_\psi$$

Indeed, it is enough to prove this formula on a coefficient of a corepresentation:

$$f=(\tau\otimes id)w$$

In order to do so, consider the following two matrices:

$$P=\left(id\otimes\int_\psi\right)w\quad,\quad Q=(id\otimes\varphi)w$$

We have then the following formulae, which all follow from definitions:

$$\left(\int_\psi*\varphi\right)f=\tau(PQ)\quad,\quad\left(\varphi*\int_\psi\right)f=\tau(QP)\quad,\quad\varphi(1)\int_\psi f=\varphi(1)\tau(P)$$

Thus, in order to prove our claim, it is enough to establish the following formula:

$$PQ=QP=\psi(1)P$$

But this latter formula follows from the fact, coming from Proposition 8.12, that $P=(id\otimes\int_\psi)w$ equals the orthogonal projection onto $Fix(w)$. Thus, we have proved our claim, namely that we have the following formula, valid for any $\varphi\in C(G)^*$:

$$\int_\psi*\varphi=\varphi*\int_\psi=\varphi(1)\int_\psi$$

Now observe that, with $\Delta f(g\otimes h)=f(gh)$, this formula can be written as follows:

$$\varphi\left(\int_\psi\otimes id\right)\Delta=\varphi\left(id\otimes\int_\psi\right)\Delta=\varphi\int_\psi(.)1$$

This formula being true for any $\varphi\in C(G)^*$, we can simply delete φ, and we conclude that $\int_G=\int_\psi$ has the required left and right invariance property, namely:

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

Finally, the uniqueness is clear as well, because if we have two invariant integrals $\int_G,\int_G'$, then their convolution equals on one hand $\int_G$, and on the other hand, $\int_G'$. □

Summarizing, we can integrate over G. In fact, we can say even more, as follows:

THEOREM 8.14. *The Haar integration over a closed subgroup $G\subset U_N$ is given by*

$$\int_G g_{i_1j_1}^{e_1}\ldots g_{i_kj_k}^{e_k}\,dg=\sum_{\pi,\nu\in D(k)}\delta_\pi(i)\delta_\nu(j)W_k(\pi,\nu)$$

for any colored integer $k=e_1\ldots e_k$ and any multi-indices i,j, where $D(k)$ is a linear basis of $Fix(v^{\otimes k})$, the associated generalized Kronecker symbols are given by

$$\delta_\pi(i)=<\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>$$

and $W_k=G_k^{-1}$ is the inverse of the Gram matrix, $G_k(\pi,\nu)=<\pi,\nu>$.

PROOF. This is something old and classical, the idea being as follows:

(1) We know from Peter-Weyl theory that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(v^{\otimes k})=span(D(k))$$

(2) Consider now the following linear map, with $D(k)=\{\xi_k\}$ being as above:

$$E(x)=\sum_{\pi\in D(k)}<x,\xi_\pi>\xi_\pi$$

(3) By a standard linear algebra computation, it follows that we have $P=WE$, where W is the inverse of the restriction of E to the following space:

$$K=span\left(T_\pi\Big|\pi\in D(k)\right)$$

(4) But this restriction is the linear map given by the Gram matrix G_k, and so W is the linear map given by the Weingarten matrix $W_k=G_k^{-1}$, and this gives the result. □

We will be back to compact groups later. Getting now to the locally compact case, we will need to prove a tough theorem, stating that any locally compact group G has indeed a uniform integration, called Haar integration. And with a subtlety appearing on the way, related to the Radon-Nikodym theorem, coming from the fact that certain interesting locally compact groups have a different dx at left and at right.

8b. Homogeneous spaces

More generally, in relation with our quest of looking for more and more general dx type beasts, that we can use in our daily computations, we can talk about homogeneous spaces G/H over the locally compact groups G, which have a Haar measure too.

8c. Ergodic theorems

Speaking group theory and Haar measures, an interesting theorem here, having countless applications, is the ergodic theorem. There are actually many possible statements here, with most of them actually needing some function space theory, that we do not know yet. However, we can at least provide here an introduction, to this interesting subject.

8d. Weingarten formula

Getting back now to the usual group case, an interesting question is that of explicitely computing the Haar measure. In the compact Lie group case, where $G\subset U_N$ is a closed subgroup, this can indeed be done, via advanced representation theory, under a certain technical "easiness" assumption on our group G. We discuss here all this, following Weyl, and Peter, Schur, Brauer, Tannaka and the other group theory greats, and then more recently following Weingarten, and Collins, Śniady and others. All this will be quite interesting, reminding a bit the random matrices that we met at the end of chapter 7, and having as well a distinct and exciting theoretical physics flavor.

But probably enough advertisement, let us get to work. In order to reach to advanced integration, we must first develop some further Peter-Weyl theory. We first have:

THEOREM 8.15 (PW3). *The dense subalgebra $\mathcal{C}(G)\subset C(G)$ generated by the coefficients of the fundamental representation decomposes as a direct sum*

$$\mathcal{C}(G)=\bigoplus_{w\in Irr(G)}M_{\dim(w)}(\mathbb{C})$$

with the summands being pairwise orthogonal with respect to the scalar product

$$<f,g>=\int_Gf\bar{g}$$

where $\int_G$ is the Haar integration over G.

PROOF. By combining the previous two Peter-Weyl results, from the beginning of this chapter, we deduce that we have a linear space decomposition as follows:

$$\mathcal{C}(G)=\sum_{w\in Irr(G)}C_w=\sum_{w\in Irr(G)}M_{\dim(w)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible representations $v,w\in Irr(G)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C_v\perp C_w$$

But this follows by Frobenius duality, by integrating. Let us set indeed:

$$P_{ia,jb}=\int_G v_{ij}\bar{w}_{ab}$$

Then P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(v,w)=\{0\}$$

Thus we have $P=0$, and this gives the result. □

Finally, we have the following result, completing the Peter-Weyl theory:

THEOREM 8.16 (PW4). *The characters of irreducible representations belong to the algebra*

$$\mathcal{C}(G)_{central}=\left\{f\in\mathcal{C}(G)\Big|f(gh)=f(hg),\forall g,h\in G\right\}$$

called algebra of central functions on G, and form an orthonormal basis of it.

PROOF. Observe first that $\mathcal{C}(G)_{central}$ is indeed an algebra, which contains all the characters. Conversely, consider a function $f\in\mathcal{C}(G)$, written as follows:

$$f=\sum_{w\in Irr(G)}f_w$$

The condition $f\in\mathcal{C}(G)_{central}$ states then that for any $w\in Irr(G)$, we must have:

$$f_w\in\mathcal{C}(G)_{central}$$

But this means that f_w must be a scalar multiple of χ_w, so the characters form a basis of $\mathcal{C}(G)_{central}$, as stated. Also, the fact that we have an orthogonal basis follows from Theorem 8.15. As for the fact that the characters have norm 1, this follows from:

$$\int_G\chi_w\bar{\chi}_w=\sum_{ij}\int_G w_{ii}\bar{w}_{jj}=\sum_i\frac{1}{M}=1$$

Here we have used the fact, coming from Frobenius duality, that the various integrals $\int_G w_{ij}\bar{w}_{kl}$ form altogether the orthogonal projection onto the following vector space:

$$Fix(w\otimes\bar{w})\simeq End(w)=\mathbb{C}1$$

Thus, the proof of our theorem is now complete. □

Moving ahead, we will need as well Tannakian duality. Let us start with:

Definition 8.17. *The Tannakian category associated to a closed subgroup $G \subset_v U_N$ is the collection $C_G = (C_G(k,l))$ of vector spaces*

$$C_G(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

where the representations $v^{\otimes k}$ with $k = \circ \bullet \bullet \circ \ldots$ colored integer, defined by

$$v^{\otimes \emptyset} = 1 \quad , \quad v^{\otimes \circ} = v \quad , \quad v^{\otimes \bullet} = \bar{v}$$

and multiplicativity, $v^{\otimes kl} = v^{\otimes k} \otimes v^{\otimes l}$, are the Peter-Weyl representations.

Let us make a summary of what we have so far, regarding these spaces $C_G(k,l)$. In order to formulate our result, let us start with the following definition:

Definition 8.18. *Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C = (C(k,l))$ of linear spaces*

$$C(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

1. *$S, T \in C$ implies $S \otimes T \in C$.*
2. *If $S, T \in C$ are composable, then $ST \in C$.*
3. *$T \in C$ implies $T^* \in C$.*
4. *$C(k,k)$ contains the identity operator.*
5. *$C(\emptyset, k)$ with $k = \circ\bullet, \bullet\circ$ contain the operator $R : 1 \to \sum_i e_i \otimes e_i$.*
6. *$C(kl, lk)$ with $k,l = \circ, \bullet$ contain the flip operator $\Sigma : a \otimes b \to b \otimes a$.*

Here the tensor power Hilbert spaces $H^{\otimes k}$, with $k = \circ \bullet \bullet \circ \ldots$ being a colored integer, are defined by the following formulae, and multiplicativity:

$$H^{\otimes \emptyset} = \mathbb{C} \quad , \quad H^{\otimes \circ} = H \quad , \quad H^{\otimes \bullet} = \bar{H} \simeq H$$

With these conventions, we have the following result, which is elementary:

Theorem 8.19. *For a closed subgroup $G \subset_v U_N$, the associated Tannakian category*

$$C_G(k,l) = Hom(v^{\otimes k}, v^{\otimes l})$$

is a tensor category over the Hilbert space $H = \mathbb{C}^N$.

Proof. We know that the fundamental representation v acts on the Hilbert space $H = \mathbb{C}^N$, and that its conjugate $\bar{v}$ acts on the Hilbert space $\bar{H} = \mathbb{C}^N$. Now by multiplicativity we conclude that any Peter-Weyl representation $v^{\otimes k}$ acts on the Hilbert space $H^{\otimes k}$, and so that we have embeddings as in Definition 8.18, as follows:

$$C_G(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

Regarding now the fact that the axioms (1-6) in Definition 8.18 are indeed satisfied, this is something that we basically already know. To be more precise, (1-4) are clear, and (5) follows from the fact that each element $g\in G$ is a unitary, which gives:

$$R\in Hom(1,g\otimes\bar{g})\quad,\quad R\in Hom(1,\bar{g}\otimes g)$$

As for (6), this is something trivial, coming from the fact that the matrix coefficients $g\to g_{ij}$ and their complex conjugates $g\to\bar{g}_{ij}$ commute with each other. □

Our purpose now will be that of showing that any closed subgroup $G\subset U_N$ is uniquely determined by its Tannakian category $C_G=(C_G(k,l))$. We first have:

Proposition 8.20. *Given a tensor category $C=(C(k,l))$ over a finite dimensional Hilbert space $H\simeq\mathbb{C}^N$, the following construction,*

$$G_C=\left\{g\in U_N\Big|Tg^{\otimes k}=g^{\otimes l}T\ ,\ \forall k,l,\forall T\in C(k,l)\right\}$$

produces a closed subgroup $G_C\subset U_N$.

Proof. This is something elementary, with the fact that the closed subset $G_C\subset U_N$ constructed in the statement is indeed stable under the multiplication, unit and inversion operation for the unitary matrices $g\in U_N$ being clear from definitions. □

We can now formulate the Tannakian duality result, as follows:

Theorem 8.21. *The above Tannakian constructions*

$$G\to C_G\quad,\quad C\to G_C$$

are bijective, and inverse to each other.

Proof. This is something quite technical, obtained by doing some abstract algebra, and for full details here, we refer to the Tannakian duality literature. □

In order to reach now to more concrete things, let us formulate:

Definition 8.22. *Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l. A collection of subsets*

$$D=\bigsqcup_{k,l}D(k,l)$$

with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*
6. *The sets $P(k,\bar{k})$ with $|k|=2$ contain the crossing partition $\slashed{X}$.*

Let us formulate as well the following definition:

Definition 8.23. *Given a partition $\pi\in P(k,l)$ and an integer $N\in\mathbb N$, we can construct a linear map between tensor powers of $\mathbb C^N$,*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb C^N$,

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and with the coefficients on the right being Kronecker type symbols,

$$\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}\in\{0,1\}$$

whose values depend on whether the indices fit or not.

To be more precise, we put the indices of i,j on the legs of π, in the obvious way. In case all the blocks of π contain equal indices of i,j, we set $\delta_\pi\binom{i}{j}=1$. Otherwise, we set $\delta_\pi\binom{i}{j}=0$. The relation with the Tannakian categories comes from:

Proposition 8.24. *The assignement $\pi\to T_\pi$ is categorical, in the sense that*

$$T_\pi\otimes T_\nu=T_{[\pi\nu]}\quad,\quad T_\pi T_\nu=N^{c(\pi,\nu)}T_{[^\nu_\pi]}\quad,\quad T_\pi^*=T_{\pi^*}$$

where $c(\pi,\nu)$ are certain integers, coming from the erased components in the middle.

Proof. This is something elementary, the first check being as follows:

$$\begin{aligned}&(T_\pi\otimes T_\nu)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\nu\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\nu]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&T_{[\pi\nu]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\end{aligned}$$

As for the other two formulae in the statement, their proof is similar. □

In relation now with the groups, we have the following result:

Theorem 8.25. *Each category of partitions $D=(D(k,l))$ produces a family of compact groups $G=(G_N)$, with $G_N\subset_v U_N$, via the formula*

$$Hom(v^{\otimes k},v^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

and the Tannakian duality correspondence.

PROOF. Given an integer $N\in\mathbb N$, consider the correspondence $\pi\to T_\pi$ constructed in Definition 8.23, and then the collection of linear spaces in the statement, namely:

$$C(k,l)=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

According to Proposition 8.24, and to our axioms for the categories of partitions, from Definition 8.22, this collection of spaces $C=(C(k,l))$ satisfies the axioms for the Tannakian categories, from Definition 8.18. Thus the Tannakian duality result, Theorem 8.21, applies, and provides us with a closed subgroup $G_N\subset_v U_N$ such that:

$$C(k,l)=Hom(v^{\otimes k},v^{\otimes l})$$

Thus, we are led to the conclusion in the statement. □

We can now formulate a key definition, as follows:

DEFINITION 8.26. *A closed subgroup $G\subset_v U_N$ is called easy when we have*

$$Hom(v^{\otimes k},v^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

for any colored integers k,l, for a certain category of partitions $D\subset P$.

As basic examples of such groups, due to Brauer, we have:

THEOREM 8.27. *We have the following results:*

1. *The unitary group U_N is easy, coming from the category $\mathcal P_2$.*
2. *The orthogonal group O_N is easy too, coming from the category P_2.*

PROOF. This is something very standard, the idea being as follows:

(1) The group U_N being defined via the relations $v^*=v^{-1}$, $v^t=\bar v^{-1}$, the associated Tannakian category is $C=span(T_\pi|\pi\in D)$, with:

$$D=<\overset{\cap}{\circ\bullet}\,,\,\overset{\cap}{\bullet\circ}>=\mathcal P_2$$

(2) The group $O_N\subset U_N$ being defined by imposing the relations $v_{ij}=\bar v_{ij}$, the associated Tannakian category is $C=span(T_\pi|\pi\in D)$, with:

$$D=<\mathcal P_2,\overset{\circ}{\underset{\bullet}{|}},\overset{\bullet}{\underset{\circ}{|}}>=P_2$$

Thus, we are led to the conclusion in the statement. □

There are many other easy groups, and as a basic example here, we have:

THEOREM 8.28. *The symmetric group S_N, regarded as group of unitary matrices,*

$$S_N\subset O_N\subset U_N$$

via the permutation matrices, is easy, coming from the category of all partitions P.

Proof. Consider the easy group $G\subset O_N$ coming from the category of all partitions P. Since P is generated by the one-block partition $\mu\in P(2,1)$, we have:

$$C(G)=C(O_N)\Big/\Big\langle T_\mu\in Hom(v^{\otimes 2},v)\Big\rangle$$

Since we have $T_\mu(e_i\otimes e_j)=\delta_{ij}e_i$, the above relations read:

$$T_\mu\in Hom(v^{\otimes 2},v)\iff v_{ij}v_{ik}=\delta_{jk}v_{ij},\forall i,j,k$$

In other words, the elements v_{ij} must be projections, and these projections must be pairwise orthogonal on the rows of $v=(v_{ij})$. We conclude that $G\subset O_N$ is the subgroup of matrices $g\in O_N$ having the property $g_{ij}\in\{0,1\}$. Thus we have $G=S_N$, as claimed. □

We can now formulate the Weingarten formula, as follows:

Theorem 8.29. *For an easy group $G\subset U_N$, coming from a category of partitions $D=(D(k,l))$, we have the Weingarten formula*

$$\int_G g_{i_1j_1}^{e_1}\ldots g_{i_kj_k}^{e_k}\,dg=\sum_{\pi,\nu\in D(k)}\delta_\pi(i)\delta_\nu(j)W_{kN}(\pi,\nu)$$

for any $k=e_1\ldots e_k$ and any i,j, where $D(k)=D(\emptyset,k)$, δ are usual Kronecker type symbols, checking whether the indices match, and $W_{kN}=G_{kN}^{-1}$, with

$$G_{kN}(\pi,\nu)=N^{|\pi\vee\nu|}$$

where $|.|$ is the number of blocks.

Proof. This follows from the abstract Weingarten formula, from Theorem 8.14. Indeed, in the easy group case the Kronecker type symbols there are then the usual ones, and the Gram matrix being as well the correct one, we obtain the result. □

There are many applications of this formula. We will be back to this.

8e. Exercises

Exercises:

Exercise 8.30.

Exercise 8.31.

Exercise 8.32.

Exercise 8.33.

Exercise 8.34.

Exercise 8.35.

Bonus exercise.

Part III

Function spaces

We are the seed of the new breed
We'll succeed, our time has come
We are the new, these words are true
Let the light of love shine through

CHAPTER 9

Banach spaces

9a. Normed spaces

Welcome to function space theory, also known as functional analysis. Although, at least in my opinion, the basics here, which are quite algebraic, rather deserve the name "functional algebra". But do not worry, we will keep things as analytic as possible.

The idea is that various types of functions form various types of infinite dimensional complex spaces, that we can study with the usual methods from linear algebra, complemented for dealing with ∞ by our favorite logic weapon, which is the Zorn lemma.

To start with, we can say a number of things about the general normed spaces, in analogy with what we know from usual linear algebra, in finite dimensions.

9b. Banach spaces

Although with this being not always the norm in advanced analysis, the most interesting normed spaces are those which are complete, called Banach spaces.

There are many interesting things that can be said about the Banach spaces. Also, examples of such Banach spaces abound, in questions coming from analysis. But more about examples later, in chapter 11 below, when discussing function spaces.

9c. Abstract results

Getting now to more advanced theory, we have many non-trivial things that can be said, about the Banach spaces, usually by using the Zorn lemma, and other tools from logic, and sometimes by using advanced linear algebra and classical analysis too.

All these statements about Banach spaces actually fall into two classes, theorems and conjectures. In this section we will discuss a number of theorems, coming with proofs, and in the next section we will discuss a number of exciting open questions.

9d. Open questions

There are many interesting open questions, regarding the Banach spaces.

9e. Exercises

Exercises:

Exercise 9.1.

Exercise 9.2.

Exercise 9.3.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

Bonus exercise.

CHAPTER 10

Hilbert spaces

10a. Hilbert spaces

We have the following definition, which will play a key role, in what follows:

Definition 10.1. *A Hilbert space is a complex vector space H with a scalar product $< x,y >$, which will be linear at left and antilinear at right,*

$$< \lambda x,y >= \lambda < x,y > \quad , \quad < x,\lambda y >= \bar{\lambda} < x,y >$$

and which is complete with respect to corresponding norm

$$||x|| = \sqrt{< x,x >}$$

in the sense that any sequence $\{x_n\}$ which is a Cauchy sequence, having the property $||x_n - x_m|| \to 0$ with $n,m \to \infty$, has a limit, $x_n \to x$.

Here our convention for the scalar products, written $< x,y >$ and being linear at left, is one among others, often used by mathematicians, and we will just use this, in the lack of a physicist with an axe around. As further comments now on Definition 10.1, there is some mathematics encapsulated there, needing some discussion. First, we have:

Theorem 10.2. *Given an index set I, which can be finite or not, the space of square-summable vectors having indices in I, namely*

$$l^2(I) = \left\{ (x_i)_{i\in I} \Big| \sum_i |x_i|^2 < \infty \right\}$$

is a Hilbert space, with scalar product as follows:

$$< x,y >= \sum_i x_i \bar{y}_i$$

When I is finite, $I = \{1,\ldots,N\}$, we obtain in this way the usual space $H = \mathbb{C}^N$.

Proof. We have already met such things before, but let us quickly recall this:

(1) We know that $l^2(I) \subset \mathbb{C}^I$ is the space of vectors satisfying $||x|| < \infty$. We want to prove that $l^2(I)$ is a vector space, that $< x,y >$ is a scalar product on it, that $l^2(I)$ is complete with respect to $||.||$, and finally that for $|I| < \infty$ we have $l^2(I) = \mathbb{C}^{|I|}$.

(2) The last assertion, $l^2(I)=\mathbb{C}^{|I|}$ for $|I|<\infty$, is clear, because in this case the sums are finite, so the condition $||x||<\infty$ is automatic. So, we know at least one thing.

(3) Regarding the rest, our claim here, which will more or less prove everything, is that for any two vectors $x,y\in l^2(I)$ we have the Cauchy-Schwarz inequality:

$$|<x,y>|\leq||x||\cdot||y||$$

(4) But this follows from the positivity of the following degree 2 quantity, depending on a real variable $t\in\mathbb{R}$, and on a variable on the unit circle, $w\in\mathbb{T}$:

$$f(t)=||twx+y||^2$$

(5) Now with Cauchy-Schwarz proved, everything is straightforward. We first obtain, by raising to the square and expanding, that for any $x,y\in l^2(I)$ we have:

$$||x+y||\leq||x||+||y||$$

Thus $l^2(I)$ is indeed a vector space, the other vector space conditions being trivial.

(6) Also, $<x,y>$ is surely a scalar product on this vector space, because all the conditions for a scalar product are trivially satisfied.

(7) Finally, the fact that our space $l^2(I)$ is indeed complete with respect to its norm $||.||$ follows in the obvious way, the limit of a Cauchy sequence $\{x_n\}$ being the vector $y=(y_i)$ given by $y_i=\lim_{n\to\infty}x_{ni}$, with all the verifications here being trivial. □

Going now a bit abstract, we have, more generally, the following result, which shows that our formalism covers as well the Schrödinger spaces of type $L^2(\mathbb{R}^3)$:

THEOREM 10.3. *Given an arbitrary space X with a positive measure μ on it, the space of square-summable complex functions on it, namely*

$$L^2(X)=\left\{f:X\to\mathbb{C}\Big|\int_X|f(x)|^2\,d\mu(x)<\infty\right\}$$

is a Hilbert space, with scalar product as follows:

$$<f,g>=\int_X f(x)\overline{g(x)}\,d\mu(x)$$

When $X=I$ is discrete, meaning that the measure μ on it is the counting measure, $\mu(\{x\})=1$ for any $x\in X$, we obtain in this way the previous spaces $l^2(I)$.

PROOF. This is something routine, remake of Theorem 10.2, as follows:

(1) The proof of the first, and main assertion is something perfectly similar to the proof of Theorem 10.2, by replacing everywhere the sums by integrals.

(2) With the remark that we forgot to say in the statement that the L^2 functions are by definition taken up to equality almost everywhere, $f=g$ when $||f-g||=0$.

(3) As for the last assertion, when μ is the counting measure all our integrals here become usual sums, and so we recover in this way Theorem 10.2. □

As a third and last theorem about Hilbert spaces, that we will need, we have:

THEOREM 10.4. *Any Hilbert space H has an orthonormal basis $\{e_i\}_{i\in I}$, which is by definition a set of vectors whose span is dense in H, and which satisfy*

$$< e_i, e_j >= \delta_{ij}$$

with δ being a Kronecker symbol. The cardinality $|I|$ of the index set, which can be finite, countable, or worse, depends only on H, and is called dimension of H. We have

$$H \simeq l^2(I)$$

in the obvious way, mapping $\sum \lambda_i e_i \to (\lambda_i)$. The Hilbert spaces with $\dim H = |I|$ being countable, such as $l^2(\mathbb{N})$, are all isomorphic, and are called separable.

PROOF. We have many assertions here, the idea being as follows:

(1) In finite dimensions an orthonormal basis $\{e_i\}_{i\in I}$ can be constructed by starting with any vector space basis $\{x_i\}_{i\in I}$, and using the Gram-Schmidt procedure. As for the other assertions, these are all clear, from basic linear algebra.

(2) In general, the same method works, namely Gram-Schmidt, with a subtlety coming from the fact that the basis $\{e_i\}_{i\in I}$ will not span in general the whole H, but just a dense subspace of it, as it is in fact obvious by looking at the standard basis of $l^2(\mathbb{N})$.

(3) And there is a second subtlety as well, coming from the fact that the recurrence procedure needed for Gram-Schmidt must be replaced by some sort of "transfinite recurrence", using standard tools from logic, and more specifically the Zorn lemma.

(4) Finally, everything at the end, regarding our notion of separability for the Hilbert spaces, is clear from definitions, and from our various results above. □

10b. Separable space

According to Theorem 10.4, there is only one separable Hilbert space, up to isomorphism. There are many interesting things that can be said, about this magic and unique Hilbert space. As a first result such result, which is of key importance, we have:

THEOREM 10.5. *The Hilbert space $H = L^2[0,1]$ is separable, having as orthonormal basis the orthonormalized version of the algebraic basis*

$$f_n = x^n$$

with $n \in \mathbb{N}$, coming from the Weierstrass density theorem. Moreover, we have extensions of this fact, based on standard measure theory.

Proof. The fact that the space $H = L^2[0,1]$ is indeed separable is clear from the Weierstrass theorem, which provides us with the algebraic basis $f_n = x^n$, which can be orthogonalized by using the Gram-Schmidt procedure, as explained in Theorem 10.4, and working out the details here is actually an excellent exercise. As for the second assertion, this is standard as well, and again, working out the details here is an excellent exercise. □

As a conclusion to all this, we are interested in one space, namely the unique separable Hilbert space H, but due to various technical reasons, it is often better to forget that we have $H = l^2(\mathbb{N})$, and say instead that we have $H = L^2(X)$, with X being a separable measured space, or simply say that H is an abstract separable Hilbert space.

There are many other interesting things that can be said about the unique separable Hilbert space, in relation with the Banach space material from chapter 9.

10c. Orthogonal polynomials

Let us go back to Theorem 10.5 and its proof. That material leads us into orthogonal polynomials, and we will discuss now this topic. There are actually countless types of orthogonal polynomials, and we will insist here on those that we will need later, in relation with quantum mechanics. For unexplained details in what follows, we refer to [9].

For the simplest compact space $X \subset \mathbb{R}$, or unit ball of $\mathbb{R}$ if you prefer, which is the interval $[-1,1]$, the orthogonal basis problem can be solved as follows:

Theorem 10.6. *The orthonormal basis of $L^2[-1,1]$ obtained by starting with the Weierstrass basis $\{x^l\}$, and doing Gram-Schmidt, is the family of polynomials $\{P_l\}$, with each P_l being of degree l, and with positive leading coefficient, subject to:*

$$\int_{-1}^1 P_k(x)P_l(x)\,dx = \delta_{kl}$$

These polynomials, called Legendre polynomials, satisfy the equation

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$$

which is the usual Legendre equation from physics, namely

$$(1-x^2)f''(x) - 2xf'(x) = \left(\frac{m^2}{1-x^2} - K\right) f(x)$$

at $m = 0$, and with $K = l(l+1)$. Moreover, we have the formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

called the Rodrigues formula for the Legendre polynomials.

PROOF. As a first observation, we are not lost somewhere in abstract math, because of the occurrence of the Legendre equation. As for the proof, this goes as follows:

(1) The first assertion is clear, because the Gram-Schmidt procedure applied to the Weierstrass basis $\{x^l\}$ can only lead to a certain family of polynomials $\{P_l\}$, with each P_l being of degree l, and also unique, if we assume that it has positive leading coefficient, with this $\pm$ choice being needed, as usual, at each step of Gram-Schmidt.

(2) In order to have now an idea about these beasts, here are the first few of them, which can be obtained say via a straightforward application of Gram-Schmidt:

$$\begin{aligned} P_0 &= 1 \\ P_1 &= x \\ P_2 &= (3x^2-1)/2 \\ P_3 &= (5x^3-3x)/2 \\ P_4 &= (35x^4-30x^2+3)/8 \\ P_5 &= (63x^5-70x^3+15x)/8 \end{aligned}$$

(3) Now thinking about what Gram-Schmidt does, this is certainly something by recurrence. And examining the recurrence leads to the Legendre equation, as stated.

(4) As for the Rodrigues formula, by uniqueness no need to try to understand where this formula comes from, and we have two choices here, either by verifying that $\{P_l\}$ is orthonormal, or by verifying the Legendre equation. And both methods work. □

As another interesting result, needed too in quantum mechanics, we have:

THEOREM 10.7. *The orthonormal basis of $L^2[0,\infty)$, with scalar product*

$$<f,g>=\int_0^\infty f(x)g(x)e^{-x}\,dx$$

obtained by starting with the Weierstrass basis $\{x^q\}$, and doing Gram-Schmidt, is the family of Laguerre polynomials $\{L_q\}$, given by the following formula,

$$L_q(x)=\frac{e^x}{q!}\left(\frac{d}{dx}\right)^q(e^{-x}x^q)$$

called Rodrigues formula for the Laguerre polynomials.

PROOF. The story here is very similar to that of the Legendre polynomials. Consider the Hilbert space $H=L^2[0,\infty)$, with the following scalar product on it:

$$<f,g>=\int_0^\infty f(x)g(x)e^{-x}\,dx$$

(1) The orthogonal basis obtained by applying Gram-Schmidt to the Weierstrass basis $\{x^q\}$ is then the basis formed by the Laguerre polynomials $\{L_q\}$.

(2) We have the explicit formula for L_q in the statement, which is analogous to the Rodrigues formula for the Legendre polynomials.

(3) Alternatively, all this follows as well by using an equation for the Laguerre polynomials, which is very similar to the Legendre equation. □

10d. Special functions

More generally, we can talk about special functions, with many interesting results.

10e. Exercises

Exercises:

Exercise 10.8.

Exercise 10.9.

Exercise 10.10.

Exercise 10.11.

Exercise 10.12.

Exercise 10.13.

Bonus exercise.

CHAPTER 11

Function spaces

11a. Function spaces

There are many interesting function spaces, including the spaces L^p with $p \in [1, \infty]$ that you probably know well, but also other spaces, like the Schwartz space $\mathcal{S}$.

11b. Distributions

Time to discuss more in detail the Dirac masses δ_x, and what can be done with them. Countless physics tricks, you would say, but the point is that the mathematicians can do tricks with them too, and this is what we want to talk about, here.

To be more precise, Schwartz developed a mathematical theory of "distributions" where, hang on, the basic step function is differentiable, with derivative δ_0.

In order to get started, we will need the following standard result:

THEOREM 11.1. *Given two functions $f, g \in C_c(\mathbb{R})$, assuming that g is differentiable, then so is $f * g$, with derivative given by the following formula:*

$$(f * g)' = f * g'$$

More generally, given $f, g \in C_c(\mathbb{R})$, and assuming that g is k times differentiable, then so is $f * g$, with k-th derivative given by $(f * g)^{(k)} = f * g^{(k)}$.

PROOF. In what regards the first assertion, with $y = x - t$, then $t = x - y$, we get:

$$\begin{aligned}
(f * g)'(x) &= \frac{d}{dx} \int_{\mathbb{R}} f(x-y)g(y)dy \\
&= \frac{d}{dx} \int_{\mathbb{R}} f(t)g(x-t)dt \\
&= \int_{\mathbb{R}} f(t)g'(x-t)dt \\
&= \int_{\mathbb{R}} f(x-y)g'(y)dy \\
&= (f * g')(x)
\end{aligned}$$

As for the second assertion, this follows form the first one, by recurrence. $\square$

Finally, getting beyond the compactly supported continous functions, we have the following result, which is of particular theoretical importance:

THEOREM 11.2. *The convolution operation is well-defined on $L^1(\mathbb{R})$, and we have:*

$$||f*g||_1 \leq ||f||_1||g||_1$$

Thus, if $f \in L^1(\mathbb{R})$ and $g \in C_c^k(\mathbb{R})$, then $f*g$ is well-defined, and $f*g \in C_c^k(\mathbb{R})$.

PROOF. In what regards the first assertion, this follows from the following computation, involving an intuitive manipulation on the double integrals, called Fubini theorem, that we will use as such here, and that we will fully clarify later on, when talking more in detail about functions of several real variables, and their integrals:

$$\begin{aligned}
\int_{\mathbb{R}} |(f*g)(x)|dx &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)g(y)|dydx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)g(y)|dxdy \\
&= \int_{\mathbb{R}} |f(x)|dx \int_{\mathbb{R}} |g(y)|dy
\end{aligned}$$

As for the second assertion, this follows from the first one, and from Theorem 11.1. □

As already mentioned, it is possible to talk about "distributions", as a continuation of this, and the basic step function is differentiable, with derivative δ_0.

11c. Fourier analysis

We discuss now the construction and main properties of the Fourier transform, which is the main tool in analysis, and even in mathematics in general. We first have:

DEFINITION 11.3. *Given $f \in L^1(\mathbb{R})$, we define a function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{ix\xi} f(x)dx$$

and call it Fourier transform of f.

As a first observation, even if f is a real function, $\widehat{f}$ is a complex function, which is not necessarily real. Also, $\widehat{f}$ is obviously well-defined, because $f \in L^1(\mathbb{R})$ and $|e^{ix\xi}| = 1$. Also, the condition $f \in L^1(\mathbb{R})$ is basically needed for construcing $\widehat{f}$, because:

$$\widehat{f}(0) = \int_{\mathbb{R}} f(x)dx$$

Generally speaking, the Fourier transform is there for helping with various computations, with the above formula $\widehat{f}(0) = \int f$ being something quite illustrating. Here are some basic properties of the Fourier transform, all providing some good motivations:

PROPOSITION 11.4. *The Fourier transform has the following properties:*

(1) *Linearity:* $\widehat{f+g} = \widehat{f} + \widehat{g}$, $\widehat{\lambda f} = \lambda \widehat{f}$.
(2) *Regularity:* $\widehat{f}$ *is continuous and bounded.*
(3) *If* f *is even then* $\widehat{f}$ *is even.*
(4) *If* f *is odd then* $\widehat{f}$ *is odd.*

PROOF. These results are all elementary, as follows:

(1) The additivity formula is clear from definitions, as follows:

$$\begin{aligned}\widehat{f+g}(\xi) &= \int_{\mathbb{R}} e^{ix\xi}(f+g)(x)dx\\ &= \int_{\mathbb{R}} e^{ix\xi} f(x)dx + \int_{\mathbb{R}} e^{ix\xi} f(x)dx\\ &= \widehat{f}(\xi) + \widehat{g}(\xi)\end{aligned}$$

As for the formula $\widehat{\lambda f} = \lambda \widehat{f}$, this is clear as well.

(2) The continuity of $\widehat{f}$ follows indeed from:

$$\begin{aligned}|\widehat{f}(\xi+\varepsilon) - \widehat{f}(\xi)| &\leq \int_{\mathbb{R}} \left|(e^{ix(\xi+\varepsilon)} - e^{ix\xi})f(x)\right| dx\\ &= \int_{\mathbb{R}} \left|e^{ix\xi}(e^{ix\varepsilon} - 1)f(x)\right| dx\\ &\leq |e^{ix\varepsilon} - 1| \int_{\mathbb{R}} |f|\end{aligned}$$

As for the boundedness of $\widehat{f}$, this is clear as well.

(3) This follows from the following computation, assuming that f is even:

$$\begin{aligned}\widehat{f}(-\xi) &= \int_{\mathbb{R}} e^{-ix\xi} f(x)dx\\ &= \int_{\mathbb{R}} e^{ix\xi} f(-x)dx\\ &= \int_{\mathbb{R}} e^{ix\xi} f(x)dx\\ &= \widehat{f}(\xi)\end{aligned}$$

(4) The proof here is similar to the proof of (3), by changing some signs. □

We will be back to more theory in a moment, but let us explore now the examples. Here are some basic computations of Fourier transforms:

PROPOSITION 11.5. *We have the following Fourier transform formulae,*

$$f = \chi_{[-a,a]} \implies \widehat{f}(\xi) = \frac{2\sin(a\xi)}{\xi}$$

$$f = e^{-ax}\chi_{[0,\infty]}(x) \implies \widehat{f}(\xi) = \frac{1}{a - i\xi}$$

$$f = e^{ax}\chi_{[-\infty,0]}(x) \implies \widehat{f}(\xi) = \frac{1}{a + i\xi}$$

$$f = e^{-a|x|} \implies \widehat{f}(\xi) = \frac{2a}{a^2 + \xi^2}$$

$$f = sgn(x)e^{-a|x|} \implies \widehat{f}(\xi) = \frac{2i\xi}{a^2 + \xi^2}$$

valid for any number $a > 0$.

PROOF. All this follows from some calculus, as follows:

(1) In what regards first formula, assuming $f = \chi_{[-a,a]}$, we have, by using the fact that $\sin(x\xi)$ is an odd function, whose integral vanishes on centered intervals:

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-a}^{a} e^{ix\xi}dx \\
&= \int_{-a}^{a} \cos(x\xi)dx + i\int_{-a}^{a} \sin(x\xi)dx \\
&= \int_{-a}^{a} \cos(x\xi)dx \\
&= \left[\frac{\sin(x\xi)}{\xi}\right]_{-a}^{a} \\
&= \frac{2\sin(a\xi)}{\xi}
\end{aligned}$$

(2) With $f(x) = e^{-ax}\chi_{[0,\infty]}(x)$, the computation goes as follows:

$$\begin{aligned}
\widehat{f}(\xi) &= \int_0^\infty e^{ix\xi - ax}dx \\
&= \int_0^\infty e^{(i\xi - a)x}dx \\
&= \left[\frac{e^{(i\xi-a)x}}{i\xi - a}\right]_0^\infty \\
&= \frac{1}{a - i\xi}
\end{aligned}$$

(3) Regarding the third formula, this follows from the second one, by using the following fact, generalizing the parity computations from Proposition 11.4:

$$F(x) = f(-x) \implies \widehat{F}(\xi) = \widehat{f}(-\xi)$$

(4) The last 2 formulae follow from what we have, by making sums and differences, and the linearity properties of the Fourier transform, from Proposition 11.4. □

We will see many other examples, in what follows. Getting back now to theory, we have the following result, adding to the various general properties in Proposition 11.4, and providing more motivations for the Fourier transform:

Proposition 11.6. *Given $f, g \in L^1(\mathbb{R})$ we have $\widehat{f}g, f\widehat{g} \in L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} f(\xi)\widehat{g}(\xi)d\xi = \int_{\mathbb{R}} \widehat{f}(x)g(x)dx$$

called "exchange of hat" formula.

Proof. Regarding the fact that we have indeed $\widehat{f}g, f\widehat{g} \in L^1(\mathbb{R})$, this is actually a bit non-trivial, but we will be back to this later. Assuming this, we have:

$$\int_{\mathbb{R}} f(\xi)\widehat{g}(\xi)d\xi = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\xi)e^{ix\xi}g(x)dxd\xi$$

On the other hand, we have as well the following formula:

$$\int_{\mathbb{R}} \widehat{f}(x)g(x)dx = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{ix\xi}f(x)g(\xi)dxd\xi$$

Thus, with $x \leftrightarrow \xi$, we are led to the formula in the statement. □

As an important result now, showing the power of the Fourier transform, this transforms the derivative, which can be a quite complicated operation, into something very simple, namely the multiplication by the variable, the precise result being as follows:

Theorem 11.7. *Given $f : \mathbb{R} \to \mathbb{C}$ such that $f, f' \in L^1(\mathbb{R})$, we have:*

$$\widehat{f'}(\xi) = -i\xi\widehat{f}(\xi)$$

More generally, assuming $f, f', f'', \ldots, f^{(n)} \in L^1(\mathbb{R})$, we have

$$\widehat{f^{(k)}}(\xi) = (-i\xi)^k\widehat{f}(\xi)$$

for any $k = 1, 2, \ldots, n$.

Proof. These results follow by doing a partial integration, as follows:

(1) Assuming that $f:\mathbb{R}\to\mathbb{C}$ has compact support, we have indeed:

$$\begin{aligned}\widehat{f'}(\xi) &= \int_{\mathbb{R}} e^{ix\xi}f'(x)dx\\ &= -\int_{\mathbb{R}} i\xi e^{ix\xi}f(x)dx\\ &= -i\xi\int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= -i\xi\widehat{f}(\xi)\end{aligned}$$

(2) Regarding the higher derivatives, the formula here follows by recurrence. □

Importantly, we have a converse statement as well, as follows:

THEOREM 11.8. *Assuming that $f\in L^1(\mathbb{R})$ is such that $F(x)=xf(x)$ belongs to $L^1(\mathbb{R})$ too, the function $\widehat{f}$ is differentiable, with derivative given by:*

$$(\widehat{f})'(\xi)=i\widehat{F}(\xi)$$

More generally, if $F_k(x)=x^kf(x)$ belongs to $L^1(\mathbb{R})$, for $k=0,1,\ldots,n$, we have

$$(\widehat{f})^{(k)}(\xi)=i^k\widehat{F_k}(\xi)$$

for any $k=1,2,\ldots,n$.

PROOF. These results are both elementary, as follows:

(1) Regarding the first assertion, the computation here is as follows:

$$\begin{aligned}(\widehat{f})'(\xi) &= \frac{d}{d\xi}\int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= \int_{\mathbb{R}} ixe^{ix\xi}f(x)dx\\ &= i\int_{\mathbb{R}} e^{ix\xi}xf(x)dx\\ &= i\widehat{F}(\xi)\end{aligned}$$

(2) As for the second assertion, this follows from the first one, by recurrence. □

As a conclusion to all this, we are on a good way with our theory, and we have:

CONCLUSION 11.9. *Modulo normalization factors, the Fourier transform converts the derivatives into multiplications by the variable, and vice versa.*

And isn't this interesting, because isn't computing derivatives a difficult task. Here is now another useful result, of the same type, this time regarding convolutions:

THEOREM 11.10. *Assuming $f,g\in L^1(\mathbb R)$, the following happens:*

$$\widehat{f*g}=\widehat{f}\cdot\widehat{g}$$

Moreover, under suitable assumptions, the formula $\widehat{fg}=\widehat{f}*\widehat{g}$ holds too.

PROOF. This is something quite subtle, the idea being as follows:

(1) Regarding the first assertion, this is something elementary, as follows:

$$\begin{aligned}
\widehat{f*g}(\xi) &= \int_{\mathbb R} e^{ix\xi}(f*g)(x)dx\\
&= \int_{\mathbb R}\int_{\mathbb R} e^{ix\xi}f(x-y)g(y)dxdy\\
&= \int_{\mathbb R} e^{iy\xi}\left(\int e^{i(x-y)\xi}f(x-y)dx\right)g(y)dy\\
&= \int_{\mathbb R} e^{iy\xi}\left(\int e^{it\xi}f(t)dt\right)g(y)dy\\
&= \int_{\mathbb R} e^{iy\xi}\widehat{f}(\xi)g(y)dy\\
&= \widehat{f}(\xi)\widehat{g}(\xi)
\end{aligned}$$

(2) As for the second assertion, this is something more tricky, and we will be back to this later. In the meantime, here is however some sort of proof, not very honest:

$$\begin{aligned}
(\widehat{f}*\widehat{g})(\xi) &= \int_{\mathbb R}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta\\
&= \int_{\mathbb R}\int_{\mathbb R}\int_{\mathbb R} e^{ix(\xi-\eta)}f(x)e^{iy\eta}g(y)dxdyd\eta\\
&= \int_{\mathbb R}\int_{\mathbb R}\int_{\mathbb R} e^{ix\eta}e^{i(y-x)\eta}f(x)g(y)dxdyd\eta\\
&= \int_{\mathbb R} e^{ix\eta}f(x)g(x)dx\\
&= \widehat{fg}(\eta)
\end{aligned}$$

To be more precise, the point here is that we can pass from the triple to the single integral by arguing that "we must have $x=y$". We will be back to this later. □

As an updated conclusion to all this, we have, modulo a few bugs, to be fixed:

CONCLUSION 11.11. *The Fourier transform converts the derivatives into multiplications by the variable, and convolutions into products, and vice versa.*

We will see applications of this later, after developing some more general theory. So, let us develop now more theory for the Fourier transform. We first have:

THEOREM 11.12. *Given $f \in L^1(\mathbb{R})$, its Fourier transform satisfies*

$$\lim_{\xi \to \pm\infty} \widehat{f}(\xi) = 0$$

called Riemann-Lebesgue property of $\widehat{f}$.

PROOF. This is something quite technical, as follows:

(1) Given a function $f : \mathbb{R} \to \mathbb{C}$ and a number $y \in \mathbb{R}$, let us set:

$$f_y(x) = f(x - y)$$

Our claim is then is that if $f \in L^p(\mathbb{R})$, then the following function is uniformly continuous, with respect to the usual p-norm on the right:

$$\mathbb{R} \to L^p(\mathbb{R}) \quad , \quad y \to f_y$$

(2) In order to prove this, fix $\varepsilon > 0$. Since $f \in L^p(\mathbb{R})$, we can find a function of type $g : [-K, K] \to \mathbb{C}$ which is continuous, such that:

$$||f - g||_p < \varepsilon$$

Now since g is uniformly continuous, we can find $\delta \in (0, K)$ such that:

$$|s - t| < \delta \implies |g(s) - g(t)| < (3K)^{-1/p}\varepsilon$$

But this shows that we have the following estimate:

$$\begin{aligned} ||g_s - g_t||_p &= \left(\int_{\mathbb{R}} \big|g(x - s) - g(x - t)\big|^p dx \right)^{1/p} \\ &< \left[(3K)^{-1}\varepsilon^p(2k + \delta) \right]^{1/p} \\ &< \varepsilon \end{aligned}$$

By using now the formula $||f||_p = ||f_s||_p$, which is clear, we obtain:

$$\begin{aligned} ||f_s - f_t||_p &\leq ||f_s - g_s||_p + ||g_s - g_t||_p + ||g_t - f_t||_p \\ &< \varepsilon + \varepsilon + \varepsilon \\ &= 3\varepsilon \end{aligned}$$

But this being true for any $|s - t| < \delta$, we have proved our claim.

(3) Let us prove now the Riemann-Lebesgue property of $\widehat{f}$, as formulated in the statement. By using $e^{\pi i}=-1$, and the change of variables $x\to x-\pi/\xi$, we have:

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{ix\xi}f(x)dx\\ &= -\int_{\mathbb{R}} e^{ix\xi}e^{\pi i}f(x)dx\\ &= -\int_{\mathbb{R}} e^{i\xi(x+\pi/\xi)}f(x)dx\\ &= -\int_{\mathbb{R}} e^{ix\xi}f\left(x-\frac{\pi}{\xi}\right)dx\end{aligned}$$

On the other hand, we have as well the following formula:

$$\widehat{f}(\xi)=\int_{\mathbb{R}} e^{ix\xi}f(x)dx$$

Thus by summing, we obtain the following formula:

$$2\widehat{f}(\xi)=\int_{\mathbb{R}} e^{ix\xi}\left(f(x)-f\left(x-\frac{\pi}{\xi}\right)\right)dx$$

But this gives the following estimate, with notations from (1):

$$2|\widehat{f}(\xi)|\leq ||f-f_{\pi/\xi}||_1$$

Since by (1) this goes to 0 with $\xi\to\pm\infty$, this gives the result. □

Quite remarkably, and as a main result now regarding Fourier transforms, a function $f:\mathbb{R}\to\mathbb{C}$ can be recovered from its Fourier transform $\widehat{f}:\mathbb{R}\to\mathbb{C}$, as follows:

THEOREM 11.13. *Assuming $f,\widehat{f}\in L^1(\mathbb{R})$, we have*

$$f(x)=\int_{\mathbb{R}} e^{-ix\xi}\widehat{f}(\xi)d\xi$$

almost everywhere, called Fourier inversion formula.

PROOF. This is something quite tricky, due to the fact that a direct attempt by double integration fails. Consider the following function, depending on a parameter $\lambda>0$:

$$\varphi_\lambda(x)=\int_{\mathbb{R}} e^{-ix\xi-\lambda|\xi|}d\xi$$

We have then the following computation:

$$\begin{aligned}(f*\varphi_\lambda)(x) &= \int_{\mathbb{R}} f(x-y)\varphi_\lambda(y)dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)e^{-iy\xi-\lambda|\xi|}d\xi dy\\ &= \int_{\mathbb{R}} e^{-\lambda|\xi|}\left(\int_{\mathbb{R}} f(x-y)e^{-iy\xi}dy\right)d\xi\\ &= \int_{\mathbb{R}} e^{-\lambda|\xi|}e^{-ix\xi}\widehat{f}(\xi)d\xi\end{aligned}$$

By letting now $\lambda\to 0$, we obtain from this the following formula:

$$\lim_{\lambda\to 0}(f*\varphi_\lambda)(x) = \int_{\mathbb{R}} e^{-ix\xi}\widehat{f}(\xi)d\xi$$

On the other hand, by using Theorem 11.12 we obtain that, almost everywhere:

$$\lim_{\lambda\to 0}(f*\varphi_\lambda)(x) = f(x)$$

Thus, we are led to the conclusion in the statement. □

There are many more things that can be said about Fourier transforms, a key result being the Plancherel formula, allowing us to talk about the Fourier transform over the space $L^2(\mathbb{R})$. Also, we can talk about the Fourier transform over the space $\mathcal{S}$ of functions all whose derivatives are rapidly decreasing, called Schwartz space.

11d. Into the waves

We have now enough accumulated material, from this chapter and the previous ones, for talking about waves. Hang on, tough mathematics and physics to come next.

11e. Exercises

Exercises:

EXERCISE 11.14. *Clarify the proof of $\widehat{fg}=\widehat{f}*\widehat{g}$, say via Fourier inversion.*

EXERCISE 11.15.

EXERCISE 11.16.

EXERCISE 11.17.

EXERCISE 11.18.

EXERCISE 11.19.

Bonus exercise.

CHAPTER 12

Linear operators

12a. Operator theory

We discuss here the theory of the linear operators $T : H \to H$ over a Hilbert space H, usually taken to be separable. Our motivation comes from both mathematics, because what we will be doing here will be a natural infinite dimensional extension of the basic linear algebra, including of the various spectral theorems known there, and from physics, and more specifically quantum mechanics. Let us start with:

THEOREM 12.1. *Given a Hilbert space H, consider the linear operators $T : H \to H$, and for each such operator define its norm by the following formula:*

$$||T|| = \sup_{||x||=1} ||Tx||$$

The operators which are bounded, $||T|| < \infty$, form then a complex algebra $B(H)$, which is complete with respect to $||.||$. When H comes with a basis $\{e_i\}_{i\in I}$, we have

$$B(H) \subset \mathcal{L}(H) \subset M_I(\mathbb{C})$$

where $\mathcal{L}(H)$ is the algebra of all linear operators $T : H \to H$, and $\mathcal{L}(H) \subset M_I(\mathbb{C})$ is the correspondence $T \to M$ obtained via the usual linear algebra formulae, namely:

$$T(x) = Mx \quad , \quad M_{ij} =< Te_j, e_i >$$

In infinite dimensions, none of the above two inclusions is an equality.

PROOF. This is something straightforward, the idea being as follows:

(1) The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S + T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(2) Regarding now the completness assertion, if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x)+T_n(y) \\ &= \lim_{n\to\infty} T_n(x)+\lim_{n\to\infty} T_n(y) \\ &= T(x)+T(y) \end{aligned}$$

Similarly, we have $T(\lambda x)=\lambda T(x)$, and we conclude that $T\in\mathcal{L}(H)$.

(3) With this done, it remains to prove now that we have $T\in B(H)$, and that $T_n\to T$ in norm. For this purpose, observe that we have:

$$\begin{aligned} \|T_n-T_m\|\leq\varepsilon\ ,\ \forall n,m\geq N &\implies \|T_nx-T_mx\|\leq\varepsilon\ ,\ \forall\|x\|=1\ ,\ \forall n,m\geq N \\ &\implies \|T_nx-Tx\|\leq\varepsilon\ ,\ \forall\|x\|=1\ ,\ \forall n\geq N \\ &\implies \|T_Nx-Tx\|\leq\varepsilon\ ,\ \forall\|x\|=1 \\ &\implies \|T_N-T\|\leq\varepsilon \end{aligned}$$

But this gives both $T\in B(H)$, and $T_N\to T$ in norm, and we are done.

(4) Regarding the embeddings, the correspondence $T\to M$ in the statement is indeed linear, and its kernel is $\{0\}$, so we have indeed an embedding as follows, as claimed:

$$\mathcal{L}(H)\subset M_I(\mathbb{C})$$

In finite dimensions we have an isomorphism, because any $M\in M_N(\mathbb{C})$ determines an operator $T:\mathbb{C}^N\to\mathbb{C}^N$, given by $<Te_j,e_i>=M_{ij}$. However, in infinite dimensions, we have matrices not producing operators, as for instance the all-one matrix.

(5) As for the examples of linear operators which are not bounded, these are more complicated, coming from logic, and we will not really need them in what follows. □

As a second basic result regarding the operators, we will need:

Theorem 12.2. *Each operator $T\in B(H)$ has an adjoint $T^*\in B(H)$, given by:*

$$<Tx,y>=<x,T^*y>$$

The operation $T\to T^*$ is antilinear, antimultiplicative, involutive, and satisfies:

$$\|T\|=\|T^*\|\quad,\quad\|TT^*\|=\|T\|^2$$

When H comes with a basis $\{e_i\}_{i\in I}$, the operation $T\to T^*$ corresponds to

$$(M^*)_{ij}=\overline{M}_{ji}$$

at the level of the associated matrices $M\in M_I(\mathbb{C})$.

Proof. This is standard too, and can be proved in 3 steps, as follows:

(1) The existence of the adjoint operator T^*, given by the formula in the statement, comes from the fact that the function $\varphi(x) =< Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\varphi(x) =< x, T^*y >$$

Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S+T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^* \quad , \quad (ST)^* = T^* S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y > \\ &= ||T^*|| \end{aligned}$$

(2) Regarding now $||TT^*|| = ||T||^2$, which is a key formula, observe that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, as desired.

(3) Finally, when H comes with a basis, the formula $< Tx, y >=< x, T^*y >$ applied with $x = e_i$, $y = e_j$ translates into the formula $(M^*)_{ij} = \overline{M}_{ji}$, as desired. □

It is convenient to upgrade our formalism, following Gelfand, as follows:

Definition 12.3. *An abstract operator algebra, or C^*-algebra, is a complex algebra A having a norm $||.||$ and an involution $*$, subject to the following conditions:*

1. *A is closed with respect to the norm.*
2. *We have $||aa^*|| = ||a||^2$, for any $a \in A$.*

In other words, what we did here is to axiomatize the abstract properties of the operator algebras $A \subset B(H)$, without any reference to the Hilbert space H. We will see later some good reasons for doing so. For the moment, $A = B(H)$ will be our main example. Getting to work now, let us develop the theory of C^*-algebras. We first have:

Theorem 12.4. *Given an element $a\in A$ of a C^*-algebra, define its spectrum as:*

$$\sigma(a)=\left\{\lambda\in\mathbb C\Big|a-\lambda\notin A^{-1}\right\}$$

The following spectral theory results hold, exactly as in the $A=B(H)$ case:

(1) *We have $\sigma(ab)\cup\{0\}=\sigma(ba)\cup\{0\}$.*
(2) *We have $\sigma(f(a))=f(\sigma(a))$, for any $f\in\mathbb C(X)$ having poles outside $\sigma(a)$.*
(3) *The spectrum $\sigma(a)$ is compact, non-empty, and contained in $D_0(||a||)$.*
(4) *The spectra of unitaries ($u^*=u^{-1}$) and self-adjoints ($a=a^*$) are on $\mathbb T,\mathbb R$.*
(5) *The spectral radius of normal elements ($aa^*=a^*a$) is given by $\rho(a)=||a||$.*

In addition, assuming $a\in A\subset B$, the spectra of a with respect to A and to B coincide.

Proof. Here the assertions (1-5), which are of course formulated a bit informally, are well-known for the full operator algebra $A=B(H)$, and the proof in general is similar:

(1) Assuming that $1-ab$ is invertible, with inverse c, we have $abc=cab=c-1$, and it follows that $1-ba$ is invertible too, with inverse $1+bca$. Thus $\sigma(ab),\sigma(ba)$ agree on $1\in\mathbb C$, and by linearity, it follows that $\sigma(ab),\sigma(ba)$ agree on any point $\lambda\in\mathbb C^*$.

(2) The formula $\sigma(f(a))=f(\sigma(a))$ is clear for polynomials, $f\in\mathbb C[X]$, by factorizing $f-\lambda$, with $\lambda\in\mathbb C$. Then, the extension to the rational functions is straightforward, because $P(a)/Q(a)-\lambda$ is invertible precisely when $P(a)-\lambda Q(a)$ is.

(3) By using $1/(1-b)=1+b+b^2+\ldots$ for $||b||<1$ we obtain that $a-\lambda$ is invertible for $|\lambda|>||a||$, and so $\sigma(a)\subset D_0(||a||)$. It is also clear that $\sigma(a)$ is closed, so what we have is a compact set. Finally, assuming $\sigma(a)=\emptyset$ the function $f(\lambda)=\varphi((a-\lambda)^{-1})$ is well-defined, for any $\varphi\in A^*$, and by Liouville we get $f=0$, contradiction.

(4) Assuming $u^*=u^{-1}$ we have $||u||=1$, and so $\sigma(u)\subset D_0(1)$. But with $f(z)=z^{-1}$ we obtain via (2) that we have as well $\sigma(u)\subset f(D_0(1))$, and this gives $\sigma(u)\subset\mathbb T$. As for the result regarding the self-adjoints, this can be obtained from the result for the unitaries, by using (2) with functions of type $f(z)=(z+it)/(z-it)$, with $t\in\mathbb R$.

(5) It is routine to check, by integrating quantities of type $z^n/(z-a)$ over circles centered at the origin, and estimating, that the spectral radius is given by $\rho(a)=\lim||a^n||^{1/n}$. But in the self-adjoint case, $a=a^*$, this gives $\rho(a)=||a||$, by using exponents of type $n=2^k$, and then the extension to the general normal case is straightforward.

(6) Regarding now the last assertion, the inclusion $\sigma_B(a)\subset\sigma_A(a)$ is clear. For the converse, assume $a-\lambda\in B^{-1}$, and set $b=(a-\lambda)^*(a-\lambda)$. We have then:

$$\sigma_A(b)-\sigma_B(b)=\left\{\mu\in\mathbb C-\sigma_B(b)\Big|(b-\mu)^{-1}\in B-A\right\}$$

Thus this difference in an open subset of $\mathbb C$. On the other hand b being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of b are equal, and in particular b is invertible in A, and so $a-\lambda\in A^{-1}$, as desired. $\square$

We can now a prove a key result, as follows:

THEOREM 12.5 (Gelfand). *If X is a compact space, the algebra $C(X)$ of continuous functions on it $f : X \to \mathbb{C}$ is a C^*-algebra, with usual norm and involution, namely:*

$$||f|| = \sup_{x \in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

Conversely, any commutative C^*-algebra is of this form, $A = C(X)$, with

$$X = \left\{ \chi : A \to \mathbb{C} \ , \ \text{normed algebra character} \right\}$$

with topology making continuous the evaluation maps $ev_a : \chi \to \chi(a)$.

PROOF. There are several things going on here, the idea being as follows:

(1) The first assertion is clear from definitions. Observe that we have indeed:

$$||ff^*|| = \sup_{x \in X} |f(x)|^2 = ||f||^2$$

Observe also that the algebra $C(X)$ is commutative, because $fg = gf$.

(2) Conversely, given a commutative C^*-algebra A, let us define X as in the statement. Then X is compact, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(3) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ decomposition formula for usual complex numbers:

$$a = \frac{a + a^*}{2} + i \cdot \frac{a - a^*}{2i}$$

Thus it is enough to prove $ev_{a^*} = ev_a^*$ for the self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, by Theorem 12.4, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

(4) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

Along the same lines, it is possible to prove various spectral theorems for the linear operators, generalizing what we know from linear algebra, in finite dimensions.

12b. Unbounded operators

Motivated by basic quantum mechanics, we must look as well into the unbounded operators, usually taken self-adjoint, and their diagonalization properties.

12c. Operator algebras

We have already seen some theory for the operator algebras, following Gelfand. Our aim here is to go beyond that, with more advanced theory, following von Neumann. To be more precise, the idea with the Gelfand C^*-algebras was that these are algebras of the form $C(X)$, with X being some sort of "quantum space". And following now von Neumann, we can axiomatize and study the algebras of type $L^\infty(X)$, with X being again a quantum space, but this time "measured", and with this being more advanced.

12d. Quantum mechanics

All the above material is very useful for the understanding of foundational quantum mechanics, following Heisenberg. Moreover, with our accumulated analytic knowledge from this book, which was often of PDE flavor, we can even go beyond that, by talking about the Schrödinger equation, and solving this equation for the hydrogen atom.

12e. Exercises

Exercises:

Exercise 12.6.

Exercise 12.7.

Exercise 12.8.

Exercise 12.9.

Exercise 12.10.

Exercise 12.11.

Bonus exercise.

Part IV

About infinity

Riders on the storm
Riders on the storm
Into this house we're born
Into this world we're thrown

CHAPTER 13

Multivariate logic

13a. Chemistry, brain

Chemistry, brain.

13b. Social science

Social science.

13c. Multivariate logic

Multivariate logic.

13d. Some applications

Some applications.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Bonus exercise.

CHAPTER 14

Infinitesimal calculus

14a. Infinitesimal calculus

Infinitesimal calculus.

14b. Stirling formula

Stirling formula.

14c. Further applications

Further applications.

14d. Some philosophy

Some philosophy.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Bonus exercise.

CHAPTER 15

Quantum landscape

15a. Quantum spaces

Quantum spaces.

15b. Quantum measures

Quantum measures.

15c. Quantum measurements

Quantum measurements.

15d. Bugs and fixes

Bugs and fixes.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Bonus exercise.

CHAPTER 16

The Yin and the Yang

16a. Some philosophy

Some philosophy.

16b. Transcedental numbers

Transcedental numbers.

16c. The physics constants

The physics constants.

16d. The Yin and the Yang

The Yin and the Yang.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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