Introduction to matrix groups

Teo Banica

Groups of matrices, Symmetries and reflections, Poisson laws, Bessel laws, Representation theory, Probability on compact groups

08/20

Foreword

These are slides written in the Fall 2020, on matrix groups.
Presentations available at my Youtube channel.

Groups of unitary matrices

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"Introduction to matrix groups", 1/6

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Groups 1/3

Definition. A group is a set G with a multiplication operation

$$(g, h) \to gh$$

satisfying the following conditions:

(1) Associativity: $(gh)k = g(hk)$.

(2) Unit: $\exists 1 \in G$, $g1 = 1g = g$.

(3) Inverses: $\forall g, \exists g^{-1}$, $gg^{-1} = g^{-1}g = 1$.

Groups 2/3

Examples.

(1) $\mathbb{R}$ with the addition operation $x + y$. Here the unit is 0 (!) and the inverses are $-x$.

(2) $\mathbb{R}^*$ with the multiplication operation xy. Here the unit is 1 and the inverses are x^{-1}.

(3) $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ with the addition operation $x + y$, and $\mathbb{Q}^*, \mathbb{C}^*$ with the multiplication operation xy.

Note that $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$ and $(\mathbb{Z}^*, \cdot)$ are not groups, because here we have no inverses.

Groups 3/3

<u>More examples</u>.

(1) The group S_N of permutations $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$. Note that we have $\sigma\tau \neq \tau\sigma$ in general, in this group.

(2) The groups $GL_N(\mathbb{Q})$, $GL_N(\mathbb{R})$, $GL_N(\mathbb{C})$ of invertible $N \times N$ matrices over $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. Here $gh = hg$ fails too, in general.

<u>Conventions</u>.

– When $ab = ba$ we say that the group is abelian.

– We usually denote the operation of an abelian group by a sum, $g + h$, the unit element by 0, and the inverses by $-g$.

– This is not a general rule. What is true, however, is that if a group is denoted $(G, +)$, then the group must be abelian.

Orthogonal groups 1/4

Notations. We use the usual scalar product and norm on $\mathbb{R}^N$:

$$< x, y >= \sum_i x_i y_i \quad , \quad ||x|| = \sqrt{< x, x >}$$

Theorem. For a matrix $U \in M_N(\mathbb{R})$, the following are equivalent, and if they are satisfied, we say that U is orthogonal:

(1) $< Ux, Uy >=< x, y >$.

(2) $||Ux|| = ||x||$.

(3) $U^t = U^{-1}$, where $(U^t)_{ij} = U_{ji}$.

(4) The rows of U form an orthonormal basis of $\mathbb{R}^N$.

(5) The columns of U form an orthonormal basis of $\mathbb{R}^N$.

Proof. All this follows from $< Ux, y >=< x, U^t y >$.

Orthogonal groups 2/4

Theorem. The set formed by the orthogonal matrices

$$O_N=\left\{U\in M_N(\mathbb R)\Big|U^t=U^{-1}\right\}$$

is a group, with the usual multiplication of the matrices.

Proof. Assuming $U,V\in O_N$, we have $UV\in O_N$, because:

$$(UV)^t=V^tU^t=V^{-1}U^{-1}=(UV)^{-1}$$

Also, $1_N\in O_N$, and $U\in O_N\implies U^{-1}\in O_N$.

Orthogonal groups 3/4

Theorem. The elements of O_2 fall into two classes:

(1) Rotations. The rotation of angle $t \in \mathbb{R}$ is given by the following formula:

$$R_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

The rotations are exactly the elements of O_2 having determinant 1, and they form a group, denoted SO_2.

(2) Symmetries. The symmetry with respect to the Ox axis rotated by $t/2 \in \mathbb{R}$ is given by the following formula:

$$S_t = \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix}$$

The symmetries are exactly the elements of O_2 having determinant -1, and they do not form a group.

Orthogonal groups 4/4

Theorem. The elements of O_N fall into two classes:

(1) Those of determinant 1, which form a group, denoted SO_N:

$$SO_N = \left\{ U \in O_N \middle| \det U = 1 \right\}$$

(2) Those of determinant -1, which do not form a group.

Proofs. For $U \in O_N$ we have $\det(UU^t) = 1$, so $\det U = \pm 1$.

The set SO_N is a group, because $\det(UV) = \det U \det V$, and its complement is not a group, because $\det(1_N) = 1$.

Finally, the various 2D formulae are well-known, and elementary.

Unitary groups 1/4

Notations. We use the usual scalar product and norm on $\mathbb{C}^N$:

$$< x, y >= \sum_i x_i \bar{y}_i \quad , \quad ||x|| = \sqrt{< x, x >}$$

Theorem. For a matrix $U \in M_N(\mathbb{C})$, the following are equivalent, and if they are satisfied, we say that U is unitary:

(1) $< Ux, Uy >=< x, y >$.

(2) $||Ux|| = ||x||$.

(3) $U^* = U^{-1}$, where $(U^*)_{ij} = \bar{U}_{ji}$.

(4) The rows of U form an orthonormal basis of $\mathbb{C}^N$.

(5) The columns of U form an orthonormal basis of $\mathbb{C}^N$.

Proof. All this follows from $< Ux, y >=< x, U^*y >$.

Unitary groups 2/4

Theorem. The set formed by the unitary matrices

$$U_N=\left\{U\in M_N(\mathbb C)\Big|U^*=U^{-1}\right\}$$

is a group, with the usual multiplication of the matrices.

Proof. Assuming $U,V\in U_N$, we have $UV\in U_N$, because:

$$(UV)^*=V^*U^*=V^{-1}U^{-1}=(UV)^{-1}$$

Also, $1_N\in U_N$, and $U\in U_N\implies U^{-1}\in U_N$.

Unitary groups 3/4

<u>Theorem</u>. The determinant of a unitary matrix $U \in U_N$ must be a number on the unit circle:

$$\det U \in \mathbb{T}$$

The unitary matrices $N \times N$ having determinant 1 form a group, denoted SU_N:

$$SU_N = \left\{ U \in U_N \middle| \det U = 1 \right\}$$

Any matrix $U \in U_N$ is proportional to a matrix in SU_N, the proportionality factor being a number $d \in \mathbb{T}$.

<u>Proof</u>. For $U \in U_N$ we have $\det(UU^*) = 1$, so $|\det U| = 1$.
The second assertion is clear from $\det(UV) = \det U \det V$.
The third assertion follows by dividing by $d = (\det U)^{1/N}$.

Unitary groups 4/4

Theorem. We have the following formula,

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \Big| \; |a|^2 + |b|^2 = 1 \right\}$$

as well as the following formula:

$$U_2 = \left\{ d \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \Big| \; |a|^2 + |b|^2 = 1, |d| = 1 \right\}$$

Proof. For $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1, $U^* = U^{-1}$ reads:

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus $c = -\bar{b}, d = \bar{a}$. Finally, $\det U = 1$ gives $|a|^2 + |b|^2 = 1$.

Subgroups 1/4

The groups that we considered so far are as follows:

$$\begin{array}{ccc} O_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ SO_N & \longrightarrow & SU_N \end{array}$$

It is possible to construct more groups along these lines:

(1) By multiplying by $\mathbb{Z}_r = \{w \in \mathbb{C} | w^r = 1\}$.

(2) By imposing the condition $(\det U)^s = 1$.

We can equally talk about the symplectic groups $Sp_N \subset U_N$.

Subgroups 2/4

Another big class of groups of matrices comes by looking at

$$U_N^{diag} = \mathbb{T}^N$$

and its subgroups. We have for instance the groups

$$\mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_N}$$

for any choice of numbers $r_1, \ldots, r_N \in \mathbb{N} \cup \{\infty\}$.

Subgroups 3/4

Importantly, the permutation groups S_N appear as well as groups of unitary matrices,

$$S_N \subset O_N \subset U_N$$

by making each $\sigma \in S_N$ act on the coordinate axes of $\mathbb{R}^N$. Indeed, this action is clearly isometric, so $S_N \subset O_N$.

Subgroups 4/4

In fact, any finite group G appears as a group of unitary matrices. Indeed, we can make act G on itself, by left multiplication,

$$G \subset S_G \quad , \quad \sigma_g(h) = gh$$

and so with $N = |G|$ we have embeddings as follows:

$$G \subset S_N \subset O_N \subset U_N$$

However, groups such as $D_N \subset O_N$ show that each finite group G has its own "privileged" embedding $G \subset U_N$.

Symmetry and reflection groups

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Finite groups 1/3

Theorem. Any finite group is a permutation group.

Proof. Given a finite group G, we have an embedding as follows:

$$G \subset S_G \quad , \quad \sigma_g(h) = gh$$

In other words, we have $G \subset S_N$, with $N = |G|$.

Finite groups 2/3

Theorem. Any finite group appears as group of orthogonal matrices.

Proof. This is true for S_N, which can be regarded as being the permutation group of the N coordinate axes of $\mathbb{R}^N$:

$$S_N \subset O_N$$

Thus, given a group G of finite order $N < \infty$, we have:

$$G \subset S_N \subset O_N$$

Finite groups 3/3

Conclusion. The following are the same thing:

(1) The finite groups.

(2) The subgroups $G \subset S_N$.

(3) The finite subgroups $G \subset O_N$.

(4) The finite subgroups $G \subset U_N$.

Problem. Given a finite group G, what is the "best" embedding of type $G \subset U_N$, say with $N \in \mathbb{N}$ being smallest possible?

Comment. This is a "representation theory" problem.

Dihedral groups 1/4

Theorem. Consider the cyclic group $\mathbb{Z}_N$.

(1) We have an embedding $\mathbb{Z}_N \subset O_2$, given by:

$$k \to \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} \quad , \quad t = \frac{2k\pi}{N}$$

(2) We have an embedding $\mathbb{Z}_N \subset O_N$, given by:

$$k \to [e_i \to e_{i+k}]$$

(3) We have an embedding $\mathbb{Z}_N \subset U_1$, given by:

$$k \to (w^k) \quad , \quad w = e^{2\pi i/N}$$

Comment. (2) is nicer than (1), and (3) beats everything.

Dihedral groups 2/4

Definition. The dihedral group D_N is the group of symmetries of a regular N-gon.

Examples.

(1) At $N = 3$ we have 3 symmetries, with respect to the 3 medians of $\triangle$, as well as 3 rotations, of angles $0^\circ, 120^\circ, 240^\circ$.

(2) At $N = 4$ we have 4 symmetries, with respect to Ox, Oy and the diagonals of $\square$, and 4 rotations, of angles $0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Dihedral groups 3/4

Theorem. The dihedral group D_N has $2N$ elements, as follows:

(1) N rotations, of angles $2k\pi/N$, with $k = 0, 1, \ldots, N-1$. These form a copy $\mathbb{Z}_N \subset D_N$ of the cyclic group $\mathbb{Z}_N$.

(2) N symmetries, with respect to the N medians when N is odd, and to the $N/2 + N/2$ symmetry axes, when N is even.

In addition, we have a formula of type $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$.

Proof. (1) and (2) are clear. Regarding the last part, D_N has the same number of elements as $\mathbb{Z}_N \times \mathbb{Z}_2$, but is not abelian. Thus, we must "twist" the product of $\mathbb{Z}_N \times \mathbb{Z}_2$ in order to obtain D_N.

Dihedral groups 4/4

Theorem. Consider the dihedral group D_N.

(1) We have an embedding $D_N \subset O_2$, given by the usual rotation and symmetry matrices:

$$R_k \to \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} \quad , \quad t = \frac{2k\pi}{N}$$

$$S_k \to \begin{pmatrix}\cos t & \sin t\\ \sin t & -\cos t\end{pmatrix} \quad , \quad t = \frac{2k\pi}{N}$$

(2) We have an embedding $D_N \subset O_N$, obtained by permuting the N-gon on the coordinate axes of $\mathbb{R}^N$, at distance 1 from 0:

$$\sigma \to [e_i \to e_{\sigma(i)}]$$

(3) We cannot have an embedding $D_N \subset U_1$, because the group U_1 is abelian, and D_N is not abelian.

Symmetric groups 1/4

Theorem. The permutation group S_N has $N!$ elements.

Proof. In order to construct a permutation $\sigma \in S_N$, we must:

(1) Choose $\sigma(1)$, and there are N choices here.

(2) Choose $\sigma(2)$, and there are $N-1$ choices left.

$\vdots$

$\vdots$

(N) Choose $\sigma(N)$, and there is 1 choice left.

Thus, we have a total of $N(N-1)\ldots 1 = N!$ choices.

Symmetric groups 2/4

Theorem. We have an embedding $S_N \subset O_N$, given by:

$$\sigma \to [e_i \to e_{\sigma(i)}]$$

By using the standard $e_{ij} : e_j \to e_i$ notation, the formula is:

$$\sigma \to \sum_i e_{\sigma(i)i}$$

In matrix notation, and with Kronecker symbols, the formula is:

$$\sigma \to [\delta_{i\sigma(j)}]_{ij}$$

Proof. The first assertion is clear, because the transformations $e_i \to e_{\sigma(i)}$ are isometries of $\mathbb{R}^N$, and the rest is clear too.

Symmetric groups 3/4

Theorem. The permutation matrices $S_N \subset O_N$ are precisely the 0-1 matrices having a 1 entry on each row and column.

Theorem. The trace of a permutation matrix $\sigma \in S_N \subset O_N$ is the number of its fixed points.

Proofs. Both these results are clear from definitions.

Symmetric groups 4/4

Theorem. The determinant of the permutation matrices

$$\det(\sigma) \in \{\pm 1\}$$

coinsides with the signature of the permutations,

$$\varepsilon(\sigma) = (-1)^c$$

where c is the number of inversions.

Proof. This is clear with any of the definitions of det.

Comment. Thus, $S_N \cap SO_N = A_N$, the alternating group.

Reflection groups 1/4

Definition. The hyperoctahedral group H_N is the symmetry group of the hypercube $\square_N \subset \mathbb{R}^N$.

Comment. Thus, we have by definition $H_N \subset S_{2^N}$.

Example. We have $H_2 = D_4$.

Problem. $|H_N| = ?$

Reflection groups 2/4

Theorem. The group H_N appears as well as the group of signed permutations of the coordinate axes of $\mathbb{R}^N$, so we have

$$H_N \subset O_N$$

with the image consisting of the $-1, 0, 1$ matrices having exactly one ± 1 entry on each row and each column. Thus we have:

$$|H_N| = 2^N N!$$

Comment. One can prove that $H_N = S_N \rtimes \mathbb{Z}_2^N$, which is also written as $H_N = \mathbb{Z}_2 \wr S_N$, wreath product.

Reflection groups 3/4

Definition. The reflection group H_N^s, depending on parameters

$$N \in \mathbb{N} \quad , \quad s \in \mathbb{N} \cup \{\infty\}$$

is the group of $N \times N$ matrices having entries in

$$\mathbb{Z}_s \cup \{0\}$$

having exactly one nonzero entry on each row and each column.

Examples. At $s = 1$ we obtain S_N, and at $s = 2$ we obtain H_N.
In general, at $s < \infty$, we have a certain finite group $H_N^s \subset U_N$.
At $s = \infty$ we have a group $K_N \subset U_N$, which is no longer finite.

Reflection groups 4/4

One can prove that the "complex reflection groups" are:

– The above groups $H_N^s = S_N \wr \mathbb{Z}_s$.

– Their subgroups H_N^{sd} given by $\det^d = 1$.

– And some exceptional examples.

Symmetric groups and Poisson laws

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Characters 1/3

Definition. A representation of a finite group G is a morphism

$$\pi : G \to U_N$$

and the character of this representation is the map

$$\chi : G \to \mathbb{C}$$

obtained by taking the trace of the images of group elements:

$$\chi(g) = Tr(\pi(g))$$

When G comes as $G \subset_\pi U_N$, we call χ the "main character".

Characters 2/3

Remark. The characters are central functions on the group, in the sense that they satisfy the following condition:

$$\chi(gh) = \chi(hg)$$

We will see later that any central function on the group is a linear combination of characters. This is something non-trivial.

Remark. We can talk, more generally, about representations and characters of compact groups, with the representations

$$\pi : G \to U_N$$

being assumed to be continuous. We will do this later on.

Characters 3/3

Problem. Given $\pi : G \to U_N$, we want to compute the law of:

$$\chi = Tr \circ \pi : G \to \mathbb{C}$$

That is, we would like to compute the following probabilities,

$$P(\chi = k) \in [0,1] \quad , \quad k \in \mathbb{C}$$

and then the complex discrete measure encoding them:

$$\mu = \sum_{k \in \mathbb{C}} P(\chi = k)\delta_k$$

There are many motivations for this question. Details later.

Fixed points 1/4

Theorem. For the symmetric group S_N, regarded as subgroup

$$S_N \subset O_N$$

permuting the coordinate axes of $\mathbb{R}^N$, the main character is

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

and its law is a discrete probability measure, supported by $\mathbb{N}$.

Proof. Each $\sigma \in S_N \subset O_N$ is a 0-1 matrix, whose trace $Tr(\chi)$ counts the 1 diagonal entries, corresponding to fixed points.

Fixed points 2/4

Theorem. The probability for a permutation $\sigma \in S_N$ to be a derangement is, in the $N \to \infty$ limit:

$$P_0 \simeq \frac{1}{e}$$

Proof. We must be outside the union $F = \bigcup_i F_i$, where:

$$F_i = \left\{ \sigma \in S_N \middle| \sigma(i) = i \right\}$$

The inclusion-exclusion principle gives:

$$F^c = N! - \sum_i |F_i| + \sum_{i<j} |F_i \cap F_j| - \sum_{i<j<k} |F_i \cap F_j \cap F_k| + \ldots$$

We obtain $P_0 = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots \simeq \frac{1}{e}$, as claimed.

Fixed points 3/4

Theorem. The probability for a permutation $\sigma \in S_N$ to have exactly $k \in \mathbb{N}$ fixed points is

$$P_k \simeq \frac{1}{e} \cdot \frac{1}{k!}$$

once again in the $N \to \infty$ limit.

Proof. We already know that the result holds at $k = 0$. In general the proof is similar, by using the inclusion-exclusion principle.

Fixed points 4/4

Theorem. The character of the standard representation

$$S_N \subset O_N$$

obtained by permuting the coordinate axes of $\mathbb{R}^N$ is given by

$$\chi(\sigma) = \#\left\{i\Big|\sigma(i) = i\right\}$$

and follows with $N \to \infty$ the following law:

$$p_1 = \frac{1}{e}\sum_k \frac{\delta_k}{k!}$$

Proof. This follows by putting together the above results.

Poisson laws 1/4

Definition. The Poisson law of parameter 1 is:

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

More generally, the Poisson law of parameter $t > 0$ is:

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

Remark. These laws have indeed mass 1.

Poisson laws 2/4

<u>Theorem</u>. We have the following formula, for any $s, t > 0$:

$$p_s * p_t = p_{s+t}$$

<u>Proof</u>. By using $\delta_k * \delta_l = \delta_{k+l}$ and the binomial formula:

$$\begin{aligned} p_s * p_t &= e^{-s} \sum_k \frac{s^k}{k!} \delta_k * e^{-t} \sum_l \frac{t^l}{l!} \delta_l \\ &= e^{-s-t} \sum_n \delta_n \sum_{k+l=n} \frac{s^k t^l}{k! l!} \\ &= e^{-s-t} \sum_n \frac{(s+t)^n}{n!} \delta_n \end{aligned}$$

Thus, we obtain the Poisson law p_{s+t}, as claimed.

Poisson laws 3/4

Theorem. The Fourier transform of p_t is given by:

$$F_{p_t}(x)=\exp\left((e^{ix}-1)t\right)$$

Proof. By using $F_f(x)=\mathbb{E}(e^{ixf})$, we obtain:

$$\begin{aligned}F_{p_t}(x) &= e^{-t}\sum_k \frac{t^k}{k!}\,e^{ikx}\\ &= e^{-t}\sum_k \frac{(e^{ix}t)^k}{k!}\\ &= \exp(-t)\exp(e^{ix}t)\end{aligned}$$

Thus, we obtain the formula in the statement.

Poisson laws 4/4

Theorem. We have the following convergence, in moments:

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

Proof. We have the following computation:

$$\begin{aligned}
F_{\delta_t}(x)=e^{itx} &\implies F_{\mu_n}(x)=\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\\
&\implies F_{\mu_n^{*n}}(x)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n\\
&\implies F_{\mu_n^{*n}}(x)=\left(1+\frac{(e^{ix}-1)t}{n}\right)^n\\
&\implies F(x)=\exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Thus, we obtain the Fourier transform of p_t.

Truncation 1/4

Problem. We know that for $S_N \subset O_N$ with $N \to \infty$, the main character follows the Poisson law p_1.

What about the general Poisson law p_t, of parameter $t > 0$? Can we obtain this law in the representation theory context?

Truncation 2/4

Definition. Given a group representation $\pi : G \to U_N$, its truncated character with respect to a parameter $t \in (0, 1]$,

$$\chi_t : G \to \mathbb{C}$$

is the map given by the following formula:

$$\chi_t(g) = \sum_{i=1}^{[tN]} \pi(g)_{ii}$$

When G comes as a group of matrices, $G \subset_\pi U_N$, we call this map χ_t the "main truncated character" of the group.

Truncation 3/4

Theorem. The main truncated character of the symmetric group

$$S_N \subset O_N$$

which permutes the coordinate axes of $\mathbb{R}^N$, is given by

$$\chi_t(\sigma) = \#\left\{i \in \{1, \ldots, [tN]\}\middle|\sigma(i) = i\right\}$$

and follows with $N \to \infty$ the Poisson law of parameter t,

$$p_t = e^{-t} \sum_k \frac{t^k}{k!}\,\delta_k$$

for any value of the parameter $t \in (0, 1]$.

Truncation 4/4

Proof. We already know that the formula holds at $t = 1$. The same method, inclusion-exclusion, gives, more generally:

$$\lim_{N\to\infty} P(\chi = k) = \frac{1}{e^t} \cdot \frac{t^k}{k!}$$

Thus, we obtain with $N \to \infty$ the Poisson law p_t, as claimed.

Comment. We will see later extensions and interpretations of all this, in the advanced representation theory context.

Complex reflections and Bessel laws

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Reflection groups 1/3

Definition. The reflection group H_N^s, depending on parameters

$$N \in \mathbb{N} \quad , \quad s \in \mathbb{N} \cup \{\infty\}$$

is the group of $N \times N$ matrices with entries in

$$\mathbb{Z}_s \cup \{0\}$$

having one nonzero entry on each row and each column.

Examples. At $s = 1$ we have the symmetric group $S_N \subset O_N$.
At $s = 2$ we have the hyperoctahedral group $H_N \subset O_N$.
At $s = 3, 4, \ldots$ we have a certain finite subgroup $H_N^s \subset U_N$.
At $s = \infty$ we have a certain infinite subgroup $K_N \subset U_N$.

Reflection groups 2/3

Theorem. We have $H_N^s=\mathbb{Z}_s\wr S_N$, wreath product decomposition.

Proof. This basically says that the elements $g\in H_N^s$ appear as permutations $\sigma\in S_N$ "decorated" with signs $\varepsilon\in\mathbb{Z}_s^N$, which is something that we already know, from the matrix picture.

Theorem. The irreducible complex reflection groups are

$$H_N^{sd}=\left\{U\in H_N^s\Big|\det U\in\mathbb{Z}_d\right\}$$

along with 34 exceptional examples.

Proof. This is something complicated, due to Shephard and Todd.

Reflection groups 3/3

Theorem. The groups H_N^s are easy, in the sense that

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

are Brauer type algebras, spanned by diagrams.

Proof. This holds indeed, with $D_{kl} \subset \mathcal{P}(k,l)$ being defined by the condition $\#\circ = \#\bullet(s)$, weighted count, in each block.

Problem. What is the law of the main character χ for H_N^s? And, what about the laws of truncated characters χ_t?

Comment. At $s = 1$, where the group is S_N, we have $\chi \sim p_1$, and more generally $\chi_t \sim p_t$, Poisson laws, with $N \to \infty$.

Real reflections 1/4

Definition. The hyperoctahedral group $H_N \subset O_N$ is:

(1) The symmetry group of the unit hypercube $\square_N \subset \mathbb{R}^N$.

(2) The group of symmetries of the N coordinate axes of $\mathbb{R}^N$.

(3) The group of permutation-like matrices with ± 1 entries.

Theory. We have $H_N = \mathbb{Z}_2 \wr S_N$, the reflection subgroups reduce to $SH_N = H_N \cap SO_N$, and we have easiness, with $D = P_{even}$.

Real reflections 2/4

Theorem. The laws of truncated characters for H_N are

$$\mathrm{law}(\chi_t) \simeq e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

for any $t \in (0,1]$, in the $N \to \infty$ limit.

Proof. Inclusion-exclusion principle, exactly as for S_N, but this time with the permutations $\sigma \in S_N$ being decorated by signs $\varepsilon \in \mathbb{Z}_2^N$.

Real reflections 3/4

Remark. The limiting truncated character law for H_N is

$$b_t = e^{-t} \sum_{k \in \mathbb{Z}} \delta_k \, f_k(t/2)$$

where f_k is the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

Due to this fact, we call b_t Bessel law, of parameter t.

Real reflections 4/4

Theorem. The Bessel laws b_t have the semigroup property

$$b_s * b_t = b_{s+t}$$

with respect to the usual convolution of real measures.

Theorem. The Bessel laws are compound Poisson laws,

$$b_t = \mathrm{law}(a - b)$$

with a, b being independent, both following the Poisson law p_t.

Proofs. Similar to the proofs for S_N, using the Fourier transform.

Bessel laws 1/4

Theorem. Given a compactly supported positive measure ν on $\mathbb{R}$, having mass $t = mass(\nu)$, the following limit converges,

$$p_\nu = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{1}{n}\nu\right)^{*n}$$

and the measure p_ν is called compound Poisson law. For

$$\nu = \sum_{i=1}^{s} t_i \delta_{z_i}$$

with $t_i > 0$ and $z_i \in \mathbb{R}$, we have the formula

$$p_\nu = \text{law}\left(\sum_{i=1}^{s} z_i \alpha_i\right)$$

whenever the variables α_i are Poisson (t_i), independent.

Bessel laws 2/4

Definition. The higher Bessel laws are the compound Poisson laws

$$b_t^s = p_{t\varepsilon_s}$$

with ε_s being the uniform measure on the s-roots of unity.

Comments. By the above, this means that we have:

$$b_t^s = \lim_{n\to\infty}\left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\varepsilon_s\right)^{*n}$$

Equivalently, we have the following formula,

$$b_t^s = \mathrm{law}\left(\sum_{r=1}^{s} w^r \alpha_i\right)$$

where $w = e^{2\pi i/s}$, and where $\alpha_i \sim p_t$, independent.

Bessel laws 3/4

Examples.

(1) At $s = 1$ we obtain the Poisson laws p_t.

(2) At $s = 2$ we obtain the Bessel laws b_t.

(3) At $s = 3, 4, \ldots$ we obtain certain discrete complex measures.

(4) At $s = \infty$ we obtain certain complex measures B_t.

Bessel laws 4/4

Theorem. The Fourier transform of b_t^s is given by:

$$F_{b_t^s}(y) = \exp\left(t\sum_{r=1}^{s}(e^{iw^r y} - 1)\right)$$

Theorem. The Bessel laws for a convolution semigroup:

$$b_t^s * b_{t'}^s = b_{t+t'}^s$$

Proofs. The first formula is clear from the $b_t^s = \mathrm{law}\left(\sum_{r=1}^s w^r \alpha_i\right)$ interpretation, and the second formula follows from it.

Complex reflections 1/4

Definition. The reflection group H_N^s, depending on parameters

$$N \in \mathbb{N} \quad , \quad s \in \mathbb{N} \cup \{\infty\}$$

is the group of $N \times N$ matrices with entries in

$$\mathbb{Z}_s \cup \{0\}$$

having one nonzero entry on each row and each column.

Examples. At $s = 1$ we have the symmetric group $S_N \subset O_N$.
At $s = 2$ we have the hyperoctahedral group $H_N \subset O_N$.
At $s = 3, 4, \ldots$ we have a certain finite subgroup $H_N^s \subset U_N$.
At $s = \infty$ we have a certain infinite subgroup $K_N \subset U_N$.

Complex reflections 2/4

Theorem. The laws of truncated characters for H_N^s are

$$\mathrm{law}(\chi_t) \simeq b_t^s$$

for any $t \in (0,1]$, in the $N \to \infty$ limit.

Proof. Inclusion-exclusion principle, exactly as for S_N, but this time with the permutations $\sigma \in S_N$ being decorated by signs $\varepsilon \in \mathbb{Z}_s^N$.

Remark. This extends and unifies all our previous results.

Complex reflections 3/4

In the order to further extend all this, a first idea would be to look at the general series of complex reflection groups:

$$H_N^{sd}=\left\{U\in H_N^s\Big|\det U\in\mathbb Z_d\right\}$$

However, this does not seem to bring new laws, at least at order 0. The study of the fluctuations is an interesting problem.

Complex reflections 4/4

Another type of extension comes by staying with H_N^s, but looking at the fluctuations of the characters

$$g \to Tr(g)$$

or of the truncated characters

$$g \to \sum_{i=1}^{[tN]} g_{ii} \quad , \quad t \in (0,1]$$

or of the Diaconis-Shahshahani variables

$$g \to Tr(g^k) \quad , \quad k \in \mathbb{N}$$

and so on. Things here are quite well understood at $s = 1, 2$.

Representations of compact groups

Teo Banica

"Introduction to matrix groups", 5/6

08/20

Representations 1/3

Definition. Given a closed subgroup $G \subset U_N$, its representations are the continuous morphisms into unitary groups:

$$\rho : G \to U_n$$

As a basic example, we have the embedding $G \subset U_N$, called fundamental representation, and denoted π.

Comment. We will assume that our representations are "smooth", in the sense that their coefficients are polynomials of g_{ij}.

Representations 2/3

Definition. The representations of G are subject to:

(1) Making sums: $\rho+\nu : g \to diag(\rho(g),\nu(g))$.

(2) Making products: $\rho\otimes\nu : g \to \rho(g)\otimes\nu(g)$.

(3) Taking conjugates: $\bar{\rho} : g \to \overline{\rho(g)}$.

Definition. Given $G\subset_\pi U_N$, its Peter-Weyl representations

$$\pi^{\otimes k} \quad , \quad k=\circ\bullet\bullet\circ\ldots$$

are the representations obtained by tensoring $\pi,\bar{\pi}$.

Representations 3/3

Definition. Given $\rho : G \to U_n$ and $\nu : G \to U_m$, we set:

$$Hom(\rho, \nu) = \left\{ T \in M_{m\times n}(\mathbb{C}) \middle| T\rho(g) = \nu(g)T \right\}$$

and we use the following conventions:

(1) $Fix(\rho) = Hom(1, \rho)$ and $End(\rho) = Hom(\rho, \rho)$.

(2) $\rho \sim \nu$ when $Hom(\rho, \nu)$ contains an invertible element.

(3) ρ is called irreducible, $\rho \in Irr(G)$, when $End(\rho) = \mathbb{C}1$.

Definition. Given $G \subset_\pi U_N$, the collection of vector spaces

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

with $k, l = \circ \bullet \bullet \circ \ldots$ is called Tannakian category of G.

Peter-Weyl 1/7

Theorem (PW1). Any representation $\rho : G \to U_n$ decomposes as

$$\rho = \rho_1 + \ldots + \rho_k$$

direct sum of irreducible representations.

Proof. Consider the intertwiner algebra of our representation:

$$A = End(\rho) \subset M_n(\mathbb{C})$$

By writing its unit as $1 = q_1 + \ldots + q_k$, with q_i being minimal projections, we obtain a decomposition as follows:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

We can now define a subrepresentation ρ_i by restricting ρ to the space $Im(q_i)$, which is invariant, and the result follows.

Peter-Weyl 2/7

Theorem (PW2). Any irreducible representation $\rho : G \to U_n$ appears inside a certain Peter-Weyl representation $\pi^{\otimes k}$.

Proof. Given a representation $\rho : G \to U_n$, consider its space of coefficients, $C_\rho = span(g \to \rho(g)_{ij})$. Then $\rho \to C_\rho$ is functorial, mapping subrepresentations into subspaces. We have:

$$< C_\pi >= \sum_k C_{\pi^{\otimes k}}$$

By smoothness, $C_\rho \subset < C_\pi >$, for certain exponents $k_1, \ldots, k_p$:

$$C_\rho \subset C_{\pi^{\otimes k_1} \oplus \ldots \oplus \pi^{\otimes k_p}}$$

Thus we have $\rho \subset \pi^{\otimes k_1} \oplus \ldots \oplus \pi^{\otimes k_p}$, and PW1 gives the result.

Peter-Weyl 3/7

Theorem. Any closed subgroup $G \subset U_N$ has a Haar measure

$$\mu(gE) = \mu(Eg) = \mu(E)$$

which can be constructed by starting with any probability measure ν, and taking the following Cesàro limit:

$$\mu = \lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} \nu^{*k}$$

Moreover, for any representation $\rho : G \to U_n$, the matrix

$$P = \left(\int_G \rho(g)_{ij} \, dg \right)_{ij}$$

is the projection onto $Fix(\rho) = \{\xi \in \mathbb{C}^n | \rho(g)\xi = \xi\}$.

Peter-Weyl 4/7

Proof. Our first claim is that given any positive mass 1 measure ν on our group G, not necessarily strictly positive, the limit

$$\int_G^\nu f = \lim_{r\to\infty}\frac{1}{r}\sum_{k=1}^r\int_G f(g)d\nu^{*k}(g)$$

exists, and for any representation $\rho : G \to U_n$, the matrix

$$P = \left(\int_G^\nu \rho(g)_{ij}\,dg\right)_{ij}$$

is the projection onto the 1-eigenspace of the matrix:

$$M = \left(\int_G \rho(g)_{ij}d\nu(g)\right)_{ij}$$

This is indeed standard algebra, on the coefficient space C_ρ.

Peter-Weyl 5/7

End of proof. Assuming now that ν is strictly positive, we must prove that $M\xi = \xi$ implies $\xi \in Fix(\rho)$. Let us set:

$$f(g) = \sum_i \left(\sum_j \rho(g)_{ij}\xi_j - \xi_i \right) \overline{\left(\sum_k \rho(g)_{ik}\xi_k - \xi_i \right)}$$

We must prove that $f = 0$. Since $\rho(g)$ is unitary, we obtain:

$$f(g) = 2\Big(||\xi||^2 - Re(< \rho(g)\xi, \xi >) \Big)$$

By using now $M\xi = \xi$, we obtain from this, by integrating:

$$\int_G f(g) d\nu(g) = 0$$

Thus we have $f = 0$, and so $\xi \in Fix(\rho)$, as desired.

Peter-Weyl 6/7

Theorem (PW3). The space $\mathcal{C}(G) = < C_\pi >$ decomposes as

$$\mathcal{C}(G) = \bigoplus_{\rho \in Irr(A)} M_{\dim(\rho)}(\mathbb{C})$$

the summands being pairwise orthogonal with respect to $\int_G$.

Proof. We must prove that for $\rho, \nu \in Irr(G)$ we have:

$$\rho \not\sim \nu \implies C_\rho \perp C_\nu$$

The matrix P given by $P_{ia,jb} = \int_G \rho_{ij} \bar{\nu}_{ab}$ is the projection onto:

$$Fix(\rho \otimes \bar{\nu}) \simeq Hom(\rho, \nu) = \{0\}$$

Thus we have $P = 0$, and this gives the result.

Peter-Weyl 7/7

Theorem (PW4). The characters of irreducible representations

$$\chi_\rho : G \to \mathbb{C} \quad , \quad g \to Tr(\rho(g))$$

belong to the algebra of "smooth central functions"

$$\mathcal{C}(G)_{central} = \left\{ f \in \mathcal{C}(G) \middle| f(gh) = f(hg) \right\}$$

and form an orthonormal basis of it.

Proof. The only tricky assertion is the norm 1 one. But:

$$\int_G \chi_\rho \bar{\chi}_\rho = \sum_{ij} \int_G \rho_{ii} \bar{\rho}_{jj} = \sum_i \frac{1}{N} = 1$$

Here we have used the fact that the integrals $\int_G \rho_{ij} \bar{\rho}_{kl}$ form the orthogonal projection onto $Fix(\rho \otimes \bar{\rho}) \simeq End(\rho) = \mathbb{C}1$.

Easiness 1/3

Theorem. The closed subgroups $G \subset U_N$ are in correspondence with the Tannakian categories $C = (C_{kl})$, via the construction

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

in one sense, and via the construction

$$G = \left\{ g \in U_N \middle| Tg^{\otimes k} = g^{\otimes l} T, \forall T \in C \right\}$$

in the other sense.

Proof. This is something quite technical, basically due to Tannaka and Krein, and heavily using the Peter-Weyl theory.

Easiness 2/3

Definition. A collection of subsets $D(k,l) \subset P(k,l)$ is called a category of partitions when it satisfies:

(1) Stability under the horizontal concatenation, $(\pi,\sigma) \to [\pi\sigma]$.

(2) Stability under vertical concatenation $(\pi,\sigma) \to [^{\sigma}_{\pi}]$ (matching).

(3) Stability under the upside-down turning $*$, with $\circ \leftrightarrow \bullet$.

(4) Each $P(k,k)$ contains the identity partition $|| \ldots ||$.

(5) Both $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ contain the semicircle $\cap$.

Easiness 3/3

Definition. A closed subgroup $G \subset U_N$ is called easy when

$$Hom(\pi^{\otimes k}, \pi^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Examples 1/2

Theorem. The basic unitary and reflection groups, namely

$$\begin{array}{ccc} O_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{array}$$

are all easy, coming from the following categories of partitions:

$$\begin{array}{ccc} P_2 & \longleftarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow \\ P_{even} & \longleftarrow & \mathcal{P}_{even} \end{array}$$

Proof. This result, due to Brauer, and also known as Schur-Weyl duality, comes from Tannaka, by working out the details.

Examples 2/2

In addition to the above, it is known that:

(1) In the continuous case, the bistochastic groups $B_N \subset O_N$ and $C_N \subset U_N$ are easy as well, coming from $P_{12}, \mathcal{P}_{12}$.

(2) In the discrete case, S_N is easy as well, coming from P itself. In fact, the reflection groups H_N^s are all easy, coming from P^s.

(3) Back to the continuous case, SU_2, SO_3 and $Sp_N \subset U_N$ are not easy. However, they are "super-easy" in a suitable sense.

(4) However, the general SO_N, SU_N, and other groups constructed using det, such as H_N^{sd}, are definitely not easy.

Probability on compact groups

Teo Banica

"Introduction to matrix groups", 6/6

08/20

Characters 1/3

Problem. Given a closed subgroup $G \subset U_N$, what is the law of

$$\chi : G \to \mathbb{C} \quad , \quad g \to Tr(g)$$

with respect to the uniform integration over G?

Characters 2/3

Motivation. The moments of χ are the dimensions

$$M_k = \dim(Fix(\pi^{\otimes k}))$$

of the fixed point spaces of tensor powers of $\pi : G \subset U_N$.

Comment. We are mostly interested in the Tannakian category

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

and by Frobenius, we have identifications as follows:

$$Hom(\pi^{\otimes k}, \pi^{\otimes l}) = Fix(\pi^{\otimes \bar{k}l})$$

Thus, the moments of χ count the dimensions $\dim(C_{kl})$.

Characters 3/3

Version. More generally, we are interested in the truncations

$$\chi_t : G \to \mathbb{C} \quad , \quad g \to \sum_{i=1}^{[tN]} g_{ii}$$

with $t \in (0,1]$ of the main character $\chi = \chi_1$.

Example. For the symmetric group $S_N \subset O_N$ we have

$$\chi \sim p_1$$

Poisson, and more generally $\chi_t \sim p_t$ for any t, with $N \to \infty$.

Finite groups 1/4

Theorem. For the cyclic group $\mathbb{Z}_N \subset O_N$ we have

$$\chi(g) = N\delta_{g0}$$

and the corresponding distribution is a Bernoulli law:

$$law(\chi) = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}\delta_N$$

Proof. The cyclic matrices have 0 on the diagonal, and so trace 0, expect for the identity, having 1 on the diagonal, and trace N.

Remark. The truncated characters and the asymptotics are not interesting. We do not have convolution semigroups.

Finite groups 2/4

Theorem. For the dihedral group $D_N \subset S_N$ we have:

$$law(\chi) = \begin{cases} \left(\frac{3}{4} - \frac{1}{2N}\right)\delta_0 + \frac{1}{4}\delta_2 + \frac{1}{2N}\delta_N & (N\ even) \\ \\ \left(\frac{1}{2} - \frac{1}{2N}\right)\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{2N}\delta_N & (N\ odd) \end{cases}$$

Proof. The dihedral group D_N consists of:

(1) N symmetries, having 1 fixed point when N is odd, and having 0 or 2 fixed points, $50 - 50$, when N is even.

(2) N rotations, having 0 fixed points, except for the identity, which has N fixed points.

Remark. The truncations and asymptotics are not interesting.

Finite groups 3/4

Theorem. For the symmetric group $S_N \subset O_N$ we have

$$\chi_t(\sigma) = \left\{ i \in \{1, \ldots, [tN]\} \middle| \sigma(i) = i \right\}$$

and we have $law(\chi_t) \simeq p_t$, Poisson laws, with $N \to \infty$.

Proof. By using the inclusion-exclusion principle, we have:

$$P(\chi = 0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^N}{N!} \simeq \frac{1}{e}$$

The same method gives succesively, by generalizing,

$$P(\chi = k) \simeq \frac{1}{e} \cdot \frac{1}{k!} \quad , \quad P(\chi_t = k) \simeq \frac{1}{e^t} \cdot \frac{t^k}{k!}$$

so we obtain in the $N \to \infty$ limit the Poisson laws p_t.

Finite groups 4/4

Theorem. For the complex reflection groups

$$H_N^s = \mathbb{Z}_s \wr S_N$$

we have $law(\chi_t) \simeq b_t^s$, Bessel laws, with $N \to \infty$.

Proof. The elements of H_N^s being usual permutations $\sigma \in S_N$ "decorated" with signs $\varepsilon \in \mathbb{Z}_s^N$, we can use the same method as before, inclusion-exclusion, and with $N \to \infty$ we are led to

$$b_t^s = \pi_{t\varepsilon_s}$$

compound Poisson laws, with ε_s being the uniform measure on $\mathbb{Z}_s$, which are called Bessel laws, due to the fact that at $s = 2$ the density is the Bessel function of the first kind.

Lie groups 1/4

Definition. The normal law of parameter 1 is:

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

These laws appear via the Central Limit Theorem (CLT).

Lie groups 2/4

Theorem. The moments of the normal laws are

$$M_k(g_t) = t^{k/2} \times k!!$$

where $k!! = 1.3.5\ldots(k-1)$, with $k!! = 0$ when k is odd.

Proof. We have the following computation:

$$\begin{aligned} M_k &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^k e^{-x^2/2t} dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} (tx^{k-1}) \left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} t(k-1)x^{k-2} e^{-x^2/2t} dx \end{aligned}$$

We obtain $M_k = t(k-1)M_{k-2}$, which gives the result.

Lie groups 3/4

Theorem. For the orthogonal group O_N we have

$$law(\chi) \simeq g_1$$

with $N \to \infty$.

Proof. By using the Brauer easiness result, we have:

$$\begin{aligned} M_k(\chi) &= \dim(Fix(\pi^{\otimes k})) \\ &= \dim(span(T_\pi | \pi \in P_2(k))) \\ &\simeq |P_2(k)| \\ &= k!! \end{aligned}$$

Thus, the main character χ has the same moments as g_1.

Lie groups 4/4

The other classical Lie groups can be investigated by using the same method, and the asymptotic law of χ is as follows:

(1) For U_N we obtain the complex Gaussian law G_1. The proof is similar, by using $M_k(G_1) = |\mathcal{P}_2(k)|$.

(2) For the bistochastic groups $B_N \subset O_N$ and $C_N \subset U_N$ we obtain shifted versions of g_1, G_1.

(3) The symplectic group $Sp_N \subset U_N$ is not exactly easy, but rather "super-easy", and we obtain the Gaussian law g_1.

Truncation 1/4

Theorem. The Haar integration over $G \subset_\pi U_N$ is given by

$$\int_G g_{i_1 j_1}^{s_1} \ldots g_{i_k j_k}^{s_k} dg = \sum_{\sigma,\tau \in D_k} \delta_\sigma(i)\delta_\tau(j) W_k(\sigma,\tau)$$

where D_k is a basis of $Fix(\pi^{\otimes k})$, $\delta_\sigma(i) =< \sigma, e_{i_1} \otimes \ldots \otimes e_{i_k} >$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\sigma,\tau) =< \sigma,\tau >$.

Proof. The integrals in the statement form the projection P onto $Fix(\pi^{\otimes k}) = span(D_k)$. Consider the following linear map:

$$E(x) = \sum_{\sigma \in D_k} < x, \sigma > \sigma$$

By linear algebra we have $P = WE$, where W is the inverse on $span(D_k)$ of the restriction of E, and this gives the result.

Truncation 2/4

Theorem. For an easy group $G_N \subset U_N$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_{G_N} g_{i_1 j_1}^{s_1} \ldots g_{i_k j_k}^{s_k} dg = \sum_{\sigma,\tau \in D(k)} \delta_\sigma(i)\delta_\tau(j) W_{kN}(\sigma,\tau)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\sigma,\tau) = N^{|\sigma \vee \tau|}$.

Proof. The vectors associated to partitions are given by:

$$T_\sigma(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\sigma \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Thus the Gram matrix and Kronecker symbols are those above.

Truncation 3/4

Application. We have the following computation,

$$\begin{aligned}
&\int_{G_N} (g_{11} + \ldots + g_{ss})^k \, dg \\
= \;& \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \int_{G_N} g_{i_1 i_1} \ldots g_{i_k i_k} \, dg \\
= \;& \sum_{\sigma,\tau \in D(k)} W_{kN}(\sigma,\tau) \sum_{i_1=1}^{s} \ldots \sum_{i_k=1}^{s} \delta_\sigma(i)\delta_\tau(i) \\
= \;& \sum_{\sigma,\tau \in D(k)} W_{kN}(\sigma,\tau) G_{ks}(\tau,\sigma) \\
= \;& Tr(W_{kN} G_{ks})
\end{aligned}$$

and the $s = [tN] \to \infty$ asymptotics can be worked out.

Truncation 4/4

Theorem. The truncated characters χ_t for the main unitary and reflection groups are as follows, in the $N \to \infty$ limit,

$$\begin{matrix} O_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{matrix} \qquad \sim \qquad \begin{matrix} g_t & \cdots\cdots & G_t \\ \vdots & & \vdots \\ b_t & \cdots\cdots & B_t \end{matrix}$$

and we have independence results as well, with $N \to \infty$.

Proof. In the discrete case, this is something that we already know. In general, this follows by using the above results.

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