Noncommutative spheres and tori

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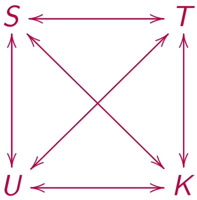
"Introduction to noncommutative geometry", 1/6

07/20

Plan

$\Longrightarrow$ No free $\mathbb{R}^N$, or free $\mathbb{C}^N$

Step 1. Axiomatize and classify the quadruplets



Step 2. Develop the geometries that you found.

Step 3. Integration theory, Riemannian aspects.

Step 4. Work more, reach to "Nash-Connes Geometry".

Real geometry 1/2

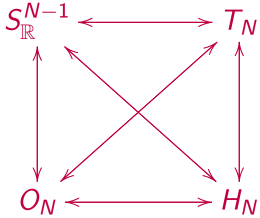
Definition. The real sphere, torus, unitary group and reflection group are:

$$\begin{aligned}
S^{N-1}_{\mathbb{R}} &= \left\{x \in \mathbb{R}^N \Big| \sum_i x_i^2 = 1\right\}\\
T_N &= \left\{x \in \mathbb{R}^N \Big| x_i = \pm\frac{1}{\sqrt{N}}\right\}\\
O_N &= \left\{U \in M_N(\mathbb{R}) \Big| U^t = U^{-1}\right\}\\
H_N &= \left\{U \in M_N(\pm 1) \Big| U^t = U^{-1}\right\}
\end{aligned}$$

These are the usual sphere, cube, orthogonal group, and hyperoctahedral group.

Real geometry 2/2

Theorem. We have a full set of correspondences, as follows,



obtained via various results from basic geometry and group theory.

Proof. This is standard, using $T_N = \mathbb{Z}_2^N$ and $H_N = \mathbb{Z}_2 \wr S_N$.

Complex geometry 1/2

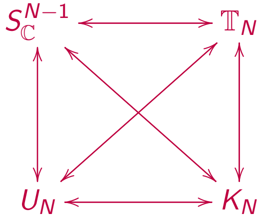
Definition. The complex sphere, torus, unitary group and reflection group are:

$$\begin{aligned} S^{N-1}_{\mathbb{C}} &= \left\{x\in\mathbb{C}^N\Big|\sum_i|x_i|^2=1\right\}\\ \mathbb{T}_N &= \left\{x\in\mathbb{C}^N\Big|\ |x_i|=\frac{1}{\sqrt{N}}\right\}\\ U_N &= \left\{U\in M_N(\mathbb{C})\Big|U^*=U^{-1}\right\}\\ K_N &= \left\{U\in M_N(\mathbb{T})\Big|U^*=U^{-1}\right\} \end{aligned}$$

These are the usual complex sphere, torus, unitary group, and complex reflection group.

Complex geometry 2/2

Theorem. We have a full set of correspondences, as follows,

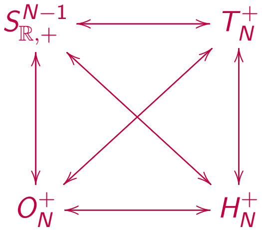


obtained via various results from basic geometry and group theory.

Proof. This is standard, using $\mathbb{T}_N = \mathbb{T}^N$ and $K_N = \mathbb{T} \wr S_N$.

Free real geometry

We will construct a diagram as follows:



– $S^{N-1}_{\mathbb{R},+}$ is defined via $x_i = x_i^*$, $\sum_i x_i^2 = 1$.

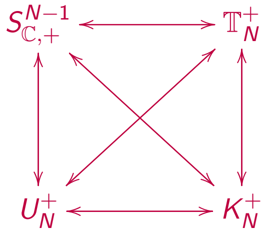
– O_N^+ is defined via $u_{ij} = u_{ij}^*$, $u^t = u^{-1}$.

– $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$ and $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$.

$\implies$ some of the correspondences are quite tricky.

Free complex geometry

We will construct a diagram as follows:



– $S^{N-1}_{\mathbb{C},+}$ is defined via $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$.

– U_N^+ is defined via $u^* = u^{-1}, u^t = \bar{u}^{-1}$.

– $\mathbb{T}_N^+ = \widehat{F_N}$ and $K_N^+ = \mathbb{T} \wr_* S_N^+$.

$\implies$ some of the correspondences are quite tricky.

Operator algebras 1/4

<u>Theorem.</u> Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra with unit $B(H)$, which:

(1) is complete with respect to $||.||$ (Banach algebra).

(2) has an involution $T \to T^*$, $< Tx, y > = < x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

<u>Proof</u>. All this is standard, and well-known for the matrices.

Operator algebras 2/4

Definition. A C^*-algebra is a complex algebra with unit A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

Theorem. (Gelfand) Any commutative C^*-algebra is the form $C(X)$, with X being a compact space.

Proof. Let $X = Spec(A)$ be the space of characters $\chi : A \to \mathbb{C}$. By basic spectral theory $ev : A \to C(X)$ is an isomorphism.

Operator algebras 3/4

Theorem. (GNS) Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. In the commutative case, $A = C(X)$, we have:

$$A \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

In general the proof is similar, by using basic spectral theory.

Operator algebras 4/4

Theorem. The finite dimensional C^*-algebras are of the form:

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

That is, they are the finite sums of matrix algebras.

Proof. This is elementary, in 5 steps, as follows:

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Spheres 1/4

Definition. Given an arbitrary C^*-algebra A, we write

$$A = C(X)$$

and call X a "noncommutative compact space".

Equivalently, the category of the noncommutative compact spaces is the category of the C^*-algebras, with the arrows reversed.

Example 1. Given a morphism $\Phi : A \to B$, we write $A = C(X)$, $B = C(Y)$, and speak of the morphism $\phi : Y \to X$.

Example 2. Given a tensor product $A = B \otimes C$, we let $A = C(X)$, $B = C(Y)$, $C = C(Z)$, and speak of $X = Y \times Z$.

Spheres 2/4

Definition. We have noncommutative spaces, as follows,

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_i x_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^*=\sum_i x_i^*x_i=1\right)$$

called free real sphere, and free complex sphere.

The above universal algebras are well-defined, because we have

$$\sum_i ||x_i||^2=\sum_i ||x_ix_i^*||\leq ||\sum_i x_ix_i^*||=1$$

and so the biggest C^*-norms on our algebras exist indeed.

Spheres 3/4

Definition. Given a noncommutative compact space X, its classical version is the subspace $X_{class}\subset X$ obtained by setting:

$$C(X_{class})=C(X)/I\quad,\quad I=<[a,b]>$$

In this situation, we say that X appears as a “liberation” of X.

In other words, X_{class} is the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe that X_{class} is indeed classical.

Spheres 4/4

Theorem. We have embeddings of NC spaces, as follows,

$$\begin{array}{ccc}
S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\
\uparrow & & \uparrow \\
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}}
\end{array}$$

and the free spheres are liberations of the classical ones.

Proof. We must establish the following isomorphisms:

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \Sigma_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\Sigma_i x_i x_i^* = \Sigma_i x_i^* x_i = 1\right)$$

But these isomorphisms are both clear, by using Gelfand.

Tori 1/4

Definition. We have noncommutative spaces, as follows,

$$C(T_N^+)=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,x_i^2=\frac{1}{N}\right)$$

$$C(\mathbb{T}_N^+)=C^*\left(x_1,\ldots,x_N\Big|x_ix_i^*=x_i^*x_i=\frac{1}{N}\right)$$

called free real torus, and free complex torus.

$\Longrightarrow$ As before, $T_N^+,\mathbb{T}_N^+$ appear as liberations of $T_N,\mathbb{T}_N$.

$\Longrightarrow$ Also, we have 4 formulae of type $T=S\cap\mathbb{T}_N^+$.

Tori 2/4

Theorem. Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by:

$$g^* = g^{-1} \quad , \quad \forall g \in \Gamma$$

The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is then a C^*-norm, and the corresponding closure of $\mathbb{C}[\Gamma]$ is a C^*-algebra, denoted $C^*(\Gamma)$.

Proof. Let $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

But this is elementary to check, and gives the result.

Tori 3/4

Theorem. When Γ is abelian, we have an isomorphism

$$C^*(\Gamma) \simeq C(G)$$

where $G = \widehat{\Gamma}$ is its dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.

Proof. Gelfand gives $A = C(X)$, with $X = Spec(A)$. But the spectrum $X = Spec(A)$, made of characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, as desired.

Definition. Given a discrete group Γ, the space G given by

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Tori 4/4

Theorem. The basic tori are all group duals, as follows,

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix} \quad = \quad \begin{matrix} \widehat{\mathbb{Z}_2^{*N}} & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \widehat{\mathbb{Z}_2^N} & \longrightarrow & \widehat{\mathbb{Z}^N} \end{matrix}$$

where F_N is the free group, and $*$ is a free product.

Proof. The diagram formed by the algebras $C(T)$ is:

$$\begin{matrix} C^*(\mathbb{Z}_2^{*N}) & \longleftarrow & C^*(\mathbb{Z}^{*N}) \\ \downarrow & & \downarrow \\ C^*(\mathbb{Z}_2^N) & \longleftarrow & C^*(\mathbb{Z}^N) \end{matrix}$$

But this gives the result, via some standard identifications.

Bugs

Problem. For non-amenable groups Γ, such as the free ones F_N,

$$C^*(\Gamma) \to C^*_{red}(\Gamma)$$

is not an isomorphism. Thus $\widehat{\Gamma}$ is multi-represented as NC space (!)

Remark. In fact this issue appeared even before, when talking about $X = Y \times Z$. Indeed, there are several tensor products.

$\implies$ What's the fix?

Fixes 1/3

Definition. The category of compact measured spaces (X, μ) is the category of the C^*-algebras with faithful traces

$$(A, \varphi)$$

with the arrows reversed. In the case where φ is not faithful, we can still talk about (X, μ), by performing the GNS construction.

$\implies$ Works for group duals, but for other spaces like the spheres this is more complicated, because we have no traces yet.

Fixes 2/3

Definition. The category of compact measured spaces (X, μ) is the category of von Neumann algebras with faithful traces

$$(A, \varphi)$$

with the arrows reversed. As basic examples, we have the algebras $L^\infty(X)$ coming by GNS construction from the algebras $C(X)$.

$\implies$ Same fix as before, better looking. Works for group duals, but for spheres and other spaces we have no traces yet.

Fixes 3/3

Definition. The category of real algebraic manifolds $X\subset S^{N-1}_{\mathbb C,+}$ is the category of the universal C^*-algebras of type

$$C(X)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

with $f_i\in\mathbb C<x_1,\ldots,x_N>$ being noncommutative polynomials, with the arrows $X\to Y$ being the $*$-algebra morphisms

$$\mathcal C(Y)\to\mathcal C(X)$$

between the corresponding $*$-algebras generated by the coordinate functions $x_1,\ldots,x_N$, mapping coordinates to coordinates.

$\implies$ This is the good fix, and we will heavily use it.