

# A NOTE ON FREE QUANTUM GROUPS

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ABSTRACT. We study the free complexification operation for compact quantum groups,  $G \rightarrow G^c$ . We prove that, with suitable definitions, this induces a one-to-one correspondence between free orthogonal quantum groups of infinite level, and free unitary quantum groups satisfying  $G = G^c$ .

## INTRODUCTION

In this paper we present some advances on the notion of free quantum group, introduced in [3]. We first discuss in detail a result mentioned there, namely that the free complexification operation  $G \rightarrow G^c$  studied in [2] produces free unitary quantum groups out of free orthogonal ones. Then we work out the injectivity and surjectivity properties of  $G \rightarrow G^c$ , and this leads to the correspondence announced in the abstract. This correspondence should be regarded as being a first general ingredient for the classification of free quantum groups.

We include in our study a number of general facts regarding the operation  $G \rightarrow G^c$ , by improving some previous work in [2]. The point is that now we can use general diagrammatic techniques from [4], new examples, and the notion of free quantum group [3], none of them available at the time of writing [2].

The paper is organized as follows: 1 contains some basic facts about the operation  $G \rightarrow G^c$ , and in 2-5 we discuss the applications to free quantum groups.

## 1. FREE COMPLEXIFICATION

A fundamental result of Voiculescu [6] states if  $(s_1, \dots, s_n)$  is a semicircular system, and  $z$  is a Haar unitary free from it, then  $(zs_1, \dots, zs_n)$  is a circular system. This makes appear the notion of free multiplication by a Haar unitary,  $a \rightarrow za$ , that we call here free complexification. This operation has been intensively studied since then. See Nica and Speicher [5].

This operation appears as well in the context of Wang's free quantum groups [7], [8]. The main result in [1] is that the universal free biunitary matrix is the free complexification of the free orthogonal matrix. In other words, the passage

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$O_n^+ \rightarrow U_n^+$  is nothing but a free complexification:  $U_n^+ = O_n^{+c}$ . Moreover, some generalizations of this fact are obtained, in an abstract setting, in [2].

In this section we discuss the basic properties of  $A \rightarrow \tilde{A}$ , the functional analytic version of  $G \rightarrow G^c$ . We use an adaptation of Woronowicz's axioms in [9].

**Definition 1.1.** *A finitely generated Hopf algebra is a pair  $(A, u)$ , where  $A$  is a  $C^*$ -algebra and  $u \in M_n(A)$  is a unitary whose entries generate  $A$ , such that*

$$\begin{aligned}\Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji}^*\end{aligned}$$

*define morphisms of  $C^*$ -algebras (called comultiplication, counit and antipode).*

In other words, given  $(A, u)$ , the morphisms  $\Delta, \varepsilon, S$  can exist or not. If they exist, they are uniquely determined, and we say that we have a Hopf algebra.

The basic examples are as follows:

- (1) The algebra of functions  $A = C(G)$ , with the matrix  $u = (u_{ij})$  given by  $g = (u_{ij}(g))$ , where  $G \subset U_n$  is a compact group.
- (2) The group algebra  $A = C^*(\Gamma)$ , with the matrix  $u = \text{diag}(g_1, \dots, g_n)$ , where  $\Gamma = \langle g_1, \dots, g_n \rangle$  is a finitely generated group.

Let  $\mathbb{T}$  be the unit circle, and let  $z : \mathbb{T} \rightarrow \mathbb{C}$  be the identity function,  $z(x) = x$ . Observe that  $(C(\mathbb{T}), z)$  is a finitely generated Hopf algebra, corresponding to the compact group  $\mathbb{T} \subset U_1$ , or, via the Fourier transform, to the group  $\mathbb{Z} = \langle 1 \rangle$ .

**Definition 1.2.** *Associated to  $(A, u)$  is the pair  $(\tilde{A}, \tilde{u})$ , where  $\tilde{A} \subset C(\mathbb{T}) * A$  is the  $C^*$ -algebra generated by the entries of the matrix  $\tilde{u} = zu$ .*

It follows from the general results of Wang in [7] that  $(\tilde{A}, \tilde{u})$  is indeed a finitely generated Hopf algebra. Moreover,  $\tilde{u}$  is the free complexification of  $u$  in the free probabilistic sense, i.e. with respect to the Haar functional. See [2].

A morphism between two finitely generated Hopf algebras  $f : (A, u) \rightarrow (B, v)$  is by definition a morphism of  $*$ -algebras  $A_s \rightarrow B_s$  mapping  $u_{ij} \rightarrow v_{ij}$ , where  $A_s \subset A$  and  $B_s \subset B$  are the dense  $*$ -subalgebras generated by the elements  $u_{ij}$ , respectively  $v_{ij}$ . Observe that in order for a such a morphism to exist,  $u, v$  must have the same size, and that if such a morphism exists, it is unique. See [2].

**Proposition 1.3.** *The operation  $A \rightarrow \tilde{A}$  has the following properties:*

- (1) *We have a morphism  $(\tilde{A}, \tilde{u}) \rightarrow (A, u)$ .*
- (2) *A morphism  $(A, u) \rightarrow (B, v)$  produces a morphism  $(\tilde{A}, \tilde{u}) \rightarrow (\tilde{B}, \tilde{v})$ .*
- (3) *We have an isomorphism  $(\tilde{\tilde{A}}, \tilde{\tilde{u}}) = (\tilde{A}, \tilde{u})$ .*

*Proof.* All the assertions are clear from definitions, see [2] for details. □

**Theorem 1.4.** *If  $\Gamma = \langle g_1, \dots, g_n \rangle$  is a finitely generated group then  $\tilde{C}^*(\Gamma) \simeq C^*(\mathbb{Z} * \Lambda)$ , where  $\Lambda = \langle g_i^{-1}g_j \mid i, j = 1, \dots, n \rangle$ .*

*Proof.* By using the Fourier transform isomorphism  $C(\mathbb{T}) \simeq C^*(\mathbb{Z})$  we obtain  $\tilde{C}^*(\Gamma) = C^*(\tilde{\Gamma})$ , with  $\tilde{\Gamma} \subset \mathbb{Z} * \Gamma$ . Then, a careful examination of generators gives the isomorphism  $\tilde{\Gamma} \simeq \mathbb{Z} * \Lambda$ . See [2] for details.  $\square$

At the dual level, we have the following question: what is the compact quantum group  $G^c$  defined by  $C(G^c) = \tilde{C}(G)$ ? There is no simple answer to this question, unless in the abelian case, where we have the following result.

**Theorem 1.5.** *If  $G \subset U_n$  is a compact abelian group then  $\tilde{C}(G) = C^*(\mathbb{Z} * \hat{L})$ , where  $L$  is the image of  $G$  in the projective unitary group  $PU_n$ .*

*Proof.* The embedding  $G \subset U_n$ , viewed as a representation, must come from a generating system  $\hat{G} = \langle g_1, \dots, g_n \rangle$ . It routine to check that the subgroup  $\Lambda \subset \hat{G}$  constructed in Theorem 1.4 is the dual of  $L$ , and this gives the result.  $\square$

## 2. FREE QUANTUM GROUPS

Consider the groups  $S_n \subset O_n \subset U_n$ , with the elements of  $S_n$  viewed as permutation matrices. Consider also the following subgroups of  $U_n$ :

- (1)  $S'_n = \mathbb{Z}_2 \times S_n$ , the permutation matrices multiplied by  $\pm 1$ .
- (2)  $H_n = \mathbb{Z}_2 \wr S_n$ , the permutation matrices with  $\pm$  coefficients.
- (3)  $P_n = \mathbb{T} \times S_n$ , the permutation matrices multiplied by scalars in  $\mathbb{T}$ .
- (4)  $K_n = \mathbb{T} \wr S_n$ , the permutation matrices with coefficients in  $\mathbb{T}$ .

Observe that  $H_n$  is the hyperoctahedral group. It is convenient to collect the above definitions into a single one, in the following way.

**Definition 2.1.** *We use the diagram of compact groups*

$$\begin{array}{ccccc} U_n & \supset & K_n & \supset & P_n \\ & & \cup & & \cup \\ & & & & \cup \\ O_n & \supset & H_n & \supset & S_n^* \end{array}$$

where  $S^*$  denotes at the same time  $S$  and  $S'$ .

In what follows we describe the free analogues of these 7 groups. For this purpose, we recall that a square matrix  $u \in M_n(A)$  is called:

- (1) Orthogonal, if  $u = \bar{u}$  and  $u^t = u^{-1}$ .
- (2) Cubic, if it is orthogonal, and  $ab = 0$  on rows and columns.
- (3) Magic', if it is cubic, and the sum on rows and columns is the same.
- (4) Magic, if it is cubic, formed of projections ( $a^2 = a = a^*$ ).

- (5) Biunitary, if both  $u$  and  $u^t$  are unitaries.
- (6) Cubik, if it is biunitary, and  $ab^* = a^*b = 0$  on rows and columns.
- (7) Magik, if it is cubik, and the sum on rows and columns is the same.

Here the equalities of type  $ab = 0$  refer to distinct entries on the same row, or on the same column. The notions (1, 2, 4, 5) are from [7, 3, 8, 7], and (3, 6, 7) are new. The terminology is of course temporary: we have only 7 examples of free quantum groups, so we don't know exactly what the names name.

**Theorem 2.2.**  *$C(G_n)$  with  $G = OHS^*UKP$  is the universal commutative  $C^*$ -algebra generated by the entries of a  $n \times n$  orthogonal, cubic, magic\*, biunitary, cubik, magik matrix.*

*Proof.* The case  $G = OHSU$  is discussed in [7, 3, 8, 7], and the case  $G = S'KP$  follows from it, by identifying the corresponding subgroups.  $\square$

We proceed with liberation: definitions will become theorems and vice versa.

**Definition 2.3.**  *$A_g(n)$  with  $g = ohs^*ukp$  is the universal  $C^*$ -algebra generated by the entries of a  $n \times n$  orthogonal, cubic, magic\*, biunitary, cubik, magik matrix.*

The  $g = ohsu$  algebras are from [7, 3, 8, 7], and the  $g = s'kp$  ones are new.

**Theorem 2.4.** *We have the diagram of Hopf algebras*

$$\begin{array}{ccccc}
 A_u(n) & \rightarrow & A_k(n) & \rightarrow & A_p(n) \\
 & & \downarrow & & \downarrow \\
 & & A_o(n) & \rightarrow & A_h(n) & \rightarrow & A_{s^*}(n)
 \end{array}$$

where  $s^*$  denotes at the same time  $s$  and  $s'$ .

*Proof.* The morphisms in Definition 1.1 can be constructed by using the universal property of each of the algebras involved. For the algebras  $A_{ohsu}$  this is known from [7, 3, 8, 7], and for the algebras  $A_{s'kp}$  the proof is similar.  $\square$

### 3. DIAGRAMS

Let  $F = \langle a, b \rangle$  be the monoid of words on two letters  $a, b$ . For a given corepresentation  $u$  we let  $u^a = u$ ,  $u^b = \bar{u}$ , then we define the tensor powers  $u^\alpha$  with  $\alpha \in F$  arbitrary, according to the rule  $u^{\alpha\beta} = u^\alpha \otimes u^\beta$ .

**Definition 3.1.** *Let  $(A, u)$  be a finitely generated Hopf algebra.*

- (1)  $CA$  is the collection of linear spaces  $\{Hom(u^\alpha, u^\beta) | \alpha, \beta \in F\}$ .
- (2) In the case  $u = \bar{u}$  we identify  $CA$  with  $\{Hom(u^k, u^l) | k, l \in \mathbb{N}\}$ .

A morphism  $(A, u) \rightarrow (B, v)$  produces inclusions  $Hom(u^\alpha, u^\beta) \subset Hom(v^\alpha, v^\beta)$  for any  $\alpha, \beta \in F$ , so we have the following diagram:

$$\begin{array}{ccc} CA_u(n) & \subset & CA_k(n) & \subset & CA_p(n) \\ & \cap & & \cap & \\ & & CA_o(n) & \subset & CA_h(n) & \subset & CA_{s^*}(n) \end{array}$$

We recall that  $CA_s(n)$  is the category of Temperley-Lieb diagrams. That is,  $Hom(u^k, u^l)$  is isomorphic to the abstract vector space spanned by the diagrams between an upper row of  $2k$  points, and a lower row of  $2l$  points. See [3].

In order to distinguish between various meanings of the same diagram, we attach words to it. For instance  $\mathfrak{m}_{ab}, \mathfrak{m}_{ba}$  are respectively in  $D_s(\emptyset, ab), D_s(\emptyset, ba)$ .

**Lemma 3.2.** *The categories for  $A_g(n)$  with  $g = ohsukp$  are as follows:*

- (1)  $CA_o(n) = \langle \mathfrak{m} \rangle$ .
- (2)  $CA_h(n) = \langle \mathfrak{m}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \rangle$ .
- (3)  $CA_{s'}(n) = \langle \mathfrak{m}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right|, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \rangle$ .
- (4)  $CA_s(n) = \langle \mathfrak{m}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right|, \cap \rangle$ .
- (5)  $CA_u(n) = \langle \mathfrak{m}_{ab}, \mathfrak{m}_{ba} \rangle$ .
- (6)  $CA_k(n) = \langle \mathfrak{m}_{ab}, \mathfrak{m}_{ba}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \begin{smallmatrix} ab \\ ab \end{smallmatrix}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \begin{smallmatrix} ba \\ ba \end{smallmatrix} \rangle$ .
- (7)  $CA_p(n) = \langle \mathfrak{m}_{ab}, \mathfrak{m}_{ba}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \begin{smallmatrix} ab \\ ab \end{smallmatrix}, \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \begin{smallmatrix} ba \\ ba \end{smallmatrix}, \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \begin{smallmatrix} a \\ a \end{smallmatrix}, \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \begin{smallmatrix} b \\ b \end{smallmatrix} \rangle$ .

*Proof.* The case  $g = ohs$  is discussed in [3], and the case  $g = u$  is discussed in [4]. In the case  $g = s'kp$  we can use the following formulae:

$$\begin{aligned} \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} &= \sum_{ij} e_{ij} \\ \left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| &= \sum_i e_{ii} \otimes e_{ii} \end{aligned}$$

The commutation conditions  $\left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \in End(u \otimes \bar{u})$  and  $\left| \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \right| \in End(\bar{u} \otimes u)$  correspond to the cubik condition, and the extra relations  $\begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \in End(u)$  and  $\begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \in End(\bar{u})$  correspond to the magik condition. Together with the fact that orthogonal plus magik means magic', this gives all the  $g = s'kp$  assertions.  $\square$

We can color the diagrams in several ways: either by putting the sequence  $xyyxyyx \dots$  on both rows of points, or by putting  $\alpha, \beta$  on both rows, then by replacing  $a \rightarrow xy, b \rightarrow yx$ . We say that the diagram is colored if all the strings match, and half-colored, if there is an even number of unmatched.

**Theorem 3.3.** *For  $g = ohs^*ukp$  we have  $CA_g(n) = \text{span}(D_g)$ , where:*

- (1)  $D_s(k, l)$  is the set of all diagrams between  $2k$  points and  $2l$  points.
- (2)  $D_{s'}(k, l) = D_s(k, l)$  for  $k - l$  even, and  $D_{s'}(k, l) = \emptyset$  for  $k - l$  odd.

- (3)  $D_h(k, l)$  consists of diagrams which are colorable  $xyyxxxyx \dots$
- (4)  $D_o(k, l)$  is the image of  $D_s(k/2, l/2)$  by the doubling map.
- (5)  $D_p(\alpha, \beta)$  consists of diagrams half-colorable  $a \rightarrow xy, b \rightarrow yx$ .
- (6)  $D_k(\alpha, \beta)$  consists of diagrams colorable  $a \rightarrow xy, b \rightarrow yx$ .
- (7)  $D_u(\alpha, \beta)$  consists of double diagrams, colorable  $a \rightarrow xy, b \rightarrow yx$ .

*Proof.* This is clear from the above lemma, by composing diagrams. The case  $g = ohsu$  is discussed in [3, 4], and the case  $g = s'kp$  is similar.  $\square$

**Theorem 3.4.** *We have the following isomorphisms:*

- (1)  $A_u(n) = \tilde{A}_o(n)$ .
- (2)  $A_h(n) = \tilde{A}_k(n)$ .
- (3)  $A_p(n) = \tilde{A}_{s^*}(n)$ .

*Proof.* It follows from definitions that we have arrows from left to right. Now since by Theorem 3.3 the spaces  $End(u \otimes \bar{u} \otimes u \otimes \dots)$  are the same at right and at left, Theorem 5.1 in [2] applies, and gives the arrows from right to left.  $\square$

Observe that the assertion (1), known since [1], is nothing but the isomorphism  $U_n^+ = O_n^{+c}$  mentioned in the beginning of the first section.

#### 4. FREENESS, LEVEL, DOUBLING

We use the notion of free Hopf algebra, introduced in [3]. Recall that a morphism  $(A, u) \rightarrow (B, v)$  induces inclusions  $Hom(u^\alpha, u^\beta) \subset Hom(v^\alpha, v^\beta)$ .

**Definition 4.1.** *A finitely generated Hopf algebra  $(A, u)$  is called free if:*

- (1) *The canonical map  $A_u(n) \rightarrow A_s(n)$  factorizes through  $A$ .*
- (2) *The spaces  $Hom(u^\alpha, u^\beta) \subset \text{span}(D_s(\alpha, \beta))$  are spanned by diagrams.*

It follows from Theorem 3.3 that the algebras  $A_{ohs^*ukp}$  are free.

In the orthogonal case  $u = \bar{u}$  we say that  $A$  is free orthogonal, and in the general case, we also say that  $A$  is free unitary.

**Theorem 4.2.** *If  $A$  is free orthogonal then  $\tilde{A}$  is free unitary.*

*Proof.* It is shown in [2] that the tensor category of  $\tilde{A}$  is generated by the tensor category of  $A$ , embedded via alternating words, and this gives the result.  $\square$

**Definition 4.3.** *The level of a free orthogonal Hopf algebra  $(A, u)$  is the smallest number  $l \in \{0, 1, \dots, \infty\}$  such that  $1 \in u^{\otimes 2l+1}$ .*

As the level of examples, for  $A_s(n)$  we have  $l = 0$ , and for  $A_{ohs'}(n)$  we have  $l = \infty$ . This follows indeed from Theorem 3.3.

**Theorem 4.4.** *If  $l < \infty$  then  $\tilde{A} = C(\mathbb{T}) * A$ .*

*Proof.* Let  $\langle r \rangle$  be the algebra generated by the coefficients of  $r$ . From  $1 \in \langle u \rangle$  we get  $z \in \langle zu_{ij} \rangle$ , hence  $\langle zu_{ij} \rangle = \langle z, u_{ij} \rangle$ , and we are done.  $\square$

**Corollary 4.5.**  $A_p(n) = C(\mathbb{T}) * A_s(n)$ .

*Proof.* For  $A_s(n)$  we have  $1 \in u$ , hence  $l = 0$ , and Theorem 4.4 applies.  $\square$

We can define a “doubling” operation  $A \rightarrow A_2$  for free orthogonal algebras, by using Tannakian duality, in the following way: the spaces  $Hom(u^k, u^l)$  with  $k - l$  even remain by definition the same, and those with  $k - l$  odd become by definition empty. The interest in this operation is that  $A_2$  has infinite level.

At the level of examples, the doublings are  $A_{ohs^*}(n) \rightarrow A_{ohs'}(n)$ .

**Proposition 4.6.** *For a free orthogonal algebra  $A$ , the following are equivalent:*

- (1)  $A$  has infinite level.
- (2) The canonical map  $A_2 \rightarrow A$  is an isomorphism.
- (3) The quotient map  $A \rightarrow A_s(n)$  factorizes through  $A_{s'}(n)$ .

*Proof.* The equivalence between (1) and (2) is clear from definitions, and the equivalence with (3) follows from Tannakian duality.  $\square$

## 5. THE MAIN RESULT

We know from Theorem 3.4 that the two rows of the diagram formed by the algebras  $A_{ohs^*ukp}$  are related by the operation  $A \rightarrow \tilde{A}$ . Moreover, the results in the previous section suggest that the correct choice in the lower row is  $s^* = s'$ . The following general result shows that this is indeed the case.

**Theorem 5.1.** *The operation  $A \rightarrow \tilde{A}$  induces a one-to-one correspondence between the following objects:*

- (1) Free orthogonal algebras of infinite level.
- (2) Free unitary algebras satisfying  $A = \tilde{A}$ .

*Proof.* We use the notations  $\gamma_k = abab \dots$  and  $\delta_k = baba \dots$  ( $k$  terms each).

We know from Theorem 4.2 that the operation  $A \rightarrow \tilde{A}$  is well-defined, between the algebras in the statement. Moreover, since by Tannakian duality an orthogonal algebra of infinite level is determined by the spaces  $Hom(u^k, u^l)$  with  $k - l$  even, we get that  $A \rightarrow \tilde{A}$  is injective, because these spaces are:

$$Hom(u^k, u^l) = Hom((zu)^{\gamma_k}, (zu)^{\gamma_l})$$

It remains to prove surjectivity. So, let  $A$  be free unitary satisfying  $A = \tilde{A}$ . We have  $CA = \text{span}(D)$  for certain sets of diagrams  $D(\alpha, \beta) \subset D_s(\alpha, \beta)$ , so we can define a collection of sets  $D_2(k, l) \subset D_s(k, l)$  in the following way:

- (1) For  $k - l$  even we let  $D_2(k, l) = D(\gamma_k, \gamma_l)$ .
- (2) For  $k - l$  odd we let  $D_2(k, l) = \emptyset$ .

It follows from definitions that  $C_2 = \text{span}(D_2)$  is a category, with duality and involution. We claim that  $C_2$  is stable under  $\otimes$ . Indeed, for  $k, l$  even we have:

$$\begin{aligned} D_2(k, l) \otimes D_2(p, q) &= D(\gamma_k, \gamma_l) \otimes D(\gamma_p, \gamma_q) \\ &\subset D(\gamma_k \gamma_p, \gamma_l \gamma_q) \\ &= D(\gamma_{k+p}, \gamma_{l+q}) \\ &= D_2(k + p, l + q) \end{aligned}$$

For  $k, l$  odd and  $p, q$  even, we can use the canonical antilinear isomorphisms  $\text{Hom}(u^{\gamma_K}, u^{\gamma_L}) \simeq \text{Hom}(u^{\delta_K}, u^{\delta_L})$ , with  $K, L$  odd. At the level of diagrams we get equalities  $D(\gamma_k, \gamma_L) = D(\delta_K, \delta_L)$ , that can be used in the following way:

$$\begin{aligned} D_2(k, l) \otimes D_2(p, q) &= D(\delta_k, \delta_l) \otimes D(\gamma_p, \gamma_q) \\ &\subset D(\delta_k \gamma_p, \delta_l \gamma_q) \\ &= D(\delta_{k+p}, \delta_{l+q}) \\ &= D_2(k + p, l + q) \end{aligned}$$

Finally, for  $k, l$  odd and  $p, q$  odd, we can proceed as follows:

$$\begin{aligned} D_2(k, l) \otimes D_2(p, q) &= D(\gamma_k, \gamma_l) \otimes D(\delta_p, \delta_q) \\ &\subset D(\gamma_k \delta_p, \gamma_l \delta_q) \\ &= D(\gamma_{k+p}, \gamma_{l+q}) \\ &= D_2(k + p, l + q) \end{aligned}$$

Thus we have a Tannakian category, and by Woronowicz's results in [10] we get an algebra  $A_2$ . This algebra is free orthogonal, of infinite level. Moreover, the spaces  $\text{End}(u \otimes \bar{u} \otimes u \otimes \dots)$  being the same for  $A$  and  $A_2$ , Theorem 5.1 in [2] applies, and gives  $\tilde{A}_2 = \tilde{A}$ . Now since we have  $A = \tilde{A}$ , we are done.  $\square$

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