Noncommutative integration theory

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"Introduction to noncommutative geometry", 5/6

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Plan

We are interested in integrating over S, T, U, K.

(1) Weingarten formula.

(2) Free probability.

(3) Free integration, $N >> 0$.

(4) Free integration, N fixed.

(5) Rotations and permutations.

(6) Riemannian aspects.

Weingarten 1/4

Theorem. Assuming that $A = C(G)$ has Tannakian category $C = (C(k,l))$, the Haar integration over G is given by

$$\int_G u_{i_1j_1}^{s_1}\ldots u_{i_kj_k}^{s_k} = \sum_{\pi,\sigma\in D_k}\delta_\pi(i)\delta_\sigma(j)W_k(\pi,\sigma)$$

where D_k is a basis of $C(\emptyset, k)$, $\delta_\pi(i) =< \pi, e_{i_1}\otimes\ldots\otimes e_{i_k} >$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\pi,\sigma) =< \pi,\sigma >$.

Proof. The integrals in the statement form the projection P onto $Fix(u^{\otimes k}) = span(D_k)$. Consider the following linear map:

$$E(x) = \sum_{\pi\in D_k} < x,\pi > \pi$$

By linear algebra we have $P = WE$, where W is the inverse on $span(D_k)$ of the restriction of E, and this gives the result.

Weingarten 2/4

Theorem. For an easy quantum group $G_N \subset U_N^+$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_{G_N} u_{i_1 j_1}^{s_1} \ldots u_{i_k j_k}^{s_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. The vectors associated to partitions are given by:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Thus the Gram matrix and Kronecker symbols are those above.

Weingarten 3/4

The above results apply to U, K. Regarding now T, we have:

Theorem. Given a discrete group $\Gamma = < g_1, \ldots, g_N >$, the integrals over the corresponding torus $T = \widehat{\Gamma}$ are given by

$$\int_T g_{i_1}^{s_1} \ldots g_{i_k}^{s_k} = \delta_{g_{i_1}^{s_1} \ldots g_{i_k}^{s_k}, 1}$$

with the Kronecker symbol being computed inside the group Γ.

Proof. This is clear, coming from the fact that $\int_T g = \delta_{g1}$ defines the Haar functional of the algebra $C(T) = C^*(\Gamma)$.

Weingarten 4/4

Finally, regarding the sphere S, we have here:

Theorem. The integration over a noncommutative sphere S, coming from a category of pairings D, is given by

$$\int_S x_{i_1}^{s_1}\ldots x_{i_k}^{s_k}=\sum_{\pi\leq\ker i}\sum_\sigma W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, where $W_{kN}=G_{kN}^{-1}$, with $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. This follows from the Weingarten formula for U, via the identification $x_i=u_{i1}$ for the coordinates of S.

Free probability 1/4

In order to process the results, we need free probability. Following Voiculescu (80s), the theory goes as follows:

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

$\Longrightarrow$ We have here models for classical and free convolution.

Free probability 2/4

Theorem (CLT). Assuming that $f_1, f_2, f_3, \ldots \in L^\infty(X)$ are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i \sim \mathcal{N}(0, t)$$

where $\mathcal{N}(0, t)$ is the Gaussian law, with density $\frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy$.

Theorem (FCLT). Assuming that $x_1, x_2, x_3, \ldots \in A$ are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where γ_t is the Wigner law, with density $\frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$.

Free probability 3/4

Theorem (PLT). We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{*n}\to p_t$$

with p_t being the Poisson law of parameter $t>0$.

Theorem (FPLT). We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{\boxplus n}\to \pi_t$$

with π_t being the Marchenko-Pastur law of parameter $t>0$,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t>0$.

Free probability 4/4

Definition. Associated to any compactly supported positive measure ρ on $\mathbb{R}$, with mass $c = mass(\rho)$, are the probability measures

$$p_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{*n}$$

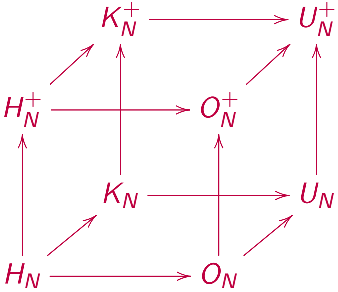
$$\pi_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0+\frac{1}{n}\rho\right)^{\boxplus n}$$

called compound Poisson and compound free Poisson laws.

$\Longrightarrow$ With $\rho = t\varepsilon_s$, we get the Bessel and free Bessel laws.

Laws of characters 1/4

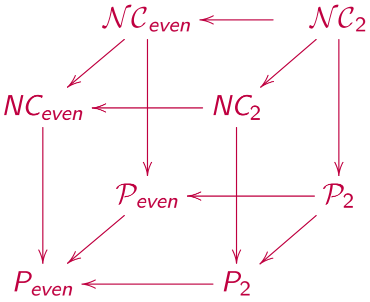
Consider the main unitary and reflection quantum groups:



(That is, real/complex, classical/free, unitary/reflection.)

Laws of characters 2/4

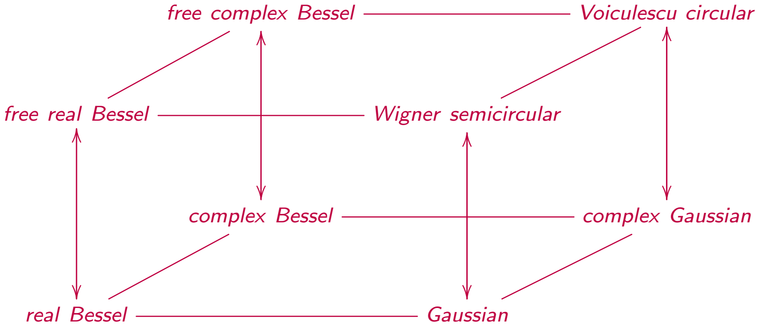
The corresponding categories of partitions are as follows,



with the calligraphic letters standing for "matching".

Laws of characters 3/4

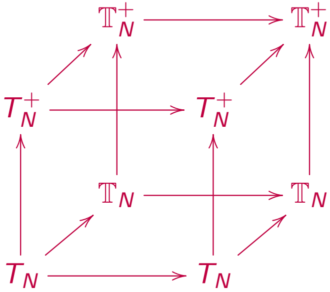
The asymptotic laws of truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ are



with the vertical arrows standing for the Bercovici-Pata bijection.

Laws of characters 4/4

Consider now the corresponding tori, which are as follows:



Here $\chi_t = \sum_{i=1}^{[tN]} g_i$ are subject to Meixner/free Meixner.

$\Longrightarrow$ We are led into the unification question for Bercovici-Pata and Meixner/free Meixner, a well-known problem.

Hyperspherical laws 1/4

Some fixed N computations, for $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$.

Theorem. The classical real spherical integral of $x_{i_1} \ldots x_{i_k}$ vanishes, unless each number $a \in \{1, \ldots, N\}$ appears an even number of times in the sequence of indices $i_1, \ldots, i_k$. We have

$$\int_{S^{N-1}_{\mathbb{R}}} x_{i_1} \ldots x_{i_k} \, dx = \frac{(N-1)!! l_1!! \ldots l_N!!}{(N + \sum l_i - 1)!!}$$

where $m!! = (m-1)(m-1)(m-5)\ldots$, and l_a is the number of occurrences of $a \in \{1, \ldots, N\}$ inside $i_1, \ldots, i_k$.

Proof. Rotation for vanishing, then polar coordinates, calculus.

Hyperspherical laws 2/4

Theorem. The half-liberated real spherical integral of $x_{i_1}\ldots x_{i_k}$ vanishes, unless each $a\in\{1,\ldots,N\}$ appears the same number of times at odd and even positions in $i_1,\ldots,i_k$. We have

$$\int_{S^{N-1}_{\mathbb{R},*}} x_{i_1}\ldots x_{i_k}\,dx=4^{\sum l_i}\frac{(2N-1)!l_1!\ldots l_N!}{(2N+\sum l_i-1)!}$$

where l_a denotes this number of common occurrences.

Proof. Follows from the previous result, by using the model

$$C(S^{N-1}_{\mathbb{R},*})\subset M_2(C(S^{N-1}_{\mathbb{C}}))$$

given by self-adjoint antidiagonal matrices.

Hyperspherical laws 3/4

The above results can be used for computing the hyperspherical laws at fixed $N \in \mathbb{N}$, in the classical and half-classical cases.

In the free case the situation is considerably more complicated, and the Weingarten formula is the only tool. We have:

Theorem. The moments of the free hyperspherical law are given by

$$\int_{S^{N-1}_{\mathbb{R},+}} x_1^{2l}\,dx = \frac{1}{(N+1)^l} \cdot \frac{q+1}{q-1} \cdot \frac{1}{l+1} \sum_{r=-l-1}^{l+1} (-1)^r \binom{2l+2}{l+r+1} \frac{r}{1+q^r}$$

where $q \in [-1,0)$ is such that $q + q^{-1} = -N$.

Hyperspherical laws 4/4

Proof. The idea is that a free spherical coordinate

$$x_1 \in C(S^{N-1}_{\mathbb{R},+})$$

has the same law as the free orthogonal coordinate

$$u_{11} \in C(O_N^+)$$

which has the same law as a certain twisted variable

$$w \in C(SU_2^q)$$

which can be in turn modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

Rotations and permutations 1/4

Theorem. The fusion rules for O_N^+ are the same as for SU_2,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

with $\dim(r_k) = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}}$, where $q^2 - Nq + 1 = 0$.

Proof. We know from easiness that we have:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC_2(k, l)\right)$$

Thus, the main character χ is semicircular:

$$\int_{S_N^+} \chi^{2p} = |NC_2(0, 2p)| = \frac{1}{p+1}\binom{2p}{p}$$

But this gives the result, using $S^3_{\mathbb{R}} \simeq SU_2$.

Rotations and permutations 2/4

Theorem. The fusion rules for S_N^+ are the same as for SO_3,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$$

with $\dim(r_k) = \frac{q^{k+1} - q^{-k}}{q-1}$, where $q^2 - (N-2)q + 1 = 0$.

Proof. We know from easiness that we have:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC(k,l)\right)$$

Thus, the main character χ is squared-semicircular:

$$\int_{S_N^+} \chi^p = |NC(0,p)| = \frac{1}{p+1}\binom{2p}{p}$$

But this gives the result, using $S^3_{\mathbb{R}} \simeq SU_2 \to SO_3$.

Rotations and permutations 3/4

Theorem. PO_n^+ is a cocycle twist of $S_{n^2}^+$, for any $n\in\mathbb{N}$.

Theorem. Let $n\geq 2$ and $w=e^{2\pi i/n}$. Then

$$\Theta(u_{ij}u_{kl})=\frac{1}{n}\sum_{a,b=0}^{n-1}w^{-a(k-i)+b(l-j)}p_{ia,jb}$$

is a trace-preserving coalgebra isomorphism $C(PO_n^+)\to C(S_{n^2}^+)$.

Theorem. The following algebras are isomorphic, via $u_{ij}^2\to X_{ij}$:

(1) The algebra generated by the variables $u_{ij}^2\in C(O_n^+)$.

(2) The algebra generated by $X_{ij}=\frac{1}{n}\sum_{a,b=1}^n p_{ia,jb}\in C(S_{n^2}^+)$.

Rotations and permutations 4/4

Definition. The noncommutative random variable

$$X(n,m,N)=\sum_{i=1}^n\sum_{j=1}^m u_{ij}\in C(S_N^+)$$

is called free hypergeometric, of parameters (n,m,N).

Theorem. The free hypergeometric variable

$$X_{ij}=\frac{1}{n}\sum_{a,b=1}^n u_{ia,jb}\in C(S_{n^2}^+)$$

has the same law as the variable $x_i^2\in C(S_{\mathbb{R},+}^{N-1})$.

Laplacians 1/2

The eigenspaces of the Laplacian for the free sphere $S^{N-1}_{\mathbb{R},+}$ can be constructed as in the classical case, by considering the spaces

$$H_k = span\left(x_{i_1}\ldots x_{i_r}\Big| i_1,\ldots,i_r \in \{1,\ldots,N\}, r \leq k\right)$$

and then by setting, for any $k \in \mathbb{N}$:

$$E_k = H_k \cap H_{k-1}^{\perp}$$

We obtain in this way the Laplacian filtration for $S^{N-1}_{\mathbb{R},+}$:

$$H = \bigoplus_{k=0}^{\infty} E_k$$

The "metric" QISO group, with respect to this filtration, is O_N^+.

Laplacians 2/2

There are many open questions regarding the free spheres, and other "easy spheres", and "easy manifolds" in general:

(1) Eigenvalues (Franz et al.).

(2) Dirac operator (probably no).

(3) Nash embedding questions.

(4) Nash-Connes Geometry.