Introduction to noncommutative geometry

Teo Banica

Spheres and tori, Isometries and reflections, Noncommutative algebraic geometry, Basic geometries, Integration theory, Easy manifolds

07/20

Foreword

This is an introduction to noncommutative geometry, from an operator algebra and quantum group viewpoint.

We discuss the basics, axiomatization and classification, then we study our manifolds using algebraic and analytic methods.

These lecture notes consist of slides written in the Summer 2020. Presentations available at my Youtube channel.

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Noncommutative spheres and tori

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"Introduction to noncommutative geometry", 1/6

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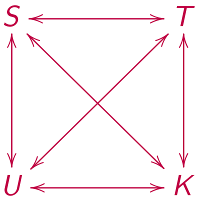
Plan

$\implies$ No free $\mathbb{R}^N$, or free $\mathbb{C}^N$

Step 1. Axiomatize and classify the quadruplets



Step 2. Develop the geometries that you found.

Step 3. Integration theory, Riemannian aspects.

Step 4. Work more, reach to "Nash-Connes Geometry".

Real geometry 1/2

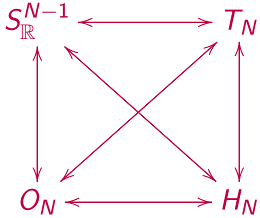
Definition. The real sphere, torus, unitary group and reflection group are:

$$\begin{aligned}
S^{N-1}_{\mathbb{R}} &= \left\{x\in\mathbb{R}^N\Big|\sum_i x_i^2=1\right\}\\
T_N &= \left\{x\in\mathbb{R}^N\Big|x_i=\pm\frac{1}{\sqrt{N}}\right\}\\
O_N &= \left\{U\in M_N(\mathbb{R})\Big|U^t=U^{-1}\right\}\\
H_N &= \left\{U\in M_N(\pm1)\Big|U^t=U^{-1}\right\}
\end{aligned}$$

These are the usual sphere, cube, orthogonal group, and hyperoctahedral group.

Real geometry 2/2

Theorem. We have a full set of correspondences, as follows,



obtained via various results from basic geometry and group theory.

Proof. This is standard, using $T_N = \mathbb{Z}_2^N$ and $H_N = \mathbb{Z}_2 \wr S_N$.

Complex geometry 1/2

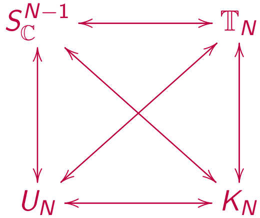
Definition. The complex sphere, torus, unitary group and reflection group are:

$$\begin{aligned} S^{N-1}_{\mathbb{C}} &= \left\{ x \in \mathbb{C}^N \Big| \sum_i |x_i|^2 = 1 \right\} \\ \mathbb{T}_N &= \left\{ x \in \mathbb{C}^N \Big| \, |x_i| = \frac{1}{\sqrt{N}} \right\} \\ U_N &= \left\{ U \in M_N(\mathbb{C}) \Big| U^* = U^{-1} \right\} \\ K_N &= \left\{ U \in M_N(\mathbb{T}) \Big| U^* = U^{-1} \right\} \end{aligned}$$

These are the usual complex sphere, torus, unitary group, and complex reflection group.

Complex geometry 2/2

Theorem. We have a full set of correspondences, as follows,

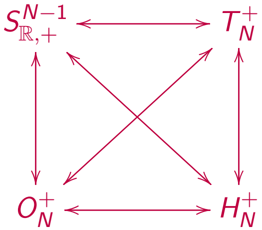


obtained via various results from basic geometry and group theory.

Proof. This is standard, using $\mathbb{T}_N = \mathbb{T}^N$ and $K_N = \mathbb{T} \wr S_N$.

Free real geometry

We will construct a diagram as follows:



– $S^{N-1}_{\mathbb{R},+}$ is defined via $x_i = x_i^*$, $\sum_i x_i^2 = 1$.

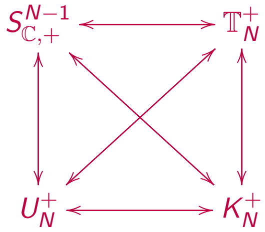
– O_N^+ is defined via $u_{ij} = u_{ij}^*$, $u^t = u^{-1}$.

– $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$ and $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$.

$\implies$ some of the correspondences are quite tricky.

Free complex geometry

We will construct a diagram as follows:



– $S^{N-1}_{\mathbb{C},+}$ is defined via $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$.

– U_N^+ is defined via $u^* = u^{-1}, u^t = \bar{u}^{-1}$.

– $\mathbb{T}_N^+ = \widehat{F_N}$ and $K_N^+ = \mathbb{T} \wr_* S_N^+$.

$\implies$ some of the correspondences are quite tricky.

Operator algebras 1/4

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra with unit $B(H)$, which:

(1) is complete with respect to $||.||$ (Banach algebra).

(2) has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Proof. All this is standard, and well-known for the matrices.

Operator algebras 2/4

Definition. A C^*-algebra is a complex algebra with unit A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

Theorem. (Gelfand) Any commutative C^*-algebra is the form $C(X)$, with X being a compact space.

Proof. Let $X = Spec(A)$ be the space of characters $\chi : A \to \mathbb{C}$. By basic spectral theory $ev : A \to C(X)$ is an isomorphism.

Operator algebras 3/4

Theorem. (GNS) Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. In the commutative case, $A = C(X)$, we have:

$$A \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

In general the proof is similar, by using basic spectral theory.

Operator algebras 4/4

Theorem. The finite dimensional C^*-algebras are of the form:

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

That is, they are the finite sums of matrix algebras.

Proof. This is elementary, in 5 steps, as follows:

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Spheres 1/4

Definition. Given an arbitrary C^*-algebra A, we write

$$A = C(X)$$

and call X a "noncommutative compact space".

Equivalently, the category of the noncommutative compact spaces is the category of the C^*-algebras, with the arrows reversed.

Example 1. Given a morphism $\Phi : A \to B$, we write $A = C(X)$, $B = C(Y)$, and speak of the morphism $\phi : Y \to X$.

Example 2. Given a tensor product $A = B \otimes C$, we let $A = C(X)$, $B = C(Y)$, $C = C(Z)$, and speak of $X = Y \times Z$.

Spheres 2/4

Definition. We have noncommutative spaces, as follows,

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_i x_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^*=\sum_i x_i^*x_i=1\right)$$

called free real sphere, and free complex sphere.

The above universal algebras are well-defined, because we have

$$\sum_i ||x_i||^2=\sum_i ||x_ix_i^*||\leq||\sum_i x_ix_i^*||=1$$

and so the biggest C^*-norms on our algebras exist indeed.

Spheres 3/4

Definition. Given a noncommutative compact space X, its classical version is the subspace $X_{class} \subset X$ obtained by setting:

$$C(X_{class}) = C(X)/I \quad , \quad I =< [a, b] >$$

In this situation, we say that X appears as a "liberation" of X.

In other words, X_{class} is the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe that X_{class} is indeed classical.

Spheres 4/4

<u>Theorem</u>. We have embeddings of NC spaces, as follows,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array}$$

and the free spheres are liberations of the classical ones.

<u>Proof</u>. We must establish the following isomorphisms:

$$C(S^{N-1}_{\mathbb{R}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \Sigma_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C}}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\Sigma_i x_i x_i^* = \Sigma_i x_i^* x_i = 1\right)$$

But these isomorphisms are both clear, by using Gelfand.

Tori 1/4

Definition. We have noncommutative spaces, as follows,

$$C(T_N^+) = C^*\left(x_1,\ldots,x_N\Big|x_i = x_i^*, x_i^2 = \frac{1}{N}\right)$$

$$C(\mathbb{T}_N^+) = C^*\left(x_1,\ldots,x_N\Big|x_ix_i^* = x_i^*x_i = \frac{1}{N}\right)$$

called free real torus, and free complex torus.

$\implies$ As before, $T_N^+, \mathbb{T}_N^+$ appear as liberations of $T_N, \mathbb{T}_N$.

$\implies$ Also, we have 4 formulae of type $T = S \cap \mathbb{T}_N^+$.

Tori 2/4

Theorem. Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by:

$$g^* = g^{-1} \quad , \quad \forall g \in \Gamma$$

The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is then a C^*-norm, and the corresponding closure of $\mathbb{C}[\Gamma]$ is a C^*-algebra, denoted $C^*(\Gamma)$.

Proof. Let $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

But this is elementary to check, and gives the result.

Tori 3/4

Theorem. When Γ is abelian, we have an isomorphism

$$C^*(\Gamma) \simeq C(G)$$

where $G = \widehat{\Gamma}$ is its dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.

Proof. Gelfand gives $A = C(X)$, with $X = Spec(A)$. But the spectrum $X = Spec(A)$, made of characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, as desired.

Definition. Given a discrete group Γ, the space G given by

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Theorem. The basic tori are all group duals, as follows,

$$\begin{array}{ccccccc}
T_N^+ & \longrightarrow & \mathbb{T}_N^+ & & \widehat{\mathbb{Z}_2^{*N}} & \longrightarrow & \widehat{F_N}\\
\uparrow & & \uparrow & = & \uparrow & & \uparrow\\
T_N & \longrightarrow & \mathbb{T}_N & & \widehat{\mathbb{Z}_2^N} & \longrightarrow & \widehat{\mathbb{Z}^N}
\end{array}$$

where F_N is the free group, and $*$ is a free product.

Proof. The diagram formed by the algebras $C(T)$ is:

$$\begin{array}{ccc}
C^*(\mathbb{Z}_2^{*N}) & \longleftarrow & C^*(\mathbb{Z}^{*N})\\
\downarrow & & \downarrow\\
C^*(\mathbb{Z}_2^N) & \longleftarrow & C^*(\mathbb{Z}^N)
\end{array}$$

But this gives the result, via some standard identifications.

Bugs

Problem. For non-amenable groups Γ, such as the free ones F_N,

$$C^*(\Gamma) \to C^*_{red}(\Gamma)$$

is not an isomorphism. Thus $\widehat{\Gamma}$ is multi-represented as NC space (!)

Remark. In fact this issue appeared even before, when talking about $X = Y \times Z$. Indeed, there are several tensor products.

$\implies$ What's the fix?

Fixes 1/3

Definition. The category of compact measured spaces (X, μ) is the category of the C^*-algebras with faithful traces

$$(A, \varphi)$$

with the arrows reversed. In the case where φ is not faithful, we can still talk about (X, μ), by performing the GNS construction.

$\implies$ Works for group duals, but for other spaces like the spheres this is more complicated, because we have no traces yet.

Fixes 2/3

Definition. The category of compact measured spaces (X, μ) is the category of von Neumann algebras with faithful traces

$$(A, \varphi)$$

with the arrows reversed. As basic examples, we have the algebras $L^\infty(X)$ coming by GNS construction from the algebras $C(X)$.

$\Longrightarrow$ Same fix as before, better looking. Works for group duals, but for spheres and other spaces we have no traces yet.

Fixes 3/3

Definition. The category of real algebraic manifolds $X \subset S^{N-1}_{\mathbb{C},+}$ is the category of the universal C^*-algebras of type

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

with $f_i \in \mathbb{C} < x_1, \ldots, x_N >$ being noncommutative polynomials, with the arrows $X \to Y$ being the $*$-algebra morphisms

$$\mathcal{C}(Y) \to \mathcal{C}(X)$$

between the corresponding $*$-algebras generated by the coordinate functions $x_1, \ldots, x_N$, mapping coordinates to coordinates.

$\Longrightarrow$ This is the good fix, and we will heavily use it.

Quantum isometries and reflections

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"Introduction to noncommutative geometry", 2/6

07/20

Plan

1. There is no free $\mathbb{R}^N$, or free $\mathbb{C}^N$.

2. We want to axiomatize the quadruplets (S, T, U, K).

3. So far we have pairs (S, T), real/complex, classical/free.

$\Longrightarrow$ We will complete with pairs (U, K), and with arrows:

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ U & \longrightarrow & K \end{array}$$

$\Longrightarrow$ Later: missing 8 arrows, main 4 cases + axiomatization.

Quantum groups 1/6

Definition. A Woronowicz algebra is a C^*-algebra A, given with a biunitary $u \in M_N(A)$ whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.

– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.

– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

Notation. Given a Woronowicz algebra A we write

$$A = C(G) = C^*(\Gamma)$$

and call G, Γ compact and discrete quantum groups.

Examples. $A = C(G)$, with $u_{ij}(g) = g_{ij}$, for $G \subset U_N$. Also $A = C^*(\Gamma)$, with $u = diag(g_i)$, for $\Gamma =< g_1, \ldots, g_N >$.

Quantum groups 2/6

Theorem. The comultiplication Δ, counit ε and antipode S satisfy the following conditions,

(1) Coassociativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

(2) Counitality: $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$.

(3) Coinversality: $m(id \otimes S)\Delta = m(S \otimes id)\Delta = \varepsilon(.)1$.

on the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the variables u_{ij}.

Proof. Clear on coordinates, and so on the $*$-algebra $\mathcal{A}$.

Remark. The square of the antipode is the identity, $S^2 = id$.

Quantum groups 3/6

Theorem. Any Woronowicz algebra has a Haar integration,

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

constructed by starting with $\varphi \in A^*$ unital positive, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation v,

$$\left(id \otimes \int_G\right) v = P$$

where P is the projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Quantum groups 4/6

Definition. A corepresentation of a Woronowicz algebra A is a biunitary matrix $v \in M_n(\mathcal{A})$ satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

where $\mathcal{A} \subset A$ is the dense $*$-subalgebra of "smooth elements".

Theorem. The following Peter-Weyl type results hold:

(1) Any corepresentation decomposes as a sum of irreducibles.

(2) The irreducibles appear inside $u^{\otimes k}$, with $k =$ colored integer.

(3) We have $\mathcal{A} = \oplus_{r \in Irr(A)} B(H_r)$, $*$-coalgebra isomorphism, $\perp$.

(4) The characters of irreps form an orthonormal basis of $\mathcal{A}_{central}$.

Quantum groups 5/6

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the following collection $C = (C(k,l))$ of vector spaces:

$$C(k,l) = Hom(u^{\otimes k}, u^{\otimes l})$$

Definition. The Woronowicz algebra associated to a Tannakian category $C = (C(k,l))$ is constructed as follows:

$$A = C^*\left((u_{ij})_{i,j=1\ldots N}\middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall T \in C(k,l)\right)$$

Theorem. These operations produce a bijection $A \leftrightarrow C$, between Woronowicz algebras, and Tannakian categories.

Quantum groups 6/6

Definition. A compact quantum group G is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Examples. The Brauer theorem says that O_N, U_N are easy, coming from $P_2, \mathcal{P}_2$: the pairings, and the matching pairings.

Unitaries 1/2

Theorem. We have quantum groups defined via

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1\ldots N}\Big| u=\bar{u}, u^t=u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1\ldots N}\Big| u^*=u^{-1}, u^t=\bar{u}^{-1}\right) \end{aligned}$$

called free orthogonal, and free unitary quantum groups.

Proof. If u is biunitary/orthogonal, so are the matrices

$$(u^\Delta)_{ij}=\sum_k u_{ik}\otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij}=\delta_{ij} \quad , \quad (u^S)_{ij}=u_{ji}^*$$

and so we can construct Δ,ε,S, by universality.

Unitaries 2/2

Theorem. The basic unitary quantum groups, namely

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

are all easy, coming from the following categories of pairings:

$$\begin{array}{ccc} NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

Proof. This comes from Tannaka (classical case: Brauer).

Permutations 1/2

The coordinates of $S_N \subset O_N$, permutation matrices, are:

$$u_{ij} = \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

A quick study of u suggests the following definition:

<u>Definition</u>. The quantum permutation group S_N^+ is defined via

$$C(S_N^+) = C^*\left((u_{ij}) \middle| u = N \times N \text{ magic}\right)$$

where "magic" = made of projections, sum 1 on rows/columns.

[the verification of the CQG axioms is routine: Wang 98]

Permutations 2/2

<u>Theorem</u>. We have the following results:

(1) The inclusion $S_N \subset S_N^+$ is an isomorphism at $N = 2, 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

(2) At $N = 4$ we have $S_4^+ = SO_3^{-1}$. At $N \geq 4$ we have $S_N^+ \sim SO_3$, same fusion rules. At $N \geq 5$ the dual $\widehat{S_N^+}$ is not amenable.

(3) The quantum groups S_N, S_N^+ are easy, coming from P, NC. The main characters are Poisson/free Poisson, with $N \to \infty$.

<u>Proof</u>. Here (1) is elementary, using $u = diag(v, w)$ at $N = 4$, (2) comes from algebra and Tannaka, and (3) from Tannaka.

Reflections 1/2

Theorem. We have quantum groups defined via

$$\begin{aligned} C(H_N^+) &= C^*\left((u_{ij})_{i,j=1\ldots N}\Big| u_{ij}=u_{ij}^*,\,(u_{ij}^2)=\mathrm{magic}\right)\\ C(K_N^+) &= C^*\left((u_{ij})_{i,j=1\ldots N}\Big| [u_{ij},u_{ij}^*]=0,\,(u_{ij}u_{ij}^*)=\mathrm{magic}\right)\end{aligned}$$

called quantum hyperoctahedral, and quantum reflection groups.

Proof. If u satisfies the above relations, then so do the matrices $u^\Delta, u^\varepsilon, u^S$. Thus we can construct Δ, ε, S by universality.

Remark. We can alternatively set $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$, $K_N^+ = \mathbb{T} \wr_* S_N^+$, in analogy with $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$.

Reflections 2/2

Theorem. The basic quantum reflection groups, namely

$$\begin{array}{ccc} H_N^+ & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{array}$$

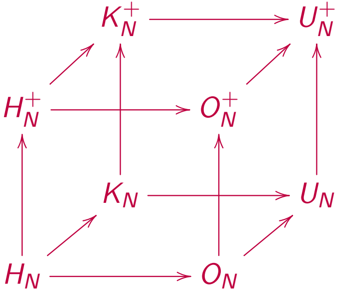
are all easy, coming from the following categories of partitions:

$$\begin{array}{ccc} NC_{even} & \longleftarrow & \mathcal{NC}_{even} \\ \downarrow & & \downarrow \\ P_{even} & \longleftarrow & \mathcal{P}_{even} \end{array}$$

Proof. This comes from Tannaka, using the results for U_N, U_N^+.

The cube

Theorem. The basic quantum unitary and reflection groups are all easy, and form a cubic diagram, as follows:



The upper objects appear as liberations of the lower objects. Also, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

Conclusion

We have basic quadruplets (S, T, U, K) as follows,

$$\begin{matrix}
S^{N-1}_{\mathbb{R},+} & - & T_N^+ \\
| & & | \\
O_N^+ & - & H_N^+
\end{matrix}
\qquad\qquad
\begin{matrix}
S^{N-1}_{\mathbb{C},+} & - & \mathbb{T}_N^+ \\
| & & | \\
U_N^+ & - & K_N^+
\end{matrix}$$

called free real and free complex, as well as

$$\begin{matrix}
S^{N-1}_{\mathbb{R}} & - & T_N \\
| & & | \\
O_N & - & H_N
\end{matrix}
\qquad\qquad
\begin{matrix}
S^{N-1}_{\mathbb{C}} & - & \mathbb{T}_N \\
| & & | \\
U_N & - & K_N
\end{matrix}$$

called classical real and complex $\implies$ construct arrows.

Plan

In the 4 main cases, real/complex and classical/free, we have arrows as follows, given by $T = S \cap \mathbb{T}_N^+$ and $K = U \cap K_N^+$:

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ U & \longrightarrow & K \end{array}$$

We will complete with the dotted arrows, $S \to U$ and $T \to K$, obtained via QISO constructions. Other arrows for later.

Affine isometries

Theorem. Given an algebraic manifold $X \subset S^{N-1}_\mathbb{C}$, the formula

$$G(X) = \left\{ U \in U_N \Big| U(X) = X \right\}$$

defines a compact group of unitary matrices (or isometries), called affine isometry group of X. As basic examples here:

(1) For $S^{N-1}_\mathbb{R}, S^{N-1}_\mathbb{C}$ we obtain the groups O_N, U_N.

(2) For $T_N, \mathbb{T}_N$ we obtain the groups H_N, K_N.

Proof. All this is clear from definitions.

Quantum isometries

Theorem. Given an algebraic manifold $X\subset S^{N-1}_{\mathbb{C},+}$, the category of closed subgroups $G\subset U_N^+$ acting affinely on X, in the sense that

$$\Phi(x_i)=\sum_a u_{ia}\otimes x_a$$

defines a morphism of C^*-algebras

$$\Phi:C(X)\to C(G)\otimes C(X)$$

has a universal object $G^+(X)$, called affine QISO group of X.

Proof. In order for Φ to exist, the variables $z_i=\sum_a u_{ia}\otimes x_a$ must satisfy the polynomial relations defining X. Thus, we can construct $G^+(X)$ by starting with U_N^+, and imposing these relations.

Quantum rotations

Theorem. We have the following QISO computations,

$$\begin{matrix} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{matrix} \quad \to \quad \begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. This is clear in the free cases. In the classical cases the trick of Bhowmick-Goswami ("apply S, relabel, process") applies.

Schur-Weyl twists

Definition. The Schur-Weyl twist of an easy quantum group

$$H_N \subset G \subset U_N^+$$

with corresponding category of partitions $\mathcal{NC}_2 \subset D \subset P_{even}$ is the quantum group $H_N \subset \bar{G} \subset U_N^+$ given by the formula

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\bar{T}_\pi \Big| \pi \in D(k,l)\right)$$

where the twisted implementation of the partitions is given by

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \bar{\delta}_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $\bar{\delta}_\pi \in \{-1, 0, 1\}$ is $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with $\tau = \ker\binom{i}{j}$, and $\varepsilon : P_{even} \to \{\pm 1\}$ being the signature.

Isometries of tori

Theorem. We have the following QISO computations,

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix} \qquad \to \qquad \begin{matrix} H_N^+ & \longrightarrow & K_N^+ \\ \vdots & & \vdots \\ \bar{O}_N & \longrightarrow & \bar{U}_N \end{matrix}$$

where $\bar{O}_N, \bar{U}_N$ are the Schur-Weyl twists of O_N, U_N.

Proof. This is similar to the computation for the spheres, with the BG trick ("apply S, relabel, process") applying as well.

Reflections of tori

Theorem. We have correspondences as follows,

$$\begin{matrix} T_N^+ & \to & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \to & \mathbb{T}_N \end{matrix} \quad \to \quad \begin{matrix} H_N^+ & \to & K_N^+ \\ \uparrow & & \uparrow \\ H_N & \to & K_N \end{matrix}$$

obtained via the operation $T \to G^+(T) \cap K_N^+$.

Proof. Follows from the previous result, by intersecting with K_N^+, because "commutation + anticommutation $\implies$ vanishing".

Conclusion

We have quadruplets (S, T, U, K) as follows,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R}} & \longrightarrow & T_N \\ \downarrow & & \downarrow \\ O_N & \longrightarrow & H_N \end{array} \qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C}} & \longrightarrow & \mathbb{T}_N \\ \downarrow & & \downarrow \\ U_N & \longrightarrow & K_N \end{array}$$

coming from $\mathbb{R}^N$, $\mathbb{C}^N$, as well as free analogues of them,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & T^+_N \\ \downarrow & & \downarrow \\ O^+_N & \longrightarrow & H^+_N \end{array} \qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C},+} & \longrightarrow & \mathbb{T}^+_N \\ \downarrow & & \downarrow \\ U^+_N & \longrightarrow & K^+_N \end{array}$$

which can be thought of as coming from $\mathbb{R}^N_+$, $\mathbb{C}^N_+$.

$\Longrightarrow$ Construct the missing arrows, axiomatize.

Noncommutative algebraic geometry

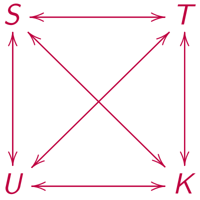
Teo Banica

"Introduction to noncommutative geometry", 3/6

07/20

Plan

We want to axiomatize the quadruplets (S, T, U, K) consisting of a noncommutative sphere, torus, unitary group and reflection group, with a full set of correspondences between them, as follows:



We have 4 main examples in mind, namely classical and free, real and complex. We will finish the work there, by constructing the missing correspondences, then we will do the axiomatization.

$\Longrightarrow$ More examples, classification, development: later.

Details

We call noncommutative sphere and torus, and quantum unitary and reflection group, the intermediate objects as follows,

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with S being an algebraic manifold, and T, U, K being compact quantum groups. Note that T must be a group dual.

$\implies$ We must axiomatize the correspondences between them.

Comments

(1) It is a good idea to mix the classical and free cases?

Yes, because there are interesting geometries between classical and free, such as the half-classical one ($abc = cba$).

(2) Is it a good idea to mix the real and complex cases?

Yes, for instance because we have $P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$. That is, "the free projective geometry is scalarless".

$\implies$ More details and comments later, on all this.

Diagonal tori 1/4

We have quadruplets (S,T,U,K) and arrows as follows,

$$\begin{array}{ccc} S^{N-1}_\mathbb{R} & \longrightarrow & T_N \\ \downarrow & & \downarrow \\ O_N & \longrightarrow & H_N \end{array} \qquad\qquad \begin{array}{ccc} S^{N-1}_\mathbb{C} & \longrightarrow & \mathbb{T}_N \\ \downarrow & & \downarrow \\ U_N & \longrightarrow & K_N \end{array}$$

coming from $\mathbb{R}^N$, $\mathbb{C}^N$, as well as free analogues of them,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & T_N^+ \\ \downarrow & & \downarrow \\ O_N^+ & \longrightarrow & H_N^+ \end{array} \qquad\qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C},+} & \longrightarrow & \mathbb{T}_N^+ \\ \downarrow & & \downarrow \\ U_N^+ & \longrightarrow & K_N^+ \end{array}$$

which can be thought of as coming from $\mathbb{R}^N_+$, $\mathbb{C}^N_+$.

Diagonal tori 2/4

Theorem. Given a closed subgroup $G \subset U_N^+$, its diagonal torus is the closed subgroup $T \subset G$ constructed as follows:

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

We have then $T = \widehat{\Lambda}$, where $\Lambda =< g_1, \ldots, g_N >$ is the discrete group generated by $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since u is unitary, $g_i = u_{ii}$ are unitaries inside $C(T)$. From $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus we have $C(T) = C^*(\Lambda)$, and so $T = \widehat{\Lambda}$, as claimed.

Diagonal tori 3/4

Theorem. The diagonal tori of the unitary quantum groups are

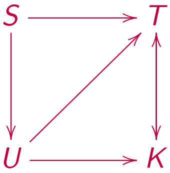
$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \quad \to \quad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

and for the reflection subgroups, we obtain the same tori.

Proof. The diagonal torus is given by $T = U \cap \mathbb{T}_N^+$, intersection computed inside U_N^+, and this gives all the results.

Diagonal tori 4/4

Theorem. We have correspondences as follows, for the 4 basic examples of quadruplets (S,T,U,K),



which are given by the following formulae:

– $U=G^+(S)$

– $K=U\cap K_N^+=G^+(T)\cap K_N^+$

– $T=S\cap\mathbb{T}_N^+=U\cap\mathbb{T}_N^+=K\cap\mathbb{T}_N^+$

Proof. This is a summary of what we have so far.

Liberation 1/4

Definition. Given $G\subset U_N^+$, let $T\subset K\subset G$ be its diagonal torus, and its reflection subgroup. The inclusion $G_{class}\subset G$ is called:

(1) A soft liberation, when $G=<G_{class},K>$.

(2) A hard liberation, when $G=<G_{class},T>$.

Remark. We have the following intersection diagram:

$$\begin{array}{ccccc}
T & \rightarrow & K & \rightarrow & G\\
\uparrow & & \uparrow & & \uparrow\\
T_{class} & \rightarrow & K_{class} & \rightarrow & G_{class}
\end{array}$$

Soft liberation means generation for the square on the right.
Hard liberation means generation for the whole rectangle.

Liberation 2/4

Theorem. The following happen:

(1) O_N^+, U_N^+ appear as soft liberations of O_N, U_N.

(2) O_N^+, U_N^+ appear as well as hard liberations of O_N, U_N.

(3) H_N^+, K_N^+ appear as soft liberations of H_N, K_N.

(4) H_N^+, K_N^+ do not appear as hard liberations of H_N, K_N.

Proof. (1) follows from (2), and (2) follows by recurrence from

$$O_N^+ =< O_N, O_{N-1}^+ >$$

which itself follows by recurrence (Chirvasitu). (3) is trivial, and (4) follows from the fact that "hard liberation stops at $H_N^{[\infty]}, K_N^{[\infty]}$".

Liberation 3/4

Theorem. We have the following formulae:

(1) $O_N =< O_N, T_N >$.

(2) $U_N =< O_N, \mathbb{T}_N >$.

(3) $O_N^+ =< O_N, T_N^+ >$.

(4) $U_N^+ =< O_N, \mathbb{T}_N^+ >$.

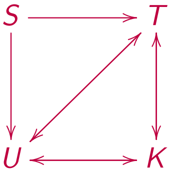
Proof. (1) is trivial. (2) follows from $\mathbb{T}O_N \subset U_N$ maximal. (3) is the hard liberation property of O_N, discussed above, and which is highly non-trivial. As for (4), this follows from (3).

Liberation 4/4

Theorem. We have correspondences as follows, obtained by adding the "soft" and "hard" generation formulae

$$U = < O_N, K > = < O_N, T >$$

to the various isometry and intersection formulae that we have,



which work for the basic quadruplets (S, T, U, K).

Spheres 1/4

Regarding $U \to S$, in the classical case the situation is very simple, because S appears by rotating $x = (1, 0, \ldots, 0)$ by the elements of U. In fact, the spheres are homogeneous spaces, as follows:

$$S^{N-1} = U_N / U_{N-1}$$

In functional analytic terms, the correspondence $U \to S$ appears, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \to u_{i1}$$

Thus, we must check if this works in the free cases too.

Spheres 2/4

Theorem. For the basic spheres, we have a diagram as follows,

$$\begin{array}{ccc} C(S) & \xrightarrow{\Phi} & C(U)\otimes C(S) \\ \downarrow^{\pi} & & \downarrow^{id\otimes\pi} \\ C(U) & \xrightarrow{\Delta} & C(U)\otimes C(U) \end{array}$$

where Φ is the affine coaction map, and where $\pi(x_i) = u_{i1}$.

Proof. This diagram commutes indeed on the standard generators.

Spheres 3/4

Theorem. We have a quotient map and an inclusion as follows,

$$U \to S_U \subset S$$

with S_U being the first column space of U, given by

$$C(S_U) =< u_{i1} >\subset C(U)$$

at the level of the corresponding algebras of functions.

Proof. We have an inclusion and a quotient map as follows:

$$C(S) \to C(S_U) \subset C(U)$$

Thus, we obtain the result, by transposing.

$\implies$ We must prove that $S_U \subset S$ is an isomorphism.

Spheres 4/4

In order to investigate the faithfulness of $S_U \subset S$, we will use the faithfulness properties of the integration over S.

Definition. We endow $C(S)$ with its integration functional

$$\int_S : C(S) \to C(U) \to \mathbb{C}$$

obtained by composing $x_i \to u_{i1}$ with the Haar integral of U.

In the real and complex classical cases, we obtain the integration with respect to the uniform measure on $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$.

Weingarten 1/4

Theorem. The integration over S has the ergodicity property

$$\left(\int_U \otimes id\right)\Phi(x) = \int_S x$$

where $\Phi : C(S) \to C(U) \otimes C(S)$ is the coaction map.

Proof. This is something non-trivial, coming from the knowledge of the integration over U, via the Weingarten formula:

$$\int_U u_{i_1 j_1}^{k_1} \ldots u_{i_p j_p}^{k_p} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

Here $\delta \in \{0,1\}$ are Kronecker type symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of the Gram matrix $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Weingarten 2/4

<u>Theorem</u>. There is a unique trace $tr : C(S) \to \mathbb{C}$ satisfying

$$(id \otimes tr)\Phi(x) = tr(x)1$$

and this is the canonical integration, constructed above.

<u>Proof</u>. Let tr be as in the statement. We have:

$$tr\left(\int_U \otimes\, id\right)\Phi(x) = \int_U (id \otimes tr)\Phi(x) = tr(x)$$

On the other hand, by ergodicity we have as well:

$$tr\left(\int_U \otimes\, id\right)\Phi(x) = tr\left(\int_S x\right) = \int_S x$$

Thus tr equals the standard integration, as claimed.

Weingarten 3/4

<u>Theorem</u>. The construction $U \to S_U$ makes correspond:

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \qquad \to \qquad \begin{matrix} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{matrix}$$

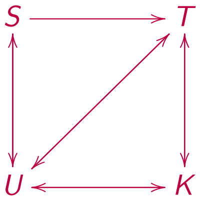
<u>Proof</u>. We use the ergodicity formula established above:

$$\left(\int_U \otimes id\right)\Phi = \int_S$$

Since $\int_U$ is faithful on $C(U)$ and $(\varepsilon \otimes id)\Phi = id$, the coaction map Φ is faithful as well. Thus $\int_S xx^* = 0$ with $x \in C(S)$ implies $x = 0$, and so $\int_S$ is faithful on $C(S)$. Thus we have $S = S_U$.

Weingarten 4/4

Theorem. We have arrows as follows, obtained by adding the first column space construction $U \to S$ to what we already have,



which work for the basic quadruplets (S, T, U, K).

$\implies$ And we'll stop here, $T \to S$ and $S \leftrightarrow K$ being non-trivial.

Axiomatization 1/2

We call noncommutative sphere and torus, and quantum unitary and reflection group, the intermediate objects as follows,

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with S being an algebraic manifold, and T, U, K being compact quantum groups. Note that T must be a group dual.

Axiomatization 2/2

A quadruplet (S,T,U,K) produces a noncommutative geometry when one can pass from each object to all the other ones,

$$\begin{array}{ccccccc} S & = & S_{<O_N,T>} & = & S_U & = & S_{<O_N,K>} \\ S\cap\mathbb{T}_N^+ & = & T & = & U\cap\mathbb{T}_N^+ & = & K\cap\mathbb{T}_N^+ \\ G^+(S) & = & <O_N,T> & = & U & = & <O_N,K> \\ G^+(S)\cap K_N^+ & = & G^+(T)\cap K_N^+ & = & U\cap K_N^+ & = & K \end{array}$$

with the usual convention that all this is up to the equivalence relation, namely isomorphism of $*$-algebras of coordinates.

Compact form

A quadruplet (S, T, U, K), between classical real and free complex,

$$(S^{N-1}_{\mathbb{R}}, T_N, O_N, H_N) \ < \ (S, T, U, K) \ < \ (S^{N-1}_{\mathbb{C},+}, \mathbb{T}_N^+, U_N^+, K_N^+)$$

produces a noncommutative geometry when

$$\begin{matrix} S & = & S_U & & \\ S \cap \mathbb{T}_N^+ & = & T & = & K \cap \mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U \\ G^+(T) \cap K_N^+ & = & U \cap K_N^+ & = & K \end{matrix}$$

up to the standard equivalence relation for algebraic manifolds.

Conclusion

(1) We have NCG axioms, and 4 basic examples, as follows:

$$\begin{array}{ccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+\\
\uparrow & & \uparrow\\
\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

(2) More examples, and classification results, coming soon.

(3) Technology used: basics, twists, liberation, Weingarten.

(4) We must develop these NCG: more manifolds, integration.

Basic noncommutative geometries

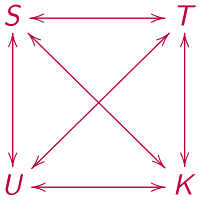
Teo Banica

"Introduction to noncommutative geometry", 4/6

07/20

Idea

There is no free $\mathbb{R}^N$, or free $\mathbb{C}^N$. We have quadruplets (S, T, U, K) consisting of a sphere, torus, unitary group and reflection group:



Such quadruplets can be axiomatized. There are 4 main examples of geometries in this sense, namely those of $\mathbb{R}^N$, $\mathbb{C}^N$, $\mathbb{R}^N_+$, $\mathbb{C}^N_+$.

Axioms

A quadruplet (S,T,U,K), between classical real and free complex,

$$\mathbb{R}^N \ < \ (S,T,U,K) \ < \ \mathbb{C}^N_+$$

produces a noncommutative geometry when

$$\begin{array}{ccccc} S & = & S_U & & \\ S\cap\mathbb{T}_N^+ & = & T & = & K\cap\mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U \\ G^+(T)\cap K_N^+ & = & U\cap K_N^+ & = & K \end{array}$$

up to the standard equivalence relation for algebraic manifolds.

Plan

We will complete the basic 4-diagram into a 9-diagram:

$$\begin{array}{ccccc}
\mathbb{R}_+^N & \longrightarrow & \mathbb{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}_*^N & \longrightarrow & \mathbb{TR}_*^N & \longrightarrow & \mathbb{C}_*^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{TR}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

Then we will discuss classification results, and extensions.

Half-liberation 1/4

Question. Is there a "standard" geometry $\mathbb{R}^N \subset \mathbb{R}^N_* \subset \mathbb{R}^N_+$?

Theorem. The algebraic manifold $S^{(k)} \subset S^{N-1}_{\mathbb{R},+}$ obtained via the relations $a_1 \ldots a_k = a_k \ldots a_1$ is as follows:
(1) At $k = 1$ we have $S^{(k)} = S^{N-1}_{\mathbb{R},+}$.
(2) At $k = 2, 4, 6, \ldots$ we have $S^{(k)} = S^{N-1}_{\mathbb{R}}$.
(3) At $k = 3, 5, 7, \ldots$ we have $S^{(k)} = S^{(3)}$.

Definition. We define the half-classical sphere via the formula

$$C(S^{N-1}_{\mathbb{R},*}) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle abc = cba \Big\rangle$$

and call the relations $abc = cba$ half-commutation relations.

Half-liberation 2/4

Definition. We define the real half-classical quadruplet

$$(S^{N-1}_{\mathbb{R},*}, T_N^*, O_N^*, H_N^*)$$

by imposing $abc = cba$ to the coordinates. We define as well

$$(S^{N-1}_{\mathbb{C},*}, \mathbb{T}_N^*, U_N^*, K_N^*)$$

by imposing $abc = cba$ to the coordinates, and their adjoints.

$\Longrightarrow$ To do: find tools for studying these objects, check our NCG axioms for them, establish some further uniqueness results.

Half-liberation 3/4

Theorem. The sphere $S^{N-1}_{\mathbb{R},*}$ has the following properties:

(1) $PS^{N-1}_{\mathbb{R},*}$ is classical, equal to $P^{N-1}_{\mathbb{C}}$.

(2) $S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ appears as the affine lift of $P^{N-1}_{\mathbb{C}}$.

(3) We have a matrix model $C(S^{N-1}_{\mathbb{R},*}) \subset M_2(C(S^{N-1}_{\mathbb{C}}))$.

(4) Similar results hold for the subspaces $X \subset S^{N-1}_{\mathbb{R},*}$.

Proof. (1) Here $\subset$ is clear, because $abc = aba$ implies $[ab, cd] = 0$, and $\supset$ follows by using the model in (3), namely:

$$x_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

(2) and the faithfulness claim in (3) are related, and follow from some algebra. As for (4), this is something more technical.

Half-liberation 4/4

<u>Theorem</u>. We have full results regarding $S^{N-1}_{\mathbb{R},*}, T_N^*, O_N^*, H_N^*$, and complex analogues as well, regarding $S^{N-1}_{\mathbb{C},*}, \mathbb{T}_N^*, U_N^*, K_N^*$.

<u>Theorem</u>. We have noncommutative geometries, as follows:

$$\begin{array}{ccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

<u>Remark</u>. It is possible to prove that O_N^* is the unique intermediate easy quantum group $O_N \subset G \subset O_N^+$. More on this later.

Hybrid geometries 1/4

An intermediate geometry $\mathbb{R}^N \subset X \subset \mathbb{C}^N$ is given by a quadruplet (S, T, U, K), whose components are subject to:

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C}}$$

$$T_N \subset T \subset \mathbb{T}_N$$

$$O_N \subset U \subset U_N$$

$$H_N \subset K \subset K_N$$

There are many solutions here, even under strong axioms, such as easiness. We will discuss here the "standard" solution.

Hybrid geometries 2/4

Theorem. We have an intermediate sphere as follows,

$$S^{N-1}_{\mathbb{R}} \subset \mathbb{T}S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{C}}$$

which appears as the affine lift of $P^{N-1}_{\mathbb{R}}$, inside $S^{N-1}_{\mathbb{C}}$.

Theorem. More generally, we have a quadruplet as follows,

$$(\mathbb{T}S^{N-1}_{\mathbb{R}}, \mathbb{T}T_N, \mathbb{T}O_N, \mathbb{T}H_N)$$

which appears in a similar way, by lifting.

Theorem. This quadruplet satisfies our NCG axioms.

$\implies$ A priori $(\mathbb{Z}_rS^{N-1}_{\mathbb{R}}, \mathbb{Z}_rT_N, \mathbb{Z}_rO_N, \mathbb{Z}_rH_N)$ are solutions too.

Hybrid geometries 3/4

Theorem. We have as well half-classical and free quadruplets,

$$(\mathbb{T}S^{N-1}_{\mathbb{R},*}, \mathbb{T}T_N^*, \mathbb{T}O_N^*, \mathbb{T}H_N^*)$$

$$(\mathbb{T}S^{N-1}_{\mathbb{R},+}, \mathbb{T}T_N^+, \mathbb{T}O_N^+, \mathbb{T}H_N^+)$$

obtained via the relations $ab^* = a^*b$.

Theorem. All the above hybrid quantum groups, namely

$$\mathbb{T}O_N, \mathbb{T}O_N^*, \mathbb{T}O_N^+ \quad , \quad \mathbb{T}H_N, \mathbb{T}H_N^*, \mathbb{T}H_N^+$$

are easy, appearing from the partition implementing $ab^* = a^*b$.

Theorem. The hybrid quadruplets satisfy our NCG axioms.

Hybrid geometries 4/4

Theorem. We have noncommutative geometries as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{T}\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

Proof. This follows by putting together what we have.

Classification 1/4

Definition. A geometry coming from a quadruplet (S, T, U, K) is called easy when both U, K are easy, and

$$U = \{O_N, K\}$$

with the operation on the right being the easy generation operation.

Remark. It is known that if G, H are easy then we have

$$< G, H > \subset < G, H >' \subset \{G, H\}$$

and both these inclusions are conjectured to be isomorphisms.

Classification 2/4

Theorem. An easy geometry is determined by a pair (D, E) of categories of partitions, which must be as follows,

$$\mathcal{NC}_2 \subset D \subset P_2$$

$$\mathcal{NC}_{even} \subset E \subset P_{even}$$

and which are subject to the following conditions,

$$D = E \cap P_2$$

$$E = < D, \mathcal{NC}_{even} >$$

and to the usual axioms for the associated quadruplet (S, T, U, K), where U, K are the easy quantum groups associated to D, E.

Proof. The conditions come from $U = \{O_N, K\}$, $K = U \cap K_N^+$.

Classification 3/4

Remark. In the context of an easy geometry, we have:

$$C(U)=C(U_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall T\in D(k,l)\right\rangle$$

$$C(K)=C(K_N^+)\Big/\left\langle T_\pi\in Hom(u^{\otimes k},u^{\otimes l})\Big|\forall k,l,\forall T\in D(k,l)\right\rangle$$

We have as well the following formula, for the dual of the torus:

$$\Gamma=F_N\Big/\left\langle g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\Big|\exists\pi\in D(k,l),\delta_\pi\begin{pmatrix}i\\j\end{pmatrix}\neq 0\right\rangle$$

As for the sphere, here the situation is a bit more complicated.

Classification 4/4

Theorem. The easy geometries are as follows:

(1) Real case: the 3 geometries that we have are unique.

(2) Classical case: uniqueness again, under an extra axiom.

(3) Other "pure" cases: uniqueness, under an extra axiom.

(4) In general: uniqueness, under an extra “slicing” axiom.

Proof. In terms of the category of pairings $\mathcal{NC}_2 \subset D \subset P_2$, the conditions $D = E \cap P_2$, $E =< D, \mathcal{NC}_{even} >$ reformulate as:

$$D =< D, \mathcal{NC}_{even} > \cap P_2$$

But this equation can be solved by using the known classification results for easy quantum groups, and related techniques.

Monomial spheres 1/2

Reminder. We have seen that the abstract construction

$$C(S^{(k)}) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle a_1 \ldots a_k = a_k \ldots a_1 \Big\rangle$$

produces in practice only 3 spheres, $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$.

Definition. A monomial sphere is a sphere $S \subset S^{N-1}_{\mathbb{C},+}$ obtained via

$$x_{i_1}^{e_1} \ldots x_{i_k}^{e_k} = x_{i_{\sigma(1)}}^{f_1} \ldots x_{i_{\sigma(k)}}^{f_k} \quad , \quad \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

with $\sigma \in S_k$, and with $e_r, f_r \in \{1, *\}$ being exponents.

Monomial spheres 2/2

Theorem. In the real case, the only monomial spheres are:

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

Proof. The idea is that the real monomial spheres are the subsets $S \subset S^{N-1}_{\mathbb{R},+}$ obtained via relations of the form

$$x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

associated to certain elements $\sigma \in G_k$, where $G = (G_k)$ is a filtered subgroup of $S_\infty = (S_k)$. But such groups can be classified.

$\implies$ The complex analogue of this is not known yet.

Projective spaces 1/2

Theorem. The projective spaces of our 9 geometries collapse to

$$\begin{array}{ccccc}
P^{N-1}_+ & \longrightarrow & P^{N-1}_+ & \longrightarrow & P^{N-1}_+\\
\uparrow & & \uparrow & & \uparrow\\
P^{N-1}_{\mathbb C} & \longrightarrow & P^{N-1}_{\mathbb C} & \longrightarrow & P^{N-1}_{\mathbb C}\\
\uparrow & & \uparrow & & \uparrow\\
P^{N-1}_{\mathbb R} & \longrightarrow & P^{N-1}_{\mathbb R} & \longrightarrow & P^{N-1}_{\mathbb C}
\end{array}$$

where P^{N-1}_+ is the free projective space, $P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$.

$\Longrightarrow$ Interesting trichotomy here, "real, complex, free".

Projective spaces 2/2

Definition. A monomial space is a subset $P \subset P^{N-1}_+$ obtained via

$$p_{i_1 i_2} \ldots p_{i_{k-1} i_k} = p_{i_{\sigma(1)} i_{\sigma(2)}} \ldots p_{i_{\sigma(k-1)} i_{\sigma(k)}}, \ \forall i \in \{1, \ldots, N\}^k$$

with σ ranging over a subset of $\bigcup_{k \in 2\mathbb{N}} S_k$, stable under $\sigma \to |\sigma|$.

Theorem. We have only 3 monomial projective spaces, namely:

$$P^{N-1}_{\mathbb{R}} \subset P^{N-1}_{\mathbb{C}} \subset P^{N-1}_+$$

$\implies$ How to axiomatize the quadruplets (P, PT, PU, PK)?

Twisting

By Schur-Weyl twisting we obtain potential geometries as follows,

$$
\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\mathbb{R}}^N_* & \longrightarrow & \mathbb{T}\bar{\mathbb{R}}^N_* & \longrightarrow & \bar{\mathbb{C}}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\mathbb{R}}^N & \longrightarrow & \mathbb{T}\bar{\mathbb{R}}^N & \longrightarrow & \bar{\mathbb{C}}^N
\end{array}
$$

but the axioms must be fine-tuned, e.g. due to QISO problems.

Intersections

An interesting problem is that of intersecting the twisted and untwisted geometries. There are $9 \times 9 = 81$ cases here.

In the real case we only have $3 \times 3 = 9$ cases. The spheres are non-smooth, "polygonal", and the QISO groups are

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
H_N & \longrightarrow & H_N^{[\infty]} & \longrightarrow & \bar{O}_N^* \\
\uparrow & & \uparrow & & \uparrow \\
H_N^+ & \longrightarrow & H_N & \longrightarrow & \bar{O}_N
\end{array}$$

where $H_N^* \subset H_N^{[\infty]} \subset H_N^+$ is the standard higher liberation of H_N.

Other extensions

Besides twisting, and taking intersections, we have:

(1) Super-easiness.

(2) Partition quantum groups.

(3) Other easiness-related theories.

(4) Other types of noncomutative spheres.

Conclusion

We have 9 main examples of geometries, as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{T}\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

The problem now is that of "developing" these geometries.

Noncommutative integration theory

Teo Banica

"Introduction to noncommutative geometry", 5/6

07/20

Plan

We are interested in integrating over S, T, U, K.

(1) Weingarten formula.

(2) Free probability.

(3) Free integration, $N >> 0$.

(4) Free integration, N fixed.

(5) Rotations and permutations.

(6) Riemannian aspects.

Weingarten 1/4

Theorem. Assuming that $A = C(G)$ has Tannakian category $C = (C(k, l))$, the Haar integration over G is given by

$$\int_G u_{i_1 j_1}^{s_1} \ldots u_{i_k j_k}^{s_k} = \sum_{\pi,\sigma \in D_k} \delta_\pi(i)\delta_\sigma(j) W_k(\pi, \sigma)$$

where D_k is a basis of $C(\emptyset, k)$, $\delta_\pi(i) =< \pi, e_{i_1} \otimes \ldots \otimes e_{i_k} >$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\pi, \sigma) =< \pi, \sigma >$.

Proof. The integrals in the statement form the projection P onto $Fix(u^{\otimes k}) = span(D_k)$. Consider the following linear map:

$$E(x) = \sum_{\pi \in D_k} < x, \pi > \pi$$

By linear algebra we have $P = WE$, where W is the inverse on $span(D_k)$ of the restriction of E, and this gives the result.

Weingarten 2/4

Theorem. For an easy quantum group $G_N \subset U_N^+$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_{G_N} u_{i_1 j_1}^{s_1} \ldots u_{i_k j_k}^{s_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Proof. The vectors associated to partitions are given by:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Thus the Gram matrix and Kronecker symbols are those above.

Weingarten 3/4

The above results apply to U, K. Regarding now T, we have:

Theorem. Given a discrete group $\Gamma = < g_1, \ldots, g_N >$, the integrals over the corresponding torus $T = \widehat{\Gamma}$ are given by

$$\int_T g_{i_1}^{s_1} \ldots g_{i_k}^{s_k} = \delta_{g_{i_1}^{s_1} \ldots g_{i_k}^{s_k}, 1}$$

with the Kronecker symbol being computed inside the group Γ.

Proof. This is clear, coming from the fact that $\int_T g = \delta_{g1}$ defines the Haar functional of the algebra $C(T) = C^*(\Gamma)$.

Weingarten 4/4

Finally, regarding the sphere S, we have here:

Theorem. The integration over a noncommutative sphere S, coming from a category of pairings D, is given by

$$\int_S x_{i_1}^{s_1}\ldots x_{i_k}^{s_k}=\sum_{\pi\leq\ker i}\sum_{\sigma}W_{kN}(\pi,\sigma)$$

with $\pi,\sigma\in D(k)$, where $W_{kN}=G_{kN}^{-1}$, with $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. This follows from the Weingarten formula for U, via the identification $x_i=u_{i1}$ for the coordinates of S.

Free probability 1/4

In order to process the results, we need free probability. Following Voiculescu (80s), the theory goes as follows:

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

$\Longrightarrow$ We have here models for classical and free convolution.

Free probability 2/4

Theorem (CLT). Assuming that $f_1, f_2, f_3, \ldots \in L^\infty(X)$ are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i \sim \mathcal{N}(0, t)$$

where $\mathcal{N}(0, t)$ is the Gaussian law, with density $\frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy$.

Theorem (FCLT). Assuming that $x_1, x_2, x_3, \ldots \in A$ are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where γ_t is the Wigner law, with density $\frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$.

Free probability 3/4

Theorem (PLT). We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{*n}\to p_t$$

with p_t being the Poisson law of parameter $t>0$.

Theorem (FPLT). We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{\boxplus n}\to \pi_t$$

with π_t being the Marchenko-Pastur law of parameter $t>0$,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t>0$.

Free probability 4/4

Definition. Associated to any compactly supported positive measure ρ on $\mathbb{R}$, with mass $c = mass(\rho)$, are the probability measures

$$p_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{*n}$$

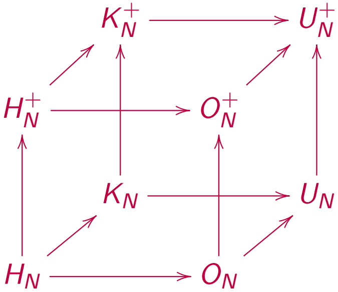
$$\pi_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{\boxplus n}$$

called compound Poisson and compound free Poisson laws.

$\Longrightarrow$ With $\rho = t\varepsilon_s$, we get the Bessel and free Bessel laws.

Laws of characters 1/4

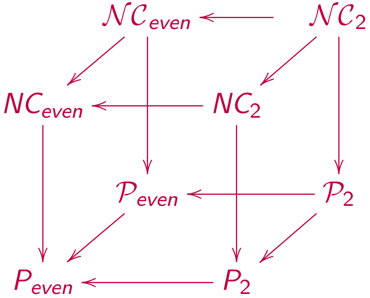
Consider the main unitary and reflection quantum groups:



(That is, real/complex, classical/free, unitary/reflection.)

Laws of characters 2/4

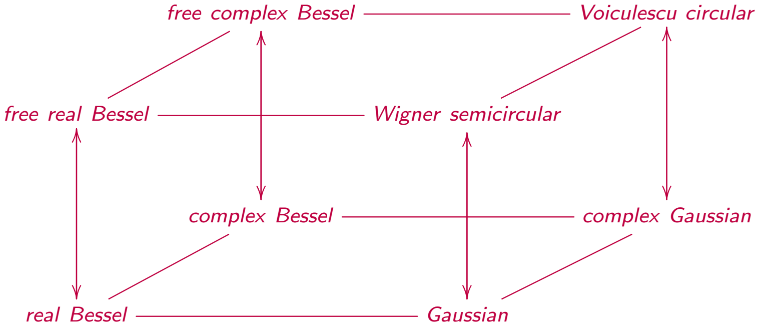
The corresponding categories of partitions are as follows,



with the calligraphic letters standing for "matching".

Laws of characters 3/4

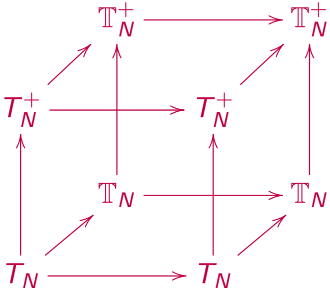
The asymptotic laws of truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ are



with the vertical arrows standing for the Bercovici-Pata bijection.

Laws of characters 4/4

Consider now the corresponding tori, which are as follows:



Here $\chi_t = \sum_{i=1}^{[tN]} g_i$ are subject to Meixner/free Meixner.

$\implies$ We are led into the unification question for Bercovici-Pata and Meixner/free Meixner, a well-known problem.

Hyperspherical laws 1/4

Some fixed N computations, for $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$.

Theorem. The classical real spherical integral of $x_{i_1} \ldots x_{i_k}$ vanishes, unless each number $a \in \{1, \ldots, N\}$ appears an even number of times in the sequence of indices $i_1, \ldots, i_k$. We have

$$\int_{S^{N-1}_{\mathbb{R}}} x_{i_1} \ldots x_{i_k} \, dx = \frac{(N-1)!! l_1!! \ldots l_N!!}{(N + \sum l_i - 1)!!}$$

where $m!! = (m-1)(m-1)(m-5)\ldots$, and l_a is the number of occurrences of $a \in \{1, \ldots, N\}$ inside $i_1, \ldots, i_k$.

Proof. Rotation for vanishing, then polar coordinates, calculus.

Hyperspherical laws 2/4

Theorem. The half-liberated real spherical integral of $x_{i_1}\ldots x_{i_k}$ vanishes, unless each $a\in\{1,\ldots,N\}$ appears the same number of times at odd and even positions in $i_1,\ldots,i_k$. We have

$$\int_{S^{N-1}_{\mathbb{R},*}} x_{i_1}\ldots x_{i_k}\,dx = 4^{\sum l_i}\frac{(2N-1)!l_1!\ldots l_N!}{(2N+\sum l_i-1)!}$$

where l_a denotes this number of common occurrences.

Proof. Follows from the previous result, by using the model

$$C(S^{N-1}_{\mathbb{R},*})\subset M_2(C(S^{N-1}_{\mathbb{C}}))$$

given by self-adjoint antidiagonal matrices.

Hyperspherical laws 3/4

The above results can be used for computing the hyperspherical laws at fixed $N \in \mathbb{N}$, in the classical and half-classical cases.

In the free case the situation is considerably more complicated, and the Weingarten formula is the only tool. We have:

<u>Theorem</u>. The moments of the free hyperspherical law are given by

$$\int_{S^{N-1}_{\mathbb{R},+}} x_1^{2l}\,dx = \frac{1}{(N+1)^l}\cdot\frac{q+1}{q-1}\cdot\frac{1}{l+1}\sum_{r=-l-1}^{l+1}(-1)^r\binom{2l+2}{l+r+1}\frac{r}{1+q^r}$$

where $q \in [-1,0)$ is such that $q+q^{-1}=-N$.

Hyperspherical laws 4/4

Proof. The idea is that a free spherical coordinate

$$x_1 \in C(S^{N-1}_{\mathbb{R},+})$$

has the same law as the free orthogonal coordinate

$$u_{11} \in C(O_N^+)$$

which has the same law as a certain twisted variable

$$w \in C(SU_2^q)$$

which can be in turn modelled by an explicit operator on $l^2(\mathbb{N})$, whose law can be computed by using advanced calculus.

Rotations and permutations 1/4

Theorem. The fusion rules for O_N^+ are the same as for SU_2,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

with $\dim(r_k) = \frac{q^{k+1}-q^{-k-1}}{q-q^{-1}}$, where $q^2 - Nq + 1 = 0$.

Proof. We know from easiness that we have:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in NC_2(k,l)\right)$$

Thus, the main character χ is semicircular:

$$\int_{S_N^+} \chi^{2p} = |NC_2(0,2p)| = \frac{1}{p+1}\binom{2p}{p}$$

But this gives the result, using $S^3_{\mathbb{R}} \simeq SU_2$.

Rotations and permutations 2/4

Theorem. The fusion rules for S_N^+ are the same as for SO_3,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$$

with $\dim(r_k) = \frac{q^{k+1}-q^{-k}}{q-1}$, where $q^2 - (N-2)q + 1 = 0$.

Proof. We know from easiness that we have:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC(k,l)\right)$$

Thus, the main character χ is squared-semicircular:

$$\int_{S_N^+} \chi^p = |NC(0,p)| = \frac{1}{p+1}\binom{2p}{p}$$

But this gives the result, using $S^3_{\mathbb{R}} \simeq SU_2 \to SO_3$.

Rotations and permutations 3/4

Theorem. PO_n^+ is a cocycle twist of $S_{n^2}^+$, for any $n \in \mathbb{N}$.

Theorem. Let $n \geq 2$ and $w = e^{2\pi i/n}$. Then

$$\Theta(u_{ij}u_{kl}) = \frac{1}{n}\sum_{a,b=0}^{n-1} w^{-a(k-i)+b(l-j)} p_{ia,jb}$$

is a trace-preserving coalgebra isomorphism $C(PO_n^+) \to C(S_{n^2}^+)$.

Theorem. The following algebras are isomorphic, via $u_{ij}^2 \to X_{ij}$:

(1) The algebra generated by the variables $u_{ij}^2 \in C(O_n^+)$.

(2) The algebra generated by $X_{ij} = \frac{1}{n}\sum_{a,b=1}^{n} p_{ia,jb} \in C(S_{n^2}^+)$.

Rotations and permutations 4/4

Definition. The noncommutative random variable

$$X(n,m,N)=\sum_{i=1}^n\sum_{j=1}^m u_{ij}\in C(S_N^+)$$

is called free hypergeometric, of parameters (n,m,N).

Theorem. The free hypergeometric variable

$$X_{ij}=\frac{1}{n}\sum_{a,b=1}^n u_{ia,jb}\in C(S_{n^2}^+)$$

has the same law as the variable $x_i^2\in C(S^{N-1}_{\mathbb{R},+})$.

Laplacians 1/2

The eigenspaces of the Laplacian for the free sphere $S^{N-1}_{\mathbb{R},+}$ can be constructed as in the classical case, by considering the spaces

$$H_k = span\left(x_{i_1}\ldots x_{i_r}\Big| i_1,\ldots,i_r\in\{1,\ldots,N\}, r\leq k\right)$$

and then by setting, for any $k\in\mathbb{N}$:

$$E_k = H_k \cap H_{k-1}^{\perp}$$

We obtain in this way the Laplacian filtration for $S^{N-1}_{\mathbb{R},+}$:

$$H = \bigoplus_{k=0}^{\infty} E_k$$

The "metric" QISO group, with respect to this filtration, is O_N^+.

Laplacians 2/2

There are many open questions regarding the free spheres, and other "easy spheres", and "easy manifolds" in general:

(1) Eigenvalues (Franz et al.).

(2) Dirac operator (probably no).

(3) Nash embedding questions.

(4) Nash-Connes Geometry.

Homogeneous spaces and easy manifolds

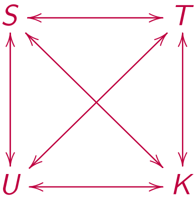
Teo Banica

"Introduction to noncommutative geometry", 6/6

07/20

Geometry

There is no free $\mathbb{R}^N$, or free $\mathbb{C}^N$. We have quadruplets (S, T, U, K) consisting of a sphere, torus, unitary group and reflection group:



Such quadruplets (S, T, U, K) can be axiomatized. Technically we need easiness, liberation, twists and the Weingarten formula.

Examples

We have seen that there are 9 main examples of geometries:

$$
\begin{array}{ccccc}
\mathbb{R}_+^N & \longrightarrow & \mathbb{T}\mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}_*^N & \longrightarrow & \mathbb{T}\mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}
$$

We must "develop" these geometries $\implies$ more manifolds.

Plan

(1) Half-classical manifolds.

(2) Quotients, partial isometries.

(3) Affine homogeneous spaces.

(4) Matrix model techniques.

Half-classical geometry 1/4

Definition. The half-classical sphere $S^{N-1}_{\mathbb{R},*}$ is defined via:

$$C(S^{N-1}_{\mathbb{R},*}) = C(S^{N-1}_{\mathbb{R},+})\Big/\Big\langle abc = cba\Big\rangle$$

The relations $abc = cba$ are called half-commutation relations.

Remark. Under suitable assumptions, this is the only sphere:

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

For instance, this is the only intermediate "monomial sphere".

Extensions. We can define T_N^*, O_N^*, H_N^* in a similar way, and under suitable assumptions, namely "easiness", we have uniqueness.

Half-classical geometry 2/4

Theorem. The sphere $S^{N-1}_{\mathbb{R},*}$ has the following properties:

(1) $PS^{N-1}_{\mathbb{R},*}$ is classical, equal to $P^{N-1}_{\mathbb{C}}$.

(2) $S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{R},+}$ appears as the affine lift of $P^{N-1}_{\mathbb{C}}$.

(3) We have a matrix model $C(S^{N-1}_{\mathbb{R},*})\subset M_2(C(S^{N-1}_{\mathbb{C}}))$.

(4) Similar results hold for the subspaces $X\subset S^{N-1}_{\mathbb{R},*}$.

Proof. (1) Here $\subset$ is clear, because $abc=aba$ implies $[ab,cd]=0$, and $\supset$ follows by using the model in (3), namely:

$$x_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

(2) and the faithfulness claim in (3) are related, and follow from some algebra. As for (4), the proof here is similar.

Half-classical geometry 3/4

Definition. The half-classical sphere $S^{N-1}_{\mathbb{C},*}$ is defined via

$$C(S^{N-1}_{\mathbb{C},*})=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle abc=cba\Big\rangle$$

with $abc=cba$ being now imposed to the variables x_i,x_i^*.

Theorem. The sphere $S^{N-1}_{\mathbb{C},*}$ has the following properties:

(1) $PS^{N-1}_{\mathbb{C},*}$ is classical, equal to $P^{N-1}_{\mathbb{C}}$.

(2) We have a model $C(S^{N-1}_{\mathbb{C},*})\subset M_2(C(S^{N-1}_{\mathbb{C}}\times S^{N-1}_{\mathbb{C}}))$.

(3) Similar results hold for the subspaces $X\subset S^{N-1}_{\mathbb{C},*}$.

Half-classical geometry 4/4

There are many other interesting examples of spheres, tori, unitary and reflection groups, between classical and free complex:

(1) Using the relations $[ab^*, cd^*] = 0$. The sphere here,

$$S^{N-1}_{\mathbb{C}} \subset S^{N-1}_{\mathbb{C},*} \subset S^{N-1}_{\mathbb{C},\times} \subset S^{N-1}_{\mathbb{C},+}$$

is the biggest one whose projective space is classical.

(2) By assuming that $\{ab^*, a^*b\}$ all commute. The sphere here,

$$S^{N-1}_{\mathbb{C}} \subset S^{N-1}_{\mathbb{C},*} \subset S^{N-1}_{\mathbb{C},**} \subset S^{N-1}_{\mathbb{C},\times} \subset S^{N-1}_{\mathbb{C},+}$$

is the biggest one whose both projective spaces are classical.

(3) And many more, cf. $U_N \subset G \subset U_N^+$ easy. Note however that the quadruplets (S, T, U, K) are excluded by our NCG axioms.

Quotient spaces 1/4

Given a quadruplet (S, T, U, K) satisfying our NCG axioms, the sphere S appears by definition as an homogeneous space over U. At the level of the algebras of functions, we have:

$$C(S) \subset C(U) \quad , \quad x_i = u_{i1}$$

The same construction with K at the place of U gives nothing interesting, because the variables $x_i = u_{i1}$ commute. Thus we always obtain the algebra $\mathbb{C}^N$, and the space $\{1, \ldots, N\}$.

$\implies$ Study the spaces $G_{NM} = G_N / G_{N-M}$, over $G = U, K$.

Quotient spaces 2/4

<u>Definition</u>. A family of compact quantum groups $G = (G_N)$, with $G_N \subset U_N^+$ for any $N \in \mathbb{N}$, is called uniform when

$$G_{N-1} = G_N \cap U_{N-1}^+$$

with respect to the standard embedding $U_{N-1}^+ \subset U_N^+$, given by:

$$u \to \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Remark 1</u>. For group duals, $G_N = \widehat{\Gamma}_N$ with $\Gamma_N =< g_1, \ldots, g_N >$, we must have $\Gamma_{N-1} = \Gamma_N / < g_N = 1 >$, projective limit situation.

<u>Remark 2</u>. Most examples are uniform. However, half-liberations are not, because $abc = cba$ with $c = 1$ gives $ab = ba$.

Quotient spaces 3/4

Theorem. For an easy quantum group $G = (G_N)$, coming from a category of partitions D, the following are equivalent:

(1) G is uniform: $G_{N-1} = G_N \cap U_{N-1}^+$ wrt. $U_{N-1}^+ \subset U_N^+$.

(2) $G_{N-1} = G_N \cap U_{N-1}^+$ wrt. all N embeddings $U_{N-1}^+ \subset U_N^+$.

(3) D is stable under removing blocks.

Application. This shows right away that the classical and free U, K are uniform, and that the half-classical ones are not.

Quotient spaces 4/4

Theorem. Given a uniform easy quantum group $G=(G_N)$, and integers $M\leq N$, consider the space $G\to G_{NM}$ given by:

$$C(G_{NM})=\left\langle u_{ij}\Big|i=1,\ldots,N,j=1,\ldots,M\right\rangle\subset C(G)$$

These spaces have then the following properties:

(1) They interpolate between S $(M=1)$ and G $(M=N)$.

(2) They are homogeneous spaces, $G_{NM}=G_N/G_{N-M}$.

(3) The uniform mesure can be computed via Weingarten.

(4) With $G=U,K$ we have BP, in the $M=[tN]\to\infty$ limit.

Proof. Here (1,2,3) go as in the sphere case, and in (4) the variables are sums of non-overlapping coordinates.

Partial isometries 1/4

(1) We have so far homogeneous spaces of the following type:

$$G_{NM} = G_N / G_{N-M}$$

(2) We will add one more parameter, $L \leq M, N$, and look at:

$$G_{NM}^L = (G_N \times G_M) \Big/ (G_{N-L} \times G_{M-L} \times G_L)$$

(3) This is a generalization ideed, because at $L = M$ we have:

$$G_{NM}^M = (G_N \times G_M) \Big/ (G_{N-M} \times G_0 \times G_M) = G_{NM}$$

(4) The spaces G_{NM}^L consist of "quantum partial isometries".

Partial isometries 2/4

Definition. Associated to any integers $L \leq M, N$ are the spaces

$$O_{NM}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{R}^M, F \subset \mathbb{R}^N, \dim_{\mathbb{R}} E = L \right\}$$

$$U_{NM}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{C}^M, F \subset \mathbb{C}^N, \dim_{\mathbb{C}} E = L \right\}$$

the notion of isometry being with respect to the usual $<,>$.

Theorem. We have identifications as follows,

$$O_{NM}^L \simeq \left\{ U \in M_{N \times M}(\mathbb{R}) \Big| U^t U = \text{projection of trace } L \right\}$$

$$U_{NM}^L \simeq \left\{ U \in M_{N \times M}(\mathbb{C}) \Big| U^* U = \text{projection of trace } L \right\}$$

by identifying partial isometries with rectangular matrices.

Partial isometries 3/4

Theorem. We have action maps as follows, which are transitive,

$$O_N \times O_M \curvearrowright O_{NM}^L \quad : \quad (A,B)U = AUB^t$$

$$U_N \times U_M \curvearrowright U_{NM}^L \quad : \quad (A,B)U = AUB^*$$

whose stabilizers are $O_{N-L} \times O_{M-L} \times O_L$, $U_{N-L} \times U_{M-L} \times U_L$.

Theorem. We have isomorphisms as follows,

$$\begin{aligned} O_{NM}^L &= (O_N \times O_M)/(O_{N-L} \times O_{M-L} \times O_L) \\ U_{NM}^L &= (U_N \times U_M)/(U_{N-L} \times U_{M-L} \times U_L) \end{aligned}$$

the quotient maps being $(A,B) \to AUB^*$, where $U = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$.

Partial isometries 4/4

Definition. Associated to any integers $L \leq M, N$ are the algebras

$$C(O_{NM}^{L+}) = C^*\left((u_{ij})_{i=1\ldots N, j=1\ldots M}\Big| u = \bar{u}, u^t u = \text{proj. trace } L\right)$$

$$C(U_{NM}^{L+}) = C^*\left((u_{ij})_{i=1\ldots N, j=1\ldots M}\Big| u^* u, u^t \bar{u} = \text{projs. trace } L\right)$$

with the trace being by definition the sum of the diagonal entries.

Theory. We have results as before, namely homogeneous space structure, Weingarten formula, and Bercovici-Pata in the

$$M = \mu N, L = \lambda N, N \to \infty$$

limit, for sums of non-overlapping coordinates. All this can be done for all our U, K groups, classical/free, and can be twisted as well.

Affine spaces 1/4

Definition. An affine homogeneous space over $G \subset U_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$, such that there exists $I \subset \{1, \ldots, N\}$ such that

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij} \quad , \quad \Phi(x_i) = \sum_j u_{ij} \otimes x_j$$

define morphisms of algebras, satisfying the ergodicity condition:

$$\left(\int_G \otimes id \right) \Phi = \int_G \alpha(.)1$$

$\implies$ The formula $(id \otimes \Phi)\Phi = (\Delta \otimes id)\Phi$ is automatic.

$\implies$ We have as well the formula $(id \otimes \alpha)\Phi = \Delta\alpha$.

Affine spaces 2/4

Theorem. When α is injective we must have $X = X_{G,I}^{min}$, where:

$$C(X_{G,I}^{min}) = \left\langle \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij} \Big| i = 1, \ldots, N \right\rangle \subset C(G)$$

Also, we must have $X \subset X_{G,I}^{max}$, as subsets of $S^{N-1}_{\mathbb{C},+}$, where

$$C(X_{G,I}^{max}) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \left\langle (Px^{\otimes k})_{i_1 \ldots i_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1 \ldots j_k \in I} P_{i_1 \ldots i_k, j_1 \ldots j_k} \right\rangle$$

with P being the orthogonal projection onto $Fix(u^{\otimes k})$.

$\implies$ In general, we have $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$, as for group algebras.

Affine spaces 3/4

Theorem. In the classical case, $G \subset U_N$, we have

$$X = G/(G \cap C_N^I)$$

where $C_N^I \subset U_N$ is the group fixing $\xi_I = \frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$.

Theorem. For group duals, $G = \widehat{\Gamma}$ with $\Gamma = < g_1, \ldots, g_N >$,

$$X = \widehat{\Gamma}_I \quad , \quad \Gamma_I = < g_i | i \in I > \subset \Gamma$$

when identifying as usual full and reduced group algebras.

Theorem. The quantum groups G_N themselves, the spaces G_{NM}, and the spaces G_{NM}^L as well, are affine homogeneous.

Affine spaces 4/4

Several questions, regarding the affine homogeneous spaces:

(1) In the easy case, we have the Bercovici-Pata bijection for sums of non-overlapping coordinates. Unify with Meixner/free Meixner.

(2) In fact, this tehnically requires the passage to product groups, $G_N \times G_N$, which are not exactly easy. New axiomatics needed.

(3) What is an easy manifold? Or a free manifold? Must get rid of G, in the definition of the AHS. Needs good Tannaka duality.

Matrix models 1/4

An idea that already appeared, in connection with half-liberation:

Definition. A matrix model for a noncommutative algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a morphism of C^*-algebras

$$\pi : C(X) \to M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

We are mostly interested in the case where T has an integration functional, and π is faithful, commuting with the traces.

Matrix models 2/4

Theorem. Let $X \subset S^{N-1}_{\mathbb{C},+}$ be algebraic, satisfying $X_{class} \neq \emptyset$. Then we have an increasing sequence of algebraic submanifolds

$$X_{class} = X^{(1)} \subset X^{(2)} \subset X^{(3)} \subset \ldots\ldots \subset X^{(\infty)} \subset X$$

with $X^{(K)}$ with $K < \infty$ being the part of X which is realizable with $K \times K$ random matrix models, and with $X^{(\infty)} = \cup_K X^{(K)}$.

Proof. Using the algebraic relations defining X, we can construct a universal $K \times K$ model space T_K, and then factorize

$$\pi_{univ} : C(X) \to M_K(C(T_K))$$

as to obtain our submanifold $X^{(K)} \subset X$. Technically, $X_{class} \neq \emptyset$ is needed. Finally, we can set $X^{(\infty)} = \cup_K X^{(K)}$, as above.

Matrix models 3/4

The available results on matrix models are as follows:

(1) $X_{half-class} \subset X^{(2)}$, cf. half-liberation.

(2) $X_{1/K-class} \subset X^{(K)}$, by cyclic extension.

(3) Hadamard models for quantum partial permutations.

(4) Many interesting results for CQG $\implies$ AHS?

Matrix models 4/4

Many interesting questions, on models, and in general:

(1) What is a free manifold? Or an easy manifold?

(2) How to unify Bercovici-Pata with Meixner/free Meixner?

(3) How to extend the CQG modelling theory to the AHS?

(4) How to unify easiness and matrix/polynomial relations?

Plus and of course, go towards Nash-Connes Geometry.

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