Hilbert spaces and operators

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"Introduction to operator algebras", 1/6

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Definition. The determinant of a system of N vectors in $\mathbb{R}^N$ is the signed volume of the associated parallelepiped

$$\det(V_1, \ldots, V_N) = \pm vol < V_1, \ldots, V_N >$$

with the sign being $+$ if you can pass from the standard basis of $\mathbb{R}^N$ to the system of vectors in V, and being $-$ otherwise.

Comment. This is the correct definition.

Calculus 2

Definition. The integral of a continuous function $f : [a, b] \to \mathbb{R}$ is the signed area below its graph. This can be computed as

$$\int_a^b f(x)dx = (b-a) \times \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

where in order to compute the average height of the function, the points $x_1, \ldots, x_N \in [a, b]$ are chosen at random.

Comment. This is the best definition.

Linear algebra 1

Definition. We say that $M \in M_N(\mathbb{C})$ has eigenvector $v \in \mathbb{C}^N$ with eigenvalue $\lambda \in \mathbb{C}$ when M dilates by λ in the v direction:

$$Mv = \lambda v$$

When M has a basis of eigenvectors $\{v_i\}$, we call it diagonalizable, and we write $M = PDP^{-1}$, with $D = diag(v_i)$.

Examples. A diagonalizable matrix, and a non-diagonalizable one:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This follows indeed from $M\binom{x}{y} = \binom{y}{x}$, and from $P\binom{x}{y} = \binom{y}{0}$.

Linear algebra 2

Theorem. The following happen:

(1) The eigenvalues of M are the roots of $f(x) = \det(M - x1_N)$.

(2) If we have N distinct eigenvalues, M is diagonalizable.

(3) If M is normal, $MM^* = M^*M$, it is diagonalizable.

(4) The diagonalizable matrices are dense inside $M_N(\mathbb{C})$.

Proof. The idea is as follows:

(1) Follows from the eigenvalue equation, $(M - \lambda 1_N)v = 0$.

(2) Vectors with different eigenvalues are linearly independent.

(3) Generalizes "$M \in M_N(\mathbb{R})$ symmetric $\implies$ diagonalizable".

(4) Because the matrices with distinct eigenvalues are dense.

Rotations 1

Definition. A scalar product $<,>$ on $\mathbb{C}^N$ must satisfy:

(1) $< x, y >$ is linear in x, antilinear in y.
(2) $\overline{< x, y >} =< y, x >$, for any x, y.
(3) $< x, x >> 0$, for any $x \neq 0$.

Examples. The usual scalar product, $< x, y >= \sum_i x_i \bar{y}_i$, that we can further complicate by adding weights, and so on.

Theorem. For a matrix $U \in M_N(\mathbb{C})$, the following are equivalent:

(1) U preserves the scalar product, $< Ux, Uy >=< x, y >$.
(2) U preserves the norm, $||Ux|| = ||x||$, where $||x|| = \sqrt{< x, x >}$.
(3) U is unitary, in the sense that $U^* = U^{-1}$, where $(U^*)_{ij} = \bar{U}_{ji}$.

Proof. All this follows from $< Mx, y >=< x, M^* y >$.

Rotations 2

Theorem 1. The unitaries in $M_2(\mathbb{C})$ of determinant 1 are

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$.

Theorem 2. The unitaries in $M_3(\mathbb{R})$ of determinant 1 are

$$U = \begin{pmatrix} x^2+y^2-z^2-t^2 & 2(yz-xt) & 2(xz+yt) \\ 2(xt+yz) & x^2+z^2-y^2-t^2 & 2(zt-xy) \\ 2(yt-xz) & 2(xy+zt) & x^2+t^2-y^2-z^2 \end{pmatrix}$$

with $x, y, z, t \in \mathbb{R}$ satisfying $x^2 + y^2 + z^2 + t^2 = 1$.

Proofs. 1 follows from $U^* = U^{-1}$, and 2 follows from 1.

Hilbert spaces

<u>Definition</u>. A Hilbert space is a complex vector space H, typically infinite dimensional, with a scalar product $< x, y >$, satisfying:

(1) $< x, y >$ is linear in x, antilinear in y.
(2) $\overline{< x, y >} = < y, x >$, for any x, y.
(3) $< x, x > > 0$, for any $x \neq 0$.
(4) H is complete with respect to $||x|| = \sqrt{< x, x >}$.

<u>Remark</u>. Here (4) is based on Cauchy-Schwarz. Basic examples:

(1) $H = \mathbb{C}^N$, with $< x, y > = \sum_i x_i \bar{y}_i$.
(2) $H = l^2(\mathbb{N})$, with $< x, y > = \sum_i x_i \bar{y}_i$.
(3) $H = L^2(X)$, with $< f, g > = \int_X f(x)\overline{g(x)}dx$.

Gram-Schmidt

Theorem. Any basis $\{f_i\}_{i \in I}$, in a dense sense, can be turned into an orthonormal basis $\{e_i\}_{i \in I}$, by using the Gram-Schmidt procedure.

Theorem. Any Hilbert space has an orthonormal basis $\{e_i\}_{i \in I}$. In other words, we have $H \simeq l^2(I)$, for some index set I.

Definition. When I is countable, and usually not finite, H is called separable. This is the same as saying that $H \simeq l^2(\mathbb{N})$.

Example. The space $H = L^2[0,1]$ is separable, because we can apply Gram-Schmidt to the basis $f_i = x^i$, with $i \in \mathbb{N}$.

Linear operators

Theorem. Let H be a Hilbert space, with basis $\{e_i\}_{i\in I}$. We have

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

with $T : H \to H$ linear corresponding to the following matrix:

$$M_{ij} = < Te_j, e_i >$$

– When $\dim(H) = N < \infty$, we obtain $\mathcal{L}(H) \simeq M_N(\mathbb{C})$.
– In the infinite separable case, we obtain $\mathcal{L}(H) \subset M_\infty(\mathbb{C})$.

Proof. The correspondence $T \to M$ is indeed linear and injective.

Comment. However, $H = L^2[0,1]$ suggests not to use all this.

Bounded operators

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, which:

(1) Is complete with respect to $||.||$ (Banach algebra).

(2) Has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Proof. Complex algebra is clear, given $\{T_n\}$ Cauchy we can set $Tx = \lim_{n\to\infty} T_n x$, the involution comes from $\varphi(x) =< Tx, y >$ which is linear, and $||TT^*|| = ||T||^2$ is by double inequality.

Operator algebras

Definition. A C^*-algebra is an algebra $A \subset B(H)$, which:

(1) Is norm closed: $T_n \in A$, $T_n \to T \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Definition. A von Neumann algebra is an algebra $A \subset B(H)$, which:

(1) Is weakly closed: $T_n \in A$, $T_n x \to Tx, \forall x \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Examples. We have $C(X)$ and $L^\infty(X)$, acting by multiplication on $L^2(X)$. We will see that when $T \in B(H)$ is normal, the algebras that it generates are of this form ("Spectral theorem").