Basic spectral theory

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Framework

Definition. An abstract C^*-algebra is a complex algebra A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

– We know that $B(H)$ is a C^*-algebra in the above sense.

– And so are all the norm closed $*$-subalgebras $A \subset B(H)$.

– We'll see later that any abstract C^*-algebra is of this form.

$\implies$ However, useful formalism, because we can construct many examples of C^*-algebras with generators and relations.

Spectra

Definition. The spectrum of an element $a \in A$ is the set

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1} \right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

Remark. For the usual matrices we obtain the eigenvalues,

$$\begin{aligned} Mv = \lambda v \quad & \iff \quad (M - \lambda)v = 0 \\ & \iff \quad M - \lambda \notin M_N(\mathbb{C})^{-1} \end{aligned}$$

or rather the eigenvalue set, with no multiplicities.

Basics

Theorem. The spectrum of any element $a \in A$ is:

(1) A compact subset of $\mathbb{C}$.

(2) Contained in the disk $D(0, ||a||)$.

Proof. The spectrum of a norm 1 element is in the unit disk. This comes from the following formula, valid for any $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

But this gives (2) by dilation, and shows as well that A^{-1} is open, and so that $\sigma(a) \subset \mathbb{C}$ is closed, and so we get (1) as well.

Nonzero

Theorem. The spectrum of any element $a \in A$ is non-empty.

Proof. Assume $\sigma(a) = \emptyset$. Pick a linear form $\varphi \in A^*$ and set:

$$f(\lambda) = \varphi\left(\frac{1}{\lambda - a}\right)$$

Then f is differentiable, so holomorphic. For $\lambda >> 0$ we have

$$\begin{aligned} \left|\left|\frac{1}{\lambda - a}\right|\right| &= \frac{1}{|\lambda|} \times \left|\left|1 + \frac{a}{\lambda} + \frac{a^2}{\lambda^2} + \ldots\right|\right| \\ &\leq \frac{1}{|\lambda| - ||a||} \end{aligned}$$

so $f(\lambda) \to 0$ with $\lambda \to \infty$. By Liouville $f = 0$, contradiction.

Products

Theorem. For any two elements $a, b \in A$ we have

$$\sigma(ab) = \sigma(ba)$$

outside $\{0\}$. Non-equality at 0 can happen.

Proof. For the equality, by dilation it is enough to prove that $\sigma(ab) = \sigma(ba)$ at $\lambda = 1$. But this follows from:

$$c = (1 - ab)^{-1} \implies 1 + cba = (1 - ba)^{-1}$$

Consider the shift $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $\delta_i \to \delta_{i+1}$. Then

$$S^*S = 1 \quad , \quad SS^* = Proj_{e_0^\perp}$$

and so $0 \notin \sigma(S^*S)$, but $0 \in \sigma(SS^*)$.

Rational functions

Definition. Given $a \in A$, and a rational function $f = P/Q$ having poles outside $\sigma(a)$, we set $f(a) = P(a)Q(a)^{-1}$. We write:

$$f(a) = \frac{P(a)}{Q(a)}$$

Theorem. We have the "rational functional calculus" formula

$$\sigma(f(a)) = f(\sigma(a))$$

valid for any $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.

Proof

<u>Case $f \in \mathbb{C}[X]$</u>. With $f(X) - \lambda = c(X - r_1) \dots (X - r_n)$:

$$
\begin{aligned}
\lambda \notin \sigma(f(a)) \quad & \iff \quad c(a - r_1) \dots (a - r_n) \in A^{-1} \\
& \iff \quad a - r_1, \dots, a - r_n \in A^{-1} \\
& \iff \quad r_1, \dots, r_n \notin \sigma(a) \\
& \iff \quad \lambda \notin f(\sigma(a))
\end{aligned}
$$

<u>Case $f \in \mathbb{C}(X)$</u>. With $f = P/Q$ and $F = P - \lambda Q$:

$$
\begin{aligned}
\lambda \in \sigma(f(a)) \quad & \iff \quad 0 \in \sigma(F(a)) \\
& \iff \quad 0 \in F(\sigma(a)) \\
& \iff \quad \exists \mu \in \sigma(a), F(\mu) = 0 \\
& \iff \quad \lambda \in f(\sigma(a))
\end{aligned}
$$

Unitaries

Theorem. The spectrum of a unitary element,

$$a^* = a^{-1}$$

is on the unit circle $\mathbb{T} \subset \mathbb{C}$.

Proof. This follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

Thus $\sigma(a)$ consists of numbers satisfying $\lambda^{-1} = \bar{\lambda}$.

Self-adjoints

Theorem. The spectrum of a self-adjoint element,

$$a = a^*$$

consists of real numbers.

Proof. This follows by using $f(z) = (z + it)/(z - it)$, with $t \in \mathbb{R}$. Indeed, for $t >> 0$ the element $f(a)$ is well-defined, and:

$$\left(\frac{a+it}{a-it}\right)^* = \frac{a-it}{a+it} = \left(\frac{a+it}{a-it}\right)^{-1}$$

Thus $f(a)$ is unitary, with spectrum contained in $\mathbb{T}$. We conclude that $f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$, and so $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$.

Spectral radius 1/2

Definition. Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. The spectral radius of a normal element,

$$aa^* = a^*a$$

equals its norm.

Proof. We already know that $\rho(a) \leq ||a||$, for any $a \in A$.

Spectral radius 2/2

For the converse, if we fix $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

(1) In the case $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of the form $n = 2^k$, and by taking n-th roots we get $\rho \geq ||a||$.

(2) In general we have $a^n(a^n)^* = (aa^*)^n$, so $\rho(a)^2 = \rho(aa^*)$. Now since aa^* is self-adjoint, $\rho(aa^*) = ||a||^2$, and we are done.